Local duality for Banach spaces

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Abstract

A local dual of a Banach space X is a subspace of the dual X^* which can replace the whole dual space when dealing with finite dimensional subspaces. This notion arose as a development of the principle of local reflexivity, and it is very useful when a description of X^* is not available.

We give an exposition of the theory of local duality for Banach spaces, including the main properties, examples and applications, and comparing the notion of local dual with some other weaker properties of the subspaces of the dual of a Banach space.

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1. Introduction

In Banach space theory, it is usual to describe the properties of a given Banach space X in terms of its dual space X^*, but many times a representation of the dual space is not available. This is the case for the space $L_\infty(\mu, X)$ of the essentially bounded, measurable, X-valued functions. However, $L_\infty(\mu, X)^*$ contains a natural copy of $L_1(\mu, X^*)$ which can replace the whole dual space for many purposes, like norming the elements of $L_\infty(\mu, X)$, or representing the duality on finite dimensional subspaces of $L_\infty(\mu, X)$. Finding those concrete subspaces of the dual space and describing their properties is the aim of the theory of local duality for Banach spaces.

As far as we know, the investigation in local duality began with the *principle of local reflexivity* (P.L.R. for short) which establishes that, when working with finite dimensional subspaces, it is possible to replace the bidual X^{**} by the original space X. The first version of that principle was obtained by Johnson, Rosenthal and Zippin [43], and the current form is due to Lindenstrauss and Rosenthal [51]. The P.L.R. has found many applications in Banach space theory. For example, it implies the existence of a basis for all separable $\mathcal{L}_p$-spaces [51] and for spaces whose dual has a basis [43]. It has also been applied in approximation theory [58], in the study of local complementation of tensor products [10] and in the study of the L-structure of $L_1(\mu)$ [38, A.6]. For an illuminating account of applications of the P.L.R., we suggest [18, Chapter 8], especially its final comments.

Proofs of this principle are given in [4], [15], [54] and [68]. Further operator versions can be found in [7] (see also [6]), [8], [55], and [61]. Moreover, the P.L.R. has been also translated to different contexts like that of Banach lattices [9], modules [14], duality of cones [24], operator ideals [60], spaces of compact operators [49] and other non-commutative settings ([22], [42], [62] and [66]).

A closed subspace Z of X^* is a *local dual* of a Banach space X if for every $\varepsilon > 0$ and every pair of finite dimensional subspaces E of X^* and F of X, there exists an operator $L\colon E \longrightarrow Z$ satisfying the following conditions:

(A) $\big|\|L(x^*)\| - \|x^*\|\big| \leq \varepsilon\|x^*\|$ for all $x^* \in E$,
(B) $L(x^*)|_F = x^*|_F$ for all $x^* \in E$,
(C) $L(x^*) = x^*$ for all $x \in E \cap Z$.

The principle of local reflexivity exactly says that X, as a subspace of X^{**}, is a local dual of X^*. A similar result, independently obtained in [46] and [70], is the *principle of local reflexivity for ultrapowers* which says that for every ultrafilter $\mathfrak{U}$, the ultrapower $(X^*)_{\mathfrak{U}}$ is a local dual of the ultrapower $X_{\mathfrak{U}}$. Note that $X = X^{**}$ if and only if X is reflexive, and $(X^*)_{\mathfrak{U}} = (X_{\mathfrak{U}})^*$ for all $\mathfrak{U}$ if and only if X is superreflexive. Further proofs and variations on the P.L.R. for ultrapowers can be found in [5], [40] and [47]; certain ultrapower operator versions can be found in [56] and [20].

The notion of local duality was inspired by both principles of local reflexivity, and has been developed in several papers like [16], [26], [32], [34], [35], [36], [37] and [67]. For a brief account of these developments we refer to [29].

The purpose of this paper is to present a detailed exposition of the theory of local dual spaces, including many examples and some applications, and emphasizing the relation between the three conditions (A), (B) and (C) that define the concept. In order to do that, we will make a parallel study of several weaker properties that a subspace Z of X^* may have: Z *norming* for X, Z *locally 1-complemented* (or ideal) in X^*, and X^* *finite dual representable* in Z. For most of the results, we include proofs that are simpler and more natural than those given in the original papers.

The paper is organized as follows. Section 2 includes some technical results that are useful to work with ε-isometries, a characterization of the norming subspaces of the dual of a Banach space, and some consequences of the Hahn-Banach separation theorem that will be applied in the proof of the principles of local reflexivity. In Section 3, we introduce the concept of local duality and the other auxiliary concepts in terms of exact conditions, and give characterizations in terms of approximate conditions. Since the P.L.R. and the P.L.R. for ultrapowers provide important examples of local dual spaces, we give a proof of both results. We also present here several concrete examples of local dual spaces, and give other examples that allow us to distinguish local duality from the weaker properties. In particular, we show that X^* finite dual representable in an ideal Z does not imply that Z is a local dual of X.

The local characterizations of Section 3 are applied in Section 4 to obtain characterizations in terms of ultrapowers, and in turn these ultrapower characterizations are applied in Section 5 to obtain characterizations of global character of the properties of subspaces of a dual space that we consider.

In general, the local characterizations are more effective than the global ones in the task of identifying examples, while the global characterizations are more efficient in finding counterexamples and in understanding the general properties of the concepts.

One important consequence of the global characterizations (Theorem 5.8) reveals a hidden symmetry in the concept of local duality: Z is a local dual of X if and only if Z^* contains a copy of X which is a local dual of Z. This result is a source of examples of local dual spaces and puts in context some natural dualities, like that between $C[0,1]$ and $L_1[0,1]$. Another consequence is the existence of the smallest local dual space for any space that does not contain copies of ℓ_1. Moreover, it is proved that a dual space X^* has a smallest local dual if it has a smallest norming subspace, in which case both smallest subspaces coincide with the only predual space of X^*. Besides, it is shown that every subspace Y of X^* is contained in a local dual Z_Y of X whose density character equals the maximum of those of Y and X.

Section 6 includes technical characterizations for finite dual representability and local duality, respectively called *polar property* and *strict polar property.* Both characterizations are of local character, and are useful in situations (like in the study of the principles of local reflexivity) where the ε-isometries satisfying (A), (B) and (C) cannot be constructed, but only shown to exist. Moreover, it is proved that if Z is a local dual of X^* then the conditions (A), (B) and (C) hold for every $\varepsilon > 0$, every finite dimensional subspace E of X^* and every reflexive subspace F of X.

A consequence of the strict polar property is that given an ultrafilter $\mathfrak{U}$ and an operator $T\colon X \longrightarrow Y$, the kernel of $(T^*)_{\mathfrak{U}}$ is a local dual of the cokernel of $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$. This result generalizes the principle of local reflexivity for ultrapowers. Besides, it entails that the kernel of $(T_{\mathfrak{U}})^*$ is finite dual representable in that of $(T^*)_{\mathfrak{U}}$, allowing to derive some duality results for operator semigroups associated to certain operator ideals. Of course, the strict polar property also produces a proof of the principle of local reflexivity.

Let X and Z be a pair of Banach spaces. Recall that if the dual of X contains an isometric copy of Z which is a local dual of X, then the dual of Z contains an isometric copy of X which is a local dual of Z (Theorem 5.8); i.e., the relation *being a local dual* is symmetric.

The following table contains the main examples of pairs of Banach spaces in local duality that appear in the paper.

$(X^*)_{\mathfrak{U}}$	$X_{\mathfrak{U}}$	Thm. 3.8
ℓ_1	$C[0,1]$	Ex. 3.11
$C(\mathbb{T})$	$L_1(\mathbb{T})$	Prop. 3.15
$L_1[0,1]$	$C[0,1]$	Prop. 3.15
$L_1(\mu, X^*)$	$L_\infty(\mu, X)$	Thm. 3.17
$L_\infty(\mu, X^*)$	$L_1(\mu, X)$	Thm. 3.17
$B_\alpha[0,1]$ with $1 \leq \alpha \leq \omega_1$	$\mathcal{M}[0,1]$	Thm. 3.18
$\ell_1(X^*)$	$\ell_\infty(X)$	Prop. 5.9
$C(\Delta)$	$L_1[0,1]$	Ex. 5.12
$X^* \otimes_\varepsilon Y^*$ (X^* with M.A.P.)	$X \otimes_\pi Y$	Prop. 5.13
$X^* \otimes_\pi Y^*$ (X^* with M.A.P.)	$X \otimes_\varepsilon Y$	Prop. 5.13
$X^* \otimes_\varepsilon L_\infty(\mu)$ (μ finite)	$L_1(\mu, X)$	Remark 5.14
$X^* \otimes_\pi \mathcal{M}(K)$ (K compact)	$C(K, X)$	Remark 5.14

We will use standard notation like in [2] and [52]. An *operator* will be a continuous linear map between Banach spaces, and B_X and S_X will be the unit ball and the unit sphere of the Banach space X. The class of all bounded operators acting between X and Y is denoted $\mathcal{B}(X, Y)$. We will introduce additional notation along the paper.

Since ultraproduct techniques play an important role in this paper, we recall some notions about ultrapowers of Banach spaces and introduce some notations (see [40] for additional details). Let I be an infinite set and let $\mathfrak{U}$ be a collection of subsets of I satisfying the following properties:

(i) $\emptyset \notin \mathfrak{U}$,
(ii) if $A \in \mathfrak{U}$ and $B \in \mathfrak{U}$ then $A \cap B \in \mathfrak{U}$,
(iii) if $A \in \mathfrak{U}$ and $A \subset B \subset I$ then $B \in \mathfrak{U}$,
(iv) for every $A \subset I$ either $A \in \mathfrak{U}$ or $I \setminus A \in \mathfrak{U}$;

If the properties (i), (ii) and (iii) are satisfied, $\mathfrak{U}$ is said to be a *filter*, and if moreover (iv) holds, $\mathfrak{U}$ is said to be an *ultrafilter*. The Axiom of Choice implies that every filter is contained by an ultrafilter.

The collection $\mathfrak{F}$ of all co-finite subsets of I is called the *Fréchet filter*. Any ultrafilter extending $\mathfrak{F}$ is called a *non-trivial* ultrafilter on I. An ultrafilter $\mathfrak{U}$ is said to be *principal* or trivial if there exists $i \in I$ such that $i \in A$ for all $A \in \mathfrak{U}$. Obviously, every ultrafilter on I is either principal or non-trivial.

The ultrafilter $\mathfrak{U}$ is said to be *countably incomplete* if there exists a countable partition $\{I_n\}_{n=1}^{\infty}$ of I disjoint with $\mathfrak{U}$. Henceforth, all ultrafilters in this paper are countably incomplete. If I is endowed with a partial order $\preceq$, the *order filter* on I is the family of all subsets $\{i \in I : j \preceq i\}$.

Given a topological space S, a family $(x_i)_{i\in I}$ of elements of S converges to $x_0 \in X$ following $\mathfrak{U}$ if for every neighborhood $\mathcal{V}$ of x, the set $\{i \in I : x_i \in \mathcal{V}\}$ belongs to $\mathfrak{U}$; it is denoted $\lim_{\mathfrak{U}} x_i = x_0$ or $x_i \underset{\mathfrak{U}}{\longrightarrow} x_0$. A subset K of S is relatively compact only if each family $(x_i)_{i\in I}$ contained in K is convergent following $\mathfrak{U}$.

The ultrapower of a Banach space X following an ultrafilter $\mathfrak{U}$ on I is the quotient space $X_{\mathfrak{U}} := \ell_\infty(I, X)/N_{\mathfrak{U}}(X)$ where $\ell_\infty(I, X)$ is the set of bounded families $(x_i)_{i\in I} \subset X$ and $N_{\mathfrak{U}}(X)$ is the subspace of families that converge to zero following $\mathfrak{U}$. The element of $X_{\mathfrak{U}}$ with representative $(x_i)_{i\in I}$ is usually denoted $[x_i]_i$ or $[x_i]$ for short; its norm is $\|[x_i]\| = \lim_{\mathfrak{U}} \|x_i\|$. The space X is isometrically contained in $X_{\mathfrak{U}}$ by means of the canonical embedding $J_{X,X_{\mathfrak{U}}}$ that maps each x to the constant class $[x]$. Note that if X is finite dimensional then $J_{X,X_{\mathfrak{U}}}$ identifies X with $X_{\mathfrak{U}}$ isometrically. The *ultraproduct following* $\mathfrak{U}$ of a family $\{C_i\}_{i\in I}$ of subsets of a Banach space X is the subset $(C_i)_{\mathfrak{U}}$ of $X_{\mathfrak{U}}$ formed by all elements $\mathbf{x}$ for which there exists $J_{\mathbf{x}} \in \mathfrak{U}$ and a representative $(x_i)_{i\in I}$ such that $x_i \in C_i$ for all $i \in J_{\mathbf{x}}$; if there is $J \in \mathfrak{U}$ such that $C_i = \emptyset$ then $(C_i)_{\mathfrak{U}} = \emptyset$. For instance, for every Banach space X, $B_{X_{\mathfrak{U}}} = (B_X)_{\mathfrak{U}}$.

The ultrapower $(X^*)_{\mathfrak{U}}$ is isometrically contained in the dual space $(X_{\mathfrak{U}})^*$ by means of the embedding $J_{X^*{}_{\mathfrak{U}},X_{\mathfrak{U}}{}^*}$ that maps each $[x_i^*]$ to $\mathbf{x}^* \in (X_{\mathfrak{U}})^*$ defined by $\langle \mathbf{x}^*, [x_i]\rangle = \lim_{\mathfrak{U}} \langle x_i^*, x_i\rangle$. Usually we will omit the canonical operators $J_{X,X_{\mathfrak{U}}}$ and $J_{X^*{}_{\mathfrak{U}},X_{\mathfrak{U}}{}^*}$; so we will write $x = [x]$, and we will understand that $[x_i^*] \in (X_{\mathfrak{U}})^*$.

The ultraproduct following $\mathfrak{U}$ of a uniformly bounded collection of operators $(T_i)_{i\in I}$ acting between X and Y is the operator $(T_i)_{\mathfrak{U}} \colon X_{\mathfrak{U}} \longrightarrow Y_{\mathfrak{U}}$ that maps each $[x_i]$ to $[T_i(x_i)]$. An operator $T \colon X \longrightarrow Y$ is extended by its ultrapower $T_{\mathfrak{U}} \colon X_{\mathfrak{U}} \longrightarrow Y_{\mathfrak{U}}$ following $\mathfrak{U}$, which is defined as $T_{\mathfrak{U}} = (T)_{\mathfrak{U}}$ and its norm equals $\|T\|$.

2. Preliminaries

Given a real number $\varepsilon > 0$, an operator $T \colon X \longrightarrow Y$ is said to be an ε-*isometry* if $1 - \varepsilon \leq \|T(x)\| \leq 1 + \varepsilon$ for all $x \in S_X$. Let us recall that a space X is said to be *finitely representable* in Y if for every finite dimensional subspace E of X and every $\varepsilon > 0$ there exists an ε-isometry $L \colon E \longrightarrow Y$.

The following results are useful to work with ε-isometries. The first two of them show that in order to estimate the norm of an operator $L \colon E \longrightarrow X$, it is enough to know the value of the norms of $L(x_i)$ for a suitable net $\{x_i\}_{i\in I}$ in S_E.

Lemma 2.1. *Let X be a Banach space and take any α-net $\{x_i\}_{i\in I}$ in B_X with $0 < \alpha < 1$. Thus for every $x \in S_X$, there is a sequence $(x_{i_n})_{n=1}^{\infty}$ in the net and a scalar sequence $(\lambda_n)_{n=1}^{\infty}$ such that, for every positive integer n,*

(i) $0 \leq \lambda_n \leq \alpha^{n-1}$, *and*

(ii) $\|x - \sum_{m=1}^{n} \lambda_m x_{i_m}\| < \alpha^n$.

Its proof can be found in [38, Lemma A.4.8].

Lemma 2.2. *Let $L\colon E \longrightarrow X$ be an operator, let $\{x_i\}_{i\in I}$ be an α-net in S_E with $0 < \alpha < 1$ and consider a real number $\delta > 0$:*

(i) if $\|L(x_i)\| \le 1 + \delta$ for all $i \in I$ then $\|L\| \le 1 + \dfrac{\alpha+\delta}{1-\alpha}$.

(ii) if $1-\delta \le \|L(x_i)\| \le 1+\delta$ for all $i \in I$ then L is a $(\alpha+\delta)(1-\alpha)^{-1}$-isometry.

Proof. (i) Let $x \in S_E$. By Lemma 2.1, there are a scalar sequence $(\lambda_n)_{n=1}^{\infty}$ and a sequence $(x_{i_n})_{n=1}^{\infty}$ in the net $\{x_i\}_{i\in I}$ such that $x = \sum_{n=1}^{\infty} \lambda_n x_{i_n}$ and $0 \le \lambda_n \le \alpha^{n-1}$. Thus, as L is bounded,

$$\|L(x)\| \le \sum_{n=1}^{\infty} \lambda_n \|L(x_{i_n})\| \le \frac{1+\delta}{1-\alpha} = 1 + \frac{\alpha+\delta}{1-\alpha}. \tag{1}$$

(ii) By *(i)*, we only need to prove $\|L(x)\| \ge 1 - (\alpha+\delta)(1-\alpha)^{-1}$ for any $x \in S_E$. To do so, given $x \in S_E$, we choose x_j in the net so that $\|x - x_j\| < \alpha$. Thus, by (1),

$$\|L(x)\| \ge \|L(x_j)\| - \|L\|\cdot\|x - x_j\| \ge 1 - \delta - \frac{1+\delta}{1-\alpha}\alpha = 1 - \frac{\alpha+\delta}{1-\alpha}$$

and the proof is done. □

Lemma 2.3. *Let E be a finite dimensional subspace of X^* and let $\varepsilon > 0$. Then there exists a finite dimensional subspace F of X such that $(1-\varepsilon)\|x^*\| \le \sup_{x\in S_F} \langle x^*, x\rangle$ for all $x^* \in E$.*

Proof. Since E is finite dimensional, given $0 < \delta := \varepsilon/3$, S_E contains a finite δ-net $\{x_i^*\}_{i=1}^{n}$. For every x_i^*, let $x_i \in S_X$ so that $1 - \delta \le \langle x_i^*, x_i\rangle$. Consider any norm one element $x^* \in E$. Selecting an element x_i^* of the net so that $\|x^* - x_i^*\| \le \delta$, we get

$$\begin{aligned} 1 = \|x^*\| &\le \|x_i^*\| + \|x^* - x_i^*\| \le 2\delta + \langle x_i^*, x_i\rangle \\ &\le 2\delta + \langle x^*, x_i\rangle + \langle x_i^* - x^*, x_i\rangle \le 3\delta + \langle x^*, x_i\rangle \end{aligned}$$

Thus $1 - \varepsilon \le \langle x^*, x_i\rangle$, and for $F := \operatorname{span}\{x_i\}_{i=1}^{n} \subset X$, we have $1 - \varepsilon \le \sup_{x\in S_F} \langle x^*, x\rangle$, which proves the lemma. □

Lemma 2.4. *Let E be a n-dimensional Banach space and let E_0 be a subspace of E with $\dim E_0 = m$. Then there exists a biorthogonal system $(x_i, f_i)_{i=1}^{n}$ of $E \times E^*$ with $E_0 = \operatorname{span}\{x_i\}_{i=1}^{m}$, $\|x_i\| = 1$ for all $1 \le i \le n$, $\|f_i\| \le 1$ for all $1 \le i \le m$ and $\|f_i\| \le 1 + m$ for all $m + 1 \le i \le n$.*

Proof. By Auerbach's lemma [52, 1.c.3], there exists a biorthogonal system $(x_i, g_i)_{i=1}^m$ in E_0 with $\|x_i\| = \|g_i\| = 1$ for all $1 \le i \le m$. For every g_i, consider a Hahn-Banach extension $f_i \in E^*$. Let $P\colon E \longrightarrow E$ be the projection that maps each x to $\sum_{i=1}^m \langle f_i, x\rangle x_i$. Note that $\|P\| \le m$, $E_0 = R(P)$ and $E_1 := N(P) = \cap_{i=1}^m N(f_i)$. Thus $E = E_0 \oplus E_1$. Again by Auerbach's lemma, there is a biorthonormal system $(x_i, g_i)_{i=m+1}^n$ in $E_1 \times E_1^*$. Consider the functionals $f_i := g_i \circ (I_E - P)$ for $i = m+1, \ldots, n$. It follows immediately that $(x_i, f_i)_{i=1}^n$ is a biorthogonal system of $E \times E^*$ satisfying the required conditions. □

Given a Banach space X, a subspace Z of X^* is said to be *norming* if for every $x \in X$, $\|x\| = \sup_{x^* \in B_Z} \langle x^*, x\rangle$. Two important examples of norming subspaces concerning local duality are the following ones:

Lemma 2.5. *Given a Banach space X and an ultrafilter $\mathfrak{U}$ on I, the following statements hold:*

(i) the natural copy of X in X^{**} is a norming subspace;
(ii) the natural copy of $(X^*)_{\mathfrak{U}}$ in $(X_{\mathfrak{U}})^*$ is a norming subspace.

Proof. Part (i) is straightforward. For part (ii), given a norm one element $\mathbf{x} \in X_{\mathfrak{U}}$, choose a family $(x_i)_{i\in I} \subset S_X$ such that $[x_i] = \mathbf{x}$, and for every x_i, take a norm one functional $x_i^* \in X^*$ such that $\langle x_i^*, x_i\rangle = 1$. Clearly, $\mathbf{z}^* := [x_i^*] \in S_{(X^*)_{\mathfrak{U}}}$ and $\langle \mathbf{z}^*, \mathbf{x}\rangle = \lim_{\mathfrak{U}} \langle x_i^*, x_i\rangle = 1$. □

Lemma 2.6. *[36] A closed subspace Z of X^* is norming if and only if for every finite dimensional subspace F of X and every $\varepsilon > 0$ there exists a normalized basis $\{x_i\}_{i=1}^n$ of F and a set $\{x_i^*\}_{i=1}^n$ in $(1+\varepsilon)B_Z$ such that $\langle x_i^*, x_j\rangle = \delta_{ij}$ for all i and j.*

Proof. The 'if' part is trivial. For the 'only if' part, assume Z is a norming subspace of X^*. By Auerbach's lemma [52, 1.c.3], there exists a pair of subsets $\{x_i\}_{i=1}^n \subset S_F$ and $\{f_i\}_{i=1}^n \subset S_{F^*}$ such that $\langle f_i, x_j\rangle = \delta_{ij}$ for all i and j. Let $J\colon F \longrightarrow X$ be the embedding of F into X. Note that $J^*(B_{X^*}) = B_{F^*}$. Thus, since Z is a norming subspace of X^*, B_Z is w^*-dense in B_{X^*}, so $\overline{J^*(B_Z)}^{w^*} = B_{F^*}$, and as $F^* \equiv X^*/F^\perp$ is finite dimensional, it follows that $J^*(B_Z)$ is norm-dense in B_{F^*}. Thus, Lemma 2.1 let us take a family $\{x_i^*\}_{i=1}^n$ in $(1+\varepsilon)B_Z$ such that $J^* x_i^* = f_i$.

Since it is clear that $\langle x_i^*, x_j\rangle = \langle x_i^*, Jx_j\rangle = \langle f_i, x_j\rangle = \delta_{ij}$, the proof is complete. □

An application of Lemma 2.5 yields the two following separation lemmas.

Lemma 2.7. *Given an operator $T\colon X \longrightarrow Y$, a vector $y \in Y$ and a real number $\eta > 1$, the set $T^{-1}(\eta B_Y + y)$ is $\sigma(X^*, X)$-dense in ${T^{**}}^{-1}(B_{Y^{**}} + y)$.*

Proof. Assume there exists $x^{**} \in {T^{**}}^{-1}(B_{Y^{**}} + y)$. Then $L := T^{-1}(\eta B_Y + y)$ is not empty. Otherwise, we would have $\eta < \|T(x) - y\|$ for all $x \in X$ and, by the Hahn-Banach theorem, there would be a norm one functional $y^* \in Y^*$ such that

$$\eta \leq \langle y^*, T(x) - y\rangle \text{ for all } x \in X,$$

and as $T^{**}(x^{**}) \in \overline{R(T)}^{w^*}$,

$$\eta \leq \langle T^{**}(x^{**}) - y, y^*\rangle \leq \|T^{**}(x^{**}) - y\|,$$

a contradiction.

Once it has been proved that $L \neq \emptyset$, assume $x^{**} \notin \overline{L}^{w^*}$. The Hahn-Banach theorem gives a norm one functional $x^* \in X^*$ and real numbers a and b such that

$$\langle x^{**}, x^*\rangle < a < b < \inf \langle x^*, \overline{L}^{w^*}\rangle.$$

Let $W := \{x \in X \colon \langle x^*, x\rangle < a\}$. Obviously, $W \cap L = \emptyset$, so $\|T(w) - y\| > \eta$ for all $w \in W$. Therefore, there exists a norm one functional $y^* \in Y^*$ such that

$$\eta \leq \langle y^*, T(w) - y\rangle \text{ for all } w \in W. \tag{2}$$

But $x^{**} \in \overline{W}^{w^*}$, so formula (2) leads to

$$\eta \leq \langle T^{**}(x^{**}) - y, y^*\rangle \leq \|T^{**}(x^{**}) - y\|$$

in contradiction with $x^{**} \in {T^{**}}^{-1}(B_{X^{**}} + y)$. □

Note that Lemma 2.7 may fail if the value $\eta > 1$ is replaced for $\eta = 1$. Indeed, given the operator $T \colon c_0 \longrightarrow \ell_\infty$ that maps each $x \in c_0$ to itself, and given $y := (2 - 1/n) \in \ell_\infty$, we have that $T^{-1}(B_{\ell_\infty} - y) = \emptyset$ but $(1, 1, 1, \ldots) \in {T^{**}}^{-1}(B_{\ell_\infty^{**}} - y)$. This observation points out a technical difference between Lemma 2.7 and its following ultrapower analogue:

Lemma 2.8. *Let $T \colon Y \longrightarrow X$ be an operator, let $\mathfrak{U}$ be an ultrafilter on I, and let w^* denote the $\sigma\big((X_{\mathfrak{U}})^*, X_{\mathfrak{U}}\big)$ topology. Then for each $\mathbf{y}^* \in (Y^*)_{\mathfrak{U}}$, the set*

$$L := B_{(X^*)_{\mathfrak{U}}} \cap (T^*)_{\mathfrak{U}}^{-1}(B_{(Y^*)_{\mathfrak{U}}} + \mathbf{y}^*)$$

is w^*-dense in $\Lambda := B_{(X_{\mathfrak{U}})^*} \cap {(T_{\mathfrak{U}})^*}^{-1}(B_{(Y_{\mathfrak{U}})^*} + \mathbf{y}^*)$.

Proof. Let $(y_i^*)_{i \in I}$ be a representative for $\mathbf{y}^*$. If $\Lambda = \emptyset$, there is nothing to be proved. Assume there exists $\mathbf{v}^* \in \Lambda$. Then we claim that $L \neq \emptyset$:

indeed, if we assume for a while that $L = \emptyset$, we would get that, for all $\mathbf{x}^* \in B_{(X^*)_{\mathfrak{U}}}$,

$$1 < \|(T^*)_{\mathfrak{U}}(\mathbf{x}^*) - \mathbf{y}^*\|; \tag{3}$$

let $(K_n)_{n=1}^\infty$ be a decreasing sequence of elements of $\mathfrak{U}$ such that $\cap_{n=1}^\infty K_n = \emptyset$, and for every $n \in \mathbb{N}$, consider the subset

$$J_n := K_n \cap \{i \in I \colon \exists\, x_i^* \in B_{X^*} \text{ such that } 1 + 1/n > \|T^*(x_i^*) - y_i^*\|\}.$$

If $J_n \in \mathfrak{U}$ for all n then $(J_n)_{n=1}^{\infty}$ would be a decreasing sequence of elements of $\mathfrak{U}$ such that $\cap_{n=1}^{\infty} J_n = \emptyset$ and therefore, for every $m \in \mathbb{N}$ and every $i \in J_m \setminus J_{m+1}$, picking $z_i^* \in B_{X^*}$ so that $1 + 1/m > \|T^*(z_i^*) - y_i^*\|$, we would have

$$\left\{ i \in I \colon 1 + \frac{1}{m} \geq \|T^*(z_i^*) - y_i^*\| \right\} \supset \bigcup_{k=m}^{\infty} J_k \setminus J_{k+1} = J_n \in \mathfrak{U}$$

which means that $1 \geq \lim_{\mathfrak{U}} \|T^*(z_i^*) - y_i^*\| = \|(T^*)_{\mathfrak{U}}([z_i^*]) - \mathbf{y}^*\|$, but $[z_i^*] \in B_{(X^*)_{\mathfrak{U}}}$, a contradiction with (3). Therefore there exists $m \in \mathbb{N}$ such that $J_m \notin \mathfrak{U}$. Thus, for every $i \in I \setminus J_m$,

$$1 + \frac{1}{m} \leq \|T^*(x^*) - y_i^*\| \quad \text{for all } x^* \in B_{X^*}$$

and by the Hahn-Banach theorem, for every $i \in I \setminus J_m$ there would exist $x_i \in S_X$ such that

$$1 + \frac{1}{m} \leq \langle T^*(x^*) - y_i^*, x_i \rangle \quad \text{for all } x^* \in B_{X^*}$$

and in turn, as $I \setminus J_m \in \mathfrak{U}$,

$$1 + \frac{1}{m} \leq \langle (T^*)_{\mathfrak{U}}(\mathbf{x}^*) - \mathbf{y}^*, [x_i] \rangle \quad \text{for all } \mathbf{x}^* \in B_{(X^*)_{\mathfrak{U}}}.$$

But by part (ii) in Lemma 2.5, $(T_{\mathfrak{U}})^*(\mathbf{v}^*) \in \overline{R\big((T^*)_{\mathfrak{U}}\big)}^{\mathrm{w}^*}$, so

$$1 \leq \langle (T_{\mathfrak{U}})^*(\mathbf{v}^*) - \mathbf{y}^*, [x_i] \rangle,$$

hence $\mathbf{v}^* \notin \Lambda$, a contradiction which proves that $L \neq \emptyset$.

Once we know that $L \neq \emptyset$, assume $\mathbf{x}^* \notin \overline{L}^{\mathrm{w}^*}$. Then the Hahn-Banach theorem provides $\mathbf{u} = [u_i] \in S_{X_{\mathfrak{U}}}$ and real numbers a and b such that

$$\langle \mathbf{x}^*, \mathbf{u} \rangle \leq a < b < \inf \langle \overline{L}^{\mathrm{w}^*}, \mathbf{u} \rangle.$$

Thus, defining $W_i := \{z^* \in X^* \colon \langle z^*, u_i \rangle \leq a\}$ for every $i \in I$, we have that

$$W := (W_i)_{\mathfrak{U}} = \{\mathbf{z}^* \in (X^*)_{\mathfrak{U}} \colon \langle \mathbf{z}^*, \mathbf{u} \rangle \leq a\}$$

hence $W \cap L = \emptyset$, which leads to

$$1 < \|(T^*)_{\mathfrak{U}}(\mathbf{z}^*) - \mathbf{y}^*\| \quad \text{for all } \mathbf{z}^* \in W \cap B_{(X^*)_{\mathfrak{U}}}. \tag{4}$$

For every $n \in \mathbb{N}$, consider the subset

$$J'_n := K_n \cap \{i \in I \colon \ \exists\, x_i^* \in W_i \cap B_{X^*} \text{ such that } 1 + 1/n > \|T^*(x_i^*) - y_i^*\|\}.$$

If every J'_n belongs to $\mathfrak{U}$, then using the same argument that proves $L \neq \emptyset$, we would obtain an element $[x_i^*] \in (W_i)_{\mathfrak{U}} \cap B_{(X^*)_{\mathfrak{U}}}$ such that $1 \geq \|(T^*)_{\mathfrak{U}}([x_i^*]) - \mathbf{y}^*\|$, in contradiction with (4). Therefore, we conclude that $J'_m \notin \mathfrak{U}$ for some m. Let $J := I \setminus J'_m \in \mathfrak{U}$ and $\eta := 1 + 1/m$. Thus, for every $i \in J$,

$$\eta < \|T^*(z^*) - y_i^*\| \quad \text{for all } z^* \in W_i \cap B_{X^*},$$

hence, using again the Hahn-Banach theorem, for every $i \in J$ there exists $g_i \in S_Y$ such that

$$\eta \leq \langle T^*(w^*) - y_i^*, g_i \rangle \quad \text{for all } w^* \in W_i \cap B_{X^*}.$$

and as $\mathbf{x}^* \in \overline{W}^{\mathrm{w}^*}$, then

$$1 < \eta \leq \langle (T_{\mathfrak{U}})^*(\mathbf{x}^*) - \mathbf{y}^*, [g_i] \rangle \leq \|(T_{\mathfrak{U}})^*(\mathbf{x}^*) - \mathbf{y}^*\|$$

which implies that $\mathbf{x}^* \notin \Lambda$, finishing the proof. $\square$

3. Local dual spaces and related concepts

Let Z be a closed subspace of the dual X^* of a Banach space X. Given finite dimensional subspaces E of X^* and F of X, a number $\varepsilon > 0$, and an operator $L \colon E \longrightarrow Z$, we consider the following conditions:

(A) $1 - \varepsilon \leq \|L(x^*)\| \leq 1 + \varepsilon$ for all $x^* \in S_E$,
(B) $\langle L(x^*) - x^*, x \rangle = 0$ for all $x^* \in E$ and all $x \in F$,
(C) $L(x^*) = x^*$ for all $x^* \in E \cap Z$.

Definition 3.1. *Let Z be a closed subspace of X^*. We say that X^* is* finite dual representable *(f.d.r. in short) in Z if for every pair of finite dimensional subspaces E of X^* and F of X, and every $\varepsilon > 0$ there exists an operator $L \colon E \longrightarrow Z$ satisfying conditions (A) and (B) with respect to F and ε.*

The notion of local complementation was introduced in [44] as follows:

Definition 3.2. *Given $\alpha \geq 1$, a closed subspace Z of Y is said to be* locally α-complemented *in Y if for every finite dimensional subspace E of Y and every $\varepsilon > 0$ there exists an operator $L \colon E \longrightarrow Z$ such that $\|L\| \leq \alpha + \varepsilon$ and $L(y) = y$ for all $y \in Z \cap E$.*

The subspace Z is said to be locally complemented *in Y if it is locally α-complemented in Y for some $\alpha \geq 1$.*

The subspace Z is said to be an ideal *in Y if it is locally 1-complemented in Y (see* [28] *and* [64]*).*

Remark 3.3. A subspace Z of X^* is an ideal in X^* if and only if for every finite dimensional subspace E of X^* and every $\varepsilon > 0$ there exists an operator $L \colon E \longrightarrow Z$ satisfying $\|L\| \leq 1 + \varepsilon$ and (C).

Remark 3.4. The construction of an operator $L\colon E \longrightarrow Z$ fixing $E \cap Z$ entails to extend the identity operator on $E \cap Z$ to an operator $E \to Z$. In fact, if P is a projection on E onto $E \cap Z$ with kernel G, then $L = P + L(I_E - P)$. Obviously, the norm of L is at least one, but its exact value depends very much on the constant of complementation of $E \cap Z$ in E. Therefore, the choice of $\alpha = 1$ in Definition 3.2 is certainly special.

Remark 3.5.

(i) if X_1 is locally α-complemented in X_2 and X_2 is locally β-complemented in X_3 then X_1 is locally $\alpha\beta$-complemented in X_3;

(ii) if X_1 is α-complemented in X_2 then X_1 is locally α-complemented in X_2;

(iii) if X^* is f.d.r. in Z then X^* is also f.d.r. in any subspace of X^* containing Z.

The notion of local duality arises merging the notions of finite dual representability and local complementation.

Definition 3.6. *A closed subspace Z of X^* is said to be a* local dual *of X if for every pair of finite dimensional subspaces E of X^* and F of X, and every $\varepsilon > 0$ there exists an operator $L\colon E \longrightarrow Z$ satisfying conditions (A), (B) and (C) with respect to F and ε.*

Local duality of Z with respect to X implies both finite dual representability of X^* in Z and local 1-complementation of Z in X^*. Nevertheless, as we shall see in Example 3.13, the converse of that implication fails.

As we mentioned in the Introduction, there are two examples of local duality which are fundamental in local theory of Banach spaces: the P.L.R. and the P.L.R. for ultrapowers. The most popular proof of the P.L.R. is the one given by Dean in [15], based on the properties of tensor products (see a detailed version in [18, Chapter 8]). Moreover, Heinrich gave a proof of the P.L.R. for ultrapowers in [40, Theorem 7.3] based on the ideas of [15]. Next we give a proof of both principles based on the ideas of [54].

Theorem 3.7 (Principle of local reflexivity). *Given a Banach space X, its natural copy in X^{**} is a local dual of X^*.*

Proof. Let E and F be a pair of finite dimensional subspaces of X^{**} and X^* respectively, and fix any real number $\varepsilon > 0$. Let $\dim E = n$, $\dim E \cap X = n - k$ and take a biorthogonal system $(y_j^{**}, h_j)_{j=1}^n$ in $E \times E^*$ such that $\|y_j^{**}\| = 1$ and $\operatorname{span}\{y_j^{**}\}_{j=k+1}^n = E \cap X$. Let us denote $y_j := y_j^{**}$ for $k+1 \le j \le n$. Thus the inclusion operator $\mathcal{J}_E\colon E \longrightarrow X^{**}$ satisfies

$$\mathcal{J}_E(x^{**}) = \sum_{j=1}^{k} \langle h_j, x^{**}\rangle y_j^{**} + \sum_{j=k+1}^{n} \langle h_j, x^{**}\rangle y_j.$$

Note that any operator of the form

$$L(x^{**}) = \sum_{j=1}^{k} \langle h_j, x^{**}\rangle v_j + \sum_{j=k+1}^{n} \langle h_j, x^{**}\rangle y_j \tag{5}$$

satisfies (C) automatically. Thus the proof will be done as soon as we find vectors $v_1, \dots v_k$ in X so that the operator L in (5) also satisfies (A) and (B) with respect to F and ε. To do so, fix $\alpha > 0$ and $\delta > 0$ small enough so that $0 < (\alpha + \delta)/(1 - \alpha) < \varepsilon$, and pick

$$\begin{aligned} &\{e_j^{**}\}_{i=1}^{N}, \text{ an } \alpha\text{-net in } S_E, \text{ and} \\ &\{u_j^{*}\}_{j=1}^{N} \text{ in } B_{X^*} \end{aligned}$$

such that $\|e^{**}\| \le (1 - \alpha)^{-1} \sup_{1\le j\le N} |\langle e^{**}, u_j^*\rangle|$ for all $e^{**} \in E$, and $\{u_j^*\}_{j=1}^{N}$ contains a basis of F. Let λ_r^j be scalars such that

$$e_j^{**} = \sum_{r=1}^{k} \lambda_r^j y_r^{**} + \sum_{r=k+1}^{n} \lambda_r^j y_r, \ 1 \le j \le N,$$

let $K := \max_{1\le j\le N} \sum_{r=1}^{n} |\lambda_r^j|$, let $W := \ell_\infty^k(X)$, and consider the subset

$$C := \left\{ (x_s)_{s=1}^{k} \in (1 + \alpha/2)B_W \colon \left\| \sum_{s=1}^{k} \lambda_s^j x_s + \sum_{s=k+1}^{n} \lambda_s^j y_s \right\| \le 1, \ 1 \le j \le N \right\}.$$

Identifying W^{**} with $\ell_\infty^k(X^{**})$, Lemma 2.7 yields that $(y_j^{**})_{j=1}^{k} \in \overline{C}^{\sigma(W^{**},W^*)}$. Consider the operator $S\colon W \longrightarrow \mathbb{R}^{kN}$ that maps each $(x_s)_{s=1}^{k}$ to $(\langle u_j^*, x_s\rangle)_{s=1}^{k}{}_{j=1}^{N}$. Thus, as $R(S)$ is finite dimensional, it follows that $S^{**}\big((y_j^{**})_{j=1}^{k}\big) \in \overline{S(C)}$ and $\overline{R(S)} = R(S) = R(S^{**})$. Hence, given $0 < \beta < \alpha/(2K)$, there exist $(c_j)_{j=1}^{k} \in C$ and $(b_j)_{j=1}^{k} \in \beta B_W$ such that

$$S^{**}\big((y_j^{**})_{j=1}^{k}\big) = S\big((c_j)_{j=1}^{k}\big) + S\big((b_j)_{j=1}^{k}\big).$$

Clearly, since the set $\{u_j^*\}_{j=1}^{N}$ contains a basis of F, the choice $v_j := c_j + b_j$ for $1 \le j \le k$ makes the operator L in (5) to satisfy (B) with respect to F. Hence, for every $1 \le j \le N$,

$$\|L(e_j^{**})\| \ge \sup_{1\le i\le N} \langle L(e_j^{**}), u_i^*\rangle = \sup_{1\le i\le N} \langle e_j^{**}, u_i^*\rangle \ge 1 - \alpha. \tag{6}$$

and as $L(y_l^{**}) = c_l + b_l$ for $1 \le l \le k$, it follows for all $1 \le j \le N$,

$$\|L(e_j^{**})\| \le \left\| \sum_{r=1}^{k} \lambda_r^j c_j + \sum_{r=k+1}^{n} \lambda_r^j y_j \right\| + \left\| \sum_{r=1}^{k} \lambda_r^j b_j \right\| \le 1 + \frac{\alpha}{2} + \beta K \le 1 + \alpha. \tag{7}$$

Therefore, (6), (7), Lemma 2.2 and the choice of α and δ show that L satisfies (A) with respect to ε. $\square$

Theorem 3.8 (Principle of local reflexivity for ultrapowers). *Given a Banach space X and an ultrafilter $\mathfrak{U}$ on I, the natural copy of $(X^*)_\mathfrak{U}$ in $(X_\mathfrak{U})^*$ is a dual local of $X_\mathfrak{U}$.*

Proof. It is similar to that of Theorem 3.7. Fix a real number $\varepsilon > 0$ and a pair of finite dimensional subspaces E and F of $(X_\mathfrak{U})^*$ and $X_\mathfrak{U}$ respectively. Let $\dim E = n$, $\dim E \cap (X^*)_\mathfrak{U} = n-k$ and $(\mathbf{x}_j^*, h_j)_{j=1}^n \subset E \times E^*$ be a biorthogonal system such that $\|\mathbf{x}_j^*\| = 1$ for all $1 \leq j \leq n$ and $E \cap (X^*)_\mathfrak{U} = \operatorname{span}\{\mathbf{x}_j^*\}_{j=k+1}^n$. For $k+1 \leq j \leq n$, we will denote $\mathbf{y}_j^* := \mathbf{x}_j^*$. Thus, any operator $T\colon E \longrightarrow (X^*)_\mathfrak{U}$ of the form

$$T(\mathbf{x}^*) = \sum_{j=1}^{k} \langle h_j, \mathbf{x}^* \rangle \mathbf{v}_j^* \; + \sum_{j=k+1}^{n} \langle h_j, \mathbf{x}^* \rangle \mathbf{x}_j^*. \tag{8}$$

satisfies (C). A sensible choice of vectors $\mathbf{v}_1^*, \ldots, \mathbf{v}_k^*$ in (8) will make the operator T to satisfy (A) and (B) with respect to F and ε.

Fix real numbers $\alpha > 0$ and $\delta > 0$ such that $0 < (\alpha + \delta)/(1 - \alpha) < \varepsilon$ and choose

$$\begin{aligned} &\{\mathbf{e}_j^*\}_{j=1}^N \text{ an } \alpha\text{-net in } S_E, \text{ and} \\ &\{\mathbf{u}_j\}_{j=1}^N \text{ in } B_{X_\mathfrak{U}} \end{aligned}$$

such that $\{\mathbf{u}_j\}_{j=1}^N$ contains a basis of F and $\|\mathbf{e}^*\| \leq (1-\alpha)^{-1} \sup_{1 \leq j \leq N} |\langle \mathbf{e}^*, \mathbf{u}_j \rangle|$ for all $\mathbf{e}^* \in E$. For every $\mathbf{u}_j$, fix a representative $(u_i^j)_{i \in I}$ so that $\mathbf{u}_j = [u_i^j]$. Let λ_r^j be real numbers so that

$$\mathbf{e}_j^* = \sum_{r=1}^{k} \lambda_r^j \mathbf{x}_j^* \; + \sum_{r=k+1}^{n} \lambda_r^j \mathbf{y}_j^*, \; 1 \leq j \leq N$$

and let $K := \max_{1 \leq j \leq N} \sum_{r=1}^n |\lambda_r^j|$. Note that given $W := \ell_1^k(X)$, $(W^*)_\mathfrak{U}$ is isometrically identified with $\ell_\infty^k\big((X^*)_\mathfrak{U}\big)$ and $(W_\mathfrak{U})^*$ with $\ell_\infty^k\big((X_\mathfrak{U})^*\big)$. Bearing in mind both identifications, consider the subset

$$C := \left\{ (\mathbf{z}_s^*)_{s=1}^k \in \left(1 + \frac{\alpha}{2}\right) B_{(W^*)_\mathfrak{U}} \colon \left\| \sum_{s=1}^{k} \lambda_s^j \mathbf{z}_s^* + \sum_{s=k+1}^{n} \lambda_s^j \mathbf{y}_j^* \right\| \leq 1, \; 1 \leq j \leq N \right\}.$$

For every $i \in I$, consider also the operator $S_i\colon \mathbb{R}^{kN} \longrightarrow \ell_1^k(X)$ given by

$$S_i\big((a_{lj})_{l=1}^N{}_{j=1}^k\big) := \left(\sum_{j=1}^{N} a_{lj} u_i^j \right)_{l=1}^k.$$

Thus its conjugate can be identified with the operator $S_i^*\colon \ell_\infty^k(X^*) \longrightarrow \mathbb{R}^{kN}$ that maps each $(x_j^*)_{j=1}^k$ to $\big((\langle x_j^*, u_i^l \rangle)_{j=1}^k{}_{l=1}^N\big)$ and $\big((S_i)_\mathfrak{U}\big)^*\colon (W_\mathfrak{U})^* \longrightarrow \mathbb{R}^{kN}$ maps each $(\mathbf{x}_j^*)_{j=1}^k$ to $(\langle \mathbf{x}_j^*, \mathbf{u}_l \rangle)_{j=1}^k{}_{l=1}^N$. Let us denote $S := \big(S_i\big)_\mathfrak{U}$. Obviously, $(S_i^*)_\mathfrak{U} = S^*|_{(W^*)_\mathfrak{U}}$.

Lemma 2.8 yields that $(\mathbf{x}_j^*)_{j=1}^k$ belongs to the $\sigma\big((W_{\mathfrak{U}})^*, W_{\mathfrak{U}}\big)$-closure of C. Thus, since S^* has finite dimensional range, it follows that $S^*\big((\mathbf{x}_j^*)_{j=1}^k\big) \in \overline{(S_i^*)_{\mathfrak{U}}(C)}$, and $\overline{R(S^*)} = R(S^*) = R\big((S_i^*)_{\mathfrak{U}}\big)$. Hence, given $0 < \beta < \alpha/(2K)$, there exist $(\mathbf{c}_j^*)_{j=1}^k \in C$ and $(\mathbf{b}_j^*)_{j=1}^k \in \beta B_W$ such that

$$S^*\big((\mathbf{x}_j^*)_{j=1}^k\big) = (S_i^*)_{\mathfrak{U}}\big((\mathbf{c}_j^*)_{j=1}^k\big) + (S_i^*)_{\mathfrak{U}}\big((\mathbf{b}_j^*)_{j=1}^k\big).$$

For every $1 \leq j \leq k$, define $\mathbf{v}_j^* = \mathbf{c}_j^* + \mathbf{b}_j^*$: thus the operator T of (8) satisfies (B) with respect to F for the vectors $\mathbf{v}_j$. The proof that T also satisfies (A) with respect to ε is formally identical to the argument given in Theorem 3.7 for the operator L. ☐

Let us state the following weak versions of conditions (A), (B) and (C):

(A') $\|L(x^*)\| \leq 1 + \varepsilon$ for all $x^* \in S_E$
(B') $\big|\langle L(x^*) - x^*, x\rangle\big| \leq \varepsilon$ for all $x^* \in S_E$ and all $x \in S_F$,
(C') $\|L(x^*) - x^*\| \leq \varepsilon$ for all norm one elements $x^* \in E \cap Z$.

We will show that Definitions 3.1, 3.2 and 3.6 can be equivalently rewritten replacing (A), (B) and (C) by (A'), (B') and (C') respectively.

Proposition 3.9. *Let Z be a subspace of a dual space X^*. Then we have:*

(a) X^* is f.d.r. in Z if and only if for every pair of finite dimensional subspaces E of X^* and F of X, and every $\varepsilon > 0$, there exists an operator $L\colon E \longrightarrow Z$ satisfying conditions (A') and (B') with respect to F and ε;

(b) Z is locally 1-complemented in X^* if and only if for every pair of finite dimensional subspaces E of X^* and F of X, and every $\varepsilon > 0$, there exists an operator $L\colon E \longrightarrow Z$ for which conditions (A') and (C') hold with respect to F and ε;

(c) Z is a local dual of X if and only if for every pair of finite dimensional subspaces E of X^* and F of X, and every $\varepsilon > 0$, there exists an operator $L\colon E \longrightarrow Z$ satisfying conditions (A'), (B') and (C') with respect to F and ε.

Proof. We need to prove only the 'if' directions of the three statements for a pair of finite dimensional subspaces E of X^* and F of X, and $\varepsilon > 0$. We denote $E_0 := E \cap Z$. By Lemma 2.3, there is no loss of generality in assuming that F is large enough so that

$$(1-\varepsilon)\|x^*\| \leq \sup_{x \in S_F} \langle x^*, x\rangle \quad \text{for all } x^* \in E. \tag{9}$$

Lemma 2.4 gives a biorthogonal system $(x_i^*, f_i)_{i=1}^n$ in $E \times E^*$ such that $E_0 = \operatorname{span}\{x_i^*\}_{i=1}^m$, $\|x_i^*\| = 1$ and $\|f_i\| \leq n+1$ for all $1 \leq i \leq n$.

Let $E_1 := \operatorname{span}\{x_i^*\}_{i=m+1}^n$, and let $P\colon E \longrightarrow E$ be the projection with $R(P) = E_0$ and $N(P) = E_1$. Denote $M := \max\{\sum_{i=1}^m |\lambda_i|\colon \|\sum_{i=1}^n \lambda_i x_i^*\| = 1\}$.

(a) Let $p := \dim F$ and choose $0 < \varepsilon_1 < \varepsilon(1+2p)^{-1}$. By hypothesis, there exists an operator $L_{\varepsilon_1} \colon E \longrightarrow Z$ with $\|L_{\varepsilon_1}\| \leq 1 + \varepsilon_1$ such that

$$|\langle L_{\varepsilon_1}(x^*) - x^*, x\rangle| \leq \varepsilon_1 \|x^*\|\|x\| \text{ for all } x^* \in E \text{ and all } x \in F.$$

By Lemma 2.6, there exists a normalized basis $\{x_i\}_{i=1}^p$ of F and a system $\{z_i\}_{i=1}^p$ in $(1+\varepsilon_1)B_Z$ such that $\langle z_i, x_j\rangle = \delta_{ij}$ for all i and j. Consider the operator $K_1 \colon E \longrightarrow Z$ given by $K_1(x^*) := \sum_{i=1}^p \langle x^* - L_{\varepsilon_1}(x^*), x_i\rangle z_i$. Note that $\|K_1\| \leq p\varepsilon_1(1+\varepsilon_1) \leq 2p\varepsilon_1$. Thus the operator $L := L_{\varepsilon_1} + K_1$ satisfies automatically condition (B) with respect to F. Therefore, for every $x^* \in E$,

$$\|L(x^*)\| \geq \sup_{x \in S_F} \langle L(x^*), x\rangle = \sup_{x \in S_F} \langle x^*, x\rangle \geq (1-\varepsilon_1)\|x^*\|,$$

and as $\|L\| \leq \|L_{\varepsilon_1}\| + \|K_1\| \leq 1 + (1+2p)\varepsilon_1 < 1 + \varepsilon$, it follows that L is an ε-isometry. Thus condition (A) is also fulfilled by L.

(b) Take $0 < \varepsilon_2 < \varepsilon/2M$. By hypothesis, there is an operator $L_2 \colon E \longrightarrow Z$ satisfying (A') and (C') with respect to ε_2, that is, $\|L_2(x^*) - x^*\| \leq \varepsilon_2\|x^*\|$ for all $x^* \in E_0$.

Define an operator $K_2 \colon E \longrightarrow Z$ by $K_2(x_i^*) = x_i^* - L_2(x_i^*)$ for $1 \leq i \leq m$ and $K_2(x_i^*) = 0$ for $m+1 \leq i \leq n$. Clearly, $L := L_2 + K_2$ maps each $x^* \in E_0$ to itself, and $\|K_2\| \leq \varepsilon_2 M$, so $\|L\| \leq 1 + \varepsilon$, and part *(b)* is done.

(c) Consider the decomposition $E = E_0 \oplus E_1$ and take $0 < \varepsilon_3 < \varepsilon/2(1+M)$. By hypothesis, there exists an operator $L_3 \colon E \longrightarrow Z$ satisfying (A'), (B') and (C') with respect to F and ε_3. By part (a), we get an operator $K_3 \colon E_1 \longrightarrow Z$ such that $\|K_3\| \leq \varepsilon_3$ and $L_3|_{E_1} + K_3$ satisfies (A) and (B) with respect to F and ε_3.

By part (b), there exists $K_4 \colon E_0 \longrightarrow Z$ such that $\|K_4\| \leq \varepsilon_3 M$ and $I_{E_0} = L_3|_{E_0} + K_4$. Let $L \colon E \longrightarrow Z$ be given by $L = (L_3 + K_3) \circ (I_{E_1} - P) + P$. Clearly, L satisfies (B) and (C) with respect to F. Moreover, as

$$L = (L_3 + K_3) \circ (I_{E_1} - P) + (L_3 + K_4) \circ P = L_3 + K_3 \circ (I_{E_1} - P) + K_4 \circ P,$$

it follows that $\|L\| \leq 1 + \varepsilon_3 + \varepsilon_3(1+M) + \varepsilon_3 M = 1 + 2\varepsilon_3(1+M) < 1 + \varepsilon$. Besides, since L satisfies (B) with respect to F, formula (9) yields that $\|L(x^*)\| \geq 1 - \varepsilon$ for all norm one element x^* in E. Therefore, L also satisfies (A) with respect to E and ε. $\square$

Remark 3.10. The weaker conditions (A'), (B') and (C') may help to understand the way that local duality works in some situations. For instance, consider a pair of finite dimensional subspaces E of ℓ_∞ and F of ℓ_1 and $\varepsilon > 0$, with F large enough so that it norms E up to ε, and for each n let $L_n \colon \ell_\infty \longrightarrow c_0$ be the operator that maps $(x_n)_{n=1}^\infty \in E$ to $(x_1, x_2, \dots, x_n, 0, 0, \overset{\infty}{\dots})$. Essentially, the ε-isometry whose existence is claimed by the principle of local reflexivity is $L_n|_E$ for some n. Indeed, if n is large enough, then $L_n|_E$ satisfies (A), (B') and (C'). In order to obtain (B) and (C), it is sufficient to perturb $L_n|_E$ adding an operator of small norm as in the proof of Proposition 3.9. This procedure will be used in Example 3.11.

Clearly no Banach space X contains a proper subspace Z such that X^{**} is f.d.r. in Z. It is also immediate that if $X \subset Z \subset X^{**}$ then X^{**} is f.d.r. in Z. However, Example 3.11 shows that it may exist a subspace Z of X^{**} such that X^{**} is f.d.r. in Z and $Z \cap X = \{0\}$.

Example 3.11. *There exists a local dual Z of ℓ_1 such that $c_0 \cap Z = \{0\}$.*

Proof. Let (t_n) be a dense sequence in $[0,1]$ with $t_1 = 0$, $t_2 = 1$, and $t_i \neq t_j$ for $i \neq j$. Let $U\colon C[0,1] \longrightarrow \ell_\infty$ be the operator defined by $U(f) := \big(f(t_k)\big)_{k=1}^{\infty}$. Clearly U is a linear isometry and $Z := U(C[0,1])$ is a closed subspace of ℓ_∞ such that $Z \cap c_0 = \{0\}$. It only remains to show that Z is a local dual of ℓ_1. To do so, fix $\varepsilon > 0$ and a pair of finite dimensional subspaces E of ℓ_∞ and F of ℓ_1.

Since F is finite dimensional, an application of Lemma 2.2 provides $n_1 \in \mathbb{N}$ such that

$$\sum_{k=n_1}^{\infty} |\gamma_k| < \varepsilon/2 \text{ for all } (\gamma_k) \in S_F.$$

For every integer $n \geq 2$ we consider the operator $P_n : \ell_\infty \longrightarrow C[0,1]$ that maps $\beta = (\beta_k)$ to the polygonal function interpolating the nodes

$$\{(t_k, \beta_k) : k = 1, \ldots, n\}.$$

Note that $\|P_n\| = 1$ and $\|P_n(\beta)\|_\infty \to \|\beta\|_\infty$ as $n \to \infty$. Since E is finite dimensional, Lemma 2.2 provides $n_2 \geq n_1$ such that $P_n|_E$ is a ε-isometry for $n \geq n_2$. Moreover, it is not difficult to see that $\|P_n U(f) - f\|_\infty \to 0$ as $n \to \infty$. Thus, arguing as before, we can find $n_3 \geq n_2$ such that $\|UP_n(\beta) - \beta\|_\infty \leq \varepsilon \|\beta\|_\infty$ for each $\beta \in E \cap Z$.

Let $L_n : E \longrightarrow Z$ be the operator defined by $L_n := UP_n|_E$. By the previous considerations, for each $n \geq n_3$, L_n is a ε-isometry that satisfies (B') and (C'), and the proof is done. $\square$

The following result shows that if $c_0 \subset Z \subset \ell_\infty$, then local complementation implies local duality for Z.

Proposition 3.12. *Each ideal Z in ℓ_∞ containing c_0 is a local dual of ℓ_1.*

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be the unit vector basis of ℓ_1, and let F_n be the subspace of ℓ_1 spanned by the first n elements of this basis. Note that in order to prove that Z is a local dual of ℓ_1 it is enough to prove that given a finite dimensional subspace E of ℓ_∞ and a positive number n, there exists an operator $L\colon E \longrightarrow Z$ satisfying (A'), (B) and (C') with respect to F_n and $1/n$. Thus, once E and n have been fixed, as Z is an ideal in ℓ_∞, there exists an operator $K\colon E \longrightarrow Z$ satisfying (A') and (C) with respect to F_n and $1/n$, that is, $\|K\| \leq 1 + 1/n$ and $K(u) = u$ for all $u \in E \cap Z$. Let $P\colon \ell_\infty \longrightarrow \ell_\infty$ be the projection that sends each $(x_i)_{i=1}^{\infty}$ to $(y_i)_{i=1}^{\infty}$ where $y_i := x_i$ for $1 \leq i \leq n$ and $y_i := 0$ for $i > n$, and let I be the identity operator on ℓ_∞. Thus $L := P + (I - P)K$ satisfies

$$\|L\| = \max\{\|P\|, \|K\|\} \leq 1 + 1/n$$

and $L(u) = u$ for all $u \in E \cap Z$, so L satisfies (A') and (C) with respect to $1/n$. Moreover, for every $1 \le m \le n$ and every $u \in E$,

$$\langle L(u), e_m \rangle = \langle P(u), e_m \rangle = \langle u, e_m \rangle$$

hence L also satisfies (B) with respect to F_n. The proof is done. $\square$

Given a subspace Z of X^*, it is straightforward that

$$Z \text{ local dual of } X \Longrightarrow X^* \text{ f.d.r. in } Z \Longrightarrow Z \text{ norming subspace of } X^*.$$

However, Examples 3.13 and 3.14 show that the reverse implications fail. Even more, in spite of Proposition 3.12, Example 3.13 shows that finite dual representability plus local complementation does not imply local duality.

Example 3.13. *The space ℓ_∞^* contains a subspace Z satisfying the following properties:*

(i) ℓ_∞^* is f.d.r. in Z;
(ii) Z is an ideal in ℓ_∞^*;
(iii) Z is not a local dual of ℓ_∞.

Proof. Recall that ℓ_∞^* can be decomposed as

$$\ell_\infty^* = \ell_1 \oplus_1 c_0^\perp. \tag{10}$$

By Dvoretzky's theorem, ℓ_2 is finitely representable in $c_0^\perp$. So we can choose a finite dimensional subspace V of $c_0^\perp$ ε-isometric to ℓ_2^n with $n > 1$ and $0 < \varepsilon < 1$ as small as we please (to be fixed later). Take a norm one element $\mathbf{v} \in V$ and let us prove that $Z := \operatorname{span} \ell_1 \cup \{\mathbf{v}\} \subset \ell_\infty^*$ satisfies properties (i), (ii) and (iii).

(i) Note that ℓ_1 is a local dual of ℓ_∞ and $\ell_1 \subset Z$, hence ℓ_∞^* is f.d.r. in Z.

(ii) Let $\mathbf{f} \in \ell_\infty^{**}$ be a norm one functional such that $\langle \mathbf{f}, \mathbf{v} \rangle = 1$. Let E be any finite dimensional subspace of ℓ_∞^*. Choose a finite dimensional subspace E_0 of ℓ_∞^* such that $E \subset E_0$ and admits a decomposition $E_0 = E_1 \oplus E_2$ with $E_1 \subset \ell_1$, $E_2 \subset c_0^\perp$ and $\mathbf{v} \in E_2$. Next, define $T\colon E_0 \longrightarrow \ell_1$ as follows: if $\mathbf{u} \in E_1$ then $T(\mathbf{u}) := \mathbf{u}$ and if $\mathbf{u} \in E_2$ then $T(\mathbf{u}) := \langle \mathbf{f}, \mathbf{u} \rangle \mathbf{v}$. Since the decomposition (10) yields $E_0 = E_1 \oplus_1 E_2$, it is straightforward that T fixes E_1 and $\|T\| = 1$. Therefore, $T|_E$ satisfies conditions (A) and (C) which proves that Z is an ideal in ℓ_∞^*.

(iii) Assume Z is a local dual of ℓ_∞^*. Then there exists an ε-isometry $L\colon V \longrightarrow Z$ such that $L(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in V \cap Z$. In particular, $L(\mathbf{v}) = \mathbf{v}$. Since ℓ_2^n is ε-isometric to V, $L(V)$ must be 3ε-isometric to ℓ_2^n. Moreover, Formula (10) gives $L(V) = \operatorname{span}\{\mathbf{v}\} \oplus_1 W$ for some $(n-1)$-dimensional subspace W of ℓ_1. But if ε is small enough then $\operatorname{span}\{\mathbf{v}\} \oplus_1 W$ cannot be 3ε-isometric to ℓ_2^n, a contradiction. $\square$

Example 3.14. *[32] The space ℓ_1^* contains a norming subspace Z such that ℓ_1^* is not finitely representable in Z, hence is not f.d.r. in Z.*

Proof. Consider the Rademacher-like sequence (x_n) in ℓ_∞ where

$$x_1 = (1, -1, 1, -1, 1, -1, \ldots)$$

and for $n \in \mathbb{N}$ the sequence x_{n+1} consists of successive repetitions of the block

$$1, \overset{(2^n)}{\ldots}, 1, -1, \overset{(2^n)}{\ldots}, -1.$$

Then (x_n) is 1-equivalent to the unit vector basis of ℓ_1. Let $\{A_n \colon n \in \mathbb{N}\}$ be an enumeration of all the finite sequences of numbers in $\{1, -1\}$ with $\operatorname{card}(A_m) \leq \operatorname{card}(A_n)$ if $m < n$, and modify each x_n in a finite number of coordinates so that the initial segment of x_n coincides with A_n. Thus (x_n) is still equivalent to the unit vector basis of ℓ_1, and its closed span is a norming subspace Z of $\ell_1^* = \ell_\infty$. However, since ℓ_∞ is not finitely representable in ℓ_1, it is not finitely representable in Z either. □

Next we give some examples of local dual spaces which are essentially different from those provided by the principles of local reflexivity. In fact, the ε-isometries fulfilling conditions (A'), (B') and (C') required by these examples are found explicitly, while in the known proofs of the mentioned principles the required ε-isometries are only shown to exist by means of Goldstine's theorem or other related results.

Here $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$ is the torus of the complex plane, $\mathcal{M}[0,1]$ denotes the space of Radon measures on $[0,1]$ and $\{I_i^k\}_{k=0}^{\infty}{}_{i=1}^{2^k}$ denotes the collection of dyadic subintervals of $[0,1]$, where $I_1^0 := [0,1]$ and for $k \geq 1$,

$$I_i^k := \begin{cases} \left[\dfrac{i-1}{2^k}, \dfrac{i}{2^k}\right) & \text{if } 1 \leq i < 2^k - 1 \\[2ex] \left[\dfrac{2^k-1}{2^k}, 1\right] & \text{if } i = 2^k \end{cases}$$

so for each k, the collection $\{I_i^k\}_{i=1}^{2^k}$ is a partition of $[0,1]$.

Proposition 3.15.

(a) *$C(\mathbb{T})$ is a local dual of $L_1(\mathbb{T})$.*

(b) *$L_1(\mathbb{T})$ is a local dual of $C(\mathbb{T})$.*

(c) *Let $\{\nu_i^k\}_{i=1}^{2^k}{}_{k=0}^{\infty} \subset \mathcal{M}[0,1]$ be positive, norm one measures with each ν_i^k concentrated on the dyadic interval I_i^k. Then the closed span Z of $\{\nu_i^k\}$ is a local dual of $C[0,1]$.*

(d) *$C[0,1]$ is a local dual of $L_1[0,1]$.*

Proof. The proofs of (b) and (d) are respectively similar to those of (c) and (a).

(a) Let $\{\psi_n\}_{n=1}^{\infty}$ be a positive summability kernel on $\mathbb{T}$ [45, Definition 2.2] such that $\psi_n(e^{it}) = \psi_n(e^{-it})$ for all $t \in [0, 2\pi]$ and all $n \in \mathbb{N}$. The following facts are well known:

(i) $\|\psi_n * f - f\|_1 \xrightarrow{n} 0$ for all $f \in L_1(\mathbb{T})$,

(ii) $\|\psi_n * g\|_\infty \leq \|g\|_\infty$ for all $g \in L_\infty(\mathbb{T})$,

(iii) $\|\psi_n * g - g\|_\infty \xrightarrow{n} 0$ for all $g \in C(\mathbb{T})$,

(iv) $\psi_n * g \in C(\mathbb{T})$ for all $g \in L_\infty(\mathbb{T})$.

Moreover statement (i), the symmetry of each ψ_n and Fubini's theorem yield:

(v) $\langle \psi_n * g, f\rangle = \langle g, \psi_n * f\rangle \xrightarrow{n} \langle g, f\rangle$ for all $g \in L_\infty(\mathbb{T})$ and all $f \in L_1(\mathbb{T})$.

Thus, for every $n \in \mathbb{N}$, statements (ii) and (iv) allows us to define the norm one operator $T_n \colon L_\infty(\mathbb{T}) \longrightarrow C(\mathbb{T})$ as $T_n(g) := \psi_n * g$. Fix finite dimensional subspaces E and F of $L_\infty(\mathbb{T})$ and $L_1(\mathbb{T})$ respectively, and fix $\varepsilon > 0$. Now, if we take real numbers $\alpha > 0$ and $\delta > 0$ small enough, and if we choose finite α-nets $\{e_i\}_{i=1}^N$ and $\{f_i\}_{i=1}^N$ in S_E and S_F, Lemma 2.3 and statements (v) and (iii) allows us to find a positive integer n large enough so that $L := T_n \mid_E$ satisfies conditions (B') and (C') with respect to F and ε. Obviously, $\|L\| \leq 1$ so (A') also holds. We have just proved that $C(\mathbb{T})$ is a local dual of $L_1(\mathbb{T})$.

(c) For every $k \in \mathbb{N}$, define the norm one projection $G_k \colon \mathcal{M}[0,1] \longrightarrow Z$ by

$$G_k(\lambda) = \sum_{i=1}^{2^k} \lambda(I_i^k)\nu_i^k. \tag{11}$$

Let χ_i^k denote the indicator function of the dyadic interval I_i^k. Given $k \in \mathbb{N}$, $f \in C[0,1]$ and $\lambda \in \mathcal{M}[0,1]$, define $M_i^k := \sup f(\overline{I_i^k})$, $m_i^k := \inf f(\overline{I_i^k})$ and

$$\rho_k(f) := \max_{1 \leq i \leq 2^k} M_i^k - m_i^k(f).$$

Let us prove that $|\langle \lambda, f\rangle - \langle G_k(\lambda), f\rangle| \leq \|\lambda\|\rho_k(f)$. Note that we can assume that λ is a positive measure.

Define the functions

$$m_f(x) := \sum_{i=1}^{2^k} m_i^k(f)\chi_i^k(x)$$

and

$$M_f(x) := \sum_{i=1}^{2^k} M_i^k(f)\chi_i^k(x).$$

Note that $\int_0^1 m_f \, d\lambda = \int_0^1 m_f \, dG_k(\lambda)$ and $\int_0^1 M_f \, d\lambda = \int_0^1 M_f \, dG_k(\lambda)$. Moreover,

$$\int_0^1 m_f \, d\lambda \leq \int_0^1 f \, d\lambda \leq \int_0^1 M_f \, d\lambda, \quad \text{and}$$
$$\int_0^1 m_f \, dG_k(\lambda) \leq \int_0^1 f \, dG_k(\lambda) \leq \int_0^1 M_f \, dG_k(\lambda).$$

Therefore, we get

$$\left|\int_0^1 f\,d\lambda - \int_0^1 f\,dG_k(\lambda)\right| \le \int_0^1 (M_f - m_f)\,d\lambda \le \sum_{i=1}^{2^k} \rho_k(f)\lambda(I_i^k) = \|\lambda\|\rho_k(f),$$

as we wanted to prove. Now, let us show that the following statements hold:

(i) $\lim_n \langle G_n(\lambda), f\rangle = \langle \lambda, f\rangle$ for all $\lambda \in \mathcal{M}[0,1]$ and all $f \in C[0,1]$;
(ii) $\lim_n \|G_n(\lambda)\| = \|\lambda\|$ for all $\lambda \in \mathcal{M}[0,1]$.

To prove (i), choose $\lambda \in S_{\mathcal{M}[0,1]}$ and $f \in C[0,1]$. By uniform continuity of f, there is a positive integer n_0 such that $\rho_{n_0}(f) < \varepsilon$. So, $|\langle \lambda - G_n(\lambda), f\rangle| < \varepsilon$ for all $n \ge n_0$.

For (ii), let $\lambda \in \mathcal{M}[0,1]$ and $\varepsilon > 0$. Choose $f \in B_{C[0,1]}$ so that $\langle \lambda, f\rangle > \|\lambda\| - 2^{-1}\varepsilon$. By statement (i), there is n_0 such that $|\langle G_n\lambda, f\rangle| > |\langle \lambda, f\rangle| - 2^{-1}\varepsilon$ for all $n \ge n_0$, so

$$\|\lambda\| - \varepsilon < |\langle G_n\lambda, f\rangle| \le \|G_n(\lambda)\| \le \|\lambda\|.$$

Now, fix finite dimensional subspaces E and F of $\mathcal{M}[0,1]$ and $C[0,1]$ and fix $\varepsilon > 0$. A similar argument of compactness as that applied in example (a) combined with (i) and (ii) yields $k \in \mathbb{N}$ large enough so that $G_k \mid_E$ satisfies (A'), (B') and (C') with respect to F and ε, which proves that Z is a local dual of $C[0,1]$. $\square$

Remark 3.16. By Proposition 3.15, both the natural copy of $L_1[0,1]$ in $\mathcal{M}[0,1]$ and the closed subspace Y generated by $\{\delta_i^k\}_{i=1}^{2^k}{}_{k=0}^{\infty}$, where δ_i^k is the Dirac delta associated with the middle point of I_i^k, are local duals of $C[0,1]$. Therefore $C[0,1]$ does not admit a smallest local dual. Besides, Y and $L_1[0,1]$ are not isomorphic because $Y \equiv \ell_1$.

Example 3.11 also gives two local duals of ℓ_1 with intersection $\{0\}$, but Proposition 5.18 will give a sufficient condition for the existence of a smallest local dual of a given space.

Given any Banach space X and any probability measure μ, the identity $L_1(\mu, X)^* = L_\infty(\mu, X^*)$ holds only when X^* has the Radon-Nikodym property with respect to μ [19, Thm. 1 in page 98]. However, the following result shows that $L_\infty(\mu, X^*)$ is always a local dual of $L_1(\mu, X)$.

Theorem 3.17. [35] *Given a Banach space X and a probability measure μ, we have:*

(a) The canonical copy of $L_1(\mu, X^*)$ in $L_\infty(\mu, X)^*$ is a local dual of $L_\infty(\mu, X)$.
(b) The canonical copy of $L_\infty(\mu, X^*)$ in $L_1(\mu, X)^*$ is a local dual of $L_1(\mu, X)$.

Sketch of the proof: Part (b) admits a similar proof to that of (a), so only (a) is sketched.

(a) Take a pair of finite dimensional subspaces E of $L_\infty(\mu, X)^*$ and F of $L_\infty(\mu, X)$, and let $\varepsilon > 0$. Let $\{\phi_i\}_{i=1}^l$ be a basis of $E_0 := E \cap L_1(\mu, X^*)$ and $\{h_i\}_{i=1}^k$ a basis of F. By Proposition 3.9, it is enough to find an operator $L\colon E \longrightarrow L_1(\mu, X^*)$ satisfying conditions (A'), (B') and (C') with respect to F and ε. In order to obtain (C'), we approximate each ϕ_i by a function $f_i = \sum_{n=1}^\infty \chi_{C_n} \otimes x_{in}^* \in L_1(\mu, X^*)$ (with $x_{in}^* \in X^*$). Thus, if $\|\phi_i - f_i\|$ is small enough and L maps ϕ_i to f_i, then (C') will be satisfied with respect to ε. In turn, the measurable sets C_n must be taken small enough so that the functions $g_i = \sum_{n=1}^\infty \chi_{C_n} \otimes x_{in} \in L_\infty(\mu, X)$ (for a suitable choice of elements $x_{in} \in X$) approximate the functions h_i in norm. Note that if the operator L satisfies (B) with respect to $\operatorname{span}\{g_i\}_{i=1}^k$, and if $\|h_i - g_i\|$ is small enough for each i, then L will satisfy condition (B') with respect to F and ε.

Next, we consider the norm one projection P on $L_\infty(\mu, X)$ given by

$$P(f) := \sum_{n=1}^\infty \mu(C_n)^{-1} \chi_{C_n} \otimes \int_{C_n} f \, d\mu.$$

Given $n \in \mathbb{N}$ and $\phi \in L_\infty(\mu, X)^*$, we define $(P^*\phi)_n \in X^*$ by

$$\langle (P^*\phi)_n, x\rangle := \langle \phi, \mu(C_n)^{-1} \chi_{C_n} \otimes x\rangle, \quad x \in X.$$

The action of duality of $P^*(\phi)$ on every $h \in L_\infty(\mu, X)$ is

$$\langle P^*(\phi), h\rangle = \Big\langle \sum_{n=1}^\infty \chi_{C_n} \otimes (P^*\phi)_n, h\Big\rangle. \tag{12}$$

Formula (12) implies $R(P^*) \subset L_1(\mu, X^*)$. Moreover, P^* maps each f_i to itself, so the restriction $P^* \mid_E$ satisfies conditions (A') and (C') with respect to ε. Besides, for every $\phi \in E$ and each g_i, formula (12) also yields $\langle L(\phi), g_i\rangle = \langle \phi, g_i\rangle$ which means, that if the values $\|h_i - g_i\|$ are small enough, then L satisfies (B') with respect to F and ε. □

Recall that $C[0,1]^*$ can be identified with the space $\mathcal{M}[0,1]$ of all Borel measures on $[0,1]$. Let ω_1 be the first uncountable ordinal. The Baire classes $B_\alpha[0,1]$, $0 \le \alpha \le \omega_1$, are defined by transfinite induction as follows. The class $B_0[0,1]$ is $C[0,1]$, and for each ordinal $1 \le \alpha \le \omega_1$, $B_\alpha[0,1]$ is the set of all the bounded functions on $[0,1]$ which are pointwise limits of sequences in $\cup_{\beta<\alpha} B_\beta[0,1]$. The class $B_{\omega_1}[0,1]$ coincides with the space of all bounded Borel measurable functions on the unit interval [3].

Each $B_\alpha[0,1]$, endowed with the supremum norm, is a Banach space that can be identified with a subspace of $\mathcal{M}[0,1]^* \equiv C[0,1]^{**}$, where the duality is given by

$$\langle f, \mu\rangle = \int_0^1 f \, d\mu \quad \text{ for } f \in B_{\omega_1}[0,1] \text{ and } \mu \in M[0,1].$$

These spaces were studied in [3, 12, 13].

Theorem 3.18. *[38] For every ordinal $1 \le \alpha \le \omega_1$, the Baire class $B_\alpha[0,1]$ is a local dual of $\mathcal{M}[0,1]$.*

The proof is rather technical, and consists of two steps. The first one is a finite dimensional version of a representation of $M[0,1]^*$ given by Mauldin [57]. This version does not use the Continuum Hypothesis, unlike Mauldin's representation. The second step is based upon the fact that the class $B_\alpha[0,1]$ with $\alpha \geq 1$ admits a dense (non-closed) subspace spanned by characteristic functions.

4. Ultrapower characterizations

A typical use of ultrapowers is the translation of local notions into global ones. For instance, the fact that a space X is finitely representable in another space Y is translated to the language of ultraproducts as follows: X is isometrically contained in some ultrapower of Y. In a similar way, in this section we will translate the local notions of finite dual representability, local duality and local complementation to the ultraproduct language.

Given a subspace Z of a dual space X^* and an ultrafilter $\mathfrak{U}$ on I, we denote by

$$\mathcal{Q}_{Z_\mathfrak{U}} \colon Z_\mathfrak{U} \longrightarrow X^*$$

the norm one operator that maps each $[z_i]$ to $\sigma(X^*, X)$-$\lim_\mathfrak{U} z_i$.

Theorem 4.1. *A subspace Z of Y is locally α-complemented in Y if and only if there exists an ultrafilter $\mathfrak{V}$ and an operator $L\colon Y \longrightarrow Z_\mathfrak{V}$ such that $\|L\| \leq \alpha$ and $L|_Z = J_{Z,Z_\mathfrak{V}}$.*

Proof. Assume Z is locally α-complemented in Y. Let $\mathfrak{V}$ be an ultrafilter refining the order filter on the set of all pairs $i \equiv (E_i, \varepsilon_i)$ with E_i a finite dimensional subspace of Y and $\varepsilon_i > 0$, where $j \preceq i$ if $E_j \subset E_i$ and $\varepsilon_j \geq \varepsilon_i$. By hypothesis, for every pair $i = (E_i, \varepsilon_i)$ there exist an operator $L_i \colon E_i \longrightarrow Z$ such that $\|L_i\| \leq \alpha + \varepsilon_i$ and $L_i(z) = z$ for all $z \in E_i \cap Z$. Thus we define $J \colon Y \longrightarrow Z_\mathfrak{V}$ by $Jy = [z_i]$, where

$$z_i = \begin{cases} L_i y, & \text{if } y \in E_i; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if $\|y\| \leq 1$, then $\|Ly\| = \lim_\mathfrak{V} \|z_i\| \leq \lim_\mathfrak{V} \alpha + \varepsilon_i = \alpha$ and for every $z \in Z$, $Lz = [z]$, concluding the proof of the direct implication.

For the reverse, assume there is an ultrafilter $\mathfrak{U}$ and an operator $L\colon Y \longrightarrow Z_\mathfrak{U}$ with $\|L\| \leq \alpha$ and $L|_Z = J_{Z,Z_\mathfrak{U}}$. Let E be a finite dimensional subspace of Y, let $\{y_k\}_{k=1}^n$ be a basis of E with $\operatorname{span}\{y_k\}_{k=m+1}^n = E \cap Z$ and fix $\varepsilon > 0$. For every y_k, let (z_i^k) be a representative of $L(y_k)$; in the case when $m+1 \leq k \leq n$, the corresponding representative is chosen to be $z_i^k = y_k$ for all i. Let $L_i \colon L(E) \longrightarrow Z$ be defined by $L_i([z_i^k]) := z_i^k$ and take $\varepsilon_1 > 0$ small enough so that $\alpha\varepsilon_1 < \varepsilon$. By [40, Lemma 7.3] or [38, Lemma A.4.12], there exists $\Lambda \in \mathfrak{U}$ such that for every $i \in \Lambda$, L_i is an ε_1-isometry. Thus, if $i \in \Lambda$, the operator $T := L_i L$ satisfies $\|T\| \leq (1+\varepsilon_1)\alpha \leq \alpha + \varepsilon$ and $T(z) = z$ for all $z \in E \cap Z$. We have just proved that Z is locally α-complemented in X^*. $\square$

The following theorem is partially proved in [36]. Part (d) for the particular case of the P.L.R. was proved in [69, Theorem 6.14].

Theorem 4.2. *Given a subspace Z of X^*, the following statements hold:*

(a) Z is a norming subspace if and only if there exists an ultrafilter $\mathfrak{U}$ such that $\mathcal{Q}_{Z_\mathfrak{U}}(B_{Z_\mathfrak{U}}) = B_{X^*}$.

(b) X^* is finite dual representable in Z if and only if there is an ultrafilter $\mathfrak{U}$ and an isometry $J \in \mathcal{B}(X^*, Z_\mathfrak{U})$ such that $\mathcal{Q}_{Z_\mathfrak{U}} J = I_{X^*}$.

(c) Z is an ideal in X^* if and only if there exists an ultrafilter $\mathfrak{V}$ and an operator $L\colon X^* \longrightarrow Z_\mathfrak{V}$ with $\|L\| = 1$ and $L\,|_Z = J_{Z,Z_\mathfrak{V}}$.

(d) Z is a local dual of X if and only if there is an ultrafilter $\mathfrak{U}$ and an isometry $J \in \mathcal{B}(X^*, Z_\mathfrak{U})$ such that $\mathcal{Q}_{Z_\mathfrak{U}} J = I_{X^*}$ and $J|_Z = J_{Z,Z_\mathfrak{U}}$.

Proof. (a) Assume Z is norming, that is, B_Z is w^*-dense in B_{X^*}. We consider the set I of triples $\alpha \equiv (E_\alpha, F_\alpha, \varepsilon_\alpha)$ where E_α runs over all finite dimensional subspaces of X^*, F_α runs over all finite dimensional subspaces of X and ε_α runs over all real numbers $0 < \varepsilon_\alpha < 1$; I is endowed with the order $\alpha \preceq \beta$ defined by $E_\alpha \subset E_\beta$, $F_\alpha \subset F_\beta$ and $\varepsilon_\alpha \geq e_\beta$, and $\mathfrak{U}$ is any ultrafilter refining the order filter on I. Let $x^* \in B_{X^*}$. For every index α we take a w^*-neighborhood $\mathcal{V}_\alpha$ of x^* such that $F_\alpha^\perp + \varepsilon_\alpha B_{X^*} \subset \mathcal{V}_\alpha$. Pick $x_\alpha^* \in \mathcal{V}_\alpha \cap B_Z$. Since for all w^*-neighborhood $\mathcal{V}$ of x^* there exists an index α such that $\mathcal{V}_\alpha \subset \mathcal{V}$, it is immediate that $\mathcal{Q}_{Z_\mathfrak{U}}([x_\alpha^*]) = x^*$.

For the converse, take a norm one element $x \in X$ and choose a norm one functional $x^* \in X^*$ such that $1 = \langle x^*, x\rangle$. Following the hypothesis, there is a family $(x_\alpha)_{\alpha\in I} \subset B_Z$ such that $\mathcal{Q}_{Z_\mathfrak{U}}([x_\alpha^*]) = x^*$. Hence $1 = \lim_{\alpha\to\mathfrak{U}}\langle x_\alpha^*, x\rangle$, so Z is norming.

Note that, by (a), in the proof of direct implications of (b) and (d) we can assume $\mathcal{Q}_{Z_\mathfrak{U}}(B_{Z_\mathfrak{U}}) = B_{X^*}$.

(b) Suppose that X^* is finite dual representable in Z and consider the same ultrafilter $\mathfrak{U}$ of the 'if' implication in the proof of (a). Then for every index α there exists a ε_α-isometry $J_\alpha\colon E_\alpha \longrightarrow Z$ that satisfies (B) with respect to F_α.

We define $J \in \mathcal{B}(X^*, Z_\mathfrak{U})$ by $Jx^* := [(Jx^*)_\alpha]$, where

$$(Jx^*)_\alpha = \begin{cases} J_\alpha x^*, & \text{if } x^* \in E_\alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly J is an isometry. Moreover, for every $x \in X$ and $x^* \in X^*$, we have

$$\langle \mathcal{Q}_{Z_\mathfrak{U}} Jx^*, x\rangle = \lim_{\alpha\to\mathfrak{U}} \langle (Jx^*)_\alpha, x\rangle = \langle x^*, x\rangle.$$

Hence $\mathcal{Q}_{Z_\mathfrak{U}} J = I_{X^*}$.

Conversely, assume that there is an ultrafilter $\mathfrak{U}$ on a set Λ and an isometry $J\colon X^* \to Z_\mathfrak{U}$ satisfying $\mathcal{Q}_{Z_\mathfrak{U}} J = I_{X^*}$. Consider a pair of finite dimensional subspaces E of X^* and F of X, and $0 < \varepsilon < 1$. Since $J\,|_E$ has finite rank, by [40, Lemma 7.3] or [38, Lemma A.4.12] there is a bounded family of operators $(L_\alpha)_{\alpha\in\Lambda}$ from $J(E)$ into Z such that $J = [L_\alpha J]$, and moreover, there exists $\Lambda_1 \in \mathfrak{U}$ so that $J_\alpha := L_\alpha J$ is an ε-isometry for all $\alpha \in \Lambda_1$.

By Lemma 2.4, there exist a normalized basis $\{x_1, \dots, x_m\}$ of F and a subset $\{x_1^*, \dots, x_m^*\}$ in X^* such that $\|x_i^*\| \le 1$ and $\langle x_i^*, x_j\rangle = \delta_{ij}$ for all i and j in $\{1, \dots, m\}$, and besides, there exist a normalized basis $\{y_1^*, \dots, y_n^*\}$ of E and elements $\{y_1^{**}, \dots, y_n^{**}\}$ in X^{**} such that $E_0 := E \cap Z = \operatorname{span}\{y_i^*\}_{i=1}^k$, $\|y_i^{**}\| \le 1 + n$ and $\langle y_i^*, y_j^{**}\rangle = \delta_{ij}$ for all i and j in $\{1, \dots, n\}$ (eventually, E_0 only plays a role in the proof of (d) as we will see). By the principle of local reflexivity, we obtain $\{y_1, \dots, y_n\}$ in $(3/2)B_X$ such that $\langle y_i^*, y_j\rangle = \delta_{ij}$ for all i and j.

Since $\mathcal{Q}_{Z_{\mathfrak{U}}} J = I_{X^*}$, we can select $\alpha \in \Lambda_1$ so that

$$|\langle J_\alpha y_i^* - y_i^*, x_j\rangle| < \frac{\varepsilon}{2mn(1+n)} \quad \text{for all } i \text{ and } j.$$

Thus, for every $y^* \in E$ and every $x \in F$, as $y^* = \sum_{i=1}^n \langle y^*, y_i\rangle y_i^*$ and $x = \sum_{j=1}^m \langle x_j^*, x\rangle x_j$, we obtain $|\langle J_\alpha y^* - y^*, x\rangle| \le \varepsilon \|y^*\|\|x\|$, and applying Proposition 3.9, the proof is done.

(c) Take $Y = X^*$ and $\alpha = 1$ in Theorem 4.1.

(d) For the direct implication, for every index α we choose a ε_α-isometry $J_\alpha \colon E_\alpha \longrightarrow Z$ satisfying conditions (A), (B) and (C) with respect to F_α and ε_α. Proceeding as in the 'if' part of (b), we get an isometry $J \colon X^* \longrightarrow Z_{\mathfrak{U}}$ such that $Q_{Z_{\mathfrak{U}}} J = I_{X^*}$. Moreover, since $J_\alpha g = g$ for all $g \in Z \cap E_\alpha$, we get $Jx^* = [x^*]$ for every $x^* \in Z$.

For the converse implication, we consider the same ultrafilter $\mathfrak{U}$ of the 'if' implication in the proof of (a) and proceed as in the converse of (b), but instead of choosing $\alpha \in \Lambda_1$, we choose $\alpha \in \Lambda_1 \cap \Lambda_2$, where

$$\Lambda_2 := \left\{ \alpha \in \Lambda : \|J_\alpha y_i^* - y_i^*\| < \frac{2\varepsilon}{3n(n+1)},\ 1 \le i \le k \right\}.$$

Notice that since $J\mid_Z$ is the natural embedding of Z into $Z_{\mathfrak{U}}$, then $\Lambda_2 \in \mathfrak{U}$. Moreover, for every norm one element $y^* = \sum_{i=1}^k \lambda_i y_i^* \in E_0$, as $|\lambda_i| = |\langle y^*, y_i\rangle| \le 3(1+n)/2$ for all $1 \le i \le k$, it follows that $\|J_\alpha(y^*) - y^*\| \le \varepsilon$. Therefore, the ε-isometry J_α on E satisfies (B') and (C') with respect to F and ε. Thus Proposition 3.9 shows that Z is a local dual of X. □

Remark 4.3. Ultrapowers allow us to avoid tedious ε-δ computations. For instance, Theorem 4.2 can be applied in order to ease some steps in Proposition 3.15 as follows:

Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{N}$ and consider the operator $J \colon L_\infty(\mathbb{T}) \longrightarrow C(\mathbb{T})_{\mathfrak{U}}$ defined by $J(g) := [T_n(g)]$. Statement (ii) gives $\|J\| \le \lim_{\mathfrak{U}} \|T_n\| = 1$, and statement (v) yields $\mathcal{Q}_{C(\mathbb{T})_{\mathfrak{U}}} J(g) = w^*\text{-}\lim_{\mathfrak{U}} T_n(g) = g$, so $\mathcal{Q}_{C(\mathbb{T})_{\mathfrak{U}}} J = I_{L_\infty(\mathbb{T})}$. Therefore J is an isometry. Finally, for every $g \in C(\mathbb{T})$, statement (iii) shows $\|J(g) - [g]\| = \lim_{\mathfrak{U}} \|T_n(g) - g\| = 0$. Thus Theorem 4.2 proves that $C(\mathbb{T})$ is a local dual of $L_1(\mathbb{T})$.

The 'only if' implication of the following theorem was proved in [44, Theorem 4.1].

Theorem 4.4. *Let Y be a Banach space and $\mathfrak{U}$ an ultrafilter on I. Given a subspace Z of Y, Z is locally α-complemented in Y if and only if $Z_{\mathfrak{U}}$ is locally α-complemented in $Y_{\mathfrak{U}}$.*

Proof. Assume $Z_{\mathfrak{U}}$ is locally α-complemented in $Y_{\mathfrak{U}}$. Thus, proceeding as in Theorem 4.2, we obtain an ultrafilter $\mathfrak{V}$ and an operator $L\colon Y_{\mathfrak{U}} \longrightarrow (Z_{\mathfrak{U}})_{\mathfrak{V}}$ such that $\|L\| \leq \alpha$ and $L(\mathbf{z}) = J_{Z_{\mathfrak{U}},(Z_{\mathfrak{U}})_{\mathfrak{V}}}(\mathbf{z})$ for all $\mathbf{z} \in Z_{\mathfrak{U}}$. In particular, as $(Z_{\mathfrak{U}})_{\mathfrak{V}}$ is isometrically identified with $Z_{\mathfrak{U}\times\mathfrak{V}}$, we can regard the restriction $T := L|_Y$ as an operator from Y into $Z_{\mathfrak{U}\times\mathfrak{V}}$ satisfying $\|T\| \leq \alpha$ and $T(z) = J_{Z,Z_{\mathfrak{U}\times\mathfrak{V}}}(z)$ for all $z \in Z$. Thus, acting again as in Theorem 4.2, we prove that Z is locally α-complemented in Y.

Assume now that Z is locally α-complemented in Y, let E be a finite dimensional subspace of $Y_{\mathfrak{U}}$ and let $\varepsilon > 0$. Let $\{\mathbf{e}_i\}_{i=1}^n$ be a basis of E such that $\{\mathbf{e}_i\}_{i=1}^m$ spans $E \cap Z_{\mathfrak{U}}$. Choose a representative $(e_i^j)_{j\in I} \subset E$ of every $\mathbf{e}_i$ such that $e_i^j \in E \cap Z$ for all $j \in I$ and all $1 \leq i \leq m$. For every $j \in I$, let $E_j := \operatorname{span}\{e_i^j\}_{i=1}^n$; by hypothesis, there exists an operator $L_j\colon E_j \longrightarrow Z$ such that $\|L_j\| \leq \alpha + \varepsilon$ and $L_j(z) = z$ for all $z \in E_j \cap Z$. Typical ultraproduct arguments yield that the ultraproduct operator $L\colon E \longrightarrow Z_{\mathfrak{U}}$ that sends each $[e_i^j]$ to $[L_j(e_i^j)]$ satisfies $\|L\| \leq \alpha + \varepsilon$ and $L(\mathbf{z}) = \mathbf{z}$ for all $\mathbf{z} \in E \cap Z_{\mathfrak{U}}$. □

A consequence of Theorem 4.4 is that Z is an ideal in Y if and only if so is $Z_{\mathfrak{U}}$ in $Y_{\mathfrak{U}}$. Compare this assertion with part (b) in the following theorem.

Theorem 4.5. *Given a subspace Z of X^* and an ultrafilter $\mathfrak{U}$ on I, the following statements hold:*

(a) X^* is finite dual representable in Z if and only if $(X_{\mathfrak{U}})^*$ is finite dual representable in $Z_{\mathfrak{U}}$.
(b) Z is an ideal in X^* if and only if $Z_{\mathfrak{U}}$ is an ideal in $(X_{\mathfrak{U}})^*$.
(c) Z is a local dual of X if and only if $Z_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$.
(d) Z is norming if and only if $Z_{\mathfrak{U}}$ is norming.

Proof. (a) The proof is essentially contained in that of (c).

(b) It follows directly from Theorem 4.4, the principle of local reflexivity for ultrapowers and statement (i) in Remark 3.5.

(c) Assume that Z is a local dual of X and fix a couple of finite dimensional subspaces F of $(X_{\mathfrak{U}})^*$ and G of $X_{\mathfrak{U}}$, and $0 < \varepsilon < 1$. Since $(X^*)_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$, there is an $\varepsilon/4$-isometry $L_1\colon F \longrightarrow (X^*)_{\mathfrak{U}}$ satisfying (B) and (C) with respect to G.

Fix a basis $\{[f_i^1],\dots,[f_i^m]\}$ of $L_1(F)$ and a basis $\{[x_i^1],\dots,[x_i^n]\}$ of G. For every $i \in I$, let $F^i := \operatorname{span}\{f_i^1 \dots f_i^n\} \subset X^*$ and $G^i := \operatorname{span}\{x_i^1 \dots x_i^n\} \subset X$.

Now, for every $i \in I$ we select an $\varepsilon/4$-isometry $L^i\colon F^i \longrightarrow Z$ satisfying (B) and (C) with respect to G^i, and define an $\varepsilon/4$-isometry $L_2\colon L_1(F) \longrightarrow Z_{\mathfrak{U}}$ by $L_2[f_i] := [L^i f_i]$.

Clearly $L := L_2 L_1\colon F \longrightarrow Z_{\mathfrak{U}}$ is an ε-isometry satisfying (B) and (C) with respect to G. Thus $Z_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$.

Conversely, assume that $Z_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$. Let $F \subset X^*$ and $G \subset X$ be finite dimensional subspaces and fix $0 < \varepsilon < 1/2$. Take $0 < \varepsilon' < 2^{-3}\varepsilon$ and choose an ε'-net $\{x_j^*\}_{j=1}^n$ in B_F and an ε'-net $\{x_j\}_{j=1}^n$ in B_G. Note that $\{x_j^*\}_{j=1}^n$ includes a basis of F. Since we can consider F and G as subspaces of $(X^*)_{\mathfrak{U}}$ and $X_{\mathfrak{U}}$, there is an ε'-isometry $L\colon F \longrightarrow Z_{\mathfrak{U}}$ satisfying (B) and (C) with respect to G.

Since L has finite rank, by [40, Lemma 7.3] there exist a set $\Lambda_1 \in \mathfrak{U}$ and a bounded family of operators $(L_i)_{i\in I}$ from F into Z such that $L = [L_i]$ and, for each $i \in \Lambda_1$, L_i is an ε-isometry. Moreover, there exists $\Lambda_2 \in \mathfrak{U}$ such that, for every $i \in \Lambda_2$,

$$|\langle L_i x_j^*, x_k\rangle - \langle x_j^*, x_k\rangle| \leq \varepsilon' \|x_j^*\|\,\|x_k\| \text{ for all } j,k = 1,\dots,n, \text{ and}$$

$$\|L_i(x_j^*) - x_j^*\| \leq \varepsilon' \|x_j^*\| \text{ for all } j = 1,\dots,n.$$

We fix $i \in \Lambda_1 \cap \Lambda_2$ and denote $T = L_i$. Thus T is a ε-isometry. Let us show that T satisfies (B') and (C') with respect to G and ε.

First, we take $x^* \in S_F$ and $x \in S_G$. We pick x_k^* and x_l so that $\|x^* - x_k^*\| < \varepsilon'$ and $\|x - x_l\| < \varepsilon'$. Thus, as

$$\langle (I-T)x^*\,,\, x\rangle = \langle (I-T)x^*\,,\, x - x_l\rangle + \langle (I-T)(x^* - x_k^*)\,,\, x_l\rangle + \langle (I-T)x_k^*\,,\, x_l\rangle$$

we obtain $|\langle (I-T)x^*\,,\, x\rangle| \leq 2(2+\varepsilon')\varepsilon' + \varepsilon' < \varepsilon$. Second, we take $x^* \in S_F \cap Z$, and pick x_k^* so that $\|x^* - x_k^*\| < \varepsilon'$. Therefore

$$\|T(x^*) - x^*\| \leq \|T(x^*) - T(x_k^*)\| + \|T(x_k^*) - x_k^*\| + \|x_k^* - x^*\| < 2\varepsilon' + \varepsilon' + \varepsilon' < \varepsilon.$$

Hence T satisfies (A), (B') and (C') with respect to G and ε, and the proof is done.

(d) Assume Z is a norming subspace of X^*. Let $[x_i] \in X_{\mathfrak{U}}$ and $\varepsilon > 0$. For each $i \in I$ there is a norm one element $x_i^* \in Z$ such that $\langle x_i^*, x_i\rangle \geq \|x_i\| - \varepsilon$, so $\langle [x_i^*], [x_i]\rangle \geq \|[x_i]\| - \varepsilon$, which proves that $Z_{\mathfrak{U}}$ is norming. For the converse, assume that $Z_{\mathfrak{U}}$ is a norming subspace of $(X_{\mathfrak{U}})^*$. Given $x \in X$, there is $[x_i^*] \in Z_{\mathfrak{U}}$ such that $\|x_i^*\| = 1$ for all i and $\|x\| = \langle [x_i^*], [x]\rangle$. Thus $\|x\| = \lim_{i\to\mathfrak{U}} \langle x_i^*, x\rangle$, hence Z is a norming subspace of X^*. □

Open problem 4.6. *Let $\mathfrak{U}$ be an ultrafilter and let X be a subspace of Y. If $X_{\mathfrak{U}}$ is complemented in $Y_{\mathfrak{U}}$, is X complemented in Y?*

Local complementability is useful in finding complemented subspaces.

Proposition 4.7. *If Z is a locally complemented subspace of Y and Z is complemented in Z^{**} then Z is complemented in Y.*

Proof. Let I be the set of all pairs $i \equiv (E_i, \varepsilon_i)$ where E_i is a finite dimensional subspace of Y and ε is a positive real number. Let us endow I with the order

$i \preceq j$ defined by $E_i \subset E_j$ and $\varepsilon_i \geq \varepsilon_j$. Let $\mathfrak{U}$ be an ultrafilter on I refining the order filter on I induced by $\preceq$.

For every $i \in I$, assume Z is locally α-complemented in Y. Then there exists an operator $T_i \colon E_i \longrightarrow Z$ such that $\|T_i\| \leq (1+\varepsilon_i)\alpha$ and $T_i(z) = z$ for all $z \in E_i \cap Z$. Let $T\colon Y \longrightarrow Z^{**}$ be the operator defined by $T(y) := \sigma(Z^{**}, Z^*)\text{-}\lim_{i\to\mathfrak{U}} y_i$ where $y_i := T_i(y)$ if $y \in E_i$ and $y_i := 0$ otherwise.

Let $P\colon Z^{**} \longrightarrow Z^{**}$ be a projection onto Z and let $J\colon Z \longrightarrow Y$ be the natural inclusion of Z into Y. Then, as PTJ is the identity operator on Z, it follows that JPT is a projection on Y whose range is Z, and the proof is done. □

A Banach space X is called *ultrasummand* [44] if for every ultrafilter $\mathfrak{U}$, the canonical copy of X contained in the ultrapower $X_{\mathfrak{U}}$ is complemented in $X_{\mathfrak{U}}$.

Proposition 4.8. *A Banach space X is complemented in X^{**} if and only if X is an ultrasummand.*

Proof. Assume X is complemented in X^{**}. Since X is an ideal in every ultrapower $X_{\mathfrak{U}}$, Proposition 4.7 proves that X is complemented in all ultrapowers of X.

For the converse, assume X is an ultrasummand. As X is a local dual of X^*, Theorem 4.2 provides an ultrafilter $\mathfrak{U}$ and an isometry $J\colon X^{**} \longrightarrow X_{\mathfrak{U}}$ that maps each $x \in X$ to the constant class $[x]$. But X is an ultrasummand, so there exists a surjection $Q\colon X_{\mathfrak{U}} \longrightarrow X$ that maps each constant class $[x]$ to x. Thus $Q \circ J \circ J_X$ is the identity operator on X, hence $J_X \circ Q \circ J$ is a projection that maps X^{**} on the canonical copy of X in X^{**}. □

5. Global characterizations

The main results of this section are the global characterizations of the notions of finite dual representability, local complementation and local duality given in subsection 5.1. These characterizations play a preeminent role in understanding the three mentioned local notions and their properties. Some of their applications is the discovery of a sort of local symmetry between Z and X when Z is a local dual of X (Theorem 5.8). They are also useful in finding examples of local duality (Subsection 5.2) and in certain results about the existence of special local duals (Subsection 5.3).

Notations: Since biduals, third and fourth duals will occur very often in this section, and as the notions of finite dual representability, local complementation and local duality depend upon the intrinsic position of the subspaces inside X^*, it is necessary to introduce precise notations and conventions in order to avoid misunderstandings.

The canonical embedding of X in X^{**} is denoted by $J_X\colon X \longrightarrow X^{**}$. Sometimes, we may write X instead of $J_X(X)$ to denote the canonical copy of X into its bidual; given $x \in X$, we also may write x or $x|_{X^*}$ rather than $J_X(x)$.

The n-th dual of X for $n \geq 3$ will be denoted $X^{(n)}$ and its elements by $x^{(n)}$, $y^{(n)}$... etc.

Given a subspace E of X, $E^{\perp}$ denotes the annihilator of E in X^*, and $E^{\perp\perp}$ denotes the annihilator of $E^{\perp}$ in X^{**}. Given a subspace F of X^*, $F_{\perp}$ denotes the annihilator of F in X.

Subspaces and their inclusion operators. Given a closed subspace E of a Banach space F, let $\mathcal{J}_E \colon E \longrightarrow F$ and $\mathcal{Q}_E \colon F \longrightarrow F/E$ denote the corresponding inclusion and quotient operators respectively. The conjugate $\mathcal{J}_E^*$ is the restriction operator given by $\mathcal{J}_E^*(x^*) = x^* \circ \mathcal{J}_E = x^* \,|\,_E$ for all $x^* \in F^*$. Moreover

$$\mathcal{J}_E^*(B_{F^*}) = B_{E^*},\ E^{\perp} = N(\mathcal{J}_E^*) = R(\mathcal{Q}_E^*),\ E^{\perp\perp} = N(\mathcal{Q}_E^{**}) = R(\mathcal{J}_E^{**}) \text{ and}$$

$$J_F(E) = E^{\perp\perp} \cap J_F(F). \tag{13}$$

Subspaces of dual spaces. The elements of a given subspace Z of X^* will be denoted by z (as element of Z) or by x^* (as element of X^*). Let Z be a subspace of X^*. Every $z^* \in Z^*$ can be written as $\mathcal{J}_Z^*(x^{**}) = x^{**} \,|\,_Z$, for some $x^{**} \in X^{**}$.

Note that $\mathcal{J}_Z^{**}$ is an isometry with range $Z^{\perp\perp} \subset X^{(3)}$. The bijective isometry from Z^{**} onto $Z^{\perp\perp}$ induced by $\mathcal{J}_Z^{**}$ will be denoted $I_{Z^{**},Z^{\perp\perp}}$ and its inverse will be denoted $I_{Z^{\perp\perp},Z^{**}} := I^{-1}_{Z^{**},Z^{\perp\perp}}$. Thus

$$\mathcal{J}_Z^{**} = \mathcal{J}_{Z^{\perp\perp}} \circ I_{Z^{**},Z^{\perp\perp}} \tag{14}$$

where $\mathcal{J}_{Z^{\perp\perp}}$ is the inclusion operator of $Z^{\perp\perp}$ in $X^{(3)}$. As it will be explained in below, the context of local duality does not admit the identification of Z^{**} with $Z^{\perp\perp}$.

Biduals as subspaces of fourth duals. As it has been already noticed, the bidual of a space X is isometrically embedded in $X^{(4)}$ in two natural ways: via $J_{X^{**}}$, which maps X^{**} onto its canonical copy in $X^{(4)}$, and via J_X^{**}, which maps X^{**} onto $J_X(X)^{\perp\perp}$. It is remarkable that both $J_{X^{**}}(X^{**})$ and $J_X(X)^{\perp\perp}$ lay in different positions as subspaces of $X^{(4)}$. Indeed, formula (13) yields $J_{X^{**}}(X^{**}) \cap J_X(X)^{\perp\perp} = J_{X^{**}}\big(J_X(X)\big)$, the canonical copy of X contained in the canonical copy of X^{**} in $X^{(4)}$.

The third dual of X is decomposed by the norm one projection $J_{X^*} \circ J_X^*$ as

$$X^{(3)} = J_{X^*}(X^*) \oplus J_X(X)^{\perp}. \tag{15}$$

This decomposition applied to $(X^*)^{(3)}$ yields

$$X^{(4)} = J_{X^{**}}(X^{**}) \oplus J_{X^*}(X^*)^{\perp}. \tag{16}$$

Besides, as $J_X^* \circ J_{X^*} = I_{X^*}$, then $J_{X^*}^* \circ J_X^{**} = I_{X^{**}}$, hence $J_X^{**} \circ J_{X^*}^*$ is a projection on $X^{(4)}$ whose range is $R\big(J_X^{**}\big) = J_X(X)^{\perp\perp}$ and its kernel is $N\big(J_{X^*}^*\big) = J_{X^*}(X^*)^{\perp}$, giving a second decomposition of $X^{(4)}$ as

$$X^{(4)} = J_X(X)^{\perp\perp} \oplus J_{X^*}(X^*)^{\perp}. \tag{17}$$

It follows from (16) and (17) that both subspaces $J_{X^{**}}(X^{**})$ and $J_X(X)^{\perp\perp}$ are complemented in $X^{(4)}$ sharing a same complement in spite of the fact that they lay in different positions inside $X^{(4)}$. However, X^{**} is a local dual of $X^{(3)}$ while formulas (15) and (17) show that $X^{(4)}$ is not even f.d.r. in $J_X(X)^{\perp\perp}$. Those considerations should prevent the reader from identifying X^{**} with $J_X(X)^{\perp\perp}$.

Extension operators. Given a subspace Z of Y, an operator $T\colon Z^* \longrightarrow Y^*$ is said to be an *extension operator* if $T(z^*)|_Z = z^*$ for all $z^* \in Z^*$. In technical words, if $\mathcal{J}_Z$ is the embedding of Z into Y, T is an extension operator when $\mathcal{J}_Z^* \circ T = I_{Z^*}$. Since this paper is mainly devoted to local duality, we will restrict ourselves to the case $Y = X^*$ although most of the results in this subsection can be easily adapted to the general case.

Proposition 5.1. *Given a subspace Z of X^*, an operator $T\colon Z^* \longrightarrow X^{**}$ is an extension operator if and only if $T^* \circ J_{X^*} \circ \mathcal{J}_Z = J_Z$.*

Proof. For the direct implication, let $x^* \in Z$ and $x^{**} \in X^{**}$. Then

$$\langle x^{**}|_Z, T^*(x^*)\rangle = \langle T(x^{**}|_Z), x^*\rangle$$

but as $x^* \in Z$, then $\langle T(x^{**}|_Z), x^*\rangle = \langle x^{**}|_Z, x^*\rangle$ and therefore,

$$\langle x^{**}|_Z, T^*(x^*)\rangle = \langle x^{**}|_Z, x^*\rangle$$

which proves that $T^*(x^*) = x^*$.

Conversely, if $x^{**} \in X^{**}$ and $x^* \in Z$ then $\langle T(x^{**}|_Z), x^*\rangle = \langle x^{**}, T^*(x^*)\rangle = \langle x^{**}, x^*\rangle$ which shows that $T(x^{**}|_Z)|_Z = x^{**}|_Z$. In other words, T is an extension operator. $\square$

Lemma 5.2. *For every extension operator $T\colon Z^* \longrightarrow X^{**}$, the following properties hold:*

(i) $\|T(z^*)\| \geq \|z^*\|$ for all $z^* \in Z^*$; in particular, T is injective;
(ii) if $x^{**} \in R(T)$ then $x^{**} = T(x^{**}|_Z)$;
(iii) $T^*|_{Z^{\perp\perp}} = I_{Z^{\perp\perp},Z^{**}}$.

Proof. (i) Trivial.

(ii) Let $x^{**} \in R(T)$ and take $y^{**} \in X^{**}$ such that $x^{**} = T(y^{**}|_Z)$. Since T is an extension operator, $x^{**}|_Z = y^{**}|_Z$. Thus $T(x^{**}|_Z) = T(y^{**}|_Z) = x^{**}$.

(iii) Since $\mathcal{J}_Z^* \circ T = I_{Z^*}$, it follows $T^* \circ \mathcal{J}_Z^{**} = I_{Z^{**}}$, and applying (14), we obtain

$$T^* \circ \mathcal{J}_{Z^{\perp\perp}} \circ I_{Z^{**},Z^{\perp\perp}} = I_{Z^{**}}$$

which proves that $T^*|_{Z^{\perp\perp}} = I_{Z^{\perp\perp},Z^{**}}$ and the proof is done. $\square$

Extension operators are associated with a particular type of projections.

Lemma 5.3. *Given a subspace Z of a dual space X^*, every extension operator $T\colon Z^* \longrightarrow X^{**}$ induces a projection $P\colon X^{**} \longrightarrow X^{**}$ with kernel $Z^\perp$ satisfying $T \circ \mathcal{J}_Z^* = P$ and vice versa.*

Proof. Let $T\colon Z^* \longrightarrow X^{**}$ be an extension operator. Then $\mathcal{J}_Z^* \circ T = I_{Z^*}$; thus it is clear that $P := T \circ \mathcal{J}_Z^*$ is a projection. Moreover, since T is injective, it follows that $N(P) = N(\mathcal{J}_Z^*) = Z^\perp$.

Let $P\colon X^{**} \longrightarrow X^{**}$ be a projection with $N(P) = Z^\perp$ and define an operator $T\colon Z^* \longrightarrow X^{**}$ as follows: given $z^* \in Z^*$, choose an element $x^{**} \in X^{**}$ such that $x^{**}\mid_Z = z^*$ and let $T(z^*) := P(x^{**})$. Note that the identity $N(P) = Z^\perp$ makes the operator T well defined and injective. Obviously, the identity $T \circ \mathcal{J}_Z^* = P$ holds by definition.

In order to see that T is an extension operator, we only need to see that $\mathcal{J}_Z^* \circ T = I_{Z^*}$. Indeed, since P is a projection, we have

$$T \circ (\mathcal{J}_Z^* \circ T) \circ \mathcal{J}_Z^* = T \circ \mathcal{J}_Z^*,$$

but T is injective and $\mathcal{J}_Z^*$ is surjective, so $\mathcal{J}_Z^* \circ T = I_{Z^*}$, as we wanted to prove. □

5.1. Global theorems.

In this subsection, Theorems 5.4, 5.6 and 5.7 describes the local notions of finite dual representability, ideal and local duality in global terms, that is to say, involving operators defined on the whole spaces Z^*, X^{**}, $X^{(3)}$, or $X^{(4)}$ instead of using operators whose domains are finite dimensional subspaces. The proofs of the aforementioned theorems offered in this article differ to a great extent from their original sources ([44], [48], [23] and [34]). The main difference between both approaches is that here, the translation from the local notions into their global counterparts is made throughout the ultrapower characterizations of Section 4. The aim of doing so is to separately trace the role played by the different hypotheses that define the local notions.

Theorem 5.4. *Given a subspace Z of X^*, the following statements are equivalent.*

(1) X^* is f.d.r. in Z,

(2) there exists a norm one operator $T\colon Z^* \longrightarrow X^{**}$ such that $T(x\mid_Z) = x$ for all $x \in X$. In particular, $J_X(X) \subset R(T)$,

(3) there exists a norm one operator $V\colon X^{(3)} \longrightarrow X^{(3)}$ such that $R(V) \subset Z^{\perp\perp}$ and $R(V - I_{X^{(3)}}) \subset J_X(X)^\perp$.

Proof. (1)⇒(2) By Theorem 4.2, there is an ultrafilter $\mathfrak{U}$ and an isometry $J\colon X^* \longrightarrow Z_{\mathfrak{U}}$ such that $\mathcal{Q}_{Z_{\mathfrak{U}}} J = I_{X^*}$. Thus the required operator is $T := J^* \circ J_{Z^*, Z_{\mathfrak{U}}^*}$, such as we will show now. First, note that for every $x^{**}\mid_Z \in Z^*$, the duality action of $T(x^{**}\mid_Z)$ on $x^* \in X^*$ is

$$\langle T(x^{**}\mid_Z), x^*\rangle = \langle [x^{**}\mid_Z], J(x^*)\rangle = \lim_{\mathfrak{U}} \langle x^{**}, z_i\rangle \tag{18}$$

where $J(x^*) = [z_i]$. Thus, given $x \in X$ and $x^* \in X^*$ (with $J(x^*) = [z_i]$), we have

$$\begin{aligned} \langle T(x\mid_Z), x^*\rangle &= \lim_{\mathfrak{U}} \langle x, z_i\rangle = \\ &= \langle x, \mathcal{Q}_{Z_{\mathfrak{U}}}([z_i])\rangle = \langle x, \mathcal{Q}_{Z_{\mathfrak{U}}} J(x^*)\rangle = \langle x, x^*\rangle, \end{aligned} \tag{19}$$

hence, $T(x|_Z) = x$, so $1 \leq \|T\| \leq \|J^*\| \cdot \|J_{Z,Z_{\mathfrak{U}}^*}\| = 1$.

(2)⇒(1). Let us assume there exists a norm one operator $T\colon Z^* \longrightarrow X^{**}$ with $T(x|_Z) = x$ for all $x \in X$. Let E and F be finite dimensional subspaces of X^* and X respectively, let $\varepsilon > 0$, and let us find an ε-isometry $L\colon E \longrightarrow Z$ satisfying conditions (A') and (B') with respect to F and ε so that Proposition 3.9 yields X^* is f.d.r. in Z.

Let $E_1 := T^*(E) \subset Z^{**}$. By the principle of local reflexivity, there exists an ε-isometry $J\colon E_1 \longrightarrow Z$ such that

$$\langle J(z^{**}) - z^{**}, x|_Z \rangle = 0 \quad \text{for all } z^{**} \in E_1 \text{ and all } x \in F.$$

Thus, the desired operator is $L := J \circ T^*|_E$. Indeed, on the one hand we have

$$\|L\| \leq \|J\|\|T^*\| \leq 1 + \varepsilon. \tag{20}$$

which fulfills condition (A') with respect to ε.

On the other hand, for every $x^* \in E$ and every $x \in F$, we have

$$\langle x^*, x\rangle = \langle x^*, T(x|_Z)\rangle = \langle T^*(x^*), x|_Z \rangle = \langle JT^*(x^*), x|_Z \rangle = \langle L(x^*), x\rangle \tag{21}$$

which proves that (B) holds with respect to F. Thus X^* is f.d.r. in Z.

(2)⇒(3) Let us assume that (2) holds and prove that $V := \mathcal{J}_Z^{**} \circ T^*$ satisfies (3). Indeed, it is immediate that $\|V\| = 1$ and $R(V) \subset R(\mathcal{J}_Z^{**}) = Z^{\perp\perp}$. Moreover, given $x^{(3)} \in X^{(3)}$ and $x \in X$,

$$\langle \mathcal{J}_Z^{**} T^*(x^{(3)}), x\rangle = \langle T^*(x^{(3)}), x|_Z \rangle = \langle x^{(3)}, T(x|_Z)\rangle = \langle x^{(3)}, x\rangle$$

which means that $V(x^{(3)}) - x^{(3)} \in J_X(X)^\perp$ for all $x^{(3)} \in X^{(3)}$.

(3)⇒(2) Assume there exists an operator $V\colon X^{(3)} \longrightarrow X^{(3)}$ satisfying (3). As $R(V) \subset Z^{\perp\perp}$, the composition $W := I_{Z^{\perp\perp},Z^{**}} \circ V$ makes sense. We will prove that statement (2) holds for $T := J_{X^*}^* \circ W^* \circ J_{Z^*}$. Indeed, fix $x \in X$. For every $x^* \in X^*$,

$$\begin{aligned}\langle T(x|_Z), x^*\rangle &= \langle W^* J_{Z^*}(x|_Z), J_{X^*}(x^*)\rangle \\ &= \langle J_{Z^*}(x|_Z), W J_{X^*}(x^*)\rangle = \langle x, V\big(J_{X^*}(x^*)\big)\rangle,\end{aligned}$$

but by hypothesis, $V\big(J_{X^*}(x^*)\big) - J_{X^*}(x^*) \in J_X(X)^\perp$, so $\langle T(x|_Z), x^*\rangle = \langle x^*, x\rangle$ for all $x^* \in X^*$, hence $T(x|_Z) = x$. Now it is clear that $\|T\| = 1$, so the proof is finished. □

According to Proposition 5.1, if the conjugate operator of T in Theorem 5.4 fixes Z then T is an extension operator.

The following result is given in [44, Theorem 3.5], [48] and [23].

Theorem 5.5. *Given a subspace Z of a Banach space Y and a real number $\alpha \geq 1$, the following statements are equivalent:*

(1) Z is locally α-complemented in Y,

(2) *there exists an extension operator* $T\colon Z^* \longrightarrow Y^*$ *such that* $\|T\| \le \alpha$,
(3) *there is a projection* $P\colon Y^* \longrightarrow Y^*$ *with* $N(P) = Z^\perp$ *and* $\|P\| \le \alpha$,
(4) *there exists a projection* $Q\colon Y^{**} \longrightarrow Y^{**}$ *such that* $R(Q) = Z^{\perp\perp}$ *and* $\|Q\| \le \alpha$.

Since we are particularly interested in the case when the hypotheses of Theorem 5.5 fit into the context of local duality, we will prove it only in the case when Y is a dual space. The attentive reader will realize that the proof given in the following theorem also works for the general case.

Theorem 5.6. *Given a subspace* Z *of* X^**, the following statements are equivalent:*

(1) Z *is an ideal in* X^*,
(2) *there is an isometric extension operator* $T\colon Z^* \longrightarrow X^{**}$,
(3) *there exists a norm one projection* $P\colon X^{**} \longrightarrow X^{**}$ *with* $N(P) = Z^\perp$,
(4) *there iss a norm one projection* $Q\colon X^{(3)} \longrightarrow X^{(3)}$ *such that* $R(Q) = Z^{\perp\perp}$.

Proof. (1)⇒(2) By Theorem 4.2, there exists an ultrafilter $\mathfrak{U}$ and an operator $J\colon X^* \longrightarrow Z_{\mathfrak{U}}$ with $\|J\| = 1$ and $J\,|_{\,Z} = J_{Z,Z_{\mathfrak{U}}}$. Given the natural inclusion $J_{Z^*,{Z_{\mathfrak{U}}}^*}\colon Z^* \longrightarrow {Z_{\mathfrak{U}}}^*$, we will prove that the required extension operator is $T := J^* \circ J_{Z^*,{Z_{\mathfrak{U}}}^*}$. Indeed, for every $x^{**} \in X^{**}$, the duality action of $T(x^{**}\,|_{\,Z})$ on $x^* \in X^*$ is given by

$$\langle T(x^{**}\,|_{\,Z}), x^*\rangle = \lim_{\mathfrak{U}} \langle x^{**}, z_i\rangle,$$

where $J(x^*) = [z_i]$. Thus, given vectors $x^{**} \in X^{**}$ and $z \in Z$, since $J(z) = z$ it follows that $\langle T(x^{**}\,|_{\,Z}), z\rangle = \langle x^{**}, z\rangle$, hence $T(x^{**}\,|_{\,Z})\,|_{\,Z} = x^{**}\,|_{\,Z}$, so T is an extension operator. Moreover, as $\|T\| \le \|T^*\| \cdot \|J_{Z^*,{Z_{\mathfrak{U}}}^*}\| \le 1$, Lemma 5.2 shows that T is an isometric extension operator.

(2)⇒(1) Assume that $T\colon Z^* \longrightarrow X^{**}$ is an isometric extension operator, and take a finite dimensional subspace E of X^* and $\varepsilon > 0$. We need to find an operator $L\colon E \longrightarrow Z$ satisfying conditions (A') and (C) with respect to ε. To do that, let $E_1 := T^*(E) \subset Z^{**}$. By the principle of local reflexivity, there exists an ε-isometry $J\colon E_1 \longrightarrow Z$ such that

$$J(z) = z \text{ for all } z \in E_1 \cap Z. \tag{22}$$

Let $L := J \circ T^*\,|_{\,E}$. Clearly $\|L\| \le \|J\|\|T\| \le 1 + \varepsilon$, so L satisfies (A') with respect to ε.

In order to check (C), take $z \in E \cap Z$. Note that (13) yields $Z = Z^{\perp\perp} \cap X^*$, and Lemma 5.2 gives $T^*(z) = z \in E_1$; thus, by (22), $L(z) = JT^*(z) = z$. The proof is done.

(2)⇔(3) It is a consequence of Lemma 5.3.

(3)⇒(4) Just take $Q := P^*$.

(4)⇒(3) Let $Q\colon X^{(3)} \longrightarrow X^{(3)}$ be a norm one projection such that $R(Q) = Z^{\perp\perp}$. Denote $J := J_{X^{**}}$, $\mathcal{J} := J_{X^*}$ and $R := \mathcal{J}^*$ (so $R(x^{(4)}) = x^{(4)}\,|_{\,\mathcal{J}(X^*)}$), and consider the quotient operator $U\colon X^{**} \longrightarrow X^{**}/Z^{\perp}$ and the operator $V\colon X^{**}/Z^{\perp} \longrightarrow X^{(4)}/Z^{\perp\perp\perp}$ that sends each $x^{**} + Z^{\perp}$ to $J(x^{**}) + Z^{\perp\perp\perp}$. As $N(Q^*) = Z^{\perp\perp\perp}$, the operator $S\colon X^{(4)}/Z^{\perp\perp\perp} \longrightarrow X^{(4)}$ that sends each $x^{(4)} + Z^{\perp\perp\perp}$ to $Q^*(x^{(4)})$ is well defined.

We need to prove that $P := RSVU$ is a norm one projection with $N(P) = Z^{\perp}$. To do so, note that for each $x^{**} \in X^{**}$,

$$P(x^{**}) = Q^*(Jx^{**})|_{\mathcal{J}(X^*)}\,. \tag{23}$$

Obviously, formulas (16) and (23) show $N(P) = Z^{\perp\perp\perp} \cap J(X^{**}) = J(Z^{\perp})$. In order to see that P is a projection, note that given any $x^{(4)} \in X^{(4)}$, if $x^{(4)}|_{\mathcal{J}(X^*)} = 0$ then $x^{(4)} \in \mathcal{J}(Z)^{\perp}$; with the help of (16), that means

$$U\big(Q^*(x^{**})|_{\mathcal{J}(X^*)}\big) = Q^*(x^{**}) + \mathcal{J}(Z)^{\perp}.$$

Therefore

$$\begin{aligned} P^2(x^{**}) &= P\big(Q^*(x^{**})|_{\mathcal{J}(X^*)}\big) \\ &= RSV\big(Q^*(x^{**}) + \mathcal{J}(Z)^{\perp}\big) = RS\big(Q^*(x^{**}) + Z^{\perp\perp\perp}\big) \\ &= R{Q^*}^2(x^{**}) = RQ^*(x^{**}) = Q^*(Jx^{**})|_{\mathcal{J}(X^*)}\,, \end{aligned}$$

which shows that $P^2 = P$ in virtue of (23). We conclude that P is a projection such that $1 \le \|P\| \le \|R\|\|S\|\|V\|\|U\| \le 1$, and the proof is done. $\square$

It follows from [40, Lemma 7.3] that a Banach space X is an ideal in each ultrapower $X_{\mathfrak{U}}$. This fact can be derived from Theorem 5.6. Indeed, it is enough to realize that the operator $T\colon X^* \longrightarrow (X_{\mathfrak{U}})^*$ that maps x^* to $[x^*]$ is a norm one extension operator.

The proof of the following characterization of local duality follows the pattern of the proofs of Theorems 5.4 and 5.6. Before giving it, some words of caution are necessary. Theorem 5.7 may leave the false impression that the decomposition (15) of $X^{(3)}$ for a Banach space X produces a new proof of the principle of local reflexivity. This is not correct because this principle is applied in the proof of Theorem 5.7.

Theorem 5.7. *Given a subspace Z of X^*, the following statements are equivalent:*

(1) Z is a local dual of X,
(2) there exists an isometric extension operator $T\colon Z^* \longrightarrow X^{**}$ such that $J_X(X) \subset R(T)$,
(3) there is a norm one projection $P\colon X^{**} \longrightarrow X^{**}$ such that $J_X(X) \subset R(P)$ and $N(P) = Z^{\perp}$.
(4) there exists a norm one projection $Q\colon X^{(3)} \longrightarrow X^{(3)}$ such that $N(Q) \subset J_X(X)^{\perp}$ and $R(Q) = Z^{\perp\perp}$.

Proof. (1)⇒(2) Since Z is a local dual of X, Theorem 4.2 provides an ultrafilter $\mathfrak{U}$ and an isometry $J \in \mathcal{B}(X^*, Z_{\mathfrak{U}})$ such that $\mathcal{Q}_{Z_{\mathfrak{U}}} J = I_{X^*}$ and $J\,|_{\,Z} = J_{Z,Z_{\mathfrak{U}}}$. Since J gathers all hypotheses of the respective implications (1) ⇒ (2) of Theorems 5.4 and 5.6, the operator $T := J^* \circ J_{Z^*, Z_{\mathfrak{U}}{}^*}$ satisfies the theses of both implications, which means that T is an isometric extension operator from Z^* into X^{**} such that $X \subset R(T)$.

(2)⇒(1). Assume there exists an isometric extension operator $T\colon Z^* \longrightarrow X^{**}$ with $X \subset R(T)$. Lemma 5.2 yields $x = T(x\,|_{\,Z})$ for all $x \in X$, hence all hypotheses of the respective implications (2)⇒(1) in Theorems 5.4 and 5.6 hold here. Thus, given a pair of finite dimensional subspaces E of X^* and and F of X and $\varepsilon > 0$, in order to find an operator $L\colon E \longrightarrow Z$ satisfying conditions (A'), (B') and (C') with respect to F and ε, we can proceed as in the mentioned implications. We consider the subspace $E_1 := T^*(E) \subset Z^{**}$. Since Z is a local dual of Z^*, we take an ε-isometry $J\colon E_1 \longrightarrow Z$ such that

$$\langle J(z^{**}) - z^{**}, x\,|_{\,Z} \rangle = 0 \quad \text{for all } z^{**} \in E_1 \text{ and all } x \in F$$
$$J(z) = z \quad \text{for all } z \in E_1 \cap Z.$$

Thus, taking $L := J \circ T^*\,|_{\,E}$, the arguments of both mentioned implications (1) ⇒ (2) prove that L satisfies conditions (A'), (B') and (C') with respect to F an ε. We have just proved that Z is a local dual of X.

(2)⇔(3) It follows from Lemma 5.3.

(3)⇒(4) It is enough to observe that $Q := P^*$ is a norm one projection with kernel $N(Q) = R(P)^\perp \subset J_X(X)^\perp$ and range $R(Q) = N(P)^\perp = Z^{\perp\perp}$.

(4)⇒(3) The projection Q satisfies all the hypotheses of statement (4) in Theorem 5.6 plus the additional hypothesis $N(Q) \subset J(X)^\perp$. The proof of (4)⇒(3) in Theorem 5.6 gives a norm one projection $P\colon X^{**} \longrightarrow X^{**}$ with $N(P) = Z^\perp$. Since $J_X(X)^{\perp\perp} \subset R(Q^*)$ by (16) and (23), we obtain $J_X(X) \subset R(P)$, as we wanted to prove. □

There is a particular step in the implication (1)⇒(2) of the above theorem that deserves some comments: the extension operator T satisfies that $T(x\,|_{\,Z}) = x$ for all $x \in X$. This implies that $\overline{T(B_{Z^*})}^{w^*} = B_{X^{**}}$, hence $T(Z^*)$ is a norming subspace of X^{**}. When Z is a subspace of Y and there exists an isometric extension operator $T\colon Z^* \longrightarrow Y^*$ such that $T(Z^*)$ is norming, Z is said to be a *strict ideal* in Y (see [50] and [64]). Therefore, each local dual of X is a strict ideal in X^*.

While local complementation is transitive, local duality enjoys a sort of symmetry.

Theorem 5.8. *Let Z be a local dual of X and let $\Upsilon := \mathcal{J}_Z^* \circ J_X\colon X \longrightarrow Z^*$. Then $\Upsilon(X)$ is a local dual of Z isometric to X.*

Since $\Upsilon(x) = x\,|_{\,Z}$ for every $x \in X$ and Z is norming, $\Upsilon\colon X \longrightarrow Z^*$ is an isometry. Moreover, by Theorem 5.7, there exists a norm one extension operator $T\colon Z^* \longrightarrow X^{**}$ such that $X \subset R(T)$.

Note that $\mathcal{J}_Z^*$ is surjective and $\mathcal{J}_Z^* \circ T$ is the identity on Z^*. Hence $T \circ \mathcal{J}_Z^*$ is a projection on X^{**} with $R(T \circ \mathcal{J}_Z^*) = R(T)$. In particular $T \circ \Upsilon = J_X$.

Let us define $\Psi\colon \Upsilon(X)^* \longrightarrow Z^{**}$ by

$$\langle \Psi f, z^* \rangle := \langle T(z^*), f \circ \Upsilon \rangle \quad \text{for all } f \in \Upsilon(X)^* \text{ and } z^* \in Z^*.$$

Obviously, $\|\Psi\| \leq 1$. Moreover, given $f \in \Upsilon(X)^*$ and $x \in X$,

$$\langle \Psi f, \Upsilon(x) \rangle = \langle T \circ \Upsilon(x), f \circ \Upsilon \rangle = \langle J_X(x), f \circ \Upsilon \rangle = \langle f, \Upsilon(x) \rangle.$$

Hence $\Psi(f)|_{\Upsilon(X)} = f$, and Ψ is a norm one extension operator. Also $Z \subset R(\Psi)$, because $\Psi(x^*|_{\Upsilon(X)}) = J_Z(x^*)$ for each $x^* \in Z \subset X^*$. Indeed, given $z^* \in Z^*$ we have

$$\langle \Psi(x^*|_{\Upsilon(X)}), z^* \rangle = \langle T(z^*), x^*|_{\Upsilon(X)} \circ \Upsilon \rangle = \langle T(z^*), x^* \rangle = \langle z^*, x^* \rangle.$$

Thus $\Upsilon(X)$ is a local dual of Z by Theorem 5.7. $\square$

The spaces $C(\mathbb{T})$ and $L_1(\mathbb{T})$ are an example of symmetric pair in the sense of Theorem 5.8. Indeed, let $\mathcal{J}$ be the natural embedding of $L_1(\mathbb{T})$ into $\mathcal{M}(\mathbb{T}) = C(\mathbb{T})^*$. As $L_1(\mathbb{T})^* = L_\infty(\mathbb{T})$, the operator $\Upsilon = \mathcal{J}^* J_{C(\mathbb{T})}\colon C(\mathbb{T}) \longrightarrow L_\infty(\mathbb{T})$ maps $C(\mathbb{T})$ into its natural copy in $L_\infty(\mathbb{T})$. Thus, as $C(\mathbb{T})$ is a local dual of $L_1(\mathbb{T})$ (Proposition 3.15 (a)), Theorem 5.8 immediately yields that $L_1(\mathbb{T})$ is a local dual of $C(\mathbb{T})$, which provides an alternative proof of part (b) in Proposition 3.15.

The Radon-Nikodym decomposition $C(\mathbb{T})^* = L_1(\mathbb{T}) \oplus_1 \mathcal{M}_{sing}(\mathbb{T})$ induces a norm-one projection $Q : C(\mathbb{T})^{**} \to C(\mathbb{T})^{**}$ with $N(Q) = L_1(\mathbb{T})^\perp$. This projection is useless to show that $L_1(\mathbb{T})$ is a local dual of $C(\mathbb{T})$ because $R(Q) = \mathcal{M}_{sing}(\mathbb{T})^\perp$ does not contain $C(\mathbb{T})$ (see Theorem 5.7).

5.2. Identification of local dual subspaces.

Although the local techniques have proved to be very powerful in finding local duals for Banach spaces (for instance, Propositions 3.15, 3.12, Example 3.11, and in particular, Theorems 3.17, 3.18, the principles of local reflexivity and the forthcoming Corollary 6.13), the global theorems of this section can be also used in the same task. In particular, Theorem 5.8 and Proposition 5.13 provide many examples (see Proposition 5.9 and the comments after Theorem 5.8).

Proposition 5.9. *Given a Banach space X, the following statements hold:*

(a) $\ell_1(X^*)$ is a local dual of $\ell_\infty(X)$,
(b) $\ell_\infty(X)$ is a local dual of $\ell_1(X^*)$.

Proof. (a) For every couple $\alpha = (E, F)$ of finite dimensional subspaces of $\ell_1(X^*), \ell_\infty(X^{**})$, we select finite dimensional subspaces E_n of X^* and F_n of X^{**}, so that $E \subset \ell_1(E_n)$ and $F \subset \ell_\infty(F_n)$. We denote $|\alpha| := \dim(E) + \dim(F)$.

For every n, the principle of local reflexivity gives an $|\alpha|^{-1}$-isometry $S_n^\alpha : F_n \longrightarrow X$ so that $\langle S_n^\alpha f, e \rangle = \langle e, f \rangle$ for every $e \in E_n$ and $f \in F_n$, and $S_n^\alpha(f) = f$ for every $f \in F_n \cap X$.

Consider the non-linear map $S^\alpha : \ell_\infty(X^{**}) \longrightarrow \ell_\infty(X)$ given by $S^\alpha(z_n) := (S_n^\alpha(z_n))$, if $(z_n) \in F$, and $S^\alpha(z_n) := 0$, otherwise. Let $\mathfrak{U}$ be an ultrafilter in the set of all couples $\alpha = (E, F)$ of finite dimensional subspaces of $\ell_1(X^*), \ell_\infty(X^{**})$ refining the order filter.

We consider the operator $\Lambda : \ell_1(X^*)^* = \ell_\infty(X^{**}) \longrightarrow \ell_\infty(X)^{**}$ defined by

$$\Lambda(z_n) := w^*\text{-}\lim_{\alpha\to\mathfrak{U}} S^\alpha(z_n),\ (z_n) \in \ell_\infty(X^{**}).$$

Note that Λ is an isometry and $\Lambda(y_n) = (y_n)$ for every $(y_n) \in \ell_\infty(X^{**})$. Therefore, Λ is an isometric extension operator. Moreover, $\Lambda((x_n)) = (x_n)$, if $(x_n) \in \ell_\infty(X)$. In particular $\Lambda(\ell_\infty(X^{**})) \supset \ell_\infty(X)$, and an application of Theorem 5.7 proves (a).

(b) It follows from part (a) and the symmetry of local duality of Theorem 5.8. $\square$

Recall that a Banach space X has the *metric approximation property* (M.A.P, for short) if for every $\varepsilon > 0$ and every compact subset K of X, there is a finite rank operator T on X such that $\|T\| \le 1$ and $\|Tx - x\| \le \varepsilon$ for every $x \in K$. Note that if X^* has the M.A.P., then so does X [19, Corollary VIII.3.9]. However, the converse implication is not valid [52, Theorem 1.e.7].

Casazza and Kalton [11] proved that for every separable Banach space X with the M.A.P., there is a sequence (T_n) of finite rank operators on X such that

$$\begin{aligned} &(a) \quad \lim_{n\to\infty} \|T_n x - x\| = 0 \text{ for all } x \in X,\\ &(b) \quad \lim_{n\to\infty} \|T_n\| = 1 \text{ and}\\ &(c) \quad T_n T_k = T_k T_n = T_{min\{k,n\}}; \end{aligned}$$

i.e., X admits a *commuting* 1-*approximating sequence* (T_n) *of finite rank operators*. Using this fact, and applying an argument similar to that of Lemma II.2 in [27], we can obtain a local dual of X.

Theorem 5.10. *Let X be a separable Banach space with the M.A.P., and let $(T_n)_{n=1}^\infty$ be a commuting 1-approximating sequence of X. Then $\overline{\bigcup_{n=1}^\infty R(T_n^*)}$ is a local dual of X with the M.A.P.*

Proof. Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{N}$. We define a map P on X^{**} by

$$Pz := w^*\text{-}lim_{k\to\mathfrak{U}} T_k^{**} z, \quad z \in X^{**}.$$

From $T_n^{**} T_k^{**} = T_k^{**} T_n^{**} = T_{min\{k,n\}}^{**}$ and the w*-continuity of the operators T_n^{**}, it follows that for every $n \in \mathbb{N}$, we have

$$T_n^{**} P = P T_n^{**} = T_n^{**}. \tag{24}$$

Hence $P^2 z = w^*\text{-}lim_{n\to\mathfrak{U}} T_n^{**} P z = Pz$. This fact and $\lim_{n\to\infty} \|T_n\| = 1$ imply that P is a norm one projection. Also, it follows from formula (24) that

$N(T_n^{**}) \supset N(P)$, for every $n \in \mathbb{N}$. Since the intersection of the kernels $N(T_n^{**})$ is contained in $N(P)$, we obtain

$$N(P) = \bigcap_{n=1}^{\infty} N(T_n^{**}).$$

In particular, $N(P)$ is w*-closed. And clearly $P(X^{**}) \supset X$.

The condition $T_n T_k = T_k T_n = T_{min\{k,n\}}$ implies that the ranges $R(T_n^*) = N(T_n^{**})_\perp$ form an increasing sequence. Therefore $N(P)_\perp = \overline{\bigcup_{n=1}^{\infty} R(T_n^*)}$, and it follows from Theorem 5.7 that $\overline{\bigcup_{n=1}^{\infty} R(T_n^*)}$ is a local dual of X.

Since $T_n^* f$ is weak*-convergent for every $f \in X^*$, and a compact operator takes weak*-convergent sequences to norm-convergent sequences, formula (24) implies

$$\lim_{k\to\infty} \|T_k^* f - f\| = \lim_{n\to\infty} \|T_n^*(T_k^* g - g)\| = 0,$$

for each $f = T_n^* g \in R(T_n^*)$. Since (T_k^*) is bounded, $\lim_{k\to\infty} \|T_k f - f\| = 0$ for every $\overline{\bigcup_{n=1}^{\infty} R(T_n^*)}$; hence $\overline{\bigcup_{n=1}^{\infty} R(T_n^*)}$ has the M.A.P. $\square$

Remark 5.11. When X has a monotone Schauder basis, the local dual of X provided by Theorem 5.10 is the subspace generated in X^* by the coefficient functionals of the basis.

The following example concerning the space of continuous functions on the Cantor set is obtained from Theorem 5.10. The Cantor set, usually denoted by Δ, can be described as $\Delta := \{0,1\}^{\mathbb{N}}$ endowed with the product topology. For a succinct description of Δ and its topology, see Remark 1.4.2 in [2].

Example 5.12. *The closed span Z of $\{\chi_{ni}\}_{n=0}^{\infty}\,{}_{i=1}^{2^n}$ in $L_\infty[0,1]$, with χ_{ni} the characteristic function of the dyadic interval I_i^n, is a local dual of $L_1[0,1]$ isometric to $C(\Delta)$.*

Proof. Consider the projections P_n on $L_1[0,1]$ defined by

$$P_n f := \sum_{i=1}^{2^n} \langle 2^n \chi_{ni}, f\rangle \chi_{ni}.$$

It is not difficult to check that (P_n) is a commuting 1-approximating sequence in $L_1[0,1]$ and $\cup_n R(P_n^*)$ is the subspace generated by the functions χ_{ni}.

Let us prove that Z is isometric to $C(\Delta)$. For each dyadic interval I_k^n with $n \geq 1$ we consider the clopen $J_{nk} := \{a_1\} \times \cdots \times \{a_n\} \times \{0,1\}^{\mathbb{N}}$ of Δ, where $k = a_1 2^{n-1} + a_2 2^{n-2} + \cdots + a_n 2^0 + 1$. The map that assigns χ_{01} to χ_Δ, and each χ_{nk} (with $n \geq 1$) to $\chi_{J_{nk}} \in C(\Delta)$ induces a linear isometry between Z and $C(\Delta)$. $\square$

Observe that $C(\Delta)$ is isomorphic but not isometric to $C[0,1]$.

Next result applies the global techniques of Theorem 5.7 to find subspaces of operators that are local dual spaces.

We denote $X \otimes_\epsilon Y$ and $X \otimes_\pi Y$ the injective and the projective tensor product of X and Y, $\mathcal{K}(X,Y)$ the compact operators, and $\mathcal{I}(X,Y)$ the integral operators.

Proposition 5.13. *Let X and Y be Banach spaces. Assume that X^* or Y^* has the M.A.P. Then*

(a) $X^* \otimes_\epsilon Y^*$ is a local dual of $X \otimes_\pi Y$,
(b) $X^* \otimes_\pi Y^*$ is a local dual of $X \otimes_\epsilon Y$.

Proof. We assume that Y^* has the M.A.P.

(a) The dual space $(X \otimes_\pi Y)^*$ can be identified with $\mathcal{B}(X, Y^*)$. Moreover, since Y^* has the M.A.P., $X^* \otimes_\epsilon Y^*$ can be identified with $\mathcal{K}(X, Y^*)$, and there exists a net (A_α) of finite rank operators on Y^* with $\|A_\alpha\| \leq 1$ such that $\lim_\alpha \|A_\alpha g - g\| = 0$ for every $g \in Y^*$. We can assume that (A_α) is $\sigma(\mathcal{K}(Y^*)^{**}, \mathcal{K}(Y^*)^*)$-convergent.

Now, given $T \in \mathcal{B}(X, Y^*)$ and $\Phi \in \mathcal{K}(X, Y^*)^*$, the expression $\Phi_T(A) := \Phi(AT)$ defines $\Phi_T \in \mathcal{K}(Y^*)^*$. Thus, defining $\Lambda : \mathcal{K}(X, Y^*)^* \longrightarrow \mathcal{B}(X, Y^*)^*$ by

$$\langle \Lambda\Phi, T\rangle := \lim_\alpha \langle \Phi, A_\alpha T\rangle = \lim_\alpha \langle A_\alpha, \Phi_T\rangle.$$

Note that for every $f \otimes g \in X^* \otimes_\epsilon Y^*$ we have

$$\langle \Lambda\Phi, f \otimes g\rangle = \lim_\alpha \langle \Phi, (A_\alpha g) f\rangle = \langle \Phi, f \otimes g\rangle.$$

So Λ is an isometric extension operator. In an analogous way we can check that for every $x \otimes y \in X \otimes_\pi Y \subset \mathcal{B}(X, Y^*)^*$, we have $\Lambda(x \otimes y\,|_{\mathcal{K}(X,Y^*)}) = x \otimes y$. Thus $X \otimes_\pi Y \subset \Lambda(\mathcal{K}(X, Y^*)^*)$, and it is enough to apply Theorem 5.7.

(b) The proof is analogous, identifying the dual space $(X \otimes_\epsilon Y)^*$ with the space $\mathcal{I}(X, Y^*)$ of all integral operators from X into Y^*. ☐

Proposition 5.13 improves a result of Lima [48] which, using an argument of Johnson in [41], shows that the space $\mathcal{K}(X, Y^*)$ of compact operators is an ideal in $\mathcal{B}(X, Y^*)$.

Remark 5.14. (a) If we assume in Proposition 5.13 that Y^* has the metric compact approximation property (defined as the M.A.P., using compact operators instead of finite rank operators), then we obtain that $\mathcal{K}(X, Y^*)$ is a local dual of $X \otimes_\pi Y$.

(b) It follows from the results of Lima [48, Theorem 13] that if Y^* has the Radon-Nikodym property and $Y^{**} \otimes_\epsilon Y^*$ is a local dual of $Y^* \otimes_\pi Y$, then Y^* has the M.A.P. So it is not enough to assume in Proposition 5.13 that X or Y has the M.A.P.

(c) Let μ be a finite measure and let K be a compact space. Since $L_1(\mu)^*$ and $C(K)^*$ have the M.A.P., we can apply Proposition 5.13 to the spaces $L_1(\mu, X) = X \otimes_\pi L_1(\mu)$ and $C(K, X) = X \otimes_\epsilon C(K)$.

Note that $X^* \otimes_\epsilon L_\infty(\mu)$ is identified with a (proper, in general) subspace of $L_\infty(\mu, X^*)$.

Given a Banach space X, let $B_1(X)$ denote the space of the first Baire class elements in X^{**}, that is, the weak* limits in X^{**} of sequences in X. Godefroy, Kalton and Saphar asked in [27, Question 10] the following question:

Open problem 5.15. *Let X be a separable Banach space. Is $B_1(X)$ an ideal in X^{**}?*

The answer is known to be positive in the following cases:

(i) X is weakly sequentially complete, because $B_1(X) = X$,
(ii) X contains no copies of ℓ_1, because $B_1(X) = X^{**}$ [59],
(iii) $X = C[0,1]$ (Theorem 3.18).

Observe that every separable space is a subspace of $C[0,1]$ and, for each subspace M of X, $B_1(M)$ can be identified with $B_1(X) \cap M^{\perp\perp}$ [17, Lemma XIII.7]. So case (iii) suggests that a general answer to Problem 5.15 could be positive.

5.3. Existence of special local dual spaces.

In this part we collect some results concerning the existence of local duals satisfying certain additional properties.

Proposition 5.16. *If X is a M-ideal in X^{**} then X^* is the only local dual of X.*

Proof. If X is a M-ideal in X^{**} then X^* is the smallest norming subspace of X^* (Corollary III.2.16 in [39]), hence X^* is the only local dual of X. □

In general, a Banach space does not have a smallest local dual (see Remark 3.16). However there are some conditions that implies its existence.

Proposition 5.17. *Assume X is isometric to a dual space and admits a smallest norming subspace Z_n. Then Z_n is the smallest local dual of X, and moreover it is the only isometric predual of X.*

Proof. By Lemma I.2 in [25], the smallest norming subspace Z_n does exist if and only if $Z_n^\perp$ equals the set $\{x^{**} \in X^{**} \colon \|x^{**} - x\| \geq \|x\| \text{ for all } x \in X\}$. In that case, $X^{**} = X \oplus Z_n^\perp$ and Z_n is the only predual of X (Theorem II.1 in [25]). Clearly, there is a projection P on X^{**} with kernel $Z_n^\perp$ and range X satisfying all the conditions of Theorem 5.7. Hence Z_n is a local dual of X, and it is the smallest one because every local dual is norming. □

A second example was studied by Godefroy and Kalton [26].

Proposition 5.18. *Let X be a Banach space containing no copies of ℓ_1. Then there is a smallest local dual Z_d of X.*

Proof. Assume that X does not contain any copy of ℓ_1. Let $\mathcal{P}_X$ denote the family of all the subspaces Y of X^{**} for which there is a norm one projection on X^{**} such that $Y = N(P)$ and $R(P) \supset X$. It was proved in [27, Proposition V.1] that $\mathcal{P}_X$ consists of w*-closed subspaces of X^{**}, and that $\mathcal{P}_X$ has a largest element L. Since, by Theorem 5.7, the local dual spaces of X are precisely the subspaces Z of X^* such that $Z^\perp \in \mathcal{P}_X$, we conclude that $Z_d := L_\perp$ is the smallest local dual of X. $\square$

Let X be a separable Banach space containing no copies of ℓ_1 such that X^* is not separable. It follows from Proposition 5.21 that the smallest local dual space Z_d provided by Proposition 5.18 is separable. Moreover, there is a smallest norming subspace Z_n in X^* [25, Lemma I.2 and Theorem II.3], and clearly Z_n is contained in the smallest local dual Z_d. Thus, the following question arises.

Open problem 5.19. *[27, Remarks V.3] Let X be a Banach space for which Z_n and Z_d exist. Does Z_n equal Z_d?*

The following result was obtained by Sims and Yost [67] (see [39, Lemmas III.4.3 and III.4.4]). Here, $\operatorname{dens} X$ stands for the *density character* of the Banach space X, defined as the smallest cardinal κ for which X has a dense subset of cardinality κ.

Proposition 5.20. *Let L be a subspace of Y and F a subspace of Y^* with $\operatorname{dens} F \le \operatorname{dens} L$. Then there is a subspace M of Y such that $\operatorname{dens} M = \operatorname{dens} L$ and $M \supset L$ for which there is an isometric extension operator $T : M^* \longrightarrow Y^*$ such that $T(M^*) \supset F$.*

We are ready to establish our next result on the existence of particular local dual spaces.

Proposition 5.21. *Every subspace L of X^* is contained in a local dual Z_L of X with $\operatorname{dens} Z_L = \max\{\operatorname{dens} L, \operatorname{dens} X\}$.*

Proof. Given a subspace L of X^*, it is easy to find a subspace L_0 of X^* so that $L \subset L_0$ and $\operatorname{dens} X \le \operatorname{dens} L_0$. If we apply Proposition 5.20 to L_0 as a subspace of X^* and X as a subspace of X^{**} we get a subspace Z_L of X^* such that $Z_L \supset L$ and $\operatorname{dens} Z_L = \max\{\operatorname{dens} L, \operatorname{dens} X\}$, for which there exists an isometric extension operator $T : Z_L^* \longrightarrow X^{**}$ such that $T(Z_L^*) \supset X$. By Theorem 5.7, this is the desired local dual of X. $\square$

6. Polar properties.

The local techniques have proved to be more powerful in finding examples of local dual spaces because they consider a single pair of finite dimensional subspaces E of X^* and and F of X, while the global techniques derived from Theorem 5.7 involve implicitly the consideration of all those pairs at once. In this sense, the local duality between $L_1(\mu, X^*)$ and $L_\infty(\mu, X)$ is a remarkable example: in fact, it was obtained in [34] using global techniques and the Continuum Hypothesis CH, while the proof in [35] based on local techniques (Theorem 3.17) does not need CH.

The local procedure can be roughly described as follows: in order to prove that X^* is f.d.r. in a subspace Z, we fix a finite dimensional subspace E of X^* and a real number $\varepsilon \in (0,1)$, take an ε'-net $\{x_i^*\}_{i=1}^n$ in S_E, and look for a family $\{z_i^\alpha\}_{\alpha\in A}$ so that $z_i^\alpha \xrightarrow[\alpha]{w^*} x_i^*$ and $\|z_i^\alpha\| \leq 1+\varepsilon'$ for all $\alpha \in A$ and all $1 \leq i \leq n$. Thus, if ε' has been chosen sufficiently small, and if for every α the map $x_i^* \mapsto z_i^\alpha$ agrees with a linear operator $L_\alpha \colon E \longrightarrow Z$, given any finite dimensional subspace F of X, we may select an index $\beta \in A$ so that L_β satisfies conditions (A') and (B') with respect to F and ε. If moreover we want to get condition (C'), then the ε'-net $\{x_i^*\}_{i=1}^n$ must contain a subset $\{x_i^*\}_{i=1}^m$ such that $\operatorname{span}\{x_i^*\}_{i=1}^m = E\cap Z$ and $L_\alpha(x_i^*) \xrightarrow[\alpha]{} x_i^*$ for all $1 \leq i \leq m$.

In all the examples of local duality offered until now, the ε-isometry L_β is explicitly found. However there are situations, like the proof of the principle of local reflexivity, where L_β cannot be constructed but only proved to exist by means of Goldstine's theorem or a related result (see the proofs in [51], [15], [54] or [68]). For those particular situations, the polar properties are powerful tools.

Let us begin with some technical definitions. Given an operator $L \in \mathcal{B}(X,Y)$, we denote by L^n the operator in $\mathcal{B}(X^n, Y^n)$ that maps $(x_i)_{i=1}^n$ to $(Lx_i)_{i=1}^n$. Also, every $k\times l$ scalar matrix $A = (a_{ij})_{i=1}^k{}_{j=1}^l$ induces an operator $A_X \in \mathcal{B}(X^l, X^k)$ defined by $A_X\big((x_j)_{j=1}^l\big) = (\sum_{j=1}^l a_{ij}x_j)_{i=1}^k$. Such operators are called *matricial*.

Proposition 6.1. *Given a $k\times l$ scalar matrix A and a Banach space X, the following properties hold:*

(a) The matricial operator A_{X^*} is the conjugate of the matricial operator $(A^*)_X$, where A^* denotes the transposed matrix of A.
(b) Given a subspace Z of X, A_X maps Z^l into Z^k.
(c) For every operator $L \in \mathcal{B}(X,Y)$, we have $L^k \circ A_X = A_Y \circ L^l$.
(d) Given an ultrafilter $\mathfrak{U}$, we have $(A_X)_{\mathfrak{U}} = A_{(X_{\mathfrak{U}})}$.
(e) The range of A_X is closed and complemented.

The proofs of properties (a), (b) and (c) are straightforward. For the proof of (d), note that $(X\times \overset{n}{\cdots} \times X)_{\mathfrak{U}}$ is isometrically identified with $X_{\mathfrak{U}}\times \overset{n}{\cdots} \times X_{\mathfrak{U}}$. For (e), it is sufficient to take into account that for any matrix A of dimension $k\times l$ there exists a matrix B of dimension $l\times k$ such that $BAB = B$.

Given a subset A of a Banach space X, $\mathring{A}$ denotes the norm interior of A.

Definition 6.2. *We say that a subspace Z of X^* has the* strict polar property *if for every $k,l\in\mathbb{N}$, every matricial operator $T\colon \ell_\infty^l(X^*) \longrightarrow \ell_\infty^k(X^*)$ and every $z \in \ell_\infty^k(Z)$, the set*

$$\ell_\infty^l(Z) \cap T^{-1}\Big(z + \mathring{B}_{\ell_\infty^k(X^*)}\Big)$$

is $\sigma\big(\ell^l_\infty(X^*), \ell^l_1(X)\big)$*-dense in* $T^{-1}\Big(z + \mathring{B}_{\ell^k_\infty(X^*)}\Big)$.

Remark 6.3. A version of the strict polar property was introduced in [37] to characterize local duality, but the proof of the equivalence was not correct. Fortunately, this mistake was not relevant for the other results contained in that paper. Definition 6.2 is weaker than the one given in [37] and, as we will show later, characterizes local duality.

Note that for any matricial operator $T\colon \ell^l_\infty(X^*) \longrightarrow \ell^k_\infty(X^*)$ and any subspace Z of X^*, the identity

$$\ell^l_\infty(Z) \cap T^{-1}\Big(z + \mathring{B}_{\ell^k_\infty(Z)}\Big) = \ell^l_\infty(Z) \cap T^{-1}\Big(z + \mathring{B}_{\ell^k_\infty(X^*)}\Big)$$

holds, but it may fail if the operator T is not matricial.

The following result offers a convenient characterization of the strict polar property in terms of closed balls and bounded sets.

Proposition 6.4. *Let Z be a subspace of a dual space X^*. Let k and l be a pair of positive integers, $z \in \ell^k_\infty(Z)$ and $T\colon \ell^l_\infty(X^*) \longrightarrow \ell^k_\infty(X^*)$ be a matricial operator. Consider the following four statements:*

(1) $\ell^l_\infty(Z) \cap T^{-1}\Big(z + \mathring{B}_{\ell^k_\infty(X^*)}\Big)$ *is w^*-dense in* $T^{-1}\Big(z + \mathring{B}_{\ell^k_\infty(X^*)}\Big)$*;*

(2) $\mathring{B}_{\ell^l_\infty(Z)} \cap T^{-1}\Big(z + \mathring{B}_{\ell^k_\infty(X^*)}\Big)$ *is w^*-dense in* $\mathring{B}_{\ell^l_\infty(X^*)} \cap T^{-1}\Big(z + \mathring{B}_{\ell^k_\infty(X^*)}\Big)$*;*

(3) for every $\eta > 1$, the set $\ell^l_\infty(Z) \cap T^{-1}\Big(z + \eta B_{\ell^k_\infty(X^*)}\Big)$ *is w^*-dense in*

$$T^{-1}\Big(z + B_{\ell^k_\infty(X^*)}\Big);$$

(4) for every $\eta > 1$, the set $B_{\ell^l_\infty(Z)} \cap T^{-1}\Big(z + \eta B_{\ell^k_\infty(X^*)}\Big)$ *is w^*-dense in*

$$B_{\ell^l_\infty(X^*)} \cap T^{-1}\Big(z + B_{\ell^k_\infty(X^*)}\Big).$$

Thus, Z has the strict polar property as a subspace of X^* if and only if any of the above four statements holds for all T and all z.

Proof. Note that the validity of (1) for all T and all z is exactly the definition of the strict polar property. To prove the remaining equivalences, we adopt the following notations: given a matricial operator $T\colon \ell^l_\infty(X^*) \longrightarrow \ell^k_\infty(X^*)$ and $z \in \ell^k_\infty(Z)$, we denote

$$O(z,T) := T^{-1}\big(z + \mathring{B}_{\ell^k_\infty(X^*)}\big) \quad \text{and} \quad P(z,T) := T^{-1}\big(z + B_{\ell^k_\infty(X^*)}\big).$$

Therefore, $P(z,T) = \bigcap_{\eta>1} O(\eta^{-1}z, \eta^{-1}T)$, and when $\lambda \neq 0$,

$$\lambda^{-1}O(z,T) = O(z,\lambda T) \quad \text{and} \quad \lambda^{-1}P(z,T) = P(z,\lambda T).$$

Moreover, denoting $\widetilde{z} := (0,z) \in \ell^l_\infty(X^*) \oplus_\infty \ell^k_\infty(X^*)$ and considering the matricial operator $\widetilde{T}\colon \ell^l_\infty(X^*) \longrightarrow \ell^l_\infty(X^*) \oplus_\infty \ell^k_\infty(X^*)$ defined as $\widetilde{T}(x^*) := (x^*, Tx^*)$, the statements (1) to (4) are respectively translated into the new notation as:

(1') the set $\ell^l_\infty(Z)\cap O(z,T)$ is w^*-dense in $O(z,T)$;
(2') the set $\ell^l_\infty(Z)\cap O(\widetilde{z},\widetilde{T})$ is w^*-dense in $O(\widetilde{z},\widetilde{T})$;
(3') for every $\eta>1$, the set $\ell^l_\infty(Z)\cap P(\eta^{-1}z,\eta^{-1}T)$ is w^*-dense in $P(z,T)$;
(4') for every $\eta>1$, the set $\ell^l_\infty(Z)\cap P(\eta^{-1}\widetilde{z},\eta^{-1}\widetilde{T})$ is w^*-dense in $P(\widetilde{z},\widetilde{T})$.

We will prove the equivalences (1')⇔(2'), (3')⇔(4') and (1')⇔(3'). First, note that

$$\begin{aligned} O(z,T) &= \bigcup_{n=1}^{\infty} n\mathring{B}_{\ell^l_\infty(X^*)} \cap O(z,T) \\ &= \bigcup_{n=1}^{\infty} n\left(\mathring{B}_{\ell^l_\infty(X^*)}\cap O\big(z,nT\big)\right) = \bigcup_{n=1}^{\infty} nO\big(\widetilde{z},n\widetilde{T}\big), \end{aligned} \tag{25}$$

and with a similar argument,

$$P(z,T)=\bigcup_{n=1}^{\infty} nP\big(\widetilde{z},n\widetilde{T}\big). \tag{26}$$

If (1') holds for all T and z, then (2') can be regarded as a particular case of (1'), so (2') also holds for all T an all z. For the reverse, assume (2') is satisfied for all T and all z. Thus, formula (25) and hypothesis (2') lead to

$$O(z,T)=\bigcup_{n=1}^{\infty} nO\big(\widetilde{z},n\widetilde{T}\big)\subset \overline{\ell^l_\infty(Z)\cap \bigcup_{n=1}^{\infty} nO\big(\widetilde{z},n\widetilde{T}\big)}^{w^*} = \overline{\ell^l_\infty(Z)\cap O(z,T)}^{w^*},$$

and (1') is proved.

Given $\eta>1$, the implication (3')⇒(4') is straightforward. Assume (4') holds for all matricial operator T and all z. Then formula (26) yields

$$\begin{aligned} P(z,T)=\bigcup_{n=1}^{\infty} nP\big(\widetilde{z},n\widetilde{T}\big) &\subset \bigcup_{n=1}^{\infty}\overline{\ell^l_\infty(Z)\cap nP(\eta^{-1}\widetilde{z},\eta^{-1}n\widetilde{T})}^{w^*} \\ &\subset \overline{\ell^l_\infty(Z)\cap\bigcup_{n=1}^{\infty} nP\Big(\eta^{-1}\widetilde{z},\eta^{-1}n\widetilde{T}\Big)}^{w^*} = \overline{\ell^l_\infty(Z)\cap P(\eta^{-1}z,\eta^{-1}T)}^{w^*}, \end{aligned}$$

obtaining (3').

For the equivalence (1')⇔(3'), fix $\eta>1$ and assume (1') holds for all T and all z. Since

$$P(z,T)\subset O(\eta^{-1}z,\eta^{-1}T)\subset P(\eta^{-1}z,\eta^{-1}T),$$

hypothesis (1') yields (3') trivially.

Assume now (3') holds for all T, z and $\eta>1$. Fix T and z, and for $x^*\in O(z,T)$, denote $\theta:=\|T(x^*)-z\|<1$. If $\theta=0$ then the proof is trivial. If $\theta>0$, then $x^*\in P(\theta^{-1}z,\theta^{-1}T)$. Thus, by hypothesis (3'), x^* belongs to the w^*-closure of $\ell^l_\infty(Z)\cap O(\theta\theta^{-1}z,\theta\theta^{-1}T)$, and the proof is done. □

Proposition 6.5. *Given a reflexive subspace R of a dual space X^*, the operator $T\colon X \longrightarrow \ell_\infty(B_R)$ that maps $x \in X$ to $(\langle f, x\rangle)_{f\in B_R}$ is weakly compact and has closed range. Moreover, if X is the dual of a Banach space Y, R is a reflexive subspace of Y and Z is a norming subspace of Y^* then T is a conjugate operator and $T(Z)$ is closed.*

Proof. Consider the factorization $T = \widetilde{T} \circ Q$ where $Q\colon X \longrightarrow X/N(T)$ is the quotient map. Since $N(T) = R_\perp$, given $x + R_\perp$, we have

$$\|\widetilde{T}(x + R_\perp)\| = \sup_{f\in B_R} \langle f, x\rangle, \tag{27}$$

and $\|x + R_\perp\| = \sup_{g\in B_{(X/R_\perp)^*}} \langle g, x\rangle$. As R is reflexive, $(X/R_\perp)^* = (R_\perp)^\perp = R$, hence $\|x + R_\perp\| = \sup_{f\in B_R}\langle f, x\rangle$, which shows, in view of (27), that $\widetilde{T}$ is isometric. Thus $\widetilde{T}$ has closed range, and so has T. In order to prove that T is weakly compact, observe that T factorizes through $X/R_\perp$, which is reflexive because its dual is isometric to R.

Let us assume now that $X = Y^*$ and $R \subset Y$. Then it is easy to check that $T = S^*$, where $S\colon \ell_1(B_R) \longrightarrow Y$ maps every $(\alpha_f)_{f\in R}$ to $\sum_{f\in B_R} \alpha_f f$.

In order to see that $T(Z)$ is closed, a similar argument to that proving T has closed range works: let $T_1 := T\,|_{\,Z}$ and consider the factorization $T_1 = \widetilde{T_1} Q_1$ where $Q_1\colon Z \longrightarrow Z/N(T_1)$ is the quotient map. If we prove that $\widetilde{T_1}$ is an isomorphism then the fact that Q_1 is surjective will trivially yield that T_1 has closed range, and in turn, $T(Z)$ is closed. In order to do so, first note that $N(T_1) = R^\perp \cap Z$. Thus, given $x^* \in Z$ such that $\|x^* + N(T_1)\| = 1$, the hypothesis that R is reflexive yields

$$\begin{aligned} 1 = \|x^* + N(T_1)\| &= \sup\left\{\langle x^*, g\rangle\colon g \in (Z/R^\perp \cap Z)^*,\ \|g\| \le 1\right\} \\ &= \sup\left\{\langle x^{**}, x^*\rangle\colon x^{**} \in (R^\perp \cap Z)^\perp,\ \|x^{**}\| \le 1\right\} \\ &= \sup\left\{\langle x^{**}, x^*\rangle\colon x^{**} \in \overline{R^{\perp\perp} + Z^\perp}^{\sigma(Y^{**},Y^*)},\ \|x^{**}\| \le 1\right\} \\ &= \sup\left\{\langle x^{**}, x^*\rangle\colon x^{**} \in \overline{R + Z^\perp}^{\sigma(Y^{**},Y^*)},\ \|x^{**}\| \le 1\right\}; \end{aligned} \tag{28}$$

moreover, $R \subset Y$ and Z is a norming subspace of Y^*, hence $Z_\perp = \{0\}$ and subsequently $R \cap Z^\perp = R \cap Z_\perp = \{0\}$. Therefore, there exists a projection P on $R + Z^\perp$ whose range is R and its kernel is $Z^\perp$. Let $K := \|P\|$. Given any $\varepsilon > 0$, the identity (28) gives a norm one element x^{**} in the $\sigma(Y^{**}, Y^*)$-closure of $R + Z^\perp$ such that $1 - \varepsilon < \langle x^{**}, x^*\rangle$. Choose a pair of elements $r_\varepsilon \in R$ and $x^{**}_\varepsilon \in Z^\perp$ such that $\|r_\varepsilon + x^{**}_\varepsilon\| \le 1$ and

$$1 - 2\varepsilon < \langle r_\varepsilon + x^{**}_\varepsilon, x^*\rangle = \langle r_\varepsilon, x^*\rangle.$$

Thus, as $\|r_\varepsilon\| \le K$, denoting $v := r_\varepsilon/\|r_\varepsilon\|$, we get

$$\frac{1 - 2\varepsilon}{K} < \langle v, x^*\rangle \le \|T_1(x^*)\|$$

hence $1/K \leq \|T_1(x^*)\|$, which proves that $\widetilde{T}_1$ is an isomorphism and the proof is done. $\square$

Observe that if R is not reflexive in Proposition 6.5, then T is not weakly compact because the conjugate operator T^* is an isomorphism on R.

The following theorem has two goals. First and most important, it shows that the strict polar property and local duality are equivalent. Second, it proves that in order to define local duality, it does not matter if we consider reflexive subspaces instead of finite dimensional subspaces F in condition (B). This fact generalizes a result of Barton and Yu [4].

Theorem 6.6. *For a closed subspace Z of X^*, the following statements are equivalent:*

(1) Z is a local dual of X,

(2) Z has the strict polar property as a subspace of X^*,

(3) for every finite dimensional subspace E of X^*, every reflexive subspace F of X and every $\varepsilon > 0$, there exists an operator $L\colon E \longrightarrow Z$ such that

(3a) L is an ε-isometry,

(3b) $\langle Lx^*, x\rangle = \langle x^*, x\rangle$ for all $x^* \in E$ and all $x \in F$,

(3c) $L(z) = z$ for all $z \in E \cap Z$.

Proof. (1)⇒(2) Assume Z is a local dual of X. Let $z = (z_i)_{i=1}^k \in \ell_\infty^k(Z)$, and $T\colon \ell_\infty^l(X^*) \longrightarrow \ell_\infty^k(X^*)$ be a matricial operator. Given an element $(f_i)_{i=1}^l \in T^{-1}(z + \mathring{B}_{\ell_\infty^k(X^*)})$, we must show that every w^*-neighborhood $\mathcal{V}$ of $(f_i)_{i=1}^l$ meets $\ell_\infty^l(Z) \cap T^{-1}(z + \mathring{B}_{\ell_\infty^k(Z)})$. In order to do that, let $\theta := \big\|T\big((f_i)_{i=1}^l\big) - z\big\|$. Take a finite subset $\{x_{ij}\}_{i=1}^{l}{}_{j=1}^{m}$ in X so that

$$\mathcal{V} \supset \{(h_i)_{i=1}^l : |\langle f_i - h_i\,,\, x_{ij}\rangle| < 1, \quad 1 \leq i \leq l, \quad 1 \leq j \leq m\}.$$

Consider the subspaces $F := \operatorname{span}\{x_{ij} : 1 \leq i \leq l,\ 1 \leq j \leq m\}$ and $E := \operatorname{span}\{f_i,\, z_j : 1 \leq i \leq l,\ 1 \leq j \leq k\}$.

Take $\varepsilon > 0$ so that $\theta(1+\varepsilon) < 1$. Thus, by the hypothesis of local duality, there exists an ε-isometry $L\colon E \longrightarrow Z$ satisfying properties (B) and (C) with respect to F. Hence $(Lf_i)_{i=1}^l \in \mathcal{V} \cap \ell_\infty^l(Z)$. Moreover, since $L^k((z_i)_{i=1}^k) = (z_i)_{i=1}^k$, Proposition 6.1 yields

$$T \circ L^l((f_i)_{i=1}^l) - (z_i)_{i=1}^k = L^k\left(T((f_i)_{i=1}^l) - (z_i)_{i=1}^k\right),$$

so

$$\|T \circ L^l((f_i)_{i=1}^l) - (z_i)_{i=1}^k\| \leq \|L\|\|T((f_i)_{i=1}^l) - z\| < (1+\varepsilon)\theta < 1.$$

Hence $(Lf_i)_{i=1}^l \in \mathcal{V} \cap T^{-1}(z + \mathring{B}_{\ell_\infty^k(Z)}) \neq \emptyset$, and the proof of (1)⇒(2) is done.

(2)⇒(3) Assume that Z has the strict polar property. In particular, Z is a norming subspace of X^*. Let E be a finite dimensional subspace of X^*, F a reflexive subspace of X and $\varepsilon > 0$. After Proposition 3.9, it will be enough to construct an operator $L\colon E \longrightarrow Z$ satisfying $\|L\| \leq 1 + \varepsilon$, (3b) and (3c).

Let $n = \dim E$ and $n-k = \dim E \cap Z$. By Lemma 2.4, E has a biorthogonal system $(y_r, h_r)_{r=1}^n$ such that $\|y_r\| = 1$ and $\|h_r\| \leq n+1$ for all $1 \leq r \leq n$ and $E \cap Z = \operatorname{span}\{y_r\}_{r=k+1}^n$. Clearly, an operator $L\colon E \longrightarrow Z$ satisfying (3c) can be written as

$$Le := \sum_{r=1}^{k} \langle h_r, e\rangle v_r \;+\; \sum_{r=k+1}^{n} \langle h_r, e\rangle y_r. \tag{29}$$

Let us find vectors $v_1, \ldots, v_k$ in Z so that L also satisfies (3b) and $\|L\| \leq 1+\varepsilon$. To do that, we take a real number $0 < \alpha < 1$ (to be fixed later) and a finite α-net $\{e_i\}_{i=1}^M$ in S_E. Let $\lambda_{ir} := \langle h_r, e_i\rangle$ for all $i \in \{1, \ldots, M\}$ and all $r \in \{1, \ldots, n\}$, so $|\lambda_{ir}| \leq n+1$ and

$$e_i = \sum_{r=1}^{n} \lambda_{ir} y_r \quad \text{for each } i \in \{1, \ldots, M\}\ .$$

Let us consider the vector

$$y := -\left(\textstyle\sum_{r=k+1}^n \lambda_{ir} y_r\right)_{i=1}^M \in \ell_\infty^M(Z)$$

and the operators $U\colon \ell_\infty^k(X^*) \longrightarrow \ell_\infty^M(X^*)$ and $S\colon \ell_\infty^k(X^*) \longrightarrow \ell_\infty^k\big(\ell_\infty(B_F)\big)$ defined by

$$U\big((f_s)_{s=1}^k\big) := \Big(\sum_{s=1}^{k} \lambda_{is} f_s\Big)_{i=1}^M \quad \text{and} \quad S\big((f_s)_{s=1}^k\big) := \big(\langle f_s, x\rangle\big)_{s=1,\ x\in B_F}^k.$$

Notice that U is a conjugate operator because it is matricial (see Proposition 6.1) and as $k \leq n$, its norm is $\|U\| \leq \max_{1\leq i\leq M} \sum_{s=1}^k |\lambda_{is}| \leq n(n+1)$.

Let $\beta > 1$ be a real number (to be fixed later) and consider the sets

$$D = B_{\ell_\infty^k(X^*)} \cap U^{-1}(y + B_{\ell_\infty^M(X^*)}) \quad \text{and} \quad C = B_{\ell_\infty^k(Z)} \cap U^{-1}(y + \beta B_{\ell_\infty^M(X^*)})$$

As Z has the strict polar property, and as a consequence of Proposition 6.4, C is w^*-dense in D. Therefore, since

$$\big\|U\big((y_s)_{s=1}^k\big) - y\big\| = \big\|(e_i)_{i=1}^M\big\| = 1,$$

it follows that $(y_s)_{s=1}^k \in D \subset \overline{C}^{w^*}$. But by Proposition 6.5, S is a weakly compact, conjugate operator with closed range because F is reflexive. Hence,

$$S(\overline{C}^{w^*}) \subset \overline{S(C)}^{w^*} = \overline{S(C)} \tag{30}$$

and

$$S(B_{\ell_\infty^k(X^*)}) \subset \overline{S(B_{\ell_\infty^k(Z)})}^{w^*} = \overline{S(B_{\ell_\infty^k(Z)})}; \tag{31}$$

moreover, since Z is a norming subspace of X^*, Proposition 6.5 also yields that $S\big(\ell_\infty^k(Z)\big)$ is closed, so $\overline{S(B_{\ell_\infty^k(Z)})} \subset \overline{S\big(\ell_\infty^k(Z)\big)} = S\big(\ell_\infty^k(Z)\big)$ and in combination with (31), we obtain

$$S\big(\ell_\infty^k(X^*)\big) = S\big(\ell_\infty^k(Z)\big). \tag{32}$$

Therefore, since $S|_{\ell^k_\infty(Z)}$ has closed range, formulas (30) and (32) show that given any $\gamma > 0$, there exist $(c_s)_{s=1}^k \in C$ and $(b_s)_{s=1}^k \in \gamma B_{\ell^k_\infty(Z)}$ such that

$$S\big((y_s)_{s=1}^k\big) = S\big((b_s)_{s=1}^k\big) + S\big((c_s)_{s=1}^k\big).$$

Let us take $v_s := b_s + c_s$ for all $s \in \{1, \dots, k\}$ in the definition of L given in (29). First, note that the identity $S\big((y_s)_{s=1}^k\big) = S\big((v_s)_{s=1}^k\big)$ yields

$$\langle y_s, x\rangle = \langle v_s, x\rangle \text{ for all } s \in \{1, \dots, n\} \text{ and all } x \in F,$$

so $\langle L(e), x\rangle = \langle e, x\rangle$ for all $e \in E$ and all $x \in F$, fulfilling condition (3b).

Moreover, for every $e_i \in \{e_i\}_{i=1}^M$, we have

$$\begin{aligned}
\|L(e_i)\| &= \left\|\textstyle\sum_{s=1}^k \lambda_{is}(b_s + c_s) \;+\; \sum_{s=k+1}^n \lambda_{is} y_s\right\| \\
&\le \left\|\Big(\textstyle\sum_{s=1}^k \lambda_{is} c_s \;+\; \sum_{s=k+1}^n \lambda_{is} y_s\Big)_{i=1}^M \;+\; \Big(\sum_{s=1}^k \lambda_{is} b_s\Big)_{i=1}^M\right\| \\
&\le \big\|U\big((c_s)_{s=1}^k\big) - y\big\| + \big\|U\big((b_s)_{s=1}^k\big)\big\| \\
&\le 1 + \beta + n(n+1)\gamma =: \delta(n, \beta, \gamma).
\end{aligned}$$

Thus, by Lemma 2.2,

$$\|L\| \le 1 + \frac{\alpha + \beta + n(n+1)\gamma}{1 - \alpha},$$

Therefore, as n and ε are fixed parameters, the values of α, β and γ can be chosen as small as we please in order to ensure that $\|L\| \le 1 + \varepsilon$ and the proof (2)⇒(3) is completed.

The implication (3)⇒(1) is trivial. □

Taking $z = 0$ in the definition of the strict polar property, we obtain the polar property.

Definition 6.7. *We say that a subspace Z of X^* has the* polar property *if for every $k, l \in \mathbb{N}$ and every matricial operator $T\colon \ell^l_\infty(X^*) \longrightarrow \ell^k_\infty(X^*)$, the set*

$$\ell^l_\infty(Z) \cap T^{-1}(\mathring{B}_{\ell^k_\infty(X^*)})$$

is $\sigma\big(\ell^l_\infty(X^*), \ell^l_1(X)\big)$-dense in $T^{-1}(\mathring{B}_{\ell^k_\infty(X^*)})$.

Recall that given a subset A of X^*, the polar set of A in X is defined as $A_\circ := \{x \in X\colon |\langle x^*, x\rangle| \le 1 \text{ for all } x^* \in A\}$. The polar property admits several equivalent forms.

Proposition 6.8. *For a closed subspace Z of X^*, the following statements are equivalent:*

(1) Z has the polar property as a subspace of X^*;

(2) for every matricial operator $T\colon \ell^l_\infty(X^*) \longrightarrow \ell^k_\infty(X^*)$, the set

$$\ell^l_\infty(Z) \cap T^{-1}(B_{\ell^k_\infty(X^*)})$$

is $\sigma\big(\ell^l_\infty(X^*), \ell^l_1(X)\big)$-dense in $T^{-1}(B_{\ell^k_\infty(X^*)})$.

(3) for every matricial operator $T\colon \ell^k_\infty(X) \longrightarrow \ell^l_\infty(X)$,

$$\overline{T(B_{\ell^k_1(X)})} = \Big(\ell^l_\infty(Z) \cap {T^*}^{-1}(B_{\ell^k_\infty(Z)})\Big)_\circ.$$

The proof follows closely that of Proposition 6.4, taking into account that for every convex symmetric subset A of X^*, its $\sigma(X^*, X)$-closure equals $(A_\circ)^\circ$, where given a subset B of X, $B^\circ := \{x^* \in X^* \colon |\langle x^*, x\rangle| \le 1 \text{ for all } x \in B\}$, the polar set of B in X^*.

As it can be observed from the proof of Theorem 6.6, the role of the element z occurring at the definition of the strict polar property is to get condition (C). Thus, letting $z = 0$ in Theorem 6.6, we obtain the following characterization of finite dual representability.

Theorem 6.9. *For a closed subspace Z of X^*, the following statements are equivalent:*

(1) X^* is f.d.r. in Z,

(2) Z has the polar property as a subspace of X^*,

(3) for every finite dimensional subspace E of X^*, every reflexive subspace F of X and every $\varepsilon > 0$, there exists an operator $L\colon E \longrightarrow Z$ such that

(3a) L is an ε-isometry,

(3b) $\langle Lx^*, x\rangle = \langle x^*, x\rangle$ for all $x^* \in E$ and all $x \in F$.

Theorems 6.6 and 6.9 must be understood in certain categorical sense: roughly speaking, they are telling us that if every finite cartesian product $\ell^n_\infty(Z)$ is, with respect to $\ell^n_\infty(X^*)$, a sort of object of the same kind as Z is with respect to X^*, then Z is a local dual of X. This description is rather vague, but it becomes precise in the proofs of the principles of local reflexivity given in the following corollary. For example, the principle of local reflexivity is derived from the fact that $\ell^k_\infty(X^{**})$ is the bidual of $\ell^k_\infty(X)$.

Corollary 6.10 (Principles of Local Reflexivity). *Given a Banach space X and an ultrafilter $\mathfrak{U}$, the two following statements hold:*

(i) The canonical copy of X in X^{**} is a local dual of X^*.

(ii) The canonical copy of $(X^*)_{\mathfrak{U}}$ in $(X_{\mathfrak{U}})^*$ is a local dual of $X_{\mathfrak{U}}$.

Proof. Let $Z := \ell^k_\infty(X)$, $Y := \ell^l_\infty(X)$ and let $T\colon Z \longrightarrow Y$ be any matricial operator.

(i) Since $\ell^k_\infty(X^{**})$ and $\ell^l_\infty(X^{**})$ are respectively identifiable with the biduals Z^{**} and Y^{**}, an application of Lemma 2.7 to T shows that X has the strict polar property as a subspace of X^{**}, and by Theorem 6.6, X is a local dual of X^*.

(ii) Analogously, note that $\ell_\infty^k((X_{\mathfrak{U}})^*)$ and $\ell_\infty^k((X^*)_{\mathfrak{U}})$ are respectively identified with $(Z_{\mathfrak{U}})^*$ and $(Z^*)_{\mathfrak{U}}$, and the pair of spaces $\ell_\infty^l((X_{\mathfrak{U}})^*)$, $\ell_\infty^l((X^*)_{\mathfrak{U}})$ are respectively identified with $(Y_{\mathfrak{U}})^*$ and $(Y^*)_{\mathfrak{U}}$. Thus, as any matricial operator from Y^* into Z^* is the conjugate of a a matricial operator $T\colon Z \longrightarrow Y$, an application of Lemma 2.8 proves that $(X^*)_{\mathfrak{U}}$ has the strict polar property as a subspace of $(X_{\mathfrak{U}})^*$ and therefore, Theorem 6.6 shows that $(X^*)_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$. □

Note that the proof of Corollary 6.10 only needs to apply Lemma 2.7 to matricial operators. The full extent of Lemma 2.7 has been applied to the investigation of duality properties of operator semigroups [65], [55], [56] and [38].

Theorem 6.12 below is an extension of the P.L.R. for ultrapowers, which can be recovered by taking $X = Y$ and $T = 0$. As in the proof of the P.R.L. (for ultrapowers), a separation lemma is needed.

Lemma 6.11. *Let $\mathfrak{U}$ be an ultrafilter on a set I and let $U\colon X \longrightarrow Y$ and $L\colon Y \longrightarrow Y$ be operators. Suppose that $L_{\mathfrak{U}}$ maps $\overline{R(U_{\mathfrak{U}})}$ into $\overline{R(U_{\mathfrak{U}})}$, and consider the operator*

$$\Lambda : \mathbf{x} + \overline{R(U_{\mathfrak{U}})} \in Y_{\mathfrak{U}}/\overline{R(U_{\mathfrak{U}})} \longrightarrow L_{\mathfrak{U}}(\mathbf{x}) + \overline{R(U_{\mathfrak{U}})} \in Y_{\mathfrak{U}}/\overline{R(U_{\mathfrak{U}})}$$

induced by $L_{\mathfrak{U}}$. Then, for every $\mathbf{g} \in N({U^*}_{\mathfrak{U}})$,

$$\overline{B_{N({U^*}_{\mathfrak{U}})} \cap \Lambda^{*-1}(\mathbf{g} + B_{N({U^*}_{\mathfrak{U}})})}^{w^*} = B_{N({U_{\mathfrak{U}}}^*)} \cap \Lambda^{*-1}(\mathbf{g} + B_{N({U_{\mathfrak{U}}}^*)}),$$

where w^* represents the $\sigma\left(N({U_{\mathfrak{U}}}^*)\,,\, Y_{\mathfrak{U}}/\overline{R(U_{\mathfrak{U}})}\right)$ topology.

The proof is rather involved, so we refer the reader to [37].

Theorem 6.12. *[37] Let $T\colon X \longrightarrow Y$ be any operator and let $\mathfrak{U}$ be an ultrafilter. Then the kernel $N({T^*}_{\mathfrak{U}})$ is a local dual of $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$.*

Proof. By Theorem 6.6, we just need to show that $N({T^*}_{\mathfrak{U}})$ has the strict polar property as a subspace of $N({T_{\mathfrak{U}}}^*) = (Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})})^*$. Let M be a matrix of order $k \times l$ and consider the induced matricial operator

$$\Delta := M_{N({T_{\mathfrak{U}}}^*)}\colon \ell_\infty^k\big(N({T_{\mathfrak{U}}}^*)\big) \longrightarrow \ell_\infty^l\big(N({T_{\mathfrak{U}}}^*)\big).$$

By Proposition 6.4, we only need to show the next identity holds for every $\mathbf{g} \in N({T^*}_{\mathfrak{U}})$:

$$\begin{aligned} &B_{\ell_\infty^k(N({T_{\mathfrak{U}}}^*))} \cap \Delta^{-1}(\mathbf{g} + B_{\ell_\infty^l(N({T_{\mathfrak{U}}}^*))}) = \\ &\qquad = \overline{B_{\ell_\infty^k(N({T^*}_{\mathfrak{U}}))} \cap \Delta^{-1}(\mathbf{g} + B_{\ell_\infty^l(N({T^*}_{\mathfrak{U}}))})}^{w^*} \end{aligned} \tag{33}$$

where w^* denotes the $\sigma\left(\ell_\infty^k\big(N(T_{\mathfrak{U}}{}^*)\big), \ell_1^k\big(Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}\big)\right)$ topology.

The proof is divided into three cases: $k = l$, $k < l$ and $k > l$.

Case $k = l$. Let us denote $U := T^k \in \mathcal{B}\big(\ell_1^k(X), \ell_1^k(Y)\big)$. Since the operator $\phi\colon \ell_1^k(Y_{\mathfrak{U}}) \longrightarrow \ell_1^k(Y)_{\mathfrak{U}}$ that sends $([y_i^j]_i)_{j=1}^k$ to $[(y_i^j)_{j=1}^k]_i$ is a bijective isometry that maps $R(U_{\mathfrak{U}})$ onto $R(T_{\mathfrak{U}})^k$, the induced operator

$$\Phi\colon \frac{\ell_1^k(Y)_{\mathfrak{U}}}{\overline{R(U_{\mathfrak{U}})}} \longrightarrow \frac{\ell_1^k(Y_{\mathfrak{U}})}{\overline{R(T_{\mathfrak{U}})}^k}$$

is also a bijective isometry, and so is $\Phi^*\colon \ell_\infty^k\big(N(T_{\mathfrak{U}}{}^*)\big) \longrightarrow N(U_{\mathfrak{U}}{}^*)$. Therefore we can identify $N(U_{\mathfrak{U}}{}^*)$ with $\ell_\infty^k\big(N(T_{\mathfrak{U}}{}^*)\big)$, and $N(U^*{}_{\mathfrak{U}})$ with $\ell_\infty^k\big(N(T^*{}_{\mathfrak{U}})\big)$.

Consider now the matricial operator $L\colon \ell_1^k(Y) \longrightarrow \ell_1^k(Y)$ associated to the matrix M^*, and the operator

$$\Lambda\colon \frac{\ell_1^k(Y)_{\mathfrak{U}}}{\overline{R(U_{\mathfrak{U}})}} \longrightarrow \frac{\ell_1^k(Y)_{\mathfrak{U}}}{\overline{R(U_{\mathfrak{U}})}}$$

defined by $\Lambda(\mathbf{x}+\overline{R(U_{\mathfrak{U}})}) := L_{\mathfrak{U}}(\mathbf{x})+\overline{R(U_{\mathfrak{U}})}$. Thus $\Lambda^*\colon N(U_{\mathfrak{U}}{}^*) \longrightarrow N(U_{\mathfrak{U}}{}^*)$, can be identified with Δ (in fact, $\Lambda^* = \Phi^{*-1} \circ \Delta \circ \Phi^*$), showing that formula (33) is equivalent to

$$\overline{B_{N(U^*{}_{\mathfrak{U}})} \cap \Lambda^{*-1}(\mathbf{g} + B_{N(U^*{}_{\mathfrak{U}})})}^{\mathsf{w}^*} = B_{N(U_{\mathfrak{U}}{}^*)} \cap \Lambda^{*-1}(\mathbf{g} + B_{N(U_{\mathfrak{U}}{}^*)}), \qquad (34)$$

where w^* denotes the $\sigma\big(N(U_{\mathfrak{U}}{}^*)\,,\, \ell_1^k(Y)_{\mathfrak{U}}/\overline{R(U_{\mathfrak{U}})}\,\big)$ topology. But the operator U satisfies all the conditions of Lemma 6.11, so formula (34) holds, which proves the case $k = l$. The cases $k < l$ and $k > l$ are easily derived from the case $k = l$. □

Given a local dual space Z of X and a separable, infinite dimensional subspace E of X^*, in general there is not an ε-isometry $L\colon E \longrightarrow Z$ $(0 < \varepsilon < 1)$, but there is an exception:

Corollary 6.13. *Let $T\colon X \longrightarrow Y$ be an operator and let $\mathfrak{U}$ be an ultrafilter. Then for every separable subspace E of $N(T_{\mathfrak{U}}{}^*)$ and every separable subspace F of $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$, there exists an isometry $L\colon E \longrightarrow N(T^*{}_{\mathfrak{U}})$ such that $\langle L(\mathbf{f})-\mathbf{f}, \mathbf{x}\rangle = 0$ for all $\mathbf{f} \in E$ and all $\mathbf{x} \in F$, and $L(\mathbf{f}) = \mathbf{f}$ for all $\mathbf{f} \in E \cap N(T^*{}_{\mathfrak{U}})$.*

Sketch of the proof. It is a direct consequence of the local duality between $N(T^*{}_{\mathfrak{U}})$ and $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$ shown in Theorem 6.12, combined with standard ultraproduct techniques: as E and F are separable, there are two increasing sequences $(E_n)_{n=1}^\infty$ and $(F_n)_{n=1}^\infty$ of subspaces such that $\overline{\cup_{n=1}^\infty E_n} = E$, $\overline{\cup_{n=1}^\infty F_n} = F$ and $\dim E_n = \dim F_n = n$. Since $N(T^*{}_{\mathfrak{U}})$ is a local dual of $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$, for every n there exists a $1/n$-isometry $L_n\colon E_n \longrightarrow N(T^*{}_{\mathfrak{U}})$ satisfying

$$\begin{aligned} &\langle L_n(\mathbf{f}) - \mathbf{f}, \mathbf{x}\rangle = 0 \qquad \text{for all } \mathbf{f} \in E_n \text{ and all } \mathbf{x} \in F_n \\ &L_n(\mathbf{f}) = \mathbf{f} \qquad \text{for all } \mathbf{f} \in E_n \cap N(T^*{}_{\mathfrak{U}}). \end{aligned}$$

We choose $\{[y_i^n]_i\}_{n=1}^\infty$ in $Y_\mathfrak{U}$ so that $\{[y_i^k]_i + \overline{R(T_\mathfrak{U})}\}_{k=1}^n$ is a normalized basis of F_n for every n. In a similar manner, we choose a system $\{\mathbf{e}_k\}_{k=1}^\infty$ in E such that for every n, $\{\mathbf{e}_k\}_{k=1}^\infty$ is a normalized basis of E_n. For every $1 \leq k \leq n$, we fix a representative $\{e_i^k\}_{i\in I}$ of $L_n(\mathbf{e}_k)$. Let $L_i^n \colon E_n \longrightarrow Y^*$ be the operator that maps each $\mathbf{e}_k$ to e_i^k.

Notice that $[f_i]$ belongs to $N({T^*}_\mathfrak{U})$ if and only if $\|T^*(f_i)\| \xrightarrow{\mathfrak{U}} 0$, that is, if and only if

$$\sup \langle f_i, T(B_X)\rangle \xrightarrow[i\to\mathfrak{U}]{} 0;$$

thus, for every n, typical ultraproduct arguments prove the existence of $H_n \in \mathfrak{U}$ such that for every $i \in H_n$,

(i) L_i^n is a $1/n$-isometry,
(ii) $\sup\langle e, T(B_X)\rangle < 1/n$, for all e in the unit sphere of $L_i^n(E_n)$,
(iii) $|\langle L_i^n(\mathbf{f}_k), y_i^l\rangle - \langle L_n(\mathbf{f}_k), [y_i^l]\rangle| < 1/n$, for all $1 \leq k \leq n$, all $1 \leq l \leq n$,
(iv) $\|L_i^n(\mathbf{f}) - f_i\| \leq 1/n$, for all $\mathbf{f} = [f_i]$ in the unit sphere of $E_n \cap (Y^*)_\mathfrak{U}$.

Since $\mathfrak{U}$ is $\aleph_0$-incomplete, we may take a decreasing sequence $(K_n)_{n=1}^\infty$ of elements of $\mathfrak{U}$ with $J_n \subset H_n$ and $\cap_{n=1}^\infty J_n = \emptyset$, which yields a countable partition $\{I_n\}_{n=0}^\infty$ of I disjoint with $\mathfrak{U}$ given by $I_0 := I \setminus J_1$ and $I_n := J_n \setminus J_{n+1}$ for $n \geq 1$. Then we define the desired operator L on each $\mathbf{f} \in \cup_{n=1}^\infty E_n$ as follows: if m is the smallest integer for which $x \in E_m$, then $L(\mathbf{f}) = [f_i]$ with $f_i := L_i^n(\mathbf{f})$ for every $n \geq m$ and every $i \in I_n$, and $f_i := 0$ for the remaining indices. Next, L is extended to the whole subspace E by continuity. Standard computations show that condition (i) yields that L is an isometry, condition (ii) shows that the range of L is contained in $N((T^*)_\mathfrak{U})$, and conditions (iii) and (iv) show respectively that conditions (B) and (C) hold with respect to F. □

Corollary 6.13 was proved in [5] for the case when T is null on X.

Applications of the polar properties.

As it may be expected from Theorem 6.12, the scope of the strict polar property goes beyond the mere unification of both principles of local reflexivity given in Corollary 6.10. Given an operator $T \colon X \longrightarrow Y$ and an ultrafilter $\mathfrak{U}$, the fact that $N\big((T^*)_\mathfrak{U}\big)$ is a norming subspace of $N\big((T_\mathfrak{U})^*\big)$ solved some questions on the duality between the semigroups of supertauberian and co-supertauberian operators ([30] and [53]). In order to extend these results to other ultrapower-stable semigroups, it was necessary to show that the kernel $N\big((T^*)_\mathfrak{U}\big)$ is finitely representable in $N\big((T_\mathfrak{U})^*\big)$. Recall that a class $\mathcal{S}$ of operators is *ultrapower-stable* if $T_\mathfrak{U} \in \mathcal{S}$ for all $T \in \mathcal{S}$ and all ultrafilters $\mathfrak{U}$.

Since $N\big((T_\mathfrak{U})^*\big)$ is not a bidual, and since $N\big((T^*)_\mathfrak{U}\big)$ coincides with the ultrapower $N\big((T^*)\big)_\mathfrak{U}$ if and only if T has closed range (Proposition 15 in [31]), neither the P.L.R. nor the P.L.R. for ultrapowers seem applicable on the question of the finite representability of the kernels of $(T_\mathfrak{U})^*$ and $(T^*)_\mathfrak{U}$. Finally, that question was positively solved in [32] by means of the polar property, which was introduced as a way of identifying when a subspace Z of a dual space X^* is norming and, at the same time, X^* is finitely representable in Z. Later, it was proved that $N\big((T^*)_{\mathfrak{U}}\big)$ has also the strict polar property [37], that is, $N\big((T^*)_{\mathfrak{U}}\big)$ is a local dual of $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$.

The ultrapower-stable semigroups we are talking about are $\mathcal{W}^{up}{}_+$, $\mathcal{U}^{up}{}_+$, $\mathcal{R}^{up}{}_+$, $\mathcal{W}^{up}{}_-$, $\mathcal{U}^{up}{}_-$ and $\mathcal{R}^{up}{}_-$, which admit the following working definitions (see [30], [32] and [33]):

(1) $T \in \mathcal{W}^{up}{}_+$ if and only if $N(T_{\mathfrak{U}})$ is superreflexive,
(2) $T \in \mathcal{U}^{up}{}_+$ if and only if c_0 is not finitely representable in $N(T_{\mathfrak{U}})$,
(3) $T \in \mathcal{R}^{up}{}_+$ if and only if ℓ_1 is not finitely representable in $N(T_{\mathfrak{U}})$,
(4) $T \in \mathcal{W}^{up}{}_-$ if and only if $T^* \in \mathcal{W}^{up}{}_+$,
(5) $T \in \mathcal{U}^{up}{}_-$ if and only if $T^* \in \mathcal{U}^{up}{}_+$,
(6) $T \in \mathcal{R}^{up}{}_-$ if and only if $T^* \in \mathcal{R}^{up}{}_+$.

The most remarkable result about the duality of the above mentioned semigroups are the following: $T \in \mathcal{W}^{up}{}_+$ (resp. $\mathcal{U}^{up}{}_+$, $\mathcal{R}^{up}{}_+$) if and only if $T^{**} \in \mathcal{W}^{up}{}_+$ (resp. $\mathcal{U}^{up}{}_+$, $\mathcal{R}^{up}{}_+$). Moreover, the ultrapower-stability of the semigroups $\mathcal{W}^{up}{}_-, \mathcal{U}^{up}{}_-$ and $\mathcal{R}^{up}{}_-$ is easily obtained via the finite representability of $N\big((T_{\mathfrak{U}})^*\big)$ in $N\big((T^*)_{\mathfrak{U}}\big)$.

Further investigation on the problem of the duality of semigroups was carried out by means of operator finite representability (see [55] and [56]). In connection with finite representability of operators, see also [8] and [21]. For more information about semigroups of operators and their relation to operator ideals, see [1] and [38].

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