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DYNAMICS OF TWO SERIALLY CONNECTED MANIPULATORS

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Abstract

The objective of this paper is to generate a closed form of the equations of motion for two serially connected manipulators. Instead of computing the coupled dynamics directly, this work constructs overall dynamic equations from previously known equations of each manipulator and coupling terms derived in this paper. This approach will reduce the number of computations significantly and show the structure of the coupling dynamics between two arms. The proposed technique has been coded in *Mathematica* for symbolic computation. As a case study, the proposed approach is applied to two examples: micro/macro manipulators and a mobile manipulator. Each case shows not only simplicity of derivation but also a reduction in computation time of at least one-third compared to the conventional direct derivation.

1. Introduction

It is becoming popular to couple two robotic systems to obtain both accuracy and mobility of motion. For example, macro/micro manipulators can provide precise motion as well as a large workspace. Also, an autonomous vehicle equipped with a manipulator has distinct advantages in mobility and versatility. Understanding the coupled dynamics between two systems is essential to control these systems. However, it has been a time consuming and recursive procedure to derive the dynamic equations of these coupled systems. The objective of this paper is to generate a closed form of the equations of motion for two serially connected manipulators. Instead of computing the coupled dynamics directly, this work constructs overall dynamic equations from the previously known equations of each manipulator and coupling terms derived in this paper. In addition, this research incorporates the flexible link dynamics of a large arm (base system) because of common characteristic of large arm. Still, this work can be applied to a rigid large arm too.

In the past, much research has been done in the formulation of the dynamic equations of motion for a manipulator. Some researchers have proposed several recursive formulations for computational efficiency using Lagrangian [Hollerbach,80], Newtonian-Euler [Luh,80], or Kane's method [Kane,83]. However, the recursive formulations fail to show the structure of dynamics which is critical information for a controller designer. With recent development of symbolic computational software, several algorithms have been developed to derive the equations of motion in symbolic forms [Burdrick,86] [Murray,841 [Cetinkunt,87]. Symbolic formulation gives the controller designer insight to the dynamic characteristics of the system. The physical interpretations and structural characteristics of the Lagrangian dynamics for rigid robots were derived by [Tourassis,85] Some research has incorporated flexible links in modeling and can be classified as three groups: (1) partial differential equation [Bejczy,88] (2) finite element method [Bayo,87] (3) assumed mode method [Book,84]. However, the Lagrangian method with the assumed mode method approach has been widely used for real world applications. [Book 84] uses homogeneous transformations for flexible link deflection, and its systematic approach for multi-link dynamics was a major contribution. [Lee,90] applied Jacobian matrices that were used in rigid dynamic formulation in [Asada,86] to the flexible link case. This method shows that the inertia matrix can be derived from the Jacobian without computing Lagrangian equations. Although most work has been developed for dynamic formulation of a single manipulator, to date, no prior work has been done to handle a coupled dynamic system such as micro/macro manipulators [Sharon,84] [Lew,91] and a mobile manipulator [Dubowsky,88] efficiently.

The outline of this paper is as follows: First, the paper reviews an idea of forward kinematics for flexible manipulators using 4 x 4 homogeneous transformations which has been introduced in [Book,84]. Second, multi-link flexible manipulator dynamics is formulated in an easy-to-understand Jacobian approach using the assumed mode method similar to [Lee,90]. Third, as a major contribution of this work, the coupled dynamics term is derived when a small arm is mounted on the large arm without direct derivation. Finally, the proposed work is applied to two examples: a two flexible link arm with a two rigid link arm, and a moving vehicle with a 3 D.O.F. manipulator.

2. Forward Kinematics of a Flexible Manipulator

Consider the kinematic structure shown in Figure 1 representing a manipulator with serial links connected by rotational joints. The elements of the manipulator are numbered, and body fixed moving coordinates are assigned as shown, where O_{XYZ} is the inertial coordinate frame. 4 x 4 homogeneous transformation matrices are used to describe the position and orientation of one coordinate frame with respect to another.

Thus, in terms of the fixed inertial coordinates of the base, the position of a point on the link i, h_i, is given as

$$h_i = W_i^o h_i^i = W_i h_i^i \tag{1}$$

where the special case of $W_i^o = W_i$. It is useful to separate the transformations due to the joint and the flexible link as shown in Figure 1:

$$W_i = W_{i-1} \mathrm{E}_{i-1} \mathrm{A}_i \tag{2}$$

where A_i = the joint transformation for joint i, and E_{i-1} = the link transformation matrix for link i-1 between joints i-1 and i. To incorporate the

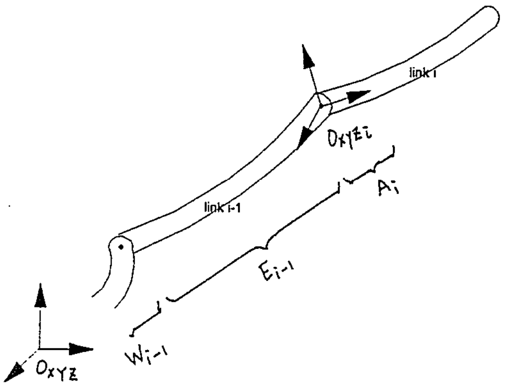

Figure 1 Joint Transformation ($\mathbf{A}_i$) and Link Transformation ($\mathbf{E}_{i-1}$) for Flexible Link Manipulators

deflection of the link, the assumed mode approach is used. The point position in link i with respect to joint i is

$$h_i^i(x) = \begin{bmatrix} x \\ 0 \\ 0 \\ 1 \end{bmatrix} + \sum_{j=1}^{n_i} q_{fij}(t) \begin{bmatrix} \chi_{ij}(x) \\ \psi_{ij}(x) \\ \phi_{ij}(x) \\ 0 \end{bmatrix} \tag{3}$$

where ψ_{ij}, ϕ_{ij} = the j-th mode shape function for the bending of the i-th link in the Y_i, Z_i direction; χ_{ij} = the j-th mode shape function for the torsion of the i-th link, but it is negligible for the most cases; q_{fij} = the time-varying amplitude of mode j of link i; and n_i = the number of modes used to described the deflection of link i.

The link transformation matrix, $\mathbf{E}_{i-1}$, must also incorporate the deflection of the link. Here the rotations as well as the translations of the deflection must be represented by a differential coordinate transformation. This is an approximation in the kinematic description. The approximation is valid to the extent that the orientation change of coordinate frame i due to deflections is small enough to justify the following approximation:

$$\sin\theta \approx \theta,$$
$$\cos\theta \approx \theta$$

Therefore, the link transformation matrix can be written as

$$\mathbf{E}_i = \mathbf{T}_i + \sum_{j=1}^{n_i} q_{fij}\mathbf{V}_{ij} \tag{4}$$

where $T_i = \begin{bmatrix} 1 & 0 & 0 & l_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $V_{ij} = \begin{bmatrix} 0 & -(\theta_z)_{ij} & (\theta_y)_{ij} & \chi_{ij} \\ (\theta_z)_{ij} & 0 & -(\theta_x)_{ij} & \psi_{ij} \\ -(\theta_y)_{ij} & (\theta_x)_{ij} & 0 & \phi_{ij} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

where $(\theta_x)_{ij}, (\theta_y)_{ij}$, and $(\theta_z)_{ij}$ = the X_i, Y_i, and Z_i rotation component of link i, respectively and can be defined as follows:

$$(\theta_x)_{ij} = \left.\frac{\partial\phi_{ij}}{\partial x_{ij}}\right|_{x_i=l_i}$$

$$(\theta_y)_{ij} = \left.\frac{\partial\psi_{ij}}{\partial x_{ij}}\right|_{x_i=l_i}$$

$$(\theta_z)_{ij} = \chi(l_i)$$

3. Dynamics of a Flexible Manipulator

In this section, dynamic equations of motion for flexible manipulators are formulated using kinematic transformations and Jacobian matrices. First, the expression for the system's kinetic energy is developed for use in Lagrange equations. Potential energy and elastic energy are also derived. Then, the Lagrange formulation shows that Jacobian matrices can be used to derive the inertia matrices. The coefficients of centrifugal and Coriolis force are derived from the inertia matrices using Christoffel's symbol.

The kinetic energy of i-th link is

$$\Delta KE_i = \frac{1}{2}\int_0^{l_i} \dot{r}_i^T \dot{r}_i \rho_i A_i dx_i \tag{5}$$

where $\dot{r}_i$ is the velocity vector of an arbitrary point in the i-th link, and ρ_i, A_i, l_i are the density, area, and the length of link i respectively. If $\dot{r}_i$ is expressed in terms of joint and flexible mode coordinates,

$$\dot{r}_i = \mathbf{J}_i \mathbf{q}_i \tag{6}$$

where $\mathbf{J}_i$ is the Jacobian matrix of the point x_i in the i-th link. Also, recall that $\mathbf{q}_i$ includes the rigid joint coordinates q_r and the flexible mode coordinates q_f up till the i-th link.

The Jacobian matrix $\mathbf{J}_i$ can be computed from the forward kinematic transformations. Since

$$\begin{aligned} h_i &= W_1^0 W_2^1 W_3^2 \cdots W_i^{i-1} h_i^i \\ &= \begin{bmatrix} R_i & r_{i-1} \\ 0^T & 1 \end{bmatrix} h_i^i \\ &= \begin{pmatrix} r_i \\ 1 \end{pmatrix} \end{aligned} \tag{7}$$

where W_i^{i-1} = a 4 x 4 homogenous transformations for joint i and flexible link i-1; R_i = a 3x3 rotational matrix; and 0 = a 3x1 vector of zeros, the Jacobian matrix for the point r_i in the i-th link is

$$J_i = \frac{\partial r_i}{\partial \mathbf{q}_i} \tag{8}$$

Assume that there are L links. From now on, substitute dm_i for $\rho_i A_i dx_i$ to simplify notation. Then, the total kinetic energy of the system is

$$\begin{aligned} KE &= \sum_{i=1}^{L} \frac{1}{2}\int_0^{l_i} \dot{r}_i^T \dot{r}_i dm_i \\ &= \frac{1}{2}\sum_{i=1}^{L}\int_0^{l_i} (\mathbf{J}_i\dot{\mathbf{q}}_i)^T(\mathbf{J}_i\dot{\mathbf{q}}_i) dm_i \\ &= \frac{1}{2}\dot{\mathbf{q}}^T \Big(\sum_{i=1}^{L}\int_0^{l_i} \mathbf{J}_i^T \mathbf{J}_i dm_i\Big)\dot{\mathbf{q}} \\ &= \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}\dot{\mathbf{q}} \\ &= \frac{1}{2}\sum_{k,j=1}^{N} M_{kj} q_k q_j \end{aligned} \tag{9}$$

where the k,j-th element of the element of the matrix M can be expressed as

$$M_{kj} = \Big(\sum_{i=1}^{L} \mathbf{J}_i^T \mathbf{J}_i dm_i\Big)_{kj} \tag{10}$$

The potential energy due to gravity is

$$U_g = \sum_{i=1}^{L}\int_0^{l_i} g^T r_i dm_i \tag{11}$$

g is the 3 x 1 gravity acceleration vector.

The potential energy due to elastic deformation is

$$U_E = \frac{1}{2}\sum_{i=1}^{L}\int_0^{l_i} E_i I_{zi}(\frac{\partial^2 u_{yi}}{\partial x_i^{\,2}})^2 + E_i I_{yi}(\frac{\partial^2 u_{zi}}{\partial x_i^{\,2}})^2 + G_i I_{xi}(\frac{\partial \theta_{xi}}{\partial x_i})^2 dx_i \quad (12)$$

where E_i is Young's modulus of elasticity, and I_i is the area moment of inertia, and G_i is the shear modulus of elasticity of the link. u_i and θ_i are the elastic deflection and rotation which can be expressed by m modes and modal coordinates.

$$\begin{aligned} u_{yi}(x,t) &= \sum_{j=1}^{m} \psi_{ij}(x)\, q_{fij}(t) \\ u_{zi}(x,t) &= \sum_{j=1}^{m} \phi_{ij}(x)\, q_{fij}(t) \\ \theta(x,t) &= \sum_{j=1}^{m} \chi_{ij}(x)\, q_{fij}(t) \end{aligned} \quad (13)$$

Therefore, the total potential energy is

$$PE = U_E + U_g$$

Using Lagrange's equation with the total kinetic energy and potential energy,

$$\frac{d}{dt}(\frac{\partial KE}{\partial \dot{q}_k}) + \frac{\partial KE}{\partial q_k} + \frac{\partial PE}{\partial q_k} = Q_i \quad (14)$$

compute the following terms to obtain the equations of motion. First, from equation (9),

$$\frac{\partial KE}{\partial \dot{q}_k} = \sum_{j=1}^{N} M_{kj}\dot{q}_j \quad (15)$$

Then, take the time derivative of equation(15).

$$\begin{aligned} \frac{d}{dt}(\frac{\partial KE}{\partial \dot{q}_k}) &= \sum_{j=1}^{N} M_{kj}\ddot{q}_j + \sum_{j=1}^{N} \dot{M}_{kj}\dot{q}_j \\ &= \sum_{j=1}^{N} M_{kj}\ddot{q}_j + \sum_{i,j=1}^{N} \frac{\partial M_{kj}}{\partial q_k}\dot{q}_i\dot{q}_j \end{aligned} \quad (16)$$

Also, the generalized coordinate derivative of the kinetic energy is

$$\frac{\partial KE}{\partial q_k} = \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial M_{ij}}{\partial q_k} q_i q_j \quad (17)$$

Finally, take the partial derivative of the potential energy.

$$\begin{aligned} \frac{\partial PE}{\partial q_k} &= \frac{\partial U_E}{\partial q_k} + \frac{\partial U_E}{\partial q_k} \\ &= \sum_{j=1}^{n_i}\int_0^{l_i} E_i I_{zi}(\frac{\partial^2 \psi_{ij}}{\partial x_i^2})^2 + E_i I_{yi}(\frac{\partial^2 \phi_{ij}}{\partial x_i^2})^2 + G_i I_{xi}(\frac{\partial^2 \psi_{ij}}{\partial x_i^2})^2 dx_i q_k + \frac{\partial U_E}{\partial q_k} \end{aligned} \quad (18)$$

where
$$\begin{aligned} \frac{\partial U_g}{\partial q_k} &= \sum_{i=1}^{N}\int_0^{l_i} g^T \frac{\partial r_i}{\partial q_k} dm_i \\ &= \sum_{i=1}^{N}\int_0^{l_i} g^T \mathbf{J}_i^{(k)} dm_i \end{aligned} \quad (19)$$

Therefore, the gravity matrix elements are defined as

$$G_k = \sum_{i=1}^{N}\int_0^{l_i} \mathbf{J}_i[j,k]dm_i \quad .(20)$$

where $\mathbf{J}_i$ [j,k] is the j-throw and k-th column of $\mathbf{J}_i$.

Finally, the generalized force Q_k can be obtained from the virtual work. The virtual work by the joint torque **t** is

$$\begin{aligned} \Delta Work &= \mathbf{t}^T \delta \mathbf{q} \\ &= \mathbf{t}^T \left[\delta q_{r1} + \sum_{j=1}^{m} \frac{\partial \psi_{1j}}{\partial x_1}\bigg|_{x_1=0} \delta q_{f1j}, \cdots\cdots, \delta q_{rL} + \sum_{j=1}^{m} \frac{\partial \psi_{Lj}}{\partial x_L}\bigg|_{x_L=0} \delta q_{fLj} \right]^T \\ &= \mathbf{t}^T \delta \mathbf{q}_r + \mathbf{t}^T \left[\sum_{j=1}^{m} \frac{\partial \psi_{1j}}{\partial x_1}\bigg|_{x_i=0} \delta q_{f1j}, \cdots\cdots, \sum_{j=1}^{m} \frac{\partial \psi_{Lj}}{\partial x_L}\bigg|_{x_L=0} \delta q_{fLj} \right] \\ &= \sum_{k=1}^{N} Q_k \delta q_k \end{aligned} \quad (21)$$

Therefore, the closed form of the dynamic equations of the L link flexible manipulator can be expressed as

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{q} + \mathbf{K}\mathbf{q} + \mathbf{G}(\mathbf{q}) = \mathbf{B}\tau \quad (22)$$

where $\mathbf{M}(\mathbf{q})$ is defined in equation(11), and the k,j-th element of the matrix $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ consists of

$$C_{kj} = \sum_{i=1}^{N}(\frac{\partial M_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial M_{ij}}{\partial q_k})\dot{q}_i \quad (23)$$

and all the elements of the matrix K are zeros except for the L+j-th diagonal element which is defined as

$$K_{(L+j)(L+j)} = \int_0^{l_i}\{\sum_{j=1}^{n_i} E_i I_{zi}(\frac{\partial^2 \psi_{ij}}{\partial x_i^2})^2 + E_i I_{yi}(\frac{\partial^2 \phi_{ij}}{\partial x_i^2})^2 + G_i I_{xi}(\frac{\partial^2 \psi_{ij}}{\partial x_i^2})^2\}dx_i \quad (24)$$

The element of the gravity $\mathbf{G}(\mathbf{q})$ is shown in equation(20), and the input matrix $\mathbf{B}$ is computed from equation(21).

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_L \\ Diag(\frac{\partial \psi_{1i}}{\partial x_i}\Big|_{x_i=0}) \\ \vdots \\ \vdots \end{bmatrix} \quad (25)$$

4. Dynamics of Serially Connected Two Manipulators

Our objective is to derive a closed form of the equations of motion for serially connected large and small manipulators using separately known dynamics of two manipulators. This approach will reduce the number of computations significantly and show the structure of coupling dynamics between two arms.

Let us assume that the dynamics of a large manipulator is known as

$$\mathbf{M}_L(\mathbf{q}_L)\ddot{\mathbf{q}}_L + \mathbf{C}_L(\dot{\mathbf{q}}_L,\mathbf{q}_L)\dot{\mathbf{q}}_L + \mathbf{K}_L\mathbf{q}_L + \mathbf{G}_L(\mathbf{q}_L) = \mathbf{B}\tau_L \quad (26)$$

where $\mathbf{q}_L$ includes the rigid joint coordinates $\mathbf{q}_r$ and the flexible mode coordinates $\mathbf{q}_f$. The large manipulator has L links. Also, assume that a small manipulator is rigid and its dynamics can be represented with the following form:

$$\mathbf{M}_S(\mathbf{q}_S)\ddot{\mathbf{q}}_S + \mathbf{C}_S(\dot{\mathbf{q}}_S,\mathbf{q}_S)\dot{\mathbf{q}}_S + \mathbf{G}_S(\mathbf{q}_S) = \tau_S \quad (27)$$

The small manipulator has S links.

Consider the kinetic energy of the i-th link which is

$$\Delta KE_i = \frac{1}{2}\int_0^{l_i} \dot{r}_i^T \dot{r}_i dm_i$$

If the small manipulator is mounted at the tip of the large manipulator, the position vector of the L+j link referenced to the fixed inertial frame will be

$$\begin{aligned} h_{(L+j)} &= W_{LL}^{0}\mathbf{E}_L W_{Sj}^{0} h_j^j \\ &= \begin{bmatrix} R_L & r_L \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} R_{Sj} & r_{Sj-1} \\ 0^T & 1 \end{bmatrix} h_j^j \\ &= \begin{pmatrix} R_L r_{Sj} + r_L \\ 1 \end{pmatrix} \end{aligned} \tag{28}$$

where W_{LL} = a 4 x 4 transformation of the large arm from the base to link L; W_{Sj} = a 4 x 4 transformation of a small arm from the base to the link j; $\mathbf{E}_L$ = a link transformation of the large arm between the joint L and the link tip where the small arm is mounted. Then the position vector r_{L+j} of an arbitrary point in the L+j th link is

$$r_{L+j} = r_L + R_L r_{Sj} \tag{29}$$

and its velocity vector is

$$\dot{r}_{L+j} = \dot{r}_L + \dot{R}_L r_{Sj} + R_L \dot{r}_{Sj} \tag{30}$$

where $$\dot{R}_L = \frac{dR_L}{dt} = \sum_{i=1}^{N} \frac{\partial R_L}{\partial q_{Li}} \dot{q}_{Li} \tag{31}$$

The second term of the equation (30) gives the portion of changes of r_{L+j} that is attributable to rotation of the reference frame of the small arm. It needs to be expressed as a function $\mathbf{q}_L$ of explicitly. Since $\dot{q}_{Li}$ is a scalar in equation (31), the second term of equation (30) can be rewritten as

$$\begin{aligned} \dot{\mathbf{R}}_L r_{Sj} &= \sum_{i=1}^{N} \frac{\partial R_L}{\partial q_{Li}} r_{Sj} \dot{q}_{Li} \\ &= \mathbf{H}_j \dot{\mathbf{q}}_L = \frac{\partial (R_L r_{Sj})}{\partial \mathbf{q}_L} \dot{\mathbf{q}}_L \end{aligned} \tag{32}$$

where $\mathbf{H}_j$ is defined as

$$\mathbf{H}_j = \begin{bmatrix} \frac{\partial R_L}{\partial q_{L1}} r_{Sj} & \frac{\partial R_L}{\partial q_{L2}} r_{Sj} & \frac{\partial R_L}{\partial q_{L3}} r_{Sj} & \cdots \end{bmatrix} = \frac{\partial (R_L r_{Sj})}{\partial \mathbf{q}_L} \tag{33}$$

Then, by substituting $\mathbf{J}_L \dot{\mathbf{q}}_r$ and $\mathbf{J}_{Sj} \dot{\mathbf{q}}_{Sj}$ into $\dot{r}_L$ and $\dot{r}_{Sj}$, we can build a quadratic form of the kinetic energy. As shown in the previous section, this quadratic form of kinetic energy makes the computation of the inertia matrix very easy. To compute the kinetic energy of serially connected two manipulators, pre-multiply the velocity term by transpose of itself.

$$\begin{aligned} \dot{r}_{L+j}{}^T \dot{r}_{L+j} &= (\mathbf{J}_L \dot{\mathbf{q}}_L + \mathbf{H}_j \dot{\mathbf{q}}_L + \mathbf{R}_L \mathbf{J}_{Sj} \dot{\mathbf{q}}_{Sj})^T (\mathbf{J}_L \dot{\mathbf{q}}_L + \mathbf{H}_j \dot{\mathbf{q}}_L + \mathbf{R}_L \mathbf{J}_{Sj} \dot{\mathbf{q}}_{Sj}) \\ &= (\dot{\mathbf{q}}_L^T \; \dot{\mathbf{q}}_{Sj}^T) \begin{bmatrix} (\mathbf{J}_L + \mathbf{H}_j)^T (\mathbf{J}_L + \mathbf{H}_j) & (\mathbf{J}_L + \mathbf{H}_j)^T \mathbf{R}_L \mathbf{J}_{Sj} \\ \mathbf{R}_L \mathbf{J}_{Sj} (\mathbf{J}_L + \mathbf{H}_j) & \mathbf{J}_{Sj}^T \mathbf{J}_{Sj} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{q}}_L \\ \dot{\mathbf{q}}_{Sj} \end{pmatrix} \end{aligned} \tag{34}$$

Recall that $\mathbf{R}_L^T \mathbf{R}_L = \mathbf{I}$ where $\mathbf{I}$ is a 3 x 3 identity matrix. Therefore, the total kinetic energy of two serially connected manipulators can be expressed as

$$\begin{aligned} KE &= \sum_{i=1}^{L} \frac{1}{2} \int_0^{l_i} \dot{r}_i^T \dot{r}_i dm_i + \sum_{j=1}^{S} \frac{1}{2} \int_0^{l_i} \dot{r}_{L+j}^T \dot{r}_{L+j} dm_i \\ &= \frac{1}{2} \dot{\mathbf{q}}_L^T \mathbf{M}_L \dot{\mathbf{q}}_L \\ &\quad + (\dot{\mathbf{q}}_L^T \; \dot{\mathbf{q}}_{Sj}^T) \sum_{j=1}^{S} \int_0^{l_j} \begin{bmatrix} (\mathbf{J}_L + \mathbf{H}_j)^T (\mathbf{J}_L + \mathbf{H}_j) & (\mathbf{J}_L + \mathbf{H}_j)^T \mathbf{R}_L \mathbf{J}_{Sj} \\ \mathbf{R}_L \mathbf{J}_{Sj} (\mathbf{J}_L + \mathbf{H}_j) & \mathbf{J}_{Sj}^T \mathbf{J}_{Sj} \end{bmatrix} dm_j \begin{pmatrix} \dot{\mathbf{q}}_L \\ \dot{\mathbf{q}}_{Sj} \end{pmatrix} \\ &= \frac{1}{2} (\dot{\mathbf{q}}_L^T \; \dot{\mathbf{q}}_{Sj}^T) \, \mathbf{M}_{L+S} \begin{pmatrix} \dot{\mathbf{q}}_L \\ \dot{\mathbf{q}}_{Sj} \end{pmatrix} \end{aligned} \tag{35}$$

and forms a quadratic expression where the inertia matrix is

$$\mathbf{M}_{L+S} = \begin{bmatrix} \mathbf{M}_L + \mathbf{M}_{L/S} & \mathbf{M}_{cp} \\ \mathbf{M}_{cp}^T & \mathbf{M}_S \end{bmatrix} \tag{36}$$

where $$\begin{aligned} M_{L/S} &= \sum_{j=1}^{S} \int_0^{l_j} (\mathbf{J}_L + \mathbf{H}_j)^T (\mathbf{J}_L + \mathbf{H}_j) \, dm_j \\ &= \sum_{j=1}^{S} \int_0^{l_j} (\mathbf{J}_L^T \mathbf{J}_L + 2\mathbf{J}_L^T \mathbf{H}_j + \mathbf{H}_j^T \mathbf{H}_j) \, dm_j \\ &= \mathbf{J}_L^T \mathbf{J}_L \sum_{j=1}^{S} m_j + 2\mathbf{J}_L^T \sum_{j=1}^{S} \int_0^{l_j} \mathbf{H}_j^T dm_j + \sum_{j=1}^{S} \int_0^{l_j} \mathbf{H}_j^T \mathbf{H}_j \, dm_j \end{aligned} \tag{37}$$

$$\begin{aligned} M_{cp} &= \sum_{j=1}^{S} \int_0^{l_j} (\mathbf{J}_L + \mathbf{H}_j)^T \mathbf{R}_L \mathbf{J}_{Sj} dm_j \\ &= \mathbf{J}_L^T \sum_{j=1}^{S} \int_0^{l_j} \mathbf{R}_L \mathbf{J}_{Sj} dm_j + \sum_{j=1}^{S} \int_0^{l_j} \mathbf{H}_j^T \mathbf{R}_j \mathbf{J}_{Sj} dm_j \end{aligned} \tag{38}$$

$\mathbf{M}_L$ and $\mathbf{M}_S$ are previously known as independent inertia matrices of the two arms. We only need to compute only terms $\mathbf{M}_{L/S}$ and $\mathbf{M}_{cp}$ defined in the Equations (37) and (38).

The newly defined terms $\mathbf{M}_{L/S}$ and $\mathbf{M}_{cp}$ have a physical meaning when two arms couple together. To investigate the effect of each term, first, immobilize the small arm, i.e., $\ddot{q}_s = 0$ and $\dot{q}_s = 0$. $\mathbf{M}_{L/S}$ represents the moment of inertia of the small arm with respect to the large arm joint coordinate. This is an inertia load that the large arm has to carry due to the attached small arm. Second, immobilize the large arm, i.e., let $\ddot{q}_L = 0$ and $\dot{q}_L = \dot{q}_S = 0$, $\mathbf{M}_{cp}$ accounts for the coupling forces of the small arm motion upon the large arm.$_{cp}$ These are interacting forces between the large and small arm due to any motion of two systems.

The nonlinear terms $\mathbf{C}_{L+S}$ also can be computed from this inertia matrix $\mathbf{M}_{L+S}$. Its k,j-th element can be obtained from equation (36).

$$\begin{aligned} (C_{L+S})_{kj} &= \sum_{i=1}^{N+S} C_{ijk} \dot{q}_i \\ &= \sum_{i=1}^{N} C_{ijk} \dot{q}_i + \sum_{i=N+1}^{N+S} C_{ijk} \dot{q}_i \end{aligned} \tag{39}$$

where C_{ijk} is known as the Christoffel symbol and defined as

$$C_{ijk} = \frac{\partial M_{L+S}^{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial M_{L+S}^{ij}}{\partial q_k} \tag{40}$$

Therefore, if we substitute the equation (36) into equation (39) and (40), and rewrite the nonlinear term in a matrix form, it will be

$$\mathbf{C}_{L+S} = \begin{bmatrix} \mathbf{C}_L + \mathbf{C}_{L/S} + \mathbf{C}_L^{cp} & \mathbf{C}_{cp1} \\ \mathbf{C}_{cp2} & \mathbf{C}_S + \mathbf{C}_S^{cp} \end{bmatrix} \tag{41}$$

where $$C_{L/S} = \sum_{i=1}^{N} \left(\frac{\partial M_{L/S}^{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial M_{L/S}^{ij}}{\partial q_k} \right) \dot{q}_i \tag{42}$$

$$(C_L^{cp})_{kj} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial M_{cp}^{ij}}{\partial q_k} \dot{q}_i \tag{43}$$

$(k = 1,2,3, \dots, N$ and $j = 1,2,3, \dots, N)$

$$(C_{cp1})_{kj} = \sum_{i=1}^{N+S} \frac{\partial M_{cp}^{kj}}{\partial q_i} \dot{q}_i - \frac{1}{2} \sum_{i=1}^{N} \frac{\partial M_{cp}^{ij}}{\partial q_k} \dot{q}_i \tag{44}$$

$$(C_S^{cp})_{kj} = -\frac{1}{2}\sum_{i=1}^{S}\frac{\partial M_{cp}^{ij}}{\partial q_{N+k}}\dot{q}_i \qquad (45)$$

$(k = 1,2,3, \ldots, N$ and $j = 1,2,3, \ldots, S)$

$$(C_{cp2})_{kj} = \sum_{i=1}^{N+S}\frac{\partial M_{cp}^{jk}}{\partial q_i}\dot{q}_i - \frac{1}{2}\sum_{i=1}^{S}\frac{\partial M_{cp}^{ji}}{\partial q_{N+k}}\dot{q}_{N+i} \qquad (46)$$

$$(C_L^{cp})_{kj} = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial M_{cp}^{ji}}{\partial q_k}\dot{q}_i \qquad (47)$$

$(k = 1,2,3, \ldots, S$ and $j = 1,2,3, \ldots, N)$

Again, If C_L and C_S were known previously, only $C_{L/S} + C_L{}^{cp}, C_{cp1}, C_{cp2}$, and $C_S{}^{cp}$ terms need to be computed to derive the nonlinear part of equation of motion for serially connected two arms. Therefore, the closed form of the dynamic equations of motion for the serially connected large and small manipulators is

$$\begin{bmatrix} \mathbf{M}_L + \mathbf{M}_{L/S} & \mathbf{M}_{cp} \\ \mathbf{M}_{cp}^T & \mathbf{M}_S \end{bmatrix}\begin{pmatrix} \ddot{\mathbf{q}}_L \\ \ddot{\mathbf{q}}_S \end{pmatrix} + \begin{bmatrix} \mathbf{C}_L + \mathbf{C}_{L/S} + \mathbf{C}_L^{cp} & \mathbf{C}_{cp1} \\ \mathbf{C}_{cp2} & \mathbf{C}_S + \mathbf{C}_S^{cp} \end{bmatrix}\begin{pmatrix} \dot{\mathbf{q}}_L \\ \dot{\mathbf{q}}_S \end{pmatrix} \qquad (48)$$
$$+ \begin{bmatrix} \mathbf{K}_L & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \mathbf{q}_L \\ \mathbf{q}_S \end{pmatrix} + \begin{pmatrix} \mathbf{G}_L \\ \mathbf{R}_L\mathbf{G}_S \end{pmatrix} = \begin{bmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{I} \end{bmatrix}\begin{pmatrix} \tau_L \\ \tau_S \end{pmatrix}$$

5. Case Study

To show an advantage of the proposed approach, two case studies are carried out in this section. The first case is a mobile manipulator. The second case is macro/micro manipulators. Dynamics of two coupled systems are derived symbolically using *Mathematica*. The main difference of the two examples is that the base system is rigid in former case but is flexible in later case.

5.1 Mobile Manipulator

It is often envisioned that an autonomous vehicle equipped with manipulators explores hostile unknown environments to collect data or provide the mobility to the robot to handle various tasks. This coupled robotic system, a vehicle with a manipulator, could be the ultimate form of robots in this decade and the next stage of robotic research topic due to its advantages over the base fixed manipulators. However, to control a complex system like a vehicle with a manipulator, it is essential to understand the coupled dynamics between two systems. Therefore, we can determine the stability of the system with the existing controllers. The proposed work can provide the coupling term in relatively simple form and explain where each term is coming from without computing the two systems' dynamics all over again.

For example, a manipulator (3 D.O.F.) is attached to a vehicle (3 D.O.F.) as shown in Figure 2. Its overall closed form equation can be derived by the proposed approach. However, this paper will show only the coupling inertia matrix due to available space. The vehicle itself is approximated as a lumped mass on the plane, and its inertia matrix with respect to the fixed frame can be obtained as:

$$\mathbf{M}_V = \begin{bmatrix} m3 & 0 & 0 \\ 0 & m3 & 0 \\ 0 & 0 & I3z \end{bmatrix}$$

where m3 is mass of the vehicle and I3z is the moment of inertia about z-axis at the mass center. The manipulator has two rigid links and three joints. Assume that we know the manipulator's inertia matrix (3 D.O.F.) by independent computations as

$$\mathbf{M}_A = \begin{bmatrix} M_{A11} & M_{A12} & M_{A13} \\ M_{A12} & M_{A22} & M_{A23} \\ M_{A13} & M_{A23} & M_{A33} \end{bmatrix}$$

where M_{Aij} is assumed to be known. Details can be found in[Lew,92]

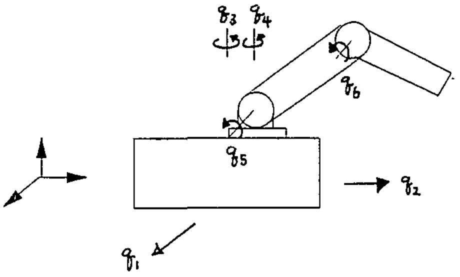

Figure 2 Mobile Manipulator

When two systems are coupled, the coupling inertia between two is form equation (38):

$$\mathbf{M}_{cp} = \begin{bmatrix} M_{cp11} & M_{cp12} & M_{cp13} \\ M_{cp21} & M_{cp22} & M_{cp23} \\ M_{cp31} & M_{cp32} & M_{cp33} \end{bmatrix}$$

where M_{CP11} = -(C5*L5c*m5 + C5*L5*m6 + C56*L6c*m6)*S34
M_{CP12} = -C34*(L5c*m5*S5 + L5*m6*S5 + L6c*m6*S56)
M_{CP13} = -C34*L6c*m6*S56
M_{CP21} = C34*(C5*L5c*m5 + C5*L5*m6 + C56*L6c*m6)
M_{CP22} = -S34*(L5c*m5*S5 + L5*m6*S5 + L6c*m6*S56)
M_{CP23} = -L6c*m6*S34*S56
M_{CP31} = C5^2*I5y + C56^2*I6y + C5*L5c*(C4*L3 + C5*L5c)*m5
+ (C5*L5 + C56*L6c)*(C4*L3 + C5*L5 + C56*L6c)*m6
+ I5x*S5^2 + I6x*S56^2
M_{CP32} = -L3*S4*(L5c*m5*S5 + L5*m6*S5 + L6c*m6*S56)
M_{CP33} = -L3*L6c*m6*S4*S56

C5 is a short form notation of Cos(q[5]). Similarly S56 is a short form for Sin(q[5]+q[6]). Lic is the distance between joint i and link i mass center, and Li is the length of link i.

The inertia of the manipulator that the vehicle carries is from equation (37):

$$\mathbf{M}_{L/S} = \begin{bmatrix} M_{LS11} & M_{LS12} & M_{LS13} \\ M_{LS12} & M_{LS22} & M_{LS23} \\ M_{LS13} & M_{LS23} & M_{LS33} \end{bmatrix}$$

where M_{LS11} = m5 + m6
M_{LS12} = 0
M_{LS13} = -(m5*(L3*S3 + C5*L5c*S34)) - m6*(L3*S3 + C5*L5*S34
+ C56*L6c*S34)
M_{LS22} = m5 + m6
M_{LS23} = C3*L3*m5 + C34*C5*L5c*m5 + C3*L3*m6 +
C34*C5*L5*m6 + C34*C56*L6c*m6
M_{LS33} = C5^2*I5y + C56^2*I6y + (L3^2 + 2*C4*C5*L3*L5c +
C5^2*L5c^2)*m5 + (L3^2 + 2*C4*C5*L3*L5 + C5^2*L5^2
+ 2*C4*C56*L3*L6c + 2*C5*C56*L5*L6c +
C56^2*L6c^2)*m6 + I5x*S5^2 + I6x*S56^2

Now, we can construct the coupled system's inertia matrix from equation (36).

5.2 Macro/Micro Manipulators

The concept of a micro manipulator mounted on the tip of a macro manipulator has been introduced to provide precise motion as well as a large workspace. This configuration comprises a large robot carrying the micro manipulator to the area of the interest and uses the micro manipulator for fine motion control necessary to eliminate positioning error. However, deriving the dynamics of these systems can be a time consuming and painful procedure. The proposed work generates a closed form of the equations of motion for serially connected macro/micro manipulators with separately known dynamics of two manipulators. This approach will reduce the number of computations significantly and show the structure of coupling dynamics between two arms.

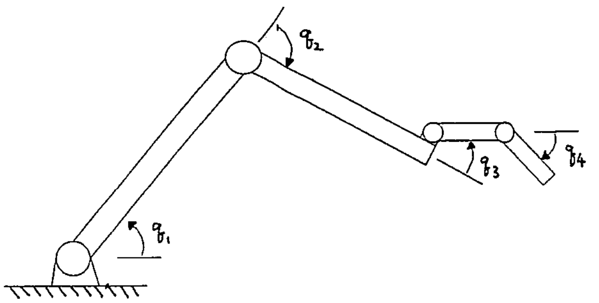

Figure 3 Micro/Macro Manipulators

For example, coupling dynamics can be investigated when SAM (2 D.O.F.) is mounted on the tip of RALF (2 D.O.F.) as shown in Figure 3. Again, this paper will show only the coupling inertia matrix due to the complexity of the equation. Assume the inertia force of RALF to be known ahead with respect to the fixed frame. The inertia matrix of RALF, which has two flexible links with one assumed mode for each link, is

$$\mathbf{M}_L = \begin{bmatrix} M_{L11} & M_{L12} & M_{L13} & M_{L14} \\ M_{L12} & M_{L22} & M_{L23} & M_{L24} \\ M_{L13} & M_{L23} & M_{L33} & M_{L34} \\ M_{L14} & M_{L24} & M_{L34} & M_{L44} \end{bmatrix}$$

where M_{Lij} is known ahead. Details can be found in [Lew,92].

Also, the inertia matrix of SAM, which has two rigid links, is known as

$$\mathbf{M}_S = \begin{bmatrix} M_{S11} & M_{S12} \\ M_{S21} & M_{S22} \end{bmatrix}$$

where Msij is defined in [Lew,92] Then, based on the equation (38) and (37), we can compute the coupling dynamics. The coupling matrices are

$$\mathbf{M}_{CP} = \begin{bmatrix} M_{CP11} & M_{CP12} \\ M_{CP21} & M_{CP22} \\ M_{CP31} & M_{CP32} \\ M_{CP41} & M_{CP42} \end{bmatrix}$$

where M_{CP11} = d3 + d4 + C25*m4*l1*l3 + C5*m4*l2*l3 + m4*l3^2 + C56*m4*l2*l4c + 2*C6*m4*l3*l4c + m4*S25*l3*qf1*she[1] + m3*S25*l3c*qf1*she[1] +m4*S256*l4c*qf1*she[1] + m4*S5*l3*qf2*she[2] + m3*S5*l3c*qf2*she[2] + m4*S56*l4c*qf2*she[2]

M_{CP12} = d4 + C6*m4*l3*l4c + m4*l4c*(C256*l1 + C56*l2 + S256*qf1*she[1] + S56*qf2*she[2])

M_{CP21} = d3 + d4 + C5*m4*l2*l3 + m4*l3^2 + C5*m3*l2*l3c + C56*m4*l2*l4c + 2*C6*m4*l3*l4c + m4*S5*l3*qf2*she[2] + m3*S5*l3c*qf2*she[2] + m4*S56*l4c*qf2*she[2]

M_{CP22} = d4 + C6*m4*l3*l4c + m4*l4c*(C56*l2 + S56*qf2*she[2])

M_{CP31} = (C25*m4*l3 + C25*m3*l3c + C256*m4*l4c)*she[1]

M_{CP32} = C256*m4*l4c*she[1]

M_{CP41} = (C5*m4*l3 + C5*m3*l3c + C56*m4*l4c)*she[2]

M_{CP42} = C56*m4*l4c*she[2]

where she[i] is a mode shape function of link evaluated at the end point. i.e., $she[i] = \psi_i(l_i)$.

The inertia of SAM that RALF has to carries is from equation (37).

$$\mathbf{M}_{L/S} = \begin{bmatrix} M_{LS11} & M_{LS12} & M_{LS13} & M_{LS14} \\ M_{LS12} & M_{LS22} & M_{LS23} & M_{LS24} \\ M_{LS13} & M_{LS23} & M_{LS33} & M_{LS34} \\ M_{LS14} & M_{LS24} & M_{LS34} & M_{LS44} \end{bmatrix}$$

where M_{LS11} = d3 + d4 + m4*l3^2 + 2*C6*m4*l3*l4c + (m3 + m4)*(l1^2 + 2*C2*l1*l2 + l2^2 + 2*S2*l2*qf1*she[1] + qf1^2*she[1]^2 - 2*S2*l1*qf2*she[2] + 2*C2*qf1*qf2*she[1]*she[2] + qf2^2*she[2]^2) + 2*(C25*m4*l1*l3 + m4*C5*l2*l3 + C25*m3*l1*l3c + m3*C5*l2*l3c + C256*m4*l1*l4c + C56*m4*l2*l4c + m4*S5*l3*qf2*she[2] + m3*S5*l3c*qf2*she[2] + m4*S56*l4c*qf2*she[2] + m4*l3*qf1*she[1]*S25 + m3*l3c*qf1*she[1]*S25 + m4*l4c*qf1*she[1]*S256)

M_{LS12} = d3 + d4 + m4*l3^2 + 2*C6*m4*l3*l4c + (m3 + m4)*(C2*l1*l2 + l2^2 + S2*l2*qf1*she[1] -S2*l1*qf2*she[2] + C2*qf1*qf2*she[1]*she[2] + qf2^2*she[2]^2)+ 2*(C25*m4*l1*l3 + m4*C5*l2*l3 + C25*m3*l1*l3c + m3*C5*l2*l3c + C256*m4*l1*l4c + C56*m4*l2*l4c + m4*S5*l3*qf2*she[2] + m3*S5*l3c*qf2*she[2] + m4*S56*l4c*qf2*she[2] + m4*l3*qf1*she[1]*S25 + m3*l3c*qf1*she[1]*S25 + m4*l4c*qf1*she[1]*S256)

M_{LS13} = (m3 + m4)*she[1]*(l1 + C2*l2 - S2*qf2*she[2])

M_{LS14} = (m3 + m4)*(C2*l1 + l2 + S2*qf1*she[1])*she[2]

M_{LS22} = d3 + d4 + m4*l3^2 + 2*C6*m4*l3*l4c + 2*(m4*C5*l2*l3 + m3*C5*l2*l3c + C56*m4*l2*l4c + m4*S5*l3*qf2*she[2] + m3*S5*l3c*qf2*she[2] + m4*S56*l4c*qf2*she[2]) + (m3 + m4)*(l2^2 + qf2^2*she[2]^2)

M_{LS23} = (m3 + m4)*she[1]*(C2*l2 - S2*qf2*she[2])

M_{LS24} = (m3 + m4)*l2*she[2]

M_{LS33} = (m3 + m4)*she[1]^2

M_{LS34} = C2*(m3 + m4)*she[1]*she[2]

M_{LS44} = (m3 + m4)*she[2]^2

Therefore, we can construct a closed form of the inertia matrix for two serially connected arms using equation (36).

5.3 Discussion on Computation Time

There is no doubt that the proposed work handles a lesser number of terms when it derives the dynamic equations. The efficiency of the proposed work can be compared with conventional direct derivation by examining the numbers of multiplication and additions. However, it is difficult to come up with a general case because the simplification procedure of kinematics is obscure for each case. Thus, computation time is used to give the idea of how the proposed work computes efficiently. Symbolic computation of two examples carried out on PC 486-33 MHz with Mathematica 2.0. For a vehicle with a manipulator case, the direct derivation took about 4 minutes, and the proposed method did in 2.5 minutes. For micro/macro manipulator case, the direct derivation took about 8 minutes, and the proposed method finished in 5 minutes. Each case shows a reduction in computation time of at least one-third compared to the conventional direct derivation. However, the authors would like to remind the readers that the simplification procedure such as 'Simplify[] ' determines the majority of the computation time rather than the multiplication and addition process. It is very important that proper usage of the simplification procedure is applied in the efficient symbolic derivation.

6. Conclusions

The proposed work generates a closed form of the equation of motion for two serially connected manipulators. Instead of computing the coupled dynamics directly, this work constructs overall dynamic equations from the previously known equations of each manipulator and coupling terms derived in this paper. This approach will reduce the number of computations significantly and show the structure of coupling dynamics between two arms. The proposed technique has been coded in *Mathematica* for symbolic computation. As a case study, the proposed approach is applied to two examples; a two link flexible arm with a two link rigid arm, and a moving vehicle with a three link arm. Each case shows not only simplicity of derivation but also a reduction in computation time of at least one-third compared to the conventional derivation.

References

(List is available upon request)