RIBBON GRAPHS AND MIRROR SYMMETRY

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Abstract. Given a ribbon graph Γ with some extra structure, we define, using constructible sheaves, a dg category $\mathrm{CPM}(\Gamma)$ meant to model the Fukaya category of a Riemann surface in the cell of Teichmüller space described by Γ. When Γ is appropriately decorated and admits a combinatorial "torus fibration with section," we construct from Γ a one-dimensional algebraic stack $\widetilde{X}_\Gamma$ with toric components. We prove that our model is equivalent to $\mathcal{P}\mathrm{erf}(\widetilde{X}_\Gamma)$, the dg category of perfect complexes on $\widetilde{X}_\Gamma$.

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1. Introduction

1.1. **Ribbon Graphs and HMS in One Dimension.** Recall from the work of Harer, Mumford, Penner and Thurston (see, e.g., [H], [P]) that ribbon graphs label cells in a decomposition of the moduli space of punctured Riemann surfaces. The graph itself is a retraction of the surface, also known as a *skeleton* or *spine.* Let X_Γ denote a punctured Riemann surface with spine Γ. We imagine a Stein structure on X_Γ so that Γ is the skeleton (the union of all stable manifolds) of the Stein function. Kontsevich in [K] conjectured that the Fukaya category of a Stein manifold can be computed locally on the skeleton and discussed applications to homological mirror symmetry for Riemann surfaces.[1] In this paper, we investigate this idea from the perspective of constructible sheaves and T-duality.

We shall define, starting from a ribbon graph Γ, a category $\mathrm{CPM}(\Gamma)$ ("constructible plumbing model"), defined using the language of constructible sheaves. $\mathrm{CPM}(\Gamma)$ serves as a stand-in for the Fukaya category $Fuk(X_\Gamma)$ of the surface X_Γ.

When the ribbon graph is appropriately decorated and carries a combinatorial version of a torus fibration with section, then we can define a "mirror" curve $\widetilde{X}_\Gamma$, an algebraic stack with toric components. Exploiting a Beilinson-Bondal-type equivalence called the coherent-constructible correspondence [B, FLTZ], we prove

$$\mathrm{CPM}(\Gamma) \cong \mathcal{P}\mathrm{erf}(\widetilde{X}_\Gamma),$$

an equivalence of dg categories. We conjecture that $\mathrm{CPM}(\Gamma) \cong Fuk(X_\Gamma)$. Together with the above result, this would prove a one-dimensional version of homological mirror symmetry.

[1] Locality in the skeleton appeared first in Kontsevich's [K], but is also part of a circle of ideas concerning the local nature of the Fukaya category of exact symplectic manifolds, prevalent the work of Abouzaid [A1] and Seidel [S2, S3] (see also [AS]), and in the relation of the Fukaya category to sheaf theory by Nadler [N1] and the last author [NZ].

1.2. **Basic Idea; Simple Example.** Recall from [NZ, N1] that the Fukaya category of a cotangent bundle $X = T^*Y$ is equivalent to constructible sheaves on the base, Y. One can specialize to sheaves that are constructible for a particular stratification $\mathcal{S}$ by only considering a subcategory of the Fukaya category generated by Lagrangians which are asymptotic to a conical Lagrangian submanifold $\Lambda_{\mathcal{S}} \subset T^*Y$ depending on $\mathcal{S}$. The conical Lagrangian Λ, encodes the "microlocal behavior" of the corresponding sheaf. When the conical Lagrangian submanifold Λ is strictly $\mathbb{R}_+$-invariant and not necessarily $\mathbb{R}$-invariant in the fibers, the corresponding subcategory of sheaves has "singular support" in Λ, a somewhat more refined microlocal condition than that defined by a stratification. For example, when $Y = S^1$, Λ will be a subset of the zero section together with a finite number of rays of cotangent fibers – i.e., a graph with valency ≤ 4 at each vertex. For instance, the category of sheaves on S^1 stratified by a point and its complement corresponds to Λ which is the union of the zero section and the cotangent fiber of the point – a graph with one 4-valent vertex and one loop. The graph is a ribbon graph, since the symplectic geometry of the cotangent of S^1 determines a cyclic ordering of the edges. It was proven in [B, FLTZ] that constructible sheaves on S^1 with this stratification are equivalent to coherent sheaves on $\mathbb{P}^1$. Since this category of constructible sheaves is equivalent to a Fukaya category on T^*S^1, by [NZ, N1], this equivalence is a form of mirror symmetry.

The model CPM is constructed by gluing pieces of a graph together, and can be inferred from the following example. Consider a family of conics C_t in $\mathbb{P}^2$ defined in homogeneous coordinates by $XY = t^2Z^2$, or in inhomogeneous coordinates by $xy = t^2$. These conics are all isomorphic to $\mathbb{P}^1$ when $t \neq 0$ but degenerate to two copies of $\mathbb{P}^1$ attached at a node when $t = 0$. For $t \neq 0$ the conics all have an open orbit C_t° of a *common* $\mathbb{C}^* \supset S^1$ defined by $\lambda : (x, y) \mapsto (\lambda x, \lambda^{-1}y)$, i.e. C_t° has an S^1 fibration over $\mathbb{R} = \mathbb{C}^*/S^1$. We can pick a $(\mathbb{C}^*)^2$-equivariant line bundle on $\mathbb{P}^2$ such as the hyperplane bundle, give it a metric (and thus compatible connection) invariant under $(S^1)^2 \supset S^1$, restrict it to C_t° and compute the monodromies over the $\mathbb{R}$ family of S^1 orbits. The resulting spectral curve, the "T-dual Lagrangian" L_t, lies in T^*S^1 (the dual S^1, actually), for $t \neq 0$. We study the degeneration $t \to 0$.

When $t \to 0$, L_t "splits" into two Lagrangians on two different cotangents of two copies of a circle stratified by a point p and its complement – but they are joined: $+\infty$ inside $T_p^*S^1$ from the perspective of one circle is glued to $-\infty$ of the other circle (just as the corresponding limits of $\mathbb{C}^*$ orbits on the two $\mathbb{P}^1$ components are joined). In terms of coordinates $(x, y \sim y+1)$ on T^*S^1, the L_t look like $y = t^2 \sinh(2x)/(1 + t^2 \cosh(2x))$. This splitting is illustrated for two values of t.

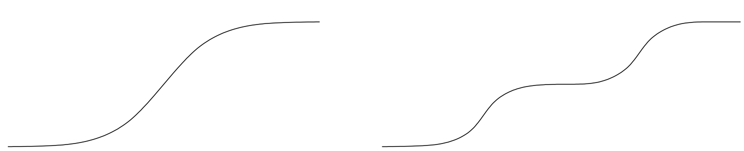

FIGURE 1. The picture represents the shape of L_t for t large, on the left, and for t small, on the right.

The picture suggests that coherent sheaves (perfect complexes, actually, [FLTZ]) on the degenerate conic can be described by pairs of constructible sheaves on S^1 such that the microlocal stalk of one sheaf along the ray to $+\infty$ in $T^*_p S^1$ equals the microlocal stalk of the other sheaf along the ray to $-\infty$. The graph Λ for such a gluing has two 4-valent vertices attached by a single edge, and two loops – a bit like a curtain rod with two rings. Extending this example, we can consider a further gluing of the toric endpoints of the two $\mathbb{P}^1$ components. It becomes clear what to do to build a constructible model for this algebraic curve: glue the curtain rod along its two ends. The graph is a circle with two circles attached at two distinct points – a combinatorial version of a torus fibration!

The algebraic curve obtained in this way is a degenerate elliptic curve – a Calabi-Yau manifold with toric components, such as what appears at the large complex structure limit point in mirror symmetry. The ribbon graph category is a model for the Fukaya category of the exact symplectic manifold at the large radius limit (the symplectic manifold gets compactified in deformations [S1] which are mirror to smoothing the elliptic curve) corresponding to the ribbon graph, and is equivalent to coherent sheaves on the algebraic mirror Calabi-Yau. Through this reasoning, then, we get a model for homological mirror symmetry of degenerate Calabi-Yau manifolds.

To summarize:

- Each four-valent vertex is modeled by the category $Sh(\mathbb{R}, \cdot)$ of constructible sheaves on $\mathbb{R}$ stratified by a point and its complement.
- Two opposing half-edges of a four-valent vertex represent the two $\mathbb{R}$ directions, while the remaining two represent the *microlocal stalks* of the constructible sheaves. More succinctly, $Sh(\mathbb{R}, \cdot)$ has four functors to chain complexes (constructible sheaves on edge intervals) defined by the stalks or microlocal stalks at the rays corresponding to the four half-edges.
- Categories are glued as "fiber products," i.e. by taking objects at each vertex category and requiring that the images of objects at two vertices agree along an edge which joins them.
- When the graph has a map to a cycle graph, with circle fibers over the vertices, we can join together the toric curves corresponding to the fibers to make a degenerate Calabi-Yau. Coherent sheaves on this Calabi-Yau will be isomorphic to the constructible plumbing model of the ribbon graph.

The discussion thus far has been quite informal. The paper is dedicated to making these observations rigorous. In Section 2, we introduce the notions needed to take fiber products and more general limits of dg categories, and review material on algebraic stacks and ribbon graphs. In Section 3, we define *chordal* ribbon graphs, which encode the structure of being "locally like $T^*\mathbb{R}$" used to relate to constructible sheaves. We review the microlocal theory of sheaves in one dimension and define $\mathrm{CPM}(\Gamma)$ from a chordal ribbon graph Γ by gluing, as discussed above. In Section 4, we define the notion of "dualizable" ribbon graph, Γ – the structure called a "combinatorial version of a torus fibration" above – which allows us to construct an algebraic curve $\widetilde{X}_\Gamma$ with toric components such that $\mathcal{P}\mathrm{erf}(\widetilde{X}_\Gamma) \cong \mathrm{CPM}(\Gamma)$.

1.3. **Influences.** Our construction is influenced by the work of many others, and we are grateful for their inspiring works. We highlight a few.

- *Abouzaid* has a plumbing construction [A1] which is equivalent to our own, in certain circumstances, and has several other parallel constructions, e.g. [A2] (see also [AS]).
- *Beilinson-Bondal* describe combinatorial models for coherent sheaves on toric varieties. Bondal's reinterpretation [B] of Beilinson's quiver in terms of constructible sheaves was an inspiration behind the coherent-constructible correspondence defined in [FLTZ].
- *Kontsevich* conjectured [K] that the Fukaya category is local in the skeleton. The rumor of this announcement inspired our search for a model using the microlocal structure of sheaf theory.
- *Nadler* has pushed the envelope in relating Fukaya categories to algebraic topology [N1] (we note also [N3, NT]).
- *Seidel* initiated the study of exact symplectic manifolds as mirrors of Calabi-Yau manifolds at their toric degeneration point (large complex structure limit), especially in his *tour de force* work [S1]. Seidel discusses the locality of the Fukaya category in [S2] and (among other things) applies these ideas to Landau-Ginzburg theories in [S3].

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2. Notation and conventions

2.1. **Background on categories and sheaves.** In this section we review material on category theory and constructible sheaves.

As we do not make any direct comparisons to the "genuine" Fukaya category, we do not work with A_∞-categories; instead, we work with dg categories. Since the plumbing perspective we take in this paper requires us to build new dg categories out of old, we have to have a good handle on "what dg categories form." The answer is contained in the work of many people, including Drinfel'd, Tabuada, Toën, and Lurie. We may summarize by saying: dg categories form a Quillen model category, which we may regard as a "∞-category" or "quasi-category" via a simplicial nerve construction. In section 2.1.1 we recall what we will need from this theory.

We work over the ground field $\mathbb{C}$.

2.1.1. *∞-and dg categories.* We refer to [Lu1] for the theory of ∞-categories, and use the same notation. An ∞-category is a simplicial set satisfying the weak Kan condition ([Lu1, Definition 1.1.2.4]). Common sources of ∞-categories are:

(1) If $\mathcal{C}$ is an ordinary category, then the nerve $N\mathcal{C}$ is an ∞-category. ([Lu1, Proposition 1.1.2.2]).
(2) If $\mathcal{C}$ is a simplicial category, i.e. a category enriched in simplicial sets, with the property that the simplicial set of maps between any two objects of $\mathcal{C}$ is a Kan complex, then the simplicial nerve $N\mathcal{C}$ is an ∞-category. ([Lu1, Proposition 1.1.5.10]).

(3) In particular if $\mathcal{C}$ is a simplicial model category and $\mathcal{C}_{cf}$ is the full subcategory spanned by objects that are both fibrant and cofibrant, then $N\mathcal{C}_{cf}$ is an ∞-category, sometimes called the "underlying ∞-category" of $\mathcal{C}$.

(4) If K and $\mathcal{C}$ are simplicial sets we write $\mathrm{Fun}(K, \mathcal{C})$ for the simplicial set given by $\Delta^n \mapsto \mathrm{Hom}(K \times \Delta^n, \mathcal{C})$. If $\mathcal{C}$ is an ∞-category then so is $\mathrm{Fun}(K, \mathcal{C})$ ([Lu1, Proposition 1.2.7.3]).

The category of $\mathbb{C}$-linear dg categories has a Quillen model structure introduced by Tabuada [Tab] and studied by Toën [To]. It has a natural simplicial enrichment [To, Section 5]. We write $\mathrm{dg}\mathcal{C}\mathrm{at}$ for the underlying ∞-category of this simplicial model category. We write $\mathrm{dg}\mathcal{C}\mathrm{at}^{\mathrm{S}} \subset \mathrm{dg}\mathcal{C}\mathrm{at}$ for the full subcategory (that is, sub-∞-category) whose objects are triangulated dg categories.

Remark 2.1. We may realize $\mathrm{dg}\mathcal{C}\mathrm{at}^{\mathrm{S}}$ as a subcategory of the ∞-category SCat_∞ of *stable ∞-categories* [Lu2], and as a full subcategory of the ∞-category $\mathrm{SCat}_\infty^{\mathbb{C}}$ of $\mathbb{C}$-linear stable ∞-categories. The latter category is defined as a full subcategory of the category of module categories for the symmetric monoidal ∞-category $\mathbb{C}$-mod [Lu3].

We have a notion of limits and colimits in ∞-categories, that behave very much like the classical 1-categorical notions [Lu1, Chapter 4]. As $\mathrm{dg}\mathcal{C}\mathrm{at}$ is the underlying ∞-category of a simplicial model category, it admits small limits and colimits [Lu1, Corollary 4.2.4.8]. Let us pay particular attention to the notion of an equalizer in $\mathrm{dg}\mathcal{C}\mathrm{at}$, for which we give below an explicit construction.

Proposition 2.2. *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$ *be a diagram in* $\mathrm{dg}\mathcal{C}\mathrm{at}$*, and let* $\mathcal{E}$ *be the dg category defined as follows:*

- *the objects of* $\mathcal{E}$ *are pairs* (C, u)*, where* $C \in \mathcal{C}$*, and* u *is a degree zero, closed morphism* $u : F(C) \cong G(C)$*, which becomes invertible in the homotopy category,*
- *the morphisms of* $\mathcal{E}$ *are pairs* $(f, H) \in \mathrm{Hom}^k(C, C') \oplus \mathrm{Hom}^{k-1}(F(C), G(C'))$*, with differential given by* $d(f, H) = (df, dH - (u'F(f) - G(f)u))$*, and componentwise composition.*

Then $\mathcal{E}$*, endowed with the natural forgetful functor* $\mathcal{E} \to \mathcal{C}$*, is a homotopy equalizer for* F *and* G*.*

Proof. Proposition 2.2 depends on the availability of an explicit model for the *path object* $P(\mathcal{D})$ of $\mathcal{D}$, which can be found in Lemma 4.1 of [Tab1]. Recall that $P(\mathcal{D})$ comes equipped with a "diagonal" functor $\Delta : P(\mathcal{D}) \to \mathcal{D} \times \mathcal{D}$, which is a fibration for Tabuada's model structure. The homotopy equalizer of F and G can be computed as the *classical* pullback[2] of the diagram

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & & P(\mathcal{D}) \\ & \searrow^{F \times G} \quad \swarrow_{\Delta} & \\ & \mathcal{D} \times \mathcal{D} & \end{array}$$

The resulting dg category is equal to $\mathcal{E}$. $\square$

[2]That is, in the ordinary category of small $\mathbb{C}$-linear categories.

2.1.2. *Grothendieck topologies and sheaves of dg categories.* We shall describe Grothendieck topologies using the language of sieves. Let $\mathcal{X}$ be a small category, and let $X \in \mathcal{X}$ be an object. A *sieve* on X is a collection of arrows $Y \to X$ with the property that if $Y \to X$ belongs to the sieve and there exists a morphism $Y_1 \to Y$ then the composite $Y_1 \to Y \to X$ also belongs to the sieve.

Example 2.3. If $\mathcal{X}$ is the partially ordered set of open subsets of a topological space X, and $\{U_i\}$ is an open cover of $U \subset X$, then the collection of all open inclusions $V \hookrightarrow U$ that factor through one of the U_i is a sieve on U.

We may regard a sieve as a full subcategory $\mathcal{U} \subset \mathcal{X}_{/X}$ of the comma category $\mathcal{X}_{/X}$. Given a sieve $\mathcal{U} \subset \mathcal{X}_{/X}$, and a morphism $f : Y \to X$, we define a sieve $f^*\mathcal{U}$ on Y by putting $U \to Y$ in $f^*\mathcal{U}$ if the composite $U \to Y \to X$ belongs to $\mathcal{U}$.

Definition 2.4. A *Grothendieck topology* on $\mathcal{X}$ consists of, for each object X of $\mathcal{X}$, a collection of sieves on X called "covering sieves," subject to the following conditions:

- If X is an object of $\mathcal{X}$, then the trivial subcategory $\mathcal{X}_{/X} \subset \mathcal{X}_{/X}$ is a covering sieve.
- If $f : Y \to X$ is a morphism and $\mathcal{U}$ is a covering sieve on X, then $f^*\mathcal{U}$ is a covering sieve on Y.
- Let X be an object of $\mathcal{X}$, let $\mathcal{U}$ be a covering sieve on X, and let $\mathcal{V}$ be an arbitrary sieve on X. Suppose that, for each $f : Y \to X$ belonging to $\mathcal{U}$, the pullback $f^*\mathcal{V}$ is a covering sieve on Y. Then $\mathcal{V}$ is a covering sieve on X.

In classical category theory, there is an order-preserving bijection between sieves on X and subobjects of the functor $\mathrm{Hom}(-, X)$ from $\mathcal{X}^{\mathrm{op}}$ to the category of sets: for all objects Y of $\mathcal{X}$ the subset of $\mathrm{Hom}(Y, X)$ corresponding to a sieve U is the set of all morphisms $Y \to X$ belonging to $\mathcal{U}$. This construction extends to the ∞-categorical setting [Lu1, Proposition 6.2.2.5]: in particular, if $j : \mathcal{X} \to \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S})$ denotes the ∞-categorical Yoneda embedding, then for each sieve $\mathcal{U}$ on X we may associate an object $j(\mathcal{U}) \in \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S})$ and a morphism $j(\mathcal{U}) \to j(X)$.

Definition 2.5. If $\mathcal{X}$ is endowed with a Grothendieck topology, then a functor $F : \mathcal{X}^{\mathrm{op}} \to \mathcal{S}$ is a *sheaf* if $\mathrm{Hom}(j(X), F) \to \mathrm{Hom}(j(\mathcal{U}), F)$ is a weak homotopy equivalence for every covering sieve $\mathcal{U} \subset \mathcal{X}_{/X}$.

Remark 2.6. The ∞-categorical Yoneda lemma [Lu1, Prop 5.1.3.1] identifies $\mathrm{Hom}(j(X), F)$ with $F(X)$. We may furthermore identify $\mathrm{Hom}(j(\mathcal{U}), F)$ with the inverse limit $\varprojlim_{(U \to X) \in \mathcal{U}} F(U)$. The sheaf condition is equivalent to the statement that the natural map

$$F(X) \to \varprojlim_{(U \to X) \in \mathcal{U}} F(U)$$

is an equivalence for every covering sieve $\mathcal{U}$—i.e. that "$F(X)$ can be computed locally."

Write $\mathrm{Shv}(\mathcal{X}) \subset \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S})$ for the full subcategory of sheaves. Let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-valued sheaf* on $\mathcal{X}$ is a functor $\mathrm{Shv}(\mathcal{X})^{\mathrm{op}} \to \mathcal{C}$ that converts small colimits in $\mathrm{Shv}(\mathcal{X})^{\mathrm{op}}$ to limits in $\mathcal{C}$. If $\mathcal{C}$ has small limits then this is equivalent to a contravariant functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{C}$

with the property that the natural map

$$F(X) \to \varprojlim_{(U \to X) \in \mathcal{U}} F(U)$$

is an equivalence whenever $\mathcal{U}$ is a covering sieve. Write $\mathrm{Shv}(\mathcal{X}, \mathcal{C})$ for the ∞-category of $\mathcal{C}$-valued sheaves on $\mathcal{X}$.

Remark 2.7. The construction $\mathrm{Shv}(\mathcal{X}, \mathcal{C})$ provides a natural ∞-categorical version of some standard constructions in sheaf theory. Let X be a topological space and let $\mathcal{X}$ be the Grothendieck site of open subsets of X. We will write $\mathrm{Shv}(X, \mathcal{C})$ instead of $\mathrm{Shv}(\mathcal{X}, \mathcal{C})$.

(1) Let $\mathbb{C}$-mod be the ∞-category associated to the Quillen model category of cochain complexes of complex vector spaces. Then $\mathrm{Shv}(X, \mathbb{C}\text{-mod})$ is an ∞-category whose homotopy category is naturally identified with $D(X; \mathbb{C})$, the unbounded derived category of sheaves of vector spaces on X.
(2) $\mathrm{Shv}(X, \mathbb{C}\text{-mod})$ is a $\mathbb{C}$-linear stable ∞-category. By remark 2.1, we may therefore identify it with a triangulated dg category via the equivalence $\mathrm{SCat}_{\infty}^{\mathbb{C}} \cong \mathrm{dg}\mathcal{C}\mathrm{at}$. The assignment $U \mapsto \mathrm{Shv}(U, \mathbb{C}\text{-mod})$, where U runs through open subsets of X, together with the restriction functors associated to inclusions $V \subset U$, assembles to an object of $\mathrm{Shv}(X, \mathrm{dg}\mathcal{C}\mathrm{at})$—the "sheaf of sheaves" on X.

A sheaf on a Grothendieck site is determined by its behavior on a "basis":

Proposition 2.8. *Let $\mathcal{X}$ be a Grothendieck site and let $\mathcal{B} \subset \mathcal{X}$ be a full subcategory with the property that each object of $\mathcal{X}$ admits a covering sieve $\{U_i \to X\}$ such that each U_i is in $\mathcal{B}$. Then the restriction functor $\mathrm{Shv}(\mathcal{X}, \mathcal{C}) \to \mathrm{Shv}(\mathcal{B}, \mathcal{C})$ is an equivalence of ∞-categories.*

2.2. **Background on algebraic stacks.** We will consider Deligne-Mumford stacks (that is, stacks with finite isotropy groups) defined over the complex numbers.

Definition 2.9. Let Gpd denote the full subcategory of the ∞-category of spaces spanned by spaces that can be obtained as the nerve of a 1-groupoid. Let $\mathrm{Sch}_{/\mathbb{C}}$ denote the category of complex algebraic schemes. A *stack* is a functor $X : \mathrm{Sch}_{/\mathbb{C}}^{\mathrm{op}} \to \mathrm{Gpd}$ that is a sheaf in the étale topology on $\mathrm{Sch}_{/\mathbb{C}}$.

We may regard the functor represented by an object S of $\mathrm{Sch}_{/\mathbb{C}}$ as valued in Gpd, by regarding each $\mathrm{Hom}(T, S)$ as a discrete groupoid, in which case it becomes a stack. We will abuse notation and denote this representable stack by S. We say that a morphism of stacks $X \to Y$ is *representable* if for each map $S \to Y$ with S representable, the fiber product $X \times_Y S$ is also representable. If P is a property of morphisms between schemes, then we will say that a representable morphism $X \to Y$ has property P if all base-changed maps $X \times_Y S \to S$ with S representable have property P.

We restrict our attention to Deligne-Mumford stacks:

Definition 2.10. A stack $X : \mathrm{Sch}_{/\mathbb{C}}^{\mathrm{op}} \to \mathrm{Gpd}$ is *Deligne-Mumford* if it satisfies the following conditions:

(1) There exists a representable étale morphism $U \to X$ where U is itself representable.
(2) The diagonal map $X \to X \times X$ is representable and finite.

We let $\mathrm{St}_{/\mathbb{C}}$ denote the full subcategory of the ∞-category of functors $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Gpd}$ spanned by Deligne-Mumford stacks.

Remark 2.11. The ∞-category $\mathrm{St}_{/\mathbb{C}}$ can be obtained from the usual 2-category of Deligne-Mumford stacks via a 2-categorical nerve construction.

A complex of quasicoherent sheaves on a scheme X is *perfect* if it is locally quasi-isomorphic to a bounded complex of vector bundles. Perfect complexes are preserved by pullback. We have descent for perfect complexes:

Proposition 2.12. *There is a functor* $\mathcal{P}\mathrm{erf} : \mathrm{Sch}^{\mathrm{op}}_{/\mathbb{C}} \to \mathrm{dg}\mathcal{C}\mathrm{at}^{\mathrm{S}}$ *whose value on a scheme* X *is the triangulated dg category of perfect complexes on* X*, and whose value on a morphism* $f : X \to Y$ *is the derived pullback functor* f^**. This functor is a sheaf in the étale topology on* $\mathrm{Sch}_{/\mathbb{C}}$*.*

The étale topology on $\mathrm{Sch}_{/\mathbb{C}}$ forms a basis for a topology on $\mathrm{St}_{/\mathbb{C}}$ which we also call the étale topology. Proposition 2.8 says that the assignment $X \mapsto \mathcal{P}\mathrm{erf}(X)$ makes sense for stacks as well as schemes. When X is a stack, objects of $\mathcal{P}\mathrm{erf}(X)$ will be called "perfect complexes" on X.

2.3. **Background on graphs and ribbon graphs.**

2.3.1. *Graphs.*

Definition 2.13.
1. A *topological graph* is a tuple (X, V_X) where X is a locally compact topological space, $V_X \subset X$ is a finite closed subset, the open set $X - V_X$ is homeomorphic to a finite disjoint union of open intervals, and each $v \in V_X$ has a neighborhood homeomorphic to the cone on a finite set. We call the components of $X - V_X$ *edges* of the topological graph. An edge is *compact* if its closure is compact, and *noncompact* otherwise.
2. A *graph* is a topological graph together with an open embedding $x_e : e \hookrightarrow \mathbb{R}$ for each edge e of X. We require that the image of each x_e is bounded.

An edge is a *loop* if it is compact and its closure contains only one vertex, or equivalently if its closure is homeomorphic to a circle. For simplicity we will assume from now on that our graphs have no loops, but we will allow noncompact and "multiple" edges—that is, we allow more than one edge to be incident with the same pair of vertices.

Definition 2.14. Let (X, V_X) be a graph. A *half-edge* of X incident with a vertex $v \in V_X$ is the germ of a connected component of a deleted neighborhood of v in X. The *degree* of a vertex is the number of half-edges incident with v, and a graph is called *locally finite* if each vertex has finite degree.

Definition 2.15. Let (X, V_X) and (Y, V_Y) be topological graphs. A *morphism* $(X, V_X) \to (Y, V_Y)$ is a continuous map $u : X \to Y$ that carries vertices to vertices, that collapses some edges to vertices, and that otherwise maps each remaining edge homeomorphically onto an edge of Y. That is, u satisfies

- $u(V_X) \subset V_Y$
- $u^{-1}(V_Y)$ is a union of vertices and edges of X

- u restricted to any edge in $X - u^{-1}(V_Y)$ is a homeomorphism onto an edge of Y.

If (X, V_X) and (Y, V_Y) are graphs, i.e. topological graphs endowed with coordinates $x_e : e \to \mathbb{R}$ on their edges, then we say that a morphism $u : X \to Y$ of topological graphs is a morphism of graphs if for each edge $e \subset X$ which maps homeomorphically onto an edge $f \subset Y$, it is of the form $x_f = ax_e + b$ for $a, b \in \mathbb{R}$. That is, a morphism of graphs is given by an affine transformation on each edge.

Remark 2.16. Though our model for them is topological, the theory of graphs and their morphisms as we have defined them is essentially combinatorial. For instance, every topological graph admits a graph structure, and and two graphs with the same underlying topological graph are isomorphic to each other by a unique isomorphism which fixes the vertices and is homotopic to the identity relative to the vertices. We will usually abuse notation and suppress the coordinate functions $\{x_e\}$ from the data of a graph $(X, V_X, \{x_e\})$.

2.3.2. *Cyclically ordered sets, and ribbon graphs.*

Definition 2.17. Let C be a finite set. A *cyclic order* on C is a ternary relation $\mathcal{R}_C \subset C \times C \times C$ satisfying the following axioms:

(1) If $(x, y, z) \in \mathcal{R}$ then $(y, z, x) \in \mathcal{R}$.
(2) No triple of the form (x, y, y) belongs to $\mathcal{R}$.
(3) If x, y, z are all distinct, then $\mathcal{R}$ contains exactly one of the triples (x, y, z) or (x, z, y).
(4) If $(x, y, z) \in \mathcal{R}$ and $(y, z, w) \in \mathcal{R}$, then $(x, y, w) \in \mathcal{R}$ and $(x, z, w) \in \mathcal{R}$.

Remark 2.18. Informally, (x, y, z) belongs to $\mathcal{R}$ if "when traveling counterclockwise around C starting at x, one encounters y before z."

Remark 2.19. A subset of a cyclically ordered set $C' \subset C$ has a natural cyclic order itself: we set $\mathcal{R}' = \mathcal{R} \cap (C' \times C' \times C')$. We call this the *induced cyclic order* on C'.

Remark 2.20. If C is a cyclically ordered set and $c \in C$, then $C - \{c\}$ has a total ordering defined by $a < b$ if $(c, a, b) \in \mathcal{R}$. Write $R(c)$ for the minimal element of the ordered set $C - \{c\}$. If C has n elements, then the map $c \mapsto R(c)$ has $R^n(c) = c$ and gives C the structure of a $\mathbb{Z}/n$-torsor. An element of the form $(c, R(c)) \in C \times C$ is called a *minimal pair*.

Definition 2.21. Let (X, V_X) be a graph in which every vertex has degree ≥ 2. A *ribbon structure* on (X, V_X) is a collection $\{\mathcal{R}_v\}_{v \in V}$ where $\mathcal{R}_v$ is a cyclic order on the set of half-edges incident with v. We call a graph equipped with a ribbon structure a ribbon graph.

Remark 2.22. A graph without vertices is necessarily homeomorphic to a disjoint union of open intervals. Such a graph has a unique ribbon structure—the cyclic orders $\{\mathcal{R}_v\}_{v \in V_X}$ are indexed by the empty set.

Remark 2.23. Let X be a graph and let $X \hookrightarrow W$ be an embedding into a surface. An orientation on W determines a ribbon structure on X.

Definition 2.24. An *oriented walk* in a ribbon graph $(X, V_X, \{\mathcal{R}_v\})$ is a totally ordered sequence of edges

$$e_1, \ldots, e_{r-1}, e_r$$

with the following properties:

- For each i the edges e_i and e_{i+1} are incident with the same vertex v_i, and $e_{i+1} = R(e_i)$ in the cyclic order $\mathcal{R}_{v_i}$.
- If e_i is a compact edge, then e_{i-1}, e_i, e_{i+1} are not all incident with the same vertex.

Let (X, V_X) be a ribbon graph, and let $Z \subset X$ be a subgraph such that all its vertices have degree 2. We consider a combinatorial analogue of a neighborhood of Z in X: this is a new ribbon graph, denoted $N_Z X$, having the same set of vertices as Z. All the edges of Z are edges of $N_Z X$ as well. Additionally $N_Z X$ has also edges of a different type. These are called *spokes*, and are defined as follows: we attach to a vertex v of Z a new non-compact edge s for each half-edge h incident with v which does not lie in Z; we stipulate that topologically $s = (0, 1)$. Note that if e is a compact edge of X which does not belong to Z but joins together two vertices lying in Z, then e defines two distinct spokes of $N_Z X$.

$N_Z X$ inherits from X a natural structure of ribbon graph. Denote $\leq^Z$ the preorder on the spokes of $N_Z X$ given by setting $e \leq^Z e'$ if there is an oriented walk in $N_Z X$ starting in e and ending in e'. The following simple observation will be useful in Section 4.

Remark 2.25. The preorder $\leq^Z$ on the set of spokes of $N_Z X$ has at most two connected components. Assume further that Z is homeomorphic to S^1. Then oriented walks yield a well defined cyclic order $\prec^Z$ on each of these components: if e, e', e'' are spokes belonging to the same component, we write $e \prec^Z e' \prec^Z e''$ if for all oriented walks in $N_Z X$ starting in e and ending in e''

$$e_1 = e, \ldots, e_{r-1}, e_r = e'',$$

there is i, $1 < i < r$, such that $e_i = e'$.

3. CPM of chordal ribbon graphs

3.1. **Microlocal sheaf theory in one dimension.** In this section we review some of the constructions of microlocal sheaf theory in the case where the base manifold is one dimensional.

Let M be a one-dimensional manifold (with or without affine structure). By a *sheaf* we will mean a cochain complex of sheaves of $\mathbb{C}$-vector spaces. Such a sheaf is *constructible* if it satisfies the following two conditions:

(1) Each stalk is quasi-isomorphic to a bounded complex of vector spaces.
(2) The sheaf is cohomologically locally constant away from a discrete closed subset of M.

Constructible sheaves on M form a dg category that we will denote by $Sh_c(M)$ or usually just by $Sh(M)$. Let $\mathcal{M}$ be the category whose objects are 1-manifolds and whose morphisms are open immersions. The assignments $M \mapsto Sh(M)$ together with pullback along open immersions define a sheaf of dg categories on $\mathcal{M}$—a subsheaf of the "sheaf of sheaves" of remark 2.7.

We can simplify the development of the microlocal theory somewhat by endowing our one-manifolds with affine structures.

Definition 3.1. Let M be a 1-manifold. An *affine structure* on M is the data of an identification $\psi_U : U \cong (a, b) \subset \mathbb{R}$ for every sufficiently small connected open subset of M, such that the transition maps $\psi_U \circ \psi_V^{-1}$ are of the form $cx + d$ with $c \neq 0$.

If M and N are affine 1-manifolds then we have evident notions of affine morphisms $M \to N$. In particular we may speak of affine $\mathbb{R}$-valued functions and their germs on M. The map $f \mapsto df$ identifies the cotangent bundle T^*M of M with the space of pairs (x, ξ) where x is a point of M and ξ is the germ of an affine $\mathbb{R}$-valued function with $\xi(x) = 0$.

Definition 3.2 (Morse groups/microlocal stalks)**.** Let M be an affine 1-manifold, let x be a point of M and let f be the germ of an affine $\mathbb{R}$-valued function on M around x. For $\epsilon > 0$ sufficiently small let A be the sublevel set $\{y \in M \mid f(y) < f(x) + \epsilon\}$ and let B be the sublevel set $\{y \in M \mid f(y) < f(x) - \epsilon\}$. We define a functor $\mu_{x,f} : Sh(M) \to \mathbb{C}$-mod to be the cone on the natural map

$$\Gamma(A; F|_A) \to \Gamma(B; F|_B).$$

Since every constructible sheaf F is locally constant in a deleted neighborhood of x, this functor does not depend on ϵ as long as it is sufficiently small.

Clearly $\mu_{x,f}$ depends only on x and df_x. When $(x, \xi) \in T^*M$ we let $\mu_{x,\xi}$ denote the functor associated to the point x and the affine function whose derivative at x is ξ.

Definition 3.3. For each $F \in Sh(M)$ we define $SS(F) \subset T^*M$, the *singular support* of F, to be the closure of the set of all $(x, \xi) \in T^*M$ such that $\mu_{x,\xi}F \neq 0$.

Definition 3.3 is a special case of Definition 5.1.2 in [KS]. As $\mu_{x,\xi} = \mu_{x,t\cdot\xi}$ when $t > 0$, the set $SS(F)$ is *conical*; that is, if $(x, \xi) \in SS(F)$ and $t \in \mathbb{R}_{>0}$, then $(x, t \cdot \xi) \in SS(F)$. If F is locally constant away from points $\{x_i\}_{i \in I}$, then $SS(F)$ is contained in the union of the zero section and the vertical cotangent spaces $T^*_{x_i}M$. In particular $SS(F)$ is 1-dimensional and therefore a Lagrangian subset of T^*M with its usual symplectic form—we say that $SS(F)$ is a *conical Lagrangian* in T^*M.

Definition 3.4. Suppose $\Lambda \subset T^*M$ is a closed, conical Lagrangian subset. Define $Sh(M, \Lambda) \subset Sh(M)$ to be the full triangulated subcategory of sheaves with $SS(F) \subset \Lambda$.

Example 3.5. Let $\Lambda = M \cup T^*_{s_1}M \cup \cdots \cup T^*_{s_n}M$ be the union of the zero section and the cotangent spaces of finitely many points $\{s_1, \ldots, s_n\}$. Then $Sh(M, \Lambda)$ is the category of sheaves that are locally constant away from $\{s_1, \ldots, s_n\}$.

3.2. **Quiver descriptions of microlocal categories.** Let M be a one-dimensional manifold and let $\Lambda \subset T^*M$ be a conical Lagrangian containing the zero section. The category $Sh(M, \Lambda)$ can be described very concretely in terms of representations of quivers, which we will recall in this section. We also discuss the Bernstein-Gelfand-Ponomarev equivalences that have some relevance for us.

We refer to the connected components of $\Lambda - M$ as the spokes of Λ. The spokes are divided into two groups depending on which component of $T^*M - M$ they fall into. Using an orientation of M we may label these groups "upward" and "downward."

Remark 3.6. We may regard Λ as a graph by letting the affine structure on M induce coordinate functions on the compact edges of Λ, and choosing arbitrary affine coordinates on the other edges. Λ has a ribbon structure coming from the orientation of T^*M, and there is an isomorphism $N_M\Lambda \cong \Lambda$. The notion of "spoke" that we use above is thus just a special case of the definition we gave in Section 2.3.2.

The conical Lagrangian Λ determines a partition P_Λ of M into subintervals (which may be open, half-open, or closed) and points. Let us describe this partition in case $M = \mathbb{R}$, the general case is similar. Each spoke of Λ is incident with a point $x \in \mathbb{R}$, which we may order $x_1 < \ldots < x_k$. We put $\{x_i\} \in P_\Lambda$ if x_i is incident with both an upward and a downward spoke. We put an interval I from x_i to x_{i+1} in P_Λ whose boundary conditions are determined by the following rules

- If x_i is incident with an upward spoke but not incident with a downward spoke, then x_i is included in I. Otherwise x_i is not included in I.
- If x_{i+1} is incident with a downward spoke but not incident with an upward spoke, then x_{i+1} is included in I. Otherwise x_{i+1} is not included in I.

We put $(-\infty, x_1)$ in P_Λ if x_1 is incident with an upward spoke and $(-\infty, x_1]$ in P_Λ if x_1 is incident with a downward spoke, and similarly we put (x_k, ∞) (resp. $[x_k, \infty)$) in P_Λ if x_k is incident with a downward (resp. upward) spoke.

Define a quiver (that is, directed graph) Q_Λ whose vertices are the elements of P_Λ and with and edge joining I to J (in that orientation) if the closure of J has nonempty intersection with I. If there are n spokes then this is a quiver of type A_{n+1} (i.e. shaped like the Dynkin diagram A_{n+1}) whose edges are in natural bijection with the spokes of Λ: an upward spoke corresponds to a left-pointing arrow and a downward spoke to a right-pointing arrow.

Theorem 3.7. *There is a natural equivalence of dg categories*

$$\Pi : Sh(M, \Lambda) \cong Rep(Q_\Lambda)$$

If (x, ξ) belongs to a spoke of Λ corresponding to an arrow f of Q_Λ, then under this equivalence the functor $\mu_{x,\xi}$ intertwines with the functor $\mathsf{C}one(f)$.

Proof. If $I \subset M$ is a sub-interval belonging to P_Λ, then Π sends the constant sheaf on I (concentrated in degree zero, and with stalks equal to $\mathbb{C}$), to the representation of Q_Λ, defined by assigning a copy of $\mathbb{C}$ to the vertex labelled by I, and 0 to all other vertices. These objects generate the respective categories, and they are simple. Therefore this assignment defines ψ uniquely. Also, it is sufficient to check the second part of the statement on these objects.

For simplicity we set $M = \mathbb{R}$. Since the statement is "local" in M this causes no loss of generality. Also, we assume that Λ has an upward spoke in $b \in \mathbb{R}$, and that $I \in P_\Lambda$, $I = (a, b]$: all other cases are similar. If $i : I \to I$ is the inclusion, then the sheaf $i_* \mathbb{C}_I$ is mapped to the representation $\cdots \mathbb{C} \overset{f=0}{\to} 0 \cdots$, and $\mathsf{C}one(f) = \mathbb{C}[1]$. On the other hand, applying Definition 3.2, we obtain that if $\xi \in T^*_b\mathbb{R}$ and $\xi < 0$ (resp. $\xi > 0$), then $\mu_{b,\xi}(i_*\mathbb{C}_I) = \mathbb{C}[1]$ (resp. $= 0$), as expected. Since both $\mathsf{C}one(f)$ and $\mu_{x,\xi}$ evaluate to zero on all other simple generators, this implies that there is an equivalence $\mathsf{C}one(f) \circ \Pi \cong \mu_{x,\xi}$. □

Example 3.8. Let $\perp\!\!\!\!+\!\!\!\top \subset T^*\mathbb{R}$ be the union of the zero section, the fiber at 0, an upward spoke at some $x_- < 0$ and a downward spoke at some $x_+ > 0$. Then

$$Sh(\mathbb{R}, \perp\!\!\!\!+\!\!\!\top) \cong Rep(\bullet \leftarrow \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet).$$

For a general quiver Q, if a is an arrow let $s(a)$ and $t(a)$ denote the source and target of a, respectively. A vertex v of Q is called a sink (resp. source) if all the arrows incident to it have $t(a) = v$ (resp. $s(a) = v$). If x is a sink or a source, then Bernstein-Gelfand-Ponomarev

[BGP] define a new quiver S_xQ obtained by reversing the orientation of all the arrows in Q incident to x.

Theorem 3.9 (Bernstein-Gelfand-Ponomarev [BGP])**.** *Let Q be a quiver, and let $x \in Q$ be a sink or a source. Then there is an equivalence of dg categories*

$$Rep(Q) \cong Rep(S_xQ).$$

If Q_1 and Q_2 are quivers with same underlying undirected graph, then $Rep(Q_1) \cong Rep(Q_2)$.

Applying this theorem to quivers of the form Q_Λ, we obtain the following.

Corollary 3.10. *Let M be a one-dimensional manifold and let Λ_1 and Λ_2 be conical Lagrangians in T^*M. Suppose that in each connected component U of $T^*M - M$, $\Lambda_1 \cap U$ and $\Lambda_2 \cap U$ have an equal number of components (i.e. Λ_1 and Λ_2 have an equal number of spokes in each group.) Then $Sh(M;\Lambda_1) \cong Sh(M;\Lambda_2)$.*

3.3. **Microlocalization and contact transformations.** One expects that a sheaf F on M has a local nature on T^*M as well as on M. This idea is made precise by Kashiwara and Schapira using the microlocal theory of sheaves. For each open subset $U \subset T^*M$ they construct a category $D^b(M;U)$, and a restriction functor from the derived category of constructible sheaves on M, to $D^b(M;U)$. If F has singular support Λ then the image of F in $D^b(M;T^*M - \Lambda)$ is zero, see [KS, Section 6.1].

We refer to [KS] for the general theory. Below, we focus on the one dimensional case, and describe a slightly enhanced version of these constructions which plays a key role in the definition of CPM.

Definition 3.11. Let M_1 and M_2 be one-dimensional manifolds, and let $U_1 \subset T^*M_1$ and $U_2 \subset T^*M_2$ be conical open subsets. A *contact transformation* from U_1 to U_2 is an open immersion $f : U_1 \to U_2$ satisfying the following properties:

(1) $f^*\omega_2 \cong \omega_1$, where ω_i is the natural symplectic form on T^*M_i
(2) f is equivariant for the $\mathbb{R}_{>0}$-actions on U_i, i.e. $f(t \cdot u) = t \cdot f(u)$ for $t \in \mathbb{R}_{>0}$.

If we endow $T^*\mathbb{R}$ with coordinates (x, ξ), such that $\xi(x) = \xi x$, and U_1 and U_2 are open subsets of $T^*\mathbb{R}$, then any contact transformation is of the form $(x, \xi) \mapsto (f(x), \xi/f'(x))$ for some smooth function f. We say that a contact transformation is *affine* if f is affine. More generally, if $f : M \to N$ is an affine, open map between affine 1-manifolds, we let C_f denote the associated contact transformation, which in local coordinates as above looks like $(f(x), \xi/f'(x))$. If $U \subset T^*M$ is connected, or more generally if the projection map $U \to M$ is injective on connected components, then every contact transformation is of the form C_f.

Remark 3.12. The simple structure of our contact transformations is a special feature of one-dimensional manifolds and their cotangent bundles. A general contact transformation need not be induced by a map between the base manifolds.

We define a category Cont in the following way:

- The objects of Cont are triples (M, U, Λ) where M is an affine 1-manifold, $U \subset T^*M$ is a conical open set, and $\Lambda \subset T^*M$ is a closed, conical Lagrangian set. If $\pi : T^*M \to M$ denotes the projection, we furthermore assume that $\pi(U) = M$, and that π defines a bijection between the connected components of U and M.

- $\mathrm{Hom}((M_1, U_1, \Lambda_1), (M_2, U_2, \Lambda_2))$ is the set of affine contact transformations $U_1 \to U_2$ that carry $\Lambda_1 \cap U_1$ homeomorphically onto an open subset of $\Lambda_2 \cap U_2$.

Remark 3.13. Note that there is a "forgetful" functor Cont $\to \mathcal{M}$ that sends a tuple (M, U, Λ) to M, and sends a contact transformation of the form C_f to f.

We endow Cont with a Grothendieck topology by letting $\{(M_i, U_i, \Lambda_i)\}$ be a covering sieve of (M, U, Λ) if the U_i cover U. Exploiting the locality of constructible sheaves over the cotangent bundle, we can define a sheaf *MSh* over Cont, having the property that for all one-dimensional manifolds M, and for all conical Lagrangians $\Lambda \subset T^*M$, $MSh(M, T^*M, \Lambda) = Sh(M; \Lambda)$.[3] We proceed as follows:

(1) The functor $\mathcal{M} \to \mathrm{dg}\mathcal{C}\mathrm{at} : M \mapsto Sh(M)$ pulls back to a functor Cont $\to \mathrm{dg}\mathcal{C}\mathrm{at}$ along the "forgetful" map Cont $\to \mathcal{M}$ of Remark 3.13. Let us call this functor $\mathcal{P}$.
(2) If we let $\mathcal{P}'(M, U, \Lambda) \subset \mathcal{P}(M, U, \Lambda) = Sh(M)$ denote the full subcategory of sheaves F with $SS(F) \cap U = \varnothing$ then the assigment $(M, U, \Lambda) \mapsto \mathcal{P}'$ defines a full subfunctor $\mathcal{P}' \subset \mathcal{P}$.
(3) The quotient construction gives us a presheaf $\mathcal{P}/\mathcal{P}'$ on Cont.[4] This functor has a further subfunctor $\mathcal{P}'' \subset \mathcal{P}/\mathcal{P}'$ with $\mathcal{P}''(M, U, \Lambda)$ given by the full subcategory spanned by sheaves F with $SS(F) \cap U \subset \Lambda \cap U$.
(4) We let *MSh* denote the sheafification of the presheaf $\mathcal{P}''$.

Remark 3.14. The sections of *MSh* can be computed explicitly in many cases. Fix a conical Lagrangian $\Lambda \subset T^*M$, then $MSh(M, U, \Lambda)$ can be described as follows:

(1) If U is of the form $\pi^{-1}(I)$ where $I \subset M$ is an open set, then the restriction functor $Sh(M, \Lambda) \to Sh(I, \Lambda \cap \pi^{-1}(I))$ induces an equivalence $MSh(M, U, \Lambda) \cong Sh(I, \Lambda \cap \pi^{-1}(I))$.
(2) If U does not contain the zero section and its intersection with Λ consists of the vertical segments $(x_1, \mathbb{R}_{>0} \cdot \xi_1), \ldots, (x_n, \mathbb{R}_{>0} \cdot \xi_n)$ then the microlocal stalk functors assemble to a map $Sh(M; \Lambda) \xrightarrow{\bigoplus \mu_{x_i, \xi_i}} \bigoplus_{i=1}^n \mathbb{C}\text{-mod}$ that induces an equivalence $MSh(U; \Lambda) \cong \bigoplus_{i=1}^n \mathbb{C}\text{-mod}$.

3.4. **Chordal ribbon graphs.** In this section we will consider a special class of ribbon graphs.

Definition 3.15. A *chordal ribbon graph* is a pair (X, Z), where

- X is a ribbon graph with vertices of valency greater than 1 and at most 4.
- Z is a closed subgraph containing each vertex of X, and such that its vertices have valency at most 2.
- Let v be a vertex of X of valency n. Then, there exists a half-edge e incident with v such that, for all $i \in \{0, \ldots, n-2\}$, $R^i(e)$ lies in Z if and only if $R^{i+1}(e)$ does not.

We refer to Z as the *zero section* of the chordal ribbon graph.

[3]For a general treatment of functors induced by contact transformations between microlocal categories of sheaves, see [KS, Chapter 7].

[4]The availability of the quotient construction for sheaves of dg categories depends on the existence of *Verdier quotients* in the dg setting, see [Dr].

Remark 3.16. For most of this paper we can assume that Z is a bivalent graph. Allowing the zero section to have vertices of degree one, however, is convenient for applications. For instance, it is often useful to deform chordal graphs in ways that alter the valency of vertices, see [Si].

Remark 3.17. If X is compact and Z is bivalent, the subset Z is a disjoint collection of circles. These circles are joined by edges of X that we might call "chords." Chordal ribbon graphs are similar to the "chord diagrams" and "string diagrams" of Chas and Sullivan [Su].

Let Chord denote the category whose objects are chordal ribbon graphs, and where $\mathrm{Hom}((U,W),(X,Z))$ is given by the set of open immersions of graphs $j : U \hookrightarrow X$ with $j(W) \subset Z$ and preserving the cyclic orders at each vertex. We endow Chord with a Grothendieck topology in the evident way.

Example 3.18. Let e be a ribbon graph with no vertices and one edge. We may endow it with two non-isomorphic chordal structures: one in which the zero section is empty and one in which the zero section is all of e.

Example 3.19. Let M be an affine 1-manifold and let $\Lambda \subset T^*M$ be a closed conical Lagrangian subset. As explained in Remark 3.6, if Λ contains the zero section, it can be given the structure of a ribbon graph. Then the pair (Λ, M) is a chordal ribbon graph.

We restrict now to a class of conical Lagrangian subsets $\Lambda \subset T^*M$, where M is homeomorphic to an open interval, that have an especially simple geometry. We say that Λ is *star-shaped* if it is connected and has at most one singular point.[5] Let $U \subset T^*M$ be a conical open subset. In a similar way to Example 3.19, we can form a chordal ribbon graph (with possibly empty zero section) by considering the pair $(\Lambda \cap U, (M \cap \Lambda) \cap U)$. We refer to chordal ribbon graphs arising in this way as *Yit-s*: note indeed that the underlying topological space of any such graph will have one of the following shapes $\mathsf{Y}, \mathsf{I}, +$. We denote Y the full subcategory of Chord spanned by Yit-s.

Lemma 3.20. *There is a faithful functor from Y to* Cont *that respects the Grothendieck topologies.*

Proof. Up to isomorphism there are only six objects in Y. Let $M = (-1, 1)$ and denote (x, ξ) the restriction of the standard coordinates on $T^*\mathbb{R} = \mathbb{R}^2$ to $T^*M \subset T^*\mathbb{R}$. For all $i \in \{1, \ldots, 6\}$ we define U_i and Λ_i as follows:

- $U_1 = T^*M, \Lambda_1 = M$,
- $U_2 = \{(x, \xi) \in T^*M | \xi > 0\}, \Lambda_2 = \{0\} \times [0, \infty)$,
- $U_3 = T^*M, \Lambda_3 = [0, 1) \times \{0\} \cup \{0\} \times [0, \infty)$,
- $U_4 = T^*M, \Lambda_4 = (-1, 1) \times \{0\} \cup \{0\} \times [0, \infty)$,
- $U_5 = T^*M, \Lambda_5 = [0, 1) \times \{0\} \cup \{0\} \times (-\infty, \infty)$,
- $U_6 = T^*M, \Lambda_6 = (-1, 1) \times \{0\} \cup \{0\} \times (-\infty, \infty)$.

We set $Y_i := (\Lambda_i \cap U_i, (M \cap \Lambda_i) \cap U_i)$. We give Y_i the structure of chordal ribbon graph by letting x induce coordinates on the edges contained in $(\Lambda \cap M) \cap U$, and assigning to the

[5]There is a maximal open dense subset $\Lambda^{reg} \subset \Lambda$, such that Λ^{reg} is a possibly non connected smooth submanifold of T^*M: by definition, the set of singular points of Λ is given by $\Lambda - \Lambda^{reg}$.

spokes of Λ some arbitrary fixed coordinate such as $(x, \xi) \mapsto (x, 1/(1+|\xi|^2))$. Every object in Y is isomorphic to one of the Y_i-s.

Let Cont^{st} be the subcategory of Cont having as objects (M, U_i, Λ_i), and as morphisms those such that the corresponding affine map $f : M \to M$ restricts to an open embedding of graphs $\Lambda_i \cap M \subset \Lambda_j \cap M$. In particular, this last condition implies also that $C_f : U_i \to U_j$ restricts to an open embedding of graphs $\Lambda_i \cap U_i \subset \Lambda_j \cap U_j$. We claim that the functor $\mathcal{I}$ given by the assignment $(M, U_i, \Lambda_i) \in \mathrm{Cont}^{st} \mapsto Y_i \in Y$, with the natural definition on morphisms, is an equivalence of categories. We have already remarked that $\mathcal{I}$ is essentially surjective, while faithfulness is a consequence of the definition of morphisms in Chord. We need to prove that $\mathcal{I}$ is full.

The cardinality of $Hom_{\mathrm{Chord}}(Y_i, Y_j)$ is finite and bounded by 4. The Hom sets are empty if $j < i$, and for the pairs $(i, j) = (1, 2), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6)$ and $(5, 6)$. If $i = j = 3$, 4 or 5 the only morphism is the identity. This leaves out a comparatively small number of cases for which fulness has to be checked, these are the pairs $(1, 1), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (6, 6)$. We explain in detail the case $(1, 6)$, the other cases being similar. The cardinality of the set $Hom_{\mathrm{Chord}}(Y_1, Y_6)$ is 4. As $\mathcal{I}$ is faithful, it is sufficient to prove that in Cont^{st} there exist four distinct morphisms between (M, U_1, Λ_1) and (M, U_6, Λ_6). These correspond to the following affine self-maps of M: $f_{m,n}(x) = \frac{(-1)^m}{2}x + \frac{(-1)^n}{2}$, for $m, n \in \{0, 1\}$.

This proves that $\mathcal{I} : \mathrm{Cont}^{st} \to Y$ is an equivalence. Choosing an inverse functor gives a faithful inclusion of Y into Cont, as desired. □

Using Lemma 3.20 we can restrict $MSh \in \mathrm{Shv}(\mathrm{Cont}, \mathrm{dg}\mathcal{C}\mathrm{at})$ to a sheaf on the subcategory Y of Chord. In Chord, Yit-s are a basis for the Grothendieck topology. By Proposition 2.8, this shows that MSh determines a sheaf of categories CPM on Chord.

Definition 3.21. We let $\mathrm{CPM} : \mathrm{Chord} \to \mathrm{dg}\mathcal{C}\mathrm{at}$ denote the sheaf of dg categories on Chord whose restriction to Y is given by MSh of Section 3.3. We call $\mathrm{CPM}(X, Z)$ the *constructible plumbing model* of the chordal ribbon graph (X, Z).

4. Dualizable Ribbon Graphs, Beilinson-Bondal, and HMS

Let (X, Z) be a chordal ribbon graph, and assume that Z is bivalent and compact. That is, as a topological space, Z is a finite disjoint union of m copies of S^1, $Z = Z_1 \amalg \cdots \amalg Z_m$. If e_1, e_2 are edges of X not belonging to Z we abuse notation and write $e_1 \leq^{Z_i} e_2$, if e_1 and e_2 are both incident to vertices of Z_i and there is an oriented walk connecting the corresponding spokes, and similarly for the cyclic order $\prec^{Z_i}$.

Definition 4.1. We say that (X, Z) is *dualizable* if the following two conditions are satisfied:

- For all $i \in \{1, \ldots, m\}$, the preorder $\leq^{Z_i}$ on the set of spokes of $N_{Z_i}X$ has exactly two connected components.
- For all triples of edges e_1, e_2, e_3 not belonging to Z, if there are $i, j \in \{1, \ldots, m\}$ such that $e_1 \prec^{Z_i} e_2 \prec^{Z_i} e_3$ and e_1 is incident to Z_j, then e_2 and e_3 are incident to Z_j as well, and $e_1 \prec^{Z_j} e_2 \prec^{Z_j} e_3$.

If (X, Z) is dualizable, we consider a topological graph B, which is a quotient of X, and is defined as follows. The vertices of B are the connected components of Z. We identify two edges e_1 and e_2 not lying in Z if there exists $i \in \{1, \ldots, m\}$ such that $e_1 \leq^{Z_i} e_2$. It is a simple consequence of Definition 4.1 that B is a well defined, bivalent graph. Also, by construction, there is a map p from X to B which collapses each component Z_i to a vertex.

As discussed in the Introduction, dualizability is a combinatorial analogue of a torus fibration with a section. In this section we show that CPM of a dualizable ribbon graph is equivalent to perfect complexes on a one-dimensional variety or orbifold—a chain of weighted projective lines ("balloons") indexed by the vertices of B.

4.1. **tcnc stacks.** We shall introduce two classes of proper one-dimensional stacks with toric components, called respectively "balloon chains" and "balloon rings," which arise as mirror partners of dualizable ribbon graphs. We refer to Deligne-Mumford stacks of any of these two kinds as "one-dimensional stacks with toric components and normal crossings" or "one-dimensional tcnc stacks" for short.

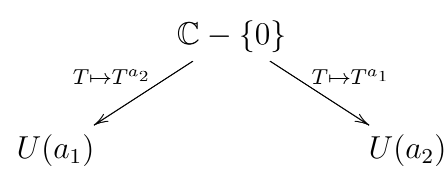
Definition 4.2. For $a \in \mathbb{Z}_{\geq 1}$, Let $\mu_a \subset \mathbb{C}^*$ be the multiplicative group of the ath roots of unity.

- We let $B(a) = [\mathrm{Spec}\,(\mathbb{C})/\mu_a]$ denote the classifying stack of μ_a.
- Let $\mathbb{C}[T]$ be a polynomial ring with coordinate variable T. We let $U(a) = [\mathrm{Spec}\,(\mathbb{C}[T])/\mu_a]$ denote the quotient stack associated to the action of μ_a on $\mathbb{C}[T]$ given
$$\zeta \cdot T = \zeta^{-1} T.$$
- Let $\mathbb{C}[T, V]/TV$ denote the coordinate ring of the T- and V-axes in the TV-plane. Let $X(a) = [\mathrm{Spec}\,(\mathbb{C}[T, V]/TV)/\mu_a]$ denote the quotient stack associated to the action of μ_a on $\mathbb{C}[T, V]/TV$ given by
$$\zeta \cdot T = \zeta T, \qquad \zeta \cdot V = \zeta^{-1} V.$$

Remark 4.3. We have open inclusions
$$U(a) \supset [\mathrm{Spec}\,(\mathbb{C}[T, T^{-1}])/\mu_a],$$
$$X(a) \supset [(\mathrm{Spec}\,(\mathbb{C}[T, T^{-1}, V, V^{-1}])/TV)/\mu_a].$$
As μ_a acts freely on $\mathbb{C} - \{0\}$, we may identify these open subsets with affine schemes $\mathrm{Spec}\,(\mathbb{C}[T^a, T^{-a}])$ and $\mathrm{Spec}\,(\mathbb{C}[T^a, T^{-a}]) \coprod \mathrm{Spec}\,(\mathbb{C}[V^a, V^{-a}])$.

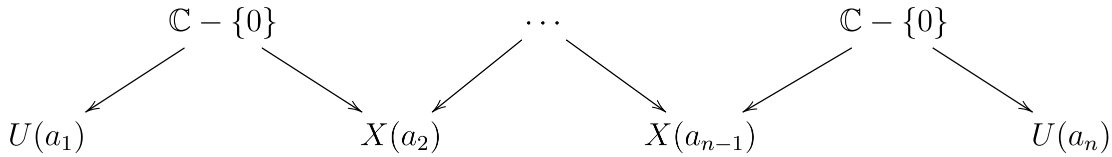
Remark 4.4. The stacks $B(1)$, $U(1)$, and $X(1)$ are representable, i.e. they are ordinary varieties. There are étale maps $B(1) \to B(a)$, $U(1) \to U(a)$, and $X(1) \to X(a)$, as well as projections to coarse moduli spaces $B(a) \to B(1)$, $U(a) \to U(1)$, and $X(a) \to X(1)$. The compositions $U(1) \to U(a) \to U(1)$ etc. are the GIT quotient maps for the actions of μ_a.

Definition 4.5. A *balloon* with indices (a_1, a_2) is the pushout of the diagram

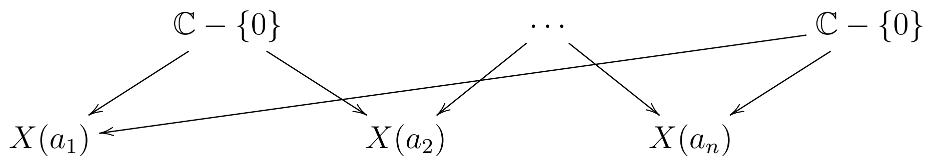


in the 2-category of Deligne-Mumford stacks. That is, a balloon is a weighted projective line that is generically representable. We will define "balloon chains" and "balloon rings" by gluing together balloons at their orbifold points.

(1) For $n \geq 2$, a *balloon chain* with indices $(a_1, \ldots, a_n)$ is the pushout of the diagram



(2) For $n \geq 2$, a *balloon ring* with indices $(a_1, \ldots, a_n)$ is the pushout of the diagram



If $A = (a_1, \ldots, a_n)$ is an n-tuple of positive integers, we will denote by $C(A)$ the balloon chain with indices A and by $R(A)$ the balloon ring with indices A.

Remark 4.6. If $A = (a_1, \ldots, a_n)$ is an n-tuple of positive integers, let $\tilde{C}(A)$ (resp. $\tilde{R}(A)$) be the disjoint union of balloons $C(a_1, a_2) \amalg \cdots \amalg C(a_{n-1}, a_n)$ (resp. $C(a_1, a_2) \amalg \cdots \amalg C(a_{n-1}, a_n) \amalg C(a_n, a_1)$). There are maps $\tilde{C}(A) \to C(A)$ and $\tilde{R}(A) \to R(A)$ that exhibit $\tilde{C}$ and $\tilde{R}$ as the normalization of the Deligne-Mumford stacks $C(A)$ and $R(A)$.

4.2. **Perfect complexes on balloon chains and rings.** For $A = (a_1, \ldots, a_n)$ an n-tuple of positive integers, set $B_C(A) = B(a_2) \amalg \cdots \amalg B(a_{n-1})$ and $B_R(A) = B(a_1) \amalg \cdots \amalg B(a_n)$. We choose two disjoint embeddings $i_1, i_2 : B_C(A) \hookrightarrow \tilde{C}(A)$ (resp. $B_R(A) \hookrightarrow \tilde{R}(A)$) lifting the inclusion $B_C(A) \subset C(A)$ (resp. $B_R(A) \subset R(A)$). We have diagrams

$$B_C(A) \rightrightarrows \tilde{C}(A) \to C(A)$$
$$B_R(A) \rightrightarrows \tilde{R}(A) \to R(A)$$

(Note that these diagrams should be understood as functors from the nerve of the category $\bullet \rightrightarrows \bullet \to \bullet$ into the ∞-category $\mathrm{St}_{/\mathbb{C}}$.) In fact we will show in Proposition 4.7 below that these functors are coequalizer diagrams in $\mathrm{St}_{/\mathbb{C}}$.

Proposition 4.7. *Let $A = (a_1, \ldots, a_n)$ be an n-tuple of positive integers. The following diagrams are equalizers in the ∞-category of dg categories.*

(1) $\mathcal{P}\mathrm{erf}(C(A)) \to \mathcal{P}\mathrm{erf}(\tilde{C}(A)) \rightrightarrows \mathcal{P}\mathrm{erf}(B_C(A))$
(2) $\mathcal{P}\mathrm{erf}(R(A)) \to \mathcal{P}\mathrm{erf}(\tilde{R}(A)) \rightrightarrows \mathcal{P}\mathrm{erf}(B_R(A))$

This is essentially proved in [La, Section 4], but let us give a proof in our current language:

Proof of Proposition 4.7. Suppose we have a diagram of stacks $X'' \rightrightarrows X' \to X$, and an étale cover $\{U_i \to X\}$ of X. Set $U_i' = U_i \times_X X'$ and $U_i'' = U_i \times_X X''$, so that for each i we have a diagram

$$U_i'' \rightrightarrows U_i' \to U_i$$

By Proposition 2.12 and basic properties of limits, to show that $\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(X') \rightrightarrows \mathcal{P}\mathrm{erf}(X'')$ is an equalizer diagram it suffices to show that $\mathcal{P}\mathrm{erf}(U_i) \to \mathcal{P}\mathrm{erf}(U_i') \to \mathcal{P}\mathrm{erf}(U_i'')$ is an equalizer diagram for each i. Applying this remark to the open covers $\{X(a_1), \ldots, X(a_n)\}$ of $R(A)$ and $\{U(a_1), X(a_2), \ldots, X(a_{n-1}), U(a_n)\}$ of $C(A)$, we see that the Proposition reduces to the claim that the map $\mathcal{P}\mathrm{erf}(X(a)) \to \mathcal{P}\mathrm{erf}(\tilde{X}(a)) \rightrightarrows \mathcal{P}\mathrm{erf}(B(a))$ is an equalizer, where $\tilde{X}(a) = U(a) \amalg U(a)$ is the normalization of $X(a)$. Another application of Proposition 2.12, this time to the étale cover $X(1) \to X(a)$, further reduces us to the case where $a = 1$.

Set $X = X(1) = \mathrm{Spec}\,(\mathbb{C}[x,y]/xy)$ and $\tilde{X} = \mathrm{Spec}\,(\mathbb{C}[x] \coprod \mathbb{C}[y])$, and $B = \mathrm{Spec}\,(\mathbb{C})$. Also let $p : \tilde{X} \to X$ be the normalization, and let $i_1, i_2 : B \to \tilde{X}$ be two distinct sections. Let $\mathbf{E}$ be the equalizer of $i_1^*, i_2^* : \mathcal{P}\mathrm{erf}(\tilde{X}) \rightrightarrows \mathcal{P}\mathrm{erf}(B)$. Recall from Proposition 2.2 that:

- The objects of $\mathbf{E}$ are pairs (V, u), where V is a complex of vector bundles on $\tilde{X}$, and u is a degree zero, closed morphism $u : i_1^*V \to i_2^*V$, which becomes invertible in the homotopy category.
- The hom complex between objects $(V, u), (W, u') \in \mathbf{E}$ is given by pairs $(f, H) \in \mathrm{Hom}^k(V, W) \oplus \mathrm{Hom}^{k-1}(i_1^*V, i_2^*W)$, with differential $d(f, H) = (df, dH - (u' i_1^*(f) - i_2^*(f)u))$.

$\mathcal{P}\mathrm{erf}(X)$ is generated by $\mathcal{O}_X$. Thus, in order to define a functor $\psi : \mathcal{P}\mathrm{erf}(X) \to \mathbf{E}$, it is sufficient to define it on $\mathcal{O}_X$ and its endomorphisms. Set $\psi(\mathcal{O}_X) = (\mathcal{O}_{\tilde{X}}, i_1^*\mathcal{O}_{\tilde{X}} = \mathbb{C} \xrightarrow{id} i_2^*\mathcal{O}_{\tilde{X}} = \mathbb{C})$, and if $f \in \mathbf{R}\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X)$, set $\psi(f) = (p^*f, 0)$. Denote $I := \psi(\mathcal{O}_X)$. We have to prove that I generates $\mathbf{E}$, and that $\psi : \mathbf{R}\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \mathbf{R}\mathrm{Hom}(I, I)$ is a quasi isomorphism.

To show that I generates we may reduce by induction to the case where $\mathcal{E}$ is a chain complex of length 1. In that case we see that $\mathcal{E} \cong I^{\oplus r}$ for some r because every vector bundle on $\tilde{X}$ is trivializable, and also any choice of $u : i_1^*\mathcal{O}_{\tilde{X}}^r \to i_2^*\mathcal{O}_{\tilde{X}}^r$ gives isomorphic objects in $\mathbf{E}$. Consider now $\psi : \mathbf{R}\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \mathbf{R}\mathrm{Hom}(I, I)$. We denote $\mathrm{Hom}^i(\bullet, \bullet)$ the i-th cohomology of $\mathbf{R}\mathrm{hom}(\bullet, \bullet)$. Observe that $\mathrm{Hom}^i(I, I) = 0$ for all $i > 0$. In fact, $\mathrm{Hom}^1(I, I)$ is isomorphic to the quotient of $\mathrm{Hom}^0(i_1^*\mathcal{O}_{\tilde{X}}, i_2^*\mathcal{O}_{\tilde{X}}) \cong \mathbb{C}$ by the image of the differential, which is surjective. Vanishing of $\mathrm{Hom}^i(I, I)$ in degrees $i > 1$ follows because $\tilde{X}$ and B are affine, and therefore all higher ext groups are trivial. We are reduced to prove that $\psi^0 : \mathrm{Hom}^0(\mathcal{O}_X, \mathcal{O}_X) \cong \mathbb{C} \to \mathrm{Hom}^0(I, I)$ is an isomorphism. We have the following short exact sequence,

$$0 \to \mathrm{Hom}^0(I, I) \to \mathrm{Hom}^0(\mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}) \to \mathrm{Hom}^0(i_1^*\mathcal{O}_{\tilde{X}}, i_2^*\mathcal{O}_{\tilde{X}}) \to 0.$$

This shows that $\mathrm{Hom}^0(I, I)$ is one dimensional, and thus that ψ^0 has to be an isomorphism. □

Remark 4.8. Using a similar argument as above it is possible to prove a more general result. That is, if X is any twisted nodal curve (see [AV] for the definition) with normalization $\tilde{X}$, B is the singular locus and $i_1, i_2 : B \to \tilde{X}$ are two disjoint sections, then $\mathcal{P}\mathrm{erf}(X)$ is the equalizer of $i_1^*, i_2^* : \mathcal{P}\mathrm{erf}(\tilde{X}) \rightrightarrows \mathcal{P}\mathrm{erf}(B)$.

4.3. **Wheels and the Beilinson-Bondal equivalence.** A *wheel* is a conical Lagrangian Λ in T^*S^1 that contains the zero section. It has a canonical chordal ribbon structure whose zero section Z is the zero section of T^*S^1. The spokes of Λ are divided into two groups. Suppose there are a spokes on one side and b spokes on the other side. Bondal [B] constructed an equivalence

$$\mathcal{P}\mathrm{erf}(C(a, b)) \cong Sh(S^1; \Lambda)$$

This equivalence is very much in the spirit of an old result of Beilinson on the derived category of projective space, which in particular showed that $\mathcal{P}\mathrm{erf}(\mathbb{P}^1)$ was equivalent to the category of representations of the quiver $\bullet \rightrightarrows \bullet$.

Theorem 4.9 (Beilinson-Bondal). *Let $\Lambda \subset T^*S^1$ be a wheel, and let $U_1 \subset T^*S^1$ and $U_2 \subset T^*S^1$ be the two connected components of $T^*S^1 - S^1$. Suppose that there are a_1 spokes in U_1 and a_2 spokes in U_2. There is a commutative diagram of triangulated dg categories*

$$\begin{array}{ccccc} \mathcal{P}\mathrm{erf}(B(a_1)) & \longleftarrow & \mathcal{P}\mathrm{erf}(C(a_1,a_2)) & \longrightarrow & \mathcal{P}\mathrm{erf}(B(a_2)) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ MSh(U_1,\Lambda) & \longleftarrow & Sh(S^1;\Lambda) & \longrightarrow & MSh(U_2,\Lambda) \end{array}$$

Proof. By Corollary 3.10 we may assume Λ has any convenient shape so long as we do not change the numbers a_1 and a_2. Let us identify the base manifold S^1 with the unit circle in $\mathbb{C}$, and let Λ be the union of S^1 together with the a_1 upward spokes placed at a_1th roots of unity and a_2 downward spokes placed at a_2th roots of unity. Let Q_Λ be the corresponding quiver. By Theorem 3.7 we only have to show that $\mathcal{P}\mathrm{erf}(C(a_1,a_2))$ is equivalent to $Rep(Q_\Lambda)$. This follows from Bondal's result. For the convenience of the reader we give an alternative proof of this fact based on [FLTZ2] and [Tr].

Theorem 7.6 of [FLTZ2] defines, in particular, an equivalence $\tilde{\kappa} : \mathcal{P}\mathrm{erf}_T(C(a_1,a_2)) \cong Sh_{cc}(\mathbb{R};\tilde{\Lambda})$. Recall that $\mathcal{P}\mathrm{erf}_T(C(a_1,a_2))$ is the category of $\mathbb{C}^*$-equivariant perfect complexes on $C(a_1,a_2)$, and $Sh_{cc}(\mathbb{R};\tilde{\Lambda})$ is the category of complexes of sheaves with *compactly supported*, constructible cohomology, and microsupport in $\tilde{\Lambda}$. The conical Lagrangian $\tilde{\Lambda} \subset T^*\mathbb{R}$ is the preimage of $\Lambda \subset T^*S^1$ along $\bar{p} : T^*\mathbb{R} \to T^*S^1$, where $\bar{p}$ is the unique lift of the universal covering $p : \mathbb{R} \to S^1$ to the contangent bundle, which locally is a symplectomorphism.

It follows from [Tr] that there is a quasi-fully faithful embedding $\kappa : \mathcal{P}\mathrm{erf}(C(a_1,a_2)) \to Sh(S^1;\Lambda)$, and a commutative diagram

$$\begin{array}{ccc} \mathcal{P}\mathrm{erf}_T(C(a_1,a_2)) & \xrightarrow{\tilde{\kappa}} & Sh_{cc}(\mathbb{R};\tilde{\Lambda}) \\ \downarrow \mathrm{forg} & & \downarrow p_! \\ \mathcal{P}\mathrm{erf}(C(a_1,a_2)) & \xrightarrow{\kappa} & Sh(S^1;\Lambda) \end{array}$$

We have to prove that κ is essentially surjective. Since $\tilde{\kappa}$ is essentially surjective, we can show equivalently that the image of $p_!$ generates $Sh(S^1;\Lambda)$. Note that we can write down explicitly a generating set for $Sh(S^1;\Lambda)$. It is given by constant sheaves supported on subintervals of S^1: see the proof of Theorem 3.7. Now, any $F \in Sh(S^1;\Lambda)$ supported on a proper subset U of S^1 lies in the image of $p_!$. In fact, pick $\tilde{U} \subset \mathbb{R}$ such that p restricted to $\tilde{U}$ is a homeomorphism, and $p(\tilde{U}) = U$. Then there exists a (unique, up to isomorphism) $\tilde{F} \in Sh_{cc}(\mathbb{R};\tilde{\Lambda})$, supported on $\tilde{U}$, such that $p_!(\tilde{F}) \cong F$. This concludes the proof. ☐

We may use this to deduce the main result of this paper. Let X be a dualizable ribbon graph and let B be the associated "base" graph defined in Section 4. If X is connected then B is either a cycle or a path—let us number the edges of B by the integers 1-n in such a way that the ith edge and the $(i+1)$st edge (and the nth and 1st edge, if B is a cycle) share a common vertex. Then let a_i be the number of edges of X that lie above the ith edge. Call $(a_1, \ldots, a_n)$ the indices associated to the dualizable ribbon graph.

Theorem 4.10 (HMS). *Let X be a dualizable ribbon graph with indices $(a_1, \ldots, a_n)$.*

(1) *If the base graph B is a path, then there is an equivalence of dg categories* $\mathrm{CPM}(X) \cong \mathcal{P}\mathrm{erf}(C(a_1, \ldots, a_n))$
(2) *If the base graph B is a cycle, then there is an equivalence of dg categories* $\mathrm{CPM}(X) \cong \mathcal{P}\mathrm{erf}(R(a_1, \ldots, a_n))$

A related result, in the case when $a_i = 1$ for all i, was announced by Kontsevich in [K].

Proof. In case (1), there is an open cover of (X, Z) by charts (W_i, Z_i), $i = 1, \ldots, n$, such that W_i is a wheel, $W_i \cap W_j = \varnothing$ if $|i - j| \geq 2$, and $W_i \cap W_{i+1}$ is a component of $W_i - Z$ (and a component of $W_{i+1} - Z$). Then by the sheaf property of CPM we have an equalizer diagram

$$\mathrm{CPM}(X) \to \mathrm{CPM}(\coprod W_i) \rightrightarrows \mathrm{CPM}(\coprod W_i \cap W_{i+1})$$

The Theorem then follows immediately from Theorem 4.9 and Proposition 4.7. Case (2) is similar. $\square$

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