Symmetry, Integrability and Geometry: Methods and Applications SIGMA **16** (2020), 069, 13 pages

On the Irreducibility of Some Quiver Varieties

Claudio BARTOCCI [†1 †2], *Ugo BRUZZO* [†3 †4 †5 †6 †7], *Valeriano LANZA* [†8] *and Claudio L.S. RAVA* [†1]

[†1] *Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy*
E-mail: *bartocci@dima.unige.it*, *clsrava@gmail.com*
URL: http://www.dima.unige.it/~bartocci/index.htm

[†2] *Laboratoire SPHERE, CNRS, Université Paris Diderot (Paris 7), 75013 Paris, France*

[†3] *SISSA (Scuola Internazionale Superiore di Studi Avanzati), Via Bonomea 265, 34136 Trieste, Italy*
E-mail: *bruzzo@sissa.it*
URL: http://www.people.sissa.it/~bruzzo/webpage/

[†4] *Departamento de Matemática, Universidade Federal da Paraíba, Campus I, João Pessoa, PB, Brasil*

[†5] *IGAP (Institute for Geometry and Physics), Trieste, Italy*

[†6] *INFN (Istituto Nazionale di Fisica Nucleare), Sezione di Trieste, Italy*

[†7] *Arnold-Regge Center for Algebra, Geometry and Theoretical Physics, Torino, Italy*

[†8] *Departamento de Análise, IME, Universidade Federal Fluminense, Rua Professor Marcos Waldemar de Freitas Reis, Niterói, RJ, Brazil*
E-mail: *vlanza@id.uff.br*

Received March 13, 2020, in final form July 10, 2020; Published online July 26, 2020
https://doi.org/10.3842/SIGMA.2020.069

Abstract. We prove that certain quiver varieties are irreducible and therefore are isomorphic to Hilbert schemes of points of the total spaces of the bundles $\mathcal{O}_{\mathbb{P}^1}(-n)$ for $n \geq 1$.

Key words: quiver representations; Hilbert schemes of points

2020 Mathematics Subject Classification: 14D20; 14D21; 14J60; 16G20

We thank Gianni Landi for the long friendship and for all collaborations that took place over the last 35 years

1 Introduction

Nakajima's quiver varieties were introduced by Hiraku Nakajima in [11] to study the moduli spaces of instantons on ALE spaces, and have been extensively studied since then, see, e.g., [8, 10, 12, 13]. They provide a modern and significant example of how algebra and geometry can be sometimes so deeply, yet surprisingly connected: in fact, their main feature is that they allow one to put in relation some moduli spaces of bundles (or torsion-free sheaves) over certain smooth projective varieties with some moduli spaces of representations of suitable algebras (the so-called *path algebras* of a quiver and quotients of them). A major example of this bridge is given by the moduli space of framed sheaves on $\mathbb{P}^2$, which can be identified with the moduli space of semistable representations of the ADHM quiver (see [12] for details).

This paper is a contribution to the Special Issue on Noncommutative Manifolds and their Symmetries in honour of Giovanni Landi. The full collection is available at https://www.emis.de/journals/SIGMA/Landi.html

The way this relation is usually looked at is the one that inspired Nakajima's first pioneering work: the philosophy is to use the algebraic data we get on one side (usually called *ADHM data*) to parameterize the geometric moduli spaces we have on the other side, i.e., the objects we are actually interested in (see for example [4, 6, 14]). But sometimes it may be useful to switch roles and use the geometric interpretation as a "tool" to prove something interesting per se on the algebraic side. For instance, this is the case when one deals with irreducibility problems: to determine whether a variety of matrices is irreducible is known to be a challenging problem (see [15] and references therein), and in the specific case of Nakajima's quiver varieties the conclusive result by Crawley-Boevey stating that *all of them* are indeed irreducible has been achieved only by using hyperkähler geometry techniques [5].

In [1] we introduced a collection of new quiver varieties $\mathcal{M}(\Lambda_n, \vec{v}_c, w_c, \vartheta_c)$, $n \geq 1$ (see below for the notation); for $n \neq 2$ they are not Nakajima's quiver varieties, as the quivers involved are not doubles. We proved that $\mathcal{M}(\Lambda_1, \vec{v}_c, w_c, \vartheta_c)$ is isomorphic to the Hilbert scheme of points of the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$, and, in particular, that it is therefore irreducible (as the Hilbert scheme is so [7]). For $n \geq 2$ we only proved a weaker result, i.e., that only a certain connected component of $\mathcal{M}(\Lambda_n, \vec{v}_c, w_c, \vartheta_c)$ can be identified with $\operatorname{Hilb}^c(\operatorname{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)))$. However, as $\mathcal{M}(\Lambda_2, \vec{v}_c, w_c, \vartheta_c)$ is a Nakajima quiver variety, its irreducibility follows from Crawley-Boevey's result, so that one only has to determine whether the varieties $\mathcal{M}(\Lambda_n, \vec{v}_c, w_c, \vartheta_c)$ are irreducible for $n \geq 3$. In this paper we prove this fact, completing the work of [1], actually showing directly that $\operatorname{Hilb}^c(\operatorname{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)))$ is isomorphic to the whole $\mathcal{M}(\Lambda_n, \vec{v}_c, w_c, \vartheta_c)$. As this technique also works for the case $n = 2$ we include it as well.

2 Some background

The quivers we are going to consider are extracted from the ADHM data for the Hilbert schemes of points of the varieties $\operatorname{Hilb}^c(X_n)$, where X_n is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$, and, in turn, the construction of the ADHM data is based on the description of the moduli spaces of framed sheaves on the Hirzebruch surfaces Σ_n in terms of monads that was given in [2]. We denote by H and E the classes in $\operatorname{Pic}(\Sigma_n)$ of the sections of the natural ruling $\Sigma_n \to \mathbb{P}^1$ that square to n and $-n$, respectively. We fix a curve ℓ_∞ in Σ_n belonging to the class H (the "line at infinity"). A framed sheaf on Σ_n is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a rank r torsion-free sheaf which is trivial along ℓ_∞, and $\theta \colon \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ is an isomorphism. A morphism between framed sheaves $(\mathcal{E}, \theta)$, $(\mathcal{E}', \theta')$ is by definition a morphism $\Lambda \colon \mathcal{E} \longrightarrow \mathcal{E}'$ such that $\theta' \circ \Lambda|_{\ell_\infty} = \theta$. The moduli space parameterizing isomorphism classes of framed sheaves $(\mathcal{E}, \theta)$ on Σ_n with Chern character $\operatorname{ch}(\mathcal{E}) = \left(r, aE, -c - \frac{1}{2}na^2\right)$, where $r, a, c \in \mathbb{Z}$ and $r \geq 1$, will be denoted $\mathcal{M}^n(r, a, c)$. We normalize the framed sheaves so that $0 \leq a \leq r - 1$.

A monad M on a scheme X is a three-term complex of locally free $\mathcal{O}_X$-modules of finite rank, having nontrivial cohomology only at the middle term (cf. [16, Definition II.3.1.1]). It was proved in [2] that a framed sheaf $(\mathcal{E}, \theta)$ on Σ_n with invariants (r, a, c) is the cohomology of a monad

$$M(\alpha, \beta)\colon \quad 0 \longrightarrow \mathcal{U}_{\vec{k}} \xrightarrow{\alpha} \mathcal{V}_{\vec{k}} \xrightarrow{\beta} \mathcal{W}_{\vec{k}} \longrightarrow 0\,, \tag{2.1}$$

where $\vec{k}$ is the quadruple (n, r, a, c), and

$$\mathcal{U}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(0, -1)^{\oplus k_1}, \qquad \mathcal{V}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(1, -1)^{\oplus k_2} \oplus \mathcal{O}_{\Sigma_n}^{\oplus k_4}, \qquad \mathcal{W}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(1, 0)^{\oplus k_3},$$

with

$$k_1 = c + \frac{1}{2}na(a-1), \qquad k_2 = k_1 + na, \qquad k_3 = k_1 + (n-1)a, \qquad k_4 = k_1 + r - a.$$

The space $L_{\vec{k}}$ of pairs in $\operatorname{Hom}(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}) \oplus \operatorname{Hom}(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}})$ fitting into (2.1), such that the cohomology of the complex is torsion-free and trivial at infinity, is a smooth algebraic variety. There is a principal $\mathrm{GL}(r, \mathbb{C})$-bundle $P_{\vec{k}}$ over $L_{\vec{k}}$ whose fibre at a point (α, β) is the space of framings for the corresponding cohomology of (2.1). The algebraic group

$$G_{\vec{k}} = \operatorname{Aut}(\mathcal{U}_{\vec{k}}) \times \operatorname{Aut}(\mathcal{V}_{\vec{k}}) \times \operatorname{Aut}(\mathcal{W}_{\vec{k}})$$

acts freely on $P_{\vec{k}}$, and the moduli space $\mathcal{M}^n(r, a, c)$ is the quotient $P_{\vec{k}}/G_{\vec{k}}$ [2, Theorem 3.4]. This is nonempty if and only if $c + \frac{1}{2}na(a-1) \geq 0$, and when nonempty, it is a smooth algebraic variety of dimension $2rc + (r-1)na^2$.

When $r = 1$ we can assume $a = 0$, and there is an identification

$$\mathcal{M}^n(1, 0, c) \simeq \operatorname{Hilb}^c(\Sigma_n \setminus \ell_\infty) = \operatorname{Hilb}^c(X_n).$$

A first step to construct ADHM data for the Hilbert schemes of points of the varieties X_n is to show that the Hilbert schemes can be covered by open subsets that are isomorphic to the Hilbert scheme of $\mathbb{C}^2$, and therefore have an ADHM description, according to Nakajima. Then one proves that these "local ADHM data" can be glued to provide ADHM data for the Hilbert schemes of X_n.

Let $P^n(c)$ be the set of collections $(A_1, A_2; C_1, \dots, C_n; e)$ in $\operatorname{End}(\mathbb{C}^c)^{\oplus n+2} \oplus \operatorname{Hom}(\mathbb{C}^c, \mathbb{C})$ satisfying the conditions

$$\text{(P1)} \quad \begin{cases} A_1 C_1 A_2 = A_2 C_1 A_1, & & \text{when } n = 1, \\ A_1 C_q = A_2 C_{q+1}, & \text{for} \quad q = 1, \dots, n-1, & \text{when } n > 1; \\ C_q A_1 = C_{q+1} A_2 & & \end{cases}$$

(P2) $A_1 + \lambda A_2$ is a *regular pencil* of matrices, i.e., there exists $[\nu_1, \nu_2] \in \mathbb{P}^1$ such that $\det(\nu_1 A_1 + \nu_2 A_2) \neq 0$;

(P3) for all values of the parameters $([\lambda_1, \lambda_2], (\mu_1, \mu_2)) \in \mathbb{P}^1 \times \mathbb{C}^2$ satisfying

$$\lambda_1^n \mu_1 + \lambda_2^n \mu_2 = 0$$

there is no nonzero vector $v \in \mathbb{C}^c$ such that

$$\begin{cases} C_1 A_2 v = -\mu_1 v, \\ C_n A_1 v = (-1)^n \mu_2 v, \\ v \in \ker e \end{cases} \qquad \text{and} \qquad (\lambda_2 A_1 + \lambda_1 A_2)\, v = 0.$$

The group $\mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C})$ acts on $P^n(c)$ according to

$$(A_i, C_j, e) \mapsto \left(\phi_2 A_i \phi_1^{-1}, \phi_1 C_j \phi_2^{-1}, e \phi_1^{-1}\right)$$

for $i = 1, 2$, $j = 1, \dots, n$, $(\phi_1, \phi_2) \in \mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C})$.

The following result expresses the fact that the collections $(A_1, A_2; C_1, \dots, C_n; e)$ satisfying conditions (P1) to (P3) are ADHM data for the varieties $\operatorname{Hilb}^c(X_n)$ (this is Theorem 3.1 in [2]).

Theorem 2.1. *$P^n(c)$ is a principal $\mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C})$-bundle over $\operatorname{Hilb}^c(X_n)$.*

3 The main result

Now we turn to the purpose of this paper, namely, proving that the Hilbert schemes of points of the varieties X_n are isomorphic to moduli spaces of representations of suitable quivers. For

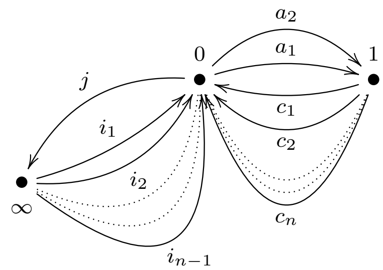


Figure 1. The quivers Q_n.

any $n \geq 2$ let Q_n be the framed quiver in Fig. 1, where ∞ is the framing vertex. Let J_n be the two sided ideal of $\mathbb{C}Q_n$ generated by the relations

$$\begin{cases} a_2 c_{q+1} - a_1 c_q = 0, \\ c_{q+1} a_2 - c_q a_1 - i_q j = 0 \end{cases} \qquad \text{for} \quad q = 1, \ldots, n-1. \tag{3.1}$$

Our purpose is to describe the spaces of representations of the quiver Q_n with relations J_n, i.e., the spaces of representations of the quotient algebra $\Lambda_n = \mathbb{C}Q_n / J_n$.

We recall some basic definitions. Given $\vec{v} = (v_0, v_1) \in \mathbb{N}^2$ and $w \in \mathbb{N}$, a $(\vec{v}, w)$-*dimensional representation of* Λ_n is the datum of a triple of $\mathbb{C}$-vector spaces V_0, V_1, W, with $\dim V_i = v_i$, $\dim W = w$, and of an element $(A_1, A_2; C_1, \ldots, C_n; e; f_1, \ldots, f_{n-1})$ in

$$\mathrm{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 2} \oplus \mathrm{Hom}_{\mathbb{C}}(V_1, V_0)^{\oplus n} \oplus \mathrm{Hom}_{\mathbb{C}}(V_0, W) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V_0)^{\oplus n-1}$$

satisfying the relations determined by equations (3.1), namely

$$\begin{cases} A_2 C_{q+1} - A_1 C_q = 0, \\ C_{q+1} A_2 - C_q A_1 - f_q e = 0 \end{cases} \qquad \text{for} \quad q = 1, \ldots, n-1. \tag{Q1}$$

The space $\mathrm{Rep}(\Lambda_n, \vec{v}, w)$ of all $(\vec{v}, w)$-dimensional representations of Λ_n is an affine variety, on which the group $G_{\vec{v}} = \mathrm{GL}(v_0, \mathbb{C}) \times \mathrm{GL}(v_1, \mathbb{C})$ acts by basis change. Indeed, we ignore the action of $\mathrm{GL}(w, \mathbb{C})$ on the vector space W attached to the framing vertex. As usual, to get a well behaved quotient space one has to perform a GIT construction by introducing a suitable notion of stability. This was done by A. King [9] and, in a slightly different way, by A. Rudakov [17]. In the case of a quiver with a framing vertex, the following definition can be shown to be equivalent to the King–Rudakov one [3, 5].

Definition 3.1. Fix $\vartheta \in \mathbb{R}^2$. A $(\vec{v}, w)$-dimensional representation (V_0, V_1, W) of Λ_n is said to be ϑ-*semistable* if, for any subrepresentation $S = (S_0, S_1) \subseteq (V_0, V_1)$, one has:

$$\text{if } S_0 \subseteq \ker e, \text{ then } \vartheta \cdot (\dim S_0, \dim S_1) \leq 0; \tag{3.2}$$

$$\text{if } S_0 \supseteq \mathrm{Im}\, f_i \text{ for } i = 1, \ldots, n-1, \text{ then } \vartheta \cdot (\dim S_0, \dim S_1) \leq \vartheta \cdot (v_0, v_1). \tag{3.3}$$

A ϑ-semistable representation is ϑ-*stable* if a strict inequality holds in (3.2) whenever $S \neq 0$ and in (3.3) whenever $S \neq (V_0, V_1)$.

Let $\mathrm{Rep}(\Lambda_n, \vec{v}, w)^{\mathrm{ss}}_{\vartheta}$ be the subset of $\mathrm{Rep}(\Lambda_n, \vec{v}, w)$ consisting of ϑ-semistable representations. By [9, Proposition 5.2], the coarse moduli space of $(\vec{v}, w)$-dimensional ϑ-semistable representations of Λ_n is the GIT quotient

$$\mathcal{M}(\Lambda_n, \vec{v}, w, \vartheta) = \mathrm{Rep}(\Lambda_n, \vec{v}, w)^{\mathrm{ss}}_{\vartheta} /\!/ G_{\vec{v}}.$$

It can be proved that the open subset $\mathcal{M}^{\mathrm{s}}(\Lambda_n, \vec{v}, w, \vartheta) \subset \mathcal{M}(\Lambda_n, \vec{v}, w, \vartheta)$ consisting of stable representations makes up a fine moduli space. Notice that, for quivers without a framing, this holds only when the dimension vector is primitive [9, Proposition 5.3], whilst this requirement is not necessary in the case of framed quivers [5]. Theorem 4.5 of [1] states that the Hilbert scheme of points $\mathrm{Hilb}^c(X_n)$ can be embedded into $\mathcal{M}(\Lambda_n, \vec{v}, w, \vartheta)$ for suitable choices of $\vec{v}$, w, and ϑ. Precisely, one has the following result:

Theorem 3.2. *For every $n \geq 2$ and $c \geq 1$ let*

$$\vec{v}_c = (c, c), \qquad w_c = 1, \qquad \vartheta_c = (2c, 1 - 2c),$$

and let $\mathcal{H}(n, c)$ be the irreducible component of $\mathcal{M}(\Lambda_n, \vec{v}_c, 1, \vartheta_c)$ given by the equations

$$f_1 = f_2 = \cdots = f_{n-1} = 0. \tag{3.4}$$

Then $\mathrm{Hilb}^c(X_n) \simeq \mathcal{H}(n, c)$.

Let $\mathrm{pr}\colon \mathrm{Rep}(\Lambda_n, \vec{v}_c, 1)^{\mathrm{ss}}_{\vartheta_c} \to \mathcal{M}(\Lambda_n, \vec{v}_c, 1, \vartheta_c)$ be the quotient map. The proof of Theorem 3.2 basically consists in proving that the counterimage $\mathrm{pr}^{-1}(\mathcal{H}(n, c)) =: Z_n(c)$ coincides with the total space of the principal fibration $P^n(c)$ we introduced in Section 2. As it is quite involved and requires a few intermediate Lemmas and Propositions, we refer the reader to [1] for further details. Here we only note that the starting point is given by the stability conditions in Definition 3.1.

Remark 3.3. The set of $(\vec{v}_c, w_c)$-dimensional representations of Λ_n which are semistable according to Definition 3.1 does not change if we let the stability parameter vary inside the open cone

$$\Gamma_c = \left\{ \vartheta = (\vartheta_0, \vartheta_1) \in \mathbb{R}^2 \,|\, \vartheta_0 > 0,\, -\vartheta_0 < \vartheta_1 < -\frac{c-1}{c}\vartheta_0 \right\}.$$

It can be shown that for any stability parameter $\bar{\vartheta}$ on the open rays

$$R_1 = \left\{ (\vartheta_0, \vartheta_1) \in \mathbb{R}^2 \,|\, \vartheta_0 > 0,\, \vartheta_0 + \vartheta_1 = 0 \right\},$$
$$R_2 = \left\{ (\vartheta_0, \vartheta_1) \in \mathbb{R}^2 \,|\, \vartheta_0 > 0,\, (c-1)\vartheta_0 + c\vartheta_1 = 0 \right\}$$

there exist representations which are $\bar{\vartheta}$-semistable, but not ϑ_c-semistable. So, Γ_c is a chamber in the space $\mathbb{R}^2_{(\vartheta_0, \vartheta_1)}$ of stability parameters and the closed rays $\overline{R_1}$, $\overline{R_2}$ are its walls. Furthermore, inside Γ_c semistability and stability are equivalent (cf. [1, Lemma 4.7]): in particular, points in $\mathcal{M}(\Lambda_n, \vec{v}_c, 1, \vartheta_c)$ can be thought of as $G_{\vec{v}_c}$-orbits of representations in $\mathrm{Rep}(\Lambda_n, \vec{v}_c, 1)$.

A full description of the chamber/wall decomposition of the space $\mathbb{R}^2_{(\vartheta_0, \vartheta_1)}$ will be the object of a future work.

We wish to prove that the component $\mathcal{H}(n, c)$ of $\mathcal{M}(\Lambda_n, \vec{v}_c, 1, \vartheta_c)$ introduced in Theorem 3.2 coincides with the whole moduli space $\mathcal{M}(\Lambda_n, \vec{v}_c, 1, \vartheta_c)$ (this will be Theorem 3.8). Let us introduce the following notation

$$\mathcal{R}(\Lambda_n, c) = \mathrm{Rep}(\Lambda_n, \vec{v}_c, 1); \qquad \mathcal{R}^{\mathrm{ss}}(\Lambda_n, c) = \mathrm{Rep}(\Lambda_n, \vec{v}_c, 1)^{\mathrm{ss}}_{\vartheta_c}.$$

Given a representation $(A_1, A_2; C_1, \ldots, C_n; e; f_1, \ldots, f_{n-1}) \in \mathcal{R}(\Lambda_n, c)$, we form the pencil $A_1 + \lambda A_2$, with $\lambda \in \mathbb{C}$. We recall that a pencil $A_1 + \lambda A_2$ is *regular* if there is a point $[\nu_1, \nu_2] \in \mathbb{P}^1$ such that $\det(\nu_1 A_1 + \nu_2 A_2) \neq 0$.

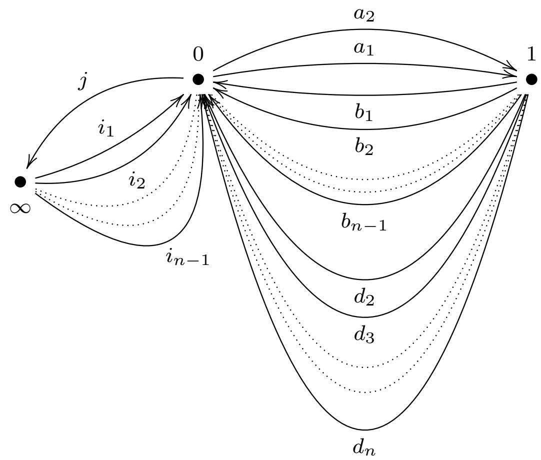


Figure 2. The quivers Q'_n for $n \geq 3$.

To prove Theorem 3.8 it is convenient to introduce "augmented" framed quivers defined as follows: let $Q'_2 = Q_2$, and, for every $n \geq 3$, let Q'_n be the framed quiver in Fig. 2. Let $J'_2 = J_2$ and, for all $n \geq 3$, let J'_n be the two sided ideal of $\mathbb{C}Q'_n$ generated by the relations

$$\begin{cases} a_2 d_{q+1} - a_1 b_q = 0, \\ d_{q+1} a_2 - b_q a_1 - i_q j = 0 \end{cases} \qquad \text{for} \quad q = 1, \dots, n-1. \tag{3.5}$$

We set $\Lambda'_n = \mathbb{C}Q'_n / J'_n$ for all $n \geq 2$. Notice that $\Lambda'_2 = \Lambda_2$; for $n \geq 3$, the algebra Λ_n can be obtained by taking the quotient of Λ'_n by a suitable ideal. Indeed, let K_n be the two sided ideal of Λ'_n generated by the relations

$$\bar{b}_q = \bar{d}_q \qquad \text{for} \quad q = 2, \dots, n-1, \tag{3.6}$$

where $\bar{x}$ is the class in Λ'_n of the element $x \in \mathbb{C}Q'_n$. Let $\tilde{p}_n \colon \mathbb{C}Q'_n \longrightarrow \mathbb{C}Q_n$ be the $\mathbb{C}$-algebra morphism determined by the assignments

$$\tilde{p}_n(a_q) = a_q, \qquad \tilde{p}_n(b_q) = c_q, \qquad \tilde{p}_n(d_q) = c_q, \qquad \tilde{p}_n(j) = j, \qquad \tilde{p}_n(i_q) = i_q. \tag{3.7}$$

It is straightforward that $\tilde{p}_n$ is surjective and that its kernel is the two sided ideal $L_n \subset \mathbb{C}Q'_n$ generated by the relations

$$b_q = d_q \qquad \text{for} \quad q = 2, \dots, n-1. \tag{3.8}$$

It follows directly from equation (3.7) that $\tilde{p}_n$ maps the set of generators of J'_n (see equation (3.5)) onto the set of generators of J_n (see equation (3.1)), so that

$$\tilde{p}_n(J'_n) = J_n.$$

Then it is not hard to check that $\tilde{p}_n$ induces a surjective morphism $p_n \colon \Lambda'_n \to \Lambda_n$, whose kernel, by equations (3.6) and (3.8), is

$$\ker p_n = L_n / (L_n \cap J'_n) = K_n.$$

In conclusion, we have proved the following lemma.

Lemma 3.4. *There is an isomorphism of $\mathbb{C}$-algebras $\Lambda'_n / K_n \simeq \Lambda_n$.*

One of the reasons to introduce the augmented quivers Q'_n is that their path algebras carry an action of the group $\mathrm{SO}(2,\mathbb{C})$ which descends to the quotient algebra Λ'_n. This action will be instrumental in proving the regularity of the pencil $A_1+\lambda A_2$.

Elements of $\mathrm{SO}(2,\mathbb{C})$ will be denoted by $\nu = \left(\begin{smallmatrix}\nu_1 & \nu_2\\ -\nu_2 & \nu_1\end{smallmatrix}\right)$. Given arrows

$$(a_1,a_2;b_1,\dots,b_{n-1};d_2,\dots,d_n;j;i_1,\dots,i_{n-1})$$

as above and $\nu\in\mathrm{SO}(2,\mathbb{C})$, we set

$$\begin{pmatrix}a'_1\\ a'_2\end{pmatrix} = \nu\begin{pmatrix}a_1\\ a_2\end{pmatrix},\qquad \begin{pmatrix}b'_q\\ d'_{q+1}\end{pmatrix} = \nu^{-1}\begin{pmatrix}b_q\\ d_{q+1}\end{pmatrix}\qquad\text{for}\quad q=1,\dots,n-1.$$

The assignment

$$\begin{aligned}&(a_1,a_2;b_1,\dots,b_{n-1};d_2,\dots,d_n;j;i_1,\dots,i_{n-1})\\ &\qquad\longmapsto (a'_1,a'_2;b'_1,\dots,b'_{n-1};d'_2,\dots,d'_n;j;i_1,\dots,i_{n-1}),\end{aligned}$$

induces an action

$$\widetilde{\Phi}_n\colon\ \mathrm{SO}(2,\mathbb{C})\to \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}Q'_n),$$

which leaves invariant the generators of the ideal J'_n, that is,

$$\widetilde{\Phi}_n(\nu)\big(J'_n\big) = J'_n.$$

So one has an induced action

$$\Phi_n\colon\ \mathrm{SO}(2,\mathbb{C})\to \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\Lambda'_n).$$

We wish now to study the space $\mathrm{Rep}(\Lambda'_n,\vec{v}_c,1) = \mathcal{R}(\Lambda'_n,c)$ of (c,c)-dimensional framed representations of Λ'_n and its open subset $\mathrm{Rep}(\Lambda'_n,\vec{v}_c,1)^{\mathrm{ss}}_{\vartheta_c} = \mathcal{R}^{\mathrm{ss}}(\Lambda'_n,c)$ of ϑ_c-semistable representations (defined analogously to Definition 3.1). For $n=2$ there is nothing new, since $\mathcal{R}(\Lambda'_2,c)=\mathcal{R}(\Lambda_2,c)$ and $\mathcal{R}^{\mathrm{ss}}(\Lambda'_2,c)=\mathcal{R}^{\mathrm{ss}}(\Lambda_2,c)$. For $n\geq 3$, $\mathcal{R}(\Lambda'_n,c)$ is the affine subvariety of the vector space

$$\mathrm{Hom}_{\mathbb{C}}(V_0,V_1)^{\oplus 2}\oplus \mathrm{Hom}_{\mathbb{C}}(V_1,V_0)^{\oplus 2n-2}\oplus \mathrm{Hom}_{\mathbb{C}}(V_0,W)\oplus \mathrm{Hom}_{\mathbb{C}}(W,V_0)^{\oplus n-1}$$

whose points $(A_1,A_2;B_1,\dots,B_{n-1};D_2,\dots,D_n,e;f_1,\dots,f_{n-1})$ satisfy the relations determined by equations (3.5), namely,

$$\begin{cases}A_2D_{q+1} = A_1B_q,\\ D_{q+1}A_2 = B_qA_1 + f_qe\end{cases}\qquad\text{for}\quad q=1,\dots,n-1. \tag{Q1$'$}$$

Lemma 3.5. *$\mathcal{R}^{\mathrm{ss}}(\Lambda'_n,c)$ is the open subset of $\mathcal{R}(\Lambda'_n,c)$ determined by the conditions:*

(Q2$'$) *for all subrepresentations $S=(S_0,S_1)$ such that $S_0\subseteq\ker e$, one has $\dim S_0\leq\dim S_1$, and, if $\dim S_0=\dim S_1$, then $S=0$;*

(Q3$'$) *for all subrepresentations $S=(S_0,S_1)$ such that $S_0\supseteq\mathrm{Im}\,f_i$, for $i=1,\dots,n-1$, one has $\dim S_0\leq\dim S_1$.*

Proof. Given a subrepresentation (S_0,S_1), we set $s_i=\dim S_i$, $i=0,1$. By substituting the definitions of $\vec{v}_c$ and ϑ_c given in Theorem 3.2 into equations (3.2) and (3.3) one gets

$$\text{if } S_0\subseteq\ker e,\text{ then } s_0\leq s_1-\frac{s_1}{2c}; \tag{3.9}$$

$$\text{if } S_0\supseteq\mathrm{Im}\,f_i \text{ for } i=1,\dots,n-1,\text{ then } s_0\leq s_1+\frac{1}{2}-\frac{s_1}{2c}. \tag{3.10}$$

Whenever $s_1>0$, one has $0<\frac{s_1}{2c}<1$; hence, equation (3.9) is equivalent to condition (Q2$'$). On the other hand, as $0\leq\frac{1}{2}-\frac{s_1}{2c}<\frac{1}{2}$, equation (3.10) is equivalent to condition (Q3$'$). ∎

Proposition 3.6. *For each point of $\mathcal{R}^{\rm ss}(\Lambda'_n, c)$ the associated matrix pencil $A_1 + \lambda A_2$ is regular.*

Proof. Let $(A_1, A_2; B_1, \dots, B_{n-1}; D_2, \dots, D_n, e; f_1, \dots, f_{n-1})$ be a point of $\mathcal{R}^{\rm ss}(\Lambda'_n, c)$, and assume that $A_1 + \lambda A_2$ is singular. If $c = 1$, then $A_1 + \lambda A_2$ is singular if and only if $A_1 = A_2 = 0$. But this implies the subrepresentation $(V_0, 0)$ does not satisfy condition (Q3′). Hence we can assume $c \geq 2$. The fact that the pencil $A_1 + \lambda A_2$ is singular implies that there is a nontrivial element

$$v(\lambda) = \sum_{\alpha=0}^{\varepsilon} (-\lambda)^\alpha v_\alpha \in V_0 \otimes_{\mathbb{C}} \mathbb{C}[\lambda] \tag{3.11}$$

such that

$$(A_1 + \lambda A_2) v(\lambda) = 0 \qquad \text{for all } \lambda \in \mathbb{C}. \tag{3.12}$$

By arguing as in the proof of [1, Lemma 4.11], one can show that the minimal degree polynomial solution $v(\lambda)$ for the pencil $A_1 + \lambda A_2$ has necessarily degree $\varepsilon > 0$. Let us inductively define the vector spaces $\{U_i\}_{i \in \mathbb{N}}$ as follows:

$$\begin{cases} U_0 = \langle v_0, \dots, v_\varepsilon \rangle, \\ U_{2k+1} = A_1(U_{2k}) + A_2(U_{2k}) & \text{for} \quad k \geq 0, \\ U_{2k} = \displaystyle\sum_{q=1}^{n-1} B_q(U_{2k-1}) + \sum_{q=2}^{n} D_q(U_{2k-1}) & \text{for} \quad k \geq 1. \end{cases}$$

Note that each U_j, with j even, is a subspace of V_0, while each U_j, with j odd, is a subspace of V_1. So, if we introduce the subspaces

$$S_0 = \sum_{k=0}^{\infty} U_{2k} \subset V_0, \qquad S_1 = \sum_{k=0}^{\infty} U_{2k+1} \subset V_1,$$

it follows that (S_0, S_1) is a subrepresentation of (V_0, V_1). We will show that this subrepresentation fails to satisfy either condition (Q2′) or condition (Q3′) of Lemma 3.5, so that one gets a contradiction.

By substituting equation (3.11) into equation (3.12) one finds out that

$$\begin{cases} A_1 v_0 = 0, \\ A_1 v_\alpha = A_2 v_{\alpha-1}, & \alpha = 1, \dots, \varepsilon, \\ A_2 v_\varepsilon = 0, \end{cases} \tag{3.13}$$

so that

$$U_1 = \langle A_1 v_1, \dots, A_1 v_\varepsilon \rangle = A_1(U_0). \tag{3.14}$$

There are two possible cases, either i) $U_0 \subseteq \ker e$, or ii) $U_0 \not\subseteq \ker e$.

i) If we suppose that $U_0 \subseteq \ker e$, equation (3.13) and condition (Q1′) imply that

$$U_2 = \sum_{q=2}^{n} D_q A_1(U_0).$$

By letting $w_{q,\alpha} = D_q A_1 v_\alpha$, $\alpha = 1, \dots, \varepsilon$, for each $q = 2, \dots, n$ we obtain an element

$$(w_{q,1}, \dots, w_{q,\varepsilon}) \in U_2^{\oplus \varepsilon}$$

such that $\sum_{\alpha=0}^{\varepsilon-1}(-\lambda)^\alpha w_{q,\alpha+1}$ is a polynomial solution for the pencil $A_1+\lambda A_2$ of degree $\varepsilon-1$. Since we have supposed ε to be minimal, one has $(w_{q,1},\dots,w_{q,\varepsilon})=0$. From that it is easy to deduce that $U_2=0$ and that $(S_0,S_1)=(U_0,U_1)$. So, since $\ker A_1\cap U_0\neq 0$ by equation (3.13), then equation (3.14) entails that (S_0,S_1) is a subrepresentation violating condition (Q2′).

ii) Suppose now that U_0 is not contained in $\ker e$. So, there is at least one $\gamma\in\{0,\dots,\varepsilon\}$ such that $e(v_\gamma)\neq 0$. Condition (Q1′) implies that

$$\operatorname{Im} f_q=\langle f_q e(v_\gamma)\rangle\subseteq U_2 \qquad \text{for all} \quad q=1,\dots,n-1. \tag{3.15}$$

To simplify computations, we may assume $\gamma=0$ and $e(v_0)=1$. Actually, one checks that the $\mathrm{SO}(2,\mathbb{C})$ action on Λ_n' induces an action on $\mathcal{R}(\Lambda_n',c)$, which commutes with the $G_{\vec{v}_c}$ action defined on the same space, and therefore it restricts to an $\mathrm{SO}(2,\mathbb{C})$ action on $\mathcal{R}^{\mathrm{ss}}(\Lambda_n',c)$. Moreover, this action preserves the regularity of the matrix pencil $A_1+\lambda A_2$. An element $\nu=\left(\begin{smallmatrix}\nu_1 & \nu_2\\ -\nu_2 & \nu_1\end{smallmatrix}\right)\in\mathrm{SO}(2,\mathbb{C})$ produces a change of basis

$$(v_0,\dots,v_\varepsilon)\mapsto \nu\cdot(v_0,\dots,v_\varepsilon)=(v_0',\dots,v_\varepsilon'),$$

so that

$$e(v_0')=\sum_{\alpha=0}^{\varepsilon}(-\nu_2)^\alpha\nu_1^{\varepsilon-\alpha}e(v_\alpha).$$

Since $(e(v_0),\dots,e(v_\varepsilon))\neq(0,\dots,0)$, there is $\nu\in\mathrm{SO}(2,\mathbb{C})$ so that $e(v_0')\neq 0$. Moreover, $e(v_0')$ can be assumed to be 1.

Next, by using condition (Q1′) and equation (3.13), along with the identity $A_2(\operatorname{Im} f_q)=\langle A_2 f_q(1)\rangle=\langle A_2 f_q(e(v_0))\rangle$, it is not hard to show that

$$A_2(\operatorname{Im} f_q)\subseteq A_1(U_2) \qquad \text{for all} \quad q=1,\dots,n-1. \tag{3.16}$$

Now we show that

$$U_{2k+1}\subseteq\sum_{l=1}^{k}A_1(U_{2l}) \tag{3.17}$$

for all $k\geq 1$. Assume $k=1$. By using equations (3.13), (Q1′) and condition $e(v_0)=1$, one gets

$$f_q e(v_\alpha)=e(v_\alpha)D_{q+1}A_1v_1$$

for $q=1,\dots,n-1$. Hence, by using equations (3.13) and (Q1′) again one shows that

$$U_2=\sum_{q=2}^{n}D_qA_1(U_0).$$

Then U_3 is spanned by the sets of vectors

$$\{A_1D_qA_1v_\alpha\}_{\substack{q=2,\dots,n\\ \alpha=1,\dots,\varepsilon}}\subseteq A_1(U_2), \qquad \{A_2D_qA_1v_\alpha\}_{\substack{q=2,\dots,n\\ \alpha=1,\dots,\varepsilon}}\subseteq A_2(U_2)$$

and it follows directly from equations (Q1′) that $A_2D_qA_1v_\alpha\in A_1(U_2)$, for $q=2,\dots,n$. So $U_3\subseteq A_1(U_2)$.

Let us now suppose that equation (3.17) holds true for $1\leq k\leq m$, with $m\geq 1$. This means that U_{2m+1} is spanned by vectors of the form A_1w with $w\in U_{2l}$, $l=1,\dots,m$. By noticing that

U_{2m+2} is spanned by vectors of the form B_pA_1w and D_qA_1w', with $w \in U_{2l}$ and $w' \in U_{2l'}$ for $l, l' = 1, \dots, m$, and by using equation (Q1′) and the inductive hypothesis one finds out that

$$U_{2m+2} \subseteq \sum_{l=1}^{m}\sum_{q=2}^{n} D_qA_1(U_{2l}) + \sum_{q=1}^{n-1} \operatorname{Im} f_q.$$

From this it follows

$$\begin{aligned} A_1(U_{2m+2}) + A_2(U_{2m+2}) = U_{2m+3} & \\ &\subseteq A_1(U_{2(m+1)}) + \sum_{l=1}^{m}\sum_{q=2}^{n} A_2D_qA_1(U_{2l}) + \sum_{q=1}^{n-1} A_2(\operatorname{Im} f_q) \\ &\subseteq A_1(U_{2(m+1)}) + A_1(U_2) + \sum_{l=1}^{m}\sum_{q=2}^{n} A_2D_qA_1(U_{2l}), \end{aligned} \tag{3.18}$$

where in the last step equation (3.16) has been used. For $q = 2, \dots, n$, equation (Q1′) implies that

$$A_2D_qA_1(U_{2l}) \subseteq A_1(U_{2(l+1)}).$$

Thus, from equation (3.18) we may conclude that

$$U_{2(m+1)+1} \subseteq \sum_{l=1}^{m+1} A_1(U_{2l}),$$

so that the inclusion (3.17) is proved.

This and equation (3.14) imply that

$$S_1 = \sum_{k=0}^{\infty} U_{2k+1} \subseteq A_1(U_0) + \sum_{k=1}^{\infty}\sum_{l=1}^{k} A_1(U_{2l}) = \sum_{k=0}^{\infty} A_1(U_{2k}) = A_1\left(\sum_{k=0}^{\infty} U_{2k}\right) = A_1(S_0).$$

But $A_1(S_0) \subseteq S_1$, so that

$$S_1 = A_1(S_0).$$

By equation (3.13), one has $\ker A_1 \cap S_0 \neq 0$, and therefore $\dim S_1 < \dim S_0$. Finally, equation (3.15) implies that the subrepresentation (S_0, S_1) violates condition (Q3′). ■

When $n \geq 3$, there is a map $\mathcal{R}(\Lambda_n, c) \longrightarrow \mathcal{R}(\Lambda'_n, c)$ given by

$$(A_1, A_2; C_1, \dots, C_n; e; f_1, \dots, f_{n-1}) \mapsto (A_1, A_2; C_1, \dots, C_{n-1}; C_2, \dots, C_n, e; f_1, \dots, f_{n-1}).$$

This map provides a $G_{\vec{v}_c}$-equivariant isomorphism of $\mathcal{R}(\Lambda_n, c)$ onto the subvariety of $\mathcal{R}(\Lambda'_n, c)$ cut by the equations

$$B_q = D_q \qquad \text{for} \quad q = 2, \dots, n-1$$

(cf. equations (3.6)). Through this isomorphism $\mathcal{R}(\Lambda_n, c)$ may be regarded as a closed subvariety of $\mathcal{R}(\Lambda'_n, c)$.

Lemma 3.7. *When $n \geq 3$, one has that*

$$\mathcal{R}^{\mathrm{ss}}(\Lambda_n, c) = \mathcal{R}^{\mathrm{ss}}(\Lambda'_n, c) \cap \mathcal{R}(\Lambda_n, c).$$

Proof. Semistability is a numerical condition which is to be checked on the set of all submodules of a given representation. Hence, it is enough to show that for any left Λ_n-module M, an abelian subgroup $N \subset M$ is a left Λ_n-submodule if and only if it is a left Λ'_n-submodule (notice that M has also a natural structure of left Λ'_n-module, induced by restriction of scalars; cf. Lemma 3.4). However, precisely because the algebra Λ_n is a quotient of Λ'_n, the category Λ_n-**mod** is a full subcategory of Λ'_n-**mod**, and this implies in particular that the set of all subobjects of a given Λ_n-module is the same in the two categories. ∎

Theorem 3.8. *The component $\mathcal{H}(n, c)$ of the moduli space $\mathcal{M}(\Lambda_n, \vec{v}_c, 1, \vartheta_c)$ defined by equations (3.4) coincides with the whole of $\mathcal{M}(\Lambda_n, \vec{v}_c, 1, \vartheta_c)$.*

Proof. For each representation $(A_1, A_2; C_1, \ldots, C_n; e; f_1, \ldots, f_{n-1}) \in \mathcal{R}^{\mathrm{ss}}(\Lambda_n, c)$, equations (3.4) hold if and only if the pencil $A_1 + \lambda A_2$ is regular (condition (P2) in Section 2 and in [1]):

- in the proof of Proposition 4.9 of [1] it has been shown that condition (P2) holds in $Z_n(c) = \mathrm{pr}^{-1}(\mathcal{H}(n, c))$; i.e., equations (3.4) imply the regularity of the pencil;
- further on, in the proof of Theorem 4.5 of [1] it has been shown that $Z_n(c)$ actually coincides with the open subset of $\mathcal{R}^{\mathrm{ss}}(\Lambda_n, c)$ (denoted $\mathcal{R}_n(c)$ in [1]) where condition (P2) is satisfied; i.e., the regularity of the pencil implies equations (3.4).

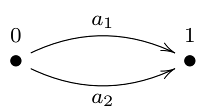
So, the orbit of a ϑ_c-semistable representation

$$(A_1, A_2; C_1, \ldots, C_n; e; f_1, \ldots, f_{n-1}) \in \mathcal{R}^{\mathrm{ss}}(\Lambda_n, c)$$

lies in $\mathcal{H}(n, c)$ if and only if the pencil $A_1 + \lambda A_2$ is regular. Then the conclusion follows from Proposition 3.6 and Lemma 3.7. ∎

4 A remark involving the 2-Kronecker quiver

We want to rephrase Proposition 3.6 is a slightly different way which involves the Kronecker quiver with two arrows Q_K



The new claim, Proposition 4.2, may be regarded as a statement in relative Geometric Invariant Theory.

The vector space of $\vec{v}_c = (c, c)$-dimensional representations of Q_K is the space $\mathrm{Rep}(Q_K, \vec{v}_c) = \mathrm{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 2}$. Since Definition 3.1 only applies to framed quivers, we need a slightly different notion of semistability. So we recall from [9, 17] that, given $\vartheta \in \mathbb{R}^2$, a $\vec{v}_c$-dimensional representation of Q_K is said to be ϑ-*semistable* if, for any proper nontrivial subrepresentation supported by $(S_0, S_1) \subseteq (V_0, V_1)$, one has

$$\frac{\vartheta \cdot (\dim S_0, \dim S_1)}{\dim S_0 + \dim S_1} \leq \frac{\vartheta \cdot \vec{v}_c}{2c}. \tag{4.1}$$

A ϑ-semistable representation is ϑ-*stable* if strict inequality holds in (4.1).

As in Section 3, we set $\vartheta_c = (2c, 1 - 2c)$.

Lemma 4.1. *A point $(A_1, A_2) \in \mathrm{Rep}(Q_K, \vec{v}_c)$ is ϑ_c-semistable if and only if the matrix pencil $A_1 + \lambda A_2$ is regular.*

Proof. Let (A_1, A_2) be a representation of Q_K supported by the pair of vector spaces (V_0, V_1), and consider a proper subrepresentation supported by (S_0, S_1). If the stability parameter is $\vartheta_c = (2c, 1-2c)$, the inequality (4.1) is equivalent to

$$\frac{2c \dim S_0 + (1-2c) \dim S_1}{\dim S_0 + \dim S_1} \leq \frac{1}{2},$$

which is in turn equivalent to

$$\dim S_0 \leq \dim S_1. \tag{4.2}$$

It is not hard to show that (4.2) implies

$$\dim(A_1(S) + A_2(S)) \geq \dim S \quad \text{for all vector subspaces } S \subseteq V_0. \tag{4.3}$$

Conversely, if condition (4.3) is satisfied, then, given any subrepresentation supported by $S = (S_0, S_1)$, one has

$$\dim S_1 \geq \dim(A_1(S_0) + A_2(S_0)) \geq \dim S_0.$$

Finally, by [1, Lemma 4.10] condition (4.3) is equivalent to the fact that the matrix pencil $A_1 + \lambda A_2$ is regular. ■

Recall that $\mathcal{R}(\Lambda_n, c)$ is the affine subvariety of

$$\operatorname{Rep}(Q_n, \vec{v}_c, 1) = \operatorname{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 2} \oplus \operatorname{Hom}_{\mathbb{C}}(V_1, V_0)^{\oplus n} \oplus \operatorname{Hom}_{\mathbb{C}}(V_0, W) \oplus \operatorname{Hom}_{\mathbb{C}}(W, V_0)^{\oplus n-1}$$

defined by equations (Q1). Let us denote by $\pi_n \colon \mathcal{R}(\Lambda_n, c) \to \operatorname{Rep}(Q_K, \vec{v}_c)$ the restriction of the natural projection $\operatorname{Rep}(Q_n, \vec{v}_c, 1) \to \operatorname{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 2} = \operatorname{Rep}(Q_K, \vec{v}_c)$.

As a straightforward consequence of Lemma 4.1, Proposition 3.6 may be rephrased in the following terms.

Proposition 4.2. *Each $(\vec{v}_c, 1)$-dimensional ϑ_c-semistable representation of Λ_n is mapped by π_n to a $\vec{v}_c$-dimensional ϑ_c-semistable representation of Q_K:*

$$\pi_n\big(\mathcal{R}^{\mathrm{ss}}(\Lambda_n, c)\big) \subseteq \operatorname{Rep}(Q_K, \vec{v}_c)^{\mathrm{ss}}_{\vartheta_c}.$$

Acknowledgements

This research was partly supported by INdAM-GNSAGA and PRIN "Geometria delle varietà algebriche". V.L. was supported by the FAPESP Postdoctoral Grants No. 2015/07766-4 (Bolsa no País) and 2017/22052-9 (Bolsa Estágio de Pesquisa no Exterior). This research was partly carried out while he was visiting the Department of Mathematics of the Northeastern University (Boston), and he wishes to thank that institution for hospitality. We thank the referees for their valuable suggestions which allowed us to substantially improve the presentation.

References

[1] Bartocci C., Bruzzo U., Lanza V., Rava C.L.S., Hilbert schemes of points of $\mathcal{O}_{\mathbb{P}^1}(-n)$ as quiver varieties, *J. Pure Appl. Algebra* **221** (2017), 2132–2155, arXiv:1504:02987.

[2] Bartocci C., Bruzzo U., Rava C.L.S., Monads for framed sheaves on Hirzebruch surfaces, *Adv. Geom.* **15** (2015), 55–76, arXiv:1205.3613.

[3] Bartocci C., Lanza V., Rava C.L.S., Moduli spaces of framed sheaves and quiver varieties, *J. Geom. Phys.* **118** (2017), 20–39, arXiv:1610:02731.

[4] Bruzzo U., Fucito F., Morales J.F., Tanzini A., Multi-instanton calculus and equivariant cohomology, *J. High Energy Phys.* **2003** (2003), no. 5, 054, 24 pages, arXiv:hep-th/0211108.

[5] Crawley-Boevey W., Geometry of the moment map for representations of quivers, *Compositio Math.* **126** (2001), 257–293.

[6] Dorey N., Hollowood T.J., Khoze V.V., Mattis M.P., The calculus of many instantons, *Phys. Rep.* **371** (2002), 231–459, arXiv:hep-th/0206063.

[7] Fogarty J., Algebraic families on an algebraic surface, *Amer. J. Math.* **90** (1968), 511–521.

[8] Ginzburg V., Lectures on Nakajima's quiver varieties, in Geometric Methods in Representation Theory. I, *Sémin. Congr.*, Vol. 24, Soc. Math. France, Paris, 2012, 145–219, arXiv:0905:0686.

[9] King A.D., Moduli of representations of finite-dimensional algebras, *Quart. J. Math. Oxford* **45** (1994), 515–530.

[10] Kuznetsov A., Quiver varieties and Hilbert schemes, *Mosc. Math. J.* **7** (2007), 673–697, arXiv:math.AG/0111092.

[11] Nakajima H., Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras, *Duke Math. J.* **76** (1994), 365–416.

[12] Nakajima H., Lectures on Hilbert schemes of points on surfaces, *University Lecture Series*, Vol. 18, Amer. Math. Soc., Providence, RI, 1999.

[13] Nakajima H., Introduction to quiver varieties – for ring and representation theorists, in Proceedings of the 49th Symposium on Ring Theory and Representation Theory, Symp. Ring Theory Represent. Theory Organ. Comm., Shimane, 2017, 96–114, arXiv:1611:10000.

[14] Nekrasov N.A., Seiberg–Witten prepotential from instanton counting, *Adv. Theor. Math. Phys.* **7** (2003), 831–864, arXiv:hep-th/0206161.

[15] Ngo N.V., Šivic K., On varieties of commuting nilpotent matrices, *Linear Algebra Appl.* **452** (2014), 237–262, arXiv:1308:4438.

[16] Okonek C., Schneider M., Spindler H., Vector bundles on complex projective spaces, *Progress in Mathematics*, Vol. 3, Birkhäuser, Boston, Mass., 1980.

[17] Rudakov A., Stability for an abelian category, *J. Algebra* **197** (1997), 231–245.