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Semistable Higgs bundles on elliptic surfaces

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Semistable Higgs bundles on elliptic surfaces

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Abstract. We analyze Higgs bundles (V, ϕ) on a class of elliptic surfaces $\pi : X \to B$, whose underlying vector bundle V has vertical determinant and is fiberwise semistable. We prove that if the spectral curve of V is reduced, then the Higgs field ϕ is *vertical*, while if V is fiberwise regular with reduced (resp., integral) spectral curve, and if its rank and second Chern number satisfy an inequality involving the genus of the base B and the degree of the fundamental line bundle of the fibration (resp., if the fundamental line bundle is sufficiently ample), then ϕ is *scalar*. We apply these results to the problem of characterizing slope-semistable Higgs bundles with vanishing discriminant in terms of the semistability of their pull-backs via maps from arbitrary (smooth, irreducible, complete) curves to X.

Contents

CHAPTER 1

Introduction

1.1. Motivation

One of the aims of the present thesis is to study a conjecture about semistable Higgs bundles with vanishing discriminant on elliptic surfaces; in doing so, we also find some structural results about Higgs bundles on elliptic surfaces, which may be of independent interest.

Let us start by introducing the conjecture. To this end, let us fix a non-singular, irreducible, projective scheme Y of dimension $n \geqslant 2$, defined over an algebraically closed field k of characteristic 0. Let us recall that the *discriminant* of a coherent sheaf F on Y, with rank r and Chern classes $c_i \in \mathrm{CH}^i(Y)$, is the characteristic class

$$\Delta(F) := 2rc_2 - (r-1)c_1^2 \in \mathrm{CH}^2(Y).$$

Let now H be an ample divisor on Y. The Bogomolov inequality (first proved in [7] for locally free sheaves in the case $k = \mathbb{C}$, $r = 2$; see [32] for the general case) asserts that, if the sheaf F is torsion-free and slope-semistable with respect to the polarization H, then its discriminant satisfies the inequality

$$\Delta(F) \cdot H^{n-2} \geqslant 0. \tag{1.1}$$

It is then natural to look for a way to characterize the slope-semistable sheaves for which the lower bound 0 for left hand side of (1.1) is attained. One of the results in this direction is the following theorem, first proved in [41], and then, independently and with a different proof, in [12]. In the theorem, which assumes $k = \mathbb{C}$, the condition $\Delta(F) \cdot H^{n-2} = 0$ is replaced by the stronger $\Delta(F) = 0$ in $\mathrm{H}^4(Y, \mathbb{Q})$; in fact, the two conditions are equivalent, as soon as F is locally free and slope-semistable with respect to the polarization H, as it can be proved by using Theorem 2 of [45].

THEOREM 1.1.1. *Let (Y, H) be a* complex *polarized variety. Then, for a locally free sheaf F on Y, the following statements are equivalent:*

(1) *F is slope-semistable with respect to H, and its discriminant vanishes in $\mathrm{H}^4(Y, \mathbb{Q})$;*
(2) *for each pair (C, f), where C is an irreducible, non-singular, projective curve, and $f : C \to Y$ a morphism, the pull-back f^*F of F to C along f is semistable.*

One also has a Higgs version of Bogomolov inequality: again, let (Y, H) be a polarized variety of dimension $n \geqslant 2$, defined over an algebraically closed field k of characteristic 0; then any torsion-free, slope-semistable Higgs sheaf (F, ϕ) on (Y, H) satisfies the inequality $\Delta(F) \cdot H^{n-2} \geqslant 0$. This was first proved by Simpson in [44] in the case $k = \mathbb{C}$ (and for stable holomorphic Higgs bundles on compact Kähler manifolds), using his *generalized*

Hitchin-Kobayashi correspondence;[1] see again [**32**] for the general case. It is then somehow natural to expect the following analogue of Theorem 1.1.1 to hold [**12, 11**]:

Conjecture 1.1.2. *Let (Y, H) be a* complex *polarized variety, and let (F, ϕ) be a Higgs bundle on Y. Then the following are equivalent:*

1. *the Higgs bundle (F, ϕ) is slope-semistable with respect to H and $\Delta(F) = 0$ in $\mathrm{H}^4(Y, \mathbb{Q})$;*
2. *for each morphism $f : C \to Y$, with C and f as in Theorem 1.1.1, the pull-back Higgs bundle $f^*(F, \phi)$ is semistable.*

The implication (1) $\Rightarrow$ (2) of 1.1.2 has been proved in [**12, 11**]. Moreover, by the Higgs version of Metha-Ramanathan theorem [**45**], any Higgs bundle satisfying item (2) of Conjecture 1.1.2 is semistable. So, what is left to be proved in order to establish the validity of the conjecture is the statement that a Higgs bundle with non-zero discriminant is unstable when pulled back to a suitable curve.

We also remark that Theorem 1.1.1 is true, more generally, for reflexive sheaves [**41**]. Thus, one may formulate a *reflexive version* of Conjecture 1.1.2. However, in the thesis we shall work with non-singular surfaces, for which there is no distinction between vector bundles and reflexive sheaves, and so we shall stick to the locally free case.

Conjecture 1.1.2 is by now known to be true for several classes of varieties, including those with nef tangent bundle [**14**], K3 surfaces [**13**], and, more generally, Calabi-Yau varieties [**9**].

As we already mentioned, the original motivation for this thesis has been an attempt to settle Conjecture 1.1.2 in the case of elliptic surfaces. We restrict throughout to the case of non-isotrivial Weierstrass fibrations with nodal singular fibers. Generalizations (e.g., to elliptic surfaces without sections, or with multiple fibers) will be the subject of future investigations. So, *in the remainder of this introduction, let us denote by $\pi : X \to B$ one such fibration, with X and B assumed to be non-singular, projective and irreducible.*

1.2. Results

Our first result on the conjecture (this is Proposition 5.5.1, whose proof uses, in particular, the equivalence of the notions of ordinary and Higgs-semistability for Higgs bundles on non-singular curves which are either rational or elliptic [**14, 16**]), shows that it is enough to prove the conjecture for Higgs bundles (V, ϕ) on X such that the vector bundle V has vertical (or even trivial) determinant, and is semistable after restriction to the fibers of π.

For this reason we mainly focus on Higgs bundles (V, ϕ) on X whose underlying bundle V has vertical determinant and is fiberwise semistable. As showed by Morgan, Friedman and Witten in [**22**], it is possible to associate to such a bundle V an effective divisor C_V on X, called the *spectral curve* of V, belonging to the the linear system $|r\Sigma + \pi^*\mu|$; here $\Sigma \subset X$ is the identity section of $\pi : X \to B$, r is the rank of V, and μ is a suitable line bundle on B, whose degree equals the second Chern number of V. Then they show that

[1]Simpson indeed proved the inequality only for stable Higgs bundles; the semistable case follows from the generalized Hitchin-Kobayashi correspondence for *semistable* Higgs bundles [**10, 36**].

the properties of the degree r morphism $C_V \hookrightarrow X \xrightarrow{\pi} B$, called the *spectral cover of* V, strongly influence those of V. For example, in the fiberwise regular case (the definition of regular bundle is recalled in Section 2.2.5) the spectral cover determines the isomorphism class of V, up to the choice of an invertible sheaf on the curve C_V.

Our results show that the properties of the spectral curve of V influence also those of the Higgs fields which V can support; let us try to explain what we mean by this, by summarizing our results on the structure of Higgs fields.

The first results in this direction (Proposition 4.3.1 and Corollary 4.3.2) concern a class of Higgs fields on bundles on elliptic surfaces which we call *vertical*. To define these, let us recall that the cotangent bundle Ω_X of the elliptic surface X possesses a distinguished invertible subsheaf, i.e., the pull-back along the projection $\pi : X \to B$ of the canonical line bundle of the base curve B. We call the Higgs fields on a bundle V on X factoring through the inclusion $V \otimes \pi^*\omega_B \hookrightarrow V \otimes \Omega_X$ *vertical*; by analyzing the restrictions of V and Ω_X to the fibers of π we show that, if V has vertical determinant and is semistable on a general closed fiber, then these are the only Higgs fields which V supports, provided that the spectral curve of V is *reduced*.

The second result is about a natural class of Higgs fields, which we call *scalar*, existing on an arbitrary sheaf defined over any variety, and parametrized by the space of global 1-forms on the variety. Using a Lemma on the relative incidence correspondence of divisors of the form $r\Sigma$ on the total space X of the elliptic fibration $\pi : X \to B$ (Lemma 4.4.2), we show (Proposition 4.4.4) that if V is fiberwise *regular*, with *reduced* (respectively, *integral*) spectral curve, and if the rank $r \geqslant 2$ and second Chern number e of V satisfy the inequality $e \geqslant (r-1)d+2g-1$ (respectively, if $d \geqslant 2g-1$), where $d \geqslant 1$ is the degree of the fundamental line bundle of π, and g the genus of B, then every Higgs field on V is necessarily scalar; in particular, for every Higgs field ϕ on V, the Higgs bundle (V, ϕ) (resp., any of its pull-backs $f^*(V, \phi)$ along some morphism $f : Y \to X$) is semistable if and only if the vector bundle V (resp., its pull-back f^*V) is.

Our results on Higgs bundles on elliptic surfaces allow us to prove (Proposition 5.5.2) that Conjecture 1.1.2 is true for Higgs bundles (V, ϕ) on X such that V has vertical determinant and is fiberwise semistable, as soon as the the rank r and second Chern number e of V satisfy the inequality $e \geqslant rd + 2g$, and the spectral curve of V is sufficiently general.

We also find (Proposition 3.2.3) that, for each degree $i \geqslant 0$, one can define in a natural way, starting from the morphism on Kähler differentials induced by the projection $\pi : X \to B$, an isomorphism between the direct image along π of the i-th symmetric power of the cotangent (respectively, tangent) bundle of the surface X, and the i-th (tensor) power of the canonical line bundle ω_B of the base curve B (respectively, of the inverse $\mathbb{L}^{-1}$ of the fundamental line bundle of the fibration π). The first isomorphism translates into a canonical isomorphism (Proposition 4.2.1) between the Hitchin base for rank r Higgs bundles on the surface X, and the Hitchin base for rank r Higgs bundles on B, suggesting the existence of a non-trivial relation between Higgs bundles on X and on B; the second shows instead that the direct image along π of a Higgs sheaf on X is an $\mathbb{L}$-valued pair on X. Thus there appears to be a picture, involving the moduli spaces M^{H}_X and M^{H}_B of semistable Higgs bundles on X and B and the moduli space $M_B^{\mathbb{L}}$ of semistable $\mathbb{L}$-valued pairs on B, which is roughly of the form

$$M_B^{\mathrm{H}} \overset{\pi^*}{\dashrightarrow} M_X^{\mathrm{H}} \overset{\pi_*}{\dashrightarrow} M_B^{\mathbb{L}}.$$

This is probably worth of further investigation.

The fiber $E = X_\eta$ of the projection $\pi : X \to B$ over the generic point η of B is a smooth curve of genus 1, defined over the function field $K := \mathcal{O}_{B,\eta}$ of B (and having the generic point of fixed zero section of π as a distinguished K-rational point). The pull-back of the cotangent bundle of X to this *generic fiber* E turns out to be isomorphic, under our assumptions on π, to the unique non-split self-extension $\mathrm{I}_2 = \mathrm{I}_2(\mathcal{O}_E)$ of the structure sheaf $\mathcal{O}_E$ on E (this is the rank 2 Atiyah bundle on E). Thus, if (V, ϕ) is a Higgs bundle on the surface X, then the pair (V_η, ϕ_η), where V_η and ϕ_η are, respectively, the pull-back of V and ϕ to E via the canonical morphism $E \to X$, is an I_2-valued pair on the elliptic curve E. Moreover, the field $\phi_\eta : V_\eta \to V_\eta \otimes \mathrm{I}_2$ satisfies the *integrability condition* $\phi_\eta \wedge \phi_\eta = 0$ in $\mathrm{Hom}(V_\eta, V_\eta \otimes \wedge^2 \mathrm{I}_2)$. Conversely, for a bundle V on X, the condition $\phi_\eta \wedge \phi_\eta = 0$ on a field $\phi : V \to V \otimes \Omega_X$ implies $\phi \wedge \phi = 0$ in $\mathrm{Hom}(V, V \otimes \Omega_X)$. This suggested to us that in order to gain some understanding of Higgs bundles on elliptic surfaces it is reasonable to take a closer look at integrable I_2-valued pairs on elliptic curves, which we do in Chapter 6.

Let E be a fixed elliptic curve, defined over a field $\mathbb{K}$ of characteristic 0. In Proposition 6.3.1 we prove that every integrable I_2-valued field on the bundle I_2 comes from a Higgs field (i.e., an endomorphism in this case) on I_2. This implies (Proposition 6.3.2) that a Higgs bundle (V, ϕ) of rank 2 and vertical determinant on the elliptic surface X, which is semistable on a general fiber and has spectral curve 2Σ, has the property that its restriction to a general closed fiber of π has zero Higgs field. This should be seen as a first attempt to complement our results on vertical Higgs fields in Section 4.3, which assume the spectral cover to be reduced.

We then go on in Section 6.3.2 to study the cone of integrable I_2-valued fields on Atiyah bundles I_r with $r \geqslant 3$. Unfortunately, starting from $r = 3$, the cone of integrable I_2-valued pairs is strictly bigger then that of Higgs fields. Thus Proposition 6.3.2 does not generalize to bundles with spectral curve $r\Sigma$ for $r \geqslant 3$.

Nonetheless we find in Section 6.3.4 that for $r = 3$ the cone of Higgs fields sits inside the locus of integrable I_2-valued fields (which is a 4-dimensional affine space) as the locus of singular points of the closed subscheme of the affine space $\mathrm{Hom}(\mathrm{I}_3, \mathrm{I}_3 \otimes \mathrm{I}_2)$ defined by the equation $\psi \wedge \psi = 0$. There should also be a relation between Higgs fields and singularities of the scheme of integrable I_2-valued pairs for ranks $r \geqslant 4$, but it is probably not so sharp as in the case $r = 3$.

An alternative description of the relation between Higgs fields and integrable I_2-valued fields on I_3 may be given in terms of the natural action of the group of automorphisms of I_3 on the locus of integrable I_2-valued fields. In fact, we see that while the action by conjugation of $\mathrm{Aut}(\mathrm{I}_r) \simeq \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{a}}^{r-1}$ on the cone of Higgs fields $\mathrm{End}(\mathrm{I}_r) \simeq \mathbb{A}^r$ is trivial for every r, this is not anymore the case for the analogous $\mathrm{Aut}(\mathrm{I}_r)$-action on $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_r \otimes \mathrm{I}_2) \simeq \mathbb{A}^{2r-1}$ if $r \geqslant 3$. In particular, in the case $r = 3$, we find that the Higgs fields are exactly the same as the fixed points of the latter action. We do not know if the equality between Higgs fields and fixed points works also in the case $r \geqslant 4$.

We conclude the chapter by showing that the locus of integrable I_2-valued fields on I_r, which has been shown in the preceding sections to be an affine space of dimension 2 (resp., 3) for $r = 2$ (resp., $r = 3$), despite being the common zero locus of a collection of $r-1$ homogeneous quadratic forms (more precisely, it is the cone over the base locus of a linear systems of quadrics of dimension $r-2$ on the projective space $\mathbb{P}\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_r \otimes \mathrm{I}_2)$), is actually a linear subvariety, of dimension $2r - 1 - \lfloor \frac{r}{2} \rfloor$, of the affine space $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_r \otimes \mathrm{I}_2)$ for any r. Starting from the knowledge of integrable I_2-valued pairs on I_r for any r, it is then easy, using the classification of bundles on elliptic curves, to determine the structure of the cone of integrable I_2-valued pairs on any *regular* bundle. The result is that this cone is again an affine space.

1.3. Structure

The structure of the thesis is as follows: besides the present chapter, it consists of two parts; each part consists of several chapters (e.g, Chapter 1), which are subdivided in sections (as in Section 1.1); finally, each section might comprise several subsections (e.g., Subsection 1.1.1). Results are numbered within each section (for example, Proposition 1.2.3 is the third proposition in Section 1.2).

1.4. Contents

Let us describe the contents of the thesis. Part 1 contains some preliminaries and it consists of two chapters. The first (Chapter 2) is about integral, projective curves of arithmetic genus 1. In it we describe the structure of the sheaf of Kähler differentials on a plane, projective, nodal cubic (Proposition 2.1.1), which will be used in Subsections 3.2.1, 3.2.4 and in Remark 3.2.2, and some properties of coherent sheaves on such curves which will be useful in Part 2. The second chapter (Chapter 3) makes precise the class of elliptic surfaces we shall work with in the thesis. In it we also prove the result about the direct image of the symmetric powers of the (co)tangent bundle of an elliptic fibration (Proposition 3.2.3), and recall some properties of fiberwise semistable bundles with vertical determinant on elliptic surfaces (in particular the construction of the spectral cover).

Part 2 contains the results about Higgs bundles on elliptic surfaces, and it consists of three chapters. In the first (Chapter 4) we prove the result on the Hitchin base (Proposition 4.2.1), and on vertical (Proposition 4.3.1 and Corollary 4.3.2) and scalar Higgs fields (Proposition 4.4.4). In the second (Chapter 5) we apply the results of Chapter 4 to the conjecture on semistable Higgs bundles with vanishing discriminant. Finally, the last chapter (Chapter 6) contains the results on I_2-valued pairs on elliptic curves and their relations with Higgs bundles on elliptic surfaces.

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1.6. Notations and conventions

We mostly use the notations of [**21**].

1.6.1. Ground field. We denote by k an *algebraically closed field of characteristic* 0, and we work in the *category of* k*-schemes* (in Section 5.5 we specialize to the case $k = \mathbb{C}$). In Chapter 6 we work over an arbitrary field $\mathbb{K}$ of characteristic 0, *non-necessarily algebraically closed*.

1.6.2. Schemes. The word *scheme* without further qualifiers will mean *separated* k*-scheme of finite-type*. Let Y be a scheme. The *structure sheaf* of Y is denoted by $\mathcal{O}_Y$. The *local ring* at a point $y \in Y$ is $\mathcal{O}_{Y,y}$, its *maximal ideal* $\mathfrak{m}_y$, and its *residue field* $k(y)$. For a scheme S, $Y(S)$ is the set of S-valued points of Y (if S is affine, say $S = \operatorname{Spec} A$ for some k-algebra A, we write $Y(A)$ instead of $Y(\operatorname{Spec} A)$).

The *Chow group* of codimension i cycles on Y modulo rational equivalence is denoted by $\mathrm{CH}^i(Y)$.

For a morphism of schemes $f : Y \to Z$, the morphism of structure sheaves $\mathcal{O}_Z \to f_*\mathcal{O}_Y$ associated to f will be denoted by f^*; if $z \in Z$ is any point, the scheme-theoretic *fiber* of f at z will be denoted by Y_z.

1.6.3. Sheaves. Coherent sheaves of modules on a scheme Y will be referred to simply as sheaves or $\mathcal{O}_Y$-modules. For a sheaf F on Y, we sometimes denote the *fiber* $F \otimes k(y)$ of F at a point $y \in Y$ as $F(y)$, and the sheaf $F \otimes \mathcal{O}_Y(D)$ as $F(D)$ if D is a Cartier divisor on Y; for a global section s of F, $\mathrm{Z}(s)$ is the *scheme of zeros* of s.

The dimensions (as k-vector spaces) of the *cohomology groups* (with respect to the Zariski topology) $\mathrm{H}^i(Y, F)$ of Y with coefficients in F will be denoted by $\mathrm{h}^i(Y, F)$; the *direct images* of F along a morphism $f : Y \to Z$ are written as $\mathrm{R}^i f_* F$.

The *Chern classes* of F are denoted by $\mathrm{c}_i(F) \in \mathrm{CH}^i(Y)$, the Chern characters by $\mathrm{ch}_i(F) \in \mathrm{CH}^i(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Given two sheaves F, G on Y, the k-vector space (resp., the sheaf) of $\mathcal{O}_Y$-linear maps $F \to G$ will be denoted by $\mathrm{Hom}_{\mathcal{O}_Y}(F, G)$ (resp. $\mathit{Hom}_{\mathcal{O}_Y}(F, G)$), and its dimension by $\mathrm{hom}(F, G)$; if $G = F$, we use the notations $\mathrm{End}(F)$, $\mathit{End}(F)$ and $\mathrm{end}(F)$, and for $G = \mathcal{O}_Y$ we write $F^\vee$. Analogously, we write $\mathrm{Ext}^i_{\mathcal{O}_Y}(F, G)$ (resp., $\mathit{Ext}^i_{\mathcal{O}_Y}(F, G)$) for the *ext groups* (resp., *sheaves*), and $\mathrm{ext}^i(F, G)$ for their dimension. For a family $f : Y \to S$, the *relative ext sheaves* will be denoted by $\mathit{Ext}^i_f(F, G)$.

The sheaf of *Kähler differentials* of a scheme Y over k (resp., of a morphism $f : Y \to Z$) is written as Ω_Y (resp., as Ω_f, or $\Omega_{X/Y}$), and its p-th exterior power as Ω^p_Y; duals of sheaves of differentials are denoted by the letter Θ; for a morphism of schemes $f : Y \to Z$, the canonical map $f^*\Omega_Z \to \Omega_Y$ is denoted by f^*; if $z \in Z$ is any point, the pull-back of a sheaf F on Y via the canonical morphism $Y_z \to Y$ is written as F_z. For a locally free sheaf V

on Y, we set $\mathbb{P}_Y(V) := \mathrm{Proj}_Y(\mathrm{Sym}\, V^\vee)$ (symmetric and exterior powers and algebras are denoted by Sym and $\wedge$ respectively).

Part 1

Preliminaries

CHAPTER 2

Curves of arithmetic genus 1

2.1. Nodal cubics

Let k be an algebraically closed field of characteristic $\neq 2, 3$, and let $\mathbb{P}^2_k$ be the projective plane over k, with homogeneous coordinates (X_0, X_1, X_2). Let $C \hookrightarrow \mathbb{P}^2_k$ be the nodal cubic with equation

$$C: \quad X_1^2 X_2 = X_0^3 + X_0^2 X_2.$$

Let us denote by $o := (0, 1, 0)$ be the point at infinity of C, and by U its open complement, so that U is the affine plane curve in

$$(\mathbb{P}^2)_{X_2} \simeq \mathbb{A}^2_k := \operatorname{Spec}(k[X, Y]), \qquad \frac{X_0}{X_2} \leftrightsquigarrow X, \quad \frac{X_1}{X_2} \leftrightsquigarrow Y$$

with equation

$$U: \quad Y^2 = \phi(X), \qquad \phi(X) := X^3 + X^2;$$

namely, $U = \operatorname{Spec}(A)$, where

$$A := \frac{k[X, Y]}{(Y^2 - \phi(X))} = k[x, y].$$

Here x, y denote the images of X, Y in A; they satisfy the equation $y^2 = \phi(x)$. The inclusion $k[X] \simeq k[x] \hookrightarrow A$ makes A into a free $k[x]$-module of rank 2, with basis $(1, y)$.

2.1.1. Normalization. Let $(\tilde{C}, \nu)$ be the normalization of C. Then $\tilde{C} \simeq \mathbb{P}^1_k$, and $\nu^{-1}(o)$ consists of 1 point, so that $\tilde{U} := \nu^{-1}(U) = \tilde{C} - \nu^{-1}(o)$ is isomorphic to the affine line $\mathbb{A}^1_k := \operatorname{Spec}(k[t])$. The base change $\hat{\nu} : \tilde{U} \to U$ of ν along the open immersion $U \hookrightarrow C$ is the morphism of affine k-schemes dual to the homomorphism of k-algebras

$$\hat{\nu}^* : A \to k[t], \qquad x \mapsto t^2 - 1, \quad y \mapsto t(t^2 - 1).$$

It is well known that the morphism of sheaves of rings on C

$$\nu^* : \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}}$$

is injective, with cokernel the (direct image of the) structure sheaf $k(p) \simeq k$ of the node $p := (0, 0, 1)$ of C; in other words, one has a short exact sequence of sheaves on C

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}} \to k(p) \to 0. \tag{2.1}$$

The map $\nu_* \mathcal{O}_{\tilde{C}} \to k(p)$ in (2.1) can be described explicitly as follows: let us denote by $i : \operatorname{Spec}(k) \hookrightarrow C$ the inclusion of the node, and by $j : \tilde{C}_p \hookrightarrow \tilde{C}$ that of the scheme-theoretic fiber $\tilde{C}_p \simeq \operatorname{Spec}(k \times k)$ of ν over p, so that the the following is a cartesian square with finite vertical arrows:

$$\begin{tikzcd} \operatorname{Spec}(k \times k) \arrow[r, hook, "j"] \arrow[d] & \tilde{C} \arrow[d, "\nu"] \\ \operatorname{Spec}(k) \arrow[r, hook, "i"] & C \end{tikzcd} . \tag{2.2}$$

Then giving an $\mathcal{O}_C$-linear map $\nu_*\mathcal{O}_{\tilde{C}} \to k(p) = i_*k$ is the same as giving a k-linear map $i^*\nu_*\mathcal{O}_{\tilde{C}} \to k$, or $\mathrm{H}^0(j^*\mathcal{O}_{\tilde{C}}) = k \oplus k \to k$; the last map is just the difference map

$$k \oplus k \to k, \qquad (a, b) \mapsto a - b.$$

This expresses the fact that a function on $\tilde{C}$ comes from one on C if and only if it has the same values at the preimages of the node, which is in turn a consequence of the fact that the nodal cubic can be seen as a projective line with 2 points identified to a node. The restriction of (2.1) to the open U is also exact; this is equivalent to the exactness of the sequence of A-modules

$$0 \to A \xrightarrow{\hat{\nu}^*} k[t] \to k \to 0,$$

in which the last map is $f(t) \mapsto f(1) - f(-1)$.

2.1.2. Kähler differentials.

Proposition 2.1.1. *The adjoint of the pull-back map on Kähler differentials $\nu^* : \nu^*\Omega_C \to \Omega_{\tilde{C}}$ is surjective, with kernel $k(p)$. In other words, there is an exact sequence of sheaves on C*

$$0 \to k(p) \to \Omega_C \to \nu_*\Omega_{\tilde{C}} \to 0. \tag{2.3}$$

Proof. The restriction of (2.3) to the complement of the node is exact, since ν induces an isomorphism $\tilde{C} - \nu^{-1}(p) \to C - \{p\}$. Thus we only need to check exactness at p. The restriction of $\Omega_C \to \nu_*\Omega_{\tilde{C}}$ to the open complement $U = C - \{o\}$ of the point at infinity o is the map of $\mathcal{O}_U = \tilde{A}$-modules corresponding to the A-linear map

$$\alpha : M \to k[t], \qquad e_1 \mapsto 2t, \quad e_2 \mapsto 3t^2 - 1,$$

where M is the A-module generated by elements e_1, e_2, subject to the relation

$$(3x^2 + 2x)e_1 - 2ye_2 = 0.$$

Thus exactness of (2.3) at $\nu_*\Omega_{\tilde{C}}$ is equivalent to α being surjective. Now, for any

$$m = (a(x) + b(x)y)e_1 + (c(x) + d(x)y)e_2 \in M \tag{2.4}$$

$(a, b, c, d \in k[x])$, we have

$$\begin{aligned}
\alpha(m) &= 2t\left[a(t^2-1)+b(t^2-1)t(t^2-1)\right]+ \\
&\quad +(3t^2-1)\left[c(t^2-1)+d(t^2-1)t(t^2-1)\right] \\
&= 2t^2(t^2-1)b(t^2-1)+(3t^2-1)c(t^2-1)+ \\
&\quad +t\left[2a(t^2-1)+(t^2-1)(3t^2-1)d(t^2-1)\right] \\
&= 2(x+1)xb(x)+(3x+2)c(x)+t\left[2a(x)+x(3x+2)d(x)\right] \\
&= r(x)+s(x)t,
\end{aligned}$$

where each of

$$r(x) := 2(x+1)xb(x)+(3x+2)c(x), \qquad s(x) := 2a(x)+x(3x+2)d(x) \tag{2.5}$$

can take on any value in $k[x]$ when a, b, c, d vary in $k[x]$, since

$$\gcd(2(x+1)x, 3x+2) = 1 = \gcd(2, x(3x+2))$$

(here it is essential to have $\mathrm{char}(k) \neq 2, 3$). One then remarks that any element $f(t) = \sum_{i\geqslant 0} a_i t^i \in k[t]$ can be written in the form $f(t) = r(x)+s(x)t$ for suitable $r, s \in k[x]$, since

$$\begin{aligned}
f(t) &= \sum_{i\geqslant 0} a_{2i}t^{2i} + \sum_{i\geqslant 0} a_{2i+1}t^{2i+1} \\
&= \sum a_{2i}(x+1)^i + t\sum a_{2i+1}(x+1)^i = r(x)+s(x)t.
\end{aligned}$$

This shows that α is surjective.

The map $\Omega_C \to \nu_*\Omega_{\tilde{C}}$ is thus a surjective map of sheaves, both having the same rank 1; so its kernel is contained in the torsion subsheaf T of Ω_C. Furthermore, one checks that the sheaf $\nu_*\Omega_{\tilde{C}}$ torsion-free; so the sheaf T is killed by $\Omega_C \to \nu_*\Omega_{\tilde{C}}$. It follows that the kernel of $\Omega_C \to \nu_*\Omega_{\tilde{C}}$ is actually equal to T. The latter sheaf is clearly supported at the node p of C. So to compute its length, it is enough to look at $\ker(\alpha)$. This consists of the elements $m \in M$ as in (2.4) such that $r(x) = s(x) = 0$ in $k[x]$, where r, s are defined in (2.5). From $r = 0$ one finds $c(x) = x(x+1)\bar{c}(x)$ for some $\bar{c} \in k[x]$; substituting back in $r = 0$ and getting rid of $x(x+1)$, we find

$$b(x) = -\frac{1}{2}(3x+2)\bar{c}(x);$$

instead, the equation $s = 0$ gives

$$a(x) = -\frac{1}{2}x(3x+2)d(x).$$

So, if m is killed by α, then necessarily

$$\begin{aligned}
m &= -\frac{1}{2}(3x+2)(xd(x)+\bar{c}(x)y)e_1 + (x(x+1)\bar{c}(x)+d(x)y)e_2 \\
&= -\frac{d(x)}{2}(x(3x+2)e_1 - 2ye_2) - \frac{\bar{c}(x)}{2}((3x+2)ye_1 - 2x(x+1)e_2) \\
&= -\frac{\bar{c}(x)}{2}\omega
\end{aligned}$$

belongs to the A-submodule $A\omega$ of M generated by

$$\omega := (3x+2)ye_1 - 2x(x+1)e_2 \in M.$$

Conversely, one checks immediately that ω is killed by α; thus, denoting by $\mathrm{ann}(\omega) := \{f \in A : f\omega = 0 \text{ in } M\}$ the annihilator of ω, we have

$$\ker(\alpha) = A\omega \simeq A/\mathrm{ann}(\omega).$$

Let us finally compute the ideal $\mathrm{ann}(\omega)$. A straightforward checking shows that x, y annihilate ω. Thus $\mathrm{ann}(\omega)$ contains the maximal ideal $\mathfrak{m} := (x, y)$. Moreover, it is easily seen that $\omega \neq 0$ in M; thus $\mathrm{ann}(\omega)$ is a proper ideal of A. By maximality of $\mathfrak{m}$, it follows that $\mathrm{ann}(\omega) = \mathfrak{m}$, and hence that $\ker(\alpha) \simeq A/\mathfrak{m} \simeq k$.

□

2.2. Some properties of coherent sheaves on curves of arithmetic genus 1

In this section, we denote by E an integral, projective curve of arithmetic genus 1, possibly singular, defined over an algebraically closed field k of characteristic 0.

2.2.1. Degree, slope, stability. The notion of *degree* can be extended to the setting of coherent sheaves $\mathcal{S}$ on E by defining

$$\deg \mathcal{S} := \chi(E, \mathcal{S}) := \mathrm{h}^0(E, \mathcal{S}) - \mathrm{h}^1(E, \mathcal{S}).$$

We remark that, for E smooth, this agrees with the usual notion of degree, by virtue of Riemann-Roch theorem. One can then define the *slope* of a sheaf on E, and the notion of *slope-(semi)stability* for torsion-free sheaves.

2.2.2. Torsion-free sheaves of rank 1 and degree 0. Let J the group of isomorphism classes of invertible sheaves of degree zero on E. The choice of a smooth point $o \in E_{\mathrm{reg}}(k)$ determines a group isomorphism

$$E_{\mathrm{reg}}(k) \to J, \qquad q \mapsto \mathcal{O}_E\left(q - o\right). \tag{2.6}$$

Now let $\bar{J}$ be the set of isomorphism classes of torsion-free sheaves of rank 1 and degree 0 on E, which can be viewed as a compactification of J [1]. The set $\bar{J}$ is, of course, the same thing as J in the non-singular case. To compute $\bar{J}$ in the singular case one uses the following (cf. the proof of Lemma 0.2 of [22]):

Proposition 2.2.1. *Let E be a projective, integral, singular curve of arithmetic genus 1, and let $\nu : \tilde{E} \to E$ be its normalization. Then any torsion-free sheaf of rank 1 and degree 0 on E is either invertible, or isomorphic to $\nu_*\mathcal{L}$ for some invertible sheaf $\mathcal{L}$ on $\tilde{E}$ (but not both).*

Let us assume E to be singular, and let us denote by $p \in E(k)$ the singular point of E. Then

$$\deg\left(\nu_*\mathcal{O}_{\tilde{E}}\right) = 1,$$

by virtue of the exact sequence of $\mathcal{O}_E$-modules

$$0 \to \mathcal{O}_E \to \nu_*\mathcal{O}_{\tilde{E}} \to k(p) \to 0. \tag{2.7}$$

Moreover, for all $a \in \mathbb{Z}$, the sheaf

$$\nu_* \mathcal{O}_{\tilde{E}}(a) = \nu_* \nu^* \mathcal{O}_E(ao) = \mathcal{O}_E(ao) \otimes \nu_* \mathcal{O}_{\tilde{E}}$$

(we recall that o is a chosen regular point of E) has degree $a+1$. Thus there exists a unique (up to isomorphism) torsion-free sheaf $\mathcal{F}$ on E of rank 1 and degree 0 which is not a line bundle, i.e.,

$$\mathcal{F} = \nu_* \mathcal{O}_{\tilde{E}}(-1) = \mathcal{O}_E(-o) \otimes \nu_* \mathcal{O}_{\tilde{E}}.$$

This shows that in the singular case, the isomorphism (2.6) extends in a unique way to a bijection of sets

$$E(k) \to \bar{J}$$

(the set $\bar{J}$ is not a group with respect to tensor product, since one can show that the sheaf $\mathcal{F}^{\otimes 2}$ is not torsion-free).

An alternative description of $\mathcal{F}$ is the following: let $\mathcal{I}_p$ be the ideal sheaf of the node, i.e., of the closed sub-scheme $\{p\}$ of E, endowed with the reduced structure. Then $\mathcal{I}_p$, being a sub-sheaf of $\mathcal{O}_E$, is torsion-free and has rank 1. Moreover, it is not a line bundle, since the Weil divisor p on E is not Cartier. Finally, it has degree -1, as follows from consideration of the exact sequence

$$0 \to \mathcal{I}_p \to \mathcal{O}_E \to k(p) \to 0. \tag{2.8}$$

This shows that

$$\mathcal{F} = \mathcal{I}_p \otimes \mathcal{O}_E(o).$$

We conclude by remarking that, by taking the long exact sequences of ext sheaves coming from (2.7) and (2.8), one obtains the short exact sequences of sheaves

$$0 \to (\nu_* \mathcal{O}_{\tilde{E}})^\vee \to \mathcal{O}_E \to k(p) \to 0,$$

and

$$0 \to \mathcal{O}_E \to \mathcal{I}_p^\vee \to k(p) \to 0,$$

from which it follows immediately that

$$(\nu_* \mathcal{O}_{\tilde{E}})^\vee = \mathcal{I}_p, \qquad \mathcal{I}_p^\vee = \nu_* \mathcal{O}_{\tilde{E}}, \qquad \mathcal{F}^\vee = \mathcal{F}.$$

2.2.3. Semistable bundles of degree 0. Every semistable bundle of degree 0 on E admits a (Jordan-Hölder) filtration of finite length with sub-quotients belonging to $\bar{J}$ (and conversely; see Theorem 1.1 of [22], which applies, more generally, to torsion-free sheaves). This filtration is not unique, but its associated graded sheaf is (up to isomorphism). If $\lambda_1, \dots, \lambda_s$ are elements of $\bar{J}$, one says that a semistable bundle $\mathcal{V}$ of degree 0 on E is *concentrated at* $\lambda_1, \dots, \lambda_s$ (and $\lambda_1, \dots, \lambda_s$ are said to be the *Jordan-Hölder constituents of* $\mathcal{V}$), if the sheaves appearing as sub-quotients in one (and hence any) Jordan-Hölder filtration of $\mathcal{V}$ are $\lambda_1, \dots, \lambda_s$.

2.2.4. Strongly indecomposable bundles. Let $\lambda \in \bar{J}$. A semistable bundle $\mathcal{I}$ of degree 0 on E, concentrated at λ, is said to be *strongly indecomposable* [22] if the equality

$$\dim(\mathrm{Hom}(\lambda, \mathcal{I})) = 1$$

holds. As the name suggests, strong indecomposability implies indecomposability (under direct sum); furthermore, if the sheaf λ is invertible, then any semistable bundle of degree 0, which is concentrated at λ and indecomposable under direct sum, is actually strongly indecomposable (and, in fact, isomorphic to the *Atiyah bundle* $\mathrm{I}_r(\lambda)$ [2] of the appropriate rank; the definition and some properties of the bundles $\mathrm{I}_r(\lambda)$ are recalled in Theorem 2.2.3).

Example 2.2.2. Let us assume the curve E to be nodal, with node p, and normalization $\nu : \tilde{E} \to E$. Let $\mathcal{F}$ be the unique non-invertible element of $\bar{J}$. As an example of strongly indecomposable bundles concentrated at $\mathcal{F}$, let us look at self-extensions of $\mathcal{F}$. These are parametrized by $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$, which can be computed by using the exact sequence

$$0 \to \mathrm{H}^1(\mathit{End}(\mathcal{F})) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{H}^0\left(\mathit{Ext}^1(\mathcal{F}, \mathcal{F})\right) \to 0$$

coming from the ext local to global spectral sequence. Here, $\mathrm{H}^1(\mathit{End}(\mathcal{F}))$ is Serre dual to

$$\mathrm{Ext}^0\left(\mathit{End}(\mathcal{F}), \omega_E^\circ\right) = \mathrm{Hom}\left(\mathit{End}(\mathcal{F}), \mathcal{O}_E\right) = \mathrm{H}^0\left(\mathit{Hom}\left(\mathit{End}(\mathcal{F}), \mathcal{O}_E\right)\right) = \mathrm{H}^0\left(\mathit{End}(\mathcal{F})^\vee\right).$$

To compute $\mathit{End}(\mathcal{F})^\vee$ one starts by observing that

$$\mathit{End}(\mathcal{F}) \simeq \mathit{End}\left(\nu_*\mathcal{O}_{\tilde{E}}\right) \simeq \nu_*\mathit{Hom}\left(\nu^*\nu_*\mathcal{O}_{\tilde{E}}, \mathcal{O}_{\tilde{E}}\right);$$

then to compute $\mathit{Hom}\left(\nu^*\nu_*\mathcal{O}_{\tilde{E}}, \mathcal{O}_{\tilde{E}}\right)$, one observes that the evaluation map $\mathrm{ev} : \nu^*\nu_*\mathcal{O}_{\tilde{E}} \to \mathcal{O}_{\tilde{E}}$ is a surjective (since ν is an affine morphism) map of rank 1 sheaves. So its kernel is a 0-dimensional sheaf, which is then contained in the torsion subsheaf T of $\nu^*\nu_*\mathcal{O}_{\tilde{E}}$. But the sheaf T is also contained in $\ker(\mathrm{ev})$, since $\mathcal{O}_{\tilde{E}}$ is torsion-free. This shows that $\ker(\mathrm{ev})$ is actually equal to T. It follows that

$$\mathit{Hom}\left(\nu^*\nu_*\mathcal{O}_{\tilde{E}}, \mathcal{O}_{\tilde{E}}\right) \simeq \mathit{Hom}\left(\nu^*\nu_*\mathcal{O}_{\tilde{E}}/T, \mathcal{O}_{\tilde{E}}\right) \simeq \mathit{Hom}\left(\mathcal{O}_{\tilde{E}}, \mathcal{O}_{\tilde{E}}\right) \simeq \mathcal{O}_{\tilde{E}},$$

and, finally, that

$$\mathit{End}(\mathcal{F}) \simeq \nu_*\mathcal{O}_{\tilde{E}}, \qquad \mathit{End}(\mathcal{F})^\vee \simeq \mathcal{I}_p.$$

Thus $\mathrm{H}^1(\mathit{End}(\mathcal{F})) \simeq \mathrm{H}^0\left(\mathit{End}(\mathcal{F})^\vee\right)^\vee \simeq \mathrm{H}^0(\mathcal{I}_p)^\vee = 0$. The ext sheaf $\mathit{Ext}^1(\mathcal{F}, \mathcal{F})$ is clearly supported at p. To compute its length, let us set

$$A := k[\![x, y]\!], \qquad I := Axy, \qquad B := A/I \simeq \hat{\mathcal{O}}_{E,p},$$

and let us denote by $u, v \in B$ the images of $x, y \in A$, and by $\mathfrak{m}$ the maximal ideal $Bu + Bv$ of B. It is then enough to compute

$$\mathrm{Ext}^1_B(\mathfrak{m}, \mathfrak{m}).$$

First of all, one has a natural short exact sequence of B-modules

$$0 \to Bu \cap Bv \to Bu \oplus Bv \to \mathfrak{m} \to 0,$$

and it is easy to check that $Bu \cap Bv = 0$. Thus the canonical map

$$Bu \oplus Bv \to \mathfrak{m}$$

is an isomorphism of B-modules. It follows that

$$\mathrm{Ext}^1_B(\mathfrak{m},\mathfrak{m}) \simeq \mathrm{Ext}^1_B(Bu,Bu) \oplus \mathrm{Ext}^1_B(Bu,Bv) \oplus \mathrm{Ext}^1_B(Bv,Bu) \oplus \mathrm{Ext}^1_B(Bv,Bv).$$

To compute the right hand side, one starts from the free resolution of Bu

$$F_\bullet : \quad 0 \leftarrow Bu \xleftarrow{u} B \xleftarrow{v} B \xleftarrow{u} B \leftarrow \cdots .$$

Applying the functor $\mathrm{Hom}_B(-,Bu)$, one gets the complex

$$\mathrm{Hom}_B\left(F_\bullet, Bu\right) : \quad 0 \to Bu \xrightarrow{v} Bu \xrightarrow{u} Bu \to \cdots ,$$

whose h^1 is

$$\mathrm{Ext}^1_B(Bu,Bu) = \frac{\ker\left(Bu \xrightarrow{u} Bu\right)}{\mathrm{im}\left(Bu \xrightarrow{v} Bu\right)} \simeq \ker\left(k[\![x]\!] \xrightarrow{x} k[\![x]\!]\right) = 0.$$

By symmetry, one has $\mathrm{Ext}^1_B(Bv,Bv) = 0$ too. Next, applying $\mathrm{Hom}_B(-,Bv)$, one obtains the complex

$$\mathrm{Hom}_B\left(F_\bullet, Bv\right) : \quad 0 \to Bv \xrightarrow{v} Bv \xrightarrow{u} Bv \to \cdots ,$$

whose first cohomology group equals

$$\mathrm{Ext}^1_B(Bu,Bv) = \frac{\ker\left(Bv \xrightarrow{u} Bv\right)}{\mathrm{im}\left(Bv \xrightarrow{v} Bv\right)} \simeq \frac{k[\![y]\!]}{\mathrm{im}\left(k[\![y]\!] \xrightarrow{y} k[\![y]\!]\right)} \simeq k\xi,$$

ξ being the class of the non-split extension

$$0 \to Bv \hookrightarrow B \xrightarrow{u} Bu \to 0.$$

By symmetry, one also has

$$\mathrm{Ext}^1_B(Bv,Bu) \simeq k\eta,$$

where η is the class of the non-split extension

$$0 \to Bu \hookrightarrow B \xrightarrow{v} Bv \to 0.$$

Thus

$$\mathrm{Ext}^1_B(\mathfrak{m},\mathfrak{m}) \simeq k\xi \oplus k\eta \simeq k^{\oplus 2}.$$

For $c \in \mathbb{P}\mathrm{Ext}^1_B(\mathfrak{m},\mathfrak{m}) \simeq \mathbb{P}^1_k$, let

$$0 \to \mathfrak{m} \to M_c \to \mathfrak{m} \to 0$$

be the extension of class c. Then M_c is isomorphic to $\mathfrak{m} \oplus B$ if c has homogeneous coordinates $(1,0)$ or $(0,1)$ with respect to the basis (ξ,η), while M_c is free of rank 2 otherwise. The previous computations show that there are two non-locally free, non-split self-extensions of $\mathcal{F}$, while all other non-split extensions are locally free sheaves of rank 2 and degree 0 concentrated at $\mathcal{F}$. If

$$\varepsilon : \quad 0 \to \mathcal{F} \to \mathcal{V} \to \mathcal{F} \to 0$$

is a locally free extension, then the natural map

$$k \cdot \mathrm{id}_{\mathcal{F}} = \mathrm{End}(\mathcal{F}) \to \mathrm{Hom}(\mathcal{F},\mathcal{V})$$

is an isomorphism (as follows from consideration of the exact sequence

$$0 \to \mathrm{End}(\mathcal{F}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{V}) \to \mathrm{End}(\mathcal{F}) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}),$$

in which the last map is injective, since it maps the identity endomorphism of $\mathcal{F}$ to the class of the non-split extension ε). This shows that $\mathcal{V}$ is strongly indecomposable.

2.2.5. Regular bundles. A semistable bundle of degree 0 on E is said to be *regular* ([22], Definition 1.12) if it isomorphic to a direct sum of strongly indecomposable bundles having pairwise distinct Jordan-Holder constituents. One can show that any S-equivalence class of semistable bundles of degree 0 on E has a unique regular representative.

2.2.6. Moduli of bundles of degree 0 with fixed rank and determinant. Let $o \in E_{\mathrm{reg}}(k)$ be the origin of the group law on $E_{\mathrm{reg}}(k)$. Let us fix a line bundle λ of degree 0 on E, say $\lambda = \mathcal{O}_E(q - o)$ for some $q \in E_{\mathrm{reg}}(k)$, and an integer $r \geqslant 1$. Then a coarse moduli space for semistable bundles on E of rank r and determinant λ is given by the complete linear system

$$|q + (r-1)o| = \mathbb{P}\mathrm{H}^0\left(E, \mathcal{O}_E\left(q + (r-1)o\right)\right) \approx \mathbb{P}_k^{r-1}.$$

In fact, if $\mathcal{V}$ is such a bundle, then the twisted bundle

$$\mathcal{V}(o) := \mathcal{V} \otimes \mathcal{O}_E(o)$$

(not to be confused with the fiber $\mathcal{V} \otimes k(o)$ at the origin o of E, which is usually denoted by the same symbol) is semistable of degree r, whence $\mathrm{h}^0\left(E, \mathcal{V}(o)\right) = r$. Moreover, the evaluation map

$$\mathrm{ev} : \mathrm{H}^0\left(E, \mathcal{V}(o)\right) \otimes \mathcal{O}_E \to \mathcal{V}(o)$$

can be shown to be an isomorphism at the generic point of E. It follows that its determinant

$$\det(\mathrm{ev}) : \mathcal{O}_E \simeq \mathcal{O}_E \otimes \det \mathrm{H}^0(E, \mathcal{V}(o)) \to \det(\mathcal{V}(o)) \simeq \mathcal{O}_E\left(q + (r-1)o\right)$$

induces a non-zero section of the line bundle $\mathcal{O}_E\left(q + (r-1)o\right)$, well defined up to multiplication by a non-zero scalar, i.e., a (k-rational) point

$$\zeta(\mathcal{V}) := \mathrm{Z}(\det(\mathrm{ev}))$$

of the projective space $|q + (r-1)o|$. More generally, if S is any scheme, and $\mathcal{V}$ a family of semistable bundles on E of rank r and determinant λ, parametrized by S, then one can define an S-valued point $\zeta(\mathcal{V})$ of $|q + (r-1)o|$ by considering the determinant of the relative evaluation map

$$p_S^* p_{S*}\left(\mathcal{V} \otimes p_E^* \mathcal{O}_E(o)\right) \to \mathcal{V} \otimes p_E^* \mathcal{O}_E(o).$$

2.2.7. Atiyah bundles, I. More details about these bundles in Chapter 6. Here we limit ourselves to the following:

THEOREM 2.2.3. *Let E be an integral, projective curve of arithmetic genus 1. Let us fix a line bundle λ of degree 0 on E and an integer $r \geqslant 1$. Then there exists a unique (up to* *isomorphism) semistable bundle* $\mathrm{I}_r(\lambda)$ *on* E *of rank* r *and degree* 0*, which is concentrated at* λ *and indecomposable under direct sum. It satisfies*

$$\det \mathrm{I}_r(\lambda) \simeq \lambda^r, \qquad \mathrm{I}_r(\lambda)^\vee \simeq \mathrm{I}_r(\lambda^{-1}),$$

and, for each $\mu \in \bar{J}$*,*

$$\hom(\mu, \mathrm{I}_r(\lambda)) = \hom(\mathrm{I}_r(\lambda), \mu) = \begin{cases} 0, & \mu \not\simeq \lambda \\ 1, & \mu \simeq \lambda \end{cases} \tag{2.9}$$

Sketch of proof. Let us sketch a proof of the existence part of the statement. It is clearly enough to build $\mathrm{I}_r := \mathrm{I}_r(\mathcal{O}_E)$, and then set $\mathrm{I}_r(\lambda) := \mathrm{I}_r \otimes \lambda$. The bundles I_r can be defined by induction on r, starting from $\mathrm{I}_1 := \mathcal{O}_E$. For $r \geqslant 2$, assuming the existence of I_{r-1} (and the equality $\mathrm{h}^1(\mathrm{I}_{r-1}) = 1$), one takes I_r as the unique non-split extension of $\mathcal{O}_E$ by I_{r-1}. For a more detailed proof the reader can consult [22], Lemma 1.7.

□

2.2.8. Degree 1 sheaves.

Theorem 2.2.4. *Let* C *be an integral, projective curve of arithmetic genus* 1*. Then, for every integer* $r \geqslant 1$*, and for every smooth point* $q \in C$*, there exists a unique (up to isomorphism) stable bundle* $\mathrm{W}_r(q)$ *of rank* r *on* C*, such that* $\det \mathrm{W}_r(q) \approx \mathcal{O}_C(q)$*. It admits a length* r *filtration by subbundles*

$$0 = W_0 \subset W_1 \subset \cdots \subset W_r = \mathrm{W}_r(q),$$

whose subquotients are isomorphic, respectively, to

$$\mathcal{O}_C, \cdots, \mathcal{O}_C, \mathcal{O}_C(q).$$

Sketch of proof. Again, let us only sketch the proof of existence, which can be done by induction on r, starting from $\mathrm{W}_1(q) := \mathcal{O}_C(q)$. So let us assume the existence of $\mathrm{W}_{r-1}(q)$, where r is $\geqslant 2$. To define $\mathrm{W}_r(q)$ one looks for non-split extensions of $\mathrm{W}_{r-1}(q)$ by $\mathcal{O}_C$, which are classified by

$$\mathbb{P}\mathrm{Ext}^1_C(\mathrm{W}_{r-1}(q), \mathcal{O}_C) \simeq \mathbb{P}\mathrm{H}^0(\mathrm{W}_{r-1}(q))^\vee,$$

a projective space over k of dimension $\mathrm{h}^0(C, \mathrm{W}_{r-1}(q)) - 1$. We have

$$\begin{aligned} \mathrm{h}^0(C, \mathrm{W}_{r-1}(q)) &= \chi(C, \mathrm{W}_{r-1}(q)) + \mathrm{h}^1(C, \mathrm{W}_{r-1}(q)) \\ &= \deg(\mathrm{W}_{r-1}(q)) + \mathrm{h}^1(C, \mathrm{W}_{r-1}(q)) \\ &= 1 + \mathrm{h}^1(C, \mathrm{W}_{r-1}(q)), \end{aligned}$$

and

$$\mathrm{h}^1(C, \mathrm{W}_{r-1}(q)) = \mathrm{ext}^0_{\mathcal{O}_C}(\mathrm{W}_{r-1}(q), \omega_C) = \hom_{\mathcal{O}_C}(\mathrm{W}_{r-1}(q), \mathcal{O}_C) = 0,$$

where the last equality is due to the fact that the bundles $\mathrm{W}_{r-1}(q)$ and $\mathcal{O}_C$ are stable, with

$$\mu(\mathrm{W}_{r-1}(q)) = \frac{1}{r-1} > 0 = \mu(\mathcal{O}_C).$$

Thus one can take $\mathrm{W}_r(q)$ as the unique non-split extension of $\mathrm{W}_{r-1}(q)$ by $\mathcal{O}_C$. More details can be found in [22], Lemma 3.1. □

CHAPTER 3

Elliptic surfaces

3.1. Elliptic Surfaces

3.1.1. Elliptic fibrations. Various aspects of the theory of elliptic surfaces can be found in [**3, 5, 8, 19, 20, 29, 34, 39**]. Let B be an integral scheme, with generic point η and function field $K = \mathcal{O}_{B,\eta}$. An *elliptic fibration over* B is a proper and flat B-scheme $X \to B$, whose geometric generic fiber $X_\eta \times_K K^{\mathrm{al}}$ is an irreducible, complete, non-singular K^{al}-curve of genus one, for some (and hence, every) algebraic closure K^{al} of K. *We assume, from now on, that the base and the total space of our elliptic fibrations are* non-singular.

3.1.2. Sections. Let $\pi : X \to B$ be an elliptic fibration. A *section* of π is a closed subscheme Σ of X such that the restriction of π to Σ is an isomorphism $\Sigma \to B$, or, equivalently, a morphism $\sigma : B \to X$ satisfying $\pi \circ \sigma = \mathrm{id}_B$.

Let Σ be a section of π. Then Σ intersects each closed fiber X_b of π transversely in the smooth point $\sigma(b)$; moreover, Σ is a (non-singular, prime) divisor of X, and thus its conormal sheaf is invertible (and canonically isomorphic to $\mathcal{O}_\Sigma(-\Sigma) := \mathcal{O}_X(-\Sigma)|_\Sigma$). The image in $\mathrm{Pic}(B)$ of $\mathcal{O}_\Sigma(-\Sigma)$ via the isomorphism $\pi|_\Sigma : \Sigma \to B$ (or, equivalently, the conormal sheaf of the closed immersion $\sigma : B \to X$) will be denoted by $\mathbb{L}$, and called the *fundamental line bundle* of the fibration. It turns out that $\mathbb{L}$ is isomorphic to $(\mathrm{R}^1\pi_*\mathcal{O}_X)^\vee$; in particular, it does not depend on the chosen section. We recall the following fact [**39, 19**]:

PROPOSITION 3.1.1. *Let $\pi : X \to B$ be an elliptic fibration with section Σ. Then:*

(1) *the map $\pi^* : \mathcal{O}_B \to \pi_*\mathcal{O}_X$ is an isomorphism;*
(2) *for each integer $r \neq 0$, the sheaf $\pi_*\mathcal{O}_X(r\Sigma)$ on B is 0 for $r < 0$, while it is (non-canonically) isomorphic to*

$$\mathbb{L}^{-2} \oplus \cdots \oplus \mathbb{L}^{-r} \oplus \mathcal{O}_B$$

for $r > 0$.

3.1.3. Weierstrass fibrations. A *Weierstrass fibration* is a pair (π, Σ), where π is an elliptic fibration *with integral fibers*, and Σ is a section of π.

Let $\mathbb{L}$ be the fundamental line bundle of a Weierstrass fibration $(\pi : X \to B, \Sigma)$. By the second item of Proposition 3.1.1, the sheaf $\pi_*\mathcal{O}_X(3\Sigma)$ is isomorphic to

$$\mathbb{L}^{-2} \oplus \mathbb{L}^{-3} \oplus \mathcal{O}_B =: \mathcal{S}_3.$$

Set $\mathbb{P} := \mathbb{P}_B(\mathcal{S}_3^\vee)$, and denote by $p : \mathbb{P} \to B$ the projection. Next, observe that

$$\mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3) \otimes p^*\mathbb{L}^6) = \mathrm{H}^0(B, \mathbb{L}^6 \otimes \mathrm{Sym}^3\mathcal{S}_3) \simeq \bigoplus_{i+j+k=3} \mathrm{H}^0(B, \mathbb{L}^{6-2i-3j})x^iy^jz^k.$$

The last vector space contains

$$\mathrm{H}^0(\mathcal{O}_B)y^2z \oplus \mathrm{H}^0(\mathcal{O}_B)x^3 \oplus \mathrm{H}^0(\mathbb{L}^4)xz^2 \oplus \mathrm{H}^0(\mathbb{L}^6)z^3.$$

It follows that, for each pair $(a_4, a_6) \in \mathrm{H}^0(\mathbb{L}^4) \oplus \mathrm{H}^0(\mathbb{L}^6)$, the expression

$$y^2z - (x^3 + a_4xz^2 + a_6z^3) \tag{3.1}$$

can be interpreted as a non-zero section of the line bundle $\mathcal{O}_{\mathbb{P}}(3) \otimes p^*\mathbb{L}^6$ on $\mathbb{P}$; thus its scheme of zeroes is an effective divisor $\bar{X}$ on $\mathbb{P}$. Moreover, if the *discriminant* section

$$\Delta := -16(4a_4^3 + 27a_6^2) \in \mathrm{H}^0(\mathbb{L}^{12}) \tag{3.2}$$

is non-zero, then the pair $(\bar{\pi}, \bar{\sigma})$, where $\bar{\pi} : \bar{X} \to B$ is the composition

$$\bar{X} \hookrightarrow \mathbb{P} \xrightarrow{p} B, \tag{3.3}$$

while $\bar{\sigma} : B \to \bar{X}$ is obtained by factoring, via $\bar{X} \hookrightarrow \mathbb{P}$, the section of p corresponding to the natural map $\mathcal{S}_3 \twoheadrightarrow \mathbb{L}^{-3}$, is a Weierstrass fibration on B. The divisor $\bar{X}$ may be singular, but a local computation shows that it is actually non-singular provided that the divisor of the discriminant section is reduced, as shown by the following:

Lemma 3.1.2. *Let us assume that the divisor of the discriminant section is reduced. Then $\bar{X}$ is a regular surface.*

Proof. Clearly, we just need to check regularity near the singular point p of a singular fiber $\bar{X}_b$; if we trivialize $\mathbb{L}$ in a neighborhood of b via an everywhere non-zero section e, we can write, in this neighborhood, $\Delta = \delta e^{12}$, where $\delta \in \mathrm{H}^0(\mathcal{O}_U)$ (for some open $U \subseteq B$ with $b \in U$) has a simple zero at b and no other zeros. Thus we can take the image of δ in $\mathcal{O}_{B,b}$ as a local uniformizer at b, and write the equation of $\bar{X}$ near p as

$$y^2 = x^3 + \alpha(\delta)x + \beta(\delta),$$

for suitable $\alpha, \beta \in \mathrm{H}^0(\mathcal{O}_U)$, satisfying the identity

$$\delta = -16(4\alpha(\delta)^3 + 27\beta(\delta)^2).$$

Taking the derivative with respect to δ we find

$$1 = -16(12\alpha^2\alpha' + 54\beta\beta'),$$

showing that $\alpha_0 := \alpha(0)$ and $\beta_0 := b(0)$ cannot simultaneously be 0; it follows that they are both non-zero, since they satisfy the equation

$$4\alpha_0^3 + 27\beta_0^2 = 0.$$

Thus $\bar{X}_b$ is the nodal cubic with equation

$$y^2 = x^3 + \alpha_0 x + \beta_0.$$

The node p of $\bar{X}_b$ has coordinates $(\delta, x, y) = (0, \bar{x}, 0)$, where $\bar{x}$ is the solution of

$$3\bar{x}^2 + \alpha_0 = 0, \qquad \bar{x}^3 + \alpha_0\bar{x} + \beta_0 = 0,$$

or

$$\bar{x}^2 = -\frac{\alpha_0}{3}, \qquad (\bar{x}^2 + \alpha_0)\bar{x} = -\beta_0,$$

which give

$$\bar{x} = -\frac{3\beta_0}{2\alpha_0}.$$

Let $f(\delta, x, y) := -y^2 + x^3 + \alpha(\delta)x + \beta(\delta)$. Then

$$\frac{\partial f}{\partial \delta}(0, \bar{x}, 0) = -\alpha'(0)\frac{3\beta_0}{2\alpha_0} + \beta'(0) \tag{3.4}$$

Solving $1 = -16(12\alpha_0^2\alpha'(0) + 54\beta_0\beta'(0))$ for $\beta'(0)$ and substituting in (3.4), we get

$$\begin{aligned}
\frac{\partial f}{\partial \delta}(0, \bar{x}, 0) &= -\alpha'(0)\frac{3\beta_0}{2\alpha_0} - \frac{1}{54\beta_0}\left(\frac{1}{16} + 12\alpha_0^2\alpha'(0)\right) \\
&= -\frac{3\alpha'(0)}{54\alpha_0\beta_0}\left(4\alpha_0^3 + 27\beta_0^2\right) - \frac{1}{864\beta_0} \\
&= -\frac{1}{864\beta_0} \neq 0.
\end{aligned}$$

□

Conversely, one has the following result [**39**, **19**]:

Proposition 3.1.3. *Let $(\pi : X \to B, \Sigma)$ be a Weierstrass fibration, and $\mathbb{L}$ its fundamental line bundle. Then the divisor 3Σ on X is very ample relatively to π, and there exists an isomorphism $\pi_*\mathcal{O}_X(3\Sigma) \simeq \mathbb{L}^{-2} \oplus \mathbb{L}^{-3} \oplus \mathcal{O}_B =: \mathcal{S}_3$, such that the image of the immersion of B-schemes $X \to \mathbb{P}_B(\mathcal{S}_3^\vee)$ corresponding to the surjection*

$$\pi^*\mathcal{S}_3 \xrightarrow{\simeq} \pi^*\pi_*\mathcal{O}_X(3\Sigma) \xrightarrow{\text{ev}} \mathcal{O}_X(3\Sigma)$$

is the divisor of zeros of (3.1) for suitable $a_i \in \mathrm{H}^0(\mathbb{L}^i)$, $i = 4, 6$. Moreover, the pair (a_4, a_6) is well defined up to the equivalence relation $\sim$ given by $(a_4, a_6) \sim (c^4a_4, c^6a_6)$ for $c \in k^\times$.

In the remainder of this thesis we will denote by B an irreducible, smooth, projective *curve* of genus $g \geqslant 0$, and by $\pi : X \to B$ a Weierstrass fibration, with section Σ, and fundamental line bundle $\mathbb{L}$. Moreover, we set

$$d := \deg(\mathbb{L}),$$

and we remark that, as a consequence of Proposition 3.1.3, we have $d \geqslant 0$, and $d = 0$ if and only if $\mathbb{L}^{12} \simeq \mathcal{O}_B$. *We will always assume that π has at worst nodal fibers.* For example, for a Weierstrass fibration as in (3.3), this is equivalent to the divisors of the sections a_4, a_6 having disjoint supports.

Let Z be the scheme of singularities of π. Then Z is a local complete intersection closed subscheme of X of dimension 0 and length $12d$, supported on the nodes of the singular fibers, and π maps Z isomorphically onto the scheme of zeros of the discriminant section Δ from (3.2). The following Lemma will be useful in the following.

Lemma 3.1.4. *For each $n \geqslant 0$ the sheaf $\mathrm{Sym}^n\mathcal{I}_Z$ is torsion-free.*

Proof. It is enough to prove the following algebraic statement: *let R be a UFD, and let I be the ideal of R generated by two relatively prime elements $f, g \in R$. Then the symmetric powers Sym^rI of I are torsion-free R-modules.* This is trivial for $r \in \{0, 1\}$.

So, let us fix an $r \geqslant 2$, and set $M := \mathrm{Sym}^r I$. The claim is then equivalent to the map $\mathrm{ev}_M : M \to M^{\vee\vee}$ being injective. From the presentation (actually, free resolution) of I

$$F_1 \xrightarrow{\beta} F_0 \xrightarrow{\alpha} I \to 0,$$

where F_0, F_1 are free R-modules with bases (e_1, e_2) and (e) respectively, while α and β are the maps

$$\alpha : \begin{cases} e_1 \mapsto f \\ e_2 \mapsto g \end{cases}, \qquad \beta : e \mapsto ge_1 - fe_2,$$

we get the presentation of M

$$G_1 \xrightarrow{\delta} G_0 \xrightarrow{\gamma} M \to 0, \tag{3.5}$$

where the modules $G_0 := \mathrm{Sym}^r F_0$ and $G_1 := F_1 \otimes \mathrm{Sym}^{r-1} F_0$ are free with bases

$$(u_i := e_1^i e_2^{r-i})_{0 \leqslant i \leqslant r}, \qquad (v_i := e \otimes e_1^i e_2^{r-1-i})_{0 \leqslant i \leqslant r-1}$$

respectively, $\gamma := \mathrm{Sym}^r \alpha$, and δ is induced by the multilinear map (symmetric in the variables $y_1, \ldots, y_{r-1}$)

$$F_1 \times F_0^{r-1} \to G_0, \qquad (x, y_1, \ldots, y_{r-1}) \mapsto \beta(x) y_1 \cdots y_{r-1};$$

thus, with respect to the bases $(u_i), (v_i)$, δ is represented by the $(r+1) \times r$ matrix with entries in R

$$\begin{pmatrix} -f & & & & \\ g & -f & & & \\ & g & \ddots & & \\ & & \ddots & -f & \\ & & & g & -f \\ & & & & g \end{pmatrix}.$$

Dualizing (3.5), we obtain the exact sequence

$$0 \to M^\vee \xrightarrow{\gamma^\vee} G_0^\vee \xrightarrow{\delta^\vee} G_1^\vee,$$

showing that $\gamma^\vee$ maps $M^\vee$ isomorphically onto the kernel of $\delta^\vee$. To compute this kernel, let us denote by K be the field of fractions of R. An easy computation shows that $\ker(\delta^\vee \otimes_R K)$ is generated by $\omega \otimes 1 \in G_0^\vee \otimes_R K$, where

$$\omega := \sum_{i=0}^r f^i g^{r-i} u_i^\vee \in G_0^\vee;$$

a standard argument (based on the assumptions that R is factorial, and that $\gcd(f, g) = 1$) then shows that $\ker(\delta^\vee) = R\omega$. Thus $M^\vee = R\bar{\omega}$, where $\bar{\omega} \in M^\vee$ is the preimage of ω via $\gamma^\vee$. This implies, in particular, that $\ker(\mathrm{ev}_M) = \ker(\bar{\omega}) = \gamma(\ker(\omega))$. Now, $\ker(\omega)$ consists of the elements $x := \sum_{i=0}^r a_i u_i$ of G_0 ($a_i \in R$) such that the pair (f, g) is a root of the (homogeneous, degree r) polynomial $\phi := \sum_{i=0}^r a_i X^i Y^{r-i} \in R[X, Y]$. An easy argument (again based on R being a UFD) shows that $\phi(f, g) = 0$ if and only if ϕ factors in $R[X, Y]$ as $(gX - fY) \cdot \psi$, for some homogeneous, degree $r-1$ polynomial ψ, say $\psi = \sum_{i=0}^{r-1} b_i X^i Y^{r-1-i}$.

Expanding the equation $\phi = (gX - fY)\cdot\psi$ then shows that the element $y := \sum_{i=0}^{r-1} b_i v_i \in G_1$ satisfies $\delta(y) = x$. Thus $\ker(\omega) \subseteq \mathrm{im}(\delta)$, and ev_M is injective, as claimed. $\square$

3.2. Differentials

3.2.1. Relative differentials and relative dualizing sheaf. Let Ω_π (resp., ω_π°) be the sheaf of relative Kähler differentials (resp., the relative dualizing sheaf) of the projection π. Let us recall the isomorphisms:

$$\Omega_\pi \simeq \mathcal{I}_Z \otimes \omega_\pi^\circ, \qquad \omega_\pi^\circ \simeq \pi^*\mathbb{L} \in \mathrm{Pic}(X).$$

They show, in particular, that Ω_π is torsion-free, with determinant $\det \Omega_\pi \simeq \omega_\pi^\circ$, and that, for each closed fiber X_b of π, the restriction $\mathcal{I}_{Z,b}$ of $\mathcal{I}_Z$ to X_b is isomorphic to the sheaf of Kähler differentials of X_b; thus, $\mathrm{h}^1(X_b, \mathcal{I}_{Z,b}) = \mathrm{h}^1(X_b, \Omega_{X_b}) = 1$. The last equality is obvious for X_b smooth, while, for X_b singular, it is a consequence of Proposition 2.1.1.

3.2.2. Higher direct images of the sheaf of ideals of the scheme of singular points of the projection. The previous considerations and Grauert's theorem imply that the coherent sheaf $\mathrm{R}^1\pi_*\mathcal{I}_Z$ on B is invertible. It can be computed explicitly using relative Serre duality [31]:

$$\begin{aligned}\mathrm{R}^1\pi_*\mathcal{I}_Z &\simeq \mathcal{E}xt^0_\pi(\mathcal{I}_Z, \omega_\pi^\circ)^\vee \simeq (\pi_*\mathcal{H}om_X(\mathcal{I}_X, \pi^*\mathbb{L}))^\vee \\ &\simeq \pi_*(\mathcal{I}_Z^\vee \otimes \pi^*\mathbb{L})^\vee \simeq \mathbb{L}^{-1} \otimes \pi_*\mathcal{O}_X \simeq \mathbb{L}^{-1}.\end{aligned}$$

The last isomorphism allows us to prove the following result.

Lemma 3.2.1. *There is a canonical isomorphism*

$$\pi_*\mathcal{I}_Z \simeq \mathbb{L}^{-12}.$$

Proof. We have the exact sequence of sheaves on X:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0;$$

thus, setting $Z' := \mathrm{Z}(\Delta)$, and taking the exact sequence of higher direct π-images, we get an exact sequence

$$0 \to \pi_*\mathcal{I}_Z \to \mathcal{O}_B \to \mathcal{O}_{Z'} \to \mathbb{L}^{-1} \to \mathbb{L}^{-1} \to 0,$$

showing that the map $\mathcal{O}_B \to \mathcal{O}_{Z'}$ is surjective. But, up to non-zero scalars, there is only one surjection $\mathcal{O}_B \to \mathcal{O}_{Z'}$; so

$$\pi_*\mathcal{I}_Z \simeq \mathcal{O}_B(-Z') \simeq \mathbb{L}^{-12},$$

as claimed. $\square$

3.2.3. The exact sequence of Kähler differentials. In the canonical exact sequence of sheaves of Kähler differentials

$$\pi^*\Omega_B \xrightarrow{\pi^*} \Omega_X \to \Omega_\pi \to 0,$$

the morphism π^* is generically injective by rank counting, so that (the sheaf $\pi^*\Omega_B$ being locally free and the scheme X integral) the sequence

$$\mathcal{E}: \quad 0 \to \pi^*\Omega_B \xrightarrow{\pi^*} \Omega_X \to \Omega_\pi \to 0 \tag{3.6}$$

is exact. In particular, one has the following special case of *Kodaira's formula* [39] for the canonical line bundle of X

$$\omega_X = \det \Omega_X \simeq \pi^*\omega_B \otimes \det \Omega_\pi \simeq \pi^*(\omega_B \otimes \mathbb{L}).$$

3.2.4. Restricting the cotangent bundle to the fibers. For a closed point b of B, the restriction

$$\mathcal{E}_b: \quad 0 \to \mathcal{O}_{X_b} \otimes_{k(b)} \Omega_B(b) \to \Omega_{X,b} \to \Omega_{X_b} \to 0$$

of the sequence (3.6) to X_b is exact (by the usual arguments); one can thus consider the *relative extension class* of (3.6), or *Kodaira-Spencer map* of π [28]. This is a global section $\xi(\mathcal{E}/B)$ of the sheaf on B

$$\begin{aligned} \mathcal{E}xt^1_\pi(\Omega_\pi, \pi^*\Omega_B) &\simeq \mathcal{E}xt^1_\pi(\Omega_\pi \otimes \pi^*\omega_B^{-1} \otimes \omega_\pi^\circ, \omega_\pi^\circ) \simeq \mathcal{E}xt^1_\pi(\pi^*(\mathbb{L}^2 \otimes \omega_B^{-1}) \otimes \mathcal{I}_Z, \omega_\pi^\circ) \\ &\simeq (\mathrm{R}^0\pi_*(\pi^*(\mathbb{L}^2 \otimes \omega_B^{-1}) \otimes \mathcal{I}_Z))^\vee \simeq \mathbb{L}^{-2} \otimes \omega_B \otimes (\pi_*\mathcal{I}_Z)^\vee \simeq \mathbb{L}^{10} \otimes \omega_B \end{aligned}$$

(the last isomorphism is due to Lemma 3.2.1), having the property that, for each $b \in B(k)$, the extension $\mathcal{E}_b$ splits if and only if the image of $\xi(\mathcal{E}/B)$ in $\mathbb{L}^{10} \otimes \omega_B \otimes k(b)$ is zero.

Proposition 2.1.1 makes clear that the sequence $\mathcal{E}_b$ is non-split if the fiber X_b is singular. It follows that if our fibration has at least one singular fiber (that is, if $d > 0$), then the relative extension class $\xi(\mathcal{E}/B)$ is a non-zero section of the line bundle $\mathbb{L}^{10} \otimes \omega_B$, and thus, for almost all $b \in B(k)$, the extension $\mathcal{E}_b$ is non-split. For such a point b, the bundle $\Omega_{X,b}$ is a then a non-split self-extension of the sheaf $\mathcal{O}_{X_b}$. Weak isomorphism classes of such extensions correspond bijectively to closed points of the projective space $\mathbb{P}\mathrm{Ext}^1_{X_b}(\mathcal{O}_{X_b}, \mathcal{O}_{X_b}) \simeq \mathbb{P}\mathrm{H}^1(\mathcal{O}_{X_b})$, which is zero-dimensional; the sheaf corresponding to its unique point is the rank 2 Atiyah bundle $\mathrm{I}_2(\mathcal{O}_{X_b})$ on X_b (cf. Proposition 2.2.3). *From now on we will assume that our fibration has at least one singular fiber, so that the restricted cotangent bundle $\Omega_{X,b}$ will be isomorphic to the rank 2 Atiyah bundle on X_b, for $b \in B(k)$* general.

Remark 3.2.2. If we restrict the sequence (3.6) to a nodal fiber X_b, we get the (non-split) extension of sheaves

$$0 \to \mathcal{O}_{X_b} \to \Omega_{X,b} \to \Omega_{X_b} \to 0 \tag{3.7}$$

(this is the same as the conormal sheaf sequence

$$0 \to \mathcal{I}_{X_b/X}/\mathcal{I}^2_{X_b/X} \to \Omega_{X,b} \to \Omega_{X_b} \to 0$$

of the closed immersion $X_b \hookrightarrow X$). The rank 2 bundle $\Omega_{X,b}$ can also be realized as a non-split extension of torsion-free sheaves. To justify this claim, let us denote by p the node of the fiber X_b, and by $\mathcal{I}_p \subset \mathcal{O}_{X_b}$ its ideal sheaf in X_b. In addition, let us write $\nu : \mathbb{P}^1 \to X_b$ for the normalization of X_b. Then, composing the map $\Omega_{X,b} \twoheadrightarrow \Omega_{X_b}$ in (3.7) with the adjoint

$$\Omega_{X_b} \twoheadrightarrow \nu_*\Omega_{\mathbb{P}^1} \simeq \nu_*(\mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \mathcal{I}_p,$$

of the map $\nu^* : \nu^*\Omega_{X_b} \to \Omega_{\mathbb{P}^1}$, we get a surjection $\Omega_{X,b} \to \mathcal{I}_p$, whose kernel $\mathcal{K}$ is a torsion-free sheaf of rank 1 and degree 1 on X_b. Moreover, the map $\mathcal{O}_{X_b} \hookrightarrow \Omega_{X,b}$ from (3.7) factors uniquely through the inclusion $\mathcal{K} \hookrightarrow \Omega_{X,b}$, and the cokernel of the resulting map $\mathcal{O}_{X_b} \hookrightarrow \mathcal{K}$ is isomorphic to the kernel of $\Omega_{X_b} \twoheadrightarrow \mathcal{I}_p$, which is the torsion subsheaf $k(p)$ of Ω_{X_b}. Thus $\mathcal{K} \simeq \nu_*\mathcal{O}_{\mathbb{P}^1}$. This shows that the desired extension of torsion-free sheaves of rank 1 is given by

$$0 \to \nu_*\mathcal{O}_{\mathbb{P}^1} \to \Omega_{X,b} \to \mathcal{I}_p \to 0.$$

By the way, the last sequence shows that the restriction of Ω_X to a singular fiber is unstable; this agrees with Theorem (1.1.1), since $\Delta(\Omega_X) = 2\mathrm{rk}(\Omega_X)\mathrm{c}_2(\Omega_X) = 4\mathrm{e}(X) = 48d \neq 0$.

3.2.5. Direct images of the symmetric powers of the (co)tangent bundle. We remark that, for each $r \geqslant 0$, there a natural map

$$\pi^*\omega_B^r \xrightarrow{\simeq} \pi^*\mathrm{Sym}^r\Omega_B \xrightarrow{\simeq} \mathrm{Sym}^r\pi^*\Omega_B \xrightarrow{\mathrm{Sym}^r\pi^*} \mathrm{Sym}^r\Omega_X. \tag{3.8}$$

We then have the following Proposition, which will be applied in Section 4.2 to the study of Higgs bundles on elliptic surfaces:

Proposition 3.2.3. *For each $r \geqslant 0$, the adjoint*

$$\omega_B^r \to \pi_*\mathrm{Sym}^r\Omega_X$$

of the map (3.8) is an isomorphism; furthermore, there is a canonical isomorphism

$$\pi_*\mathrm{Sym}^r\Theta_X \simeq \mathbb{L}^{-r}.$$

The proof of Proposition 3.2.3 we will make use of the following elementary lemma, which will also be applied several times in the sequel:

Lemma 3.2.4. *Let F be a torsion-free sheaf on X. Then the direct image π_*F is a locally free sheaf on B (possibly zero).*

Proof. Since B is a regular scheme of dimension 1, it is enough to show that the torsion subsheaf of π_*F is 0. So let us fix a subsheaf G of π_*F of dimension $\dim G \leqslant 0$, and let us factor the adjoint $\pi^*G \to F$ of the inclusion $G \hookrightarrow \pi_*F$ as

$$\pi^*G \twoheadrightarrow H \hookrightarrow F.$$

The dimension of the sheaf π^*G, and with it also that of its quotient H, does not exceed 1, since G has dimension $\leqslant 0$ and π has relative dimension 1. This forces $H = 0$, since F is torsion-free of dimension 2; thus the map $\pi^*G \to F$ and its adjoint $G \hookrightarrow \pi_*F$ are 0, showing that $G = 0$ and concluding the proof. $\square$

Proof of Proposition 3.2.3. We will prove the claim by induction on r. The case $r = 0$ is just the statement that the map $\pi^* : \mathcal{O}_B \to \pi_*\mathcal{O}_X$ is an isomorphism, which we already discussed (see Proposition 3.1.1). So, let us fix an integer $r \geqslant 1$, and let us assume the claim to be true for $r-1$. Starting from (3.6), we get a canonical exact sequence of sheaves on X:

$$\mathrm{Sym}^{r-1}\Omega_X \otimes \pi^*\omega_B \to \mathrm{Sym}^r\Omega_X \to \mathrm{Sym}^r\Omega_\pi \to 0, \tag{3.9}$$

in which the first two sheaves are locally free or rank r and $r+1$, respectively, while the third

$$\mathrm{Sym}^r\Omega_\pi \simeq \mathrm{Sym}^r(\mathcal{I}_Z \otimes \pi^*\mathbb{L}) \simeq \mathrm{Sym}^r\mathcal{I}_Z \otimes \pi^*\mathbb{L}^r$$

it torsion-free of rank 1 by Lemma 3.1.4. The usual arguments then show that the first map in (3.9) is injective. We thus have the short exact sequence of torsion-free sheaves on X

$$0 \to \mathrm{Sym}^{r-1}\Omega_X \otimes \pi^*\omega_B \to \mathrm{Sym}^r\Omega_X \to \mathrm{Sym}^r\Omega_\pi \to 0.$$

Taking direct images we get an exact sequence of (locally free, by Lemma 3.2.4) sheaves on B

$$0 \to \omega_B \otimes \pi_* \mathrm{Sym}^{r-1}\Omega_X \to \pi_* \mathrm{Sym}^r\Omega_X \to \mathbb{L}^r \otimes \pi_* \mathrm{Sym}^r\mathcal{I}_Z.$$

All the bundles appearing here actually have rank 1. This is clear for $\mathbb{L}^r \otimes \pi_* \mathrm{Sym}^r\mathcal{I}_Z$. For $\omega_B \otimes \pi_* \mathrm{Sym}^{r-1}\Omega_X$ and $\pi_* \mathrm{Sym}^r\Omega_X$ one uses the fact that the restriction of Ω_X to X_b is isomorphic to I_2 for b general, as discussed in Subsection 3.2.4; thus

$$(\mathrm{Sym}^r\Omega_X)_b \simeq \mathrm{Sym}^r\mathrm{I}_2 \simeq \mathrm{I}_{r+1},$$

the Atiyah bundle of rank $r+1$ on X_b (see Proposition 6.2.2 for the last isomorphism), which satisfies $\mathrm{h}^0(X_b, \mathrm{I}_{r+1}) = 1$ by (2.9). An easy argument now shows that the map

$$\omega_B \otimes \pi_* \mathrm{Sym}^{r-1}\Omega_X \to \pi_* \mathrm{Sym}^r\Omega_X$$

is an isomorphism. The inductive hypothesis guarantees that the first arrow in

$$\omega_B^r = \omega_B \otimes \omega_B^{r-1} \to \omega_B \otimes \pi_* \mathrm{Sym}^{r-1}\Omega_X \to \pi_* \mathrm{Sym}^r\Omega_X,$$

is an isomorphism. This completes the proof of the first claim, since the last composition coincides with the adjoint of the map (3.8).

The second claim is proved by taking the direct image of the isomorphism

$$\mathrm{Sym}^r\Theta_X \simeq \mathrm{Sym}^r(\Omega_X \otimes \omega_X^{-1}) \simeq \mathrm{Sym}^r\Omega_X \otimes \pi^*(\mathbb{L} \otimes \omega_B)^{-r},$$

and then using the projection formula and the first claim. □

3.3. Vector bundles on elliptic surfaces

Some references on the topic of vector bundles on elliptic fibrations and elliptic surfaces are [4, 17, 18, 19, 22, 24]. Let $(\pi : X \to B, \Sigma)$ be a non-isotrivial Weierstrass fibration with nodal singular fibers. We will be interested in vector bundles V on X whose restrictions to the closed fibers of π have trivial determinant, and are semistable (here and in the following by semistable we mean slope-semistable; the slope function on a singular fiber is defined by the using notion of degree recalled in Subsection 2.2.1). Following [22], we will recall how it is possible to associate, to each such V, a finite map $C_V \to B$ of degree equal to the rank of V, called the *spectral cover of* V. Properties of C_V such as, for instance, reducedness and integrality, strongly influence those of V; in particular, we will see in the next chapter (Chapter 4) that they restrict the types of Higgs fields which V can support.

3.3.1. Spectral covers. Let V be a rank r bundle on X, and let us assume that the restriction of V to each closed fiber X_b of π is semistable and has trivial determinant. Then, first of all, the sheaf $\delta := \pi_* \det V$ on B is invertible, and the natural map

$$\pi^*\delta = \pi^*\pi_* \det V \to \det V$$

is an isomorphism (one also says that $\det V$ is *vertical* in this case). Moreover, the twisted bundle

$$V(\Sigma) := V \otimes \mathcal{O}_X(\Sigma)$$

restricts to a semistable bundle of degree r on each fiber X_b, whence

$$\mathrm{h}^0(X_b, V(\Sigma)_b) = r, \qquad \mathrm{h}^1(X_b, V(\Sigma)_b) = 0.$$

Thus the sheaf $\mathrm{R}^0\pi_*V(\Sigma)$ on B is locally free of rank r, while the sheaf $\mathrm{R}^1\pi_*V(\Sigma)$ is 0. Let us recall the following result from [22]:

PROPOSITION 3.3.1. *For each closed point b of B, the restriction of the natural map*

$$\mathrm{ev} : \pi^*\pi_*V(\Sigma) \to V(\Sigma) \tag{3.10}$$

to the fiber X_b of π over b is an isomorphism at the generic point of X_b.

This implies, in particular, that the determinant of the map (3.10) is a non-zero map of line bundles on X, with source

$$\det \pi^*\pi_*V(\Sigma) \simeq \pi^* \det \pi_*V(\Sigma)$$

and target

$$\det V(\Sigma) \simeq \mathcal{O}_X(r\Sigma) \otimes \det V \simeq \mathcal{O}_X(r\Sigma) \otimes \pi^*\delta;$$

thus, setting

$$\mu := (\det \pi_*V(\Sigma))^{-1} \otimes \delta = (\det \pi_*V(\Sigma))^{-1} \otimes \pi_* \det V \in \mathrm{Pic}(B), \tag{3.11}$$

one can consider the map $\det(\mathrm{ev})$ as a non-zero section of the line bundle $\mathcal{O}_X(r\Sigma) \otimes \pi^*\mu$ on X, well defined up to multiplication by an element of $k^\times$; or as a curve

$$C_V := \mathrm{Z}(\det(\mathrm{ev})) \in |r\Sigma + \pi^*\mu|$$

on X, called the *spectral curve* of V (here $|r\Sigma + \pi^*\mu| \simeq \mathbb{P}\mathrm{H}^0(\mathcal{O}_X(r\Sigma)\otimes\pi^*\mu)$ is the linear system of the line bundle $\mathcal{O}_X(r\Sigma)\otimes\pi^*\mu$). The curve C_V is, in general, neither reduced nor irreducible; the restriction of the projection π to C_V is a finite morphism of degree r from C_V to B, called the *spectral cover* of V.

We remark that a slight modification of the previous construction allows one to define a spectral cover also in the case in which V is assumed to be semistable not on every fiber, but only on a general fiber.

3.3.2. Some relations of the spectral curve with the second Chern number. Set $\mathcal{S}_r := \mathrm{R}^0\pi_*\mathcal{O}_X(r\Sigma) \simeq \bigoplus_{i\in\{2,\dots,r,0\}}\mathbb{L}^{-i}$. Then, by virtue of the isomorphism

$$\mathrm{H}^0(X,\mathcal{O}_X(r\Sigma)\otimes\pi^*\mu) \simeq \mathrm{H}^0(B,\mathcal{S}_r\otimes\mu),$$

the map $\det(\mathrm{ev})$ can also be considered as a section of the rank r vector bundle on B

$$\mu\otimes\mathcal{S}_r \simeq \bigoplus_{i\in\{2,\dots,r,0\}} \mu\otimes\mathbb{L}^{-i};$$

or, equivalently, as an r-tuple

$$(s_2,\dots,s_r,s_0) \in \bigoplus_{i\in\{2,\dots,r,0\}} \mathrm{H}^0(B,\mu\otimes\mathbb{L}^{-i}),$$

well defined up to an overall scaling by a non-zero element of k. It follows that the vector space $\mathrm{H}^0(B,\mu\otimes\mathbb{L}^{-i})$ is non-zero for some $i\in\{2,\dots,r,0\}$; thus, setting

$$e := \deg(\mu),$$

we have that $0 \leqslant \deg(\mu\otimes\mathbb{L}^{-i}) = e - id$, or $e \geqslant id$. In particular, the integer e is non-negative. The following lemma shows that the inequality $e \geqslant 0$ is a Bogomolov-type inequality (see Section 5.3 for more details on the Bogomolov inequality). In [22] the same result is proved with different techniques under the assumption of regularity on almost all fibers (the notion of regular bundle has been recalled in Subsection 2.2.5):

LEMMA 3.3.2. *Let V be a locally free sheaf on X, fiberwise semistable and with vertical determinant. Then the degree e of the line bundle μ defined in* (3.11) *satisfies*

$$e = \mathrm{c}_2(V).$$

PROOF. By (3.11), we have

$$e = \deg(\delta) - \deg(\det\pi_*V(\Sigma)) = \deg(\delta) - \deg(\pi_*V(\Sigma)), \tag{3.12}$$

The degree of the bundle $\pi_*V(\Sigma)$ can be computed, for example, by using Grothendieck-Riemann-Roch theorem in the form

$$\mathrm{ch}(\pi_!V(\Sigma))\mathrm{td}(B) = \pi_*(\mathrm{ch}(V(\Sigma))\mathrm{td}(X)).$$

This produces the quality $\deg(\pi_*V(\Sigma)) = \deg(\delta) - \mathrm{c}_2(V)$, which, substituted in (3.12), returns the claim $e = \mathrm{c}_2(V)$.

□

Let us finally remark that if the second Chern number of V satisfies the inequality

$$\mathrm{c}_2(V) = e < id$$

for some $0 < i \leqslant r$, then one has

$$\deg(\mu \otimes \mathbb{L}^{-j}) = e - jd \leqslant e - id < 0$$

for all indexes $j \geqslant i$. Thus, the component $s_j \in \mathrm{H}^0(\mu \otimes \mathbb{L}^{-j})$ of the section $\det(\mathrm{ev}) = (s_2, \dots, s_r, s_0)$ vanishes for $j \geqslant i$, and so the spectral curve of V belongs to the image of the canonical injection of linear systems

$$|(i-1)\Sigma + \pi^*\mu| \hookrightarrow |r\Sigma + \pi^*\mu|\,, \qquad D \mapsto D + (r-i+1)\Sigma.$$

In particular, for $i = r$, one finds that $C_V \geqslant \Sigma$ is non-integral, while for $i = r-1$, $C_V \geqslant 2\Sigma$ is non-reduced. Let us collect the previous remarks for future reference in the following:

Lemma 3.3.3. *The following implications hold:*

(i) *if C_V is integral, then $\mathrm{c}_2(V) \geqslant rd$;*
(ii) *if C_V is reduced, then $\mathrm{c}_2(V) \geqslant (r-1)d$.*

3.3.3. Fiberwise regular bundles. Let us conclude the chapter by recalling two properties, both from [**22**], of fiberwise regular bundles with vertical determinant on X, which will be useful in Part 2. The first of these will be used in the proof of Proposition 4.4.4:

Proposition 3.3.4. *Let V be a vector bundle of rank r on X, which has vertical determinant and is fiberwise regular. Let $\varphi : C \to B$ be the spectral cover of V. Then there is an isomorphism of rank r bundles on B (and of sheaves of $\mathcal{O}_B$-algebras)*

$$\pi_* \mathit{End}(V) \simeq \varphi_* \mathcal{O}_C.$$

Finally, the following fact, which is a sufficient condition of regularity on a fiber, will play a role in the proof Proposition 5.5.2:

Proposition 3.3.5. *Let V be a fiberwise semistable bundle on X with vertical determinant, and let $C \to B$ be the spectral cover of V. Furthermore, let $b \in B(k)$ be such that the curve C is non-singular at all points lying over b. Then the restriction of V to X_b is regular. In particular, if C is non-singular, then V is fiberwise regular.*

Part 2

Higgs bundles on elliptic surfaces

CHAPTER 4

Higgs bundles on elliptic surfaces

4.1. Generalities on Higgs bundles

4.1.1. Higgs bundles. Some references on Higgs bundles are [**25**, **26**, **45**, **46**, **47**]. Let Y be a non-singular scheme of dimension $\geqslant 1$, and let V be a locally free sheaf on Y. A *Higgs field* on V is a morphism of $\mathcal{O}_Y$-modules $\phi : V \to V \otimes \Omega_Y$ (or a global section of the bundle $\mathit{End}(V) \otimes \Omega_Y$) satisfying the *integrability condition* (automatic if Y is a curve)

$$\phi \wedge \phi = 0 \quad \text{in} \quad \operatorname{Hom}(V, V \otimes \Omega_Y^2),$$

where $\phi \wedge \phi : V \to V \otimes \Omega_Y^2$ denotes the composition

$$V \xrightarrow{\phi} V \otimes \Omega_Y \xrightarrow{\phi\otimes 1} V \otimes \Omega_Y \otimes \Omega_Y \xrightarrow{1\otimes\wedge} V \otimes \Omega_Y^2;$$

a *Higgs bundle* on Y is a pair (V, ϕ), where V is a locally free sheaf on Y, and ϕ a Higgs field on V. The dual of an $\mathcal{O}_Y$-linear map $\phi : V \to V \otimes \Omega_Y$ can also be regarded as a map $\Theta_Y \to \mathit{End}(V)$, which induces a morphism of sheaves of (non-necessarily commutative) $\mathcal{O}_Y$-algebras from the tensor algebra $\mathrm{T}\Theta_Y := \bigoplus_{i\in\mathbb{Z}_{\geqslant 0}} \Theta_Y^{\otimes i}$ of Θ_Y to $\mathit{End}(V)$. The integrability condition for ϕ is then equivalent to the requirement that the last morphism factors through the projection $\mathrm{T}\Theta_Y \to \operatorname{Sym}\Theta_Y$, thus defining a structure of $\operatorname{Sym}\Theta_Y$-module on V.

This definition can be generalized, in a straightforward way, to that of *Higgs sheaf*.

4.1.2. Operations on Higgs bundles. We now briefly recall some operations on Higgs bundles which will be useful in the following sections of the chapter. First of all, one has a natural notion of *pull-back* for Higgs bundles: let $f : Z \to Y$ be a morphism of non-singular schemes, and let $\mathcal{V} = (V, \phi)$ be a Higgs bundle on Y; then one sets $f^*\mathcal{V} := (f^*V, \psi)$, where $\psi : f^*V \to f^*V \otimes \Omega_Z$ is the composition

$$f^*V \xrightarrow{f^*\phi} f^*(V \otimes \Omega_Y) \xrightarrow{\simeq} f^*V \otimes f^*\Omega_Y \xrightarrow{1\otimes f^*} f^*V \otimes \Omega_Z.$$

There is then a natural notion of *tensor product* of two Higgs bundles, say $\mathcal{V} = (V, \phi), \mathcal{W} = (W, \psi)$, defined on the same scheme Y: one sets

$$\mathcal{V} \otimes \mathcal{W} := (V \otimes W, \phi \otimes 1 + 1 \otimes \psi),$$

where the map $\theta := \phi \otimes 1 + 1 \otimes \psi : V \otimes W \to V \otimes W \otimes \Omega_Y$ is defined according to the following recipe: let (s_α), (t_A) be local frames for V and W, respectively, defined on the same open subscheme U of Y. Then one can write

$$\phi\,(s_\alpha) = s_\beta \otimes \phi^\beta_\alpha, \qquad \psi\,(t_A) = t_B \otimes \psi^B_A,$$

where $\phi^\beta_\alpha, \psi^B_A \in \Gamma\,(U, \Omega_Y)$. The action of θ on $s_\alpha \otimes t_A$ is defined as

$$\theta\,(s_\alpha \otimes t_A) := -s_\beta \otimes t_A \otimes \phi^\beta_\alpha + s_\alpha \otimes t_B \otimes \psi^B_A,$$

where the minus sign is needed for θ to satisfy the integrability condition. In fact,

$$\begin{aligned}(\theta\otimes 1)\circ\theta\,(s_\alpha\otimes t_A) &= -\left(-s_\gamma\otimes t_A\otimes\phi^\gamma_\beta + s_\beta\otimes t_C\otimes\psi^C_A\right)\otimes\phi^\beta_\alpha\\ &\quad+\left(-s_\gamma\otimes t_B\otimes\phi^\gamma_\alpha + s_\alpha\otimes t_C\otimes\psi^C_B\right)\otimes\psi^B_A,\end{aligned}$$

and the left hand side has image

$$s_\gamma\otimes t_A\otimes\phi^\gamma_\beta\wedge\phi^\beta_\alpha - s_\beta\otimes t_B\otimes\left(\psi^B_A\wedge\phi^\beta_\alpha+\phi^\beta_\alpha\wedge\psi^B_A\right)+s_\alpha\otimes t_C\otimes\psi^C_B\wedge\psi^B_A=0$$

in $V_1\otimes V_2\otimes\Omega^2_Y$, since the integrability condition for ϕ (resp., ψ) amounts to $\phi^\alpha_\beta\wedge\phi^\beta_\gamma=0$ for all α,γ (resp., $\psi^A_B\wedge\psi^B_C=0$ for all A,C), and since $\psi^B_A\wedge\phi^\beta_\alpha+\phi^\beta_\alpha\wedge\psi^B_A=0$ (the ϕ^β_α and ψ^B_A being 1-forms).

Finally, one can define the *dual* of a Higgs bundle $\mathcal{V}=(V,\phi)$ on a scheme Y as $\mathcal{V}^\vee:=(V^\vee,-\psi)$, where $\psi: V^\vee\to V^\vee\otimes\Omega_Y$ is the composition

$$V^\vee \xrightarrow{\simeq} V^\vee\otimes\mathcal{O}_Y \xrightarrow{1\otimes\mathrm{tr}^\vee} V^\vee\otimes\Omega_Y^\vee\otimes\Omega_Y \xrightarrow{\phi^\vee\otimes 1} V^\vee\otimes\Omega_Y.$$

4.1.3. The cone of Higgs fields on a vector bundle. Let $E:=\mathit{End}(V)$. Then the map

$$\mathrm{H}^0(E\otimes\Omega_Y)\to\mathrm{H}^0(E\otimes\Omega^2_Y),\qquad \phi\mapsto\phi\wedge\phi,$$

factors as

$$\mathrm{H}^0(E\otimes\Omega_Y)\xrightarrow{\phi\mapsto\phi^2}\mathrm{Sym}^2\mathrm{H}^0(E\otimes\Omega_Y)\xrightarrow{\ell}\mathrm{H}^0(E\otimes\Omega^2_Y),$$

where ℓ is k-linear. The image of the dual map $\ell^\vee:\mathrm{H}^0(E\otimes\Omega^2_Y)^\vee\to\mathrm{Sym}^2\mathrm{H}^0(E\otimes\Omega_Y)^\vee$ defines a linear system of quadrics in the projective space $\mathbb{P}\mathrm{H}^0(E\otimes\Omega_Y)$; let $\mathfrak{B}$ be its base scheme. Then the set $\mathcal{H}_V$ of Higgs fields on V is just the set of closed points of the affine cone over the projective scheme $\mathfrak{B}$. The cone $\mathcal{H}_V$ might very well be a vector subspace of $\mathrm{H}^0(E\otimes\Omega_Y)$, and in fact this is what will happen in the cases we will analyze in the following.

4.2. The Hitchin base

Let (V,ϕ) be a Higgs bundle on a complete, non-singular scheme Y, and let $r\geqslant 1$ be the rank of V. The characteristic polynomial $\det(T-\phi)$ of the twisted endomorphism $\phi\in\mathrm{H}^0(\mathit{End}(V)\otimes\Omega_Y)$ has the form

$$T^r+a_1T^{r-1}+\cdots+a_{r-1}T+a_r,$$

with $a_i\in\mathrm{H}^0(\mathrm{Sym}^i\,\Omega_Y)$. The affine space $\mathcal{B}_{Y,r}$ associated to the vector space

$$\bigoplus_{i=1}^r\mathrm{H}^0(\mathrm{Sym}^i\Omega_Y)$$

is called the *Hitchin base* (for rank r Higgs bundles on Y). Let $M^{\mathrm{H}}_{Y,r}$ be the moduli space of semistable Higgs bundles of rank r on Y [46]. Then the association $(V,\phi)\mapsto(a_1,\dots,a_r)$ induces a morphism

$$M^{\mathrm{H}}_{Y,r}\to\mathcal{B}_{Y,r},$$

called the *Hitchin fibration*. It is a fundamental tool for the study of $M^{\mathrm{H}}_{Y,r}$.

Our first result concerning Higgs bundles on elliptic surfaces is the following Proposition, which suggests a strong relation between Higgs bundles on the total space and on the base of an elliptic surface, and whose proof follows immediately from Proposition 3.2.3:

PROPOSITION 4.2.1. *Let $\pi : X \to B$ be a non-isotrivial Weierstrass fibration with nodal singular fibers. Then, for every integer $r \geqslant 1$, there is a canonical isomorphism*

$$\pi^* : \mathcal{B}_{B,r} \to \mathcal{B}_{X,r}. \tag{4.1}$$

Let us remark that, if we use the isomorphism (4.1) to identify $\mathcal{B}_{X,r}$ with $\mathcal{B}_{B,r}$, we have that the operation of pull-back via π induces a rational map of $\mathcal{B}_{B,r}$-schemes

$$M^{\mathrm{H}}_{B,r} \dashrightarrow M^{\mathrm{H}}_{X,r}.$$

Moreover, using again Proposition 3.2.3, we get an isomorphism of $\mathcal{O}_B$-algebras

$$\pi_* \operatorname{Sym} \Theta_X \simeq \operatorname{Sym} \mathbb{L}^{-1};$$

thus, the operation of push-forward via π induces a rational map from the moduli space of semistable Higgs bundles on X to the moduli space of semistable $\mathbb{L}$-valued pairs on B (see, e.g., [**42, 27**] for the notions of semistable pairs and their moduli).

The theme of the relation between objects of various types defined on the total space X and on the base B of an elliptic fibration $X \to B$ has already been treated in the literature. For example, in [**49**] one finds a relation between Higgs bundles with vanishing discriminant and vertical determinant on X and parabolic Higgs bundles on B.

4.3. Structure of Higgs fields, I

4.3.1. Vertical Higgs fields. From now on, let $(\pi : X \to B, \Sigma)$ be a non-isotrivial Weierstrass fibration with nodal singular fibers. Let V be a vector bundle on X. Then, starting from a linear map $\psi : V \to V \otimes \pi^*\omega_B$, we get a linear map $\phi : V \to V \otimes \Omega_X$, by composing ψ on the left with

$$1 \otimes \pi^* : V \otimes \pi^*\omega_B \to V \otimes \Omega_X.$$

Moreover, the map ϕ obtained in this way satisfies the integrability condition, i.e., it is a Higgs field on V; and the association $\psi \mapsto \phi$ is an injective k-linear map, since it can be regarded as the map on global sections induced by the injection

$$1 \otimes \pi^* : \mathcal{E}nd(V) \otimes \pi^*\omega_B \to \mathcal{E}nd(V) \otimes \Omega_X. \tag{4.2}$$

It follows that any bundle V on an elliptic fibration admits a family of Higgs fields parametrized by the vector space $\operatorname{Hom}(V, V \otimes \pi^*\omega_B)$, which we shall refer to as *vertical* Higgs fields.

4.3.2. The results. We now show that, under suitable assumptions on V, the only Higgs fields on V are the scalar ones. More precisely, one has the following:

PROPOSITION 4.3.1. *Let V be a rank r vector bundle on X with vertical determinant, and let E be its sheaf of endomorphisms. Suppose that V satisfies the following two assumptions:*

(1) *V is semistable on a general closed fiber of π;*
(2) *the spectral cover of V is reduced.*

Then the natural map

$$\pi_*(1 \otimes \pi^*) : \pi_*(E \otimes \pi^*\omega_B) \to \pi_*(E \otimes \Omega_X) \tag{4.3}$$

is an isomorphism of rank r bundles on B.

This immediately implies the following:

Corollary 4.3.2. *Let V be as in Proposition 4.3.1. Then the only Higgs fields on V are vertical.*

Proof. This follows immediately from Proposition 4.3.1 by taking the map on global sections induced by the isomorphism (4.3), and noting that this map is the same as the map on global sections induced by (4.2). ☐

Proof of Proposition 4.3.1. Tensoring the sequence (3.6) by the locally free sheaf E, we get the short exact sequence of torsion free sheaves on X

$$E \otimes \mathcal{E} : \quad 0 \to E \otimes \pi^*\omega_B \to E \otimes \Omega_X \to E \otimes \Omega_\pi \to 0;$$

taking direct images, we then get the exact sequence of (locally free, by Lemma 3.2.4) sheaves on B

$$0 \to \pi_*(E \otimes \pi^*\omega_B) \to \pi_*(E \otimes \Omega_X) \to \pi_*(E \otimes \Omega_\pi). \tag{4.4}$$

The reducedness of the spectral cover implies that, on a general closed fiber X_b of π, the bundle V_b is isomorphic to a direct sum of r pairwise distinct line bundles of degree zero on X_b, say

$$V_b \simeq \lambda_1 \oplus \cdots \oplus \lambda_r.$$

This, together with the fact that the sheaves $\pi^*\omega_B$ and Ω_π are trivial on a general closed fiber, entails immediately that the bundles $\pi_*(E \otimes \pi^*\omega_B)$ and $\pi_*(E \otimes \Omega_\pi)$ have rank r. Finally, using the fact that $\Omega_{X,b}$ is isomorphic to the Atiyah bundle $\mathrm{I}_2(\mathcal{O}_{X_b})$ on X_b for $b \in B(k)$ general, we find

$$(E \otimes \Omega_X)_b \simeq \bigoplus\nolimits_{i,j=1}^{r} \lambda_i \otimes \lambda_j^{-1} \otimes \mathrm{I}_2(\mathcal{O}_{X_b});$$

thus,

$$\mathrm{h}^0(X_b, (E \otimes \Omega_X)_b) = \sum_{i,j=1}^{r} \mathrm{h}^0(X_b, \lambda_i \otimes \lambda_j^{-1} \otimes \mathrm{I}_2(\mathcal{O}_{X_b})) = \sum_{i,j=1}^{r} \delta_{ij} = r,$$

showing that $\pi_*(E \otimes \Omega_X)$ has rank r too. It follows that (4.4) is an exact sequence of rank r bundles, and this implies immediately that the first map is an isomorphism, as claimed. ☐

4.3.3. Remarks on the assumptions of Proposition 4.3.1. We remark that both of the assumptions in 4.3.1 are necessary. To show that assumption (1) is necessary, let us set $\ell := \mathcal{O}_X(\Sigma)$, $V_1 = \ell \oplus \ell^{-1}$. Then V_1 is a rank 2 bundle with trivial determinant, but it is unstable on every fiber; moreover, the sheaf of endomorphisms $E_1 := \mathit{End}(V_1)$ satisfies $E_1 \simeq \ell^2 \oplus \mathcal{O}_X^{\oplus 2} \oplus \ell^{-2}$. Thus, the bundle

$$\pi_*(E_1 \otimes \pi^*\omega_B) \simeq \omega_B \otimes (\mathcal{O}_B^{\oplus 3} \oplus \mathbb{L}^{-2})$$

has rank 4; instead, the bundles $\pi_*(E_1 \otimes \Omega_X)$ has rank 6, since for $b \in B(k)$ general, we have, setting $p := \sigma(b) \in X_b$,

$$\mathrm{h}^0((E_1 \otimes \Omega_X)_b) = \mathrm{h}^0((\mathcal{O}_{X_b}(2p) \oplus \mathcal{O}_{X_b}^{\oplus 2} \oplus \mathcal{O}_{X_b}(-2p)) \otimes \mathrm{I}_2) = 4 + 2 + 0.$$

To show that assumption (2) is necessary, consider a rank $r \geqslant 2$ bundle V_2 on X whose restriction to a general fiber is isomorphic to the rank r Atiyah bundle I_r (such bundles exist; for $r = 2$, one could take, e.g., $V_2 = \Omega_X$, or $V_2 = \Theta_X$), so that V_2 is semistable (in fact, regular) and has trivial determinant on a general fiber, but it has non-reduced spectral cover $r\Sigma$. Let $E_2 := \mathit{End}(V_2)$. Then the bundle $\pi_*(E_2 \otimes \pi^*\omega_B)$ has rank r, while one shows easily that the bundle $\pi_*(E_2 \otimes \Omega_X)$ has rank $2r - 1 \neq r$.

4.4. Structure of Higgs fields, II

4.4.1. Scalar Higgs fields. Let Y be a non-singular, complete scheme; let V be a locally free sheaf on Y of rank $r \geqslant 1$, and E its sheaf of endomorphisms. The linear map $\mathcal{O}_Y \to E$ corresponding to the identity section of E is injective; thus, tensoring by Ω_Y, and then taking global sections, we get an injective k-linear map

$$\mathrm{H}^0(\Omega_Y) \hookrightarrow \mathrm{H}^0(E \otimes \Omega_Y), \qquad \alpha \mapsto \phi_\alpha.$$

This map factors through the inclusion of the cone $\mathcal{H}_V$ of Higgs fields on V into $\mathrm{H}^0(E \otimes \Omega_Y)$. In fact, for each global 1-form α on Y, the field ϕ_α acts on a local section s of V, defined on a Zariski open $\mathcal{U}$ of Y, as $\phi_\alpha(s) = s \otimes \alpha|_{\mathcal{U}}$; from this it follows immediately that ϕ_α satisfies the integrability condition $\phi_\alpha \wedge \phi_\alpha = 0$, and that, moreover, the matrix of ϕ_α with respect to a local frame $(e_1, \dots, e_r) : \mathcal{O}_{\mathcal{U}}^{\oplus r} \xrightarrow{\simeq} V|_{\mathcal{U}}$ for V is the (scalar) matrix of 1-forms on $\mathcal{U}$

$$\mathrm{diag}(\alpha|_{\mathcal{U}}, \dots, \alpha|_{\mathcal{U}});$$

thus the Higgs fields on V obtained in this way might be called *scalar* Higgs fields. This shows that V admits a family of Higgs fields of dimension $\mathrm{h}^0(\Omega_Y)$. For example, for a Weierstrass fibration $\pi : X \to B$ (satisfying the usual assumptions), one has, using the case $r = 1$ of Proposition 3.2.3,

$$\mathrm{h}^0(X, \Omega_X) = \mathrm{h}^0(B, \pi_*\Omega_X) = \mathrm{h}^0(B, \omega_B) = g,$$

where g is the genus of the base curve B.

We remark that scalar Higgs fields pull-back to scalar Higgs fields; more precisely, for a morphism $f : Z \to Y$, and a vector bundle V on Y, one has a commutative diagram

$$\begin{tikzcd} \mathrm{H}^0(\Omega_Y) \arrow[r, hook] \arrow[d] & \mathcal{H}_V \arrow[d] \\ \mathrm{H}^0(\Omega_Z) \arrow[r, hook] & \mathcal{H}_{f^*V} \end{tikzcd}$$

where the vertical arrows are the pull-back maps on global 1-forms and on Higgs fields.

Our next goal is to show that, under suitable assumptions, a vector bundle on a Weierstrass fibration supports only scalar Higgs fields.

4.4.2. The Universal Spectral Cover. We fix a Weierstrass fibration $(\pi : X \to B, \Sigma)$, and an integer $r \geqslant 2$. We recall that the sheaf $\mathcal{S} = \pi_*\mathcal{O}_X(r\Sigma)$ is a rank r vector bundle on B, by Proposition 3.1.1. The morphism $p : \mathbb{P}_B(\mathcal{S}) \to B$ is then a $\mathbb{P}^{r-1}$-bundle on B, while $\tilde{p} : \mathbb{P}_X(\pi^*\mathcal{S}) \to X$ is a $\mathbb{P}^{r-1}$-bundle on X; and there is a canonical morphism $\tilde{\pi} : \mathbb{P}_X(\pi^*\mathcal{S}) \to \mathbb{P}_B(\mathcal{S})$ such that

$$\tilde{\pi}^*\mathcal{O}_{\mathbb{P}_B(\mathcal{S})}(1) \simeq \mathcal{O}_{\mathbb{P}_X(\pi^*\mathcal{S})}(1), \tag{4.5}$$

and the square

$$\begin{tikzcd} \mathbb{P}_X(\pi^*\mathcal{S}) \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{p}"] & \mathbb{P}_B(\mathcal{S}) \arrow[d, "p"] \\ X \arrow[r, "\pi"] & B \end{tikzcd}$$

is cartesian. The evaluation map

$$\mathrm{ev} : \pi^*\mathcal{S} = \pi^*\pi_*\mathcal{O}_X(r\Sigma) \to \mathcal{O}_X(r\Sigma)$$

is surjective, since the line bundle $\mathcal{O}_X(r\Sigma)$ is relatively globally generated, r being $\geqslant 2$. Thus its kernel $\mathcal{K} := \ker(\mathrm{ev})$ is a vector bundle on X of rank $r-1$, whence its projectivization $\mathcal{C} := \mathbb{P}_X(\mathcal{K})$ is a $\mathbb{P}^{r-2}$-bundle on X (this is just the *relative incidence correspondence* of the family of complete linear systems on the fibers of π associated to the line bundle $\mathcal{O}_X(r\Sigma)$). $\mathcal{C}$ embeds into $\mathbb{P}_X(\pi^*\mathcal{S})$ as a non-singular, prime divisor, and its associated invertible sheaf $\mathcal{O}_{\mathbb{P}_X(\pi^*\mathcal{S})}(\mathcal{C})$ is given by

$$\begin{aligned} \mathcal{O}_{\mathbb{P}_X(\pi^*\mathcal{S})}(\mathcal{C}) &\simeq \mathcal{O}_{\mathbb{P}_X(\pi^*\mathcal{S})}(1) \otimes \tilde{p}^*\mathcal{O}_X(r\Sigma) \\ &\simeq \tilde{\pi}^*\mathcal{O}_{\mathbb{P}_B(\mathcal{S})}(1) \otimes \tilde{p}^*\mathcal{O}_X(r\Sigma), \end{aligned}$$

as results from the following lemma, the exact sequence

$$0 \to \mathcal{K} \to \pi^*\mathcal{S} \to \mathcal{O}_X(r\Sigma) \to 0,$$

and the isomorphism (4.5).

LEMMA 4.4.1. *Let Y be a scheme, and let*

$$0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{L} \to 0$$

be an exact sequence of locally free sheaves on Y, with $\mathcal{L}$ invertible. Then $\mathcal{D} := \mathbb{P}_Y(\mathcal{V}')$ is an effective Cartier divisor on $\mathbb{P} := \mathbb{P}_Y(\mathcal{V})$, and

$$\mathcal{O}_{\mathbb{P}}(\mathcal{D}) = \mathcal{O}_{\mathbb{P}}(1) \otimes \varpi^*\mathcal{L},$$

where $\varpi : \mathbb{P} \to Y$ is the projection.

PROOF. The epimorphism $\mathcal{V} \to \mathcal{L}$ is an element of

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{V}, \mathcal{L}) &= \mathrm{H}^0(Y, \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{V}, \mathcal{L})) = \mathrm{H}^0(Y, \mathcal{V}^\vee \otimes_{\mathcal{O}_Y} \mathcal{L}) \\ &= \mathrm{H}^0(Y, \mathcal{L} \otimes_{\mathcal{O}_Y} \varpi_*\mathcal{O}_{\mathbb{P}}(1)) = \mathrm{H}^0(Y, \varpi_*(\mathcal{O}_{\mathbb{P}}(1) \otimes_{\mathcal{O}_{\mathbb{P}}} \varpi^*\mathcal{L})) \\ &= \mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes_{\mathcal{O}_{\mathbb{P}}} \varpi^*\mathcal{L}), \end{aligned}$$

i.e., it can be regarded as a global section s of the line bundle $\mathcal{O}_{\mathbb{P}}(1) \otimes_{\mathcal{O}_{\mathbb{P}}} \varpi^*\mathcal{L}$ on $\mathbb{P}$; and it is easy to check that the scheme of zeros $\mathrm{Z}(s)$ of s is exactly $\mathcal{D}$, whence

$$\mathcal{O}_{\mathbb{P}}(\mathcal{D}) = \mathcal{O}_{\mathbb{P}}(\mathrm{Z}(\sigma)) = \mathcal{O}_{\mathbb{P}}(1) \otimes \varpi^*\mathcal{L},$$

as claimed. $\square$

The morphism $\tilde{p} : \mathbb{P}_X(\pi^*\mathcal{S}) \to X$ restricts to $\mathcal{C}$ to give the bundle projection $\mathbb{P}_X(\mathcal{K}) \to X$, while the restriction of $\tilde{\pi}$ to $\mathcal{C}$ is a finite, degree r morphism

$$\varphi : \mathcal{C} \to \mathbb{P}_B(\mathcal{S}).$$

The following commutative diagram summarizes the situation:

$$\begin{tikzcd} & & X \arrow[dr, "\pi"] & \\ \mathcal{C} \arrow[r, hook] \arrow[urr, bend left] \arrow[drr, bend right, "\varphi"'] & \mathbb{P}_X(\pi^*\mathcal{S}) \arrow[ur, "\tilde{p}"] \arrow[dr, "\tilde{\pi}"'] & & B \\ & & \mathbb{P}_B(\mathcal{S}) \arrow[ur, "p"'] & \end{tikzcd}.$$

The morphism φ is called the *universal spectral cover*, for the following reason: let V be a rank r bundle on X, with vertical determinant and semistable on a general fiber of π. We saw that the map $\det(\mathrm{ev} : \pi^*\pi_*V(\Sigma) \to V(\Sigma))$ can be considered as a section of $\mathcal{S} \otimes \mu$, for a suitable element μ of $\mathrm{Pic}(B)$, or as a morphism $\mathcal{S}^\vee \to \mu$, surjective at the generic point of B. Thus it gives rise to a rational section of $p : \mathbb{P}_B(\mathcal{S}) \to B$, which then extends to a global section $A_V : B \to \mathbb{P}_B(\mathcal{S})$ (since the scheme B is a non-singular curve, and the scheme $\mathbb{P}_B(\mathcal{S})$ is complete). One shows that the spectral cover $\varphi_V : C_V \to B$ of V is isomorphic (as a B-scheme) to the base change of the universal spectral cover via the section A_V; in other words, there exists a closed immersion $C_V \hookrightarrow \mathcal{C}$ such that the square

$$\begin{tikzcd} C_V \arrow[r, hook] \arrow[d, "\varphi_V"'] & \mathcal{C} \arrow[d, "\varphi"] \\ B \arrow[r, "A_V"] & \mathbb{P}_B(\mathcal{S}) \end{tikzcd}$$

is cartesian.

4.4.3. A useful lemma. The following lemma will be used in the proof of Proposition 4.4.4.

LEMMA 4.4.2. *There is an exact sequence of vector bundles on* $\mathbb{P}_B(\mathcal{S})$*:*

$$0 \to \mathcal{O}_{\mathbb{P}_B(\mathcal{S})} \to \varphi_* \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathbb{P}_B(\mathcal{S})}(-1) \otimes p^*(\mathcal{S}^\vee \otimes \mathbb{L}^{-1}). \tag{4.6}$$

PROOF. Let us set $\mathbb{P} := \mathbb{P}_B(\mathcal{S})$, $\mathbb{P}^* := \mathbb{P}_X(\pi^*\mathcal{S})$, and let us denote by ι the closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}^*$. Then, there is a short exact sequence of sheaves on $\mathbb{P}^*$:

$$0 \to \mathcal{O}_{\mathbb{P}^*}(-\mathcal{C}) \to \mathcal{O}_{\mathbb{P}^*} \to \iota_* \mathcal{O}_{\mathcal{C}} \to 0.$$

Taking the induced exact sequence of higher direct images along $\tilde{\pi} : \mathbb{P}^* \to \mathbb{P}$, one obtains an exact sequence of sheaves on $\mathbb{P}$:

$$0 \to \tilde{\pi}_* \mathcal{O}_{\mathbb{P}^*}(-\mathcal{C}) \to \tilde{\pi}_* \mathcal{O}_{\mathbb{P}^*} \to \tilde{\pi}_* \iota_* \mathcal{O}_{\mathcal{C}} \to \mathrm{R}^1 \tilde{\pi}_* \mathcal{O}_{\mathbb{P}^*}(-\mathcal{C}). \tag{4.7}$$

Let us compute the sheaves in the last sequence. We have

$$\begin{aligned} \mathrm{R}^i \tilde{\pi}_* \mathcal{O}_{\mathbb{P}^*}(-\mathcal{C}) &\simeq \mathrm{R}^i \tilde{\pi}_* (\tilde{\pi}^* \mathcal{O}_{\mathbb{P}}(-1) \otimes \tilde{p}^* \mathcal{O}_X(-r\Sigma)) \\ &\simeq \mathcal{O}_{\mathbb{P}}(-1) \otimes \mathrm{R}^i \tilde{\pi}_* (\tilde{p}^* \mathcal{O}_X(-r\Sigma)) \\ &\simeq \mathcal{O}_{\mathbb{P}}(-1) \otimes p^* \mathrm{R}^i \pi_* \mathcal{O}_X(-r\Sigma), \end{aligned}$$

where the sheaf $\mathrm{R}^i \pi_* \mathcal{O}_X(-r\Sigma)$ is zero for $i = 0$, while for $i = 1$ it is isomorphic to

$$\begin{aligned} \mathcal{E}xt^0_\pi(\mathcal{O}_X(-r\Sigma), \omega^\circ_\pi)^\vee &\simeq (\pi_*(\mathcal{O}_X(r\Sigma) \otimes \pi^* \mathbb{L}))^\vee \\ &\simeq \mathcal{S}^\vee \otimes \mathbb{L}^{-1}; \end{aligned}$$

it follows that

$$\tilde{\pi}_* \mathcal{O}_{\mathbb{P}^*}(-\mathcal{C}) = 0, \qquad \mathrm{R}^1 \tilde{\pi}_* \mathcal{O}_{\mathbb{P}^*}(-\mathcal{C}) \simeq \mathcal{O}_{\mathbb{P}}(-1) \otimes p^*(\mathcal{S}^\vee \otimes \mathbb{L}^{-1}).$$

Finally, we have

$$\tilde{\pi}_* \mathcal{O}_{\mathbb{P}^*} \simeq \tilde{\pi}_* \tilde{p}^* \mathcal{O}_X \simeq p^* \pi_* \mathcal{O}_X \simeq p^* \mathcal{O}_B \simeq \mathcal{O}_{\mathbb{P}},$$

and

$$\tilde{\pi}_* \iota_* \mathcal{O}_{\mathcal{C}} = (\tilde{\pi} \circ \iota)_* \mathcal{O}_{\mathcal{C}} = \varphi_* \mathcal{O}_{\mathcal{C}}.$$

Substituting in (4.7), we get the sequence (4.6). □

REMARK 4.4.3. The sequence (4.6) is actually part of the longer exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_B(\mathcal{S})} \to \varphi_* \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathbb{P}_B(\mathcal{S})}(-1) \otimes p^*(\mathcal{S}^\vee \otimes \mathbb{L}^{-1}) \to p^* \mathbb{L}^{-1} \to 0.$$

4.4.4. The result. Using lemma 4.4.2, we can prove the following proposition, where, as usual, we denote by $(\pi : X \to B, \Sigma)$ a non-isotrivial Weierstrass fibration (with smooth base and total space) with nodal singular fibers; moreover, d is the degree of the fundamental line bundle $\mathbb{L}$ of (π, Σ), and g the genus of the (complete, irreducible) curve B.

PROPOSITION 4.4.4. *Let V be a vector bundle on X of rank $r \geqslant 2$, which is fiberwise* regular *and has vertical determinant. Let $\varphi_V : C_V \to B$ be the spectral cover of V, and let us assume that at least one of the following two assumptions is satisfied:*

(i) *C_V is integral and $d \geqslant 2g - 1$;*

(ii) *C_V is reduced and* $\mathrm{c}_2(V) \geqslant (r-1)d + 2g - 1$.

Then every Higgs field on V is scalar.

Proof. Let us start by remarking that, under either of the assumptions (i) or (ii), we can apply to V Proposition 4.3.1 and its Corollary 4.3.2. Thus, any map $V \to V \otimes \Omega_X$ is a Higgs field, and these are the same as the global sections of the vector bundle on B

$$\pi_*(\pi^*\omega_B \otimes \mathcal{E}nd\, V) \simeq \omega_B \otimes \pi_* \mathcal{E}nd\, V.$$

Moreover, V being fiberwise regular, the bundle $\pi_* \mathcal{E}nd\, V$ is isomorphic to $\varphi_{V*}\mathcal{O}_{C_V}$, by Proposition 3.3.4.

Let us set $\mathcal{S} := \pi_*\mathcal{O}_X(r\Sigma) \simeq \bigoplus_{i\in\{2,\dots,r,0\}} \mathbb{L}^{-i}$, and let us denote by $p : \mathbb{P} \to B$ the projectivization of $\mathcal{S}$, and by $A_V : B \to \mathbb{P}$ the section of p corresponding to V; recall that we have a cartesian square with finite vertical arrows

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\varphi_V"'] & \mathcal{C} \arrow[d, "\varphi"] \\ B \arrow[r, hook, "A_V"] & \mathbb{P} \end{tikzcd},$$

where $\varphi : \mathcal{C} \to \mathbb{P}$ is the universal spectral cover. Thus, for every sheaf F on $\mathcal{C}$, the natural map

$$A_V^*\varphi_*F \to \varphi_{V*}(F \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{O}_C)$$

is an isomorphism (see, e.g., Lemma 5.6 of [22]). In particular, taking $F = \mathcal{O}_\mathcal{C}$, we get an isomorphism

$$A_V^*\varphi_*\mathcal{O}_\mathcal{C} \simeq \varphi_{V*}\mathcal{O}_{C_V}.$$

It follows that if we pull the exact sequence (4.6) back to B using $A_V : B \to \mathbb{P}$, we get an exact sequence of vector bundles on B

$$0 \to \mathcal{O}_B \to \varphi_{V*}\mathcal{O}_{C_V} \to \mu^{-1} \otimes \mathcal{S}^\vee \otimes \mathbb{L}^{-1},$$

where $\mu \simeq A_V^*\mathcal{O}_\mathbb{P}(1)$ is the line bundle (3.11); tensoring with ω_B we obtain an exact sequence

$$0 \to \omega_B \to \omega_B \otimes \varphi_{V*}\mathcal{O}_{C_V} \to \mathcal{S}^\vee \otimes \mathbb{L}^{-1} \otimes \mu^{-1} \otimes \omega_B. \tag{4.8}$$

The last bundle splits as $\lambda_2 \oplus \cdots \oplus \lambda_r \oplus \lambda_0$, where

$$\lambda_i := \mathbb{L}^{i-1} \otimes \mu^{-1} \otimes \omega_B \in \mathrm{Pic}(B).$$

Now one recalls that $d \geqslant 1$, and that $e := \deg(\mu)$ satisfies $e = \mathrm{c}_2(V)$ by Lemma 3.3.2, and the inequalities

$$e \geqslant \begin{cases} rd & , \text{ if } C_V \text{ is integral} \\ (r-1)d & , \text{ if } C_V \text{ is reduced} \end{cases}$$

by Lemma 3.3.3. It follows that, for each $i \in \{0, 2, \dots, r\}$, the degree

$$\deg(\lambda_i) = (i-1)d - e + 2g - 2$$

satisfies

$$\deg(\lambda_i) \leqslant (r-1)d - e + 2g - 2,$$

and the right and side of this inequality is negative under any of the assumptions (i) and (ii). Thus

$$\mathrm{H}^0(\mathcal{S}^\vee \otimes \mathbb{L}^{-1} \otimes \mu^{-1} \otimes \omega_B) = \bigoplus_i \mathrm{H}^0(\lambda_i) = 0,$$

and the arrow $\omega_B \to \omega_B \otimes \varphi_{V*}\mathcal{O}_{C_V}$ in (4.8) induces an isomorphism on global sections. It follows that the natural map $\mathrm{H}^0(\Omega_X) \hookrightarrow \mathcal{H}_V = \mathrm{Hom}(V, V \otimes \Omega_X)$ is an injection between vector spaces of the same dimension g, thus an isomorphism. □

4.5. Base point freeness

Proposition 4.4.4 is useful in conjunction with the following sufficient condition for the base-point freeness of linear systems on X of the form $|r\Sigma + \pi^*\mu|$ ($r \in \mathbb{Z}$, $\mu \in \mathrm{Pic}(B)$):

PROPOSITION 4.5.1. *Let r be an integer $\geqslant 2$, and let μ be a line bundle on B of degree $e \geqslant rd + 2g$. Then the linear system on X*

$$|r\Sigma + \pi^*\mu|$$

is base-point free.

PROOF. For each point $b \in B(k)$, the line bundle $\lambda := \mathcal{O}_X(r\Sigma) \otimes \pi^*\mu$ on X restricts on X_b to $\mathcal{O}_{X_b}(r\sigma(b)) \in \mathrm{Pic}(X_b)$, and the linear system $|r\sigma(b)|$ on X_b is base-point free, since r is assumed to be $\geqslant 2$. It is then enough to check that the restriction map

$$\mathrm{res} : \mathrm{H}^0(X, r\Sigma + \pi^*\mu) \to \mathrm{H}^0(X_b, r\sigma(b))$$

is surjective. Its cokernel injects into $\mathrm{H}^1(X, \lambda - X_b) = \mathrm{H}^1(X, r\Sigma + \pi^*(\mu - b))$; by the Leray spectral sequence we have

$$\mathrm{h}^1(X, \lambda - X_b) = \mathrm{h}^1(B, \mathrm{R}^0\pi_*(\lambda \otimes \mathcal{O}_X(-X_b))) + \mathrm{h}^0(B, \mathrm{R}^1\pi_*(\lambda \otimes \mathcal{O}_X(-X_b))),$$

with $\mathrm{R}^i\pi_*(\lambda \otimes \mathcal{O}_X(-X_b)) = \mathrm{R}^i\pi_*\mathcal{O}_X(r\Sigma) \otimes \mu \otimes \mathcal{O}_B(-b)$; the last sheaf is zero for $i = 1$, while for $i = 0$ it is isomorphic to $\bigoplus_{j \in \{2,\ldots,r,0\}} \mathbb{L}^{-j} \otimes \mu \otimes \mathcal{O}_B(-b)$. Thus we find

$$\begin{aligned} \mathrm{h}^1(X, \lambda - X_b) &= \sum_j \mathrm{h}^1(\mathbb{L}^{-j} \otimes \mu \otimes \mathcal{O}_B(-b)) \\ &= \sum_j \mathrm{h}^0(\omega_B \otimes \mathbb{L}^j \otimes \mu^{-1} \otimes \mathcal{O}_B(b)), \end{aligned}$$

and each of the summands in the last sum is zero, since

$$\deg(\omega_B \otimes \mathbb{L}^j \otimes \mu^{-1} \otimes \mathcal{O}_B(b)) = 2g - 2 + jd - e + 1 \leqslant rd + 2g - e - 1 \leqslant -1 < 0.$$

This shows that the map res is surjective, as claimed. □

CHAPTER 5

Applications to the conjecture

5.1. Stability

Some references on (semi)stable sheaves and their moduli are [**28**, **30**, **33**, **43**]. Let us recall that the notions of degree, slope, and slope-semistability for a sheaf on a projective scheme Y require the choice of an ample divisor H on Y (or of an open ray $\mathbb{R}_{>0} \cdot H$ in the ample cone of Y), which in this context is called a *polarization* on Y. A *polarized variety* is then a pair (Y, H), where Y is a non-singular, irreducible, projective scheme, and H a polarization on Y.

Let (Y, H) be a polarized variety, with $n := \dim(Y) \geqslant 1$. The *degree* of a coherent sheaf F on Y is the integer

$$\deg(F) := \mathrm{c}_1(F) \cdot H^{n-1};$$

if F has positive rank, the *slope* of F is the rational integer

$$\mu(F) := \frac{\deg(F)}{\mathrm{rk}(F)}.$$

A torsion-free sheaf F on Y is said to be *slope-semistable* if, for each non-zero, proper subsheaf S of F, one has

$$\mu(S) \leqslant \mu(F) \tag{5.1}$$

Replacing (5.1) with the inequality $\mu(S) < \mu(F)$ one obtains the definition of *slope-stable sheaf*. Since this is the only notion of stability that we shall consider, we will sometimes shorten slope-semistable to *semistable*.

5.2. The discriminant

Let Y be an irreducible, non-singular, projective scheme, and let F be a coherent sheaf of rank r on Y. The *discriminant* of F is the characteristic class

$$\Delta(F) := 2r\mathrm{c}_2(F) - (r-1)\mathrm{c}_1(F)^2 \in \mathrm{A}^2(Y).$$

(here $\mathrm{A}^2(Y)$ is the Chow group of codimension 2 cycles on Y modulo rational equivalence). If F is reflexive, one has the equality

$$\Delta(F) = \mathrm{c}_2(\mathit{End}(F)) = \mathrm{c}_2(F^\vee \otimes F) \tag{5.2}$$

(whose expansion in terms of the Chern roots of F might help explaining the choice of the name discriminant). In fact, in this case, F is locally free in codimension 2; thus, the natural map $F \otimes F^\vee \to \mathit{End}(F)$ is an isomorphism in codimension 2, whence $\mathrm{c}_i(\mathit{End}(F)) =$

$c_i(F \otimes F^\vee)$ for $i \in \{1,2\}$. This explains the second equality in (5.2), and shows that $c_1(End(F)) = ch_1(F \otimes F^\vee) = 0$. The vanishing of $c_1(End(F))$, implies, in turn, that

$$\begin{aligned} c_2(End F) &= -ch_2(F^\vee \otimes F) \\ &= -\left(2ch_0(F)\, ch_2(F) - ch_1^2(F)\right) \\ &= -\left(r\left(c_1^2(F) - 2c_2(F)\right) - c_1^2(F)\right) \\ &= 2rc_2(F) - (r-1)c_1^2(F) = \Delta(F), \end{aligned}$$

which is the first equality in (5.2).

It follows from the previous remarks (again assuming the sheaf F to be reflexive) that

$$\Delta(End(F)) = 2r^2 c_2(End(F)) = 2r^2\Delta(F).$$

5.3. The Bogomolov inequality

If Y has dimension $n \geqslant 2$ and is polarized by the ample divisor H, one can multiply the discriminant of F by a suitable power of H, to obtain an integer, called the *Bogomolov number* of F (with respect to H), and denoted by $B(F)$:

$$B(F) := \Delta(F) \cdot H^{n-2}.$$

When Y is a surface the integer $B(F)$ is independent of the polarization, and it is just the image of the discriminant $\Delta(F)$ in $\mathbb{Z}$ under the degree homomorphism $A^2(Y) = A_0(Y) \to \mathbb{Z}$.

The Bogomolov number is the subject of a remarkable theorem, the *Bogomolov inequality*, proved for the first time in [7] for a vector bundle on a complex surface (see [32] for a proof of the general statement):

THEOREM 5.3.1 (Bogomolov inequality). *Let (Y, H) be a polarized variety, and let F be a torsion-free coherent sheaf on Y. Then, if F is slope-semistable, the Bogomolov number of F with respect to H is non-negative.*

It is then natural to try to find a characterization of the class of slope-semistable sheaves with vanishing Bogomolov number. The main result in this direction is the following theorem [41, 12]:

THEOREM 5.3.2. *Let (Y, H) be a* complex *polarized variety. Then, for a locally free sheaf F on Y, the following are equivalent:*

(1) *F is slope-semistable and has vanishing discriminant in $H^4(Y, \mathbb{Q})$;*
(2) *for every irreducible, non-singular, projective curve C, and for every morphism $f : C \to Y$, the pull-back f^*F of F to C along f is semi-stable.*

Following [13], we call a sheaf F satisfying (2) of Theorem 5.3.2 *curve-semistable*. Then Theorem 5.3.2 can be paraphrased by saying that, on a complex polarized variety, the semistable locally free sheaves with vanishing discriminant are exactly the locally free sheaves that are curve-semistable.

5.4. The Higgs case and the conjecture

There is a notion of slope-semistabilty adapted to torsion-free Higgs sheaves on a polarized variety (Y, H): a torsion-free Higgs sheaf (F, ϕ) on Y is slope-semistable (with respect to H) if it satisfies the inequality (5.1) for the non-zero, proper subsheaves S of F that are *invariant* under the action of the Higgs field ϕ (i.e., such that the restriction of ϕ to S factors through the injection $(S \hookrightarrow F) \otimes \Omega_Y$). Then one has a version of Bogomolov inequality for Higgs sheaves (proved in [44] for *stable* Higgs bundles over *complex* varieties, and in [32] in the general case):

THEOREM 5.4.1 (Bogomolov inequality for Higgs sheaves). *Let (Y, H) be a polarized variety, and let (F, ϕ) be a torsion-free Higgs sheaf on Y. If (F, ϕ) is slope-semistable, then the Bogomolov number of F is non-negative.*

Once again, it is then natural to try to characterize the torsion-free, slope-semistable Higgs sheaves with vanishing Bogomolov number. In view of Theorem 5.3.2, the following statement appears to be natural. In it we use the notion of curve-semistability for a Higgs bundle (F, ϕ) on a polarized variety (Y, H): the Higgs bundle (F, ϕ) is *curve-semistable* if, for each pair (C, f), where C is an irreducible, non-singular, projective curve and $f : C \to Y$ a morphism, the pull-back Higgs sheaf $f^*(F, \phi)$ is semistable.

CONJECTURE 5.4.2. *Let (Y, H) be a* complex *polarized variety, and let (F, ϕ) be a Higgs bundle on Y. Then the following are equivalent:*

(1) *(F, ϕ) is semistable with vanishing discriminant in $\mathrm{H}^4(Y, \mathbb{Q})$;*
(2) *(F, ϕ) is curve-semistable.*

The implication (1) $\Rightarrow$ (2) of the previous conjecture was proved in [11, 12]. Moreover, the Higgs version [45] of the Metha-Ramanatan restriction theorem [37, 38] implies that a curve-semistable Higgs bundle is semistable. So, what is left to be proved is the statement that a curve-semistable Higgs bundle has vanishing discriminant, or, equivalently, that a torsion-free Higgs bundle with non-zero discriminant is unstable (i.e., non-semistable) when pulled back to a suitable curve.

It has been proved over the last few years that the conjecture is true for some classes of varieties, including those with nef tangent bundle [14], K3 surfaces [13], and, more generally, Calabi-Yau varieties [9]. Thus, if we restrict ourselves to the case of surfaces, we can consider the conjecture as proved for surfaces of Kodaira dimension $\leqslant 0$. The next case which is then natural to examine is that of surfaces of Kodaira dimension 1, the so-called honest elliptic surfaces. In the next section, we will use the results on Higgs bundles proved in the previous chapter (Chapter 4) to make some progress in the study of the conjecture in the case of elliptic surfaces.

5.5. Study of the conjecture on elliptic surfaces

Assume $k = \mathbb{C}$ in this section. Let $(\pi : X \to B, \Sigma)$ be a non-isotrivial Weierstrass fibration with nodal singular fibers. As always, we denote by the $\mathbb{L} \in \mathrm{Pic}^d(B)$ $(d \geqslant 1)$ the fundamental line bundle of X, and by g the genus of B. Let us fix a polarization H on X. For each Higgs bundle $\mathcal{V} = (V, \phi)$ on X, let us denote by $\mathcal{P}_{\mathcal{V}}$ the following claim:

if $\Delta(V)$ is non-zero, then there exists a pair (C, f), where C is a non-singular, irreducible, projective curve and $f : C \to X$ a morphism, such that the pull-back Higgs bundle $f^*\mathcal{V}$ is unstable.

Then, by our previous remarks, Conjecture 5.4.2 holds if and only if claim $\mathcal{P}_{\mathcal{V}}$ is true for every Higgs bundle $\mathcal{V}$ on X. Our first result states that it is enough to check the validity of $\mathcal{P}_{\mathcal{V}}$ for Higgs bundles $\mathcal{V}$ on X whose underlying locally free sheaf has trivial determinant and is semistable on the fibers of π:

PROPOSITION 5.5.1. *Assume claim $\mathcal{P}_{\mathcal{V}}$ to be true for Higgs bundles $\mathcal{V} = (V, \phi)$ on X such that V is fiberwise semistable and has trivial determinant. Then claim $\mathcal{P}_{\mathcal{V}}$ is true for every Higgs bundle $\mathcal{V}$ on X.*

PROOF. Let $\mathcal{V} = (V, \phi)$ be a Higgs bundle on X of rank $r \geqslant 2$, satisfying the condition $\Delta(V) \neq 0$. Let us denote the Higgs bundle $\mathcal{V} \otimes \mathcal{V}^\vee$ by $\mathcal{W} = (W, \psi)$ (see Section 4.1.2 for the definitions of tensor products and duals of Higgs bundles). Then the sheaf $W = V \otimes V^\vee \simeq \mathit{End}(V)$ is locally free of rank $r^2 \geqslant 4$ and trivial determinant, so it satisfies

$$\Delta(W) = 2r^2 c_2(W) = 2r^2 \Delta(V) \neq 0.$$

If the restriction $W_b \simeq V_b \otimes V_b^\vee$ of W to some closed fiber X_b of π is unstable, then so is the restriction V_b. In fact, one knows that the slope-semistability of a bundle is equivalent to that of its dual, and the tensor product of two slope-semistable bundles is slope-semistable. Moreover, if the restriction V_b is unstable, then so is the pull-back f^*V, where $f : C \to X$ is the composition

$$C \xrightarrow{\nu} X_b \hookrightarrow X,$$

$\nu : C \to X_b$ being the normalization of X_b. But, since the curve C has genus $\leqslant 1$, the instability of the bundle f^*V implies that of the Higgs bundle $f^*\mathcal{V}$ [**14, 16**], and claim $\mathcal{P}_{\mathcal{V}}$ is true.

Let us then assume that W is fiberwise semistable. By our assumption, claim $\mathcal{P}_{\mathcal{W}}$ is true, and hence the pull-back $f^*\mathcal{W} \simeq f^*\mathcal{V} \otimes f^*\mathcal{V}^\vee$ is unstable for some curve C, and some morphism $f : C \to X$; thus, so is the pull-back $f^*\mathcal{V}$ (by the properties of duals and tensor products of semistable *Higgs bundles* analogous to those valid for ordinary semistable bundles), showing that claim $\mathcal{P}_{\mathcal{V}}$ is true in this case too. □

Let us then focus our attention on Higgs bundles $\mathcal{V} = (V, \phi)$ on X such that V has rank $r \geqslant 2$ and trivial (or, more generally, vertical) determinant, and is fiberwise semistable. Such a bundle V has $\Delta(V) = 2r c_2(V)$, and $c_2(V) \geqslant 0$ (by Proposition 3.3.2). Thus, the assumption $\Delta(V) \neq 0$ in claim $\mathcal{P}_{\mathcal{V}}$ is actually equivalent to $\Delta(V) > 0$, or to $c_2(V) > 0$. Using the results from the previous section, we are able to prove that claim $\mathcal{P}_{\mathcal{V}}$ is true, as soon as $c_2(V)$ (or $\Delta(V)$) is enough big, and the spectral cover of V is sufficiently general. We see this as a strong indication that Conjecture 5.4.2 (or, at least, some *generic* version of it) is true.

PROPOSITION 5.5.2. *Let (r, e) be a pair of integers satisfying $r \geqslant 2$ and $e \geqslant rd + 2g$. Let us fix a line bundle μ of degree e on B, and let us denote by C a general element of the linear series $\mathbb{P} := |r\Sigma + \pi^*\mu|$ on X. Let $\mathcal{V} = (V, \phi)$ be a Higgs bundle on X whose underlying locally free sheaf V has rank r, vertical determinant, $\mathrm{c}_2(V) = e$, and is fiberwise semistable with spectral curve $C_V = C$. Then claim $\mathcal{P}_{\mathcal{V}}$ is true.*

PROOF. By Proposition 4.5.1, the linear series $\mathbb{P}$ is base-point free. Thus, the curve C can be assumed to be smooth (hence reduced). Proposition 3.3.5 then implies that the bundle V is fiberwise regular. Next, point (2) of Proposition 4.4.4 allows us to conclude that the Higgs field ϕ of $\mathcal{V}$ is scalar. Finally, by Theorem 5.3.2, the vector bundle V is not curve-semistable. So one can pick a curve D, and a morphism $f : D \to X$, such that the vector bundle f^*V is unstable; but then the Higgs bundle $f^*\mathcal{V}$ is also unstable, since its Higgs field is scalar (because ϕ is scalar). Thus $\mathcal{P}_{\mathcal{V}}$ holds true, as claimed. □

5.6. Bundles of fiber degree 1

The goal of this section is to show that Higgs bundles $\mathcal{V}$ on X of fiber degree 1 satisfy the conjecture, as soon as the line bundle $\mathbb{L}$ of the elliptic fibration is sufficiently ample (more precisely, as soon as $\deg \mathbb{L} \geqslant g$). This case is interesting because here one is able to produce explicitly a curve C and a morphism $f : C \to X$ such that the pull-back $f^*\mathcal{V}$ is unstable. Let us start with the relative version of Theorem 2.2.4:

THEOREM 5.6.1. *For every integer $r \geqslant 1$, and for every section $s : B \to X$ of $\pi : X \to B$, there exists a vector bundle $\mathcal{W}_r(s)$ on rank r on X such that, for every $b \in B$,*

$$(\mathcal{W}_r(s))_b \simeq \mathrm{W}_r(s(b))$$

(here, the left hand side denotes, as always, the restriction of $\mathcal{W}_r(s)$ to the fiber X_b of π over b, while the right hand side is, as in Therem 2.2.4, the unique stable bundle of rank r on X_b satisfying $\det \mathrm{W}_r(s(b)) = \mathcal{O}_{X_b}(s(b))$). The bundle $\mathcal{W}_r(s)$ is unique up to tensoring with the pull-back of a line bundle on B, and it admits a length r filtration

$$0 = \mathcal{W}_0 \subsetneq \mathcal{W}_1 \subsetneq \cdots \subsetneq \mathcal{W}_r = \mathcal{W}_r(s), \tag{5.3}$$

whose quotients are isomorphic, respectively, to

$$\pi^*\mathbb{L}^{r-1}, \dots, \pi^*\mathbb{L}, \mathcal{O}_X(s(B)).$$

REMARK 5.6.2. From the filtration (2.2.4) one can compute the Chern classes and the discriminant of $\mathcal{W}_r(s)$:

$$\det \mathcal{W}_r(s) = \mathcal{O}_X(s(B)) \otimes \prod_{i=1}^{r-1} \pi^*\mathbb{L}^i = \pi^*\mathbb{L}^{\frac{r(r-1)}{2}} \otimes \mathcal{O}_X(s(B)),$$

$$(\mathrm{c}_1(\mathcal{W}_r(s))^2) = (s(B)^2) + 2\left(\mathcal{O}_X(s(B)) \cdot \pi^*\mathbb{L}^{\frac{r(r-1)}{2}}\right) = -d + dr(r-1) = d\left(r^2 - r - 1\right),$$

$$\mathrm{c}_2\left(\mathcal{W}_r(s)\right) = \sum_{i=1}^{r-1}(\pi^*\mathbb{L}^i \cdot \mathcal{O}_X(s(B))) = d\frac{r(r-1)}{2},$$

$$\Delta(\mathcal{W}_r(s)) = dr^2(r-1) - d(r-1)\left(r^2 - r - 1\right) = d\left(r^2 - 1\right).$$

In particular, one sees that the discriminant is positive for $r \geqslant 2$.

Proof of Theorem 5.6.1. As usual, the existence is by induction on r. For $r = 1$, one can take

$$\mathcal{W}_1(s) = \mathcal{O}_X(s(B)).$$

It satisfies

$$\mathrm{R}^1\pi_*\left(\mathcal{W}_1(s)^\vee\right) \simeq \left(\mathrm{R}^0\pi_*\left(\mathcal{O}_X(s(B)) \otimes \pi^*\mathbb{L}\right)\right)^\vee \simeq \mathbb{L}^{-1}.$$

Now let us fix an integer $r \geqslant 2$, and let us assume that the bundle $\mathcal{W}_{r-1}(s)$ has been defined, and that, in addition, it satisfies $\mathrm{R}^1\pi_*\left(\mathcal{W}_{r-1}(s)^\vee\right) \simeq \mathbb{L}^{-(r-1)}$. To define $\mathcal{W}_r(s)$, we look for an extension of $\mathcal{W}_{r-1}(s)$ by a line bundle on X trivial on every fiber of π (thus of the form $\pi^*\lambda$ for some $\lambda \in \operatorname{Pic}(B)$), such that the image of its extension class

$$\xi \in \operatorname{Ext}^1_X\left(\mathcal{W}_{r-1}(s), \pi^*\lambda\right) \simeq \mathrm{H}^1\left(X, \mathcal{W}_{r-1}(s)^\vee \otimes \pi^*\lambda\right)$$

in

$$\mathrm{H}^0\left(B, \mathrm{R}^1\pi_*\left(\mathcal{W}_{r-1}(s)^\vee \otimes \pi^*\lambda\right)\right) \simeq \mathrm{H}^0\left(B, \lambda \otimes \mathbb{L}^{-(r-1)}\right)$$

is an everywhere non-zero section. The last requirement forces $\lambda \simeq \mathbb{L}^{r-1}$. Thus, let us set $\lambda := \mathbb{L}^{r-1}$. Then, the exact sequence

$$\begin{aligned} 0 \;\to\; & \mathrm{H}^1\left(B, \mathrm{R}^0\pi_*\left(\mathcal{W}_{r-1}(s)^\vee \otimes \pi^*\lambda\right)\right) \to \mathrm{H}^1\left(X, \mathcal{W}_{r-1}(s)^\vee \otimes \pi^*\lambda\right) \to \\ \to\; & \mathrm{H}^0\left(B, \mathrm{R}^1\pi_*\left(\mathcal{W}_{r-1}(s)^\vee \otimes \pi^*\lambda\right)\right) \to 0 \end{aligned}$$

and the equality

$$\mathrm{R}^0\pi_*\left(\mathcal{W}_{r-1}(s)^\vee \otimes \pi^*\lambda\right) \simeq \lambda \otimes \mathrm{R}^0\pi_*\left(\mathcal{W}_{r-1}(s)^\vee\right) = 0$$

give an isomorphism

$$\begin{aligned} \mathrm{H}^1\left(X, \mathcal{W}_{r-1}(s)^\vee \otimes \pi^*\lambda\right) &\simeq \mathrm{H}^0\left(B, \mathrm{R}^1\pi_*\left(\mathcal{W}_{r-1}(s)^\vee \otimes \pi^*\lambda\right)\right) \\ &\simeq \mathrm{H}^0\left(B, \mathcal{O}_B\right) = k. \end{aligned}$$

Thus there exists a unique non-split extension $\mathcal{W}_r(s)$ of $\mathcal{W}_{r-1}(s)$ by $\pi^*\mathbb{L}^{r-1}$, restricting to $\mathrm{W}_r(s(b))$ on every fiber X_b.

As to the uniqueness, let $\mathcal{W}$ be a vector bundle of rank r on X whose restriction to every fiber X_b is isomorphic to the unique stable bundle on X_b of rank r and determinant $\mathcal{O}_{X_b}(s(b))$. Then the sheaf $\pi_*\mathcal{H}om\left(\mathcal{W}, \mathcal{W}_r(s)\right)$ is a line bundle μ on B, and

$$\pi_*\left(\mathcal{H}om\left(\mathcal{W} \otimes \pi^*\mu, \mathcal{W}_r(s)\right)\right) \simeq \pi_*\left(\pi^*\mu^{-1} \otimes \mathcal{H}om\left(\mathcal{W}, \mathcal{W}_r(s)\right)\right) \simeq \mathcal{O}_B.$$

The (unique up to scalars) non-zero element of

$$\mathrm{H}^0\left(B, \mathcal{O}_B\right) \simeq \operatorname{Hom}\left(\mathcal{W} \otimes \pi^*\mu, \mathcal{W}_r(s)\right)$$

is then a morphism $\mathcal{W} \otimes \pi^*\mu \to \mathcal{W}_r(s)$ which restricts on every fiber X_b to the unique (up to scalars) isomorphism $\mathcal{W}_b \to \mathrm{W}_r(s(b))$, thus an isomorphism. $\square$

We shall also need the following result (this is Corollary 4.8 of [22]):

Proposition 5.6.3. *The pull-back of the filtration (5.3) along the morphims $s : B \to X$ splits. Thus there is an isomorphism*

$$s^*\mathcal{W}_r(s) \simeq \mathbb{L}^{r-1} \oplus \cdots \oplus \mathbb{L} \oplus \mathbb{L}^{-1}.$$

We are now ready to prove that Higgs bundles of fiber degree 1 satisfy the conjecture:

PROPOSITION 5.6.4. *Let us assume that the fundamental line bundle $\mathbb{L}$ of the Weierstrass fibration $\pi : X \to B$ satisfies the inequality $d = \deg \mathbb{L} \geqslant \mathrm{g}(B) = g$. Let $(\mathcal{W}, \phi)$ be a Higgs bundle on X of rank $r \geqslant 2$ and fiber degree 1, and let $s : B \to X$ be the section of π such that, for each $b \in B$, the restriction of $\mathcal{W}$ to the fiber of π over b is isomorphic to the bundle $\mathrm{W}_r(s(b))$ on X_b. Then the pull-back of $(\mathcal{W}, \phi)$ along s is unstable.*

PROOF. By Theorem 5.6.1, the bundle $\mathcal{W}$ is isomorphic to $\mathcal{W}_r(s(b)) \otimes \pi^*\mu$ for some $\mu \in \mathrm{Pic}(B)$; thus, using Proposition 5.6.3, one obtains

$$s^*\mathcal{W} \simeq s^*\mathcal{W}_r(s) \simeq \mathbb{L}^{r-1} \oplus \cdots \oplus \mathbb{L} \oplus \mathbb{L}^{-1}.$$

It follows that the bundle $s^*\mathcal{W}$ has slope

$$\mu(s^*\mathcal{W}) = \frac{1}{r}\left(d\frac{r(r-1)}{2} - d\right) = \frac{d(r+1)(r-2)}{2r}.$$

This is strictly greater then the slope $\mu(\mathbb{L}^{-1}) = -d$ of its quotient $\mathcal{W} \twoheadrightarrow \mathbb{L}^{-1}$, showing that $s^*\mathcal{W}$ is unstable. In addition, the assumption $d \geqslant g$ guarantees that

$$\mathrm{Hom}(\mathbb{L} \oplus \cdots \oplus \mathbb{L}^{r-1}, \mathbb{L}^{-1} \otimes \omega_B) = 0.$$

Thus the kernel $\mathbb{L} \oplus \cdots \oplus \mathbb{L}^{r-1}$ of the projection $\mathcal{W} \twoheadrightarrow \mathbb{L}^{-1}$ is ϕ-invariant, so that $\mathbb{L}^{-1}$ is a Higgs quotient and the Higgs bundle $s^*(\mathcal{W}, \phi)$ is unstable, as claimed. ☐

CHAPTER 6

I_2-valued pairs on elliptic curves

6.1. Motivation and generalities on I_2-valued pairs

As usual, let $(\pi : X \to B, \Sigma)$ be a non-isotrivial Weierstrass fibration with nodal singular fibers. Let us denote by η the generic point of the base curve B, and by $K = \mathcal{O}_{B,\eta}$ its function field. Then the pair (E, o), where $E := X_\eta$ is the generic fiber of π, and $o \in E(K)$ the generic point of the section Σ, is an elliptic curve over K.

Now let V be a vector bundle on X. Then the operation of pull-back to the generic fiber gives us a k-linear map

$$\mathrm{Hom}_X(V, V \otimes \Omega_X) \to \mathrm{Hom}_E(V_\eta, V_\eta \otimes \mathrm{I}_2), \qquad \phi \mapsto \phi_\eta$$

where $\mathrm{I}_2 \simeq (\Omega_X)_\eta$ is the unique non-split self-extension of $\mathcal{O}_E$. The image ϕ_η of a Higgs field ϕ on V under this map satisfies $\phi_\eta \wedge \phi_\eta = 0$ in $\mathrm{End}_E(V_\eta)$, where $\phi_\eta \wedge \phi_\eta$ is the composition

$$V_\eta \xrightarrow{\phi_\eta} V_\eta \otimes \mathrm{I}_2 \xrightarrow{\phi_\eta \otimes 1} V_\eta \otimes \mathrm{I}_2 \otimes \mathrm{I}_2 \xrightarrow{1 \otimes \wedge} V_\eta \otimes \bigwedge^2 \mathrm{I}_2 \simeq V_\eta.$$

Conversely, if $\phi \in \mathrm{Hom}_X(V, V \otimes \Omega_X)$ is such that its image ϕ_η in $\mathrm{Hom}_E(V_\eta, V_\eta \otimes \mathrm{I}_2)$ satisfies $\phi_\eta \wedge \phi_\eta = 0$ in $\mathrm{End}_E(V_\eta)$, then ϕ turns out to be a Higgs field on V. This suggests that, in order to study Higgs bundles on X, it might useful to take a closer look at pairs (W, ψ), where W is a vector bundle on E, and

$$\psi : W \to W \otimes \mathrm{I}_2$$

an $\mathcal{O}_E$-linear map satisfying the *integrability condition* $\psi \wedge \psi = 0$ in $\mathrm{End}_E(W)$. Let us call such pairs *integrable* I_2*-valued pairs.*

One has a natural notion of slope-(semi)stability for (integrable) I_2-valued pairs, which is completely analogous to the one we used for Higgs bundles: a (non-necessarily integrable) pair (W, ψ) is *slope-semistable* (resp., *slope-stable*) if and only if W is torsion-free and, for every non-zero, proper, ψ-invariant subsheaf S of W, one has $\mu(S) \leqslant \mu(W)$ (resp., $\mu(S) < \mu(W)$).

As in the case of Higgs bundles on elliptic curves, the slope-semistability of an I_2-valued pair (W, ψ) is equivalent tho the slope-semistability of its underlying bundle W. This is due to the fact that the value bundle I_2 is semistable of degree 0 [14].

6.2. The Atiyah bundles, II

Let us briefly recall the definition and basic properties of the Atiyah bundles on an elliptic curve (E, o), defined over a (non-necessarily algebraically closed) field $\mathbb{K}$ of characteristic 0. First of all, one has an analogue of Theorem 2.2.3: for every integer $r \geqslant 1$, there exists a unique vector bundle $\mathrm{I}_r = \mathrm{I}_r(\mathcal{O}_E)$ on E of rank r and degree 0, which is indecomposable (with respect to $\oplus$) and has $\mathrm{H}^0(\mathrm{I}_r) \neq 0$. I_r is called the rank r *Atiyah bundle* on E.

Let us recall the inductive construction of the I_r, starting from $\mathrm{I}_1 := \mathcal{O}_E$. To define I_2, one looks for non-split extensions of $\mathcal{O}_E$ by I_1, which are parametrized by

$$\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_E, \mathrm{I}_1) = \mathbb{P}\mathrm{H}^1(\mathcal{O}_E) \simeq \mathbb{P}^0_{\mathbb{K}}.$$

Let us denote by $\mathcal{E}$ the only such extension, and by I_2 its total space:

$$\mathcal{E}: \quad 0 \to \mathrm{I}_1 \xrightarrow{i} \mathrm{I}_2 \xrightarrow{p} \mathcal{O}_E \to 0. \tag{6.1}$$

In the exact sequence of cohomology groups induced by $\mathcal{E}$,

$$0 \to \mathrm{H}^0(\mathrm{I}_1) \xrightarrow{\mathrm{H}^0(i)} \mathrm{H}^0(\mathrm{I}_2) \xrightarrow{\mathrm{H}^0(p)} \mathrm{H}^0(\mathcal{O}_E) \xrightarrow{d} \mathrm{H}^1(\mathrm{I}_1),$$

the connecting homomorphism d sends the function $1 \in \mathrm{H}^0(\mathcal{O}_E) = \mathbb{K} \cdot 1 \simeq \mathbb{K}$ to the class $0 \neq c \in \mathrm{Ext}^1_E(\mathcal{O}_E, \mathrm{I}_1) \simeq \mathrm{H}^0(\mathcal{O}_E)^\vee = \mathbb{K} \cdot 1^\vee \simeq \mathbb{K}$ of the extension $\mathcal{E}$, so it is an isomorphism. It follows that the map $\mathrm{H}^0(i)$ is also an isomorphism. In particular, one has $\mathrm{h}^0(\mathrm{I}_2) = \mathrm{h}^0(\mathrm{I}_1) = 1$. Observe now that, by (6.1), one has $\det(\mathrm{I}_2) \simeq \mathrm{I}_1 \otimes \mathcal{O}_E \simeq \mathcal{O}_E$, so that

$$\mathrm{I}_2^\vee \simeq \mathrm{I}_2 \otimes \det(\mathrm{I}_2)^{-1} \simeq \mathrm{I}_2.$$

Thus $\mathrm{h}^1(\mathrm{I}_2) = \mathrm{h}^0(\mathrm{I}_2^\vee) = 1$ (the equality $\mathrm{h}^1(\mathrm{I}_2) = 1$ can also be deduced from the long exact sequence of cohomology groups associated to $\mathcal{E}$, as we will shortly do for I_3). Finally, we claim that I_2 is indecomposable. In fact, if it were decomposable, then by virtue of the isomorphism $\det(\mathrm{I}_2) \simeq \mathcal{O}_E$, it would be necessarily of the form $\ell \oplus \ell^{-1}$ for some $\ell \in \mathrm{Pic}(E)$. Taking global sections and using $\mathrm{h}^0(\mathrm{I}_2) = 1$, one would have $\mathrm{h}^0(\ell) = 1$ and $\mathrm{h}^0(\ell^{-1}) = 0$, or $\mathrm{h}^0(\ell) = 0$ and $\mathrm{h}^0(\ell^{-1}) = 1$. Let us assume, e.g., that we are in the first situation. Then

$$\deg(\ell) = \chi(\ell) = \mathrm{h}^0(\ell) - \mathrm{h}^0(\ell^{-1}) = 1,$$

so that $\ell \simeq \mathcal{O}_E(q)$ for some $q \in E(\mathbb{K})$. But the bundle $\mathcal{O}_E(q) \oplus \mathcal{O}_E(-q)$ does not possess any subbundle isomorphic to $\mathcal{O}_E$ (in fact, the cokernel of its non-zero section is isomorphic to the sheaf $k(q) \oplus \mathcal{O}_E(-q)$, which is not torsion free), so it cannot be isomorphic to I_2. Alternatively, one could argue that I_2, being an extension of semistable bundles of the same slope, is slope-semistable, while $\mathcal{O}_E(q) \oplus \mathcal{O}_E(-q)$ is unstable, since $1 = \mu(\mathcal{O}_E(q)) > \mu(\mathcal{O}_E(q) \oplus \mathcal{O}_E(-q)) = 0$ (which shows that the subbundle $\mathcal{O}_E(q)$ is destabilizing).

REMARK 6.2.1 (Trivializing I_2). Before passing to the definition of I_3, and in order to fix notations, let us construct a trivializing open cover for I_2, which will be useful in our subsequent computations. First of all, let us set

$$x := i(1) \in \mathrm{H}^0(\mathrm{I}_2);$$

this is an everywhere non-zero section of I_2. Next, let $\mathcal{U} = \{U, V, \dots\}$ be an affine open cover of E. Then, for each open $U \in \mathcal{U}$, the restriction of $\mathcal{E}$ to U splits, since, U being affine, one has $\mathrm{Ext}^1_U(\mathcal{O}_U, \mathcal{O}_U) = \mathrm{H}^1(U, \mathcal{O}_U) = 0$. So one can pick a section s_U of $p|_U$:

$$\mathcal{E}|_U: \quad 0 \to \mathcal{O}_U \xrightarrow{i|_U} \mathrm{I}_2|_U \underset{p|U}{\overset{s_U}{\leftrightarrows}} \mathcal{O}_U \to 0$$

Let $y_U := s_U(1) \in \mathrm{H}^0(U, \mathrm{I}_2)$. Then the pair

$$\mathbf{e}_U := (x|_U, y_U)$$

is a local frame for I_2 on U, i.e., the map of $\mathcal{O}_U$-modules $\mathcal{O}_U^{\oplus 2} \to \mathrm{I}_2|_U$ corresponding to the pair of sections $\mathcal{O}_U \underset{y_U}{\overset{x|_U}{\rightrightarrows}} \mathrm{I}_2|_U$ is an isomorphism. Thus, the cover $\mathcal{U}$ trivializes I_2. To compute the transition matrices for I_2 relative to $\mathcal{U}$, let us fix $U, V \in \mathcal{U}$. Then

$$p\left(y_V|_{U\cap V} - y_U|_{U\cap V}\right) = p(s_V(1)) - p(s_U(1)) = 1 - 1 = 0,$$

so that

$$y_V|_{U\cap V} - y_U|_{U\cap V} = c_{UV} \cdot x|_{U\cap V}$$

for a suitable function $c_{UV} \in \mathrm{H}^0(\mathcal{O}_{U\cap V})$. Thus

$$\mathbf{e}_V|_{U\cap V} = \mathbf{e}_U|_{U\cap V} \cdot g_{UV},$$

where

$$g_{UV} = \begin{pmatrix} 1 & c_{UV} \\ 0 & 1 \end{pmatrix} \in \mathrm{H}^0(U \cap V, \mathrm{GL}_2(\mathcal{O}_E)).$$

The family

$$g := (g_{UV})_{(U,V)\in\mathcal{U}^2} \in \check{\mathrm{Z}}^1(\mathcal{U}, \mathrm{GL}_2(\mathcal{O}_E))$$

is the cocycle for I_2 associated to the trivializing open cover $\mathcal{U}$. The cocycle condition for g is equivalent to

$$(c_{UV})_{(U,V)\in\mathcal{U}^2}$$

being an $\mathcal{O}_E$-valued Čech 1-cocycle relative to the cover $\mathcal{U}$; its image in $\check{\mathrm{H}}^1(E, \mathcal{O}_E) \simeq \mathrm{H}^1(\mathcal{O}_E)$ is the class $c \neq 0$ of the extension (6.1).

Let us go back to the definition of I_3: one looks for non-split extensions of $\mathcal{O}_E$ by I_2. Again, there is only one such non-split extension (up to weak isomorphism), since

$$\mathrm{Ext}^1(\mathcal{O}_E, \mathrm{I}_2) \simeq \mathrm{H}^1(\mathrm{I}_2) \simeq \mathbb{K};$$

let I_3 be its total space:

$$0 \to \mathrm{I}_2 \to \mathrm{I}_3 \to \mathcal{O}_E \to 0. \tag{6.2}$$

Then I_3 is a vector bundle of rank 3, with determinant $\det(\mathrm{I}_3) \simeq \det(\mathrm{I}_2) \otimes \mathcal{O}_E \simeq \mathcal{O}_E$. Moreover, in the long exact sequence of cohomology groups corresponding to (6.2),

$$0 \to \mathrm{H}^0(\mathrm{I}_2) \to \mathrm{H}^0(\mathrm{I}_3) \to \mathrm{H}^0(\mathcal{O}_E) \xrightarrow{d} \mathrm{H}^1(\mathrm{I}_2) \to \mathrm{H}^1(\mathrm{I}_3) \to \mathrm{H}^1(\mathcal{O}_E) \to 0,$$

the map $d : \mathrm{H}^0(\mathcal{O}_E) \to \mathrm{H}^1(\mathrm{I}_2)$ is a non-zero (again, it sends $1 \in \mathrm{H}^0(\mathcal{O}_E)$ to the class $0 \neq d(1) \in \mathrm{H}^1(\mathrm{I}_2) \simeq \mathrm{Ext}^1(\mathcal{O}_E, \mathrm{I}_2)$ of the extension (6.2)) $\mathbb{K}$-linear map between 1-dimensional vector spaces, thus an isomorphism. It follows that the maps $\mathbb{K} \simeq \mathrm{H}^0(\mathrm{I}_2) \to \mathrm{H}^0(\mathrm{I}_3)$ and $\mathrm{H}^1(\mathrm{I}_3) \to \mathrm{H}^1(\mathcal{O}_E) \simeq \mathbb{K}$ are isomorphisms too. Finally, one proves that I_3 is indecomposable.

One then defines I_4 as the unique non-split extension of $\mathcal{O}_E$ by I_3 (again, uniqueness is due to $\mathrm{ext}^1(\mathcal{O}_E, \mathrm{I}_3) = \mathrm{h}^1(\mathrm{I}_3) = 1$), and so on.

Let us remark that I_r could have also been defined inductively as the unique non-split extension *of* I_{r-1} *by* $\mathcal{O}_E$:

$$0 \to \mathcal{O}_E \to \mathrm{I}_r \to \mathrm{I}_{r-1} \to 0.$$

Uniqueness then allows one to prove that each I_r is *self-dual* (i.e., isomorphic to its own dual $\mathrm{I}_r^\vee$).

6.2.1. Atiyah bundles and symmetric powers of I_2. The following result has been already used in the proof of Proposition 3.2.3, and it will be useful in decomposing the tensor products of the Atiyah bundles in the next section. As a side remark, we point out that it shows an alternative path to the definition of the bundles I_r.

PROPOSITION 6.2.2. *For each $r \geqslant 0$ we have an isomorphism*

$$\mathrm{Sym}^r \mathrm{I}_2 \simeq \mathrm{I}_{r+1}.$$

PROOF. The cases $r \in \{0, 1\}$ are clear. So let us fix an integer $r \geqslant 2$, and let us assume the isomorphism $\mathrm{Sym}^{r-1}\mathrm{I}_2 \simeq \mathrm{I}_r$. From the extension (6.1) we get, in a canonical way, an exact sequence

$$\mathrm{Sym}^{r-1}\mathrm{I}_2 \otimes \mathcal{O}_E \to \mathrm{Sym}^r \mathrm{I}_2 \to \mathrm{Sym}^r \mathcal{O}_E \to 0,$$

or, using the assumption $\mathrm{Sym}^{r-1}\mathrm{I}_2 \simeq \mathrm{I}_r$,

$$\mathrm{I}_r \to \mathrm{Sym}^r \mathrm{I}_2 \to \mathcal{O}_E \to 0.$$

By the usual arguments (rank counting), the first map is injective. Thus the bundle $\mathrm{Sym}^r\mathrm{I}_2$ appears as an extension of $\mathcal{O}_E$ by I_r:

$$0 \to \mathrm{I}_r \to \mathrm{Sym}^r \mathrm{I}_2 \to \mathcal{O}_E \to 0.$$

One can prove that this extension is non-split, showing that $\mathrm{Sym}^r\mathrm{I}_2 \simeq \mathrm{I}_{r+1}$, as claimed. ☐

6.2.2. Decomposing the tensor products of Atiyah bundles. In order to understand the cone of integrable I_2-valued pairs on regular on elliptic curves, we will need to know how to decompose the tensor products of Atiyah bundles. In fact, a tensor product such as $\mathrm{I}_r \otimes \mathrm{I}_s$ is not, in general, indecomposable, but Proposition 6.2.2 in conjunction with the Clebsch-Gordan decomposition, allows one to decompose it:

PROPOSITION 6.2.3. *For each pair of integers $r, s \geqslant 1$ there is a canonical isomorphism*

$$\mathrm{I}_r \otimes \mathrm{I}_s \simeq \bigoplus_{i=1}^{\min(r,s)} \mathrm{I}_{|r-s|+2i-1} = \mathrm{I}_{|r-s|+1} \oplus \mathrm{I}_{|r-s|+3} \oplus \cdots \oplus \mathrm{I}_{r+s-3} \oplus \mathrm{I}_{r+s-1}. \tag{6.3}$$

PROOF. We have $\det \mathrm{I}_2 \simeq \mathcal{O}_E$, and, by Proposition 6.2.2,

$$\mathrm{I}_r \otimes \mathrm{I}_s \simeq \mathrm{Sym}^{r-1}\mathrm{I}_2 \otimes \mathrm{Sym}^{s-1}\mathrm{I}_2.$$

Thus, one can apply the Clebsch-Gordan decomposition, stating that for any rank 2, locally free sheaf V with trivial determinant, defined on an arbitrary scheme Y, one has, for each choice of nonnegative integers a, b, a canonical isomorphism of $\mathcal{O}_Y$-modules

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V \simeq \mathrm{Sym}^{|a-b|} V \oplus \mathrm{Sym}^{|a-b|+2} V \oplus \cdots \oplus \mathrm{Sym}^{a+b-2} V \oplus \mathrm{Sym}^{a+b} V.$$

It follows that

$$\begin{aligned} \mathrm{I}_r \otimes \mathrm{I}_s &\simeq \mathrm{Sym}^{|r-s|}\mathrm{I}_2 \oplus \mathrm{Sym}^{|r-s|+2}\mathrm{I}_2 \oplus \cdots \oplus \mathrm{Sym}^{r+s-4}\mathrm{I}_2 \oplus \mathrm{Sym}^{r+s-2}\mathrm{I}_2 \\ &\simeq \mathrm{I}_{|r-s|+1} \oplus \mathrm{I}_{|r-s|+3} \oplus \cdots \oplus \mathrm{I}_{r+s-3} \oplus \mathrm{I}_{r+s-1}, \end{aligned}$$

as claimed.

□

REMARK 6.2.4. (1) For $r = s = 2$ the isomorphism

$$\mathrm{I}_2^{\otimes 2} \simeq \mathrm{I}_1 \oplus \mathrm{I}_3 \simeq \bigwedge^2 \mathrm{I}_2 \oplus \mathrm{Sym}^2\mathrm{I}_2$$

is the same as the decomposition,valid for any locally free sheaf V,

$$V^{\otimes 2} \simeq \bigwedge^2 V \oplus \mathrm{Sym}^2 V.$$

This decomposition is obtained by looking at the eigenbundles of the involution of $V^{\otimes 2}$ defined by $v \otimes w \mapsto w \otimes v$;

(2) for any pair of integers $r, s \geqslant 1$, the projection

$$\mu : \mathrm{I}_r \otimes \mathrm{I}_s \twoheadrightarrow \mathrm{I}_{r+s-1}$$

relative to the splitting (6.3) is induced by the multiplication of the symmetric algebra of I_2 [23]:

$$\mathrm{I}_r \otimes \mathrm{I}_s \simeq \mathrm{Sym}^{r-1}\mathrm{I}_2 \otimes \mathrm{Sym}^{s-1}\mathrm{I}_2 \twoheadrightarrow \mathrm{Sym}^{r+s-2}\mathrm{I}_2 \simeq \mathrm{I}_{r+s-1}.$$

We will be particularly interested in the case $r \geqslant 2, s = 2$ of the splitting (6.3), by virtue of its applications to the study of I_2-valued pairs. In this case, (6.3) becomes

$$\mathrm{I}_r \otimes \mathrm{I}_2 \simeq \mathrm{I}_{r-1} \oplus \mathrm{I}_{r+1}.$$

More concretely, one has a split short exact sequence

$$0 \to \mathrm{I}_{r-1} \xrightarrow{\iota^-} \mathrm{I}_r \otimes \mathrm{I}_2 \xrightarrow{\mu} \mathrm{I}_{r+1} \to 0, \tag{6.4}$$

where μ is the multiplication map, while the map $\iota^- : \mathrm{I}_{r-1} \hookrightarrow \mathrm{I}_r \otimes \mathrm{I}_2$ can be described explicitly as follows: the affine open cover $\mathcal{U} = \{U, V, \dots\}$ of E from Remark 6.2.1 trivializes not only I_2, but also all of the bundles $\mathrm{I}_s \simeq \mathrm{Sym}^{s-1}\mathrm{I}_2$ (and their tensor products); in fact, for each affine open $U \in \mathcal{U}$, the local frame $(x|_U, y_U)$ for I_2, which we will denote for simplicity just by (x, y), induces a local frame

$$(x^{s-1}, x^{s-2}y, \dots, xy^{s-2}, y^{s-1})$$

for I_s on U, for all $s \geqslant 2$. A computation then shows that the local maps $\mathrm{I}_{r-1} \to \mathrm{I}_r \otimes \mathrm{I}_2$ defined, for each $U \in \mathcal{U}$, by

$$x^{r-2-i}y^i \mapsto x^{r-1-i}y^i \otimes y - x^{r-2-i}y^{i+1} \otimes x, \qquad 0 \leqslant i \leqslant r-2,$$

patch to give the map ι^- (we remark that at least one has

$$\begin{aligned} \mu(x^{r-1-i}y^i \otimes y - x^{r-2-i}y^{i+1} \otimes x) &= x^{r-1-i}y^i y - x^{r-2-i}y^{i+1}x \\ &= x^{r-1-i}y^{i+1} - x^{r-1-i}y^{i+1} = 0). \end{aligned}$$

The map $\iota^-|_U$ can be represented more effectively in matrix form, using the $r \times (r-1)$ matrix with coefficients in the abelian group (or $\mathrm{H}^0(\mathcal{O}_U)$-module) $\mathrm{H}^0(\mathcal{O}_U)x \oplus \mathrm{H}^0(\mathcal{O}_U)y$ (here and in the following we regard the sections of $\mathrm{I}_s \otimes \mathrm{I}_2$ as I_2-valued sections of I_s):

$$I^- := \begin{pmatrix} y & & & \\ -x & y & & \\ & -x & \ddots & \\ & & \ddots & y \\ & & & -x \end{pmatrix}. \tag{6.5}$$

Let us now describe a splitting $\iota^+ : \mathrm{I}_{r+1} \to \mathrm{I}_r \otimes \mathrm{I}_2$ of the sequence (6.4): again, one shows that the local maps $\mathrm{I}_{r+1} \to \mathrm{I}_r \otimes \mathrm{I}_2$ defined by the formula

$$x^{r-i}y^i \mapsto \frac{1}{r}\left(ix^{r-i}y^{i-1} \otimes y + (r-i)x^{r-i-1}y^i \otimes x\right), \qquad 0 \leqslant i \leqslant r,$$

or, in matrix form, by the $r \times (r+1)$, $\mathrm{H}^0(\mathcal{O}_U)x \oplus \mathrm{H}^0(\mathcal{O}_U)y$-valued matrix

$$I^+ := \begin{pmatrix} x & \frac{1}{r}y & & & 0 \\ 0 & \frac{r-1}{r}x & \frac{2}{r}y & & \\ & & \frac{r-2}{r}x & \ddots & \vdots \\ \vdots & & & \ddots & \frac{r-1}{r}y & 0 \\ 0 & & & & \frac{1}{r}x & y \end{pmatrix}, \tag{6.6}$$

patch to give a well defined section ι^+ of μ.

6.2.3. Endomorphisms of Atiyah bundles. For every integer $r \geqslant 1$, the algebra of endomorphisms $\mathrm{End}(\mathrm{I}_r)$ of the Atiyah bundle I_r is isomorphic (as a $\mathbb{K}$-algebra) to the truncated polynomial algebra $\mathbb{K}[t]/(t^r)$. This is obvious for $r = 1$, while for $r \geqslant 2$ an explicit isomorphism of $\mathbb{K}$-algebras $\mathbb{K}[t]/(t^r) \to \mathrm{End}(\mathrm{I}_r)$ can be established by making the class of t correspond to the composition

$$\nu : \mathrm{I}_r \overset{\varpi}{\twoheadrightarrow} \mathrm{I}_{r-1} \overset{\iota}{\hookrightarrow} \mathrm{I}_r \tag{6.7}$$

(in particular, the family $(1, \nu, \ldots, \nu^{r-1})$ constitutes a basis of $\mathrm{End}(\mathrm{I}_r)$ as a $\mathbb{K}$-vector space). Here:

- the injection $\iota : \mathrm{I}_{r-1} \hookrightarrow \mathrm{I}_r$ is the one coming from the inductive definition of the Atiyah bundles; equivalently, it is the map of multiplication by the fixed non-zero section $x : \mathcal{O}_E \to \mathrm{I}_2$ of I_2:

$$\iota : \mathrm{I}_{r-1} \xrightarrow{\simeq} \mathrm{I}_{r-1} \otimes \mathcal{O}_E \xrightarrow{1\otimes x} \mathrm{I}_{r-1} \otimes \mathrm{I}_2 \xrightarrow{\mu} \mathrm{I}_r.$$

  Explicitly, with respect to our local frames, one has

$$\iota(x^{r-2-i}y^i) = x^{r-1-i}y^i, \qquad 0 \leqslant i \leqslant r-2;$$

thus the restrictions of ι to the open sets of the trivializing cover $\mathcal{U}$ are represented by the $r \times (r-1)$ matrix

$$I := \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{pmatrix}; \tag{6.8}$$

- the surjection $\varpi : \mathrm{I}_r \twoheadrightarrow \mathrm{I}_{r-1}$ is, by definition, $r-1$ times the composition

$$\mathrm{I}_r \xrightarrow{\kappa_r} \mathrm{I}_r^\vee \overset{\iota^\vee}{\twoheadrightarrow} \mathrm{I}_{r-1}^\vee \xrightarrow{\kappa_{r-1}^{-1}} \mathrm{I}_{r-1},$$

where, for each $s \geqslant 1$, $\kappa_s : \mathrm{I}_s \xrightarrow{\simeq} \mathrm{I}_s^\vee$ is the isomorphism defined locally, with respect to the frames $(e_i := x^{s-1-i}y^i)_{0 \leqslant i \leqslant s-1}$ for I_s and $(e_i^\vee)$ for $\mathrm{I}_s^\vee$, by the formula

$$\kappa_s(e_i) := \frac{(-1)^{s-1-i}}{\binom{s-1}{i}} e_{s-1-i}^\vee, \qquad 0 \leqslant i \leqslant s-1.$$

A computation shows that the map ϖ is locally described by the $(r-1) \times r$ matrix

$$P := \begin{pmatrix} 0 & 1 & & & \\ 0 & & 2 & & \\ \vdots & & & \ddots & \\ 0 & & & & r-1 \end{pmatrix}. \tag{6.9}$$

It follows that $\nu = \iota \circ \varpi$ is locally represented by the $r \times r$ matrix $N := IP$, which is the nilpotent matrix of order r obtained from P by adding an r-th row of zeros:

$$N := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & r-1 \\ & & & & 0 \end{pmatrix}. \tag{6.10}$$

6.2.4. Maps of Atiyah bundles. Finally, let us point out that, for each pair of integers $r, s \geqslant 1$, there is an isomorphism of $\mathbb{K}$-vector spaces

$$\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_s) \simeq \mathrm{End}(\mathrm{I}_t),$$

where

$$t := \min(r, s).$$

(observe that this at least agrees with the computation

$$\begin{aligned} \hom(\mathrm{I}_r, \mathrm{I}_s) &= \mathrm{h}^0(\mathcal{H}om(\mathrm{I}_r, \mathrm{I}_s)) = \mathrm{h}^0(\mathrm{I}_r^\vee \otimes \mathrm{I}_s) = \mathrm{h}^0(\mathrm{I}_r \otimes \mathrm{I}_s) \\ &= \mathrm{h}^0\left(\bigoplus_{i=1}^t \mathrm{I}_{|r-s|+2i-1}\right) = \sum_{i=1}^t \mathrm{h}^0(\mathrm{I}_{|r-s|+2i-1}) = t = \mathrm{end}(\mathrm{I}_t)). \end{aligned}$$

For example, for every integer $r \geqslant 2$, there are isomorphisms

$$\mathrm{End}(\mathrm{I}_{r-1}) \to \mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r-1}), \qquad \phi \mapsto \phi \circ \varpi$$

and

$$\mathrm{End}(\mathrm{I}_{r-1}) \to \mathrm{Hom}(\mathrm{I}_{r-1}, \mathrm{I}_r), \qquad \phi \mapsto \iota \circ \phi,$$

where $\varpi : \mathrm{I}_r \twoheadrightarrow \mathrm{I}_{r-1}$ and $\iota : \mathrm{I}_{r-1} \hookrightarrow \mathrm{I}_r$ are the maps appearing in (6.7).

6.3. Integrable I_2-valued fields on Atiyah bundles

6.3.1. Integrable I_2-valued fields on I_2. Let (E, o) be an elliptic curve over a field $\mathbb{K}$ of characteristic 0. It turns out that the integrable I_2-valued fields on I_2 are essentially the same as the Higgs fields (i.e., endomorphisms) of I_2. More precisely, one has the following:

PROPOSITION 6.3.1. *The $\mathbb{K}$-linear map*

$$- \otimes x : \mathrm{End}(\mathrm{I}_2) \to \mathrm{Hom}(\mathrm{I}_2, \mathrm{I}_2^{\otimes 2}), \tag{6.11}$$

sending a Higgs field ϕ on I_2 to its tensor product

$$\mathrm{I}_2 \xrightarrow{\simeq} \mathrm{I}_2 \otimes \mathcal{O}_E \xrightarrow{\phi \otimes x} \mathrm{I}_2 \otimes \mathrm{I}_2$$

with the non-zero section $x : \mathcal{O}_E \to \mathrm{I}_2$ of I_2, maps $\mathrm{End}(\mathrm{I}_2)$ bijectively onto the cone of integrable I_2-valued pairs on I_2.

PROOF. We will use the notations from Remark 6.2.1. An I_2-valued field $\psi : \mathrm{I}_2 \to \mathrm{I}_2^{\otimes 2}$ can be viewed as a family $(\psi_U)_{U \in \mathcal{U}}$ of local maps

$$\psi_U = \gamma_U \otimes x|_U + \delta_U \otimes y_U \in \mathrm{H}^0(U, \mathcal{H}om(\mathrm{I}_2, \mathrm{I}_2^{\otimes 2})), \qquad \gamma_U, \delta_U \in \mathrm{H}^0(U, \mathcal{E}nd(\mathrm{I}_2))$$

agreeing on the overlaps $W := U \cap V$. We have

$$\begin{aligned} \psi_V|_W &= \gamma_V|_W \otimes x|_W + \delta_V|_W \otimes y_V|_W \\ &= \gamma_V|_W \otimes x|_W + \delta_V|_W \otimes (y_U|_W + c_{UV} x|_W) \\ &= (\gamma_V|_W + c_{UV}\delta_V|_W) \otimes x|_W + \delta_V|_W \otimes y_U|_W, \end{aligned}$$

and this equals $\psi_U|_W = \gamma_U|_W \otimes x|_W + \delta_U|_W \otimes y_U|_W$ if and only if

$$\gamma_V|_W + c_{UV}\delta_V|_W = \gamma_U|_W, \qquad \delta_V|_W = \delta_U|_W.$$

The second condition tells us that the local endomorphisms δ_U of I_2 patch, i.e., there exists an element $\delta \in \mathrm{End}(\mathrm{I}_2)$ such that $\delta_U = \delta|_U$ for each $U \in \mathcal{U}$; the first condition can then be rewritten as

$$\gamma_V|_W + c_{UV}\delta|_W = \gamma_U|_W. \tag{6.12}$$

Set $\tau_U := \mathrm{tr}(\gamma_V) \in \mathrm{H}^0(\mathcal{O}_U)$, $\sigma := \mathrm{tr}(\delta) \in \mathrm{H}^0(\mathcal{O}_E) \simeq \mathbb{K}$. Taking the trace of (6.12), we find

$$\tau_V|_W + c_{UV}\sigma = \tau_U|_W.$$

This forces $\sigma = 0$; otherwise, one would have

$$c_{UV} = \left(-\frac{\tau_V}{\sigma}\right)\Big|_W - \left(-\frac{\tau_U}{\sigma}\right)\Big|_W,$$

and so the image of (c_{UV}) in $\mathrm{H}^1(\mathcal{O}_E)$ would be 0. Now we recall (Subsection 6.2.3) that $\mathrm{End}(\mathrm{I}_2)$ has a $\mathbb{K}$-basis $(1, \nu)$, with ν nilpotent, hence traceless. Thus $\mathfrak{sl}(\mathrm{I}_2) = \mathbb{K} \cdot \nu$ (here we need $\mathrm{char}(\mathbb{K}) \neq 2$), whence $\delta = \ell \cdot \nu$ for some $\ell \in \mathbb{K}$. Summing up, we have

$$\psi_U = \gamma_U \otimes x|_U + \ell \cdot \nu|_U \otimes y_U, \tag{6.13}$$

with

$$\ell \in \mathbb{K}, \qquad \gamma_V|_W - \gamma_U|_W = -\ell c_{UV}\nu|_W. \tag{6.14}$$

Next, the condition $\psi \wedge \psi = 0$ is equivalent to $\psi_U \wedge \psi_U = 0$ for each $U \in \mathcal{U}$; in turn, the equation $\psi_U \wedge \psi_U = 0$ is equivalent to $\ell \cdot [\gamma_U, \nu|_U] \otimes (x|_U \wedge y_U) = 0$ (here and in the following $[f, g]$ will denote the commutator $fg - gf$ of two endomorphisms or matrices), or to

$$\ell \cdot [\gamma_U, \nu|_U] = 0.$$

Let us show that this forces $\ell = 0$. In fact, if ℓ were $\neq 0$, the last condition would be equivalent to $[\gamma_U, \nu|_U] = 0$. Let us write γ_U as a 2×2, $\mathrm{H}^0(\mathcal{O}_U)$-valued matrix

$$G_U = \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix},$$

and let us recall (6.10) that $\nu|_U$ is represented by the matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then we have the chain of equivalences

$$[\gamma_U, \nu|_U] = 0 \Leftrightarrow [G_U, N] = 0 \Leftrightarrow \begin{pmatrix} -c_U & a_U - d_U \\ 0 & c_U \end{pmatrix} = 0 \Leftrightarrow (c_U = 0, \quad a_U = d_U)\,.$$

Thus, if ℓ were $\neq 0$, we would have $G_U = a_U + b_U N$, or $\gamma_U = a_U + b_U \nu|_U$, and the equality in (6.14) would become

$$a_V|_W - a_U|_W + (b_V|_W - b_U|_W)\nu|_W = -\ell c_{UV}\nu|_W,$$

which again would give the contradiction $c_{UV} = \left(-\frac{b_V}{\ell}\right)\big|_W - \left(-\frac{b_U}{\ell}\right)\big|_W$. This shows that $\ell = 0$, as claimed. If we substitute this in the equation of (6.14), we get that the local endomorphisms γ_U patch, say $\gamma_U = \gamma|_U$ for some $\gamma \in \mathrm{End}(\mathrm{I}_2)$. Finally, substituting in (6.13), we obtain $\psi_U = (\gamma \otimes x)|_U$, or $\psi = \gamma \otimes x$, showing that ψ comes from the Higgs field γ, as claimed. $\square$

The previous proposition has the following consequence for Higgs bundles on elliptic surfaces:

Proposition 6.3.2. *Let $(\pi : X \to B, \Sigma)$ be a non-isotrivial Weierstrass fibration with nodal singular fibers. Let (V, ϕ) be a rank 2 Higgs bundle on X with vertical determinant. Suppose that V is regular on a general fiber of π, and has spectral curve equal to 2Σ (so the restriction of V to a general fiber X_b of π is isomorphic to the bundle I_2 on X_b). Then the restriction of (V, ϕ) to a general fiber X_b has the form $(V_b, 0)$ (i.e., the Higgs field induced by ϕ on a general fiber vanishes).*

6.3.2. Integrable I_2-valued fields on higher rank Atiyah bundles. Let us investigate integrable I_2-valued fields on higher rank Atiyah bundles. Let r be an integer $\geqslant 2$, and let us set

$$\mathcal{H} = \mathcal{H}_r := \mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_r \otimes \mathrm{I}_2), \qquad \mathcal{E} = \mathcal{E}_r := \mathrm{End}(\mathrm{I}_r).$$

The integrable I_2-valued fields on I_r are the elements of the isotropic cone of the $\mathcal{E}$-valued quadratic form on $\mathcal{H}$

$$\mathcal{H} \to \mathcal{E}, \qquad \phi \mapsto \phi \wedge \phi.$$

(here we are identifying $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_r \otimes \wedge^2 \mathrm{I}_2)$ with $\mathcal{E}$). Let us fix some convenient bases for $\mathcal{H}$ and $\mathcal{E}$. First of all, by Subsections 6.2.2 and 6.2.4, we have

$$\begin{aligned} \mathcal{H} &\simeq \mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r-1} \oplus \mathrm{I}_{r+1}) \simeq \mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r-1}) \oplus \mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r+1}) \\ &\simeq \mathrm{End}(\mathrm{I}_{r-1}) \oplus \mathrm{End}(\mathrm{I}_r) \end{aligned}$$

(in particular, $\mathcal{H}$ is a $\mathbb{K}$-vector space of dimension $(r-1) + r = 2r - 1$). Here,

- the isomorphism $\mathcal{H} \xleftarrow{\simeq} \mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r-1}) \oplus \mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r+1})$ is induced by the inclusions $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r\pm1}) \hookrightarrow \mathcal{H}$ obtained by composition with the injections $\iota^{\pm} : \mathrm{I}_{r\pm1} \hookrightarrow \mathrm{I}_r \otimes \mathrm{I}_2$, which are locally represented by the matrices (6.5) and (6.6). If we denote by 1_s (resp., 0_s) the $s \times s$ identity matrix (resp., the 0 column vector of length s), and define the $s \times s$ diagonal matrices

$$\Delta_s^- := \mathrm{diag}(1, 2, \ldots, s), \qquad \Delta_s^+ := \mathrm{diag}(s, s-1, \ldots, 1),$$

we can rewrite I^- and I^+ as the I_2-valued block matrices

$$I^- = -\begin{pmatrix} 0_{r-1}^{\mathrm{t}} \\ 1_{r-1} \end{pmatrix} x + \begin{pmatrix} 1_{r-1} \\ 0_{r-1}^{\mathrm{t}} \end{pmatrix} y, \qquad I^+ = \begin{pmatrix} \Delta_r^+ & 0_r \end{pmatrix} x + \begin{pmatrix} 0_r & \Delta_r^- \end{pmatrix} y$$

(the exponent t denotes transposition);
- the isomorphism $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r-1}) \oplus \mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r+1}) \simeq \mathrm{End}(\mathrm{I}_{r-1}) \oplus \mathrm{End}(\mathrm{I}_r)$ is the direct sum of the isomorphisms $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r-1}) \simeq \mathrm{End}(\mathrm{I}_{r-1})$ and $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_{r+1}) \simeq \mathrm{End}(\mathrm{I}_r)$, obtained by composition with the maps

$$\varpi : \mathrm{I}_r \twoheadrightarrow \mathrm{I}_{r-1}, \qquad \iota : \mathrm{I}_r \hookrightarrow \mathrm{I}_{r+1},$$

from Subsection 6.2.3. These are locally represented by the matrices (6.9) and (6.8), which we rewrite here, respectively, as

$$P = \begin{pmatrix} 0_{r-1} & \Delta_{r-1}^- \end{pmatrix}, \qquad I = \begin{pmatrix} 1_r \\ 0_r^{\mathrm{t}} \end{pmatrix}.$$

Finally, for any integer $s \geqslant 1$, we have a $\mathbb{K}$-basis $(1, \nu_s, \ldots, \nu_s^{s-1})$ of the algebra $\mathrm{End}(\mathrm{I}_s)$, where the map $\nu_s \in \mathrm{End}(\mathrm{I}_s)$ is locally represented by the matrix (6.10), which can be rewritten as

$$N_s := \begin{pmatrix} 0_{s-1} & \Delta_{s-1}^- \\ 0 & 0_{s-1}^{\mathrm{t}} \end{pmatrix}.$$

Putting all of this together, we can write a direct sum decomposition

$$\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+,$$

where the vector subspaces $\mathcal{H}_-$ and $\mathcal{H}_+$ have $\mathbb{K}$-bases

$$(\alpha_i := \iota^- \circ \nu_{r-1}^i \circ \varpi)_{0\leqslant i\leqslant r-2}, \qquad (\beta_i := \iota^+ \circ \iota \circ \nu_r^i)_{0\leqslant i\leqslant r-1},$$

respectively. Let us compute the matrix representations of α_i and β_i:

- β_i is represented by the matrix $B_i := I^+ I N_r^i$. We have

$$\begin{aligned} I^+I &= \begin{pmatrix}\Delta_r^+ & 0_r\end{pmatrix}\begin{pmatrix}1_r\\ 0_r^{\mathrm{t}}\end{pmatrix} x + \begin{pmatrix}0_r & \Delta_r^-\end{pmatrix}\begin{pmatrix}1_r\\ 0_r^{\mathrm{t}}\end{pmatrix} y \\ &= \Delta_r^+ x + \begin{pmatrix}N_r & re_r\end{pmatrix}\begin{pmatrix}1_r\\ 0_r^{\mathrm{t}}\end{pmatrix} y = \Delta_r^+ x + N_r y.\end{aligned}$$

  (here we used the equality

$$\begin{pmatrix}0_r & \Delta_r^-\end{pmatrix} = \begin{pmatrix}N_r & re_r\end{pmatrix}$$

  where e_r is the column vector of length r given by $e_r = (0\cdots 01)^{\mathrm{t}}$); thus

$$B_i = \Delta_r^+ N_r^i x + N_r^{i+1} y.$$

  In particular, the map $\gamma := \beta_{r-1}$ is represented by the matrix

$$C := B_{r-1} = \Delta_r^+ N_r^{r-1} x + N_r^r y = rN_r^{r-1} x,$$

  since $N_r^r = 0$;
- Next, α_i is represented the matrix

$$\begin{aligned} A_i := I^- N_{r-1}^i P &= -\begin{pmatrix}0_{r-1}^{\mathrm{t}}\\ 1_{r-1}\end{pmatrix} N_{r-1}^i \begin{pmatrix}0_{r-1} & \Delta_{r-1}^-\end{pmatrix} x + \begin{pmatrix}1_{r-1}\\ 0_{r-1}^{\mathrm{t}}\end{pmatrix} N_{r-1}^i \begin{pmatrix}0_{r-1} & \Delta_{r-1}^-\end{pmatrix} y \\ &= -\begin{pmatrix}0_{r-1}^{\mathrm{t}}\\ 1_{r-1}\end{pmatrix}\begin{pmatrix}0_{r-1} & N_{r-1}^i\Delta_{r-1}^-\end{pmatrix} x + \begin{pmatrix}1_{r-1}\\ 0_{r-1}^{\mathrm{t}}\end{pmatrix}\begin{pmatrix}0_{r-1} & N_{r-1}^i\Delta_{r-1}^-\end{pmatrix} y \\ &= -\begin{pmatrix}0 & 0_{r-1}^{\mathrm{t}}\\ 0_{r-1} & N_{r-1}^i\Delta_{r-1}^-\end{pmatrix} x + \begin{pmatrix}0_{r-1} & N_{r-1}^i\Delta_{r-1}^-\\ 0 & 0_{r-1}^{\mathrm{t}}\end{pmatrix} y \\ &= -M_i x + N_r^{i+1} y.\end{aligned}$$

  Here, we introduced the notation

$$M_i := \begin{pmatrix}0 & 0_{r-1}^{\mathrm{t}}\\ 0_{r-1} & N_{r-1}^i\Delta_{r-1}^-\end{pmatrix},$$

  and we used the equality

$$\begin{pmatrix}0_{r-1} & N_{r-1}^i\Delta_{r-1}^-\\ 0 & 0_{r-1}^{\mathrm{t}}\end{pmatrix} = N_r^{i+1}.$$

Let us denote by

$$(a_0, a_1, \ldots, a_{r-2}, b_0, b_1, \ldots, b_{r-2}, c)$$

the basis of $\mathcal{H}^\vee$ dual to the basis

$$(\alpha_0, \alpha_1, \ldots, \alpha_{r-2}, \beta_0, \beta_1, \ldots, \beta_{r-2}, \gamma)$$

of $\mathcal{H}$, so that an arbitrary field $\phi \in \mathcal{H}$ can be written as

$$\phi = \sum_{i=0}^{r-2}(a_i\alpha_i + b_i\beta_i) + c\gamma.$$

This is represented by the matrix (here we write for simplicity N for N_r, and Δ^+ for Δ_r^+)

$$\begin{aligned}
\Phi &:= \sum_{i=0}^{r-2}(a_iA_i + b_iB_i) + cC \\
&= rcN^{r-1}x + \sum_{i=0}^{r-2} a_i(-M_ix + N^{i+1}y) + b_i(\Delta^+N^ix + N^{i+1}y) \\
&= x\left(rcN^{r-1} + \sum_{i=0}^{r-2}(-a_iM_i + b_i\Delta^+N^i)\right) + y\sum_{i=0}^{r-2}(a_i+b_i)N^{i+1} \\
&= \Phi_1x + \Phi_2y,
\end{aligned}$$

where

$$\Phi_1 := rcN^{r-1} + \sum_{i=0}^{r-2}(-a_iM_i + b_i\Delta^+N^i), \qquad \Phi_2 := \sum_{i=0}^{r-2}(a_i+b_i)N^{i+1}.$$

Thus $\phi \wedge \phi$ is represented by

$$\Phi \wedge \Phi = [\Phi_1, \Phi_2]x \wedge y$$

Now,

$$[\Phi_1, \Phi_2] = \sum_{i,j=0}^{r-2} -a_i(a_j+b_j)[M_i, N^{j+1}] + b_i(a_j+b_j)[\Delta^+N^i, N^{j+1}],$$

with

$$[M_i, N^j] = -jN^{i+j}, \qquad [\Delta^+N^i, N^j] = jN^{i+j};$$

substituting, we get

$$\begin{aligned}
[\Phi_1, \Phi_2] &= \sum_{i,j=0}^{r-2}(j+1)a_i(a_j+b_j)N^{i+j+1} + (j+1)b_i(a_j+b_j)N^{i+j+1} \\
&= \sum_{i,j=0}^{r-2}(j+1)(a_i+b_i)(a_j+b_j)N^{i+j+1} \\
&= \sum_{\ell=0}^{r-2}N^{\ell+1}\sum_{i=0}^{\ell}(i+1)(a_i+b_i)(a_{\ell-i}+b_{\ell-i}) \\
&= \sum_{\ell=0}^{r-2}Q_\ell(a_0+b_0, a_1+b_1, \ldots, a_\ell+b_\ell)N^{\ell+1},
\end{aligned}$$

where, for each $\ell \geqslant 0$, we denoted by $Q_\ell \in \mathbb{K}[X_0, X_1, \ldots, X_\ell]$ the quadratic form (with integer coefficients) defined by

$$Q_\ell(X_0, X_1, \ldots, X_\ell) := \sum_{i=0}^{\ell}(i+1)X_iX_{\ell-i}. \tag{6.15}$$

Thus

$$\phi \wedge \phi = \sum_{\ell=1}^{r-1} q_{\ell-1}(\phi)\nu_r^\ell$$

(we remark that the endomorphism $\phi \wedge \phi$ is traceless, because its component along $1 = \nu_r^0$ vanishes), where, for each integer $0 \leqslant \ell \leqslant r-2$, we denoted by $q_\ell : \mathcal{H} \to \mathbb{K}$ the quadratic form

$$\phi = \sum_{i=0}^{r-2}(a_i\alpha_i + b_i\beta_i) + c\gamma \mapsto q_\ell(\phi) := Q_\ell(a_0 + b_0, a_1 + b_1, \ldots, a_\ell + b_\ell).$$

In particular, a field $\phi \in \mathcal{H}$ is integrable (i.e., $\phi \wedge \phi = 0$) if and only if

$$q_0(\phi) = q_1(\phi) = \cdots = q_{r-2}(\phi) = 0.$$

Thus, the set

$$\mathcal{K}_r := \{\phi \in \mathcal{H}_r : \phi \wedge \phi = 0\}$$

of integrable I_2-valued fields on I_r coincides with the set of $\mathbb{K}$-rational points of the affine cone in $\mathcal{H} \simeq \mathbb{A}_{\mathbb{K}}^{2r-1}$ over the base locus of the linear system of quadrics in $\mathbb{P}\mathcal{H} \simeq \mathbb{P}_{\mathbb{K}}^{2r-2}$ generated by $q_0, q_1, \ldots, q_{r-2}$.

6.3.3. The case $r = 2$ again. Let us show that the previous computations give us back, for $r = 2$, Proposition 6.3.1: in this case, the vector space $\mathcal{H}_2 = \mathrm{Hom}(\mathrm{I}_2, \mathrm{I}_2 \otimes \mathrm{I}_2)$ has dimension $2r - 1 = 3$, and it has a basis $(\alpha, \beta, \gamma) := (\alpha_0, \beta_0, \gamma)$, with corresponding linear coordinates $(a, b, c) := (a_0, b_0, c)$; instead, the space $\mathcal{E}_2 = \mathrm{End}(\mathrm{I}_2)$ is 2-dimensional, with basis $(1, \nu) := (1, \nu_2)$. For each field $\phi = a\alpha + b\beta + c\gamma \in \mathcal{H}_2$, we have

$$\phi \wedge \phi = q_0(\phi)\nu = Q_0(a + b)\nu = (a + b)^2\nu,$$

which vanishes if and only if $(a + b)^2 = 0$, or $a + b = 0$. Thus the cone $\mathcal{K}_2$ of integrable I_2-valued fields on I_2 is the 2-dimensional vector subspace of $\mathcal{H}_2$ with equation $a + b = 0$. The natural injection $\mathcal{E}_2 \hookrightarrow \mathcal{H}_2$ obtained by tensoring with the section x of I_2 is easily checked to factor through the inclusion $\mathcal{K}_2 \hookrightarrow \mathcal{H}_2$. This is true in general: for every vector bundle V on the elliptic curve E, we have a natural injective linear map

$$\mathrm{End}(V) \hookrightarrow \mathrm{Hom}(V, V \otimes \mathrm{I}_2), \qquad \phi \mapsto \phi \otimes x, \tag{6.16}$$

whose image is contained in the cone of integrable I_2-valued fields on V; observing that the elements of the source space $\mathrm{End}(V)$ of (6.16) are nothing but the Higgs fields on V, one could say that every vector bundle on E admits a natural family of integrable I_2-valued fields, parametrized by the space of Higgs fields on it. Now, the linear map $\mathcal{E}_2 \hookrightarrow \mathcal{K}_2$ induced by the map $\mathcal{E}_2 \hookrightarrow \mathcal{H}_2$ is an injection between vector spaces of the same dimension, thus an isomorphism, confirming Proposition 6.3.1. To see this even more explicitly, observe that, using for example the linear coordinates (b, c) on the space $\mathcal{K}_2$, an arbitrary element $\phi = -b\alpha + b\beta + c\gamma$ of $\mathcal{K}_2$ is locally represented by the 2×2, I_2-valued matrix

$$\Phi = (2cN_2 + b(M_0 + \Delta_2^+))x,$$

with

$$M_0 + \Delta_2^+ = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_1^- \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2;$$

so one has $\Phi = 2(b+cN_2)x$, showing that ϕ is the image of the endomorphism $\psi := 2(b+c\nu)$ of I_2 via the map (6.11), namely, $\phi = \psi \otimes x$.

Let us point out another consequence of the equality between Higgs fields and integrable I_2-valued pairs on I_2: the pairs (I_2, ϕ), $\phi \in \mathcal{K}_2$, are pairwise non-isomorphic, because so are the Higgs bundles (I_2, ψ), $\psi \in \mathcal{E}_2$. In fact, first of all, for each integer $r \geqslant 2$, the action of the (affine, commutative) algebraic $\mathbb{K}$-group

$$G_r := \mathrm{Aut}(\mathrm{I}_r) = \mathcal{E}_r^\times$$

(i.e., the group of units of the $\mathbb{K}$-algebra $\mathcal{E}_r = \mathrm{End}(\mathrm{I}_r)$) on $\mathcal{E}_r$ by conjugation is trivial, since the algebra $\mathcal{E}_r$ is commutative. Next, using the isomorphism $\mathcal{E}_2 \xrightarrow{\simeq} \mathcal{K}_2$, one sees that the natural action of G_2 on $\mathcal{K}_2$ is the action by *conjugation on the first factor*, i.e., writing an element $\phi \in \mathcal{K}_2$ as $\phi = \psi \otimes x$ with $\psi \in \mathcal{E}_2$, one has, for every $g \in G_2$,

$$g \cdot \phi = (g \circ \psi \circ g^{-1}) \otimes x = \phi,$$

showing that this action is trivial too. We will see in the next section that this is not anymore the case for $r = 3$ (and presumably also for $r \geqslant 4$).

We finally remark that the closed subscheme of $\mathcal{H}_2 \simeq \mathbb{A}^3_{\mathbb{K}}$ defined by the equation $\phi \wedge \phi = 0$ (i.e., the affine cone over the double line in $\mathbb{P}\mathcal{H}_2 \simeq \mathbb{P}^2_{\mathbb{K}}$ defined by the rank 1 quadratic form $q_0 : \mathcal{H}_2 \to \mathbb{K}$), is isomorphic to $\mathbb{I}_{\mathbb{K}} \times \mathbb{A}^2_{\mathbb{K}}$, where $\mathbb{I}_{\mathbb{K}} := \mathrm{Spec}(\mathbb{K}[t]/(t^2))$. This can be seen, for example, by introducing the new set of linear coordinates on $\mathcal{H}_2$:

$$(s, d, c) := (a + b, a - b, c).$$

In these coordinates, we have

$$\mathrm{Spec}\left(\frac{\mathbb{K}[a,b,c]}{((a+b)^2)}\right) \simeq \mathrm{Spec}\left(\frac{\mathbb{K}[s,d,c]}{(s^2)}\right) \simeq \mathrm{Spec}\left(\frac{\mathbb{K}[s]}{(s^2)}\right) \times \mathrm{Spec}(\mathbb{K}[d,c]) \simeq \mathbb{I}_{\mathbb{K}} \times \mathbb{A}^2_{\mathbb{K}}.$$

Probably this non-reduced scheme structure (and the ones we will find in the case $r \geqslant 3$) has some interpretation in terms of the fibers of the natural surjective map $(V, \phi) \mapsto V$ from the moduli space of semistable I_2-valued pairs of rank 2 and trivial determinant on E, to the moduli space $|2o| \simeq \mathbb{P}^1_K$ of semistable bundles on E of rank 2 and trivial determinant.

6.3.4. The case $r = 3$. Let us move on to the case $r = 3$. In this case, the vector spaces $\mathcal{H}_3$ and $\mathcal{E}_3$ have dimension $2r - 1 = 5$ and $r = 3$, respectively; moreover we have bases $(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma)$ (with dual basis (a_0, a_1, b_0, b_1, c)) for $\mathcal{H}_3$, and $(1, \nu, \nu^2)$ for $\mathcal{E}_3$ (here $\nu := \nu_3$). For every field $\phi \in \mathcal{H}_3$, the endomorhism $\phi \wedge \phi \in \mathcal{E}_3$ can be written as

$$\phi \wedge \phi = Q_0(a_0 + b_0)\nu + Q_1(a_0 + b_0, a_1 + b_1)\nu^2,$$

where

$$Q_0(X_0) = X_0^2 \in \mathbb{Z}[X_0], \qquad Q_1(X_0, X_1) = X_0X_1 + 2X_1X_0 = 3X_0X_1 \in \mathbb{Z}[X_0, X_1].$$

Thus the cone $\mathcal{K}_3$ of integrable I_2-valued fields on I_3 is the set of $\mathbb{K}$-rational points of the closed subscheme of $\mathcal{H}_3 \simeq \mathbb{A}^5_{\mathbb{K}} = \mathrm{Spec}(\mathbb{K}[a_0, \ldots, c])$ defined by the ideal (s_0^2, s_0s_1), where $s_i := a_i + b_i$ $(i \in \{0, 1\})$. This is the same as the set of $\mathbb{K}$-rational points of the codimension

1 vector subspace of $\mathcal{H}_3$ defined by the linear form $s_0 = a_0 + b_0$. Thus we see that in this case the injection

$$\mathcal{E}_3 \hookrightarrow \mathcal{K}_3, \qquad \psi \mapsto \psi \otimes x$$

is not an isomorphism (since $\dim \mathcal{E}_3 = 3$, while $\dim \mathcal{K}_3 = \dim \mathcal{H}_3 - 1 = 5 - 1 = 4 > 3$). This means that there exist integrable I_2-valued fields on I_3 which are not induced by Higgs fields on I_3 via multiplication by $x \in \mathrm{H}^0(\mathrm{I}_2)$.

Nonetheless, it is possible to find an interpretation of the image of the map $\mathcal{E}_3 \hookrightarrow \mathcal{K}_3$ in terms of the geometry of the closed subscheme $\mathcal{S}_3$ of $\mathcal{H}_3$ defined by $\phi \wedge \phi = 0$. Introducing, as before, new linear coordinates (s_0, s_1, d_0, d_1, c) on $\mathcal{H}_3$, where

$$s_i := a_i + b_i, \qquad d_i := a_i - b_i, \qquad i \in \{0, 1\},$$

one sees that the scheme $\mathcal{S}_3$ is isomorphic to

$$\operatorname{Spec}\left(\frac{\mathbb{K}[s_0, s_1]}{(s_0^2, s_0 s_1)}\right) \times \operatorname{Spec}(\mathbb{K}[d_0, d_1, c]) \simeq T \times \mathbb{A}^3_{\mathbb{K}},$$

where we denoted by T the (irreducible, non-reduced) closed subscheme $\operatorname{Spec}\left(\frac{\mathbb{K}[s_0,s_1]}{(s_0^2,s_0s_1)}\right)$ of $\operatorname{Spec}(\mathbb{K}[s_0, s_1]) \simeq \mathbb{A}^2_{\mathbb{K}}$, supported on the s_1-axis

$$\operatorname{Spec}\left(\frac{\mathbb{K}[s_0, s_1]}{(s_0)}\right) \simeq \operatorname{Spec}(\mathbb{K}[s_1]) \simeq \mathbb{A}^1_{\mathbb{K}} \hookrightarrow \mathbb{A}^2_{\mathbb{K}}.$$

Its intersection with the principal open subscheme $U := \{s_1 \neq 0\} \simeq \operatorname{Spec}(\mathbb{K}[s_0, s_1^{\pm 1}])$ of $\mathbb{A}^2_{\mathbb{K}}$ is the regular scheme given by

$$T \cap U \simeq \operatorname{Spec}\left(\frac{k[s_0, s_1^{\pm 1}]}{(s_0)}\right) \simeq \operatorname{Spec}(k[s_1^{\pm 1}]) \simeq \mathbb{A}^1_k - \{\mathbb{O}\},$$

where $\mathbb{O} := (s_1)$ is the origin of $\mathbb{A}^1_{\mathbb{K}}$; and $T = U \cup \{\hat{\mathbb{O}}\}$, where $\hat{\mathbb{O}} := (s_0, s_1)$ is the origin of $\mathbb{A}^2_{\mathbb{K}}$. This is a singular point of T, since the local ring $\mathcal{O}_{T,\hat{\mathbb{O}}}$ contains non-zero nilpotents. Thus, the singular scheme of $\mathcal{S}_3$ is supported on the 3-dimensional vector subspace of $\mathcal{H}_3$ with equations $s_0 = s_1 = 0$. But this subspace is exactly the image of the map $\mathcal{E}_3 \hookrightarrow \mathcal{K}_3$, namely the locus of I_2-valued fields coming from Higgs fields (for higher odd integers $r \geqslant 5$, the image of the map $\mathcal{E}_r \hookrightarrow \mathcal{K}_r$ will still be contained in the singular locus of the closed subscheme $\mathcal{S}_r$ of $\mathcal{H}_r$ defined by the equation $\phi \wedge \phi = 0$, but probably with a strict inclusion). In fact, for an arbitrary $r \geqslant 2$, by the exactness of the sequence of vector spaces

$$0 \to \mathcal{E}_r \to \mathcal{H}_r \to \mathcal{E}_r,$$

the image of the map $\mathcal{E}_r \hookrightarrow \mathcal{K}_r$ (which is the same as that of $\mathcal{E}_r \hookrightarrow \mathcal{H}_r$) coincides with the kernel of the map $\mathcal{H}_r \to \mathcal{E}_r$. Using the previous computations, one sees that the last map can be written as

$$\sum_{i=0}^{r-2} (a_i \alpha_i + b_i \beta_i) + c\gamma \mapsto \sum_{i=0}^{r-2} (a_i + b_i) \nu_r^{i+1};$$

thus its kernel has equations $a_i + b_i = 0$, $i \in \{0, 1, \ldots, r-2\}$. So we showed that *the Higgs fields on* I_3 *appear as the singular locus of the scheme of integrable* I_2*-valued fields.*

6.3.5. The action of $G_3 = \mathrm{Aut}(\mathrm{I}_3)$ on $\mathcal{K}_3$. First of all, let us recall the notion of morphism between two I_2-valued pairs defined on the same elliptic curve E. Given two such pairs $\mathcal{V} = (V, \phi), \mathcal{W} = (W, \psi)$, a morphism $\mathcal{V} \to \mathcal{W}$ is an $\mathcal{O}_E$-linear map $\alpha : V \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & V \otimes \mathrm{I}_2 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\otimes 1} \\ W & \xrightarrow{\psi} & W \otimes \mathrm{I}_2 \end{array} \tag{6.17}$$

Composition of morphisms of pairs is the same as composition of morphism of sheaves; in particular, an isomorphism of pairs $\alpha : (V, \phi) \to (W, \psi)$ is the same as an isomorphism $\alpha : V \to W$ of the underlying sheaves such that the diagram (6.17) commutes. It follows that, for a vector bundle V, the set of isomorphism classes of integrable pairs $\mathcal{V} = (V, \phi)$ having V as their underlying vector bundle is the same as the set of orbits for the action of the group (or group-scheme) $\mathrm{Aut}(V)$ of automorphisms of V on the set (or scheme) $\mathcal{K}_V := \{\phi \in \mathrm{Hom}(V, V \otimes \mathrm{I}_2) : \phi \wedge \phi = 0\}$ of integrable I_2-valued pairs on V, defined by

$$\alpha \cdot \phi := \alpha \otimes 1 \circ \phi \circ \alpha^{-1}, \qquad (\alpha, \phi) \in \mathrm{Aut}(V) \times \mathcal{K}_V.$$

We saw in Section 6.3.3 that this action is trivial in the case $V = \mathrm{I}_2$. Let us see what the action looks like in the case $V = \mathrm{I}_3$. An endomorphism $c_0 + c_1\nu + c_2\nu^2$ of I_3 (here $\nu := \nu_3$, $c_i \in \mathbb{K}$) is invertible iff $c_0 \neq 0$. Thus

$$G_3 := \mathrm{Aut}(\mathrm{I}_3) = \left\{ \sum_{i=0}^{2} c_i \nu^i \,\middle|\, c_i \in \mathbb{K}, c_0 \neq 0 \right\} \simeq \mathrm{Spec}\,\mathbb{K}[c_0^{\pm 1}, c_1, c_2]$$

(as variety; as an algebraic group, $G_3 \simeq \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{a}$), and, for any element $g = \sum c_i \nu^i = c_0 \left(1 + \frac{c_1}{c_0}\nu + \frac{c_2}{c_0}\nu^2\right) \in G_3$, we have

$$\begin{aligned} g^{-1} &= c_0^{-1}\left(1 + \frac{c_1}{c_0}\nu + \frac{c_2}{c_0}\nu^2\right)^{-1} \\ &= c_0^{-1}\left(1 - \frac{c_1}{c_0}\nu - \frac{c_2}{c_0}\nu^2 + \left(\frac{c_1}{c_0}\right)^2 \nu^2\right) \\ &= c_0^{-1}\left(1 - \frac{c_1}{c_0}\nu + \left(\frac{c_1^2}{c_0^2} - \frac{c_2}{c_0}\right)\nu^2\right). \end{aligned}$$

On

$$\mathcal{K}_3 = \mathcal{K}_{\mathrm{I}_3} = \left\{ \sum_{i=0}^{1} (a_i\alpha_i + b_i\beta_i) + c\gamma \,\middle|\, a_i, b_i, c \in \mathbb{K}, a_0 + b_0 = 0 \right\}$$

we have linear coordinates (b_0, a_1, b_1, c) (we just omitted a_0); the element

$$\phi = b_0(-\alpha_0 + \beta_0) + a_1\alpha_1 + b_1\beta_1 + c\gamma$$

of $\mathcal{K}_3 \simeq \mathbb{A}^4_\mathbb{K} = \mathrm{Spec}\,\mathbb{K}[b_0, a_1, b_1, c]$ with coordinates (b_0, a_1, b_1, c) is locally represented by the 3×3, I_2-valued matrix $\Phi = \Phi_1 x + \Phi_2 y$, where

$$\Phi_1 = 3cN^2 + b_0(M_0 + \Delta^+) - a_1 M_1 + b_1 \Delta^+ N, \qquad \Phi_2 = (a_1 + b_1)N^2.$$

Here

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Delta^+ = \mathrm{diag}(3,2,1), \quad M_i = \begin{pmatrix} 0 & 0_2^{\mathrm{t}} \\ 0_2 & N_2^i \Delta_2^- \end{pmatrix}, \; i = 0, 1,$$

where $N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\Delta_2^- = \mathrm{diag}(1,2)$; thus

$$N^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_0 + \Delta^+ = 3, \quad M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Delta^+ N = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix},$$

and Φ_1, Φ_2 can be rewritten as

$$\Phi_1 = 3cN^2 + 3b_0 - a_1 M_1 + b_1 \Delta^+ N, \qquad \Phi_2 = (a_1 + b_1) N^2.$$

Now, for $g = \sum c_i \nu^i \in G_3$, the element $g \cdot \phi \in \mathcal{K}_3$ is represented by the matrix

$$\Gamma \Phi_1 \Gamma^{-1} x + \Gamma \Phi_2 \Gamma^{-1} y,$$

where

$$\Gamma := \sum c_i N^i = \begin{pmatrix} c_0 & c_1 & 2c_2 \\ 0 & c_0 & 2c_1 \\ 0 & 0 & c_0 \end{pmatrix}, \quad \Gamma^{-1} = \begin{pmatrix} \frac{1}{c_0} & -\frac{c_1}{c_0^2} & 2\left(\frac{c_1^2}{c_0^3} - \frac{c_2}{c_0^2}\right) \\ 0 & \frac{1}{c_0} & -2\frac{c_1}{c_0^2} \\ 0 & 0 & \frac{1}{c_0} \end{pmatrix}.$$

Let us compute $\Gamma \Phi_i \Gamma^{-1}$: obviously $\Gamma \Phi_2 \Gamma^{-1} = \Phi_2$, while

$$\Gamma \Phi_1 \Gamma^{-1} = 3cN^2 + 3b_0 - a_1 \Gamma M_1 \Gamma^{-1} + b_1 \Gamma \Delta^+ N \Gamma^{-1}, \tag{6.18}$$

where

$$\Gamma M_1 \Gamma^{-1} = M_1 + \frac{c_1}{c_0} N^2, \quad \Gamma \Delta^+ N \Gamma^{-1} = \Delta^+ N - \frac{c_1}{c_0} N^2.$$

Substituting the last two equalities in (6.18), we finally get

$$\Gamma \Phi_1 \Gamma^{-1} = \Phi_1 - \frac{c_1}{c_0} a_1 N^2 - \frac{c_1}{c_0} b_1 N^2 = \Phi_1 - \frac{c_1}{c_0}(a_1 + b_1) N^2,$$

and

$$\Gamma \Phi_1 \Gamma^{-1} x + \Gamma \Phi_2 \Gamma^{-1} y = \Phi - \frac{c_1}{c_0}(a_1 + b_1) N^2 x.$$

The matrix $-\frac{c_1}{c_0}(a_1 + b_1) N^2 x$ is the local representative of the element $-\frac{c_1}{3c_0}(a_1 + b_1)\gamma$ of $\mathcal{K}_3$, which has (b_0, a_1, b_1, c)-coordinates equal to $(0, 0, 0, -\frac{c_1}{3c_0}(a_1 + b_1))$. Thus the action of of G_3 on $\mathcal{K}_3$ is the map of $\mathbb{K}$-variaties dual to the map of $\mathbb{K}$-algebras

$$\mathbb{K}[b_0, a_1, b_1, c] \to \mathbb{K}[c_0^{\pm 1}, c_1, c_2] \otimes \mathbb{K}[b_0, a_1, b_1, c] = \mathbb{K}[c_0^{\pm 1}, c_1, c_2, b_0, a_1, b_1, c]$$

defined by

$$f(b_0, a_1, b_1, c) \mapsto f\left(b_0, a_1, b_1, c - \frac{c_1}{3c_0}(a_1 + b_1)\right)$$

From this formula it is clear that the fields $\phi \in \mathcal{K}_3$ coming from Higgs fields $\psi \in \mathcal{E}_3$ are fixed points of the action (this was already mentioned before). Conversely, a fixed point (b_0, a_1, b_1, c) of the action satisfies $-\frac{c_1}{3c_0}(a_1 + b_1) = 0$ for all $c_0 \in \mathbb{K}^\times$, $c_1 \in \mathbb{K}$, or $a_1 + b_1 = 0$.

This shows that *the fixed points of the G_3-action on $\mathcal{K}_3$ are exactly the I_2-valued fields coming from Higgs fields.*

From the previous formula it is also clear that the G_3-action is induced from the $\mathbb{G}_{\mathrm{a}}$-action on $\mathcal{K}_3$ with coaction

$$\begin{array}{rcl} \mathbb{K}[b_0,a_1,b_1,c] & \to & \mathbb{K}[t]\otimes\mathbb{K}[b_0,a_1,b_1,c]=\mathbb{K}[t,b_0,a_1,b_1,c] \\ f(b_0,a_1,b_1,c) & \mapsto & f\left(b_0,a_1,b_1,c-\frac{t}{3}(a_1+b_1)\right) \end{array},$$

(here t is the coordinate on the group variety $\mathbb{G}_{\mathrm{a}}\simeq\mathbb{A}^1_{\mathbb{K}}=\operatorname{Spec}\mathbb{K}[t]$) via the morphism of algebraic $\mathbb{K}$-groups

$$\theta: G_3\to\mathbb{G}_{\mathrm{a}},\qquad c_0+c_1\nu+c_2\nu^2\mapsto\frac{c_1}{c_0}$$

(this is a morphism $\mathbb{K}$-varieties, but also of groups, since for any $g=\sum c_i\nu^i,\bar{g}=\sum\bar{c}_i\nu^i\in G_3$, one has

$$\theta(g\cdot\bar{g})=\theta(c_0\bar{c}_0+(c_0\bar{c}_1+c_1\bar{c}_0)\nu+\cdots)=\frac{c_0\bar{c}_1+c_1\bar{c}_0}{c_0\bar{c}_0}=\theta(g)+\theta(\bar{g})).$$

The $\mathbb{G}_{\mathrm{a}}$-action is associated to the linear representation

$$\mathbb{G}_{\mathrm{a}}\to\mathrm{GL}_4,\qquad t\mapsto\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-\frac{t}{3}&-\frac{t}{3}&1\end{pmatrix}.$$

A non-fixed point $p=(b_0,a_1,b_1,c)$ of this action (i.e., a point with $a_1+b_1\neq0$) has trivial stabilizer, and its orbit is the line through p parallel to the c-axis (in particular, all orbits are closed, and the stabilizers are either trivial or infinite; these are general features of $\mathbb{G}_{\mathrm{a}}$-actions on affine schemes [15]); the ring of $\mathbb{G}_{\mathrm{a}}$-invariants is the subalgebra

$$\mathbb{K}[b_0,a_1,b_1]\subset\mathbb{K}[b_0,a_1,b_1,c]$$

(which is finitely generated, even though the group $\mathbb{G}_{\mathrm{a}}$ is non-reductive; again this is a general feature of $\mathbb{G}_{\mathrm{a}}$-actions on affine schemes [15]), and the map

$$\operatorname{Spec}\mathbb{K}[b_0,a_1,b_1,c]\to\operatorname{Spec}\mathbb{K}[b_0,a_1,b_1,c]^{\mathbb{G}_{\mathrm{a}}}$$

does not separate the orbits of the fixed points lying on the same line parallel to the c-axis (this is due to the non-reductivity of the group $\mathbb{G}_{\mathrm{a}}$).

6.3.6. The case of general r. The system of homogeneous quadratic equations

$$\left\{\begin{array}{c} Q_0(s_0)=0\\ Q_1(s_0,s_1)=0\\ \vdots\\ Q_{r-2}(s_0,s_1,\ldots,s_{r-2})=0\end{array}\right.\tag{6.19}$$

(here $s_i:=a_i+b_i$) defining the locus $\mathcal{K}_r$ of integrable I_2-valued pairs on I_r inside $\mathcal{H}_r=\operatorname{Hom}(\mathrm{I}_r,\mathrm{I}_r\otimes\mathrm{I}_2)$ can be solved explicitly for each $r\geqslant2$. In fact the quadratic forms (6.15) can be rewritten for ℓ even (say $\ell = 2m$) as

$$Q_\ell(X_0, \dots, X_\ell) = (m+1)\left(X_m^2 + 2\sum_{i=0}^{m-1} X_i X_{\ell-i}\right), \quad \ell = 2m,$$

and for ℓ odd (say $\ell = 2m+1$) as

$$Q_\ell(X_0, \dots, X_\ell) = (\ell+2)\sum_{i=0}^{m} X_i X_{\ell-i}, \qquad \ell = 2m+1$$

(thus a suitable multiple of Q_ℓ is the quadratic form $X^{\mathrm{t}} A X$, where

$$X = \begin{pmatrix} X_0 \\ \vdots \\ X_\ell \end{pmatrix}, \qquad A = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \vdots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}).$$

It follows that the system (6.19) is equivalent to

$$s_i = 0, \qquad i = 0, 1, \dots, \lfloor \frac{r}{2} \rfloor - 1.$$

(for example, the systems for $r = 4$ and $r = 5$, which are given, respectively, by

$$\begin{cases} s_0^2 = 0 \\ s_0 s_1 = 0 \\ s_1^2 + 2 s_0 s_2 = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} s_0^2 = 0 \\ s_0 s_1 = 0 \\ s_1^2 + 2 s_0 s_2 = 0 \\ s_0 s_3 + s_1 s_2 = 0 \end{cases},$$

are both equivalent to $s_0 = s_1 = 0$). In particular, it turns out that the locus $\mathcal{K}_r$, even though defined by quadratic equations, is a linear subvariety of $\mathcal{H}_r$.

This implies that the locus of Higgs fields on a vector bundle $V \to X$, where X is an elliptic surface of the type considered in the thesis, turns out to be a vector space, as soon as V has vertical determinant and is regular on a general fiber.

6.4. Examples

In this concluding section we will present some examples of Higgs bundles on elliptic surfaces in which we apply the technique of restriction to the generic fiber developed in the previous sections. More precisely, we will consider examples of Higgs bundles whose underlying vector bundle has vertical determinant and is semistable on a general fiber, but fails to have reduced spectral cover or to be fiberwise regular.

As usual, we will denote by B a non-singular, irreducible, projective curve of genus g, and by $\pi : X \to B$ a non-isotrivial Weierstrass fibration with nodal singular fibers, section $\Sigma \subset X$ and fundamental line bundle $\mathbb{L} \in \mathrm{Pic}^d(B)$. The surface X will always be assumed to be non-singular.

6.4.1. Bundles with spectral cover 2Σ. We recall the following proposition from [22]:

PROPOSITION 6.4.1. *Let V be a vector bundle of rank 2 and vertical determinant on X, and assume that the restriction of V to a general fiber X_b of π is isomorphic to the rank 2 Atiyah bundle of X_b. Then V fits into a non-split short exact sequence of the form*

$$0 \to \pi^*\lambda \to V \to \pi^*\mu \otimes \mathcal{I}_S \to 0, \tag{6.20}$$

where λ, μ are line bundles on B, and $\mathcal{I}_S \subseteq \mathcal{O}_X$ is the ideal is the sheaf of a 0-dimensional closed subscheme of $S \subset X$ (or $\mathcal{I}_S = \mathcal{O}_X$). Moreover, V satisfies

$$\det V \simeq \pi^*(\lambda \otimes \mu), \qquad \mathrm{c}_2(V) = \dim_k \mathrm{H}^0(S, \mathcal{O}_S).$$

We remark that, for a vector bundle V as in (6.20), one has isomorphisms

$$\pi_* \operatorname{Sym}^i V \simeq \lambda^i \tag{6.21}$$

for each non-negative integers i.

REMARK 6.4.2. Let us fix two line bundles λ, μ on B, and a local complete intersection 0-cycle $S \subset X$. Then the non-split extensions of $\mathcal{I}_S \otimes \pi^*\mu$ by $\pi^*\lambda$ are parametrized by the projective space $\mathbb{P}\mathrm{Ext}^1_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)$. The vector space $\mathrm{Ext}^1_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)$ fits into the exact sequence

$$\begin{aligned} 0 \;&\to\; \mathrm{H}^1(\mathcal{E}xt^0_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)) \to \mathrm{Ext}^1_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda) \to \\ &\to\; \mathrm{H}^0(\mathcal{E}xt^1_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)) \to \mathrm{H}^2(\mathcal{E}xt^0_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)), \end{aligned} \tag{6.22}$$

coming from the local to global ext spectral sequence. It is known (see for example [19]) that the rank 2 sheaf corresponding to a point $k\xi \in \mathbb{P}\mathrm{Ext}^1_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)$ is locally free if and only if the image of the extension class $\xi \in \mathrm{Ext}^1_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)$ under the map $\mathrm{Ext}^1_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda) \to \mathrm{H}^0(\mathcal{E}xt^1_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda))$ of (6.22) is a globally generating section of the sheaf [1]

$$\mathcal{E}xt^1_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda) \simeq \pi^*\mu^{-1} \otimes \mathcal{E}xt^1_X(\mathcal{I}_S, \mathcal{O}_X) \otimes \pi^*\lambda \simeq \det(\mathrm{N}_{S/X}) \otimes \pi^*(\lambda\mu^{-1}) \simeq \mathcal{O}_S.$$

Thus a locally free extension exists if, for example, the last group in (6.22) is zero. But

$$\mathcal{E}xt^0_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda) \simeq \pi^*\mu^{-1} \otimes \mathcal{H}om_X(\mathcal{I}_S, \mathcal{O}_X) \otimes \pi^*\lambda \simeq \pi^*(\lambda\mu^{-1}),$$

so that

$$\mathrm{H}^2(\mathcal{E}xt^0_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)) \simeq \mathrm{H}^0(\pi^*(\lambda^{-1}\mu) \otimes \omega_X)^\vee \simeq \mathrm{H}^0(\lambda^{-1}\mu\mathbb{L}\omega_B)^\vee \simeq \mathrm{H}^1(\lambda\mu^{-1}\mathbb{L}^{-1}).$$

and the space $\mathrm{H}^1(\lambda\mu^{-1}\mathbb{L}^{-1})$ is in general non-zero, as shown in the following example.

[1]Here $\mathrm{N}_{S/X}$ is the normal bundle of S in X, which is a locally free sheaf of rank 2 on S, since S has been assumed to be a locally complete intersection of codimension 2. Actually the bundle $\mathrm{N}_{S/X}$ is (globally) trivial, since the underlying topological space of the scheme S is discrete. Finally, the line bundle $\det(\mathrm{N}_{S/X}) \otimes \ell \in \mathrm{Pic}(S)$ is also trivial, for any $\ell \in \mathrm{Pic}(X)$.

Example 6.4.3. Let us assume that $\deg_B(\lambda) \leqslant \deg_B(\mu)$ (this is a necessary condition for any bundle V fitting into an exact sequence of the form (6.20) to be slope-semistable with respect to a divisor of the form $H_n = \Sigma + n\Phi$, where Φ is the numerical equivalence class of a closed fiber of π and n is an integer $> d$ [2]). Then

$$\deg_B(\lambda\mu^{-1}\mathbb{L}^{-1}) = \deg_B(\lambda) - \deg_B(\mu) - d \leqslant -d < 0,$$

so that

$$\mathrm{h}^1(\lambda\mu^{-1}\mathbb{L}^{-1}) = -\deg_B(\lambda\mu^{-1}\mathbb{L}^{-1}) - 1 + g \geqslant g,$$

showing that $\mathrm{H}^2(\mathit{Ext}^0_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)) \neq 0$ for $g \geqslant 1$.

We also remark that dimension of the first vector space in (6.22) can be written, by the Leray spectral sequence, as

$$\begin{aligned}
\mathrm{h}^1(\mathit{Ext}^0_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)) &= \mathrm{h}^1(\pi^*(\lambda\mu^{-1})) \\
&= \mathrm{h}^1(\mathrm{R}^0\pi_*\pi^*(\lambda\mu^{-1})) + \mathrm{h}^0(\mathrm{R}^1\pi_*\pi^*(\lambda\mu^{-1})) \\
&= \mathrm{h}^1(\lambda\mu^{-1}) + \mathrm{h}^0(\lambda\mu^{-1}\mathbb{L}^{-1})
\end{aligned}$$

This can be explicitly computed for example when $\deg_B(\lambda) < \deg_B(\mu)$ (a necessary condition for V to be slope-stable with respect to H_n). One obtains

$$\mathrm{h}^1(\mathit{Ext}^0_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)) = -\deg_B(\lambda\mu^{-1}) - (1-g) = \deg_B(\mu) - \deg_B(\lambda) - 1 + g \geqslant g.$$

Nonetheless, even when $\mathrm{H}^2(\mathit{Ext}^0_X(\mathcal{I}_S \otimes \pi^*\mu, \pi^*\lambda)) \neq 0$, one knows (see, e.g., [19]) that if the scheme S is assumed to be reduced, then a locally free extension of $\mathcal{I}_S \otimes \pi^*\mu$ by $\pi^*\lambda$ exists if and only if S has the Cayley-Bacharach property with respect to the linear system

$$|\pi^*\mu \otimes \pi^*\lambda^{-1} \otimes \omega_X| \simeq \mathbb{P}\mathrm{H}^0(\mu\lambda^{-1}\mathbb{L}\omega_B).$$

We recall that this means the following: for any curve $C \in |\pi^*\mu \otimes \pi^*\lambda^{-1} \otimes \omega_X|$, and for any point $x \in S$, if the curve C passes through all the points of $S - \{x\}$, then it passes through the point x as well.

Let V be a vector bundle on X as in Proposition 6.4.1. We want to study the locus of Higgs fields on V, that is, the come

$$\mathcal{H}_V := \{\phi \in \mathbf{H}_1 | \, \phi \wedge \phi = 0 \text{ in } \mathbf{H}_2\}\,, \tag{6.23}$$

where

$$\mathbf{H}_i := \mathrm{Hom}(V, V \otimes \wedge^i\Omega_X) \simeq \mathrm{H}^0(B, \pi_*(\mathit{End}(V) \otimes \wedge^i\Omega_X)). \tag{6.24}$$

In our case, the sheaf $\mathit{End}(V)$ can be written as

$$\begin{aligned}
\mathit{End}(V) &\simeq V^\vee \otimes V \\
&\simeq V^{\otimes 2} \otimes (\det V)^{-1} \\
&\simeq (\mathrm{Sym}^2 V \oplus \wedge^2 V) \otimes (\det V)^{-1} \\
&\simeq \mathrm{Sym}^2 V \otimes \pi^*(\lambda^{-1}\mu^{-1}) \oplus \mathcal{O}_X. \qquad (6.25)
\end{aligned}$$

[2] The condition $n > d$ ensures that the divisor H_n is ample.

REMARK 6.4.4. Let V be a vector bundle on X of rank r and vertical determinant, and assume that V is semistable on a general closed fiber of $\pi : X \to B$, with spectral cover $\varphi : C \to B$. We know that if V is fiberwise regular, then there is an isomorphism

$$\pi_* \mathcal{E}nd(V) \simeq \varphi_* \mathcal{O}_C \tag{6.26}$$

of vector bundles of rank r on X. Using the previous computations we will now show that the isomorphism (6.26) is in general not valid if one does not assume V to be regular on *every* fiber, even if V happens to be regular on a general fiber.

To this end, let us assume that V satisfies the assumptions of Proposition 6.4.1, and let us write V as an extension of the form (6.20). Then, using (6.25) and (6.21), one finds

$$\pi_* \mathcal{E}nd(V) \simeq \mathcal{O}_B \oplus \lambda\mu^{-1}. \tag{6.27}$$

On the other hand, the bundle V is semistable (in fact regular) on a general closed fiber, with spectral curve $2\Sigma \hookrightarrow X$. To compute the rank 2 bundle $\varphi_* \mathcal{O}_{2\Sigma}$, one starts from the exact sequence

$$0 \to \mathcal{O}_X(-2\Sigma) \to \mathcal{O}_X \to \mathcal{O}_{2\Sigma} \to 0,$$

and takes the induced exact sequence of higher direct π-images, which looks like[3]

$$0 \to 0 \to \mathcal{O}_B \to \varphi_* \mathcal{O}_{2\Sigma} \to \mathbb{L} \oplus \mathbb{L}^{-1} \to \mathbb{L}^{-1} \to 0 \to 0.$$

This shows that $\varphi_* \mathcal{O}_{2\Sigma}$ fits into an exact sequence

$$0 \to \mathcal{O}_B \to \varphi_* \mathcal{O}_{2\Sigma} \to \mathbb{L} \to 0. \tag{6.28}$$

Next, one observes that the morphism $\sigma : B \to X$ corresponding to the section Σ factors through the immersion $2\Sigma \hookrightarrow X$, thus producing a section $s : B \to 2\Sigma$ of the spectral cover $\varphi : 2\Sigma \to B$, and hence a splitting

$$\varphi_* s^* : \varphi_* \mathcal{O}_{2\Sigma} \to \varphi_* s_* \mathcal{O}_B \simeq \mathcal{O}_B$$

of the sequence (6.28). The conclusion is an isomorphism

$$\varphi_* \mathcal{O}_{2\Sigma} \simeq \mathcal{O}_B \oplus \mathbb{L}. \tag{6.29}$$

Now, the right hand sides of (6.27) and (6.29) are not isomorphic vector bundles in general. For example, if $\deg(\lambda) < \deg(\mu)$ and $d > 2g - 2$, one has

$$\mathrm{h}^0(\mathcal{O}_B \oplus \lambda\mu^{-1}) = 1 < 1 + (d + 1 - g) = \mathrm{h}^0(\mathcal{O}_B \oplus \mathbb{L}).$$

A side remark: the previous computations show that the inequality $\deg(\lambda) < \deg(\mu)$ implies V to be simple. In fact, one has $\dim_k \mathrm{End}(V) = \mathrm{h}^0(\pi_* \mathcal{E}nd(V)) = \mathrm{h}^0(\mathcal{O}_B \oplus \lambda\mu^{-1}) = 1$ for $\deg(\lambda) < \deg(\mu)$. This agrees with the fact the a slope-stable bundle is simple.

Related to the previous remark is the observation that the inequality $2g - 2 < d$, which appears as one of the assumptions of Proposition 4.4.4, is the same as the inequality

[3]We recall that, by relative Serre duality, one has an isomorphism

$$\mathrm{R}^1 \pi_* \mathcal{O}_X(-n\Sigma) \simeq (\pi_*(\mathcal{O}_X(n\Sigma) \otimes \omega_\pi^\circ))^\vee \simeq \mathbb{L}^{-1} \otimes \bigoplus_{i \in \{0,2,\dots,n\}} \mathbb{L}^i$$

for each integer $n < 0$.

$\deg(\lambda) < \deg(\mu)$ applied to the cotangent sheaf of X, as follows from consideration of the exact sequence[4]

$$0 \to \pi^*\omega_B \to \Omega_X \to \mathcal{I}_Z \otimes \pi^*\mathbb{L} \to 0.$$

As a final remark we point out that the failure of the bundle V of (6.20) to be regular on some fibers is not tied to the presence of the sheaf $\mathcal{I}_S$, but it can occur also in the case $S = \emptyset$. In fact, let us fix two line bundles λ, μ on B such that $\lambda\mu^{-1}\mathbb{L}^{-1} \not\simeq \mathcal{O}_B$ and $\mathrm{h}^0(\lambda\mu^{-1}\mathbb{L}^{-1}) \neq 0$ (e.g., one could take $\mu = \mathcal{O}_B$ and λ any sufficiently ample line bundle $\not\simeq \mathbb{L}$), and consider extensions of the form

$$0 \to \pi^*\lambda \to V \to \pi^*\mu \to 0. \tag{6.30}$$

Then the restriction V_b of the bundle V to a fiber X_b of π is either trivial, hence non-regular, if the restriction of the sequence (6.30) to X_b splits, or isomorphic to I_2, hence regular, otherwise. Thus V_b is non-regular if and only if the image of the extension class $\xi \in \mathrm{Ext}^1_X(\pi^*\mu, \pi^*\lambda) \simeq \mathrm{H}^1(X, \pi^*(\lambda\mu^{-1}))$ of (6.30) in $\mathrm{H}^0(\mathit{Ext}^1_\pi(\pi^*\mu, \pi^*\lambda)) \simeq \mathrm{H}^0(\mathrm{R}^1\pi_*\pi^*(\lambda\mu^{-1}))$, via the surjective map

$$\mathrm{H}^1(X, \pi^*(\lambda\mu^{-1})) \to \mathrm{H}^0(B, \mathrm{R}^1\pi_*\pi^*(\lambda\mu^{-1}))$$

coming from the Leray spectral sequence, vanishes at the point $b \in B$. But $\mathrm{R}^1\pi_*\pi^*(\lambda\mu^{-1}) \simeq \lambda\mu^{-1}\mathbb{L}^{-1}$. Thus, if we fix arbitrarily a non-zero element

$$s \in \mathrm{H}^0(B, \mathit{Ext}^1_\pi(\pi^*\mu, \pi^*\lambda)) \simeq \mathrm{H}^0(\mu^{-1}\lambda\mathbb{L}^{-1})$$

then we see that the bundle V corresponding to any extension of the form (6.28) whose class maps to the section s will fail to be regular over $\deg(\lambda\mu^{-1}\mathbb{L}^{-1}) > 0$ points (with multiplicities).

Now let us go back to the spaces $\mathbf{H}_i$ in (6.24) and the quadratic map

$$\mathbf{H}_1 \to \mathbf{H}_2, \qquad \phi \mapsto \phi \wedge \phi, \tag{6.31}$$

whose isotropic cone is the set $\mathcal{H}_V$ in (6.23). First of all, using (6.27) we find for $\mathbf{H}_2 \simeq \mathrm{H}^0(B, \pi_*(\mathit{End}(V) \otimes \omega_X))$ an isomorphism

$$\begin{aligned} \mathbf{H}_2 &\simeq \mathrm{H}^0(\omega_B\mathbb{L} \otimes (\mathcal{O}_B \oplus \lambda\mu^{-1})) \\ &\simeq \mathrm{H}^0(\omega_B\mathbb{L}) \oplus \mathrm{H}^0(\lambda\mu^{-1}\omega_B\mathbb{L}). \end{aligned} \tag{6.32}$$

Next, for $\mathbf{H}_1 \simeq \mathrm{H}^0(B, \pi_*(\mathit{End}(V) \otimes \Omega_X))$ we find, using (6.25),

$$\mathbf{H}_1 \simeq \mathrm{H}^0(\lambda^{-1}\mu^{-1} \otimes \pi_*(\Omega_X \otimes \mathrm{Sym}^2 V)) \oplus \mathrm{H}^0(\omega_B). \tag{6.33}$$

To compute $\pi_*(\Omega_X \otimes \mathrm{Sym}^2 V)$, one starts from the sequence of torsion-free sheaves, easily shown to be exact, deduced from (6.20):

$$0 \to \pi^*\lambda \otimes V \to \mathrm{Sym}^2 V \to \mathrm{Sym}^2(\pi^*\mu \otimes \mathcal{I}_S) \to 0.$$

[4] We recall that we denote by Z the scheme of singular points of the projection π.

Tensoring it with the locally free sheaf Ω_X produces another sequence of torsion-free sheaves, still exact:

$$0 \to \pi^*\lambda \otimes V \otimes \Omega_X \to \Omega_X \otimes \mathrm{Sym}^2 V \to \Omega_X \otimes \mathrm{Sym}^2(\pi^*\mu \otimes \mathcal{I}_S) \to 0. \tag{6.34}$$

The pull-back to the generic fiber of π of the non-zero sheaves appearing here (equivalently, their restriction to a general closed fiber of π) are isomorphic, respectively, to

$$\mathrm{I}_2 \otimes \mathrm{I}_2 \simeq \mathrm{I}_1 \oplus \mathrm{I}_3, \quad \mathrm{I}_2 \otimes \mathrm{Sym}^2\mathrm{I}_2 \simeq \mathrm{I}_2 \otimes \mathrm{I}_3 \simeq \mathrm{I}_2 \oplus \mathrm{I}_4, \quad \mathrm{I}_2.$$

It follows that by applying the functor π_* to (6.34), one obtains an exact sequence of vector bundles on B

$$0 \to \pi_*(\pi^*\lambda \otimes V \otimes \Omega_X) \to \pi_*(\Omega_X \otimes \mathrm{Sym}^2 V) \to \pi_*(\Omega_X \otimes \mathrm{Sym}^2(\pi^*\mu \otimes \mathcal{I}_S)), \tag{6.35}$$

in which the non-zero bundles have rank $2, 2$ and 1 respectively. Thus the first non-zero arrow in (6.35) is an isomorphism, showing that

$$\pi_*(\Omega_X \otimes \mathrm{Sym}^2 V) \simeq \pi_*(\pi^*\lambda \otimes V \otimes \Omega_X) \simeq \lambda \otimes \pi_*(V \otimes \Omega_X).$$

Substituting this in (6.33), we obtain

$$\mathbf{H}_1 \simeq \mathrm{H}^0(\mu^{-1} \otimes \pi_*(\Omega_X \otimes V)) \oplus \mathrm{H}^0(\omega_B) \tag{6.36}$$

Understanding the rank 2 bundle $\pi_*(\Omega_X \otimes V)$ in full generality does not seem so easy. In the next section we treat the special case $V = \Omega_X$, which can be understood completely.

6.4.2. The case of the cotangent bundle. We know that the cotangent bundle Ω_X satisfies the assumptions of Proposition 6.4.1. In fact, it fits into an exact sequence of torsion-free sheaves on X of the form

$$0 \to \pi^*\omega_B \to \Omega_X \to \Omega_\pi \to 0, \tag{6.37}$$

where $\Omega_\pi \simeq \mathcal{I}_Z \otimes \pi^*\mathbb{L}$ is the sheaf of relative differentials of π, and Z the scheme of singular points of the morphism π (which under our assumptions is the reduced subscheme of X consisting of the nodes of the fibers of π). We have isomorphisms

$$\pi_*\Omega_X^{\otimes 2} \simeq \pi_*(\mathrm{Sym}^2\Omega_X \oplus \wedge^2\Omega_X) \simeq \omega_B^2 \oplus \omega_B\mathbb{L}. \tag{6.38}$$

Thus, setting $(V, \lambda, \mu) = (\Omega_X, \omega_B, \mathbb{L})$ in (6.36) and (6.32), we obtain

$$\mathbf{H}_1 \simeq \mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B^2\mathbb{L}^{-1}), \qquad \mathbf{H}_2 \simeq \mathrm{H}^0(\omega_B^2) \oplus \mathrm{H}^0(\omega_B\mathbb{L}). \tag{6.39}$$

To see how the map (6.31) reads in terms of the isomorphisms (6.39), we look at what happens on the generic fiber of π: we know (see Subsection 6.3.3) that, with respect to suitable bases (α, β, γ) on $\mathrm{Hom}(\mathrm{I}_2, \mathrm{I}_2 \otimes \mathrm{I}_2)$ and $(1, \nu)$ on $\mathrm{Hom}(\mathrm{I}_2, \mathrm{I}_2 \otimes \wedge^2\mathrm{I}_2) \simeq \mathrm{End}(\mathrm{I}_2)$, the map

$$\mathrm{Hom}(\mathrm{I}_2, \mathrm{I}_2 \otimes \mathrm{I}_2) \to \mathrm{End}(\mathrm{I}_2), \qquad \psi \mapsto \psi \wedge \psi$$

can be written as

$$a\alpha + b\beta + c\gamma \mapsto (a+b)^2\nu.$$

This shows that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B^2\mathbb{L}^{-1}) \arrow[r] \arrow[d, "\simeq"', "(6.39)"] & \mathrm{H}^0(\omega_B^2) \oplus \mathrm{H}^0(\omega_B\mathbb{L}) \arrow[d, "(6.39)"', "\simeq"] \\
\mathbf{H}_1 \arrow[r, "\phi\mapsto\phi\wedge\phi"] & \mathbf{H}_2
\end{tikzcd},$$

in which the top horizontal arrow is the map

$$(a, b, c) \mapsto ((a - b)^2, 0);$$

here $(a + b)^2$ is the image of $(a + b) \otimes (a + b) \in \mathrm{H}^0(\omega_B) \otimes \mathrm{H}^0(\omega_B)$ under the multiplication map $\mathrm{H}^0(\omega_B) \otimes \mathrm{H}^0(\omega_B) \to \mathrm{H}^0(\omega_B^2)$.

Thus the locus $\mathcal{H}_{\Omega_X} \subseteq \mathbf{H}_1$ of Higgs fields on Ω_X corresponds, under the isomorphism $\mathbf{H}_1 \simeq \mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B^2\mathbb{L}^{-1})$, to the vector subspace

$$\{(a, -a, c) : a \in \mathrm{H}^0(\omega_B), c \in \mathrm{H}^0(\omega_B^2\mathbb{L}^{-1})\}.$$

The last space is isomorphic to $\mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B^2\mathbb{L}^{-1})$ via $(a, -a, c) \leftrightsquigarrow (a, c)$; so $\mathcal{H}_{\Omega_X}$ is itself a vector space, of dimension

$$\dim_k \mathcal{H}_{\Omega_X} = g + \mathrm{h}^0(\omega_B^2\mathbb{L}^{-1}).$$

From the last formula we see that, under the assumption $d > 4g - 4$, the vector space $\mathcal{H}_{\Omega_X}$ and its vector subspace formed by the scalar Higgs fields[5] both have the same dimension g, so they coincide. Thus we arrive at the (by now familiar) conclusion that the there are no Higgs fields on Ω_X besides the scalar ones. Compare this with Proposition 4.4.4, which says that any fiberwise regular vector bundle V on X of rank r and vertical determinant supports only scalar Higgs fields, if either its spectral curve is integral and $d > 2g-2$, or if its spectral curve $C \in |r\Sigma + \pi^*D|$ is reduced and $\mathrm{c}_2(V) > (r-1)\deg D + 2g - 2$.

Remark 6.4.5. We remark that the bundle Ω_X

(1) fails to be regular on some fibers;
(2) has non-reduced spectral cover.

Nonetheless we saw that, under the assumption $d > 4g-4$, it satisfies the same conclusion as that of Proposition 4.4.4. This could either be a coincidence, or else it could point the way towards a suitable generalization of Proposition 4.4.4, valid for vector bundles of rank 2 and vertical determinant, regular on a general fiber, over surfaces with sufficiently ample $\mathbb{L}$.

In the next section we will see with some simple examples showing that there is no way to obtain the same conclusion without the assumption of regularity on a general fiber.

[5]We recall (see Subsection 4.4.1) that the scalar Higgs fields on a vector bundle V on X are the Higgs fields $V \to V \otimes \Omega_X$ of the form $s \mapsto s \otimes \alpha$, where α is a global 1-form on X.

6.4.3. Trivial bundles. Let Y be a non-singular, projective, irreducible k-scheme, and let V be a trivial vector bundle of rank r on Y. Set $\mathbf{H} := \mathrm{H}^0(V)$, and identify V with $\mathbf{H} \otimes \mathcal{O}_Y$ via the evaluation map $\mathbf{H} \otimes \mathcal{O}_Y \xrightarrow{\simeq} V$. Then one has an isomorphism $\mathrm{Hom}(V, V \otimes \wedge^i \Omega_Y) \simeq \mathrm{End}(\mathbf{H}) \otimes \mathrm{H}^0(\wedge^i \Omega_Y)$ for each $i \geqslant 0$, and a commutative diagram

$$\begin{tikzcd} \mathrm{End}(\mathbf{H}) \otimes \mathrm{H}^0(\Omega_Y) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{End}(\mathbf{H}) \otimes \mathrm{H}^0(\wedge^2 \Omega_Y) \arrow[d, "\simeq"] \\ \mathrm{Hom}(V, V \otimes \Omega_Y) \arrow[r, "\phi \mapsto \phi \wedge \phi"] & \mathrm{Hom}(V, V \otimes \wedge^2 \Omega_Y) \end{tikzcd},$$

where the top horizontal arrow is the quadratic map associated to the symmetric bilinear map

$$\begin{array}{ccc} \mathrm{Sym}^2(\mathrm{End}(\mathbf{H}) \otimes \mathrm{H}^0(\Omega_Y)) & \to & \mathrm{End}(\mathbf{H}) \otimes \mathrm{H}^0(\wedge^2 \Omega_Y) \\ f \otimes \alpha \cdot g \otimes \beta & \mapsto & [f, g] \otimes \alpha \wedge \beta \end{array}.$$

Here $[f, g]$ is the commutator $f \circ g - g \circ f$ of the linear maps $f, g \in \mathrm{End}(\mathbf{H})$, while $\alpha \wedge \beta$ is the image of $\alpha \otimes \beta \in \mathrm{H}^0(\Omega_Y)^{\otimes 2}$ via the multiplication map $\mathrm{H}^0(\Omega_Y)^{\otimes 2} \to \mathrm{H}^0(\wedge^2 \Omega_Y)$.

In the case of an elliptic surface $\pi : X \to B$ things become particularly simple, because the map $\mathrm{H}^0(\Omega_X)^{\otimes 2} \to \mathrm{H}^0(\omega_X)$ is zero. Thus in this case one has an equality

$$\mathcal{H}_V = \mathrm{Hom}(V, V \otimes \Omega_X) \simeq \mathrm{End}(\mathbf{H}) \otimes \mathrm{H}^0(\omega_B).$$

One sees that the locus $\mathcal{H}_V$ is again a vector space (of dimension $r^2 g$), but for $r \geqslant 2$ and $g \geqslant 1$ the inclusion $\mathrm{H}^0(\Omega_X) \hookrightarrow \mathcal{H}_V$ is proper (not all Higgs fields on V are scalar). This shows the importance of the assumption of fiberwise regularity (or of regularity on a general fiber), since the bundle V has trivial determinant and is fiberwise semistable, but it is non-regular on every fiber.

6.4.4. The case $V = \Omega_X \oplus \mathcal{O}_X$. Up to this point, we have only seen cases of bundles V for which the cone of Higgs fields $\mathcal{H}_V$ is a vector subspace, perhaps proper, of the space $\mathrm{Hom}(V, V \otimes \Omega_X)$. We now analyze the bundle $V := \Omega_X \oplus \mathcal{O}_X$ (which has vertical determinant ω_X and is non-regular on every fiber), which exhibits a more interesting behavior. First of all we write the spaces $\mathbf{H}_i := \mathrm{Hom}(V, V \otimes \wedge^i \Omega_X)$ as spaces of 2×2 matrices, with entries belonging to suitable vector spaces of bundle maps:

$$\mathbf{H}_i \simeq \begin{pmatrix} \mathrm{Hom}(\Omega_X, \Omega_X \otimes \wedge^i \Omega_X) & \mathrm{Hom}(\mathcal{O}_X, \Omega_X \otimes \wedge^i \Omega_X) \\ \mathrm{Hom}(\Omega_X, \wedge^i \Omega_X) & \mathrm{Hom}(\mathcal{O}_X, \wedge^i \Omega_X) \end{pmatrix} =: \begin{pmatrix} A & B \\ C & D. \end{pmatrix} \tag{6.40}$$

Here $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \simeq A \oplus B \oplus C \oplus D$ is the space of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $(a, b, c, d) \in A \oplus B \oplus C \oplus D$. Let us compute the spaces A, B, C, D in the cases $i \in \{1, 2\}$:

- in the case $i = 1$ we have:
  - $A = \mathrm{Hom}(\Omega_X, \Omega_X \otimes \Omega_X) \simeq \mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B^2 \mathbb{L}^{-1})$ by the first isomorphism in (6.39);
  - $B \simeq \mathrm{H}^0(\Omega_X^{\otimes 2}) \simeq \mathrm{H}^0(\omega_B^2) \oplus \mathrm{H}^0(\omega_B \mathbb{L})$ by (6.38);
  - $C = \mathrm{End}(\Omega_X) \simeq \mathrm{H}^0(\mathcal{O}_B) \oplus \mathrm{H}^0(\omega_B \mathbb{L}^{-1})$ by (6.27) applied to the case $(V, \lambda, \mu) = (\Omega_X, \omega_B, \mathbb{L})$;

– $D \simeq \mathrm{H}^0(\Omega_X) \simeq \mathrm{H}^0(\omega_B)$.

Substituting in (6.40) we arrive at

$$\mathbf{H}_1 \simeq \begin{pmatrix} \mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B) \oplus \mathrm{H}^0(\omega_B^2 \mathbb{L}^{-1}) & \mathrm{H}^0(\omega_B \mathbb{L}) \oplus \mathrm{H}^0(\omega_B^2) \\ \mathrm{H}^0(\mathcal{O}_B) \oplus \mathrm{H}^0(\omega_B \mathbb{L}^{-1}) & \mathrm{H}^0(\omega_B) \end{pmatrix}; \tag{6.41}$$

- in the case $i = 2$ we have:
  – $A \simeq \mathrm{H}^0(\omega_B^2) \oplus \mathrm{H}^0(\omega_B \mathbb{L})$ by the second isomorphism in (6.39);
  – $B \simeq \mathrm{H}^0(\Omega_X \otimes \omega_X) \simeq \mathrm{H}^0(\pi_*(\Omega_X \otimes \pi^*(\omega_B \mathbb{L})) \simeq \mathrm{H}^0(\omega_B^2 \mathbb{L})$;
  – $C = \mathrm{Hom}(\Omega_X, \omega_X) \simeq \mathrm{H}^0(\Omega_X^\vee \otimes \omega_B) \simeq \mathrm{H}^0(\Omega_X) \simeq \mathrm{H}^0(\omega_B)$;
  – $D \simeq \mathrm{H}^0(\omega_X) \simeq \mathrm{H}^0(\omega_B \mathbb{L})$.

  Substituting in (6.40) we obtain

$$\mathbf{H}_2 \simeq \begin{pmatrix} \mathrm{H}^0(\omega_B \mathbb{L}) \oplus \mathrm{H}^0(\omega_B^2) & \mathrm{H}^0(\omega_B^2 \mathbb{L}) \\ \mathrm{H}^0(\omega_B) & \mathrm{H}^0(\omega_B \mathbb{L}) \end{pmatrix}. \tag{6.42}$$

In order to read the map (6.31) in terms of the isomorphisms (6.41) and (6.42), we again move to the generic fiber $E := X_\eta$ of π. The pull-back of V to E is isomorphic to $W := \mathrm{I}_2 \oplus \mathcal{O}_E$, so we look at the map

$$\mathbf{K}_1 \to \mathbf{K}_2, \qquad \psi \mapsto \psi \wedge \psi, \tag{6.43}$$

where $\mathbf{K}_i := \mathrm{Hom}(W, W \otimes \wedge^i \mathrm{I}_2)$. We again write the $\mathbf{K}_i$ as vector spaces (over the function field $K := \mathcal{O}_{B,\eta}$ of the base B) of 2×2 matrices with coefficients in spaces of bundle maps:

$$\mathbf{K}_i \simeq \begin{pmatrix} \mathrm{Hom}(\mathrm{I}_2, \mathrm{I}_2 \otimes \wedge^i \mathrm{I}_2) & \mathrm{Hom}(\mathcal{O}_E, \mathrm{I}_2 \otimes \wedge^i \mathrm{I}_2) \\ \mathrm{Hom}(\mathrm{I}_2, \wedge^i \mathrm{I}_2) & \mathrm{Hom}(\mathcal{O}_E, \wedge^i \mathrm{I}_2) \end{pmatrix} =: \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}. \tag{6.44}$$

Let us consider separately the cases $i \in \{1, 2\}$ of (6.44):

- in the case $i = 1$ the spaces A', B', C', D' in (6.44) are of the form $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_s \otimes \mathrm{I}_2)$ for $r, s \in \{1, 2\}$. The elements of the space $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_s \otimes \mathrm{I}_2)$ can be represented, locally, by $s \times r$, I_2-valued matrices $\Phi_1 x + \Phi_2 y$, where Φ_1, Φ_2 are $s \times r$, $\mathcal{O}_E$-valued matrices and (x, y) is the usual local frame on I_2. Let us write down the form of the elements of each of the spaces A', B', C', D':
  – we saw already in Subsection 6.3.3 that the space $A' = \mathrm{Hom}(\mathrm{I}_2, \mathrm{I}_2 \otimes \mathrm{I}_2)$ has linear coordinates (a, b, c) with respect to which its elements are locally represented by the matrices $\Phi_1 x + \Phi_2 y$, where

$$\Phi_1 = \begin{pmatrix} 2b & 2c \\ 0 & -a+b \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} 0 & a+b \\ 0 & 0 \end{pmatrix};$$

  – the space $B' = \mathrm{Hom}(\mathcal{O}_E, \mathrm{I}_2 \otimes \mathrm{I}_2)$ satisfies $B' \simeq \mathrm{H}^0(\mathrm{I}_2^{\otimes 2}) \simeq \mathrm{H}^0(\mathrm{Sym}^2 \mathrm{I}_2 \oplus \wedge^2 \mathrm{I}_2) \simeq \mathrm{H}^0(\mathrm{I}_3) \oplus \mathrm{H}^0(\mathcal{O}_E) \simeq K^{\oplus 2}$, so it has a K-basis $(x \wedge y, x^2)$, where $x \wedge y$ and x^2 denote, respectively, the maps $\mathcal{O}_E \to \mathrm{I}_2^{\otimes 2}$ locally of the form $1 \mapsto x \otimes y - y \otimes x$ and $1 \mapsto x \otimes x$; they are locally represented by the matrices

$$\begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} y, \qquad \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x.$$

Thus an arbitrary element $dx \wedge y + ex^2$ of B' ($d, e \in K$) is locally represented by

$$\begin{pmatrix} e \\ -d \end{pmatrix} x + \begin{pmatrix} d \\ 0 \end{pmatrix} y;$$

– $C' = \mathrm{End}(\mathrm{I}_2)$ has dimension 2 and a basis (id, ν), where the map $\nu : \mathrm{I}_2 \to \mathrm{I}_2$ is locally of the form $x \mapsto 0, y \mapsto x$. The maps id, ν are locally represented, respectively, by the matrices

$$\begin{pmatrix} x & y \end{pmatrix} \qquad \begin{pmatrix} 0 & x \end{pmatrix}.$$

Thus an arbitrary element of $f\mathrm{id} + g\nu$ of C' ($f, g \in K$) is represented locally by the matrix

$$\begin{pmatrix} f & g \end{pmatrix} x + \begin{pmatrix} 0 & f \end{pmatrix} y;$$

– finally, $D' \simeq \mathrm{H}^0(\mathrm{I}_2) = Kx$. We write its elements as hx ($h \in K$).

The previous computations show that the space $\mathbf{K}_1 \simeq K^{\oplus 8}$ has linear coordinates $(a, b, \ldots, h)$ with respect to which an arbitrary element $\psi \in \mathbf{K}_1$ can be written locally as a 3×3, I_2-valued matrix $\Psi = \Psi_1 x + \Psi_2 y$, where

$$\Psi_1 = \begin{pmatrix} 2b & 2c & e \\ 0 & -a+b & -d \\ f & g & h \end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix} 0 & a+b & d \\ 0 & 0 & 0 \\ 0 & f & 0 \end{pmatrix}; \tag{6.45}$$

- in the case $i = 2$ the spaces A', B', C', D' in (6.44) are of the form $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_s)$ for $r, s \in \{1, 2\}$ (since $\wedge^2 \mathrm{I}_2 \simeq \mathcal{O}_E$). The elements of the space $\mathrm{Hom}(\mathrm{I}_r, \mathrm{I}_s)$ can be written locally as $s \times r$, $\mathcal{O}_E$-valued matrices. Thus $\mathbf{K}_2 \simeq K^{\oplus 5}$ has linear coordinates $(\alpha, \beta, \gamma, \delta, \varepsilon)$ with respect to which an arbitrary element can be written locally as a 3×3, $\mathcal{O}_E$-valued (in fact, constant) matrix

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & 0 \\ 0 & \delta & \epsilon \end{pmatrix}.$$

Now, for an element $\psi \in \mathbf{K}_1$ represented by the matrix $\Psi = \Psi_1 x + \Psi_2$, with Ψ_1, Ψ_2 as in (6.45), the element $\psi \wedge \psi \in \mathbf{K}_2$ is represented by

$$[\Psi_1, \Psi_2] = \Psi_1 \Psi_2 - \Psi_2 \Psi_1 = \begin{pmatrix} -df & (a+b)^2 + ef - dg & d(a+3b-h) \\ 0 & -df & 0 \\ 0 & f(2a+h) & 2df \end{pmatrix},$$

from which one can read the coordinate representation of the map (6.43), namely

$$\begin{cases} \alpha = -df \\ \beta = (a+b)^2 + ef - dg \\ \gamma = d(a+3b-h) \\ \delta = f(2a+h) \\ \epsilon = 2df \end{cases}. \tag{6.46}$$

The previous computations show the existence of a commutative diagram

$$\begin{tikzcd}
\begin{pmatrix} \mathrm{H}^0(\omega_B^{\oplus 2}\oplus\omega_B^2\mathbb{L}^{-1}) & \mathrm{H}^0(\omega_B\mathbb{L}\oplus\omega_B^2) \\ \mathrm{H}^0(\mathcal{O}_B\oplus\omega_B\mathbb{L}^{-1}) & \mathrm{H}^0(\omega_B) \end{pmatrix} \arrow[r] \arrow[d, "\simeq"', "(6.41)"] & \begin{pmatrix} \mathrm{H}^0(\omega_B\mathbb{L}\oplus\omega_B^2) & \mathrm{H}^0(\omega_B^2\mathbb{L}) \\ \mathrm{H}^0(\omega_B) & \mathrm{H}^0(\omega_B\mathbb{L}) \end{pmatrix} \arrow[d, "(6.42)"', "\simeq"] \\
\mathbf{H}_1 \arrow[r, "\phi\mapsto\phi\wedge\phi"] & \mathbf{H}_2
\end{tikzcd},$$

where the top horizontal arrow is the map

$$\begin{pmatrix} (a,b,c) & (d,e) \\ (f,g) & h \end{pmatrix} \mapsto \begin{pmatrix} (-df,(a+b)^2+ef-dg) & d(a+3b-h) \\ f(2a+h) & 2df \end{pmatrix}.$$

So the integrability condition $\phi\wedge\phi=0$ is equivalent to the system

$$\left\{\begin{array}{ccc} df & = & 0 \\ f(2a+h) & = & 0 \\ d(a+3b-h) & = & 0 \\ (a+b)^2+ef-dg & = & 0 \end{array}\right. . \tag{6.47}$$

Example 6.4.6. Assume the base B to be elliptic. Then

$$\mathrm{H}^0(\omega_B^i)\simeq\mathrm{H}^0(\mathcal{O}_B)=k,$$
$$\mathrm{H}^0(\omega_B^i\mathbb{L}^{-1})\simeq\mathrm{H}^0(\mathbb{L}^{-1})=0,$$

(since we are assuming $\mathbb{L}\not\simeq\mathcal{O}_B$), and

$$\mathrm{H}^0(\omega_B\mathbb{L})\simeq\mathrm{H}^0(\mathbb{L})\simeq k^{\oplus\deg\mathbb{L}}.$$

Thus in this case

$$\mathbf{H}_1\simeq\left\{\begin{pmatrix} (a,b) & (d,e) \\ f & h \end{pmatrix}\middle| a,b,e,f,h\in k, d\in\mathrm{H}^0(\mathbb{L})\right\}=\begin{pmatrix} k\oplus k & \mathrm{H}^0(\mathbb{L})\oplus k \\ k & k \end{pmatrix}\simeq k^{\oplus(\deg\mathbb{L}+5)},$$

and the equations (6.47) are equivalent to the union of the following three systems

$$\left\{\begin{array}{ccc} b & = & -a \\ d & = & 0 \\ f & = & 0 \end{array}\right., \quad \left\{\begin{array}{ccc} b & = & -a \\ f & = & 0 \\ h & = & -2a \end{array}\right., \quad \left\{\begin{array}{ccc} d & = & 0 \\ h & = & -2a \\ (a+b)^2+ef & = & 0 \end{array}\right. .$$

So the locus $\mathcal{H}_V$ appears as the union of three irreducible components: two vector subspaces, one of dimension 3 and one of dimension $2+\deg\mathbb{L}$, and a third component isomorphic to the affine cone over a quadric of rank 3 in $\mathbb{P}^3_k$. These three components pairwise intersect each other in the same 2-dimensional vector subspace.

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