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Citation: Chaos **20**, 023111 (2010); doi: 10.1063/1.3396283
View online: https://doi.org/10.1063/1.3396283
View Table of Contents: http://aip.scitation.org/toc/cha/20/2
Published by the American Institute of Physics

Fluctuation theorem, nonlinear response, and the regularity of time reversal symmetry

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(Received 9 October 2009; accepted 26 March 2010; published online 28 May 2010)

The Gallavotti–Cohen fluctuation theorem (FT) implies an infinite set of identities between correlation functions that can be seen as a generalization of Green–Kubo formula to the nonlinear regime. As an application, we discuss a perturbative check of the FT relation through these identities for a simple Anosov reversible system; we find that the lack of differentiability of the time reversal operator implies a *violation* of the Gallavotti–Cohen fluctuation relation. Finally, a brief comparison to Lebowitz–Spohn FT is reported. © *2010 American Institute of Physics*. [doi:10.1063/1.3396283]

At present time, there is no universally accepted microscopic theory of the steady state of nonequilibrium systems; we call *nonequilibrium steady state* the stationary state of a dissipative system. However, for small external forcings some properties of nonequilibrium steady states can be inferred by considering them as "small perturbations" of equilibrium states, which are well described by equilibrium statistical mechanics. As a result, the response to a small external forcing can be expressed, at least formally, in terms of quantities which are computed at equilibrium, through the so-called *linear response theory*. Obviously, the linear response theory loses its predictive power when the forcings are not small, that is when the assumption of closeness to an equilibrium state is no longer justified; the goal of nonequilibrium statistical mechanics is to provide relations between physical observables in the nonlinear regime that reduce, in particular, to the usual linear response theory in the limit of zero forcings. Analogous to what happens at equilibrium with the ergodic hypothesis, one can attempt a description of nonequilibrium steady states by making some assumptions on the dynamics and then checking if the theoretical predictions which are eventually implied by the assumptions are consistent with the experiments; this is what Gallavotti and Cohen proposed with their *chaotic hypothesis* (CH). This hypothesis consists of an apparently very strong claim on the dynamics of the particles at the steady state; it assumes that for the purpose of studying macroscopic quantities a system exhibiting chaotic motions may be considered chaotic in a "maximal" sense (which can be made mathematically precise). Accepting this, if the law of motion is reversible under a suitably regular time reversal operation, then it is possible to rigorously prove a nontrivial large deviation law for the rate of entropy production, which is the content of the so-called Gallavotti–Cohen (GC) fluctuation theorem (FT); this law has been successfully checked in many experiments, most of which numerical, in the past 15 years. Remarkably, in the limit of zero forcings the GC FT implies the Green–Kubo (GK) formula, a well-known result of linear response theory; in this paper we show that with little efforts, the GC FT allows to compute the higher order corrections to linear response. As an application, we perform a check of the GC fluctuation relation by explicitly evaluating the first correction to linear response in a very simple model, whose dynamics is reversible under a continuous but not differentiable time reversal operation; we find that the lack of differentiability of the time reversal operator implies a violation of the fluctuation relation. However, in the considered case a different large deviation rule is true, which is essentially equivalent to the Lebowitz–Spohn FT.

I. INTRODUCTION

Despite many proposals have been advanced, a general theory of the steady state of dissipative systems is still lacking. Nevertheless, it is a remarkable fact that under *suitable hypothesis* something can be said and physical predictions can be made; for example, CH,[1] stating that for the purpose of studying macroscopic properties, a system exhibiting chaotic motions may be regarded as a transitive hyperbolic (that is Anosov) one, see Ref. 2, implies two remarkable results: the GC FT,[3,4] holding for *reversible* systems, and a formula describing the linear response of nonequilibrium systems due to Ruelle.[5–7] These results have been and are still widely studied in the physical literature; see Ref. 8 (where a relation which inspired the FT was empirically discovered) and Refs. 9–13, for instance. In this note we focus on the first of these two results; in particular, in Sec. II we show that FT has some nice implications on nonlinear response (in Ref. 14 it has been already pointed out that FT implies the usual linear response theory, that is GK formula and Onsager reciprocity relations), while in Sec. III we discuss a check of the FT relation in a simple Anosov reversible system. Interestingly, a simple perturbative calculation shows that the lack of differentiability of the time reversal operator implies a *violation* of the GC fluctuation relation. Finally, in Sec. IV we show that in the presence of a nondifferentiable time reversal transformation an identity equivalent to a result proved by Lebowitz and Spohn in Ref. 15 is true.

The connection between FT and nonlinear response consists of the fact that FT implies identities between correlation

1054-1500/2010/20(2)/023111/5/$30.00

functions of physical observables in a nonequilibrium steady state; this has been pointed out first in Ref. 16, where the authors considered systems whose evolution was stochastic and ruled by a master equation, and independently in Ref. 17 in the context of deterministic systems satisfying CH. The results of Ref. 17 are presented in Secs. II and III of this paper.

Before turning to our results, we spend a few words on some of the main features of Anosov systems; we refer the interested reader to Ref. 2 for a modern introduction to the subject. Consider a generic discrete dynamical system $x_k = S^k x_0$ (this is not a lack of generality, since S can be thought as the map arising from the Poincaré section of a system evolving in continuous time, see Ref. 19) and assume that S is Anosov; informally, this means that given a point x the nearby points separate exponentially fast from x in the future and in the past, except when located on a surface $W_s(x)$ (*stable* manifold) or $W_u(x)$ (*unstable* manifold), respectively, for the future and for the past.

It is a well-known result that Anosov systems admit an invariant measure μ_+, the so-called *Sinai–Ruelle–Bowen* (SRB) *measure*;[18] in fact, given a sufficiently regular observable $F(x)$ the following equality holds:

$$\lim_{T\to+\infty}\frac{1}{T}\sum_{j=0}^{T-1}F(S^j x) = \int \mu_+(dx)F(x), \tag{1.1}$$

apart from a set of points of zero volume measure. Remarkably, the SRB measure admits in principle an *explicit* representation, similar to the equilibrium Gibbs distribution.

Notice that at equilibrium, that is when the system is stationary and nondissipative, the chaotic hypothesis implies the ergodic hypothesis, in the sense that assuming CH the SRB measure is the Liouville one; but in general, when dissipation is present the SRB measure is *singular* with respect to the volume, that is it is concentrated on a zero volume set.

To conclude, the SRB measure verifies a large deviation theorem (see, for example, Ref. 19 for a proof of this statement for a special choice of F and in the more complex case of Anosov flows). In fact, consider the finite time average $f = \tau^{-1}\sum_{j=-\tau/2}^{\tau/2-1}F(S^j x)$, where $F(x)$ is Hölder continuous in x; then, it is possible to prove that there are values f_1, f_2 such that if $[a,b] \in (f_1, f_2)$, then $\text{Prob}_{\mu_+}(f\in[a,b]) \sim e^{\tau\zeta_F(f)}$ in the sense that

$$\lim_{\tau\to+\infty}\frac{1}{\tau}\log \text{Prob}_{\mu_+}(f\in[a,b]) = \max_{f\in[a,b]}\zeta_F(f) \tag{1.2}$$

and $\zeta_F(f)$ is analytic and convex in (f_1, f_2).

II. FLUCTUATION THEOREM AND NONLINEAR RESPONSE

Currently, no universally accepted definition of entropy for a dissipative system has been given. Nevertheless, the *rate of entropy production* is a well-defined quantity and is proportional to the work per unit time made by the thermostats on the system; the proportionality factor is the inverse of the temperature of the thermostats (setting to 1 the Boltzmann constant). In particular, for a special class of thermostats, the *Gaussian* ones, the entropy production rate corresponds to the *phase space contraction*, that is to minus the divergence of the equations of motion, see Ref. 20; this fact can be taken as a general definition of entropy production if one assumes that the steady state of a large system is not affected by the details of the thermostatting mechanism which ensures the existence of the steady state.

In the case of a system evolving in discrete time, which is the case that we want to consider, the entropy production rate σ is given by $\sigma(x) = -\log|\det \partial S(x)|$, where ∂S is the Jacobian of the time evolution S. Now, assume that the system is dissipative, i.e., that $\sigma_+ \equiv \int \mu_+(dx)\sigma(x) > 0$, let p be the adimensional average over a time τ of $\sigma(x)$, that is $p = (\tau\sigma_+)^{-1}\sum_{j=-\tau/2}^{\tau/2-1}\sigma(S^j x)$, and call $\zeta(p)$ the large deviation functional of σ, as defined in Eq. (1.2); assume that CH holds, and that the system is reversible, in the sense that there exists a differentiable isometry I such that $I\circ S = S^{-1}\circ I$, $I\circ I = 1$. Then, as proven by Gallavotti and Cohen, see Ref. 3 or Ref. 21 for a detailed proof from a formal viewpoint, the following result holds.

Fluctuation Theorem: *There is* $p^* \geq 1$ *such that for* $|p| < p^*$,

$$\zeta(-p) = \zeta(p) - p\sigma_+. \tag{2.1}$$

This result has an interesting corollary. Setting $\pi_\tau(q)dq = \text{Prob}_{\mu_+}(p\in[q, q+dq])$, define $\lambda(\beta)$ as

$$\lambda(\beta) = \lim_{\tau\to+\infty}\frac{1}{\tau}\log\int e^{\tau(q-1)\sigma_+\beta}\pi_\tau(q)dq. \tag{2.2}$$

Clearly, $\lambda(\beta)$ is related to $\zeta(p)$ through a Legendre transform, that is

$$-\zeta(p) = \max_\beta(\beta\sigma_+(p-1) - \lambda(\beta)) \tag{2.3}$$

and, moreover, $\lambda(\beta)$ admits the following expansion:

$$\lambda(\beta) = \sum_{n\geq 2}\sum_{t_1,\ldots,t_{n-1}=-\infty}^{+\infty}\langle\sigma(\cdot)\sigma(S^{t_1}\cdot)\cdots\sigma(S^{t_{n-1}}\cdot)\rangle_+^T\frac{\beta^n}{n!}$$
$$\equiv \sum_{n\geq 2}C_n\frac{\beta^n}{n!}, \tag{2.4}$$

where by $\langle\cdots\rangle_+^T$ we denote the *cumulant* with respect to the SRB measure μ_+. It is straightforward to see that the FT implies an identity for the generating functional $\lambda(\beta)$, see Refs. 13 and 15–17 for instance: it follows that as a consequence of the relation $\pi_\tau(p) \sim e^{\tau p\sigma_+}\pi_\tau(-p)$, valid under the hypothesis of FT,

$$\lambda(\beta) = \lambda(-1-\beta) - \sigma_+(2\beta+1). \tag{2.5}$$

Notice that the generating functional of the cumulants is usually defined as (see Refs. 13, 15, and 16) $\tilde{\lambda}(\beta) = \lim_{\tau\to+\infty} -\tau^{-1}\log\langle e^{-\tau p\sigma_+\beta}\rangle$, and with this definition the relation (2.5) is replaced by the more familiar $\tilde{\lambda}(\beta) = \tilde{\lambda}(1-\beta)$; but the two definitions are equivalent since $\lambda(\beta) = -\beta\sigma_+ - \tilde{\lambda}(-\beta)$. Formula (2.5) translates immediately in a relation for σ_+; in fact, Eq. (2.5) evaluated at $\beta=0$ becomes

$$0=\lambda(-1)-\sigma_+\Rightarrow\sigma_+=\sum_{n\geq 2}C_n\frac{(-1)^n}{n!}. \qquad (2.6)$$

Assuming that the entropy production $\sigma(x)$ has the form $\sigma(x)=\Sigma_i G_i J_i^{(0)}(x)+O(G^2)$, where $\{G_i\}$, and $\{J_i^{(0)}(x)\}$ are, respectively, the forcing parameters and the corresponding currents, it has been shown in Ref. 14 that the identity

$$\sum_{i,j}G_iG_j\partial_{G_iG_j}\sigma_+\Big|_{\underline{G}=\underline{0}}=\frac{1}{2}\sum_{i,j}G_iG_j\partial_{G_iG_j}C_2|_{\underline{G}=\underline{0}}, \qquad (2.7)$$

which is nothing else than Eq. (2.6) at second order, is equivalent to the GK formula, stating that

$$L_{ij}\equiv\partial_{G_j}\langle J_i^{(0)}\rangle_+|_{\underline{G}=\underline{0}}=\frac{1}{2}\sum_{t=-\infty}^{+\infty}\langle J_i^{(0)}(S^t\cdot)J_j^{(0)}(\cdot)\rangle_0, \qquad (2.8)$$

where by $\langle\cdot\rangle_0$ we denote the expectation with respect to the invariant measure at zero forcing (which by CH is the Liouville measure). Hence, formula (2.6) can be seen as a *generalization* of GK formula to the nonlinear regime, being an identity for σ_+ valid for $\underline{G}\neq\underline{0}$. Moreover, by taking derivatives with respect to β in the right-hand side (rhs) and left-hand side (lhs) of Eq. (2.5) we find that

$$C_n=\sum_{k\geq 0}\frac{(-1)^{k+n}C_{k+n}}{k!},\quad n\geq 2, \qquad (2.9)$$

which is a nontrivial identity for the cumulants valid at $\underline{G}\neq\underline{0}$. Finally, these identities can be considerably extended by using a generalized version of FT. Consider a generic observable O odd under time reversal, i.e., such that $O(Ix)=-O(x)$, assume that the system is dissipative and that $O_+\equiv\int\mu_+(dx)O(x)\neq 0$, let w be the adimensional average over a time τ of $O(x)$, that is $w=(\tau O_+)^{-1}\Sigma_{j=-\tau/2}^{\tau/2-1}O(S^jx)$, and call $\zeta(p,w)$ the large deviation functional of the joint probability distribution $\pi_\tau(p,w)$ of p and w; then, under the same hypothesis of FT, the following result holds as a special case of a much more general result in Ref. 22.

Generalized Fluctuation Theorem (GFT): *There are* $w^*\geq 1$, $p^*\geq 1$ *such that for* $|p|<p^*$ *and* $|w|<w^*$,

$$\zeta(-p,-w)=\zeta(p,w)-p\sigma_+. \qquad (2.10)$$

One can define the generating functional $\lambda(\beta_1,\beta_2)$ of the mixed cumulants of O, σ in a way analogous to Eq. (2.2),

$$\lambda(\beta_1,\beta_2)=\lim_{\tau\to+\infty}\frac{1}{\tau}\log\int e^{\tau(q-1)\sigma_+\beta_1+\tau(t-1)O_+\beta_2}\pi_\tau(q,t)dqdt. \qquad (2.11)$$

Again, $\lambda(\beta_1,\beta_2)$ is related to $\zeta(p,w)$ through a Legendre transform, and moreover, it can be expressed as

$$\lambda(\beta_1,\beta_2)=\sum_{k\geq 2}\sum_{\substack{m,n\geq 0,\\ m+n=k}}\frac{\beta_1^n\beta_2^m}{n!m!}C_{n,m}, \qquad (2.12)$$

where $C_{n,m}=\partial_{\beta_1}^n\partial_{\beta_2}^m\lambda(\beta_1,\beta_2)|_{\underline{\beta}=\underline{0}}$, that is $C_{n,m}$ is given by a sum over times of mixed cumulants of $\sigma(S^{t_i}x)$, $O(S^{t_j}x)$, $0\leq i\leq n$, and $0\leq j\leq m$. Then, in full analogy with what has already been discussed, it is easy to show that GFT translates into an identity for the generating functional, namely,

$$\lambda(\beta_1,\beta_2)=\lambda(-1-\beta_1,-\beta_2)-(2\beta_1+1)\sigma_+-2\beta_2O_+, \qquad (2.13)$$

which implies the following relations:

$$O_+=\sum_{k\geq 2}\frac{(-1)^k}{2^{k-1}}\sum_{l=0,2l\leq k-1}\frac{C_{k-(2l+1),2l+1}}{(2l+1)!(k-(2l+1))!}, \qquad (2.14)$$

$$C_{l,n-l}=\sum_{k\geq 0}\frac{(-1)^{n+k}}{k!}C_{k+l,n-l},\quad n\geq 2,\ \ l\leq n. \qquad (2.15)$$

Equation (2.14) is obtained setting $\beta_1=\beta_2=-\frac{1}{2}$ in Eq. (2.13), while Eq. (2.15) can be proved differentiating with respect to β_1, β_2 the rhs and the lhs of Eq. (2.13). It is interesting to see what happens to formula (2.14) in the linear regime. Assuming that $\sigma(x)=\Sigma_iG_iJ_i^{(0)}(x)+O(G^2)$ we can rewrite Eq. (2.14) as

$$\begin{aligned}O_+&=\sum_{k\geq 2,k\ \text{even}}\frac{1}{(k-1)!}\frac{(-1)^k}{2^{k-1}}C_{1,k-1}\\&\quad+\sum_{k\geq 3,k\ \text{odd}}\frac{1}{k!}\frac{(-1)^k}{2^{k-1}}C_{0,k}+O(G^2)\\&=\frac{1}{2}C_{1,1}+\sum_{k\geq 4,k\ \text{even}}\frac{1}{(k-1)!}\frac{(-1)^k}{2^{k-1}}C_{1,k-1}\\&\quad+\sum_{k\geq 3,k\ \text{odd}}\frac{1}{k!}\frac{(-1)^k}{2^{k-1}}C_{0,k}\\&=\frac{1}{2}C_{1,1}+\sum_{k\geq 3,k\ \text{odd}}\frac{1}{k!}\frac{(-1)^k}{2^{k-1}}\left(C_{0,k}-\frac{1}{2}C_{1,k}\right)+O(G^2),\end{aligned} \qquad (2.16)$$

and from Eq. (2.15) we find that for k odd $C_{0,k}-\frac{1}{2}C_{1,k}=O(G^2)$. Hence,

$$O_+=\frac{1}{2}C_{1,1}+O(G^2)=\frac{1}{2}\sum_{t=-\infty}^{\infty}\langle\sigma(S^tx)O(x)\rangle_++O(G^2), \qquad (2.17)$$

which gives

$$\partial_{G_i}O_+|_{\underline{G}=\underline{0}}=\frac{1}{2}\sum_{t=-\infty}^{\infty}\langle J_i^{(0)}(S^tx)O(x)\rangle_0. \qquad (2.18)$$

Formula (2.18) describes the linear response of a generic observable odd under time reversal; this result can be seen as a special case of the remarkable linear response formula obtained by Ruelle,[5,6] valid in a much more general context.

III. CHECK OF FLUCTUATION THEOREM FOR A SIMPLE ANOSOV SYSTEM

In this section we will perform a check of the fluctuation relation (2.1) for a simple Anosov model, the *perturbed Arnold cat map*, starting from the identities (2.6) and (2.9); as we are going to see, the lack of differentiability of the time reversal operator implies a violation of Eq. (2.1). Consider the following discrete evolution on the bidimensional torus $\mathbb{T}^2$:

$$\underline{x}_k = S_\varepsilon^k \underline{x}_0 \mod 2\pi, \tag{3.1}$$

where

$$S_\varepsilon \underline{x} = S\underline{x} + \varepsilon \underline{f}(\underline{x}), \quad S \equiv \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \underline{f}(\underline{x}) = \begin{pmatrix} f_1(\underline{x}) \\ f_2(\underline{x}) \end{pmatrix}, \tag{3.2}$$

and $f_1(\underline{x})$, $f_2(\underline{x})$ are trigonometric polynomials. The map S is the so-called Arnod cat map, which is the simplest example of Anosov map: in fact, the eigenvalues λ_+, λ_- of S are such that $\lambda_+ > 1$, $\lambda_- < 1$. Moreover, the map S is reversible, that is there exists I such that $I \circ S = S^{-1} \circ I$, $I \circ I = 1$, where

$$I = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{3.3}$$

Notice that since S is conservative ($\det S = 1$) $\sigma_+ = 0$, which makes meaningless the fluctuation relation (2.1) if $\varepsilon = 0$ (the adimensional quantity p is not defined); but one can derive the analogous of Eq. (2.1) for the *dimensional* quantity $p' = \sigma_+ p$, and this relation becomes trivial if the evolution is conservative because in this case $p' = 0$.

Consider now $\varepsilon \neq 0$. By the structural stability of Anosov systems, the evolution generated by S_ε is still Anosov *provided* ε is chosen small enough. In fact, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ a *conjugation* H_ε, defined by the identity $S_\varepsilon \circ H_\varepsilon = H_\varepsilon \circ S$, can be explicitly constructed through a convergent power series in ε, and it turns out that $H_\varepsilon(\underline{x})$ is Hölder continuous in $\underline{x}$.[2] In an analogous way, the SRB measure can be explicitly constructed, and it follows that the expectations of Hölder continuous functions exist and are analytic in ε.[2] In particular it follows that, generically,

$$\sigma_+ \equiv \langle -\log|\det \partial S_\varepsilon| \rangle_+ > 0, \tag{3.4}$$

which means that the system is dissipative, so that the invariant measure is singular with respect to the volume; hence, the check of the fluctuation relation (2.1) is nontrivial in this case. Notice that the proof of FT (Ref. 21) requires that the evolution is reversible, and in particular that the time reversal operator is differentiable; in our specific case the existence of H_ε implies that $I_\varepsilon = H_\varepsilon \circ I \circ H_\varepsilon^{-1}$ verifies $I_\varepsilon \circ S_\varepsilon = S_\varepsilon^{-1} \circ I_\varepsilon$, *but* due to the mild regularity properties of H_ε, $I_\varepsilon(\underline{x})$ is likely to be not differentiable. Here, the time reversal operator I_ε associated to the map S_ε is perturbatively constructed as a convergent power series in ε starting from I; time reversal symmetry is not destroyed by small perturbations. However, it is interesting to notice that there exist classes of dissipative Anosov *flows* for which time reversal symmetry holds for *any* value of the forcings; see the "Gaussian thermostat case" of Ref. 23 for instance, where the time reversal operation is simply the inversion of the velocities. The continuous time dynamics of the Anosov flows can be reduced to discrete ones by repeating the analysis of Ref. 19.

One can ask what happens to the fluctuation relation in the simple case we are considering; to understand this, we make the choice $f_1(\underline{x}) = \sin(x_1) + \sin(2x_1)$, $f_2(\underline{x}) = 0$. An explicit computation shows that the linear response is still valid, as expected, and that

$$C_3 = -12\varepsilon^3 + O(\varepsilon^4), \tag{3.5}$$

i.e., formula (2.9) with $n=3$ is false at the lowest nontrivial order in perturbation theory, since it tells that $C_3 = C_4/2 + O(\varepsilon^5)$ and $C_4 = O(\varepsilon^4)$. This is enough to say that Eq. (2.1) is violated; in fact, from Eq. (2.3) it follows that[17] (using the linear response relation $\sigma_+ = (1/2)C_2 + O(\varepsilon^3)$):

$$\zeta(p) = -\frac{(p-1)^2}{4}\left[\sigma_+ + O(\varepsilon^3)\right] + \frac{(p-1)^3}{48}\left[C_3 + O(\varepsilon^4)\right] + O((p-1)^4\varepsilon^4), \tag{3.6}$$

that is

$$\zeta(p) - \zeta(-p) = p\left[\sigma_+ + O(\varepsilon^3)\right] + p^3\left[\frac{C_3}{24} + O(\varepsilon^4)\right] + O(p^5\varepsilon^5), \tag{3.7}$$

and Eq. (3.5) implies that the difference $\zeta(p) - \zeta(-p)$ is not linear in p [the coefficient of p^3 in Eq. (3.7) is nonzero]. This result also shows that as expected, I_ε cannot be differentiable: a check that would be not so easy without using the FT.

IV. CONCLUSIONS

The GC FT implies nontrivial identities between correlation functions valid at nonzero forcing $\underline{G}$ which reduce to the usual linear response in the limit $\underline{G} \to \underline{0}$. Through a check of these identities in a simple Anosov system, we have shown that it is essential that the time reversal transformation be (continuously) differentiable for the fluctuation relation to hold. Indeed, we considered a model that admits a time reversal transformation which is only Hölder continuous, and we have found that the identities implied by the fluctuation relation are not true at the third order in the forcing (while the linear response, corresponding to the second order, is still valid, as expected); this is enough to prove that the fluctuation relation cannot hold in the case we considered. Notice, however, that in physical applications the time reversal transformation is usually regular; for instance, it can correspond to the inversion of velocities (or to more subtle permutations of coordinates, see Ref. 24).

To conclude, it is interesting to note that in the presence of a nondifferentiable time reversal operator I a different FT holds, the Lebowitz–Spohn one, see Ref. 15; strictly speaking this theorem has been proved in the context of general Markov processes, but it can be understood also in the case of deterministic chaotic dynamics. This result applies in particular to systems that are (suitably small) perturbations of reversible Anosov ones, since, as it has been pointed out in Sec. III, by the structural stability of Anosov dynamics the time reversal symmetry is not destroyed by the perturbation (although it will be in general only Hölder continuous in x).

The fact that a fluctuation relation still holds is a natural consequence of the Gibbsian nature of the invariant measure describing the steady state; this has been pointed out in Ref. 25 where a FT for the one-dimensional Ising model in an external field was derived, and then systematized in Ref. 26, where large deviation rules for Gibbs states involving transformations different from time reversal were discussed.

Following step by step the proof of GC FT,[21] one can see that the large deviation functional $\tilde{\zeta}(\tilde{p})$ of the dimensionless quantity $\tilde{p}=(\tau\tilde{\sigma}_+)^{-1}\sum_{j=-\tau/2}^{\tau/2-1}\tilde{\sigma}(S^j x)$, where $\tilde{\sigma}(x)=-\lambda_u(x)+\lambda_u(Ix)$ and $\lambda_u(x)$ is the sum of the positive Lyapunov exponents corresponding to the local unstable manifold $W_u(x)$ of S, verifies

$$\tilde{\zeta}(-\tilde{p})=\tilde{\zeta}(\tilde{p})-\tilde{p}\tilde{\sigma}_+ \tag{4.1}$$

(of course under the physical restriction on $\tilde{p}$ to vary within the analyticity interval of $\tilde{\zeta}$, see Ref. 21), which reduces to Eq. (2.3) if I is differentiable, since in this case $\lambda_u(Ix)=-\lambda_s(x)$. Formula (4.1) is equivalent to the Lebowitz–Spohn FT; in fact, $\tilde{\sigma}(x)$ is proportional to the logarithm of the ratio of the SRB probabilities of the trajectories $x_{-\tau/2},x_{-\tau/2+1},\ldots,x_{\tau/2-1}$ and of its time reversed, which is precisely the "action functional" introduced in Ref. 15.

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