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Abstract

The quotient group concept is a difficult for many students getting started in abstract algebra (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Melhuish, Lew, Hicks, & Kandasamy, 2020). The first study in this thesis explores an undergraduate, a first year graduate, and second year graduate student's representational fluency as they work on a "collapsing structure" quotient task across multiple registers: Cayley tables, group presentations, Cayley to Schreier coset digraphs, and formal-symbolic mappings. The second study characterizes the (partial) make-up of the two graduate students' example-based intuitions related to orbit-stabilizer relationships induced by group actions. The (partial) make-up of an intuition as a quantifiable object was defined in this thesis as a point viewed in $\mathbb{R}^{17}$ with: 12 attribute values collected with a new prototype instrument called The Non-Creative versus Creative Forms of Intuition Survey (NCCFIS), 2 values for confidence in truth value, and 3 additional values for error to non-error type, unique versus common, and network thinking. The revised Fuzzy C-Means Algorithm (FCM) by Bezdek et al. (1981) was used to classify students' reported intuitions into fuzzy sets based on attribute similarity.

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Chapter 1

Introduction

1.1 Motivation

A considerable amount of work, over the past two decades, has been put forth to develop a research-based curriculum for introductory group theory under the Teaching Abstract Algebra for Understanding (TAAFU) research program. TAAFU affiliated researchers continue to emphasize the need for educators to transition towards active inquiry-instructional styles for undergraduate Abstract Algebra courses (Larsen, Johnson, & Bartlo, 2013; Johnson, Keller, Peterson, & Fukawa-Connelly, 2019). Topics that the TAAFU curriculum covers are: groups, subgroups, isomorphisms, homomorphisms, and quotient groups with classroom activities that reflect the principles of guided reinvention and Realistic Mathematics Education (RME). The following is a brief description of this curriculum from the TAAFU group: "Each unit begins with a reinvention phase in which students develop concepts based on their intuitions, informal strategies, and prior knowledge. The end product of the reinvention phase is a formal definition (or definitions) constructed by students and a collection of conjectures..." (Lockwood, Johnson, & Larsen, 2013, p. 777).

Even with NSF funding and training support on how to incorporate this curriculum in the classroom, there has been "almost no uptake" (Fukawa-Connelly, Johnson, & Keller, 2016, p. 280). Traditional lecturing is still the dominant method. Recent work has uncovered that Abstract Algebra instructors' apprehension towards inquiry and active learning is complex. It involves an entanglement of many factors such as: perceived lack of preparation time, difficulty to maintain coverage of intended course content, concerns that it would not work, and limited awareness of resources (Fukawa-Connelly, Johnson, & Keller, 2016; Johnson, Keller, Peterson, & Fukawa-Connelly, 2019).

Another concern is the lack of sufficient evidence to either support or reject the efficacy of this curriculum compared to traditional lecture approaches. Evaluative studies are still needed to determine the efficacy of this curriculum from multiple perspectives compared to other forms of instruction (Johnson, Andrews-Larson, Keene, et al., 2020; Kuster, Johnson, Rupnow, & Wilhelm, 2019). In an early small-scale assement administered by the IOAA group on quotient groups tasks, IOAA students significantly outperformed non-IOAA students. However, "...more work will need to be done to establish efficacy of the curriculum. For example, these tasks may be specifically suited to students who have engaged with the IOAA curriculum. It is possible that different tasks could capture deficiencies in IOAA students' understanding that are not typically present in students who learn group theory via more traditional approaches" (Larsen, Johnson, & Bartlo, 2013, p. 710). Another descriptive contrast to more traditional ways of teaching Abstract Algebra, was also given by Carter (2009). He emphasized experimentation with Cayley digraphs towards formal results. Carter's textbook was characterized as, "a nonstandard approach to group theory..." with "more than 300 images, and average of more than one per page. The most used visualization tool is Cayley diagrams because they represent group structure clearly and faithfully" (Carter, 2009, p. 1).

To address the tug of war between inquiry and more traditional instructional styles, Kuster, Johnson, Rupnow, and Wilhelm (2019) published the Inquiry Oriented Instructional Measure (IOIM), a quantitative instrument that measures how much classroom instruction resembles Inquiry-Oriented instruction. This instrument has opened new doors for efficacy research. Kuster et al. (2019) stated that "The IOIM represents our attempt to characterize instruction - with the idea that this characterization will allow for further research into the relationship between instructional practice and student learning" (p. 201). An aspect of student learning of central importance and long-standing interest in mathematics culture is the development of students' intuition (Bruner, 1960; Wilder, 1967; Fischbein, 1987; Burton, 1999; Bubp, 2014; Thomas, 2015). By leveraging the new advance of the IOIM, the following motivating hypothesis was formulated, with the IOIM score as the independent variable and student gains related to intuition associated factors as the dependent variable.

> Motivating hypothesis: Learners in environments characterized by higher IOIM scores have more significant gains on evaluations for intuition associated factors than those that learn in environments characterized by lower

IOIM scores.

A quasi-experimental study that can be used to test the above hypothesis is still not accessible for researchers at this time. Even though instruments like the IOIM that characterize the instructional environment exist, too many theoretical and methodological obstacles remain with respect to the dependent variable. For example, theoretically characterizing the intuition construct and quantifying intuition at a single point in time has been an open problem in mathematics education research for at least half a century. Moreover, the phrase ‘intuition associated factors’ is ambiguous. Bruner (1960) stated “it is certainly clear that procedures or instruments are needed to characterize and measure intuitive thinking, and that the development of such instruments should be pursued vigorously” and that “precise definitions in terms of observable behavior is not readily within our reach at this time...” (p. 61).

To date, there are no accepted quantitative or mixed-methods techniques that have been specifically designed to evaluate facets of mathematical intuition with the exception of Fischbein, Tirosh, and Melamed’s (1981) measure of intuitive acceptance of geometry problems at a middle to high-school level. A major hurdle among many to overcome is to provide a “workable definition” or framework to guide university level mathematics education research that is focused on the intuition construct and is powerful enough to extend qualitative case to mixed-methods studies (Noddings & Shore, 1984; Fischbein, 1987; Bubp, 2014). Mathematical intuition, as a research construct, continues to be a “slippery concept that carries a heavy load of mystery and ambiguity” (Davis, Hersh, & Marchisotto, 2012, p. 433; Bubp, 2014).

In order to move forward, the previously stated motivating hypothesis, like a conjecture, was decomposed into sub-problems and more fundamental research questions that if figured out could provide new insights and tools that could be used to eventually return to the motivating hypothesis. At times, it felt like the theories and methodologies in mathematics education, in their current state, were not enough to handle this pursuit. Reading papers or books in psychology was like taking a trip to the future of what things could look like and that what was difficult was not impossible. For example, in the preface of “Foundations for Tracing Intuition”, Glöckner and Witteman (2010) wrote,

> The study of intuition is a burgeoning research topic in psychology and beyond. While the area has the potential to radically transform our conception of the mind and decision making, the procedures used for establishing empirical conclusions have often been vaguely formulated and obscure. This

> book fills a gap in the field by providing a range of methods for exploring intuition experimentally and thereby enhancing the collection of new data...by introducing the different methods and their applications in a step-by-step manner, this text is an invaluable reference for individual research projects. It is also very useful as a course book for advanced decision-making... (para. 1-4).

Glöckner and Witteman's (2010) course book comes after the psychology fields intensely focused efforts on intuition research for over seven decades beginning with de Groot's (1946) investigations of chess players and Noble prize winning work that laid the foundations for behavioral economics (Kahneman, 2003).

Today, no such course book specialized for investigations of mathematical intuition in university level education research exists. This along with a community of researchers, who were interested in how to 'draw out mathematics learners' intuitions', motivated a goal of this thesis: to bring together theories and illustrate methodology that could support research on intuition within a group theory setting. The general Networking of Theories Group (2014) methodology text for mathematics education research was a major resource and guide. The Networking Theories approach of placing multiple theoretical lenses on the same set of data seemed to be a key for dealing with the multifaceted nature of the intuition construct. Research on intuition, at the least, involves a mix of semiotics, creativity, and affect.

1.2 Outline of chapters

Chapter 2 covers the past and current state of research in the area of mathematical intuition with an integrative literature review. This review included a $N = 781$ papers from the four highest quality ranked university level mathematics education journals: Journal for Research in Mathematics Education (JRME), Educational Studies in Mathematics (ESM), Journal of Mathematical Behavior (JMB), and ZDM (Williams & Leatham, 2017). Seventy additional papers identified as highly relevant were also included. The review addressed the research question: How should mathematical intuition be represented as a research construct? A major breakthrough made while working on this chapter, was viewing mathematical intuition as a fuzzy class-concept and framing the coarsest theoretical classification between non-creative and creative forms of intuition in terms of various attributes. This resulted in new explicated connections between the research areas of mathematical creativity and mathematical intuition along with the incorporation of semiotics. Collectively, this integrative literature review, interviews with mathematicians, and personal experiences as a mathematics learner led to a theoretical framework that organized four intuition-associated factors: representational fluency, example-based intuitions, counter-example stance, and affective impacts.

Following the integrative literature review on intuition, an additional literature review was carried out in chapter 3 to cover prominent types of research that have occurred for introductory Abstract Algebra education. This included: instructional design research with teaching experiments, evaluative investigations of learners' thinking related to group theory topics, and efficacy-based studies that compare Inquiry-Oriented Abstract Algebra instruction (IOAA) to non-IOAA instruction. Evaluative investigations of learners' thinking can inform instructional design research by identifying their strengths and difficulties with respect to particular mathematical concepts. Evaluative investigations can also provide groundwork for articulating units of analysis and designing instruments for efficacy-based studies.

Chapter 4 introduces inner frameworks for the four previously stated intuiton associated factors. The factors for which the most progress was made was the representational fluency and example-based intuition factor. The inner framework for representational fluency combined three theoretical lenses: the Theory of Semitic Representations (Duval, 2017), uni-modal versus multi-modal register use (Arzarello, 2006), and metarepresentational competence and sanctioned versus non-sanctioned modes of aquisition (diSessa, 2004). The non-creative versus creative distinction, initially made by Policastro (1995) and Dane and Pratt (2009), was added to the example-based intuition factor to further typify and analyze a pool of students example-based intuitions.

This distinction was of primary focus due to its potential use in characterizing what it means to develop intuition. For example, if the content of an intuition is fixed the transition from a creative form at an initial point to a non-creative form at a subsequent point is a sign of memory consolidation. The activation of non-creative forms allows for more cognitive energy to explore more complex or different angles of a problem. This provides an opportunity for new creative intuitions to be produced by non-conscious processes (Bastick, 1982; Dane & Pratt, 2009; Hogarth, 2010). While the distinction of non-creative and creative forms is important, just looking at differences in the frequency of each type of intuition to infer something about the development of intuition is not enough. It is necessary to eventually pay attention to transitions from a creative to non-creative intuiting states or vice a versa, the content, along with additional attributes, and how affective factors like motivation and anxiety act on the learner's intuiting state.

Under the expansive umbrella of intuition covered in previous chapters, this study narrowed in on and targeted two intuition associated factors: representational fluency and example-based intuitions. Chapter 5 details methodology and instrument development for a multi-part mixed case study design. Two tasks, the collapsing structure and adding structure task, were designed to collect data on learners' representational fluency for quotients, direct-product and semi-direct product constructions across multiple registers: Cayley tables, Cayley to Schreir digraphs, group presentations, formal-symbolic mappings, and objects of symmetry. The three participants in this study, an undergraduate, a first year graduate, and second year graduate student, represented three consecutive levels. Data analysis for the collapsing structure task was predominantly done through a Duvalian semiotic lens with a four-level analytic framework developed in this thesis. These four levels were: 1) collapsing or losing structure interpretation themes, 2) strategies, 3) register use, 4) pseudo-semiotic triangles, semiotic triangles, and conversions. Two additional lenses: a default-interventionist (Kahneman, 2011) and affective lens (Fischbein, 1987), were also used. The following research questions were addressed for the collapsing structure task:

1. What are learners' difficulties and strengths with quotients or homomorphisms, detected through a semiotic lens during the collapsing structure task?

   1a. What are learners' interpretation of the losing or collapsing structure?

   1b. Do learners produce semiotic representations that are inconsistent with quotient maps or homomorphism across multiple registers? Which registers?

   1c. Did learners make quotient map conversions from registers that they natural started in to a prompted register for valid Z_2 or $Z_2 \times Z_2$ strategies?

2. What are learners' difficulties and strengths on the quotient task detected through additional intuition-associated lenses?

Researchers who have made progress in the area of mathematical intuition in the past have stressed the importance and need to better understand the construct (Fischbein, 1987). This need along with my experience as a psychology and mathematics learner activated associations between how I tried to understand a class of mathematical objects and how I might go about trying to understand learners' intuitions as a class of cognitive objects. At the point of conception, a mathematical object and a cognitive object seemed to be more alike than different. I wrestled with this thought for a long time trying to understand how to understand a cognitive object like intuition. The first step was to figure out what a subset of theoretically relevant invariant and variable attributes attached to the objects were. The second step was to figure out how to practically extract these attributes attached to a given object. The third step was to develop or find existing tools that could be used to classify these objects in various ways by paying attention to different choices of attributes or restrictions. The fourth step would be to ask additional questions, hypothesize and explore the validity of implication statements regarding these objects by calling on their attributes. Classification problems have been explored by many disciplines and are considered to be of great value. Biologists classify species. In mathematics, one may classify irreducible linear representations or other mathematical objects. What would a classification problem for learners' intuitions look like and would it be of value to education researchers?

Both quantitative and qualitative data attached to learners' reported example-based intuitions related to group actions was obtained. A first prototype version, of a self-report survey instrument was created to extract numerical values for the (partial) make-up of each reported example-based intuition. The (partial) make-up of a learner's example-based intuitions, as a quantifiable object, was defined in this thesis as a point in $\mathbb{R}^{17}$, 12 attribute values collected with the Non-Creative versus Creative Forms of Intuition Survey (NCCFIS), 2 values for the confidence in truth value, and 3 additional variables: error to non-error type, unique versus common content, and network thinking. The NCCFIS was administered with a baseline questionnaire, student self-report intuition logs and journals to collect additional qualitative data. The revised Fuzzy C-Means Clustering Algorithm (FCM) by Bezdek et al. (1981) was used to classify the (partial) make-up of learners' reported intuitions into fuzzy sets according to attribute similarity. The following research questions were addressed:

3. Is it possible to quantitatively characterize the (partial) make-up of learners' example-based intuitions during a group actions task?

   3a. What is the (partial) quantitative make-up of learners' reported intuitions related to orbit-stabilizer relationships induced by group actions?

   3b. Is it possible to measure intuitive feelings of rightness or wrongness? Are learners' feelings of rightness and wrongness working properly?

4. What is the qualitative content of learners' intuitions related to orbit-stabilizer relationships?

Chapter 6 presents results from the collapsing structure, quotient fluency, task and learner's reported example-based intuitions for group actions. Chapter 7 discusses findings in the context of relevant literature and answers research questions. Finally, chapter 8 provides concluding remarks, a list of contributions, and revisits the motivating hypothesis with directions for future research.

Chapter 2

Integrative Literature Review: Building a Conceptual Structure for Mathematical Intuition as a Research Construct

2.1 Introduction

On September 1959, a group of 35 educators, physicists, biologists, chemist, historians, mathematicians, and psychologists attended the Woods Hole Conference, the first educational reform conference held in the United States. *The Role of Intuition in Learning and Thinking* was a theme that this group regarded as imminent for working group discussions. A chronicle of these discussions were published in *The Process of Education* (Bruner, 1960). The Woods Hole group stood firm that intuition played a major role in student learning. They came to the consensus that to facilitate intuition educators should: increase the connectedness of knowledge, teach heuristics, guide learners to formulate conjectures peaked by their own interests and questions, evaluate the impact grading systems of rewards and punishments on intuition, and provide a safe environment for learners to experiment with their ideas and gain confidence (Bruner, 1960). In 1976, the International Group for the Psychology of Mathematics Education (PME) was founded. At the second PME conference in 1978, *Intuitive and Reflective Processes in Mathematics* was a conference theme.

In contrast to the early emphasis on intuition in mathematics education, a lack of emphasis was signaled in 1999 with the question: “Why is intuition so important to mathematicians but missing from mathematics education?" (p. 27). This was followed by an eyebrow raising claim made by Burton (1999), based on interviews with 70 mathematicians:

> ...with the notable exception of the work of Fischbein, accounts of the deliberate nurturing of intuition and insight is absent from the mathematics education literature, even from process based research, and, despite the claim for the centrality of it to mathematical work, it is equally absent from practices with students I would like to encourage mathematicians, indeed anyone who has the responsibility for the learning of mathematics, to open mathematical activity to include the subjectivity of intuitions, to model their own intuitive processes, to create the conditions in which learners are encouraged to value and explore their own and their colleagues' intuitions and the means that they use to gather them (p. 32).

Fast forward in time to the current year. Education research with a focus on intuition at the university level has disappeared from a visible and collaborative platform of conference working groups or special issues in journals. Moreover, a comprehensive work in this area has not been disseminated since the foundational text *Intuition in science and mathematics: An educational approach* (Fischbein, 1987). This text pre-dates: the first special session on Research in Undergraduate Mathematics Education (RUME) at JMM in 1991, the first RUME conference in 1996, the formation of the Special Interest Group of the MAA, RUME, in 2001, and the surge of advances generated by this growing research community. There is now a need in the field to establish some internal consistency (Darragh, 2016; Czocher & Weber, 2020). More recently, researchers have expressed repeated concerns with the lack of a coherent representation of constructs situated in mathematics education research. In the area of identity research, Darragh (2016) asked "Without the knowledge of the perspectives taken, how can we engage in a productive conversation within this area? How can we build appropriately on others' ideas and develop greater understanding about this topic without a common language or definition of the term (p. 20)?"

The purpose of this chapter is to provide a vantage point of what has transpired in the research area of intuition within the 1987 to 2020 gap. The specific aim is to provide an updated response to the questions: 1) How can mathematical intuition be represented as a research construct? 2) What are instructional practices that may enhance the development of learners' intuition?

With regard to the first question, Fischbein (1987) cautioned that:

> The attempt to find a common definition for this great variety of meanings, features and connotations seems to represent an impossible task. Intuitive knowledge seems to cover the whole domain cognition (p. 7).

Reflecting on Fischbein's work, Keene, Hall, and Duca (2014) stated:

> Fischbein seems to be alluding to the fact that while we all know we have intuition regarding various subjects, pinpointing exactly what constitutes our intuition is altogether more difficult to illustrate.

While Fischbein's (1987) text has been cited for general properties of intuition, such as self-evidence and immediacy, researchers are not currently using his theorized types of intuitions. Instead researchers are producing new types rooted in different theoretical frameworks. This is a point at which new perspectives diverge away from Fischbein's theoretical core and away from one another. For example, Bubp (2014) experienced first-hand issues due to insufficient theory on intuition specific to mathematics at the university level and explicitly stated that in the future, "theoretical research on developing a standard working definition of intuition in mathematics would benefit the mathematics education community" (p. 243). Bubp (2014) went on to introduce six additional types of mathematical intuition. Likewise, Semadeni (2008) moved away from Fischbein's (1987) types and into the older theoretical core of concept image/concept definition to term "deep intuition" out of necessity to describe cases of intuition he encountered in proof construction that Fischbein's (1987) framework could not account for (p. 9).

Efforts must be made to find ways to connect perspectives in a way that renders an organized conception of the constructs researchers work with. From stable theoretical cores, new ideas are sparked and researchers fan out in all directions. At some point, various collections of diverging perspectives must be strung together to establish new cores with common threads of language. The process of many researchers fanning out from stable cores to enter new theoretical territory, connecting back up at points of compatibility, and production of resultant many-to-one compressions must repeat to avoid "an exponential theoretical inflation" (Prediger, Bikner-Ahsbahs, & Arzarello, 2008, p. 170-172). For intuition research in mathematics education a past compression was Fischbein's (1987) work. Examples of past compressions in the psychology literature are Kahneman (2011), Glöckner and Witteman (2010), Dane and Pratt (2007, 2009), and Bastick (1982). What would a compression for mathematical intuition, that accounts for advances post 1987 look like and how can some progress be made towards this?

2.2 An integrative literature review methodology and descriptive statistics

An integrative literature review methodology was chosen to make some initial progress towards an updated compression. An "integrative literature review is a form of research the reviews, critiques, and synthesizes representative literature on a topic in an integrated way such that new frameworks and perspectives on the topic are generated" (Torraco, 2005, p. 356). Integrative literature reviews are typically conducted on "emerging topics or mature topics" that have stimulated a high volume of research output (Torraco, 2016, p. 404). A major requirement of an integrative literature review, that distinguishes it from more traditional literature reviews, is that it must contain a methodology section that details the reviewing procedure: how literature will be searched, collected, analyzed, and integrated (Cooper, 1982; Russel, 2005; Torraco, 2016).

2.2.1 Literature search and sampling procedure

Literature search and collection for this review was carried out in three phases: an (1) exploratory search phase, (2) a systematic follow-up search to obtain high quality literature that may have been missed or blocked due to a focus induced by the exploratory sample, and (3) an ongoing phase. During the exploratory phase, several sources where used: Google Scholar, PsychInfo, coursework materials, notes taken at national mathematics education conferences, and literature recommendations made by faculty mathematics education researchers and cognitive psychologists. During this phase, the author underwent an extended period of immersion in the literature with publication dates that ranged from 1911 to 2019 and followed the hermeneutic literature review process framed by Boell and Cecez-Kecmanovic (2014). As a result, two exploratory review drafts were written, one in July 2018 and an overhauled version on April 2019. These drafts reflected a gradual progression from scattered information and across-field segregation to an organization of connected components with some across-field connections between mathematics education and psychology research. In total, 49 mathematics education and 30 psychology references were collected and stored in EndNote. Notes of possible gaps in the literature were documented during the exploratory phase that could not be sufficiently backed up due to a lack of citation or evidence to confirm the gap. Following this exploratory review phase, a systematic literature search and collection was conducted. This would later provide a larger sample that could be used to more accurately test whether or not possible noted gaps were actually gaps.

The online databases JSTOR Arts and Sciences IV, ScienceDirect, and SpringerLink

Table 2.1: Total publication counts in JRME, ESM, JMB, and ZDM from 1987-2020 on March 11, 2020

Search criteria	JRME	ESM	JMB	ZDM	Total
intuition in title	1	9	2	3	15
intuition in abstract	5	19	9	11	44
intuition in article	116	297	179	189	781

Journals Complete were used to systematically search mathematics education literature published from 1987 to 2020. This search was restricted to four high quality journals: the Journal for Research in Mathematics Education (JRME), Educational Studies in Mathematics (ESM), The Journal of Mathematical Behavior (JMB), and Zentralblatt fur Didaktik der Mathematik (ZDM) (Williams & Leatham, 2017). References for journal articles that contained the word "intuition" were exported as a RIS file or from the online databases and imported into EndNote with abstracts. PDFs for each journal article were attached to their reference in EndNote. Abstracts that were not automatically imported were attached manually. Table 2.1 gives a summary count of the journal papers collected in this systematic search.

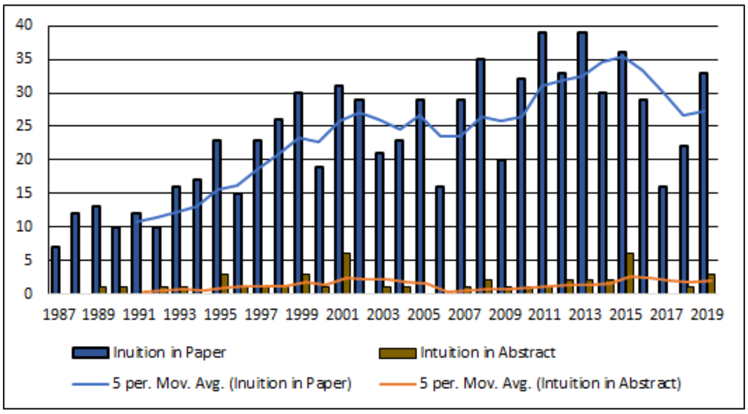


Figure 2.1: Total number of journal papers that contain intuition in paper and in abstract from 1987 to Dec. 2019.

Figure 2.1 shows the collective number of journal papers in the systematic sample published per year with intuition in the paper or more strictly in the abstract. The moving average shows an overall gradual uptrend in the number of papers using the term intuition. This uptrend may appear more prominent between 1987 and 1997 because JMB began to publish in 1994 and ZDM began in 1997. This figure also shows a large gap between the number of papers that used the term intuition and papers that

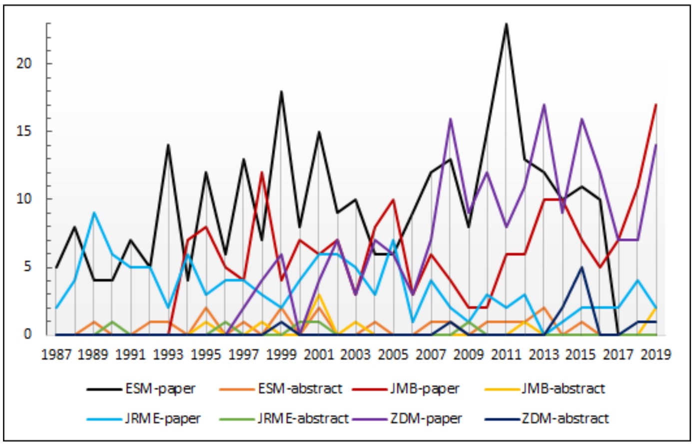


Figure 2.2: Number of JRME, ESM, JMB, and ZDM papers and abstracts per year containing the term intuition from 1987 to Dec. 2019.

placed a major focus on intuition. Figure 2.2 separates this information for each journal, JRME, ESM, JMB, and ZDM.

2.2.2 Methodological design trends

The initial methodological norm in undergraduate mathematics education research was experimental quantitative designs. Over time, this research community began to regard quantitative methods as inferior, they washed out important details in the data (Asiala et al., 2007). At the time of this transition, from quantitative to qualitative methods, Asiala et al. (1997) stated, "In the last decade or so, there has been a growing concern with the impossibility of really meeting the conditions required to make application of statistical tests to mathematics education valid" and "developing understanding that the fundamental mechanisms of learning mathematics are not as simplifiable and controllable..." (p.2). Asiala et al. (1997) pushed for qualitative methods to be the prominent method used by researchers and called for the development of new qualitative approaches.

To capture a more general climate of this field, Hart et al. (2009) conducted a systematic review of research methods in mathematics education to determine the coverage of different research designs published between 1995 and 2005 in journals such as *Educational Studies in Mathematics (ESM)*, *Journal for Research in Mathematics Education (JRME)*, and *Proceedings of the International Group for the Psychology of Mathematics Research*. Hart et al. (2009) found that within the collective *ESM*, *JRME*, and Proceedings sample (N=571), 28 % of the studies were mixed designs, 58 % were qualitative only, and 14 % were quantitative only. While qualitative case studies have dominated the last few decades, research advisory committee's and funding organizations have put pressure on this research community to consider mixed designs to evaluate instructional practices or learning environments (Hart et al., 2009; Ross & Onwuegbuzie, 2012). In a follow up to Hart et al.'s (2009) evaluation, Ross and Onwuegbuzie (2012) found that the percentage of mixed methods studies in *JRME* and *The Mathematics Educator (TME)* decreased from 40 % in 2002 to 20 % in 2004 and increased to 30 % in 2006. Ross and Onwuegbuzie (2014) also found that the quantitative analysis portions of studies remained on the less complex end of their scale which ranged from descriptive, univariate, and multivariate analysis. They speculated that "...the mathematics education researchers' use of lower level statistical analyses might suggest that mathematics education researchers are not asking the most complex questions" (p. 71). This statement was made without any mention of the complexity of the qualitative portion that these researchers may also be juggling, such as grounded theory, or limitation in resources. However, the circumstance of a small number of participants is not completely impervious to the application of more advanced quantitative techniques such as longitudinal case studies of individuals with dynamic factor model analysis (Ferrer & Nesselroade, 2003).

To capture a more current climate within the scope of intuition research, 44 journal papers from the systematic sample that contained intuition in the abstract were examined from the angle of methodology. Out of the 42 papers in English, 8 (19.05 %) of the papers were theoretical and 2 (4.76 %) systematic reviews. The systematic reviews examined literature on school-aged learners' inhibitory control and their detection of errored intuitive judgements (Lubin et al., 2015; Star & Pollack, 2015). From the remaining 32 empirical studies: 17 (40.7 %) were coded as solely qualitative, 6 (14.29 %) quantitative, and 9 (21.43 %) mixed using Creswell and Creswell's (2018) research design framework.

Using Ross and Onwuegbuzie's (2014) 8 level *Quantitative Analysis Complexity Continuum* framework, 9 (60.00 %) of the combined mixed and quantitative papers were coded as level 1-descriptive summary stats, 4 (26.67 %) level 2-univariate, 2 (13.33 %) level 3-multivariate. Przenioslo (2004) was the only mixed study in the sample with university level learners. She used elements of the qualitative grounded theory approach to classify learners' concept images of the limit of functions in terms of "intuitive conception" combinations (p. 104). The notion of intuitive conception was described by Fischbein (1987) as a mental representation of a concept that comes to the mind quickly, feels obvious, and viewed as true. Przenioslo (2004) identified several specific intuitive conceptions under broader themes such as limits in terms of neighborhoods. She turned frequencies of qualitative classification codes, based on prior research and data of 420 university students, into descriptive statistics. No quantitative data collection instruments for mathematical intuition were found, with the exception of Fischbein, Tirosh, and Melamed's (1981) measure for intuitive acceptance.

2.2.3 Empirical studies at the university level

A quarter, 10 (22.73 %), of the 44 papers with intuition in the abstract included participants at an undergraduate level or beyond: 8 (18.18 %) undergraduate level, 1 with graduate students in an educational studies program and first year psychology students taking statistics (Lem, 2015), and 1 (2.27 %) with mathematicians. The 10 undergraduate level studies focused on the limit concept in calculus (Keene, Hall, & Duca, 2014), derivatives (Kidron, 2011), optimization of area and discrete algorithms (Malaspina & Font, 2010), complex analysis (Hancock, 2019), the limit of a function for metric spaces generalized to topological spaces (Przenioslo, 2004), differential equations (Rasmussen, 2001), and Bertrand's paradox (Wilensky, 1995). Lem (2015) examined students' applications of the law of large numbers in statistics. Sriraman (2009) interviewed five mathematicians' on their creative processes and expanded on Hadamard's (1954) investigation. Sriraman (2009) found that these mathematicians' creative processes coincided the four stage Gestalt model preparation, incubation, illumination, and verification. Intuition was a theme that surfaced in these interview transcripts after periods of preparation and incubation.

Bubp's (2014) and Adiredja (2018) stood out in the exploratory sample. Next, the sample of journal papers that incorporated intuition in the theoretical background sections without the strict criteria that intuition be in the title or abstract were re-examined. Five more papers that stood out were included: Kidron and Dreyfus (2014), Kaisari and Patronis (2010), Antonini (2019), and Hanna, de Bruyn, Sidoli, and Lomas (2004), and Alcock and Simpson (2004). A memo was kept for each of the 16 remaining empirical studies, that were in English or had English translation, to log how researchers have grappled with mathematical intuition as a research construct. Memos included: notes on the mathematical topic area, research questions, aim, method, participant sample, key constructs and theoretical frames.

A key practice taken by many researchers in these studies was to refrain from explicitly asking participants to prove a given statement and phrase problems in a:

> ...not quite conventional way in order to avoid automatic solutions. For example, there were no problems directly asking the student to prove that a given number is a limit of a given function or sequence. In a pilot study,

> such problems were found to trigger attempts to recall some standard solution method, which obstructed constructive thinking (Przeniosolo, 2004, p. 108).

Amount of mixing across mathematics education research areas

Next, the computer assisted qualitative data analysis software NVivo 12 was used to further evaluate the 781 JRME, ESM, JMB, and ZDM journal papers that contained the keyword intuition. Separate text search queries for the terms intuition, proof, representational shift, semiotics, visualization, and creativity were conducted. These were identified in the exploratory review phase as broad areas that were related to intuition. The text search query for intuition was used to determine the amount of focus on intuition in each of the papers. The remaining text search queries were used to measure the amount of mixing among the broad research areas within each of the papers. The amount of focus and mixing was measured by word coverage and frequency of the word representatives for the research areas. Query settings “with stemmed words”.

Out of the 781 journal papers, 333 (42.58 %) used the term intuition ≤ 2 times. Over half of the journal sample used intuition as an ambiguous term leaving the reader to interpret it without citing a nominal, conceptual or operational description. Overall, many researchers are using the term intuition in their writing but few conduct empirical studies that place an intentional focus on it. Figure 2.3 fixes the top 150 journal papers with respect to intuition frequency coverage and visually displays the amount of focus of intuition in each paper along with the amount of across area mixing. In total, 447 of the 781 journal papers met the text search criteria for proof 447 (57.23 %), 651 (83.35 %) for representation, 431 (55.19 %) for visualization, 219 (28.04 %) for creativity, and 118 (15.11 %) for semiotics.

Table 2.2: Empirical studies on intuition at the university level.

Study title and Figure 2.3 ID	Math Topic	Aim	Method	Sample	KC and Frames
Teaching proof in the context of physics (Hanna, de Bruyn, Sidoli, & Lomas, 2004)[121]	Statics in Physics & Euclidean Geometry	To explore how student leverage principles in physics to construct proofs in geometry; to investigate students performance and preferences for static proofs using principles in physics and geometric proofs	qual+quan descriptive stats	19 students from class with 11th and 12th graders; ages ranged from 16-18	Physical intuition (Hanna & Janhnke, 2002); physical enactive proof (Tall, 1999); convincing and explanatory proofs (Hanna, 1990)
Building on "Misconceptions" and Student's Intuitions (Adiredja, 2018)	Limit definition in Calculus	Meeting student's where they are at to guide them towards constructing the definition of a limit	QUAL -¿ descriptive stats	An instrumental case study from a study of 24 undergraduates	Sociopolitical perspective (Guitiérrez, 2013) and Knowledge in Pieces (KiP; diSessa, 1993)
Intuitive acceptance of proof by contradiction (Antonini, 2019)[20]	Euclidean Geometry	Investigate reasoning of geometric figures that are impossible according to mathematical theory; explore whether or not proof by contradiction and indirect argumentation are intuitively accepted as self-evident in a pencil-and-paper and dynamic geometry environment	QUAL	6 cases selected from 40 secondary students (grades 10-13), university students, and teachers from various stem disciplines	Self-evidence and intuitive acceptance (Fischbbein, 1987); indirect argumentation (Freudenthal, 1973); model to analyze proof by contradiction (Antonini & Mariotti, 2008); Figural concepts (Fischbein, 1993)
To Prove or Disprove (Bubp, 2014)	function concepts: injectivity, inc., dec., composition, and equivalence relations	Typify students' intuitive errors and track analytic thinking vs. intuition in proof construction	QUAL-¿ descriptive statistics	12 undergraduate math, secondary mathematics education, and economics majors	Dual-process theory and Systematic intuitive errors (Kahneman, 2011 Semantic and Syntactic Reasoning (Weber & Alcock, 2004)
The role of intuition in the solving of optimization problems (Malaspina & Font, 2010)[6]	Optimization of Area and Discrete Algorithms	To validate or invalidate the existence of optimizing intuition	QUAL	38 first year undergrad engineer majors	Onto-semiotic approach (Godino, Batanero, & Font, 2007); Epistemic and cognitive configuration and Optimizing intuition (Malaspina & Font, 2009, 2010)

Table 2.3: Empirical studies on intuition at the university level.

Study title and Figure 2.3 ID	Math Topic	Aim	Method	Sample	KC and Frames
Convergence of sequences as series: interactions between visual reasoning and the learner's beliefs (Alcock & Simpson, 2004)[236]	Real-analysis concepts: bounded, convergent sequences, and series convergence	Unpack approaches of visual and non-visual reasoners assessment of the truth value of a statement and justifications	QUAL	18 students in a first semester first course in Real-analysis	Visual theorems (Davis, 1993); concept image/concept definition (Tall & Vinner, 1981)
(Kaisari & Patronis, 2010)[509]	Transforming Hilbert's axioms of Euclidean into axioms in Elliptic Geometry	To better understand how students develop or modify their interpretations of points, lines, and planes through interactions with peers	QUAL	5 university students	Project method of teaching with a main theme of historical conflict (Frey, 2002); socially negotiated meaning (Cobb, 1986)
Proof Image (Kidron & Dreyfus, 2014)[52]	An exercise in Real-Analysis and Bifurcations	Investigate intuitive and logical thinking towards proof image and contrast theoretical frames to analyze proof constructions	QUAL	2 mathematicians	Abstraction in Context (AiC) (Dreyfus, Hershkowitz, & Schwarz, 2001); Proof Image (Kidron & Dreyfus, 2014)
Paradox, programming, and learning probability (Wilensky, 1995)[78]	Bertrand's probability paradox	investigate how student develops notion of randomness, detects and deals with conflict	QUAL	a case study of a computer professional	Connected Mathematics (Papert, 1991)
The intuitiveness of the law of large numbers (Lem, 2015)[13]	law of large numbers hospital problem in statistics	to address conflicting claims about students' intuition on the hospital problem	QUAN	65 grad students in educational studies MA program and 213 first year psych majors	Dual-process theory, heuristics and biases (Kahneman & Tversky, 1972)
Sequence limits in calculus: using design research and building on intuition to support instruction (Keene, Hall, & Duca, 2014)[26]	Limit concept	to investigate the efficacy of design based research on students' intuition for the limit concept	QUAL-¿ descriptive statistics (coding frequencies)	28 undergraduate students in a calculus class training to become elementary school teachers	Instructional design theory of RME, guided reinvention (Gravemeijer & Doorman, 1999)

Table 2.4: Empirical studies on intuition at the university level.

Study title and Figure 2.3 ID	Math Topic	Aim	Method	Sample	KC and Frames
Images of the limit of function (Przenioslo, 2004)[111]	Real-analysis and topological notions for a limit of a function	to investigate how students' generalize the limit of a real-valued function to metric and topological spaces and to categorize student's concept images for limits of a function	QUAL*Grounded Theory -¿ descriptive statistics (frequencies of codes)	238 Year 3-5 and 182 graduating university mathematics students	Concept images (Tall & Vinner, 1981) and intuitive conceptions (Fischbein, 1987)
Tacit models, treasured intuitions and the discrete-continuous interplay (Kidron, 2011)[17]	limit definition of a derivative (continuous), Euler's numerical method (discrete), differential equations, and bifurcation values	exploring conflicts between formal and intuitive representations, tacit models, and awareness of conflicts	QUAL instrumental case studies	three advanced mathematics learners	Concept image and definition (Tall & Vinner, 1981), tacit models (Fischbein, 2001), and false "treasured intuitions" (Stewart, 2001)
The role of multimodal uncertainty in collective argumentation regarding complex integration (Hancock, 2019)[359]	integrating complex-valued functions	to investigate how met-befores of real numbers influence reasoning in a complex number setting and social argumentation during integration of complex functions tasks	QUAL case studies	2 pairs of students enrolled in undergraduate level complex variables course	Toulmin's (2003) model of argumentation and Three Worlds (Tall, 2013)
New directions in differential equations (Rasmussen, 2001)[31]	realizing solutions as functions to differential equations, equilibrium solutions, and direction fields	to illustrate a framework for students' reasoning about differential equations under two umbrellas: 1) interpretations of functions as solutions and 2) images and intuitions	QUAL case studies	six students enrolled in intro differential equations course	intuition as a mini-theory (Fischbein, 1987)
The characteristics of mathematical creativity (Sriraman, 2009)[109]	mathematicians' research process	to abstract qualities of mathematical creativity	QUAL semi-structured interviews	five tenured mathematicians	Gestalt model of creativity (Hadamard, 1945)

Infrequent mixed area conversations and a blind spot: intuition, semiotics, and creativity

After removing 12 duplicates between the exploratory mathematics education and systematic journal search and collection phases, 851 unique references remained for further evaluation. The sample needed to be reduced further in order to narrow in on journal papers that placed a focus on mathematical intuition within more dominant areas such as proof. At the same time faint links between mathematical intuition and areas, such as creativity and semiotics needed to be preserved and explored further. Creative intuition, a type of intuition that occurs after an incubation period, was highlighted in several psychology references (Policastro, 1995; Raidl & Lubart, 2001; Eubanks, Murphy, & Mumford, 2010; Sadler-Smith, 2015; Gilhooly, 2016). Viewing creative intuition as a type of intuition with incubation as a variable attribute that separates creative intuiton from other forms, such as the quick default intuitive judgements studied under the Heuristics and Biases Program, seems to be a blind spot to mathematics education researchers. Out of the entire mathematics education sample, 781 journal papers plus the exploratory collection, only one paper, Krummheuer (2007), mentioned creative intuition a single time without a description or citation for the term. This was the only time intuition was mentioned in the entire paper. A slightly more frequent occurrence of incubation was found. 10 journal papers discussed incubation, all within the context of creativity research (Savic, 2015; Sriraman, Haavold, & Lee, 2013).

Tanguay and Venant (2016) provided a statement that mixed semiotic theory and intuition. The more ways one can represent a mathematical object the better informed their intuition is and "the process of amalgamation described by Arzarello is of a synthetic nature, and allows the different meanings to be aggregated with the help of analogies and metaphors, glued, condensed so to be more easily internalized and mobilized by intuition" (Tanguay & Venant, 2016, p. 889). To preserve these faint or less frequent links between mathematical intuition and mixed areas the decision was made to include all 851 journal papers in further analysis. NVivo text search query was used at various stages to narrow in further on a more relevant and manageable subset of papers within the big set. That is an inclusion function ι from the big set of papers to itself with different word restrictions W was applied at different times to render different images of ι. The lens of focus shifted many times throughout the analysis and synthesis process.

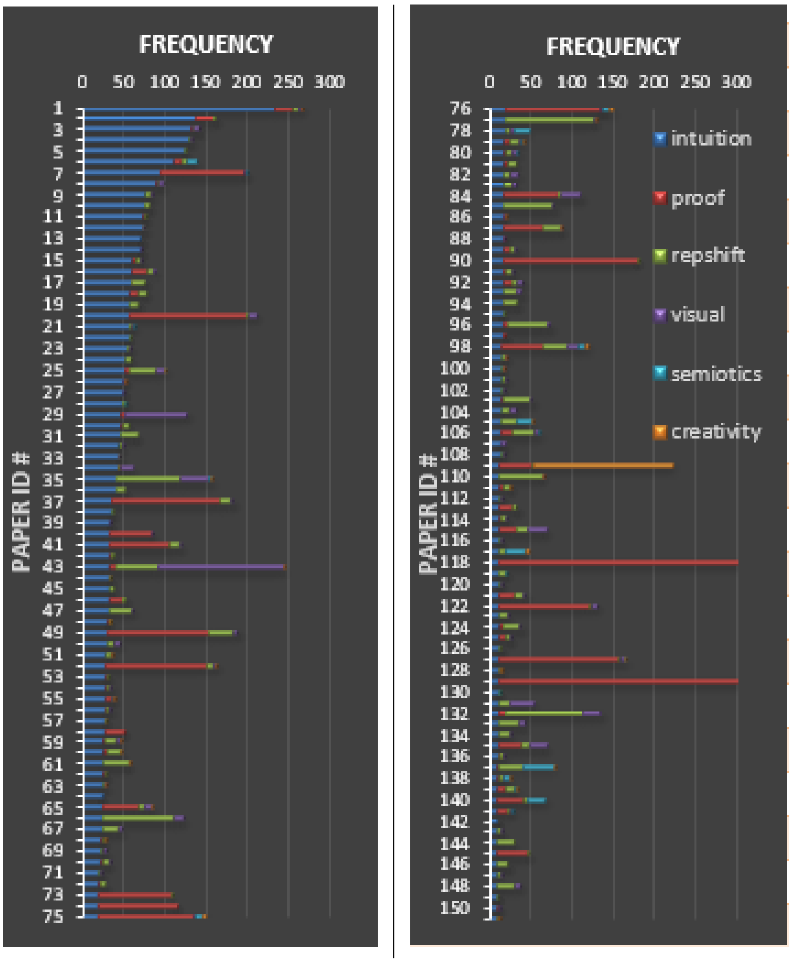


Figure 2.3: Across area mixing in the top 150 intuition papers out of 781 journal paper sample based on frequency.

The more dominant conversation: intuition and proof

In contrast to the more numerous theoretical, philosophical, or historical accounts of intuition in proof (Brouwer, 1913; Poincaré, 1969; Heyting, 1966; Otte, 1990; Dummett, 2000), empirical education studies on intuition in proof have been regarded as needles in a haystack. Kidron and Dreyfus (2014) claimed that, “only a small part of this research [on proof] has addressed the question of how people, mathematicians, and students go about constructing a proof and even less has explored about the intuitive aspects of that process” (p. 302). Out of the sample of journal papers in the systematic sample that contained ‘intuition’ in the abstract, 97.73% of them also contained ‘proof’, ‘prove’, or ‘proving’ in the paper. This dropped to 15.91 % with the stricter criterion that proof terms need to be in the abstract. The self-evident property was often a point of focus in papers on intuition and proof.

Antonini (2019) studied the self-evident property of intuition and used the term "evident" to also include "obvious to sight; recognizable at a glance", "clear to the understanding or the judgement; obvious; plain" (Murray et al. 1961, p. 346-347, cited in Antonini, 2019, p. 795). He then investigated the conflict between the logic of proofs by contradiction and self-evident truths suggested by indirect argumentation based on drawings of geometric figures. "Indirect argumentation" as defined by Freduenthal (1973) as an argument of the form "...if it were not so, it would happen that..." (p. 629). A proof by contradiction is an indirect argumentation, but an indirect argumentation is looser or more hand wavy than a proof. Stated logically, if P is a proof by contradiction then it is an indirect argumentation but the converse of this statement need not hold. Antonini (2019) explored learners' indirect argumentation based on drawings of geometric figures was of the form: if a statement is not so then, the geometric figure X must exist, but the existence of this geometric figure X is an impossible case or optical illusion.

Mariotti and Pedemonte's (2019) study on intuition and proof in high school geometry and definition of "constructive argumentation, that contributes to the construction of the conjecture and commonly precedes the formulation of the statement; and a structurant argumentation, that justifies a conjecture reached as an intuition and commonly comes afterwards" provided insight into a resonating point (p. 761). Intuition reached as a conjecture, a conjectural intuition, may be arrived at through simple observations and without a constructive argumentation. This leads to a statement that 'what' is arrived at is felt as true but does not illuminate to the student 'why' it is true. On the other hand, a constructive argumentation reveals aspects of 'the why'. To put this into the context of proof construction in abstract algebra, take Lagrange's Theorem and a common activity where students write down examples of groups, subgroups, and a list of their orders to arrive at the conjecture: If H is a subgroup of G then $|H|$ divides the $|G|$. These numerical observation lists tell you what the conjecture is, but does not reveal hints as to why it should be true. In other words, the constructive argument in such an activity is missing prior to the formulation of the conjecture. A 'why' in this case is embedded in understanding the orbits of the action in which H acts on G and that orbits partition the set being acted on.

Collectively, this microcosm of empirical investigations on intuition in proof emphasized that learners' difficulties associated with intuition in proof construction stems from cognitive aspects: lack of language needed to externally express an intuition, not familiar with proof-based techniques to go from intuitions to a formal proof or a formally stated conjecture; difficulty with detection and rejection of errored intuitions, struggling to see or express connections between an intuition and a given formal associate; difficulty moving towards a structurant argumentation; and absence of a prior constructive argumentation. Data from Stewart, Thompson and Brady's (2017) study surrounding a geometer's teaching episodes in an algebraic topology course at the graduate level uncovered that "there were alot of questions about how to pass from an intuition to a formal proof (many of these examples used techniques/results from quotient spaces) (p. 2262)." Kidron and Dreyfus (2014) found that it can be difficult to go from intuitions to more formal communicable products at all mathematical levels. The distance between intuition and a formal proof product can vary depending on the type of intuition, the context, the learners background, and additional factors.

For example, the transition from intuition(s) to a formal proof product may also be slowed or blocked as a result of emotion-based factors such as "local affect" or "global affect". Goldin (1998) described "local affect" as emotional reactive response or "changing states of feelings that problem solvers experience, and utilized during problem solving" (p. 154). For example, Weber and Alcock (2009) found that undergraduate learners, who felt strong feelings of contentment in the truth of an informal argument or visual observations, did not feel the need to develop a more formally rigorous argument in a syntactic system. Along these lines with a specific focus on intuition Fischbein (1987) found that intuitions are often "directly acceptable" before a formal proof product is obtained. This may cause the prover to feel so confident that their intuition is true that they do not feel the need to give a formal justification (p. 200). In contrast the previous examples, studies have also shown that emotion-based factors, such as intrinsic motivation and accompanied positive emotions, can support the continuation of work towards a formal proof product (Kidron & Dreyfus, 2010; 2014; Fischbein, 1987). In other words, emotion-based affective factors can open or constrict the passage way the enables the prover to go from an informal representation or intuition(s) towards a corresponding proof product.

2.3 Method of synthesis

It has become apparent that investigations on mathematical intuition requires an integration of a wide range of theoretical frames and compression of multiple areas of mathematics education research. As an added factor that increases research complexity, mathematical intuition as a construct tends to behave like a fuzzy class-concept in which the membership of an intuition to a particular class can be partial. The characterization of intuition as a fuzzy class-concept was supported by the literature and a study on abstract algebra learners' intuitions, conducted and presented later on in this thesis. According to the psychology literature, intuition represents an enormous class of cognitions and mathematics education researchers, like psychologists, have not been studying the same psychological phenomenon under the label of intuition. Instead, they are studying related phenomena that may take on the same values for a common set of attributes, but can take on many sometimes far away values or even polar opposite values for attributes outside a common set (Policastro, 1995; Dane & Pratt, 2007, 2009; Glöckner & Witteman, 2010). Moreover, other related constructs such as affective stressors like time pressure in high-stake situations leads to an increased production in quick reactive types of intuition. On the other hand, the same affective stressors may completely shut off a type of intuition that is tied to new discoveries. It is important to make clear what phenomenon exist under the label of 'intuition' and how other psychological constructs act on the production state, the on and off switch, for different types of intuition (Fischbein, 1987; Dane & Pratt, 2007, 2009: Sinclair, 2010).

To work towards an updated representation of mathematical intuition as a research construct a series of guiding stages for assembling a conceptual structure of a class-concept were followed: establishing boundary properties, identification, grouping, classification, ordering, formation, and maintenance. The stages are predominantly based on Piaget's (1968) text *Structuralism* and Skemp (1979). According to Piaget, elements that are part of or live in a structure are "bounded" by laws or properties and these properties that signify the whole (p. 14). Boundary properties are invariant attributes among a collection. During the boundary stage for assembling a conceptual structure of mathematical intuition, general boundary properties for what intuition is and what it is not, as well as, boundary properties for mathematical intuition are collected from the literature. After invariant attributes are collected up, the identification stage can begin with respect to a boundary. Identification is the stage of identifying whether or not something satisfies already existing properties that roughly bound a class of objects or if it resembles a prototypical member of a class. If an object possesses boundary properties or if it resembles a prototypical member then they are mapped into the corresponding existing structure. If properties are not already established then the grouping stage is entered.

Grouping is the stage where aggregates are formed through detection of variable attribute values that are shared by certain elements yet distinct from other elements of the structure. It is in this stage that one is looking for patterns among the elements. In the construction of class-concepts Skemp (1979) explained:

> from concepts of particular objects which are alike in some way we may

> derive a class concept; e.g., from seeing a number of different chairs, we abstract certain regular features from which we form the concept chair (as against this or that particular chair). In the same way we form other class concepts such as bookcase, desk, bed. Continuing the process, by abstracting certain features common to chair, bookcase, desk, bed,..., we can form a new concept, furniture; and from concepts such as furniture, cameras, gardening equipment, motor vehicles we can derive another concept: consumer durables. And so on (Skemp, 1979, p. 120).

This process is referred to as "successive abstraction" (p. 119-120). Successive abstraction involves reflecting on the aggregates and abstracting attributes that the aggregates have in common to form new boundaries of the whole or to form new intermediate nested classes.

Taking the collection I_M of mathematical intuition and specifying a relation in terms of variable attributes one can obtain a classification. A necessary condition for defining a relation for mathematical intuition is that varying attributes among the collection of intuitions must exist and the researcher must be able to find and state what these attributes are. For example, to define a relation to classify a set of blocks you must know that variable attributes such as color exist among the collection. This allows one to define a relation that two blocks are in the same class if they have the same color. The more variable attributes that exist the more ways there are to classify or shuffle around the blocks. Ultimately, "The major challenge facing intuition research is to classify intuitive phenomena in ways that can lead to more precise and useful questions" (Hogarth, 2010, p. 339).

The classification stage begins when varying attributes are specified that are common for some descriptions of intuition but not all. Classification techniques at have been used in the past to tackle the intuition construct. For example, Beth and Piaget (1966) and Fischbein (1987) both made attempts to classify types of intuition. Fischbein (1987) departed from Beth and Piaget's (1966) classification and stated that, "It is difficult to follow Piaget's classification because of the generality he confers on the term intuition" (p. 58). Fischbein (1987) went on to establish 4 broad classes of intuition: affirmatory, conjectural, anticipatory, and conclusive. He also distinguished between three sub-classes within the class of affirmatory intuitions. These classifications were made according to the role of the intuition and its relationship to the solution. He also makes a second classification, primary and secondary intuitions, according to a combination of the origin of an intuition and Beth and Piaget's (1966) stages of cognitive development: pre-operational, concrete operational, and formal operational.

The ordering stage is reached when one can organize existing disparate subclasses by ordering or mapping them into less restrictive classifications. Formation is memory consolidation of structures constructed in the previous stages and these consolidated structures form one's mental representation of a concept. During its development, a conceptual structure may be problematic in at least two ways. It may be superficial in that it does not contain much intermediate structure or nesting within the boundary of the whole. In other words, the properties of the outer frame of the whole is there but the substance within the frame is missing; it is as if the conceptual structure is just a shell without the yolk. In addition, a conceptual structure may be dissonant meaning logical inconsistencies arise and the conceptual structure is flawed with conflicting boundaries, identifications, classification, or orderings.

To resolve problematic structures one continually undergoes maintenance through "assimilation" and "accommodation" as one cycles back through former stages (Piaget, 1968, p. 63). When new structures are simply added with existing structure preserved this is assimilation and it resolves a superficial structure. Accommodation occurs when already existing formations are incorrect and then corrected to resolve dissonance between the representation of the concept and new incoming information. Through accommodation the conceptual structure is restructured and this resolves dissonance. In order for accommodation to occur one must at least be able to detect the dissonance and be open to change in order to correct the errors or re-organize their conceptual structure.

Along with cycling through the above guiding stages, the current assembly involved memoing and multiple rounds of thematic coding to: identify the current boundary and develop the yolk of the conceptual structure. NVivo 12 was used as a selective and open coding aid. Selective codes used in this integrative literature review were informed by the initial exploratory literature reviews. These selective codes included: boundary properties for intuition and mathematical intuition, existing classifications, collections of attributes assigned to these classifications, and additional attributes.

2.4 Synthesis results

2.4.1 The general boundary of intuition

Within the field of cognitive psychology, intuition is a matter of information activation and retrieval or generation. Non-conscious processing activates internal knowledge structures based on stimuli in the surrounding situation and either information that has been consolidated and re-consolidated over and over (familiar knowledge) is retrieved from memory or newly generated knowledge is constructed by non-conscious processes and delivered to the conscious mind. While the formation of intuitions are dependent on prior work across many learning experiences, the actual conscious experience of an intuition comes to the intuiter quickly and without much mental strain in the moment of its arrival, like a fruit of labor (Fischbein, 1987; Betsch & Glöckner, 2010). Kahneman described intuition as “something that happens to you”, similiar to “perceptions” in that “when we see the world we don’t decide to see it”, and “it comes from somewhere and we are not the author of it” (Google Talks, 2011).

General boundary properties that separate what intuition is from what it is not have been given by Bastick (1982), Dane and Pratt (2007, 2009), and Epstein (2010) to name a few. Collectively, dual-process theories are the major source of these boundary properties. While there are many variations of dual-process theories, the “generic version” separates intuitive Type 1 processes “thinking fast” from analytic Type 2 processes “thinking slow” (Evans, 2019, p. 21; Stanovich, West, Toplak, 2014; Kahneman, 2011). Type 1 processes are automatically engaged, involuntary, fast, responses to stimuli in the environment. Type 1 processes select or configure a best fit option based on a vault of associations aquired through past experiences. They operate at a non-conscious level, not within the bounds of mental awareness. This type of process is unconstrained meaning it can handle massive amounts of data both internal stores in memory and external stimuli in the environment. Type 1 processes are thought to suck up little mental energy compared to Type 2 process. This leaves the intuiter with more energy to explore higher levels of complexity (Betsch & Glöckner, 2010).

While type 1 processes work without you being aware of it at no cost to the working memory, they are not as reliable as type 2 processes. Type 2 processes are intentionally engaged and consciously controlled, reflective, evaluative, slower, effortful, serial, and capacity limited. They deplete attentional resources and puts severe demands on working memory that lead to mental fatigue (Stanovich, West, Toplak, 2014; Kahneman, 2011). Aside from the more general consensus in the two types of processing, there is an ongoing debate of whether or not the interactive nature between Type 1 and 2 processing is strictly sequential, parallel or both (Evans, 2006; Epstein, 2010).

Dual-process theorists in favor of the sequential interaction view are default-interventionists (Kahneman & Frederick, 2002; Evans, 2006). Default-interventionists take the stance that making a decision begins with an automatic default mental representation in the form of an intuitive judgement, hypothesis, or mental model that is activated implicitly with only surface-level analytic processing of the task. This default intuitive response is influenced by cues in the decision making environment and prior experiences in which similar cues were present, current goals, background knowledge, and task features. The default intuitive response may be influenced by how a task or problem is framed, this is called the framing effect.

Evidence of the framing effect was reported by Tversky and Kahneman's (1981) experiment. In this experiment participants were give a prompt that told them that 600 people in the United States are expected to die of a disease. Two conditions were set up a positive framing condition and negative framing condition. In the positive framing condition participants were told to quickly choose, without performing any explicit calculations, between two options to mitigate the disease outbreak: if the first option is taken then 200 people will be saved, if the second option is taken then there is a 1/3 probability that 600 people will be saved and a 2/3 probability that no one will be saved. In the negative framing condition, participants were told to choose between two options: if the first option is taken then 400 people will die, and if the second option is taken 1/3 probability that no one will die and 2/3 probability that 600 people will die. Despite the fact that the two framings are logically equivalent for the first and second options, participants when presented with the positive framing chose the first options, risk aversion, and switched to the second option, risk taking, in the negative framing condition. This provide evidence that was is considered logically equivalent may not be equivalent according to type 1 intuitive processes, especially when emotion, risk and uncertainty are involved (Tversky & Kahneman, 1981).

Kahneman and Frederick (2002) reported on the complacency of the mind to accept intuitive judgements from type 1 processes without further evaluation using type 2 processes. For example, participants were given the following prompt: "A bat and a ball cost $\$1.10$ in total. The bat costs $\$1$ more than the ball. How much does the ball cost?" (p. 7). Many participants errored by impulsively stating that the ball cost 10 cents without checking. Poor judgements, like those found in the previous experiments, are due to the failure of both types of processing systems. Type 1 for producing an errored or nonsensical response and type 2 for accepting the intuitive response as satisfactory and not intervening to evaluate and correct the errored response (Kahneman & Frederick, 2002). Based on the "satisficing principle" of hypothetical thinking and decision theory, analytic processes have a proclivity to accept an intuitive response generated by type 1 processes as satisfactory and follows its lead with minimal oversight. The analytic processes will not modify or completely reject an intuitive response, unless it is presented with substaintial reason to do so (Evans, 2006, p. 380). And even when counter information is available, the analytic system may still accept an errored or nonsensical type 1 response (Kahneman & Frederick, 2002). A decision-maker's analytic process is more likely to kick in if they have higher connectivity among units of knowledge, if they have received instruction that trains abstract or logical reasoning, and if there is more time available (Evans, 2006; Cowan, Chen & Rouder, 2004).

The default-interventionist perspective has been supported by numerous studies from the heuristics and biases program (Kahneman, Slovic, & Tversky, 1982; Kahneman, 2011) and neuroscience (Lieberman, 2000: Lieberman et al., 2004; Lieberman, 2007). But some theorists still hold the view that the two systems may also interact in parallel or run at the same time. Epstein (2010) is one such theorist who argued that the interactive nature is both sequential and parallel. His argument is framed using a particular dual-systems theory, the *Cognitive-Experiential Self-Theory* (CEST). According to CEST the experiential-intuitive system prefers images, metaphors, scenarios, narratives, and concrete information. The rational analytic system encodes abstract symbols, words and numbers.

In addition to the dual-process and systems distinction, psychologists have compiled various definitions of intuition and analyze similarities and dissimilarities among them (Dane & Pratt, 2007, 2009; Gore & Sadler-Smith, 2011). They found that these definitions designated intuition as a "process" or "outcome" or a combination of both. For example, Dane and Pratt (2007) presented Raidl and Lubart's (2000, 2001) definition of intuition as "A perceptual process, constructed through a mainly subconscious act of linking disparate elements of information" (p. 35). Outcome descriptions refer to intuition as "expectations", "hunches", "bodily awareness", "viscerally(gut feelings) and less specific feeling states", and "mental images (visual, auditory)" that emerge into conscious awareness (Dane & Pratt, 2007, 2009; Sadler-Smith, 2016, p. 1081). Given that the intuiting process is non-conscious, it is not possible for an intuiter to introspect directly on their own non-conscious processes and the associations this process has built up over time (Kahneman, 2003). Simon (1986) explained that vague or matter of fact descriptions of how one arrived at a decision is a hallmark of intuition:

> ...if we present a physician with a set of symptoms and the physician immediately (e.g., within two seconds) makes the diagnosis "measels" and if, when we ask him or her her how the decision was reached, we receive only answers like, "I used my intuition", "I don't know", "It's a simple matter of medical judgement", "...then we may say that the judgement has been reached intuitively. We will be particularly inclined to say it was intuitive if the subject cannot provide a veridical account of the steps of the problem solving, question answering, or recognition process that were used to arrive

at the response (p. 242).

Sinclair (2010) also found a process-outcome distinction when she examined theories with ties to intuition. Dual-process theories were viewed as more process oriented and theories such as the Cognitive Continuum Theory proposed by Hammond (1996) were viewed as more outcome oriented. She concluded that:

> The distinction between process and outcome theories in conceptualizing intuition is therefore critical, the notion of which extends to the way we attempt to measure the construct. In practical terms, it is imperative to differentiate between intuiting as non-conscious information processing and intuition as its consciously registered outcome (p. 379).

Overall, it has been repeated in the psychology literature that further work is needed to uncover the adaptive purpose intuition serves, determine the interactive principles between intuition and analytic processing, and determine process and outcome attributes. There is also consensus that the boundary properties for the class of intuitions are still too "loose" (Epstein, 2010, p. 297). Loose, in this context, means that the complement of all analytic processes, which includes dreams, irrational fears, and motor reflexes, goes beyond the bounds of what psychologists consider intuition to be. It also means that many kinds of cognitions satisfy the current list of type 1 process attributes with a consciously registered outcome, however not all such cognitions should be considered an intuition (Dane & Pratt, 2007, 2009; Glöckner & Witteman, 2010; Epstein, 2010; Sinclair, 2010; Hogarth, 2010). The next section, seeks a tighter domain-specific boundary by collecting additional boundary properties associated with mathematical intuition within the sample of 781 journal papers pulled from JRME, ESM, JMB, and ZDM and exploratory sample.

2.4.2 The boundary for mathematical intuition

Within the mathematics education literature, boundary properties or descriptions have been given by Bruner (1960), Poincaré (1969), Fischbein (1987), Malaspina and Font (2010), Davis, Hersh, and Marchisotto (2012), Thurston (1994), Otte (1990), Chiu (1996), Burton (1999), and more cryptically by Duval (2000, 2017). Bruner (1960) referred to the Webster Dictionary definition of intuition: "immediate apprehension or cognition" as a starting point towards a working definition in the field (p. 60). He went on to give a mathematical description:

> Intuition implies the act of grasping the meaning, significance, or structure of a problem or situation without explicit reliance on the analytic apparatus of one's craft. The rightness or wrongness of an intuition is finally decided not by intuition itself but by the usual methods of proof. It is the intuitive mode, however, that yields hypothesis quickly, that hits on combinations of ideas before their worth is known (p. 60).

Poincaré (1969) described intuition as foresight. Fischbein (1987) covered mathematical intuition as "immediate knowledge" and proposed the following list of boundary properties: self-evidence, immediacy, intrinsic certainty, perseverance, coerciveness, theory status, extrapolativeness, globality and implicitness (Fischbein, 1987, p.6). Intuitions are "self-evident" meaning that the intuiter strongly feels that initially their intuition is true or probably true before they have a proof (Fischbein, 1987, p. 196). Building from a discussion of several examples he stated that, "one may learn that there are, in mathematics (and in science in general), statements which appear to be acceptable directly, self-evident, while for other statements, a proof is necessary in order to accept them as true" (Fischbein, 1999, p. 18). This leads to the hypothesis that there are: pre-proof "cognitions which appear directly acceptable as self-evident. These are intuitive cognitions" and "A category of cognitions which are accepted indirectly on the basis of a certain explicit, logical proof. These are logical, or logically-based cognitions" (Fischbein, 1999, p. 18). Fischbein (1987) referred to self-evidence as the "main attribute" of intuition which begins to separate intuition from what it is not (p. 21). Out of the sample of all 781 journal papers, 82 papers (10.50%) referenced the boundary property self-evidence or self-evident. Out of the sample of 44 journal papers with intuition in the abstract, 17 papers (38.64%) referenced self-evidence or self-evident. And out of the 15 journal papers that contained intuition in the title, 14 (93.33%) referenced self-evidence or self-evident. Stout (1934) summarized the nuances of self-evident cognitions and wrote:

> What is self-evident need not be obvious; and what seems self-evident need not be true. It need not be obvious because in order to be able to perceive it we must sufficiently understand the proposition with which we are dealing; and this may require long previous preparation. What seems self-evident need not be true, because we are liable to errors of inadvertence and confusion, through which we fail to distinguish between what really is self-evident and something else which is not so (p. 393).

Stout preferred the phrase "indeterminately self-evident" (p.393). While feelings of self-evidence can cause illusion in the case of an errored intuition that leads to a conjecture that is later falsified; it may also provide emotional support to keep pushing forward (Hadamard, 1954; Fischbein, 1987). Mathematicians and learners must believe that their intuitions contain at least some truth or be plausible in order to stay motivated and keep working on a problem (Noddings & Shore, 1984; Fischbein, 1987). A common pattern of mathematical progress has been recorded in history in cases where self-evident intuitions are mathematically false. In the latter case, the clash between self-evidence and the realization of false intuition leads to a surge of attention and effort to reconcile dissonance between collective intuition and mathematical reality (Wilder, 1967).

Historically, the term self-evidence can be traced back to the axiomatic foundations of mathematics and early cultural practices. Frege stipulated two criteria that must be met to establish an axiom: an axiom must be self-evident *einleuchten* and it must not be reducible from nor equivalent to other axioms or propositions. Frege also used the term *selbstverstandilich* meaning that primitive truths such as axioms are the blocks that are not provable from which non-primitive truths are built upon (Jeshion, 2001; Davis, Hersh, & Marchisotto, 2012). Fischbein (1999) concluded that "...the formal world of mathematics-as they are accepted by the scientific community - contradict in many respects, our natural, self-evident interpretations, our intuitions" (Fischbein, 1999, p. 13). Paradoxes that arise in mathematics exemplify the clash that can occur between formal primitive systems, formal non-primitive systems and intuition. For example, take the Banach Tarski' paradox, it can be proved that a sphere can be broken into a finite number of pieces, non-measurable sets, and reconfigured using the axiom of choice into two spheres both with the same size as the original sphere.

The following additional boundary properties were proposed by Fischbein (1987, 1999). Intrinsic certainty is a certainty derived from the self with little interference from external factors such as a person of authority. Perseverance refers to intuitions to be mental representations that are not easily changed in the mind of their possessor even when faced with evidence that counters it and suggests that it is errored. People tend to want to hold onto their intuitions. Coerciveness means that "intuitively accepted" statements cannot be easily reconciled as being false even if the learner is aware of counterexamples or statements that contradict their belief. Intuition is "theory-like" and represents an individual's personal theory to a mathematical problem (Fischbein, 1990, p. 41). Griffiths (2013) defined intuition as "...the development of a personal theory which is not the result of explicit tuition but may arise either spontaneously or via some activity that is related only indirectly to the theory in question" (p. 81-82).

Unlike formal theory, intuition is more like a hypothesis that has not undergone the necessary transformations to produce a formal proof that can be either verified or rejected by the mathematical community. The extrapolitiveness property of intuition states that intuition calls on information that is outside the realm of being directly perceptible. The property of globality is that intuitions are representations that encompass the problem-solving situation as a whole. It is not a controlled act of breaking down a problem into finer parts which is termed analytic thinking (Bruner, 1960). Bruner (1960) described analytic thinking as step-by-step and “intuitive thinking rests on familiarity with the domain of knowledge involved and with its structure, which makes it possible for the thinker to leap about, skipping steps and employing short cuts in a manner that requires a later rechecking...” (p. 58). The final property of implicitness means that an intuition as an outcome is generated by non-conscious processes. Fischbein (1987) stated that:

> Although apparently self-evident, intuitions are in fact based on complex mechanisms of selection, globalisation and inference. But this activity is generally unconscious and the individual is aware only of the final product, the apparently self-evident, intrinsically consistent cognitions. The tacit character of intuitive elaborations explains the difficulty of controlling and influencing them (p. 201).

Building from Fischbein (1987) boundary properties and intuitions that lead to axioms or primitive truths, Malaspina and Font (2010) characterized intuition as a process consisting of idealization, generalization, and argumentation.

Davis, Hersh, and Marchisotto (2012) also gave boundary properties which they admitted to be “vague” characterizations of intuition. They stated that intuition is “the opposite of rigorous”, “visual”, “plausible, or convincing in the absence of proof...”, “incomplete...”, “based on a physical model or on some special examples...”, “holistic or integrative as opposed to detailed or analytic...” (p. 433-434). The meaning of “rigorous is never given precisely...”, but an attempt was made to shed some light on the difference between a rigorous and intuitive argument (p. 433). Davis, Hersh, and Marchisotto (2012) stated:

> To be rigorous, we must justify our conclusion deductively, by a chain of reasoning where each step can be defended from criticism, ad where the first step is considered known, and the last step is the desired result. If the chain of reasoning is extremely long and complicated, the rigorous proof

> may leave the reader still subject to serious doubt and misgiving; in a genuine sense, it may be less convincing than an intuitive argument, which can be grasped as a whole, and which uses implicitly the assumption that mathematics as a whole is coherent and reasonable (p. 434).

To try to clarify the property the intuitive means visual and is not rigorous they juxtaposed “intuitive” and “rigorous” topology or geometry:

> ...the intuitive version has a meaning, a referent in the domain of visualized curves and surfaces, which is excluded from the rigorous (i.e, formal or abstract) version. In this respect, the intuitive is superior; it has a quality that the rigorous version lacks. On the other hand, the visualization may lead us to regard as obvious or self-evident statements which are dubious or even false (p. 433).

Thurston (1994) described intuition as “...sensing something without knowing where it comes from...words, logic, and detailed pictures rattling around can inhibit intuitions and associations” (p.5).

Researchers have also provided additional attempts to articulate differences between formalism and intuition. For example, Otte (1990) described intuition and the process of combining, refining, and translating intuitions as something that could not be replicated or fully recovered after they pass. He described a formal proof as a product that could be replicated in full by machine and followed. Chiu (1996) pulled the notion of a formal concept and intuition apart along six dimensions: “origin”, “societal support”, “internal structure”, “explication”, “systematicity”, and “justification” (p. 480). For the first dimension, the origin of formal concepts are experts and the origin for intuition is the self. For the second dimension, formal concepts are supported by a teacher but learners’ intuitions have little to no support from others. For the third dimension, formal concept have an analytic structure meaning the concept is decomposable into finer parts which can be strung together in a step by step or algorithmic fashion. Intuition is holistic. For the fourth dimension a formal concept is explicated precisely where as intuition is often not well articulated. For the fifth dimension, a formal concept is considered “densely connected” where as intuition is “sparsely connected” (p. 480). For the sixth dimension, formal concepts are justified by authority and estabilished systems, but justification for intuition is derived from personal experience.

Shortly after Chiu’s (1996) work, Duval (2000) proposed a cognitive dual systems framework called the Cognitive Architecture for Mathematical Thinking that paralleled early dual-process theories in psychology. This model distinguished between the automatic non-conscious system and the intentional system used to flex between various representations of mathematical objects. While Duval goes on to carve out the intentional system into subsystems termed registers, which paved the foundations for the Theory of Registers of Semiotic Representations (Duval, 2000; 2017), he leaves the details of the automatic system behind. He characterized the automatic system obscurely as a system that provides "direct and immediate access to objects (for which we often use the general and plurivocal word intuition)..." (2017, p. 1). Since Duval's earlier work, psychologists have articulated different types of non-conscious processes within the automatic system to classify different forms of intuition (Glöckner & Witteman, 2010). The carving out of different types of non-conscious processes within the automatic system will be returned to later during the classification stage of this assembly.

2.4.3 Intuiting methods

Despite some parallels between mathematics education and psychology theories, there is a major component of mathematical intuition that is not covered by the process-outcome distinction. Recall that within the psychology culture intuition is a process-outcome object in which the intuiting process component involves a non-conscious processing of inputs and knowledge structures followed by a consciously realized outcome (Betsch & Glöckner, 2010). But in mathematics culture the intuiting process that comes before a consciously realized outcome is not always strictly non-conscious. Mathematicians often attach the label 'intuitive thinking or activity' during learners proof constructions to be: drawing sketches, using visualizations to examine an extract information, constructing visual models, informal rephrasing of formal objects and testing examples (Wittmann, 1981; Fischbein, 1987; Gray, Pinto, Pitta, & Tall, 1999; Raman, 2002; Dawkins, 2015). This kind of activity is said to be useful for stimulating intuition.

One reason for this major difference that is not emphasized by psychologists may boil down to a difference in decision-making environments being investigated. When psychologists investigate intuition such as that of chess players, firefighters, air-fighter pilots, and nurses they are studying subjects that are already and naturally surrounded by a rich external scene of cues for the intuiting process to operate. The chess players in de Groot's (1946) studies can see the entire board, the arrangement of pieces, and possible moves. The firefighters in studies covered by Klein (1993) are attuned to sensorial cues such as wind shifts and particular smell which allows them to predict that roof is going to collapse just moments before it collapses. The air-fighter pilot can quickly intuit the best possible maneuver for survival based on cues in the dynamic (changing environment) in the aerial combat zone. The value of expert intuition in these studies is its speed and accuracy under time pressured situations. In a study by Crandall and Getchell-Reiter (1993) expert neonatal intensive care unit (NICU) nurse's were able to intuit patient diagnoses and outcomes. These expert nurses' were attuned to perceptual cues that could not be found in medical texts or the training literature such as poor muscle tone and the appearance of the eyes.

In all of these mentioned studies, the external cues are physical sensorial cues that are already present in the environment and the intuiting process can immediately act on these readily available cues to activate associated internal knowledge structures to generate the intuition outcome. But mathematics learners must paint their own scene, often times it is not externally in front of them. Initially, there are not very many cues to feed the intuiting process. When proving something there are little to no initial physical cues. Initial physical cues may only include the statement of what needs to be proved. In some cases the statement of what needs to be proved may not exist yet. Thus, learners, at times, must generate their own cues and stimuli to provide inputs that support the intuiting process. The more registers they have access to, the greater the variety of cues.

So what does the intuiting process prefer to eat? According to the Epstein (1994) the intuiting process is governed by the *Experiential System* which implies that it encodes concrete images, metaphors and narratives. And this is what intuitive mathematical activity supplies. Intentional and consciously controlled strategies that are used to generate cues and support the intuiting process have been called 'intuitive methods' (Noddings & Shore, 1984). Intuiting methods are "attempts to provide a holistic picture, visual displays and hands-on experiences", but noted that these "methods" are not sufficient for the "faculty of intuition" to be able to operate to generate an intuition outcome. When approaching a proof one may not have any clear ideas of what to do. Noddings and Shore (1984) described:

> a stage that may precede the first clear intuition [outcome]. During this stage the intuition [as a process] is looking, but as though in a darkened room or at a great distance. The feeling that accompanies this stage is discomfort...We are impatient, uncertain that anything will be accomplished, afraid that interruptions will occur, edgy, doubting our own capacities...this is the stage of mental torture, in which no subjective certainty sustains us (p. 88).

One must relax and drop into an “intuitive mode”, a state of mind in which the goal is to obtain understanding through exploration and following curiosities that come from within rather than producing a refined formalized product (Noddings & Shore, 1984, p. 69, 81).

To add substance to the methods component of mathematical intuition, a thematic analysis of the literature with open coding was conducted to collect strategies that are used to generate pertinent cues and facilitate intuiting processes. A text search query was used with the search terms ‘gain intuition’, ‘develop intuition’, ‘build intuition’, ‘stimulate intuition’ or ‘facilitate intuition’ with exact matches, stemmed words and synonyms. 27 out of the 781 journal papers met this search criteria. The themes: leveraging interactive visual software environment and computation tools, tactile model constructions, representational shifting, informal paraphrasing, use of metaphors, experimenting with examples, and switching mathematical scenes were extracted from the 27 papers. Analogic reasoning was a theme informed by Fischbein’s work. In a follow up search within the systematic journal intuition in abstract sample of 44 papers, 5 (11.36%) papers contained ‘analogic reasoning’ or ‘analogy’, 165 (21.13%) contained ‘analogic reasoning’ or ‘analogy’ within the entire journal paper sample.

Leveraging visual software environment and computation tools

Computer tools that offer a visual environment to explore patterns and develop intuition in school mathematics, teacher training, and in mathematics research done by mathematicians (Leung, 2008; Watson & Callingham, 2013; Martínez, Guíñez, Zamora, Bustos, Rodríguez, 2020; Kortenkamp, Monaghan, & Trouche, 2016). In school geometry the visual environment provided by SmartBoard helps to “...develop students geometrical intuition for identifying quadrilaterals and the interrelationship among them...” and “strengthen[s] their intuitions by varying progressively the dimensions of the figures with onscreen live illustration of dynamic images accompanied by the narrative explanation of the teacher” (Leung, 2008, p. 1009). Leung (2008) used animations and static images in and interactive SmartBoard (SB) environment to give a compact view of a parallelogram, rectangle, rhombus, square, kite, and trapezoid. Students, age 9, then explored these shapes and color coded assigned properties to each of them on the visual SB environment. At a glance of all the shapes and their properties can be viewed. The stimuli organized in the SB environment naturally caused students to look at different groupings of shapes rather than looking at them in isolation. Using hoola hoops to make Venn diagrams and physical cut outs of the shapes, as artefacts, these students were able to grasp the notion of relaxing and adding properties. They also were able to draw set-theoretic classification lattices. This stimulated conjectures such as a rectangle is a parallelogram and a rhombus is a parallelogram. The tactile stimuli also provided clues on how to proceed with a deductive argument (Leung, 2008).

For mathematicians, software such as Maple Interface in a Dynamic Geometry Environment (DGE) is a laboratory where experimentation takes place, data is produced, patterns become readily apparent, and conjectures are formulated. However, the proof is not always so revealing in a DGE (Kortenkamp et al., 2016). In a case study by Mariotti and Pedemonte (2019), a DGE Cabri-Geometry helped students to formulate conjectures based on "perceptive facts - what is perceived as true, but not felt as necessarily true". They also found, for one student, that "Dynamic intuition", intuition induced by a DGE, was "an obstacle to the construction of a proof" (p. 762). In this case, the student was able to produce an initial conjecture. This student's explorations in the DGE led to the recognition of seemingly unrelated perceptive facts and conjectures. This student seemed to be confined to perceptual features in the DGE and never developed a "structurant argumentation, that justifies a conjecture reached as an intuition" and is produced after the conjecture is formulated. This prevented further progress towards a formal proof product (Mariotti & Pedemonte, 2019, p. 761).

Model constructions and embodied cognition

Models may be consciously and deliberately produced to gain access to additional sensorial information that the intuiting process can work with or models maybe remain tacit, outside of awareness, but still influence behavior (Fischbein, 1987). In the context of geometry Kaisari and Patronis (2010) defined a model to be "a particular set of geometric entities in 3D space (representing points, lines, or planes), which satisfy the axioms of a theory" (p. 254). They investigated university students' use of models to modify Hilbert's axioms of Euclidean geometry to arrive at a new set of axioms for elliptic geometry, identifying antipodal points, and double elliptic geometry, not identifying antipodal points, on a spherical surface.

In some cases abstract mathematical concepts are materialized into this world as tactile physical models. In other cases one takes the reverse direction in which physical bodily representations, interactions with the physical world, and gestural models are used to arrive at abstractions. In the area of embodied cognition, researchers focus on the later direction. Everyday gestures are often so frequent that they go unnoticed if not intentionally reflected upon. Embodied cognition researchers explore how learners' gestures and corpus can be used to physically act out mathematical concepts in the classroom so that they can connect to the mathematics on a more intimate level (Núñez, Edwards, & Matos, 1999). It is theorized that the sensory-motor system which is often attributed to procedural or rote actions actually plays a role in conceptual understanding and abstraction (Gallese & Lakoff, 2005).

Representational shifting, informal paraphrasing and metaphors

Several papers included representational shifting, informal paraphrasing, or metaphor use as general methods that are used to gain or facilitate intuition (Cangelosi et al., 2013; Keene, Hall & Duca, 2014; Dawkins, 2015; Hegg et al., 2018; Pinto, 2019). During the total move towards a formal proof product, several localized de-mechanizations may occur from a formal representation to an informal or intuitive representation. Dawkins (2015) termed an "intuitive paraphrase" as informal paraphrasing of formal definitions, axioms, and theorem statements to be proved in the *Representational System of Mathematical Proof* to a representation that is less formal and more diagrammatic or imagistic.

Pinto (2019) investigated variations between the informal content covered in class by different instructors teaching a proof-based Real-Analysis course. One of the instructors, Amit who frequently drew pictures of definitions stood out. "Before getting into examples, I would like us to gain some intuition. We are going to unpack this definition and make a drawing of it, to see what it really means" (p. 8). At one point Amit even discussed that from the drawings this was "clear" that one property was stronger than the other. "After repeating his last explanation at a request of a student, Amit asserted that the geometric interpretation they just learned can be used to gain intuition as to 'why differentiability is in fact stronger than continuity'..."(p. 8). These geometric drawings and loose verbal explanations of them motivated how one may go about constructing a formal proof for differentiability implies continuity. Amit then modeled how to construct the formal proof by leveraging the geometric interpretation.

While it is important for instructors to demonstrate informal paraphrasing maneuvers and shifts into geometric registers during lecture, it is not sufficient to ensure that students are able to use or make such maneuvers to make progress towards a proof product (Pinto, 2019; Dawkins, 2015). In a curriculum design experiment for a course in neutral axiomatic geometry, Dawkins (2015) found that students never gave or used an intuitive paraphrase of the formal textbook definition of 'betweeness of points' even though the professor provided intuitive paraphrases of the definition by illustrating his spatial reasoning and drawing a euclidean line diagram during lecture.

The extent to which intuitive paraphrases of definitions and axioms are built into a curriculum is under debate in school geometry. The Common Core State Standards in Mathematics approach stressed that the definition of congruence should be phrased more intuitively as “two figures in the plane are said to be congruent if there is a sequence of rigid motions that takes one figure onto the other” rather than axiomatically (Common Core Standards Writing Team, 2016, p. 15; cited in Hegg, Papadopoulos, Katz, & Fukawa-Connelly, 2018, p. 56). However, it also may be the case that moving to such an intuitive paraphrase that lends itself to “tactile investigations” may make increase the distance needed to travel in the other direction towards a formal proof, making it much more difficult for students to access a formal proof compared to the axiomatic approach (p. 57). Hegg et al. (2018) investigated students preference to use Triangle Congruence Criteria (TCC, i.e., SAS, ASA, SSS, and AAS rules) versus the transformation definition of congruence to construct proofs. They found that 5 out of 6 students could successfully construct proofs using either TCC or transformations and they thought it was easier to use TCC and so they liked this approach more. However, what is easier or cognitively less demanding may not always be what is better for the student’s long term intuition development.

Experimenting with examples and building example spaces

In a study by Alcock and Inglis (2008) of two doctoral students’ example use was investigated as they proved number theoretic statements about prime, abundant, perfect, and deficient numbers. Data indicated that one of the student’s Chris formulated an initial default hunch that a given conjecture, supplied by the study, was presumably true and gave a very informal argument of why the conjecture was probably true, but later expressed that he was not sure if the conjecture was true because he had not investigated examples yet. Alcock and Inglis (2008) concluded that “even though Chris initially thought the conjecture was probably true, he did not immediately attempt to construct a general proof, but rather returned to his earlier strategy of trying to develop his intuition by exploring the relevant example spaces” (p. 125).

Watson and Mason (2005) stated that “...examples learners produce arise from a small pool of ideas that simply appear in response to particular tasks in particular situations. We call these pools example spaces” (p. ix). They offered several metaphors to illustrate the use and accessibility of examples in the example space. Watson and Mason (2005) gave a “pasture” metaphor (p. 60-61):

> ...the spatial sense of example space suggests a landscape, with some very familiar examples acting like easily accessible pastures in the valley, whereas less familiar or more complex examples are like pastures higher up on the slopes or hidden behind hedges and hence more difficult to see and reach.

Individuals example spaces have an internal mental organization in which examples are related, grouped, and "can be explored or extended by searching for situationally peculiar examples as doorways to new classes, by being given further constraints to focus on particular characteristics of examples..." (Watson & Mason, 2005, p. 50-51). The literature signaled that several example types within the example space are used to set-up, gain, or strengthen intuition. These types are: start-up examples (Michener, 1978; Mills, 2014), examples used for pattern exploration to generalization (Fischbein, 1987; Bills & Rowland, 1999), generic examples and non-generic examples (Rowland, 2002), side by side comparisons of examples and non-examples, boundary examples (Mason & Watson, 2001) and example chains (Antonini, 2011).

Start-up examples are used to illustrate a theory that is being learned for the first time and should be accessible to the first time learner and accompanied with an uncomplicated picture that is easily interpreted. If possible start-up examples should be contextualized and concrete, something that students can associate with some aspect of everyday reality (Michener, 1978). "These start-up examples help one get started in a new subject by motivating basic definitions and results, and setting up useful intuitions" (Michener, 1978, p. 366). Unlike start-up examples that are used to induce intuitions, counter-examples are said to "add focus and limits to one's intuitions" and are used as an initial check to see if an intuition that has already been produced can be refined (Michener, 1978, p. 369).

Generic examples are used for "seeing the general through the particulars" and are a "carrier of the general" (Mason & Pimm, 1984, p. 287). Balacheff (1998a) described generic examples as an example that stands for an entire class of objects. A generic example can also be described as an example with an argument that can hold in general for an entire class of objects and can be turned into a proof with slight modifications to notation, replacement of the example with a general object, and the incorporation of lemmas, such proofs are called "generic proofs" (Leron & Zaslavsky, 2013, p. 24). Leron and Zaslavsky (2013) provide several generic proofs for results in Abstract Algebra. Research on students use of generic examples to aid in proving activity is scant and warrants further attention (Aricha-Metzer & Zaslavsky, 2017).

In one study relevant to generic, non-generic example use and intuition, Aricha-Metzer and Zaslavsky (2019) investigated how a group of middle school, high school, and undergraduates used generic and non-generic examples and whether the use of these examples led to a productive or non-productive reasoning towards the construction of a proof or pieces of a proof. Summarizing the results of this study they stated that “While empirical, non-generic, example-use could be helpful for making sense of a conjecture and gaining an intuition regarding its validity (as in Case 8), interestingly, our analysis didn’t reveal any cases in which a non-generic use of examples was productive for proving” (p. 320). In Case 8, a 9th grade student generated and examined the examples 16 and 25 and non-examples 18 and 30 when asked to prove that a perfect square has an odd number of distinct factors. This student was able to for a correct conjecture that the perfect squares 16 and 25 had an odd number of factors by listing the factors and counting them. However, he could not use these examples to see why and make advancing steps towards a proof for the conjecture. It seems that the novice learner was able to produce valuable empirical cues by listing the factors 1, 2, 4, 8, 16 and 1, 5, 25, but this learner’s intuiting process was not attuned to the more specific pattern among these cues of their being a bijection between a divisor d and $16/d$ on either side of the $\sqrt{16}$ in the factor list and similarly for 25. Just because key empirical cues in the environment have been generated and are focused on does not guarantee that the intuiting process will pick up on patterns suggested by the cue set.

Aricha-Metzer and Zaslavsky (2019) concluded that case 8 may have not been able to pick up on the more specific structural pattern because the representation of factors in a list was not as revealing as the representation of pairs of factors. While a change in representation from a list of factors to pairs may have offered a more revealing cue set such a representational shift may still not have been enough for Case 8 to proceed towards a proof. One possible alternative interpretation of Case 8, is that the failure to advance further towards a proof was not due to a lack of engagement in proper intuiting methods (i.e. the use of examples and representational shifting), rather it was a failure of the intuiting process for not being attuned to or able to pick up on the more specific structural patterns in the cue sets for a number of reasons. Some reasons include that the learner’s internal knowledge stores in memory do not contain enough experiences working with bijective functions and so the intuiting process can not match a cue set to something it does not have memory of. It could also be the case that memory stores do contain experiences with bijective functions, but other forms of interference prevent or block the matching.

Boundary examples are examples that show why the assumptions in a claim are necessary. It is similiar to the notion of modifying assumptions in a theorem, but rather than reasoning with just the abstract assumption a concrete example that fits the mold of the modified assumptions is inserted to see what happens to the claim. Mason and Watson (2001) described the use of boundary examples and stated:

> You start with maximum freedom (generality) and then you impose constraints. Another form of it is weakening a constraint to reveal a class of solutions, which sometimes enables a solution to be found to weaker constraints while the tighter constrain remains unsolved. When students are offered examples to illustrate theorems, and even where these are boundary examples because they show why constraints are required (p. 126).

Goldenberg and Mason (2008) noted that, "Learners who are aware of looking for what is the same and what is different as a learning an problem solving strategy develop rich and extended example spaces, while those who try to master the examples they are given as templates are condemned to a restricted and potentially misleading example space" (p. 192). Some ways to extend example spaces that are also intuiting methods that generate cues to the intuiting process are: building chains of examples from a few initial examples by tweaking a previous example in the chain slightly and involves performing cognitive transformations on "semiotic sets" (Arzarello, 2006, p. 281; Antonini, 2011). A semiotic set consists of: a set of semiotic representations, rules the interpreter uses to produce sign-interpretation pairs, and performing cognitive transformations, and "a set of relationships among these signs and their meanings embodied in an underlying meaning structure" (Arzarello, 2006, p. 281; Peirce, 1991).

Overall, examples are used to produce data filled with perceptual patterns, regularities and invariants. Such patterns, regularities, and invariants have the potential to be detected by the intuiting process and generalized to produce an conjecture, but goes beyond the observable data within a finite set of examples (Fischbein, 1987). Studies have shown that the choice of certain examples matters. Some examples can lead to appropriate generalizations while others lead to inappropriate generalizations (Zazkis, Liljedahl, & Chernoff, 2008). Example use after generating a conjecture or being presented with a conjecture has been regarded as a means to 'strengthen' one's intuition about the validity of a conjecture before making an attempt to construct a formal proof (Fischbein, 1987; Alcock & Inglis, 2008; Aricha-Metzer & Zaslavsky, 2019).

Switching mathematical scenes

Intuiting methods at a more advanced graduate level and beyond also include the use of functors to switch from one's current mathematical scene in a particular area of mathematics to another scene in another area of mathematics in order to activate a fresh set of cues or access to additional tools to work with or to reduce noise. A paper from CERME 10, in the exploratory sample, offered data through a case study. This case study found that

> mathematicians often translate a problem from one area of mathematics (e.g., Topology) to another (e.g., Algebra)...When a statement of a problem is translated from one language to another, some of the details may get lost in the translation. Perhaps this loss of information has an unexpected benefit; the simpler formulation of the problem in a new language might allow for new insights or intuitions to be gained, and perhaps even for a solution...(Stewart, Thompson, & Brady, 2017, p. 2264)

Creating analogies

Fischbein (1987) described an analogy between two entities as a "similarity of structure, a stock of common structured properties" (p. 127). He stated "Two objects, two systems are said to be analogical if, on the basis of a certain partial similarity, one feels entitled to assume that the respective entities are similiar in other respects as well" (p. 127). Creating analogies is thought to lead the mind to formulate an intuition as a "mini-theory" (Fischbein, 1987, p. 129). Experts analogies between different mathematical domains or systems may be difficult for students to comprehend or find useful. Clement (2008) has found that both experts and undergraduates produced a variety of analogies in physics and engineering tasks. He also found that analogies are not arrived at with ease, it takes a considerable amount of time and persistent effort to create them.

2.4.4 Classifying types of intuition

In previous sections, intuition was characterized as a method-process-outcome object along with boundary attributes of what intuition is versus what it is not from the dual perspective in psychology and the perspective mathematics education researchers. The assembly of a conceptual structure for mathematical intuition as a research construct is continued in this section and the classification stage is entered to explicate different types of intuition. A central issue brought up by psychologists is the common label of intuition is used in research, but researchers are not studying the same phenomenon under this label. Rather they investigate non-conscious processes with very different sometimes opposing attributes such as: divergent and convergent processing; processing that occurs within an incubation period and processing that occurs with no incubation period; and processing that produces an outcome with a high-intensity emotional response or low-intensity to no emotional response (Policastro, 1995; Dane & Pratt, 2007, 2009; Glöckner & Witteman, 2010; Gore & Sadler-Smith, 2011). Likewise, similar issues in mathematics education have also been raised. Bubp (2014) affirmed in the limitations of her thesis that "The limited research on intuition in mathematics indicates that there may be a variety of types of intuition that need to be distinguished along with differentiating intuition from analysis" (p. 243). Table 2.5 is a confirmed record of scattered types of mathematical intuition termed from 1911 to 2017 and the theoretical frames they are situated in.

Current trends that are present among researchers that investigate mathematical intuition includes: the label of 'intuition' is used but the meaning attached to this label is left ambiguous, intuition is equated with informal reasoning (Bubp, 2014), a new type of intuition is defined with some comparison among other types, or a new type is defined in isolation without a review of or comparison among other existing theoretical types. Additional types defined have provided an additional window into what mathematics educators mean by intuition. Looking through this window one sees that the label of intuition spans many qualitatively different cognitions: from self-evident axioms and "common-sense" judgements such as "every natural number has a successor" (Fischbein, 1982, p. 9) to "instantiations" as prototypical representations of a mathematical concept that come to mind quickly in response to a mathematical situation. "For example, an instantiation of a convergent sequence might consist of a graph of a 'prototypical' convergent sequence" that is experienced regularly in a mathematical environment (Weber & Alcock, 2004, p. 211); to a "correlation-hypothesis" which is intuition in the form of a mental boom across distinct fields and is a first glimpse into never before seen mathematics. It may take years and in some recorded cases decades for intuitions of this form to be coherently expressed, verified through proof and accepted by the mathematics community (Grosholz, 1978). Grosholz (1978) described this phenomenon:

> When two fields are put in correlation, it may happen that a new field or fields are generated at their interface. The correlation-hypothesis...makes possible the solution of problems which neither field on its own could have solved...ways of sharing information between the two fields are established, a new field, with hybrid items and problems, may crystallize around the correlation-hypothesis, explaining, extending, and warranting it (p. 56).

Different types of intuition are also being classified according to various mathematical subjects or topic areas. For example, geometric intuition in geometry (Fujita, Jones

& Yamamoto, 2004), intuition of infinity (Fischbein, Tirosh, & Hess, 1979), optimizing intuition for area and algorithm optimization problems (Malaspina & Font, 2010), probabilitic and combinatorial intuition (Fischbein & Grossman, 1997). In this section, classification is done with respect to method, process, and outcome attributes rather than classification that is done with respect to the mathematical subject or topic area that an intuition is experienced in.

Table 2.5: Types of mathematical intuition

Type	Paper	Theoretical Frame
Dynamic Intuition	Mariotti & Pedemonte (2019)	Cognitive Unity [Garuti et al., 1998]; Toulmin's Model [Toulmin, 2003]
Covariation intuition and correspondence intuition between sets	Wilkie (2019)	Covariational reasoning [Carlson et al., 2002]
Embodied intuition	Stewart, Thompson, & Brady (2017)	Three Worlds [Tall, 2013]
Intuitive paraphrase of axiom, definition, or theorem	Dawkins (2015)	RSP [Weber & Alcock, 2009]
"Naive"-errored	Thomas (2015)	Concept image/definition (Tall & Vinner, 1981)
Empirical and abstract intuition	Zagorianakos & Shvarts (2015)	Husserl (1970a)
Proof image	Kidron & Dreyfus (2014)	AiC+RBC (Kidron & Dreyfus, 2010)
Memory, property, similiarity, understanding visualization-based, example-based, and unjustified	Bubp (2014)	Dual-process Theory (Kahneman, 2011) Semantic/Syntactic Proof (Weber & Alcock, 2009)
Four-dimensional intuition	Davis, Hersh, & Marchisotto (2012)	N/A
Optimizing intuition	Malaspina & Font (2010)	Onto Semiotic Approach (Font & Contreras, 2008); Cognitive science of mathematics (Lakoff and Núñez, 2000)
Deep intuition	Semadeni (2008)	Concept image and definition (Tall & Vinner, 1981)
Technical-algebraic, visual-spatial intuition	Selden & Selden (2007)	Informed from Empirical Observation

Table 1.1 (Continued)

Type	Paper	Theoretical Frame
Geometrical intuition	Fujita, Jones & Yamamoto (2004) Davis, Hersh, & Marchisotto (2012)	Concept image/definition (Tall & Vinner, 1981)
Spatial intuition of number and number line	Raftopoulos (2002)	Object files (Kahneman, Treisman, & Gibbs, 1992)
Combinatorial intuition	Fischbein & Grossman (1997)	Schemata (Inhelder & Piaget, 1958)
Primary, secondary, affirmatory, conjectural, anticipatory, conclusive	Fischbein (1987)	*Intuition in Science and Mathematics Education [Fischbein, 1987]*
Affirmatory-semantic, Affirmatory-relational, Affirmatory-inferential	Fischbein (1987)	*Intuition in Science and Mathematics Education (Fischbein, 1987)*
Ground (common), individual (unique)	Fischbein (1987)	*Intuition in Science and Mathematics Education (Fischbein, 1987)*
Lay, expert	Fischbein (1987)	*Intuition in Science and Mathematics Education (Fischbein, 1987)*
Generalization by induction	Fischbein (1987)	Scientific Method
Probabilistic intuition	Fischbein (1975)	Stimulus Sampling Theory (Estes, Burke, Atkinson, & Frankmann, 1957)
Collective intuition	Wilder (1967)	Historical Cases
Axiomatic intuition	Young et al., 1911	N/A

In the field of Psychology, what has been included under the intuition construct has really been stretched. It has been suggested that non-conscious autonomic processes can range from simple retrieval of already acquired action scripts and automated actions to the complex generation of new mental representations (Glöckner & Witteman, 2010;

Policastro, 1995). The following subsections summarize four classes of non-conscious processes initially put forth by psychologists Glöckner and Witteman (2010): simple associative, matching, accumulative, and constructive and their corresponding outcomes. This section continues to synthesize the psychology and mathematics education literature by collecting relevant constructs and types from the mathematics education literature and mapping them into these four classes.

2.4.4.1 Simple associative

Simple associative forms of intuition include affective-bodily arousal, guiding feelings, and behavioral reactions (Glöckner & Witteman, 2010). Behavioral schemas, a construct studied in the mathematics education literature, was mapped into this section as it is most relevant to simple forms of intuition (Selden, McKee, & Selden, 2010). Simple forms are acquired and reinforced through more primitive modes of association learning such as implicit learning, signal learning, classical conditioning and operant conditioning. This section begins with somatic markers which are included under simple forms of intuition as affective-bodily arousal (Glöckner & Witteman, 2010).

Somatic markers play an important role in narrowing the option search space before you have any conscious awareness of decision guiding bodily-feelings or even that a decision needs to be made. Neuroscientist Damasio's (1996) theory of the somatic marker hypotheses was focused on innate neurobiological mechanisms that aid humans in decision-making. He was particular interested in a group of patients that were unable to make adaptive decisions leading to successful outcomes. He found that the inability to make productive decisions was not due to

> defects in (a) pertinent knowledge; (b) intellectual ability; (c) language; (d) basic working memory; or (e) basic attention (Damasio, 1996, p. 1414).

But rather the inability to make productive decisions was caused by damage to the ventro-medial prefrontal cortex of these patients. This led to evidence that the autonomic, automatic non-conscious, neural processes of the ventro-medial prefrontal cortex are vital to adaptive decision-making and that emotion plays a major role. Somatic markers are physiological bodily-emotion states linked to past learning experiences (Damasio, 1996).

> The hypothesis thus suggests that somatic markers normally help constrain the decision-making space by making that space manageable for logic-based, cost-benefit analyses...In the absence of a somatic marker, options

> and outcomes become virtually equalized and the process of choosing will depend entirely on logic operations over many option-outcome pairs. The strategy is necessarily slower and may fail to take into account previous experience (Damasio, 1996, p. 1415).

That is, before awareness that a decision needs to be made autonomic processes sort through these markers discarding ones linked to negative behavioral outcomes such as punishment and is likely to select those linked with positive outcomes such as rewards and ultimately guides behavior in an advantageous direction (Demasio, 1996). Without the activation of somatic markers due to neurological damage patients often become so overwhelmed with too many options to choose from that they could not make any decision at all. They would switch conscious attention erratically from option to option analyzing every aspect of each option without settling on any decision. This places enormous demands on attention and working memory capacity. Alternatively some patients would avoid the cognitive strain of analysis and make non-sensical random impulsive decisions. If every option is equalized then any random choice will do (Damasio, 1996).

The somatic marker hypothesis has gained interest in mathematics education research (Brown & Reid, 2006). Brown and Reid (2006) summarized this hypothesis stating:

> As we go through life some of our behaviours afford events that we experience as pleasurable. That experience changes our bodily structures in ways that mean that the behaviour becomes marked, so that in similar circumstances we are likely to behave in similar ways. Other events we experience as unpleasant, and then our bodily structure changes in ways that mean the behaviours we associate with those events are less likely to occur in the future (p. 180).

Demasio (1996) emphasized that somatic markers guide behavior in an advantageous direction (Demasio, 1996). However, Brown and Reid's (2006) work found that mathematics learners' somatic markers sometimes guide them in disadvantageous directions. For example, a learner named Frank, when given a mathematical task, had a tendency to grab for his calculator before even thinking consciously about the problem. When interviewed he stated that he does that when he gets panicked and even though he knows that he shouldn't first grab his calculator he does it anyway. Panic can be an indication of over activation of the amygdala specifically the anterior cingulate cortex. When the anterior cingulate cortex is overly activated our ability to identify dissonance between what the mind is prompting you to do (i.e. don't grab the calculator) and what the body (feelings) is prompting you to do (i.e., grab the calculator) shuts down and negative bodily-emotion over-rides the intuiting process and produces the disadvantageous outcome (Demasio, 1996).

Guiding bodily-emotions can either drive behavior without conscious awareness of feelings or feelings may be consciously registered. If guiding feelings are consciously registered then one has a sense of liking or disliking, familiarity or unfamiliarity, rightness or wrongness about a situation without the explicit mental representation of the factual knowledge experienced in the past (Damasio, 1996; Brown & Reid, 2006) and such feelings are low in intensity. That is, we may have a sense that something feels right or wrong but what we can't pin point exactly why it feels right or wrong. It has been recorded that undergraduate mathematics learners are sometimes aware of their own guiding feelings but more often they do not have guiding feelings. Mathematicians and more advanced learners seem to be more in-tune with these feelings than novices meaning they are more aware of their guiding feelings and they are more likely to investigate why they have occurred (Selden, Selden, & McKee, 2008). According to Reber (1989):

> To have an intuitive sense of what is right and proper, to have a vague feeling of the goal of an extended process of thought, to "get the point" without really being able to verbalize what it is that one has gotten, is to have gone through an implicit learning experience and have built up the requisite representative knowledge base to allow for such judgement (p. 233).

Reber (1989) summarizes the findings of several experiments that illustrated implicit learning, also called tacit knowledge acquisition. In these experiments implicit learning is viewed as a form of pattern and rule abstraction from a complex artificial language systems without awareness. Participants in these experiments based on responses show that they had learned rules of the language systems without explicit exposure to what the rules were or that rules even existed and without conscious reflection on the symbol strings they were exposed to. The underlying rules of the system remained outside the participants realm of awareness yet their action responses and predictions were not random in this system. After feeding participant symbol strings from the system, participants made responses and predictions consistent with the underlying rules of the system.

Another simple form of intuition which may or may not be mixed with guiding feelings are behavioral schemas. Behavioral schemas are "<situation,action>" pairs or situations which prompt an automated behavior. The automated behavior is completely reflexive and bypasses conscious awareness of an internally held mental representation. This means that internally held mental representation containing pieces of ones concept images or other knowledge structures are not expressed consciously. Furthermore, the automated behavior is often implemented without any conscious evaluation of other behavioral alternatives. These are called "habits of the mind" and are retrieved from procedural memory and difficult to change (Klein, 1993; Kahneman & Klein, 2009; Glöckner & Witteman, 2010; Selden, Mckee, & Selden, 2008, 2010; Lim & Selden, 2010, p. 1).

For example, if you are driving and see a green light you reflexively push the gas pedal without even thinking about it. A situation is made up of cues (i.e. stimuli in the environment) and we react to them. In the previous example the most important cue for prompting you to take your foot off the break and push down on your pedal was the green light. Other cues in the environment were not perceived as important for example trees that can be viewed out of the window. The intuiting process in a situation receives the cues and is attuned to what it perceives to be the most important and asks if these perceptual cues then what is the best action response. This is a matter of implicit propositional logic, 'If these cues then implement this automated action'. The non-conscious probabilistic calculations of the best action response is dependent on the frequencies of actions paired with the cues perceived to be important and reinforced in the past (Klein, 1993; Kahneman & Klein, 2009).

It is possible for a learner to develop an appropriate behavioral schema that had been reinforced over and over on past occasion for a particular type of task. But when the task slightly changes to a new task the same reinforced behavioral schema may be applied due to an overlap in perceptual or other sensorial cues across past tasks and the new task without detection of dissimilar cues. This phenomenon is termed stimulus generalization. On the other hand if there was a detection of dissimilar cues this is termed stimulus discrimination and the action associated with the cue sets in past tasks would not be taken (Pearce, 1994; McLaren, Forrest, & McLaren, 2012).

According to Selden and Selden (2013):

> behavioral schemas are always available-they do not have to be searched for or recalled. Behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action. One becomes aware of the resulting action of a behavioral schemas as it occurs or immediately afterwards (p. 3).

Behavioral schemas allow us to act quickly in a way that bypasses conscious awareness and uses little to no attentional and working memory resources (Simon & Chase, 1973; Prietula & Simon, 1989; Selden, Selden, & McKee, 2008, 2010). According to Skemp (1979):

> once a particular process has been mastered, it is a great advantage if it can be repeated on subsequent occasions without having to devote our fully conscious attention to it...the more difficult the task, the more do we need the help which comes from routinizing that which can be routinized, so that as much as possible of our consciousness is free to concentrate on the new and problematic aspects of the situation (p. 159).

In other words, cognitive resources are reserved for more novel aspects of a problem-solving situation. It is crucial that the learner be able to bring the conceptual meaning attached to the automated action if a situation arises in which one must correct or adapt his or her automated actions (Skemp, 1979). Correcting an inappropriate behavioral schema is very difficult because behavioral schemas are "habits of the mind" that are resistant to change (Lim & Selden, 2010, p. 1). Instructional applications should aim to increase the value and adaptiveness of behavioral schemas by: 1) reinforcing relational links between automated procedure and concepts 2) increasing the overlap between the cue set a student is attending to with the cues of a particular task that are most pertinent and 3) encouraging learners to be more aware of the presence or absence of their own guiding feelings and how it impacts their actions (Selden, Selden, & Mckee 2008; Selden & Selden, 2013). After studying literature related to simple associative intuition aggregates inside the class simple associative forms emerged.

A feature of simple associative forms that seems to distinguish it from other forms and bounds it as a class is that the outcome is not accompanied with awareness of internally held mental representations or images. In other words, the mental-representation is not brought to mind, it bypasses conscious awareness straight to a behavioral reaction. While psychology recognizes behavioral schemas as a form of intuition. It is not clear whether or not the mathematical community would include or exclude such forms from the class-concept of mathematical intuitions.

2.4.4.2 Matching associative

This type of intuition involves matching stimuli in a current situation to "exemplars, prototypes, images, and schemas" in order to make categorical judgements. For example, seeing a problem and categorizing it as this or that type of problem or situation; or

Table 2.6: Simple associative forms of intuition with respect to the process and outcome components.

Simple Associative Forms	Process-Outcome Description
Aggregate A	Intuiting Process: autonomic activation of somatic markers
	Intuition Outcome: physiological bodily-feeling states that guides behavior; may remain below the threshold of conscious awareness; or manifested as guiding feelings of rightness or wrongness are experienced ; or manifested as feelings of likeness or dislike
Aggregate B	Intuiting Process: processing of situational cues and activation of best fit behavioral schema
	Intuition Outcome: experienced as an immediate and automated behavioral reaction; may or may not be mixed with awareness of guiding feelings of rightness or wrongness and feelings of likeness or dislike

the current situation caused some previously learned exemplars or prototype to come to mind. This form of intuition involves matching stimuli in a current situation, often called a probe, to already acquired exemplars and prototypes held in multiple-trace memory (Glöckner & Witteman, 2010, p. 8; Sinclair, 2010). In the Minerva 2 simulation model for multiple-trace memory, memory is separated into primary memory (PM) and secondary memory (SM). Primary memory holds the current representation being attended to, this representation is a stimulus probe. Secondary memory holds partial recordings of previous experiences called “traces” and “...each experience with an object is separately stored in memory as a single trace. Intuition, in the sense of a feeling towards an option, is an “echo” that results from automatically comparing the current object or situation to all similar experiences stored in memory” (Glöckner & Witteman, 2010, p. 9). Traces stored in SM are activated if they share features of the probe in PM. Hintzman (1986) defined the similarity between a stored memory trace denoted $T(i,j)$ with fixed index i running across all $j = 1, \cdot n$ features and a current probe $P(j)$ running across all j features as $S(i) = (1/N_r) \sum_{j=1}^{n} P(j)T(i,j)$ and the intensity and content of an echo are functions that are dependent on the similarity $S(i)$ across all traces (p. 413-414). The echo is a consciously registered mental representation abstracted from multiple traces (Hintzman, 1986).

In the mathematics education literature, "Intuition in mathematical reasoning certainly includes the capacity to directly grasp patterns: different geometrical figures belonging to the same category; different problems related to the same mathematical structure and leading to analogous solutions" (Fischbein, 1990, p. 42). Seeing a problem and categorizing it as this or that type of problem requires familiarity and recognition of the currently presented problem in terms of certain information bits accumulated in the past. Carlson and Bloom (2005) found that this type of recognition is common during the "orientation stage" of mathematical problem solving, which is embedded in proving (Carlson & Bloom, 2005, p. 68). During the orientation stage, experts "spontaneously accessed their concepts, facts, and algorithms as needed to represent the problem situation. Their constructions were also aided by heuristics such as categorizing the problem as an X kind of problem and working backwards" (Carlson & Bloom, 2005, p. 68).

Selden and Selden (2013) observed that "An individual who has reflected on a number of problems is likely to have seen (perhaps tacitly) similarities between some of them. He or she might recognize (not necessarily explicitly or consciously) several overlapping problem situations, each arising from problems with similiar features" (p. 306). These features may or may not have labels or names that a learner has assigned to them, but they do evoke "tentative solution starts", certain examples, previously learned theorems, and an appraisal of how easy or difficult the problem is from a type of knowledge structure termed "a problem situation image" (Selden & Selden, 2013, p. 307).

Certain cues in a problem situation may also bring to mind "instantiations" as prototypical representations of a mathematical concept that come to mind quickly in response to a mathematical stimulus. Weber and Alcock (2004) defined an instantiation as a, "systematically repeatable way that an individual thinks about a mathematical object, which is internally meaningful to that individual" (p. 210). They found that algebraists approached problems first by intuitively feeling them out and classified their intuitive thoughts as instantiations. For example, when a problem involves an arbitrary convergent sequence a representative instantiation of a particular convergent sequence may come to mind. And two groups having the same multiplication tables after some ordering of the columns is an instantiation for group isomorphisms (Weber & Alcock, 2004). Instantiations are said to be part of one's concept image, which includes all acquired informal representatives of a mathematical concept and is contrasted with the concept definition which is formal definition of the concept (Weber & Alcock, 2004; Tall & Vinner, 1981). The concept image develops through various experiences and includes all, “mental pictures and associated properties and proesses” and “can be considered as part of intuition”. It is “evoked in our mind in an intuitive way. They are the immediate reaction of our mind to the concept name that we hear or see” (Vinner, 1997, p. 63-78). Weber & Alcock (2004) stated that “Effective intuitive understanding of a concept” is reached whenever the intuiter displays many accurate and related instantiations or intuitive representations of a mathematical concept, can translate these instantiations into formal language rapidly, and use them to support formal analytic reasoning (p. 229).

Table 2.7: Matching associative forms of intuition with respect to the process and outcome components.

Matching Forms	Process-Outcome Description
Aggregate C	Intuiting Process: matching of an abstract mathematical concept in a problem situation to prototypes, exemplars, or intantiations from one’s concept image
	Intuition Outcome: prototypical representatives and examples of a mathematical concept
Aggregate D-G	Intuiting Process: feature matching between current problem and memory traces of problems experienced in the past; retrieval of traces from situation image/multiple trace memory.
Aggregate D	Intuition Outcome: bringing to mind facts such as theorems or previously learned results that feel relevant
Aggregate E	Intuition Outcome: categorization of a problem as X type of problem
Aggregate F	Intuition Outcome: a tentative solution start; an automatic default approach to the problem at hand; the intuiter’s go-to response derived from their already acquired bag of tricks
Aggregate G	Intuition Outcome: an appraisal of the difficulty level of the problem at hand

2.4.4.3 Accumulative and constructive

Additional types of intuiting processes are accumulative and constructive processes. Accumulative processes add up evidence and cues to orient and guide which direction to take or when to make a turn. Constructive processes borrow from the information gathered by accumulative processes. Constructive processes are more gradual than simple associative or matching associative processes and samples from and integrates a wider range of memory stores, accumulated evidence over time, and newly formed links between knowledge structures (Hogarth, 2010; Glöckner & Witteman, 2010). Moreover, constructive processes are not merely a retrieval of already acquired knowledge that draws from the past and present. It allows for simulations of future scenarios and predictions of what may be possible, but is not yet known to be possible. As constructive processes continue "relevant pieces of information are fitted into the 'solution picture' in a seemingly haphazard way, similar to assembling a jigsaw puzzle. When the assembled pieces start making sense, the big picture suddenly appears, frequently accompanied by a feeling of certitude or relief" (Sinclair & Ashkanasy, 2005, p. 357). In mathematics education terminology, constructive intuiting process translates to a gradual sampling of matching-associative forms of intuition such as recognition of relevant already acquired situation images and concept images that are integrated with accumulated evidence and newly constructed knowledge.

A prediction of what may be possible is akin to a guess or conjectural intuition and occur prior to the proof image. After many episodes of conscious work and non-conscious work during breaks a the "proof image" is formed in the mind. It is accompanied with feelings of enlightenment;"When learners have a proof image, they are fully confident that they can produce the detailed steps of the proof according to the image they have" (Kidron & Dreyfus, 2014, p. 304). The proof image comes into conscious awareness slowly initially as a faint mental representation and later as fully coherent image of the proof as a solution held in the mind. The proof image precedes the externalization and transformation of the total intuitive representation into a written formal deductive proof product. "Intuition might have been gained from examples but the entity characteristic of a proof image implies a complete image of the proof rather than specific instantiations of the mathematical object being explored..."(p. 304).

In other words, many intuitions, of different types, are gradually integrated into a summative "intuitive representation" to form a proof image. In case studies with mathematicians, Kidron and Dreyfus (2014) used the *Nested Epistemic Model for Abstraction in Context* (Hershkoqitz, Schwarz, & Dreyfus, 2001) to frame three epistemic actions *recognition*, *building with* and *constructing* to organize each mathematician's multiple construction episodes during their research towards a proof image. They found that, recognition of old constructs drives a motivation or need to construct something new. Emergence occurs when the learner experiences some degree of awareness of a new construct which is not fully coherent and, "is often fragile and context dependent" (p. 4). There is a shift towards increased coherency when a new construct is applied as a

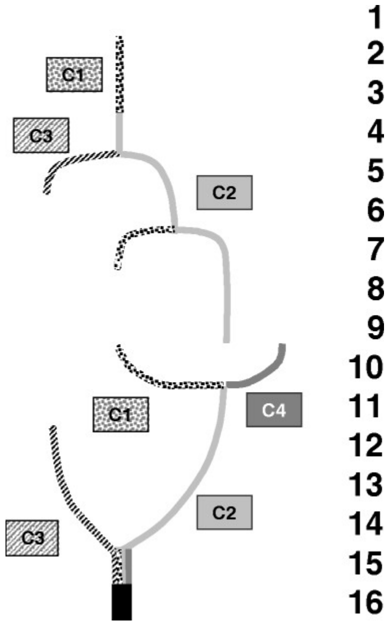


Figure 2.4: A mathematicians' construction episodes and proof image (Kidron & Dreyfus, 2014, p. 313).

"building block" in other situations for which it is relevant (p. 11). After repeated use of the new construct to "build-with" the construct may become consolidated in memory. Signs that a new construct has been consolidated include: immediacy, self-evidence, confidence, flexibility, and awareness (Dreyfus and Tsamir, 2004). "Constructing (C) consists of assembling and integrating previous constructs in a specific context in such a way that a new (to the intuiter) construct emerges" (Kidron & Dreyfus, 2014, p. 308). Figure 2.4 represents a mathematician's constructions through 16 episodes of work on a proof (p. 59).

For constructions labeled C_1, C_2, C_3 and C_4 were linked in a non-linear fashion. For example, in episode 10 a new construction emerged C_4 and the mathematician recovered C_1 and estabilished a new link. At this point the mathematician reported feelings of "enlightenment" and a sense of "completeness" even though only a portion of the image had been formed (Kidron & Dreyfus, 2014, p. 314). Another way of phrasing this is in terms of a psychological embedding. In mathematics, an object X is embedded in another object Y if there is some structure preserving map $f : X \hookrightarrow Y$. The object $C_1 - C_4$ and C_2 was embedded in another object $C_1 - C_4 - C_2$ in which the initial structure was preserved and some some structure, a link, was added.

After a formal proof product is completed and the prover may experience aftershocks called "conclusive intuitions" (Fischbein, 1987, p. 117). Conclusive intuitions offer a compact summary of the key ideas of a solution mixed with new interpretations of the behavior of certain mathematical objects, additional underyling forces that are driving the previously proved result, or how it may relate to other mathematical phenomenon (Fischbein, 1987). Conclusive intuitions are like hindsight after the proving event has occurred. This hindsight brings to light something that was missed, leads to a deepened appreciation, and perhaps new ideas to pursue. It seems that a certain level of intrinsic motivation and interest are needed to experience such aftershocks. To date, no studies in the Endnote library of 852 references have explicitly investigated ‘conclusive intuitions’ in undergraduate learners or what factors may contribute to presence or absence of this type of intuition.

2.4.4.4 Non-creative versus creative forms of intuition

The previously covered forms of intuition: simple associative, matching, accumulative, and constructive can be roughly collapsed to define two more general classes of intuition that differ in the non-conscious processing, incubation periods, and emotional responses that co-occur with an intuition outcome (Dane & Pratt, 2007, 2009; Gore & Sadler-Smith, 2011; Glöckner & Witteman, 2010). These two classes are non-creative and creative intuition which represents the coarsest theoretical classification of intuition found in this integrative review from the exploratory Psychology literature sample.

Non-Creative Intuition: this form of intuition can be acquired through repeated practice and “involves a process whereby current situations are viewed in terms of their similarity or differences with past experiences” and includes simple-associative and matching-associative forms of intuition (Dane & Pratt, 2007, 2009, p. 6). Based on Dane and Pratt (2007, 2009) work, psychologists Gore and Sadler-Smith (2011) described problem solving (non-creative) intuition as:

> ...a response to a tightly structured problem based on non-conscious processing information, activated automatically, eliciting matching of complex patterns of multiple cues against previously acquired prototypes and scripts held in long-term memory (p. 308, 2011).

A defining feature of non-creative intuition is that the intuiting process is, “reproductive rather than productive; it uses currently available cues to retrieve representations stored on past occasions when similar cues were present (Smith & DeCoster, 1999, p. 328).” Non-creative intuition encompasses early characterizations of intuition made by Simon (1992):

> The situation has provided a cue: This cue has given the expert access to

> information stored in memory, and the information provides the answer. Intuition is nothing more and nothing less than recognition (p. 155).

Another description that coincided with more non-creative forms was given by Kahneman (2003):

> Thoughts and preferences that come to mind quickly without much reflection (p. 697).

Kahneman and Klein (2009) posed three conditions that are necessary for this form of intuition to be "skilled" : 1) "the environment must provide adequately valid cues to the nature of the situation", 2) the learner must have the opportunity to notice and achieve awareness of what the salient cues are, and 3) there has to be an appropriate level of regularity or consistency between experiences so that the mind can identify patterns in the environment (p. 520). Identifying cues that prompts experts intuition outcomes are of considerable interest to psychologists who want to unveil the hidden source of experts intuition. If the cues that experts intuiting processes are attuned to can be identified then they may be able to take steps towards instruction that allows others to become attuned to these pertinent cues.

Creative Intuition: Psychologist Policastro (1995) contrasted non-creative and creative intuition. She stated that:

> Implicit learning and certain forms of intuition seem to operate as tacit pattern recognition, among experiences that have a similar structure. On the other hand, creative intuition seems to operate as a form of tacit pattern generation, entailing the organization of novel structure (p. 110).

Creative intuition involves combining information in novel ways to create new formations (Dane & Pratt, 2007, 2009; Eubanks, Murphy, & Mumford, 2010). A similar interpretation is given by Poincaré:

> If I may be permitted a crude comparison, let us represent future elements of our combinations as something resembling Epicurus's hooked atoms. When the mind is in complete repose these atoms are immovable; they are, so to speak, attached to the wall.... On the other hand, during a period of unconscious work, some of them are detached from the wall and set in motion. They plough through space in all directions, like a swarm of gnats, for instance, or, if we prefer a more learned comparison, like the gaseous molecules in the kinetic theory of gases. Their mutual impacts

> may then produce new combinations. What is the part to be played by the preliminary conscious work? Clearly it is to liberate some of these atoms, to detach them from the wall and set them in motion. We think we have accomplished nothing, when we have stirred up the elements in a thousand different ways to try to arrange them...(Poincaré, 1914, p. 61).

2.4.4.5 A set of attributes that separates non-creative and creative forms of intuition

The non-creative and creative intuition distinction was estabilished by psychologists Dane and Pratt (2007, 2009) with early work by Policastro (1995) that signaled the need for this distinction. Policastro (1995) drew from autobiographies, history, psychometric assessments and experimental studies to argue that creative intuition should be regarded as a viable research construct. Dane and Pratt pointed out that a defining feature of non-creative forms is that the intuition outcome experienced by the intuiter occurs rapidly after exposure to associated cues with little to no incubation period. Moreover, if the outcome is mixed with a local affective emotional response, the level of intensity of this emotional response is low. On the other hand, creative intuition is delayed intuition that comes after a lengthy amounts of deliberation and incubation. It is important to note that creative and non-creative classes are general classes meaning they are not specific to a particular field or domain. General psychological frameworks do not include more domain-specific aspects of these two classes. Domain-specific aspects need to be clarified (Dane & Pratt, 2007, 2009; Sadler & Smith, 2011).

Although many characterizations of intuition and findings in the mathematics education literature align with the creative and non-creative distinction, this distinction has not been explicated in the mathematics education literature. Out of the entire mathematics education sample, 781 journal papers plus the exploratory collection, only one paper (0.128%), Krummheuer (2007), mentioned 'creative intuition'. It was mentioned a single time without a description or citation for the term and was the only time intuition was mentioned in the entire paper. A slightly more frequent occurrence of incubation was found. 10 journal papers (1.28 %) discussed incubation, all within the context of creativity research (Savic, 2015; Sriraman, Haavold, & Lee, 2013).

The mathematics education literature will now be mapped into the general non-creative and creative classification frame. While integrating the psychology and mathematics education literature 7 attributes that separate non-creative and creative forms of intuition emerged: personal novelty, incubation level, emotional valence and intensity, ease of externalization, use of sanctioned or non-sanctioned semiotic representations to produce cues prior to an intuition outcome, the presence of sanctioned or non-sanctioned semiotic representations contained in the externalized intuition outcome, and network thinking. To the best of my knowledge, this is the first time that these 7 attributes have been combined in order to characterize non-creative and creative forms of intuition for use in mathematics education research.

Personal novelty. Personal novelty refers to the degree to which an intuiter feels like their intuition is something that they have or have not experienced or thought about before. Non-creative forms of intuition are associated with simple-associative and matching associative intuiting processes. These type of processes match stimuli in the environment to already acquired action scripts, prototypes and exemplars stored in long-term memory resulting in an intuition outcome that feels familiar due to recovery of old information. And this is contrasted with creative forms of intuition in which the outcome feels novel to the intuiter like something they have not experienced before and along with strong shifts in conceptions (Policastro, 1995; Glöckner & Witteman, 2010). Creative intuition is "...synthesis in which disparate elements are fused together in novel combinations" (Dane & Pratt, 2009, p. 9). Aside from intuition, the mathematics education literature associates mathematical creativity with novelty (Sriraman, 2009).

Incubation period. The roles of incubation in the creative process of discovery, invention, and decision-making has been described by various theories in psychology and summarized in the existing literature. Some of these key psychological theories that describe the roles of incubation include "unconscious and conscious work theory", "remote association", "fatigue recovery", and "opportunistic" assimilation (Hélie & Sun, 2010, p. 996-997; Gilhooly, 2016). According to remote association theory often when a decision making task or problem is first presented, the intuiter matches features of the presented problem to a repository in long-term memory that contain problems with similiar features that the intuiter has found solutions to in the past. This is a matter of simple and matching associative processes. However, these initial associative processes ignore features in the current problem situation that may be different and thus require a new approach to obtain a solution. The incubation period reduces the activation of similarity-based associations and allows for new approaches that steer 'off the beaten path' to formulate in the non-conscious mind and eventually reach conscious awareness.

A defining feature of creative intuition is that the outcome occurs after long periods of conscious work on a problem and an incubation period, a period whenever conscious work directed on the problem at hand is put on pause (Dane & Pratt, 2009;

Hélie & Sun, 2010). Thus far, incubation periods have been measured in terms of length of time or occurrence characterization in the psychology literature to separate non-creative and creative forms of intuition: little to no incubation period for problem non-creative and moderate to lengthy for creative intuition (Dane & Pratt, 2007, 2009).

Accounts of incubation in mathematics research have been documented by Hadamard (1954) and investigated further by Savic (2015) through case studies. In the mathematics education creativity literature and in the context of mathematical proving, Savic (2015) defined incubation to be "a period of time, following an attempt to construct a least part of a proof, during which similiar activity (e.g., work on that same proof) does not occur" (p. 9). Results from his study indicated that 6 out of 9 mathematicians were cognizant of when to enter a period of incubation and get over episodes of being stuck. Savic argued that the length of the incubation period is not a sufficient condition for creativity, and viewed that an intentional act of taking a break to let one's ideas simmer was more of a marker of creativity than incubation period length. He stated:

> What matters is not the exact length of time, or the discovery of an error, but the prover's awareness that the argument has not been progressing and requires a new direction or new ideas that are not forthcoming (p. 8).

Emotional valence and intensity. Another attribute of a creative intuition, is that the outcome is mixed with high emotional intensity and is positively valenced (Dane & Pratt, 2007, 2009). This is consistent with Hadamard's (1954) case studies of mathematicians' process of discovery. These emotionally charged mental representations occurred during a stage of creative work termed "Illumination", defined as "the appearance of the 'happy idea' together with the psychological events which immediately preceded and accompanied that appearance" of an idea or solution (Wallas, 1926, p. 80). Similarly, Kidron and Dreyfus (2014) found that a mathematician experienced feeling of satisfaction and enlightenment after linking two disjoint proof construction episodes and arriving at a proof image. Unlike Creative forms, Non-creative forms tend to be associated with matter-of-fact non-emotional reactions (Dane & Pratt, 2009; Gore & Sadler-Smith, 2011).

Ease of externalization. Non-creative forms of intuition are also quicker to externalize and mechanize into common formal language, as in the case of matching associative forms in which the intuition outcome contains a prototypical representation that the intuiter has already acquired the language to express it. In contrast, creative intuition is often murky and additional indicators of creative form are that they are difficult to coherently externalize (Policastro, 1995). It is stated that ”metaphors, theoretical hunches, and syncretic sketches might represent attempts at articulating creative intuitions into plausible descriptions. These primitive representations...prefigure the beginnings of a complex problem-solving stage, because they require exhaustive elaborations (over long periods of time) in order to become fully articulated into valuable final products” (Policastro, 1995, p. 102). Evidence of the difficulty to articulate mathematical intuition was found in mathematics education studies. Kidron and Dreyfus (2014) found that,

> People may have a proof image in their mind but be unable to write it down or explain it in words to somebody else or even to themselves; a proof image may be definite but transitory and disappear again, and the person may be worried about that possibility (p. 314-315).

Presence of sanctioned or non-sanctioned semiotic representations contained in intuition outcome. When an intuitive representation or intuition experienced by the prover has never been experienced before it is like a foreign entity and figuring out how to express it through any form of language can be extremely difficult and slow (Kidron & Dreyfus, 2014). It is inferred that when creative intuitions are externalized that these externalization would predominantly contain non-sanctioned semiotic representations in the form of metaphors or drawings where as non-creative forms may predominantly contain sanctioned representations. The non-sanctioned and sanctioned distinction was given by diSessa (2004).

Use of sanctioned or non-sanctioned semiotic representations to produce cues. When reasoning with mathematical objects, the intuiter may draw their intuition from socially sanctioned representations that are commonly taught as part of a curriculum within a particular culture (i.e., matrices, graphs, objects of symmetry, etc.). The intuiter may also independently re-invent sanctioned representations that have not been taught to them by others or invent non-sanctioned representations that have yet to be adopted by a particular mathematical community as a culturally agreed upon way to think about an object. For example, Cayley tables were first publicly introduced by Arthur Cayley (1854). Cayley is considered to be an inventor of cayley tables which he used to think about groups as a collection of permutation functions under composition. For cayley this would have been considered a non-sanctioned representation at the time. For mathematics learners today that have had exposure to cayley tables, their adopted use of a cayley table would be considered sanctioned representation use. Young diagrams also come to mind, as a representation that has become sanctioned and the use of them is considered a part of common practices within certain mathematical cultures or areas of research.

Attempts to stimulate the intuiting process prior to awareness of an intuition outcome may involve more non-sanctioned representation use or independent re-invention of sanction representations without prior exposure in the case of more creative forms. diSessa et al. (1991) viewed student's reinvention of conventional sanctioned representations as "genuine and creative work" (p. 117). On the other hand, cue production through sanctioned representations, that the intuiter has already been taught how to use, is more indicative of non-creative forms.

According to the Creativity-in-Progress Rubric (CPR), a general creativity assessment in undergraduate mathematics, creative gains are marked by the learner being able to reason with multiple representations and relating multiple representations (Savic, Karakok, Tang, El Turkey, & Naccarato, 2017). The use of multiple representations and relating these representation prior to an intuition outcome may also be a marker of more creative forms of intuition. At this time, the CPR does not incorporate the non-sanctioned or sanctioned distinction nor specify any conditions for creativity with respect to representation type in terms of representations that are either directly acquired through enculturation, invented, or re-invented.

Network thinking. In a comparison review of psychologist E. Toulouse's extensive case study of Henri Poincaré, Gestalt psychologist Max Wertheimer's examination of the development of Einstein's theory of relativity, Miller (1992) noted that Poincaré and Einstein engaged in a process termed "network thinking" (p. 386). "In network thinking, concepts from apparently disparate disciplines are combined by proper choice of a mental image or metaphor...This nonlinear thought process can occur unconsciously and is not necessarily in real time" (Miller, 1992, p. 386). Furthermore, a creative intuition outcome may prompt divergent thinking that leads one to generate multiple routes that may lead to a solution rather than convergent thinking that is fixed towards a single route (Policastro, 1995; Dane & Pratt, 2009).

2.4.4.6 Additional attributes

Aside from the previously covered major classes of intuition and the variable attributes associated with them, researchers may also isolate a single attribute or subset of attributes of interest. This section covers additional attributes that were collected and isolated: unique versus common, errored versus non-errored, error type, the level of confidence in truth value, and obviousness. At this time, we consider these attributes to be independent of the attributes that separate the non-creative and creative classes of intuition. Unique versus common can be viewed as binary attribute with two values that an intuition can take on. The researcher must determine a cut-off for what frequency of a particular intuition within a specified population is to be considered common or unique. The researcher may also place a collection of intuitions on a spectrum from unique to common. Errored, non-errored, or unknown is first viewed as a ternary attribute, however one may go further inside of the class of errored intuition and assign additional values for error types which partitions the class or errored intuitions further.

Unique versus common. Intuitions that are common among members of a mathematical culture reflect shared experiences. On the other hand, intuitions that are unique within a culture reflect an individual's personal experiences that are deviations from the cultural norm (Wilder, 1967; Witteman, 1981; Fischbein, 1987; Thurston, 1994). For example, Weber and Alcock (2004) asked mathematicians, "Describe for me in your own words what it means intuitively for two groups to be isomorphic?" Responses to the second question were consistent among algebraists who said, "same structure", "algebraically the same", and the same up to re-labeling (p. 217). This is a shared form of intuition common among algebraists today. Several historical examples of common collective and unique intuitions were contrasted by Wilder (1967). In one such example he recalls a time when a majority of the mathematical community believed that a real continuous function that was no where differentiable could not exist, a case of a common errored intuition. Wilder (1967) stated, "...probably almost every mathematician felt intuitively that such a function could not exist; this intuition had become a cultural attitude, a common belief" (p. 606). However, Weierstrass did not accept this common delusion and gave an example of a function that was real continuous and no where differentiable. Wilder (1967) concluded that, "...there can be expected to exist a kind of intuition that is common to most members of the mathematical community. But as soon as one goes beyond these concepts to mathematical specialities-particularly to their frontiers-then the intuition becomes a quite individual affair; and it is this intuition that is of immediate importance in creative work" (p. 606). Based on this statement, it may be inferred that the attribute of unique within certain cultures is highly correlated with attributes in the creative intuition cluster, however this has not been statistically verified or refuted.

Errored versus non-errored and error types. An errored intuition can be further typified according to the type of process error. Types of process errors termed "Systematic intuitive errors", intuitive errors that are pervasive across populations, have been organized in the field of psychology. Big umbrella error types are: *attribute substi-*

tution and *relevance errors* (Kahneman & Frederick, 2002). An attribute substitution error is said to occur when a attribute of a current problem or object is substituted with an attribute that is more accessible to the mind. A relevance error occurs when relevant cues within a problem solving environment are ignored or when irrelevant features are of major focus (Kahneman & Frederick, 2002). Bubp (2014) adapted this typification of systematic intuitive errors, from the angle of dual-process theory, to analyze the type of errors mathematics students make on prove or disprove tasks. One such task was the "Monotonicity Task". This task asked participants to:

> Prove or disprove: If $f : R \longrightarrow R$ and $g : R \longrightarrow R$ are decreasing on the interval I, then the composite function $f \circ g$ is increasing on I (p. 77).

Within a participant sample size of 12 undergraduates, that included mathematics, mathematics education, and economics majors, Bubp (2014) found that: 8 students made relevance errors on the monotonicity task and 3 students were able to detect and recover from relevance errors. In particular, they were not attuned to the interval restriction as being an important piece and ignored it as if it was irrelevant. The interval restriction is needed to recognize and cook up a counterexample in order to correctly assert that the statement is false. Some students also committed the attribute substitution error that "two negatives make a positive" applies to functions so that any two decreasing functions make an increasing function. The attribute that "two negatives make a positive" for numbers was substituted as an attribute of functions in the monotonicity task by 7 students. 4 of the 3 students were able to overcome this error. Overall, many students' analytic Type 2 processes accepted their initial and errored intuitive Type 1 response without any further evaluation, signaling of an error, or correction from Type 2 processes. Instead, students analytic Type 2 processes accepted the intuition as true and were used to form deductive arguments that backed up the errored intuition (Bubp, 2014).

Leron and Hazzan (2009) also applied dual-process theory to study mathematics students' intuitive errors. They gave 113 university level computer science majors the prompt: " A student wrote in an exam, Z_3 is a subgroup of Z_6. In your opinion, is the statement true, partially true, or false? Please explain your answer" (p. 268). A desirable logical deductive response to this answer would be:

> The 'converse' of Lagrange's theorem stated as, if d is a divisor of the order of a group G then there exists a subgroup H of order d in G. This converse holds for special cases like cyclic groups. Since 3 is a divisor of

> Z_6 then there exists a subgroup of order 3 in Z_6, namely $\{0, 2, 4\}$ and Z_3 is isomorphic to the subgroup $\{0, 2, 4\}$ in Z_6.

However, 73 gave an incorrect answer and 27% of these students gave the quick response that the statement in the prompt is true, 3 divides 6. Such an intuitive error without logical deduction is reminiscent of the baseball and bat example studied by Kahneman & Fredrick (2002). Leron and Hazzan's (2009) findings were re-examined by Melhuish's (2018) replication study with several different framings of the original prompt.

Overall, it is important that researchers continue to try to understand why an errored intuition has occurred, typify Type 1 process errors further, and be able to diagnose which process error was involved in order to identify and more efficiently address students' needs (Bubp, 2014). Investigations of students' intuitive errors are also tied to an area of research that is primarily concerned with the development of students' "inhibitory control mechanisms" which are reflective or analytic strategies that encourage students to be skeptics of and their intuitions. How instructors approach students' intuitive errors or "misconceptions" through classroom instruction is still open for debate (Adiredja, 2018, p. 60; Thomas, 2015, p. 873; Attridge & Inglis, 2015). Thomas (2015) stated that "It seems that intuitive thinking is common place in mathematics and while often invaluable, it can sometimes lead to erroneous ideas, and hence the need for inhibition" (p. 867). Adiredja (2018) made the case that there is a tendency for cognitive researchers to negatively frame student's intuitive thinking using terms such as 'naive', 'inability', and 'unsatisfactory understanding' rather than seeing such instances as an opportunity for growth (p. 60). Formal mathematical knowledge is often viewed as "power" while students' preformal ideas and intuitions are viewed as weak by some researchers. Adiredja (2018) refered to materializations of this view in the literature as "deficit narratives" (p. 60).

Confidence in truth value and obviousness. Fischbein, Tirosh, and Melamed (1981) conducted an empirical study that combined the degree of confidence C and the degree of obviousness O to give a measure of intuitive acceptance I as a geometric mean $I = \sqrt{C \times O}$. In this study they ask students in grades 8 and 9 to solve 8 problems, 7 of these 8 problems were directly tied to the concept of infinity. First participants were told to solve the problem and explain their solutions. For each of the solutions, the students answered 3 yes or no questions for the confidence. one such question was "Have you doubts with the correctness of your answer" and 3 questions for obviousness, one such question was "Did you need some effort in order to completely agree with your answer"? A Guttman scale was used since these questions had a hierarchical structure in which the questions were increasingly more specific meaning the questions determined how extreme their confidence and obviousness with respect to their solutions. The scale had a 0 to 6 score range for the variable confidence and a 0 to 6 score range for obviousness. For each problem, the responses were grouped together if they were the same. Next, the mean confidence and mean obviousness score was calculated for each type of response across participants. Finally, the intuitive acceptance of each of the solutions was computed by the research by taking the geometric mean of the average confidence and obviousness scores. The results indicated that there were 3 kinds of problems in the 8 problem sample: (1) problems that had high correct solutions and high intuitive acceptance, (2) problems that had intuitive acceptance close to 3 and with almost equal correct and incorrect solutions, and (3) problems that had a high frequency of incorrect solutions with high intuitive acceptance of a solution (Fischbein, et al., 1981).

2.5 Instructional environments that may enhance the development of mathematical intuition

This section covers instructional environments that may enhance the development of students' mathematical intuition at the university level compared to traditional lecture based environments based on theoretical assumptions. There are many, not exactly well-defined, ways to think about what it means to enhance the overall development of mathematical intuition. To enhance could mean: to increase the frequency of creative intuitions, to increases the occurrences in which an intuition with fixed content in the outcome passes from a creative to non-creative state, reduce the frequency of non-creative forms of intuitions that are errored and accepted by the intuiter as true without self-correction, to improve representational fluency (i.e., increased size in fluency digraphs and reduction in the number of pseudo-conversions, subsection 4.2.3), and the use of counterexamples to build and refine intuition, to increase network thinking across mathematical systems, to improve affective factors such as intrinsic motivation or enjoyment, and to reduce negative affective impacts such as anxiety, fear of being wrong, and learning to manage strong feelings of certainty that dissuade work towards a proof.

2.5.1 Realistic mathematics education (RME) and guided reinvention

Realistic mathematics education (RME) is an instructional design theory. One key principal of RME is "...to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Doorman, 1999). A second principle is to frame mathematics in real life contexts so that the student can see how it might be useful to them. RME supports guided reinvention and opposes "anti-didactical inversion", a routine behavior of the instructor to start their lessons with a refined end result such as axioms, formal definitions, and theorems that took many mathematicians many years to reach (Gravemeijer & Doorman, 1999, p. 116). Guided reinvention is an instruction format where activities are designed to guide students to reinvent important mathematical definitions, concepts, and results via personal explorations and interactions with peers. RME and guided reinvention principles are implemented in: Lakatos' style of instruction, the project method of teaching, and inquiry-oriented instruction.

2.5.2 Lakatosian style of instruction

Lakatos' (1976) text *Proofs and Refutations* advocated for mathematics learning environments to be set up in a way that gave students the opportunity to experiment with mathematical objects, to follow their curiosities, make guesses and use their intuition to arrive at conjectures peaked by their own interests (Lakatos, 1963; 1976). His pedagogical approach to proof, is not what is meant by proof in the formal product sense, but rather it is viewed as a process in which counterexample generation serves a major role. Lakatos (1963) viewed proof as: "a rough thought experiment or a quasi-experiment which suggests a decomposition of the original conjecture into sub conjectures or lemmas, thus embedding it in a quite distant body of knowledge" (p. 10).

To illustrate how the above definition of proof might play out in the classroom, Lakatos (1976) provided a fiction of interactions between a teacher and students. These interactions followed a historical trajectory of mathematical problems like the classification of polyhedra problem posed by Euler (1758a) and the evolution of this problem in papers such as Abel (1809), Cayley (1861), Lhuilier (1890b), and Jonquieres (1890a, 1890b), to name a few. In this interaction the teacher poses the problem of whether or not their is a relationship between the number of vertices, edges, and faces of a regular polyhedra. By experimenting with several examples students come to a starting conjecture that $V - E + F = 2$. Next, students try to refute this conjecture with counterexamples but it seems that the conjecture still holds true. Students are given space to try to prove the conjecture, but the following day in class no one gives a proof. So the teacher presents a visual proof for a special case, namely the tetrahedron and inserts "rubber-sheet geometry" which inserts hidden assumptions that certain properties of spaces are not affected by continuous deformations (Lakatos, 1976, p. 7). While one student is quick to accept the visual proof for the example case as a general proof of the conjecture, several students are cautious and questioned the validity for some of the steps in the proof for all cases. This led to the formulation of two additional conjectures that would need to be proved as lemmas to go from a proof of a special case to the general case.

Lakatos (1976) categorized counterexamples according to whether they actually falsify a conjecture or are "pathological" counter-examples that signal an ill-defined underlying definition that is lurking around. "Pathological" counter-examples are called "monsters" (p. 14). Students in Lakatos' fiction were not told the formal and refined classifications of polyhedra that is known today. This was an intentional pedagogical decision made by the teacher which aligns with an opposition to anti-didactical inversion, a principle of RME defined earlier. The intentional masking of a clear or refined definition of polyhedra gave students space to wrestle with the ambiguity (Lakatos, 1976, p. 14, 41). Larsen and Zandieh (2008) connected guided reinvention from RME and Lakatos pedagogical approach to proof. They extracted methods of proof from Lakatos' text and gave a condensed and tractable frame that mathematics education researchers could use to examine student thinking during proof construction activities. These methods included "monster-barring" and "exception-barring" (Lakatos, 1976, p. 14, 41).

Monster-barring is the process of getting rid of monsters to the main conjecture by modifying definitions. In Lakatos' fiction students erupted in a turbulent cycle of generating a monster, polyhedra that did not have Euler characteristic 2, and getting rid of the monster by making a perturbation to their definition of polyhedra, generating another monster and making another perturbation, and so on until they move closer to the modern day notion of convex polyhedra as polyhedra that can be 'pumped into a ball', more formally polyhedra that are homeomorphic to a ball.

Exception-barring is an additional tactic of coming up with a collection of counterexamples to a conjecture and making restrictions to the assumption clause of a conjecture. Restrictions are made so that this list of examples are not counterexamples to the newly adjusted conjecture (Lakatos, 1976). Larsen and Zandieh (2008) defined exception-barring to include, "any response that results in a modification of the conjecture to exclude a counterexample without reference to the proof...from simply listing counterexamples as exceptions, to reformulating the conjecture by restricting its domain to exclude the counterexample" (p. 208). The difference between monster-barring and exception-barring is that the focus of modification on monster-barring is a definition and the other is concerned with adding qualifiers in the assumption clause of the conjecture (Larsen & Zandieh, 2008).

2.5.3 Project method of teaching

Kaisari and Patronis illustrated the "project method" at the university level in which students "work on the same theme continuously during the semester and discuss their work as a team, under the teacher-researcher's guide" (p. 254). It allows for students to experience immersion and getting wrapped up in a big problem for an extended period much like Lakatos (1967) approach. The project method begins with an introduction to a starting "theme" to set the scene for a "main theme" (p. 258). The starting theme was geometry of the earth's surface. Students were given texts such as La Bela by Denis Guedj, that mixes relatable stories of navigation with mathematics problems of historical importance. The main theme was to draw analogies between euclidean geometry and elliptic and double elliptic geometry. Students transformed definitions of points, lines, and planes from euclidean geometry to spherical geometry. Next, they transformed Hilbert's axioms for euclidean geometry into axioms for elliptic and double elliptic geometry for the surface of a sphere.

Kaisari and Patronis (2010) stated criteria for the main theme. One criteria is that the theme needs to relate or dissociate familiar elementary mathematics, such as entry calculus or euclidean geometry, to more advanced mathematical systems. This may help to highlight habitual thinking from school geometry that leads to rigid thinking and errors in more advanced abstract settings. Another criteria for a chosen theme is that it should give rise to multiple ideas about what might be going or what the meaning of an object is in different systems. For example, the meaning of a line in planar geometry versus the meaning of a line on the surface of a sphere, or interpretations of axioms in one system that either translate or don't translate to another system, and theorems that are known in one system but fail to hold in another system.

2.5.4 Inquiry-oriented instruction

Inquiry-Oriented Instruction (IOI) has been developed for: Introductory Abstract Algebra (IOAA), Linear Algebra (IOLA), and Differential Equations (IODE) (Wawro et al., 2012; Larsen, Johnson, & Bartlo, 2013; Rasmussen et al., 2006; Keene, Hall, &

Duca, 2014). The design of IOI is based on the principles realistic mathematics education (RME) and implements guided reinvention activities. The following is a brief description of this curriculum from members of the IOAA group:

> Each unit begins with a reinvention phase in which students develop concepts based on their intuitions, informal strategies, and prior knowledge. The end product of the reinvention phase is a formal definition (or definitions) constructed by students and a collection of conjectures... (Lockwood, Johnson, & Larsen, 2013, p. 777).

The development of guided reinvention instructional activities towards a curriculum is labor intensive. It is not practical for an individual instructor to develop them from scratch, test them, and work with mathematicians to refine how the actually implementation of them works. Research teams, for example under IOAA, have gone through several stages over an 11 year-period, from 2002 to 2013, to develop curriculum materials as well as guidelines they refer to as “instructor support materials” for using these activities in the classroom (p. 778). These “support materials”, based on research findings, inform instructors of how student’s reason during these activities, the possible routes they take, along with points to watch out for that students struggled with.

Based on a culmination of the IOI research team, Kuster, Johnson, Rupnow, and Wilhelm (2019) published the Inquiry Oriented Instructional Measure (IOIM), a quantitative measure of how much a classroom learning environment resembles inquiry-oriented instruction. The IOIM is an observational rubric that is used to evaluate audio video recorded classroom episodes. It is used to score how much teachers and students’ actions satisfy seven IOI practices. The IOI team advocated that taking a route towards quantitative measures has improved coherency and communication of what an IOI environment looks like. The IOI team described the inquiry in IOI as two-way, the student investigates an instructor’s questions or statements and the instructor investigates their students’ thoughts. The job of the instructor to reflect on student responses and build instructional plans from students’ ideas is a major tenant. Kuster et al. (2019) described the work of an inquiry-oriented instructor:

> Starting with research-based tasks designed to promote rich mathematical thinking and reasoning, IO-instructors elicit student generated contributions, and through inquiry interpret them, decide which are useful, and then determine how to use them to move the classroom toward developing the lesson’s intended mathematical idea...IOI teachers listen to students’

> contributions (e.g., reasoning, methods, and justifications) and, when appropriate, use these contributions as a springboard for follow up questions and further exploration by students (p. 185, 188).

The pairing of the IOIM with other mixed methods instruments associated with mathematical intuition could help to illuminate what instructional environments, relative to IOIM scores, are more or less supportive when it comes to students' intuition development. However, ways to quantitatively characterize a mathematics student's intuition at a point in time and intuition development across time are either extremely scarce or non-existent.

2.6 Summary

This chapter has synthesized a sizable sample of university level mathematics education and psychology literature to give a combined view of the construct mathematical intuition. This chapter has illustrated that intuition is an enormous class of cognitions with many attributes running around. Several boundary properties and distinct classifications of intuition were brought together in this chapter. The coarsest classification of non-creative and creative forms of intuition seemed to be a significant finding from the psychology literature that has been a blind spot in the mathematics education literature and has the potential to increase coherency and open new areas of research. In addition several tactics that are thought to enhance mathematical intuition such as representational shifting, experimentation with examples, embodiment and creating analogies were found in the literature. Instructional environments that are thought to enhance the development of leaners' intuition were also discussed. However, it remains unclear and ambiguous as to whether these environment are truly developing learners' intuition. Despite the importance of intuition in learning, there are still no estabilished education research programs on mathematical intuition.

Chapter 3

Literature Background: Instructional Design Research in Abstract Algebra, Evaluative Investigations, and Efficacy

3.1 Introduction

Introductory Abstract algebra is an advanced undergraduate typically taken at the junior or senior level and is a primer for abstract algebra at the graduate level. This chapter gives background on three facets of education research for abstract algebra: (a) instructional design research, (b) investigations of student thinking that inform instructional design research, and (c) studies the evaluate the efficacy of experimental instruction. This chapter covers abstract algebra topics that were incorporated into tasks used to investigate learner's representational fluency and example-based intuitions and concentrated on in this thesis. These topics are: homomorphisms, quotient groups, isomorphisms, and group actions with emphasis on Cayley's theorem, Lagrange's theorem, and the orbit-stabilizer theorem.

3.2 Instructional design research

Early instructional design research was based on APOS theory and the activities, class discussions, and exercises (ACE) cycle. Asiala et al. (1997) developed the APOS theory and ACE approach and designed an experimental abstract algebra course to improve learning outcomes for equivalence relations, cosets, normality, and quotient groups. The course incorporated programming activities during computer lab periods that learners worked on in teams, lecture and class discussion to reflect on the mathematical concepts and standard textbook exercises, and exams with unlimited time. The computer lab activity for cosets, guided learners to program a function that defines a binary operation on a group and returns four scenarios: (1) an element binary operation an element, (2) an element binary operation a subgroup, (3) a subgroup binary operation an element, and (4) all possible combinations of an element binary operation an element from two subgroups. Once learners are able to program this function they have an action or procedural view of a coset. After achieving an action view, learners are guided to input various groups and subgroups into their program. They look for patterns and explore why the patterns they find exist. A process view of cosets occurs when the learner can describe a coset as a list of elements. A leaner achieves an object view of cosets when they can identify patterns like cosets all have equal sizes and are disjoint and what causes such patterns to exist. A more evolved object view occurs when leaners are able to use these findings to prove statements like Lagrange's theorem.

More recent and ongoing instructional design research is being conducted under the Teaching Abstract Algebra for Understanding program (TAAFU). The materials created for the TAAFU research program are based on an accumulation of "local instructional theories" (LIT) that follow the principles of guided reinvention, Realistic Mathematics Education (RME), constructivism, and elements of a Lakatosian style of proof instruction (Larsen, Johnson, & Bartlo, 2013). LITs are products of a three stage process: (1) creating an initial design for a sequence of in class instructional activities, (2) carrying out teaching experiments to gather information on how learners interact with eachother, their instructor, and the mathematics when the design is implemented, and (3) retrospective analysis by a team of researchers and instructors (Gravemeijer, 2004). The first stage is prospective, the designers anticipates how learners might think about concepts and what they might struggle with, alternative ways that a concept could be introduced in the classroom to address learners' needs, and hypothetical interactions between the learners and instructor. The designers make informed assumptions based on prior research for how learners approach the concept of interest or they may conduct an initial set of interviews to see how students are thinking. The second stage is focused on documenting how things actually unfold when the design is implemented with a small group of students and, after further preparation, how things unfold in the full classroom setting. After the first experiment, the design is improved and subsequent experiments are conducted for modified designs. The third stage is focused on analyzing data collected during the teaching experiments, determining what aspects of the implementation worked, and presenting an LIT for a particular concept that other educators can incorporate into their teaching (Larsen & Lockwood, 2013; Gravemeijer, 2004).

Currently, there are published LITs for the group, homomorphism, isomorphism, and quotient group concept (Larsen, Johnson, & Bartlo, 2013; Larsen & Lockwood, 2013), in addition to work on guided reinvention materials for rings, integral domains, and fields (Cook, 2012, 2018) and symmetry groups for molecules in chemistry (Bergman, 2020). Larsen and Lockwood's (2013) LIT instructional activities for the quotient group concept incorporated Cayley tables of the dihedral group. Learners were guided to notice similarities between how the odd and even integers behaved and how the reflections and rotations of the dihedral group behaved. Next, they found subsets of elements of the dihedral group that could be used to build a new group with set multiplication as the binary operation. Through questions and in class activities, the instructor guided learners to make important observations like one of the subsets must act like an identity and be a subgroup. A classroom activity that was part of this LIT shown in Figure 3.1, was designed along the way from a learners' insights in the class, a key feature of Inquiry Oriented Instruction.

During the session learners investigated examples of various collections of subsets with set multiplication to make conjectures about what subsets acted like an identity and stronger necessary properties that the identity subgroup needed to have in order for the set of subsets to be a group. They also recognized that once the identity subgroup is chosen that the rest of the partition can be formed without additional information. Towards the end learners made observations and counter arguments that moved them towards the condition of normality. There was evidence that learners noticed that the right cosets and left cosets of the identity subgroup must be equal. In addition to Larsen and Lockwood's (2013) guided reinvention sequence for the quotient group, learners may also benefit from Mena-Lorca and Parraguez (2016) recommendation to reinforce the fundamental homomorphism theorem of sets and design classroom activities that encourage them to draw comparisons between this theorem for sets and groups. To supplement instructional sequences, the IOAA team has conducted additional research that investigated mathematicians' input on the implementation of the TAAFU curriculum. They also developed resources for instructors that gives foresight into how students respond to the activities (Johnson, Caughman, Fredericks & Gibson, 2013; Lockwood, Johnson, & Larsen, 2013).

There are currently no Local Instructional Theories for direct or semi-direct products, however Bergman (2019, 2020) has made some initial progress related to this. She has designed and piloted activities that guide learners to find all of the symmetries

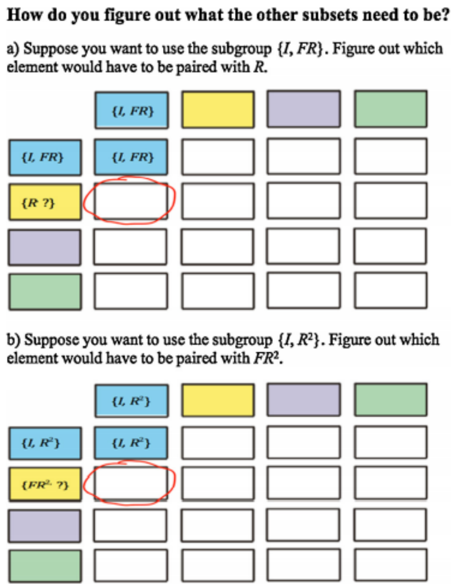


Figure 3.1: Activity built from student's idea in the LIT sequence for the quotient group concept (Larsen & Lockwood, 2013, p. 738).

of basic molecules like water, ammonia and ethane and derive a generalized procedure for finding all symmetries of basic introductory molecules. On a few occasions during these activities, graduate level participants arrived at distinctions between direct product and semi-direct products.

3.3 Investigations of student thinking

Homomorphisms, Quotient groups, and Isomorphisms

Dubinsky et al. (1994) assessed 24 high school teachers responses to a questionnaire that covered topics in a first introduction to group theory course. Topics included: binary operations, group properties, subgroup properties, cosets, normal subgroups, and quotient groups. They found that groups are often viewed as a set, subgroups as a subset, and focus on the size of the set rather than the binary operation. For subgroups a subset of elements is chosen that are similiar in type, for example all the flips in a dihedral group, but have difficulty checking that necessary properties of a group such as closure hold. In some cases, necessary properties are not considered. These difficulties persisted past the fifth week of the course for some. Performance on the quotient group question, that involved the dihedral group of order six, was lower than for other topics assessed in the questionnaire. Half of the teachers that were interviewed had not consolidated what a normal subgroup is or its significance and one successfully constructed a quotient group with minimal guidance (Dubinsky et al., 1994).

In a small sample study Weber (2001) found that four undergraduate learners' were able to successfully prove homomorphism statements that were basic consequences of the formal definition of a homomorphism. However, for more challenging proofs that involved formal facts, like the first isomorphism theorem and Lagrange's theorem, either these learners knew the formal facts that were needed but did not recover them during the proving process or they had not aquired such facts yet. "Strategic Knowledge", knowing what facts to apply and arriving at efficient proofs quickly while avoiding overly complicated wrong turns, was a major source of difficulty (p. 111-114). Weber (2001) also found that proofs that involved a quotient group to be constructed were of considerable struggle for 3 out of 4 undergraduates in a study. Two of the undergraduates did not have knowledge of quotient groups, but did have knowledge but did not bring it to mind during this proof even when cues like "It is known that K is a normal subgroup of S_4 where present in the to be proved statement. Weber (2001) called for additional larger sample studies to determine the generalizability of his findings and the most prominent sources of difficulties for undergraduate abstract algebra learners. While comparing undergraduate and doctoral learners' solutions, Weber (2001) noted that doctoral learners efficiency and sound strategic knowledge for homomorphism proof was developed over time and going down the wrong inefficient path was likely part of the process. He suggested that

> Identifying activities that are useful for constructing effective strategic knowledge, as well as activities that lead students to acquire faulty strategic knowledge, would be valuable research that could help shape future instruction (p. 115).

Since Weber's (2001) study, others have tried to pin down the source of undergraduates difficulties with the homomorphism and quotient group concept. During a TAAFU quotient group unit Lockwood, Johnson & Larsen (2013) found that during a task where students were asked to form various partitions of the dihedral group of order eight and come up with necessary conditions for the partition to form a group, students would see that this would occur when the subsets were inverses of themselves using set multiplication. One difficulty was that students would attribute this finding as a general necessary condition to form a group in all cases. They found that the octagon counter-example was a useful way to counteract this.

Melhuish et al. (2020) probed their understanding of more fundamental underlying concepts such as functions and equivalence relations as a collapsing. The "collapsing metaphor" for a homomorphism, refers to the identification of elements in a group into cosets of equal size, these cosets form a partition of the group's elements (p.6). The task in the study displayed a set of domain to range function mapping diagrams from an arbitrary group G of order 3 or 6 to an arbitrary group H of order 6 to investigate students' proficiency with some necessary conditions of a homomorphism through the collapsing process such as: the underlying equivalence relation that partitions G into cosets, of equal size, along with function requirements like well-definedness. Within a sample of six undergraduate participants that had completed an abstract algebra course, they found that some had correctly viewed function properties as being necessary conditions for a homomorphism, but had not integrated the collapsing metaphor into their conception of a homomorphism and others continued to display more fundamental difficulties in dealing with functions. The necessary condition S be a normal subgroup or a coset of a normal subgroup of G was irrelevant in this task which they referred to as the "puddle diagram task" (p.6). This suggests that a difficulty seeing that every homomorphism can be thought of as running through a quotient map may be exacerbated by a lack of mastery for the more fundamental set-theoretic and function concepts.

Other studies have shown that when proficiency for fundamental concepts is reached, undergraduates do not pay attention to or have trouble handling additional layers of complexity when going from thinking about sets to thinking about groups. The transition from the fundamental question of 'how does one partition a set?' to 'how does one partition a set into subsets so that it is guaranteed that the subsets are of equal size?' and arriving at 'how does one partition a group into subsets so that the subsets viewed as elements with some binary operation form a group?', has been a documented challenge for many undergraduates (Dubinsky, et al., 1994; Mena-Lorca and Parraguez, 2016). Mena-Lorca and Parraguez (2016) investigated undergraduates comparisons between the set isomorphism theorem and group isomorphism theorem. They found that the learners who made progress going beyond the set isomorphism theorem to construct the group isomorphism theorem were exceptions. The few who were able to arrive at aspects of the group isomorphism theorem showed more of a mastery of the isomorphism theorem for sets. Mena-Lorca and Parraguez referred back to previous work which illustrated that many theorems related to homomorphisms have corresponding set-theoretic theorems and that a categorical perspective to teaching group theory is a direction for education researchers that is worth investigating further. Undergraduate learners have fundamental difficulties with quotient maps, specifically viewing elements as sets of elements. This has been a recurring impasse to the first isomorphism theorem that continues to be documented (Nardi, 2000; Melhuish, et al., 2020).

Rupnow (2021) investigated two instructors use of metaphors related to homomorphisms and isomorphisms in the classroom and during semi-structured interviews. The collapsing metaphor was used frequently by one instructor towards the later part of the homomorphism unit and throughout the interview. Compared to the instructors use of other metaphors under “sameness”, “mapping”, and “formal defintion” categorizations, the instructors use of the collapsing metaphor was most prevalent (p. 9). The other instructor used the metaphor while covering the fundamental homomorphism theorem, but focused more on formal definition references such as non-bijective isomorphism in an interview setting. Rupnow found that learners in the first instructors class transitioned from more formal definition language when describing a homomorphism to the incorporation of informal metaphors for an equivalence relations such as collapsing vocabulary, but still had difficulty with the structure preserving metaphor. Rupnow (2021) concluded that questions remain regarding how students adopt and interpret the language that their instructors use in reference to homomorphisms and isomorphisms.

Cayley’s theorem, Lagrange’s theorem, the orbit-stabilizer theorem and group actions

The key idea of Cayley’s theorem is viewing elements of a group as permutation functions and viewing the binary operation of the group as being synonymous with function composition. In a guided reinvention task for the group concept students investigate the symmetries of a triangle and use algebraic notation R and F or cycle notation to express the symmetries (Larsen, 2013). Once they transform symmetries as elements into algebraic notation they come up with a binary operation to combine elements. They organize the outcome of taking an element and hitting it with a binary operation in a Cayley table. Next, students try to generate a list of observed properties from the table. Larsen (2013) found the students are good at noticing properties like each row and column contain every element exactly once called the “Sudoku property” (p. 720).

Learner’s have difficulty noticing that every element has its own inverse. Associativity is the most difficult to notice. He noted that

> issues related to the associative property can be explained by the fact that the operation of combining symmetries can be thought of as a string of actions rather than as a binary operation...a pedagogical challenge is to make

> sure that students explicitly attend to their use of regrouping so that it is included in their emerging list of rules (p. 719).

Once learners arrive an observed property they try to formalize it by providing a proof. The Sudoku property is a key observation to the idea behind Cayley's theorem. This in turn motivates the need to define a set of necessary axioms that allow a proof of the Sudoku property, this is where the need for the inverse axiom comes in.

Zazkis, Dubinsky, and Dautermann (1996) investigated 32 first year abstract algebra students' reasoning about dihedral group of order eight. Students received the experimental APOS theory and ACE experimental course. Two registers of thinking were visualizing symmetries of a square and multiplying permutations that were written in cycle notation. They referred to first of these two approaches as"visual" and the second "analytic". Analytic thinking referred to cases where "symbols are taken to be markers for mental objects and manipulated entirely in terms of their meaning or according to syntax rules" and if "the nature of the symbols themselves or their configurations is used then we would consider it to be an act of visualization" (p. 442). They found that students would take at least one of the approaches, but had difficulty in using both approaches and realizing that the two approaches were two ways of doing the same thing. They suggested that the visualization and analytic approaches should be gradually integrated.

There are several researchers who have documented students' understanding of Lagrange's theorem. Asiala et al. (1997) investigated how students connected the underlying objects of cosets to Lagrange's theorem. During clinical interviews they found that some students knew the word coset and the notation aS, but could not take it things any further. In another case learners could make a word association that Lagrange's theorem may have involved cosets, but that they couldn't remember. In a third type of case a learner was able to remember a process for partitioning a group into subsets using a normal subgroup of equal sizes and that this process was related to Lagrange's theorem, but they couldn't quite reconstruct the formal proof. The average score on an exam question to discuss key details of the proof was a 76%.

In another study, Hazzan and Leron (1996) investigated how students used Lagrange's theorem to solve application problems. They found that 73 out of 113 undergraduates answered the true or false, with explanation, prompt "Z_3 is a subgroup of Z_6" with error. 20 of the students incorrectly used the converse of Lagrange's theorem, which is not true in general, to justify the statement. However, technically a special case of the converse of Lagrange's known as Cauchy's theorem states: If G is a finite group and p is a prime number such that $p||G|$ then G has an element of order p which implies G has a subgroup of order p. There is only one group of order 3 up to isomorphism and by Cauchy's theorem Z_3 is isomorphic to a subgroup of Z_6. Hazzan (1999) did not mention if Cauchy's theorem was relevant to students responses or not for this prompt. However, they also documented that 25 out of 108 students misapplied Lagrange's theorem for the prompt "S_4 is a subgroup of S_5" giving answers like "S_4 is a subgroup of S_5 because 24 divides 120" (p. 6). Students' applications of the converse of Lagrange's theorem could lead to productive discussion of special cases in which the converse holds. Melhuish (2018) replicated Hazzan and Leron's (1996) study with some modification to the question format and found that 7.4% of students ($N = 349$) proved the answer yes with valid reasoning to the "Z_3 is a subgroup of Z_6" open-ended prompt, 58.5% said yes with invalid reasoning, and 26.1% provided no valid reasoning. Melhuish also incorporated a closed-ended prompt to try to identify the source of invalid reasoning. She concluded that learners may not be distinguishing the mod 3 binary operation of Z_3 with the mod 6 binary operation. Technically, Z_3 is not literally a subgroup of Z_6, but Z_3 is isomorphic to a subgroup in Z_6.

Several of the Teaching for Abstract Algebra Understanding activities, developed by IOAA researchers, incorporate groups actions on objects of symmetry (Lockwood, Johnson, & Larsen, 2013; Larsen, 2013). Learners are exposed to some aspects of group actions through activities in a first introductory undergraduate course, the term group action and formal definition is not made explicit. Little is known regarding how undergraduates that advance on to take abstract algebra courses, beyond a first year undergraduate course, develop a deeper understanding of and link Cayley's theorem, Lagrange's theorem, and the orbit-stabilizer theorem within the setting of group actions.

3.4 Efficacy studies

The efficacy of abstract algebra instruction on student learning has become an international concern (Fukawa-Connelly, Johnson, & Keller, 2016; Agustyaningrum et al., 2021). Agustyaningrum et al. (2021) used a questionnaire to collect data on internal and external learning difficulties of 30 students in a mathematics education department in Indonesia. Internal difficulties related to affective constructs such as attitude towards the subject and motivation as well as cognitive constructs such as prior knowledge. External difficulties related to Lecturer's knowledge of the material, teaching style, and instructional materials. They found the most concerning student responses for attitude towards the subject, prior knowledge, and teaching materials. Half of the students strongly agreed or agreed that "abstract algebra courses are difficult to understand" (p. 854). A student reported that the all of the new terminology in abstract algebra was overwhelming and he had difficulty with "memorizing" and remembering what the new terms mean (p. 858). There was also evidence that some were attributing the fast pace of lectures to internal deficiencies like being a "slow in understanding abstract algebra material" (p. 858). Students responses about the teaching method were more split with over half that disagreed that the method, in this case a combination of lecture and group work, supported their learning of the material. One drawback to group work that was mentioned by a student was that members of the group don't prepare prior to lecture or participate to the best of their abilities.

Ioannou and Nardi (2009) examined the effects of lecture based instruction on emotional states, visualisation, and engagement. The participants were 78 undergraduates in a second year Abstract Algebra course that was characteristic of lecture-based instruction with little interaction between the instructor and students. They found the theme that learners engagement waned in the course, despite eagerness and excitement at the start of the course. Another theme was that learners expressed a need to for visualisations, but rarely used the visualisations show in class. They lacked confidence in visualisations.

Despite evidence of student difficulties, over a decade of abstract algebra education research, and non-lecture instructional materials being produced under the TAAFU program, these materials are not being used (Fukawa-Connelly, T., Johnson, W., & Keller, R., 2016). To find out why, Fukawa-Connelly, Johnson, and Keller (2016) surveyed 131 undergraduate abstract algebras instructors, with a large majority coming from universities that offer Master's and PhD programs. The survey showed that 85% teach use a lecture format, 44% were open to incorporate instructional formats besides lecture and the rest were not. Instructors responded to the question "I will never switch from lecture because...": "It's not appropriate for my students" (17), "I think it would go poorly" (26), and "I need to cover a certain amount of content" (32) (p. 278). In a follow-up, they stated for those open to other instructional formats besides lecturing:

> ...1 finds PRIMUS or the MAA Notes series very influential, only 1 finds mathematics education research literature very influential; only 6 find talks, workshops or conferences about teaching very influential; and only 4 find participating in communities like Project NExT very influential. It is our belief that this is not because the materials themselves are not useful, but rather those who need them most are not utilizing them (p. 280).

Currently, there are few large scale studies that compare IOAA to non-IOAA groups or groups that receive instruction from environments with high and low IOIM scores in relation to students' learning outcomes. In fact such studies are only now becoming accessible to researchers due to the recent development of quantitative instruments such as the GTCA and IOIM (Melhuish, 2015; 2019). Melhuish (2015) designed a Group Theory Content Assessment (GTCA) for binary operations, group properties, isomorphism, homomorphism, Lagrange's theorem, and quotient groups among other introductory topics. The seventeen open question pilot GTCA was turned into a closed-ended multiple-choice assessment that used common student responses from the pilot as answer choices. During the pilot the quotient group question was problematic for several reasons, one being that it took to long and learners we need to search for normal subgroups. It was replaced with a computational question regarding the size of coset $2 + H$ in Z_{12} that also evaluated whether or not students viewed sets as elements. This was the only quotient question in the GTCA.

With these new tools researchers have started to evaluate the efficacy of IOAA compared to other forms of abstract algebra instruction (Johnson, Andrews-Larson, Keene, et al., 2020). Johnson et al. (2020) conducted a study that compared a group of abstract algebra learner's that had instructors who participated in the TIMES IOAA project and a group that had non-participating instructors. The TIMES IOAA project provided training on IOAA instruction. Both groups, a total of 522 students, took the GTCA. There was no statistically significant difference between the two groups GTCA scores. Gender differences were also examined. Males scored significantly better on the GTCA in the TIMES group than males in the non-TIMES group, there were no gender differences between females across groups. Additional analysis controlled for other potential confounds, but found no significant differences between groups on GTCA scores. They concluded that equity issues in an inquiry environment needs further research attention.

3.5 Reflections on the literature

From a theoretical point of view it seems that instructional materials designed under the TAAFU program are doing the right things with respect to facilitating learners' intuitions as discussed in the previous chapter. However, from a more empirical data driven end there is not enough evidence to pick a side IOAA under the TAAFU program or non-IOAA abstract algebra instruction. What is clear is that there are issues with the current system, the TAAFU program has offered alternatives, however there are potential benefits and issues with these alternatives.

While the GTCA serves a particular purpose, it is not a sufficient instrument alone nor was it intended to address the effectiveness of instruction on the facilitation of intuition associated constructs such as representational fluency or affective impacts. For every local instructional theory, I believe that there should be many local quasi-experimental studies to evaluate the efficacy of a LIT sequence compared to the more standard treatment. What I mean by this is if there is a local instructional theory for the quotient concept then instruments need to be specifically developed to address whether or not students who receive that quotient LIT outperform learners who receive other forms of instruction with respect to some dependent variable like fluency. Moreover, it is important that efficacy studies for instructional designs investigate dependent variables related to students and instructors as well. For example, instructional changes may impact instructors feelings of autonomy. Loss of autonomy could also have negative consequences and lead to heightened tensions. After many efficacy studies have been performed for various dependent variables, meta-analysis can be conducted to compile these studies and determine the costs and benefits. This is a standard line of research practice in medicine and psychology, but occurs after the field or area of research is fairly mature.

However, there are dissenting opinions to mine. The need for efficacy studies that evaluate instructional sequences, born out of instructional design research, is quite controversial. For instance Prediger, Gravemeijer, and Confrey (2015) stated:

> We may counter, however, that design research is grounded in the assertion that classrooms are complex ecologies where one cannot sufficiently control all variables to draw causal conclusions. This calls for a different approach than the reductionist approach of natural science like psychology in which phenomena are disassembled in individual variables whose interdependencies can be researched systematically - essentially by testing hypothesis (p. 883).

To counter back, quasi-experimental studies used to compare different instruction types are meant to inform and improve instruction further. The process of working towards such studies, aiming for goals of hypothesis testing, and eventually meta-analysis may still lead to important discussions regarding the variables in play. Experimental design research represents genuine attempts to document and understand what is working and what is not. I believe that such pursuits have the potential to deepen our understanding of these "complex ecologies". I am not advocating for a strict quasi-experimental or reductionist approach at the expense or devaluation of others. I am only advocating for its inclusion alongside the more dominant qualitative tradition of instructional design research.

This thesis takes the philosophical stance that one must understand and articulate the individual units within a system in order to better understand the system as a whole and attempts to illustrate the potential value in what some may view as a reductionist approach in education research. The next chapter decomposes intuition in intuition-associated factors with a focus on representational fluency and example-based intuition. Representational fluency is decomposed further into smaller units of analysis such as modes of semiotic production, modes of semiotic aquisition, conversions and pseudo-semiotic representations. Example-based intuitions are decomposed further into creative and non-creative forms using variables associated with these different forms.

Chapter 4

Theoretical Framework

4.1 Introduction

This chapter presents an outer framework of four prominent intuition associated themes that emerged after an integrative literature review and interviews with three mathematicians. These themes are: (a) representational fluency, (b) example-based intuitions, (c) counter-example stance, and (d) affective impacts. The first inferred prominent theme related to intuition was representational fluency. The larger the learner's representational repertoire for a mathematical object is, that is consistent with mathematical reality, and the more flexible they are in shifting between various representations, the more informed their intuition may be (Cangelosi et al., 2013; Hegg et al., 2018; Pinto, 2019). The development of learners' intuition derived from examples was another central theme. By focusing on a subset of examples, the learner uses their intuition to find patterns and develop conjectures. The next theme, counter-example stance, namely the stance to accept a counter-example as an intriguing case to explore or to reject a counter-example, came from Lakatosian influences. The last abstracted theme was affective impacts. Affective impacts refers to the various ways that affective constructs such confidence and anxiety can impact the intuiting system. The following sections provide progress on inner frames for each theme in Figure 4.1. A goal during the development of this outer framework, and subsequent inner frameworks, was to arrive at units of analysis for empirical studies.

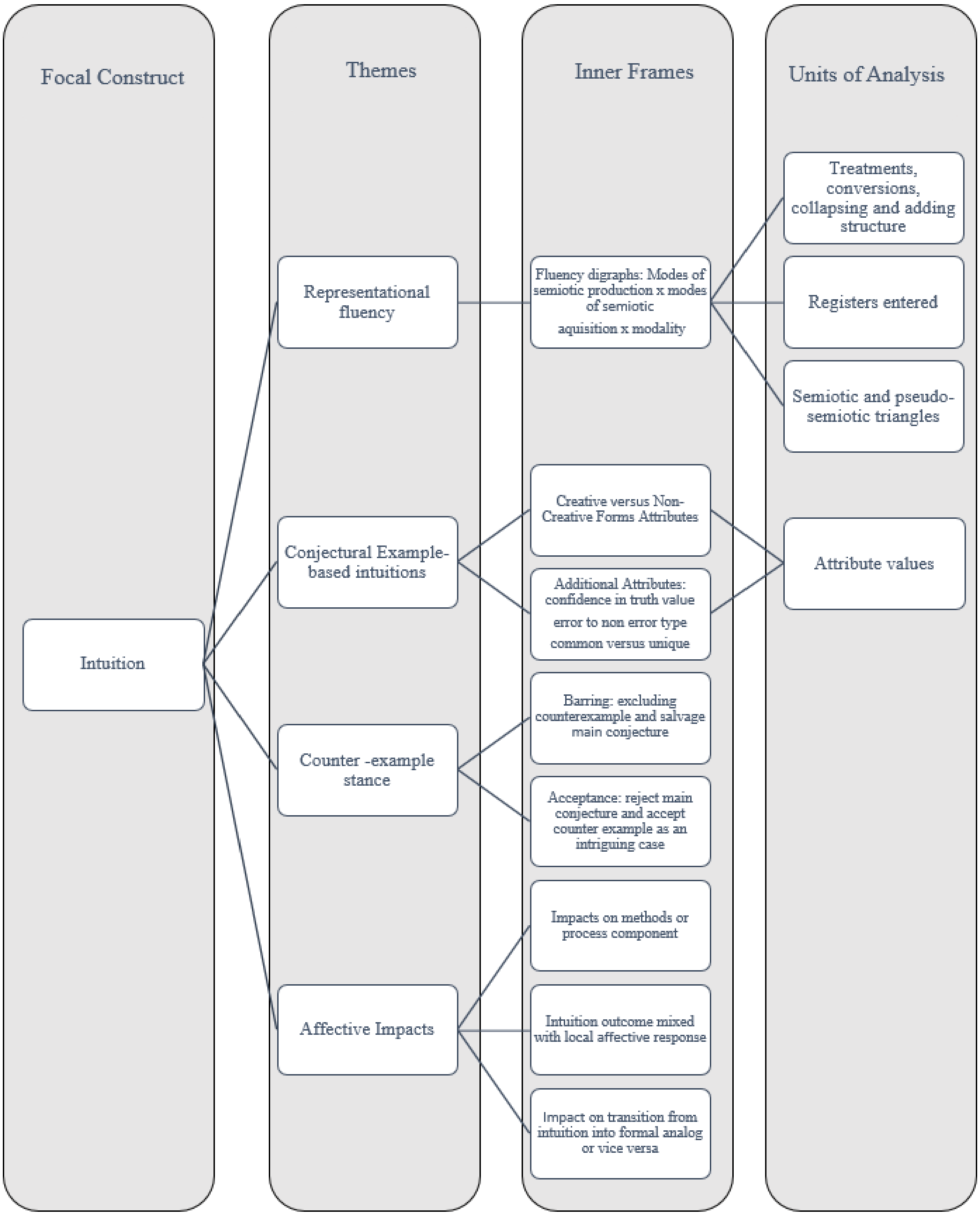


Figure 4.1: Intuition Framework

4.2 Representational fluency

Representational fluency is "the ability to create, interpret, translate between, and connect multiple representations" of mathematical objects (Fonger, 2019, p. 1). If one is able to represent the same object in multiple ways the content extracted from each of the representations are added up, "the process of amalgamation described by Arzarello is of a synthetic nature, and allows the different meanings to be aggregated with the help of analogies and metaphors, glued, condensed so to be more easily internalized and mobilized by intuition" (Tanguay & Venant, 2016, p. 889). Pulling from Duvalian semiotic theory, a mathematical object can be realized as an invariant of multiple semiotic representations (SR) (Duval, 2017). According to Peirce's (1992) triadic model, a semiotic representation of a particular object o is made up of three constituents: a sign, an interpretation of the meaning of the sign by an interpreter, and the object being signified.

In an attempt to make progress towards a unit of analysis for students' representational fluency, the theoretical knowledge structure termed a fluency digraph was derived in Lajos and Stewart (2020). A fluency digraph contains the semiotic representations that an individual has accumulated over many learning experiences, a typification of these SRs in terms of *modes of production* (i.e., the register used) and *modes of acquisition*, and paths, (i.e., conversions), the individual has made between various SRs for the same object. The *Networking strategy* "combining and coordinating" was taken to arrive at the fluency digraph.

> ...combining and coordinating are typical for conceptual frameworks, which do not necessarily aim at a coherent complete theory but at the use of different analytical tools for the sake of a practical problem or the analysis of a concrete empirical phenomenon...We use the word coordinating when a conceptual framework is built by well fitting elements from different theories (Prediger, Bikner-Ahsbahs & Arazarello, 2008, p. 172-173).

In particular three theoretical lenses were combined and coordinated: (1) the Theory of Registers of Semiotic Representation (TRSR) (Duval, 2000a, 2006, 2017), which includes "modes of production" (i.e., registers used to produce a semiotic representation) and "conversion fluency" (Duval, 2017, p. 110-111), (2) the uni-modal and multi-modal distinction (Arzarello, 2006b), and (3) metarepresentational competence and modes of acquisition (diSessa, 2004; Gravemeijer & Doorman, 1999).

4.2.1 Theory of Registers of Semiotic Representation

A mathematical object is an invariant of multiple semiotic representations. That is, there are multiple ways to represent the same mathematical object. When working with a mathematical object, it is important to consider the "content" of the semiotic representation (SR), the "system" and "register" the SR is produced in, and the "represented object" (Duval 2000a, 2006, 2017, p. 27). The content of the SR refers to the particular facets or qualities of an object that an SR that stand out. For example, let the object be a normal linear operator on a vector space B, where V is a finite dimensional inner product space over the complex field. Since the operator is normal, the operator can be expressed as a diagonal matrix with respect to some basis. A sign for the operator that is a diagonal matrix makes the eigenvalues, as content, clear-cut to see. A sign that takes on the form of a matrix and is not diagonal, nor lower or upper-triangular, hides the eigenvalues from view. In general, different signs may naturally highlight or suppress certain facets or qualities of a mathematical object. There are often inconsistencies and differences among individuals in how they interpret or assign meanings to the same sign (Duval, 2017). Based on Peirce's view, "The sign, or representation, is something that stands for something to somebody in some respect or capacity" (Peirce; cited in Duval, 2017, p. 14). There is a great deal of variability in undergraduate, graduate learners' and mathematicians' interpretations of mathematical objects through various signs (Thurston, 1994; Duval, 2017).

Systems and registers

Cognitive systems involved in the generation of representations are: the automatic (non-conscious) neural system and the intentional system (consciously controlled). The automatic system generates intuitions or recovers mental images from memory. The intentional system brings, "...into play the semiotic system (mentally or materially)" to produce a semiotic representation that "Denotes the represented object" through non-vocal verbalizations, externalized speech, writings, drawing, or computers (Duval, 2017, 2000a, p. 66). While one may pass into the intentional system to produce representations for communication one may also pass into the intentional system to produce representations that are used to feed stimuli to the intuiting process within the automatic system. The intentional system contains sub-systems called registers.

Registers are like representation systems. Goldin (1998) defined "representational systems or representational modes" as a, "system of spoken symbols, written symbols, static figural models or pictures, manipulative models, and real world situations" (p. 143). Goldin (1998) also stressed that ambiguity arises when defining the functionality of a representational systems in terms of how the system interacts with other systems, how "higher-level structures" organize reasoning within and across representation systems, and the syntax rules govern how letters or symbols called "characters" are defined and put together as "permissible configurations" (Goldin, 1998, p. 143). Some general registers-systems of representation used by researchers that build from Duval's work include: verbal, written, gestural, geometric, and algebraic (Sandoval & Possani, 2016; Fonger, 2019). Specialized mathematical object or mathematical domain-specific registers can also be found in the literature (Ely, 2017).

Semiotic representations have been theoretically classified according to the registers used to produce them (Duval, 2000a, Duval, 2017). Duval defined four main registers: discursive, non-discursive, multifunctional and monofunctional. The discursive register is used to represent mathematical objects using spoken words or written symbols. The non-discursive register is used to represent mathematical objects through a non-spoken language such as visual images, geometric shapes, graphs, and diagrams. A multifunctional register is used to produce SRs consisting of vernacular language, writings, and drawings that are free from the strict confines of the formal mathematical language. A SR produced in this register by a learner is predominantly a reflection of that learner's intimate and unique view of a mathematical object. In contrast to the multifunctional register, a monofunctional register is used to produce SRs that are restricted to writings in formal mathematical language that such as logical quantifiers, algebraic symbols, formal axioms, definitions, and theorems. A SR produced in this register is predominantly a reflection of standard mathematical practices (Duval, 2006, 2017).

Treatments and conversions

In order to survive in a mathematical environment, the learner must at least be able to perform cognitive processes termed cognitive transformations of semiotic representations (CTSRs) (Duval, 2017). A CTSR is a mental mapping from one semiotic representation to another. "In any semiotic transformation, it is necessary to distinguish between the starting representation and new representation produced, i.e. the arrival representation" (Duval, 2017, p. 43). In some cases, the starting SR and target SR are produced using the same register. In other cases, the "starting" SR and "target" SR are produced through different registers (Duval, 2006, p. 112). To theoretically separate these two cases, Duval (2006) made the distinction between two types of CTSRs: treatments and conversions. A treatment is a change in SR without a change in register. An example of a treatment is solving algebraic equations while staying in a monofunctional $\times$ discursive register. The more involved and cognitively demanding mental process of keeping the object the same and switching from a starting to target SR through different registers is called a conversion (Duval, 2006). A conversion becomes increasingly difficult for school and entry undergraduate learners because a change in register means that the semiotic representations are perceptually dissimilar, but represent the same mathematical object (Duval, 2000a, 2006, 2017).

A conversion from one SR produced in one register to a SR produced in a different register may be algorithmic and performed through step by step procedures. For example there are step by step procedures that can be used to go from a symbolic SR of a function to drawing a graph of a function. Procedures can be acquired through "instrumental" (memorization of an already formulated procedure) or "relational learning" (independent creation of the procedure) (Skemp, 1978, 1979, p. 285-286). In contrast with an algorithmic move, a non-algorithmic move from one register to another has no formulaic mapping (Duval, 2017).

As learners progress from entry undergraduate to more advanced courses there is a rising expectation for them to (1) produce multiple representations that are sanctioned and not sanctioned for a mathematical object and (2) perform CTSRs, especially conversions, on their own with and without the aid of an algebraic mediator. However, the conversions undergraduate learners are expected to do and what they do at a point in time do not always align. In a recent study, Sandoval and Possani (2016) found that a majority of undergraduate learners in an introductory to linear algebra course faced challenges performing geometric treatments adding vectors and conversions between verbal, algebraic, and geometric registers for vectors and planes in $\mathbf{R}^3$. They observed that, "One important result shows that, once students choose a register to solve a task, they seldom make transformations between different registers, even thought this facilitates solving the task at hand" (p. 109).

Conversion fluency

At some stage, the learner must reach the learning milestones of "conversion fluency" for a variety of mathematical objects (Dreyfus, 1991; Duval, 2017, p. 110-111). Duval (2017) affirmed that, "without any explicit training of conversion fluency, there is no possible mathematics learning for most students..."(p. 110). The construct of conversion fluency (CF) is a form of representational fluency (RF). RF is "the ability to create, interpret, translate between, and connect multiple representations" of a mathematical object (Fonger, 2019, p. 1). Conversion fluency of with respect to a particular mathematical object is achieved if the learner can: (a) recognize and generate multiple semiotic representations of a mathematical object in a variety of registers, (b) realizes that the semiotic representation is used to represent an object and is not the object itself, (c) can begin with an initial semiotic representation and arrive at target for a variety of initial and target choices, and (d) integrate the semantic bindings attached to multiple representations (Dreyfus, 1991; Duval, 2017).

> As a result of this process [integration], one has available what is best described as multiple-linked representations, a state that allows one to use several of them simultaneously, and efficiently switch between them at appropriate moments as required by the problem or situation one thinks about (Dreyfus, 1991, p. 32)

4.2.2 Metarepresentational competence and modes of acquisition

From the Duvailian perspective, semiotic representations were classified according to the mode of production or the register the representation was produced in. From a broader socio-cultural perspective, semiotic representations can be alternatively classified in terms of how they are acquired, either through social interactions or independently. Sanctioned representations are representations commonly taught and adopted from a particular mathematical culture. Non-sanctioned representations are representations that are created independently by a person to reason with and understand the objects that they work with (Johansen & Misfeldt, 2018; diSessa, 2004).

Johansen and Misfeldt (2018) emphasized a related social angle of representation acquisition through "enculturation", a process where an individual interacts socially with others that are part of a culture and adopts the cultures preferred practices, language, representations and ways of thinking. They also found that mathematicians often invented their own representations as a source of ideas and a sense-making tool, but were sometimes hesitant to publish them or use them to communicate publicly:

> If you use your own idiosyncratic representations, it will be difficult for you to share ideas and collaboration with other mathematicians, not only because they might not understand the conventions of your representational language, but also because they might not understand you at a more fundamental cognitive level (p. 3737).

Research has shown that representations arrived at independently that are not standard within a culture are naturally suppressed by some mathematicians when communicating with others. However, the use of their own novel representations and externalized inscriptions of them are amplified when the mathematician is trying to figure out something for themselves (Misfeldt, 2018).

Without intervention, this publicly encountered norm, of suppressing one's own non standard representations or unique ways of thinking, may permeate into instruction where standard representational language is emphasized to such an extent that students become less likely to create and value their own non-sanctioned representations. Fortunately, researchers have developed modes of instruction that encourage students to invent their on personal representations and then transform them for public discourse (Enyedy, 2005; diSessa et al., 1991). diSessa (2004) affirmed that a goal of mathematics instruction is to foster metarepresentational competence (MRC), representational competence apart from proficiency with standard sanctioned representation use taught by the curriculum. The learning milestone of metarepresentational competence (MRC) is reached when the learner can (diSessa, 2004, p. 93): Invent or design their own representations, choose effective representations and judging the quality of representations for a given task, understand the purpose and strengths of certain representations in various contexts, explain the information given by a representation, and learn new representations with little guidance from others.

4.2.3 Combined and coordinated lens: technical construction of a fluency digraph

Based on Duval's (2000a; 2006; 2017) theoretical perspectives the first classification will be termed class 1 transformations (C1Ts), which includes treatments and conversions. We also consider a second classification (C2Ts), which goes beyond treatments and conversions. A C2T involves: varying a mathematical object by manipulating its structure or properties and necessarily varying the semiotic representation because the object being denoted has changed. In addition to having a change from an initial to a target semiotic representation(s) there is a change from an initial object(s) to a related target object(s). It follows that this second classification includes two sub-classes: adding structure and/or adding properties to obtain more specialized object ($C2T^{+}$) which necessarily includes going from a general object to example cases; and forgetting structure or ignoring properties to obtain a more general-looser object ($C2T^{-}$) which necessarily includes going from example cases to a general object. Some notions of $C2T^{-}$ are generalization as a many to one map from several initial example cases to a general object that is a distills invariants in the collection of initial examples and the notion of preserving some but not all of properties or structure. A third classification would be leveraging an object in one register (or mathematical system) to gain information about another object in another register.

Drawing from the three previously stated lenses, we now term a fluency digraph F_o of a specified mathematical object o to be a type of knowledge structure that that a learner has built up, consolidated and re-consolidated over many experiences. It consists of many semiotic representations (SRs) (vertices) used to denote the specified object o and class 1 cognitive transformations (C1Ts) (arrows) between the semiotic representations of the same object o. The fluency digraph can be viewed as a directed graph where the underlying graph may or may not be connected and: the vertex set is a collection of semiotic representations S_o, a relation $R \subseteq S_o \times S_o$ of (initial, target) semiotic representation pairs, and an arrow set $A(F_o) \subseteq R$. An element (s^i_o, s^j_o) in the arrow set is viewed as an arrow pointing from s^i_o to s^j_o and as a short-hand will be denoted $s^i_o s^j_o$.

Graduate learners and mathematicians are known to have fluency for numerous mathematical objects (Duval, 2017). The size of their fluency digraphs quantified in terms of $|V(F_o)|$ and $|A(F_o)|$ may be quite large. However, during a mathematical task they do not explicitly use all semiotic representations and directed connections that they have consolidated in the past. To account for this, we term a sub-graph of an individual's fluency digraph that is semiotically denoted and explicitly used during mathematical activity a 'container' c^o.

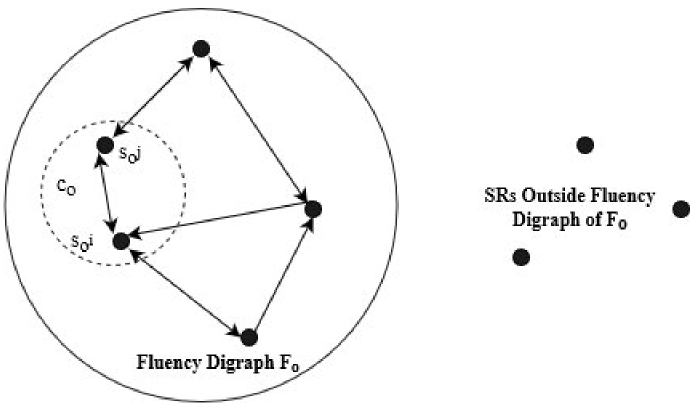


Figure 4.2: Fluency Digraph, Container, C1Ts, and SRs.

An example is the container that one may use for working with the object 'a linear representation' in the mathematical area of representation theory of finite groups. Some denote 'a linear representation' semiotically as $\rho : G \longrightarrow GL(V)$, the amalgamation of three mathematical objects: a finite group G, a group homomorphism ρ, and the group of linear invertible transformations $GL(V)$. Others prefer to denote a 'linear representation' as an FG-module. These semiotic views are definitionally "equivalent" (Dummit & Foote, 2004, p. 843); there is a sense of sameness in the mathematical object but variability in the semiotic representations. Sometimes one may switch back and forth between two such semiotic representations depending on the mathematical situation. But why do mathematicians prefer or choose one container, a subset of a fluency digraph of a mathematical object, over the other?

Let's look at an example of an intentional choice of a container versus a random choice at a graduate-level. Take the object with the following SR: 'the Borel subgroup B of the group of invertible upper-triangular 2x2 matrices which consists of matrices of the form $\begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}$ with $\beta \in \mathbf{F}_q^\times$, $\alpha, \gamma \in \mathbf{F}_q$, $q = p^r$ for p prime'. As a graduate student the size of my fluency digraph for this object increased to include two additional SRs among others from assigned prior homework exercises in an abstract algebra course. These two additional SRs were:

(1) '$B \cong Z \rtimes P$ where Z is the subgroup of matrices of the form $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$' and P is the subgroup of matrices of the form $\begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}$'. And

(2) '$B \cong U \rtimes D$ where U is the subgroup of unipotent matrices of the form $\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$ and D is the subgroup of diagonal matrices of the form $\begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}$'.

As a graduate student, I intentionally selected SRs of this Borel subgroup as these semi-direct products where the normal sub-groups Z and U were abelian to be part of my container at a moment in time because: (a) they were fresh on my mind from lecture notes that followed Piatetski-Shapiro (1983), (b) they were efficient choices for the task I was performing. To classify and construct all irreducible representations for B, and (c) I knew of a tool for constructing and classifying irreducible representations for a group using Serre's (1977) 8.2 construction, which can be used when the group is expressed as a semi-direct product where the normal subgroup is abelian (p. 62). But the decision between which of the two semi-direct product SRs to use, was a random choice. I switched from using one to using the other only because I got stuck with the proof.

When analyzing conversion fluency in empirical data it is important, for logical soundness, that a researcher chooses a theoretical demarcation of registers (Sandoval & Possani, 2016); and states whether or not the chosen demarcation are the general duvalian registers or if they are specialized object specific registers either sitting inside general register or viewed apart from general registers. Assign each register in a chosen demarcation a number $r_1, r_2, r_3, \ldots r_n$. Duval (2017) gives a general theoretical demarcation. Different demarcations of more general or specialized subject-specific registers for mathematics can also be found in the literature (Ely, 2017). Production of semiotic representations can be typified in terms of the register used (Duval, 2017). Recall that production may also be typified as "multi-modal"- multiple registers are used simultaneously or "uni-modal"- a single predominant register is used (Arzarello, 2006).

To integrate the multi-modal (multiple registers used in tandem) and uni-modal (a single predominant register is used with serial processing) case, take the power set Σ of all letters (i.e., all possible combinations of registers). Take Σ^+ to be the set of all words generated by elements of $\Sigma = \{r_1, \ldots r_n, \{r_1, r_2\}, \ldots \{r_1, r_2, \ldots r_n\}\}$. A word from Σ^+ is used to move from an initial to target semiotic representation. In which the first letter stands for the register combination used to denote the first SR and the last letter stands for the target SR. We visualize the registers a planes sitting inside a fluency digraph. In some cases, a register plane is not needed to successfully complete a specific task, but the leaner still has access to it. It may be the case that a learner, in general, prefers certain registers over others and this may be indicated by the size of a particular register plane. Moreover, certain registers such a geometric register may be more heavily weighted when it comes to the development of intuition in terms of representational fluency within certain mathematical contexts. In some cases register planes may be empty for an object o meaning $|V(F_o)|$ restricted to some register plane p is zero.

To integrate the mode of aquisition lens, the cross-product between the two-tuple $<$sanctioned and non-sanctioned$>$ and the power set of registers was taken. Now we can think about typifying representation according to their "mode of production" (i.e., duvalian registers or specialized object-specific registers) and the mode of acquisition simultaneously. A fluency digraph, depicted in Figure 4.4, is a product of three merged theoretical lenses, was intended to provide a finer frame to analyze individual's representational fluency for a single object or a collection of objects.

Within this integrated frame a semiotic representation can be typified beyond the Duvalian flavor of registers and in terms of more specialized object specific registers used to produce the representation. In terms of other compatible theories, mode of aquisition was also viewed in this thesis as the three-tuple ¡directly told by instruction or authority, reinvention, and invention¿. In addition the fluency digraph frame may help to formulate some partial numerical measure of the stimuli that an intuiter could generate to feed their intuiting process. These measures could be defined in terms of the total size of the vertex set $|V|$ of a fluency digraph F_o summed over many objects o, the size of the vertex set restricted to various register planes p, modes of aquisition a, or quadrants q with different weights, and the size of the edge sets that across register planes and the non-sanctioned sanctioned axis. For example a numerical characterization of representational fluency (RF) could be defined as in (4.1) and (4.2) below.

$$RF_{|V|_{p\cap q}} = \sum_{o} \sum_{p_i,k} w_{p_i \cap q_k} |V(F_o)|_{p_i \cap q_k} \tag{4.1}$$

$$RF_{C1T}(|E|_p) = \sum_{o} \sum_{p_i,p_j} w_{p_i \to p_j} |E(F_o)|_{p_i \to p_j} \tag{4.2}$$

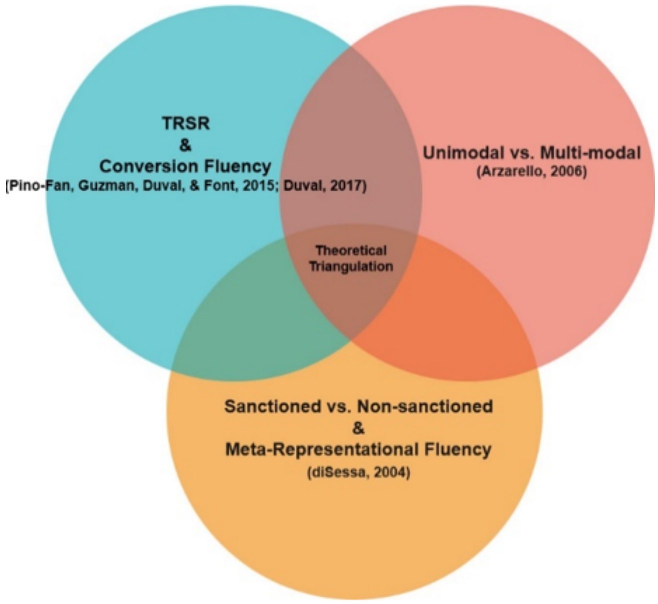
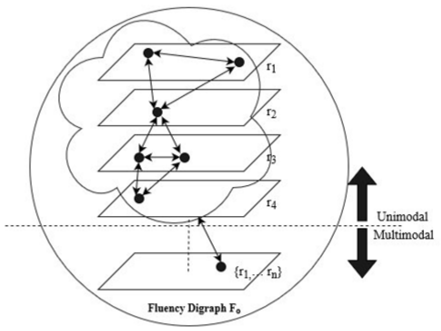
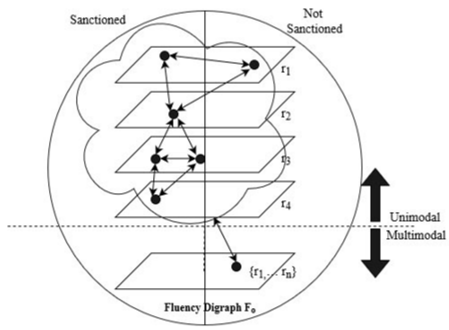


Figure 4.3: Combined Theoretical Lenses.

These measures could be taken at some initial point in time and repeated at distant points in time to obtain a difference measure of fluency growth with respect a specified mathematical object or collection of objects. One could also consider class 2 transformations (C2T), a mental path of how to travel from one fluency digraph F_{o_i} to another F_{o_j}. These paths may involve adding, collapsing or ignoring structure if o_i and o_j are in fact different objects. It may also be the case that o_i and o_j are perceived as different objects in the learners mind, but are mathematically the same object. When this sameness is realized F_{o_i} starts to merge with F_{o_j} through class 1 transformations (C1Ts).



Register planes are the power set of a chosen demarcation of registers



Take the 2-tuple (sanctioned, not sanctioned) x power set of registers

Figure 4.4: Fluency digraph construction.

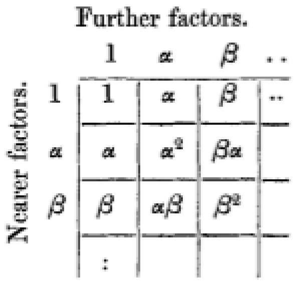
4.2.4 Introductory object-specific registers for finite groups

To begin application of the fluency digraph construct to evaluate undergraduate and first year graduate students' representational fluency for introductory finite groups, the general fluency digraph frame was specialized further according to inserted registers where the SR's sign constituent is fixed, but the interpretation constituent varies: multiplication table, cayley graph, Schreier coset graph, object of symmetry, cycle graphs, and the group presentation. A group may also be represented as a direct product or semi-direct product by the student. The visual data offered by the sign constituent in an SR may be dependent on an initial choice of generators (e.g., Cayley color graphs and Schreier coset graphs) or it may exist independent from an initial choice of generators (e.g., Cayley tables and cycle graphs). Semiotic representations for groups that have been incorporated in empirical educational studies and the TAAFU curriculum are: Cayley tables, symmetry objects, and group presentations (Larsen, Johnson, & Bartlo, 2013).

Arthur Cayley (1854) paper *On the theory of groups as depending on the symbolic equation* $\theta^n = 1$ is considered the first published formulation of a Cayley table in Figure 4.5. In this paper, Cayley views the binary operation of a group as a function that reshuffles the elements of a group. He goes on to define a group as

> A set of symbols,
>
> $$1, \alpha, \beta...$$
>
> all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself belongs to

> the set, is said to be a *group**. It follows that if the entire group is multiplied by any one of the symbols, either as further or nearer factor, the effect is simply to reproduce the group; or what is the same thing, that if the symbols of the group are multiplied together as to form a table, thus:-


Further factors.

Nearer factors.	1	α	β	..
1	1	α	β	..
α	α	α^2	$\beta\alpha$	
β	β	$\alpha\beta$	β^2	
	:			

Figure 4.5: Arthur Cayley's definition of a group.

> that as well each line as each column of the square will contain all the symbols 1, α, β... It also follows that the product of any number of the symbols, with or without repetitions, and in any order whatever, is a symbol of the group (p. 41).

In this definition there is a mention of identity, associativity, and closure properties, but the mention of inverses is missing. Figure 4.6 illustrates the view that elements of the group can be represented as bijective functions. Going back to the absent inverse property of groups in Cayley's definition, the question then becomes does the inverse of these bijective functions also live in the Cayley table? The visual patterns offered by Cayley tables provided hints to what would later be proved as *Cayley's theorem: every group is isomorphic to a subgroup of a symmetric group*.

$\cdot$	a	b	c	d	inputs
τ_a	a·a	a·b	a·c	a·d	outputs of τ_a
τ_b	b·a	b·b	b·c	b·d	outputs of τ_b
τ_c	c·a	c·b	c·c	c·d	outputs of τ_c
τ_d	d·a	d·b	d·c	d·d	outputs to τ_d

Figure 4.6: Viewing elements of a group as permutation functions.

A group can also be viewed as symmetries of a geometric object such as crystals, molecules, polygons and platonic solids. Polygons have been used by education researchers to set up exploration activities in which students identify symmetries and using these symmetries to construct a compressed list of algebraic rules that represents a minimal coding of the structure of dihedral groups. This minimal coding is called a group presentation, and is a list of generators for a group and relations among the generators. After working with several examples of polygons students then abstract and construct the definition of a group without prior exposure to the textbook definition. Through ongoing experimentation and observation, the learner discovers uniqueness of inverses and the cancellation law (Larsen, 2013).

Another type of sign used to represent groups are cayley color graphs. The vertex set of the Cayley color graph for a group G has size $|G|$, one vertex for each element in the group. For a choice of generators $\{s_i\}_{i=1}^{n}$, each generator is assigned a color. To construct a directed Cayley color graph each directed edge is of the form $(g, s_i g)$ and is colored with the color assigned to s_i. Cayley's (1878) paper *The Theory of Groups: Graphical Representation* provides the earliest and most well known documentation of representing groups as graphs. These graphs provide a visual way to detect and illustrate what it means for groups to be isomorphic. In fact, two groups are isomorphic if their Cayley graphs are identical for some choice of generators (Tao, 2010; Carter, 2009). Cayley digraphs with the addition of a metric space later gave rise to geometric group theory (Tao, 2010). Another modification of Cayley digraphs are Schreier coset graphs. For Schreier coset graphs the vertex set are cosets of the group with respect to a specified subgroup. The final introductory sign for groups that this study considers are Cycle graphs which display the cyclic subgroups of a group.

Despite the utility of Cayley to Schreier coset digraphs as visual representations of forming quotient groups, there seems to be little education research that investigates students' reasoning with them. This was corroborated through personal communication with an IOAA researcher. Moreover, none of the education papers gathered in the integrative review on intuition sample incorporated task-based activities with Cayley to Schreier digraphs. This was confirmed using NVivo 12 text search software and double checked using EndNote text search. Only one study, Weber and Alcock (2004), briefly mentioned digraphs. It came up as a piece of data from an interview question that asked a group of four algebraists about how they represented groups. One algebraist stated that they used a, "directed graph, symmetries of a geometrical object, and Venn-like diagram of a set partitioned by a sub-group and its cosets" (p. 218). Out of the four undergraduates that participated in Weber and Alcock's (2004) case study, none of the undergraduates gave representations of groups or isomorphisms other than the formal textbook definitions. Moreover, "The undergraduates were uniform in their responses. Each said that they would first compare the order of the two groups. If the groups had the same order, they would look at bijective mappings between the two groups and see if they were isomorphisms. The undergraduates claimed that they would not use their intuition to determine whether the two groups were isomorphic" (Weber & Alcock, 2004, p. 220).

4.3 Example-based intuitions towards conjectures

Griffiths (2013) defined intuition as "...the development of a personal theory which is not the result of explicit tuition but may arise either spontaneously or via some activity that is related only indirectly to the theory in question" (p. 81-82). The source is personal, the self, it is not something that is taken directly from instruction, a person of authority or text (Chiu, 1996). Intuitions are also characterized as cognitions that are "plausible or convincing in the absence of proof" (Davis, Hersh, & Marchisotto, 2012).

Example-based intuitions, are intuitions that are based on students' observations as they work with example cases of mathematical object. This form of intuition may involve implicit matching between external visual stimuli in the decision making environment and internal knowledge structures that results in a consciously registered pattern (Bubp, 2014; Kahneman & Klein, 2009). Kahneman and Klein (2009) who stated three general and necessary conditions that must be present for the intuiting process to detect patterns: sensorial cues must be present in the environment, the intuiting process must select pertinent cues, and there has to be a sufficient level of regularity or predictability in the environment so that matching associative processes can run and patterns can be recognized. Forms of intuition that are both example-based and visualization-based are investigated in this thesis.

This was done by providing students with various examples of groups along with various semiotic representations for each example object. Fischbein (1987) stated:

> After one has found that a certain number of elements (objects, substances, individuals, mathematical entities etc.) have certain properties in common one tends intuitively to generalize and to affirm that the whole category of elements possesses that property...According to Poincaré "generalization by induction, copied, so to speak from the procedures of experimental sciences" is one of the basic categories of intuitions (p. 59).

Conjectural intuitions were termed by Fischbein (1987) as, "an assumption about future events, about the course of a certain phenomenon etc. Such a conjecture is an intuition only if it is associated with a feeling of confidence" (p. 60). However, an intuiter is not always confident that there conjectural intuitions are true. In an interview, I pressed Dr. M about whether or not he felt confident that his conjectural intuitions in the context of his research contained truth or were at least plausible. He responded with,

Dr. M: I'm pretty cautious in terms of my conjectures even to myself so if I'm going to conjecture something to myself I probably have some kind of strong feeling about it like I've seen enough evidence or it to nice not to be true kind of feeling. Then most of the time when I have these ideas like that, there's no way I could ever answer it one way or the other. It's like just too broad. The best I'd be able to do is to be able to tackle one specific case in a particular nice situation and maybe say something about that.

He went on to explain why he was cautious.

Dr. M: Often times my conjectures are wrong because I'm making a broad generalization and then I have to go reconcile what I thought was a general picture with a specific thing that contradicts in a paper or a talk that I hear.

Based on interview data collected, self-report of confidence that a conjectural intuition is true will be characterized in this study as an attribute of varying degree, not a property of all conjectural intuitions. Given a pool of intuitions one can also apply the Non-creative versus creative forms attributes in 2.4.4.4 along with attributes outside this attribute set. Focusing on a subset of attributes may be a fruitful way to navigate the labyrinth of mathematical intuition.

4.4 Counter-example stance

Recall that Exception-barring and monster-barring are both subsumed under barring, which more generally, is the use of a counter-example or a "pathological case" to make modifications that turns a counter-example or pathological case into something that is neither a counter-example nor an example (Lakatos, 1976, p. 15; subsection 2.5.2). Monster-barring is the activity of refining a loose definition to make the pathological case, which Lakatos (1976) called a "monster", into a non-example of the main definition (p. 15). In exception-barring some collection of counter-examples are used to make modifications to the conjecture in a way that restricts the assumption clause. Larsen and Zandieh (2008) produced the framework for monster-barring and exception-barring in Table 4.1.

Table 4.1: Lakatosian methods of mathematical discovery framed by Larsen and Zandieh (2008, p. 209).

Type of activity	Focus of activity	Outcome of activity
Monster-barring	Counter-example & underlying definition	Modification or clarification of an underlying definition
Exception-barring	Counter-example & conjecture	Modification of the conjecture
Proof-analysis	The proof, the counter-example, & the conjecture	Modification of the conjecture & sometimes a definition for a new proof-generated concept

Based on Lakatos (1976) two categorizations of counter-example use with modification to the main conjecture emerged: barring or accepting which were not covered by Larsen and Zandieh (2008). The following steps show why exception-barring turns the counter-examples into something that is neither a counter-example nor example. First, for each counter-example in a collection, create a list of properties that the counter-example does not posses (i.e., anti-properties of counter-example). Next, take the smallest subset of properties that contains the intersection of the property lists and at least one property from each list. Once these properties are added as qualifiers to the assumption clause of the main conjecture the initial collection of counter-examples are no longer counter-examples and do not turn into examples.

In contrast to behavior of barring, accepting refers to viewing the counter-example as an interesting case that should be kept around and making modifications that turn this counter-example into an example that exemplifies a conjecture. Unlike "monster-barrers who contracted concepts" namely considering polyhedra only implicitly in the sense of convex polyhedra, "it is refutationists who expanded them" or "stretch them" to accept polyhedra beyond the notion of convex (p. 83-84). The refutationists refute the main conjecture when faced with legitimate counter-examples and take sides with the counter-examples. Refutationists look to make a new conjecture that is not a mere modification to an underlying definition or assumption clause of the main conjecture.

For example, rather than monster-barring to tighten the definition of polyhedra to convex polyhedra to save the conclusion clause of the main conjecture : " if P is a polyhedra then, $\chi = V - E + F = 2$", from the counter-example of non-convex Kepler polyhedra. Cayley accepted the non-convex Kepler-Poinsot counter as an interesting case and stretched the concept of Euler's characteristic formula to a similar formula that held for convex and non-convex Kepler-Poinsot polyhedra. This is an instance where "Counter-examples are turned into new examples-new fields of inquiry open up" (Lakatos, 1976, p. 127). It is inferred that those who take multiple stances, both barring and acceptance, towards counter-examples may develop a stronger intuition than those who just take a barring stance. In the end the main conjecture and its proof are never totally scrapped it is just massaged and modified, either restricted or extended, and whatever is left at then end will be a proof of some truth (Lakatos, 1976). The distinction being barring and acceptance of counter-example behavior is framed in Table 4.2.

Table 4.2: Teasing out barring and acceptance stances towards counter-examples and a main conjecture illustrated by Lakatos (1976)

Types of counter-examples	Signals a need to
Counter-examples to conjectures	Modify the main conjecture, a local conjecture or both
Pathological counter-examples	Modify an underlying definition or making hidden implicit assumptions explicit

Stance towards counter examples to conjecture	Use of counter examples	Type of modification	Counter-examples after modification
Barring - exclude counter-example and salvage main conjecture	Anti-properties of counter-examples are incorporated to modify conjecture or underlying definition	Restrictive modification to the assumption clause of a conjecture by adding more qualifiers or clarifying an underlying definition or hidden assumption	No longer a counter-example to the pursued conjecture nor an example
Acceptance - reject main conjecture and accept counter example	Counter examples are accepted as interesting cases and their properties are incorporated	Modifications to the assumption clause and conclusion clause are made to formulate and pursue a new conjecture	Turns into an example of the newly pursued conjecture

4.5 Affective impacts

There is a need for researchers to expand on the fields knowledge of intuition in relation to the affective domain (Fischbein, 1987). Goldin (2002) affirmed that "when individuals are doing mathematics, the affective system is not merely auxillary to cognition-it is central" (p. 60). In the specific case of intuition, affect can not be disregarded; it is impossible to understand one without the other because they co-occur (Sinclair, 2010). In the mathematics education literature the affective domain spans many constructs including beliefs, values, attitudes, self-efficacy, emotions, identity, motivation, engagement, etc (Grootenboer & Marshman, 2016). Although the entire affective domain is likely involved in the facilitation or interference of intuition it is beyond the scope of this research. This section briefly outlines some aspects of global and local affect that interact with the components of intuition: method, process, and outcome.

Global affect encompasses a person's stable beliefs, values, and attitudes related to a mathematics subject. Global affect may or may not be directed at the self. For example, one may have a consistent attitude that abstract algebra is fun. Such an attitude is directed outwards toward mathematics as a subject. On the other hand one may consistently lack confidence in their own mathematical abilities, this is stable and directed inwards toward the self. Local affect is defined as the shifting emotional states that occur as a reactive response to a particular experience, for example, excitement, comfort, fear, and anxiety (Goldin, 1998; Grootenboer & Marshman, 2016). Affect can impact intuition before the outcome is formed or it can be mixed with the outcome or in the case of guiding feelings it is the outcome (Sinclair, 2010). Using the language of intuition as a method-process-outcome triple in the integrative literature review may provide a basis for organizing different types interactions between affective factors and intuition.

The first of these interactions is that global and local affect can impact the intuiting methods or process component. Fischbein (1987) asserted that the affective combination of the "need for structure", "the fear of invalidity" and "preference for desirable conclusions" negatively impacts the intuiting process prior to the conscious intuition outcome (Fischbein, 2002, p. 196). This affective combination cuts the intuiting process short; a phenomenon termed "premature closure" (Fischbein, 2002, p. 195). "Closure" is adopted in this context from Gestalt Psychology. Whenever one perceives a physical object figures, basic lines and shapes, are encoded and the rest is filled in by an implicit matching process to relevant schemas (if they exist) to create a whole. Similarly, when presented with a mathematics problem, "we tend automatically to fill in, intuitively, the gap, to close the figure and reduce the uncertainties" (Fischbein, p. 195). The second interaction is that the intuition outcome may be accompanied or mixed with a local affect responses. For example, a conjectural intuition may be accompanied with strong positive emotions or non-emotional responses. The intensity the accompanied emotion is theorized to be a marker that separates creative and non-creative forms of intuition (Dane & Pratt, 2009). This was covered in 2.4.4.4. The third interaction is that local affect response linked to the intuition outcome can impact a learner's transition from an intuition outcome into a formal analog. Weber and Alcock (2009) used the framework representational system of mathematical proof (RSP) and non-RSP system to look at how learners were transitioning from formal and informal proof products. They found that undergraduate learner's who felt extremely satisfied with an informal argument formed in the non-RSP (informal system) had strong feelings of contentment and did not see the point in moving to the RSP system to formalize it.

Along these lines with a specific focus on intuition Fischbein (1987) stated that intuitions are self-evident cognitions meaning they are mental representations that are "directly acceptable" by their possessor prior to , "extrinsic justification-a formal proof or empirical support" and the prover's strong feelings of acceptance that accompany the experienced intuition may cause the prover to feel so confident that their intuition is true that they do not seek formal justification (p. 200). In contrast the previous examples, studies have also shown that emotion-based factors can support the move from intuition to a formal proof product (Kidron & Dreyfus, 2014; Fischbein, 1987). In other words, emotion-based factors can open or constrict the passage way the enables the prover to go from an informal representation or intuition(s) to a formal proof.

4.6 Summary

Several areas of mathematics education research were brought together in this chapter to create a framework to manage intuition research. This framework was comprised of an outer frame of intuition associated themes, inner frames for these themes, and units of analysis. Out of the four stated themes, the most progress was made on inner frames for representational fluency and example-based intuitions. The inner frame for representational fluency was the fluency digraph. It was a blend of Duval's (2017) modes of production, modes of aquisition which were emphasized by diSessa (2004), and modality which came from work by Arzarello (2006b). Modes of production refered to the registers that learners use to represent mathematical objects. Modes of aquisition was concerned with how learners acquire the representations and mathematical language that they use. And modality refered to two processing modes that learners may use, the use of one register at a time or multiple registers simultaneously. This inner frame led to an organization of several relevant units of analysis. Some of these units of analysis involved: the number of registers that learners use, the number of ways they can represent a particular mathematical object, if they acquire these representations from the instructor directly or through their own explorations, if they can see multiple instances of the same object in various contexts, and if they can add or collapse structure of a given object to access a new object.

In addition to the fluency frame, the inner frame for example-based intuitions offered a theoretical view of leaners' intuitions as singleton objects that take on various attribute values. This view of intuition was a product of several classification systems for the intuition construct that were developed by psychologists (Dane & Pratt, 2009; Glöckner & Witteman, 2010) and characterizations of mathematical intuition (Fischbein, 1987; Bubp, 2014; Kidron & Dreyfus, 2014). The next chapter puts these inner frames and units of analysis to practical use by aligning them with a methodology to investigate abstract algebra learners' fluency for several objects encountered in a first year group theory course and the make-up of their conjectural example-based intuitions related to group actions.

Chapter 5

Methodology

5.1 Introduction

A major goal of this thesis was to develop a methodology that could be used to investigate learners' representational fluency for quotient maps and example-based intuitions for group actions. This chapter begins with a review of different types of mixed methods research designs and case study research. Next, the mixed case study design that was chosen for the multi-part study in this thesis is presented. Data collection and analysis procedures are described for each part separately.

5.2 Review of mixed methods research designs

Tashakkori and Creswell (2007) defined mixed methods designs as "research in which the investigator collects and analyzes data, integrates the findings, and draws inferences using both qualitative and quantitative approaches or methods in a single study or program of inquiry" (p. 4). Narrative, phenomenology, grounded theory, ethnography, and case studies are categorized as qualitative methods (Creswell & Poth, 2018). Quantitative methods include experimental designs, longitudinal designs, and non experimental designs that incorporate instruments in which the data is numerically quantified. Researchers that choose qualitive methods, "support a way of looking at research that honors and inductive style, a focus on individual meaning, and the importance of reporting the complexity of a situation" (Creswell & Creswell, 2018, p. 4). Researchers that apply quantitative methods are focused on statistically measuring and evaluating relationships between pre-defined variables, their goal is often to replicate and generalize results (Creswell & Creswell, 2018).

In a mixed methods design, data collection is done systematically through quantitative means, such as surveys, and qualitative means, such as journaling, interviews, or observation notes. During analysis the researcher pays attention to both types of data to develop a more robust and informed interpretation of the data as a whole. The researcher utilizes qualitative analysis techniques such as inductive thematic analysis of verbal transcriptions and text. In addition, quantitative analysis such as descriptive statistics, t-tests, analysis of variance and covariance, factor analysis, or multidimensional scaling are carried out (Ross & Onwuegbuzie, 2014; Creswell & Plano Clark, 2018). Mixed methods designs have a long history in education research.

Creswell and Plano Clark (2018) layed out the historical development of mixed methods design in education beginning with the formative period from the 1950s to 1980s through the reflection and refinement period from 2003 to the present day. The formative period was illustrated by a transition from the first use of several quantitative measures to evaluate the same set of traits to the first studies that combined qualitative and quantitative methods. It ended with the advocation of the mixed qualitative and quantitative approach. A reflection and refinement period ensued and consisted of the organization of mixed methods studies through systematic reviews, clarification of different types of mixed methods designs, benefits and drawbacks of these designs, how to teach mixed methods to a new generation of researchers, characteristics of novice versus expert mixed methods researcher, and the skills needed to become more expert (Guetterman, 2017; Plano Clark, 2019).

General methodological trends in mathematics education research at the university level as well as specific methodological trends in the area of intuition research were examined in subsection 2.2.2 and subsection 2.2.3. This examination indicated that a majority of empirical studies rooted in theory on mathematical intuition applied solely qualitative data collection and analysis techniques. If quantitative analysis was done, it was limited to descriptive or univariate statistics. Despite the general emphasis and push for mixed methods studies in education research, there were only a few mixed studies, on university level participants, that placed an intentional focus on the intuition construct. A mixed design was pursued in this thesis to address this shortage. The remainder of this section covers different types of mixed designs in order to describe the multi-part study in this thesis.

Mixed methods designs are broadly typified as fixed or emergent. In a fixed design the decision of which qualitative and quantitative techniques to use is made prior to the start of the data collection process with participants (Creswell & Plano Clark, 2018). The variables that the researcher pays attention to and the administration of instruments such as surveys, semi-structured interviews, or manipulation of variables, in the case of experimental design, is planned prior to data collection. In an emergent design, the study begins as either a qualitative study or solely quantitative study. It is not until after the interpretation of results process begins, that the researcher decides to utilize the other to enhance their study. For example, a researcher may use qualitative techniques to collect data and use qualitative techniques like thematic analysis and later decide to turn themes into numerical data (Creswell & Plano Clark, 2018).

Along with the fixed and emergent characterizations, their are several taxonomies of mixed designs. These taxonomies were formed by methodologists to address questions such as: Is there a predominant qualitative or quantitative focus?, Where has the mixing occured, during data collection or analysis?, Are the qualitative and quantitative aspects carried out sequentially or in parallel?, How are the two types of data linked or how does one influence the existence of the other? (Creswell & Plano Clark, 2018; Leech & Onwuegbuzie, 2009). Leech and Onwuegbuzie (2009) provided a taxonomy of designs, in which the designs were characterized with respect to three facets: amount of mixing, timing of the quantitative and qualitative "concurrent versus sequential", and the amount of weight given to the qualitative or quantitative part (p. 268). Creswell and Plano Clark (2018) identified a taxonomy of three "core" mixed designs: convergent, explanatory sequential, and exploratory sequential (p. 65-67).

The convergent design consists of both quantitative and qualitative data collection instruments and analysis procedures. The two forms of data collection are done simultaneously within the same session. Either separate qualitative and quantitative data collection instruments are used or qualitative data is turned into numerical data. The analysis of the qualitative and quantitative data sets are done apart using techniques that are exclusive to each of the two traditions. The results compare, contrast, or join the two sets of analysis together.

An explanatory sequential design is employed when the variables of interest are known and quantitative instruments to measure these variables are already available. In an explanatory sequential design, the researcher starts with selection of already built quanitiative instruments and proceeds with quantitative data collection and analysis. Once the quantitative portion is completed the researcher follows up with qualitative data collection which seeks to add substance to interesting quantitative findings or to fill in gaps the quantitative portion could not fully capture. "The primary intent of this design is to use a qualitative strand to explain initial quantitative results" (Creswell & Plano Clark, 2018, p. 77).

In contrast to the explanatory sequential design, the exploratory sequential design is pursued in situations where there is no accompanied theory or quantitative instruments that can sufficiently assist the researcher with tackling their research questions. The researcher must begin with preliminary qualitative work to obtain a frame and figure out what the relevant variables might be. The exploratory sequential design starts with a qualitative design, Creswell and Plano Clark (2018) suggested grounded theory, to obtain a guiding frame and determine variables of interest. This frame is used to inform the construction of quantitative instruments. After quantitative instruments are constructed the quantitative data collection and analysis period begins. In an exploratory sequential design "Integration involves using the qualitative results (e.g., themes and significant statements) to build a new quantitative feature that is grounded in the culture and perspectives of participants. This new feature is then quantitatively tested" on a second, more sizable, sample of participants (Creswell & Plano Clark, 2018, p. 87).

The three basic core designs: convergent, explanatory or exploratory, can be layed into a more dominant qualitative or quantitative design. Such embedded designs are called higher-order or complex mixed methods designs. For example, a mixed methods case design lays core design in to the case study paradigm. One option would be to choose a convergent core design. When a core convergent is layed into a case design, the qualitative and quantitative data is collected and analyzed separately then brought together during the write up of the results to describe a case or compare multiple cases.

> a good joint display of the integration results would be the statistics by theme approach. Alternatively, cases might be placed on a quantitative scale along with quotes that suggest differences in the cases (Creswell & Plano Clark, 2018, p. 244)

Given the lack of mixed studies in the area of intuition, a need to understand, and the pilot like stage of this research, a mixed design was layed into a case study design.

5.3 Review of case study research designs

Case study research provides a concentrated and detailed view of an organization or group with common interests and goals, an instructional program, or individuals. Creswell and Poth (2018) defined case study research as,

> a qualitative approach in which the investigator explores a real-life, contemporary bounded system (a case) or multiple bounded systems (cases)

> over time, through detailed, in-depth data collection involving multiple sources of information (e.g., observations, interviews, audiovisual material, and documents and reports), and reports a case description and case themes (p. 97).

Case studies are characterized further according to the number of cases and type of analysis. The number of cases refers to how many individuals or cases are being studied by the researcher, one or several. Type of analysis refers ways in which the researcher examines or cross-examines cases and how findings in these cases are presented. Creswell and Poth (2018) summarized three types: an instrumental case study, collective comparative case study, and intrinsic case study. An instrumental case study is one in which the researcher choosing one case to highlight a particular phenomenon of interest. In a collective comparative case study several case studies are conducted and presented to portray phenomenon of interest. In an intrinsic case study several cases are examined and a spotlight is put an outlier case that exhibits a "unique" presentation of the phenomenon of interest (p. 74).

While discussing strengths of case study research, Merriam and Tisdell (2015) stated:

> It offers insights and illuminates meanings that expand its readers' experiences. These insights can be construed as tentative hypotheses that help to structure future research; hence, case study plays an important role in advancing a field's knowledge base (p. 51)...it can bring about the discovery of new meaning, extend the reader's experience, or confirm what is known (p. 44).

The written products of case study research, often stimulates deep conversations by drawing the readers in to interpret the presentation of a case or the phenomenon that the case brings into view, and to share their own experiences that are reactivated by reading about the case, things that they relate to, and points they either agree or disagree with. These conversations lead to possible generalizations, not in the experimental sense of generalizable results, but in the exploratory sense of abstracting commonalities from a few cases to formulate guesses about what might be the cause of certain local trends and eventually arrive at a hypothesis to investigate such trends within a larger participant population. In the setting of education published case studies are not just for the education researchers, they can reach many audiences and be used as training materials that inform educators of how learners at certain levels might be thinking about a concept and acts as a stimulus in a relaxed space for educators to brainstorm about how they might introduce a concept or improve a learner's experience (Merriam & Tisdell, 2015).

5.4 Design background for this three-part study

This three-part study used a mixed methods case study design to explore ways in which the two intuition associated factors: representational fluency and example-based intuitions could be evaluated. A single case in this context refered to an individual learner. Out of the approaches to analysis of cases, a comparative analysis was the main approach. Instances of unique presentations were also highlighted. The intent of this design was to: address the shortage of mixed studies that placed an intentional focus on the intuition construct at the university level, obtain rich qualitative data that tells a detailed story of individuals' representational fluency as well as their example-based intuitions with a restricted context, and start to develop and integrate quantitative techniques. The complex mixed design for this study was crafted by laying various spin-offs of the core convergent mixed design type into a more dominant case study design. Spin-offs of the core convergent design used were: data-transformative and parallel-convergent. Part I and II of this three part study was predominantly focused on the representational fluency factor of mathematical intuition. Part III focused on the example-based intuition factor described in chapter 4.

The plan for Part I was to incorporate a data-transformation mixed case design to draw out Abstract Algebra students' representational repertoire for small order finite groups. A card sorting task was used in this study as a data collection instrument. A design is transformative if qualitative data is transformed into numerical data. Creswell and Plano Clark (2018) described a data-transformation design:

> ...the researcher uses procedures to quantify the qualitative findings (e.g., creating a new variable based on qualitative themes). The transformation allows the results from the qualitative data set to be combined with quantitative data" (p. 73).

Several types of quantitative analysis for card sort data are available. Card sort data can be paired with similarity measures, multidimensional scaling and factor analysis, to name a few (Capra, 2005; Whaley & Longoria, 2009).

Part II used solely qualitative data collection and analysis techniques to investigate students' proficiency with collapsing and adding mathematical structure across various registers. Specific tasks were designed with semi-structured interviews to collect data. In these tasks collapsing mathematical structure involved defining an equivalence relation and quotient maps. Adding mathematical structure involved combining two groups operations to obtain a more complex group, possibly by forming a direct product or semi-direct product. Data analysis involved several rounds of thematic coding according to different theoretical lenses or perspectives. According to the *Networking Theories Group* a researcher may start of with a plan and a particular lense chosen prior to data collection. As data is collected the researcher may start to automatically associate pieces of data with other theoretical lenses that they have studied. These additional lenses may be tacked on to provide a more thorough analysis and get more mileage out of the data (Maracci, 2008; Bikener-Ahsbahs & Prediger, 2014). The initial round of thematic coding for Part II was focused on learners' representational fluency and was done according to a new four-level representational fluency coding framework. A second round of coding used the well estabilished *Nested Epistemic Actions Model for Abstraction in Context* (AiC) framework by Dreyfus, Hershkowitz and Schwarz (2015). The final rounds of data analysis for Part II were brief and brought in the default-interventionist and affective perspectives on intuition.

Part III used a parallel-databases convergent mixed case design to examine students' example-based intuitions as they worked with several examples of group actions and explored relationships between orbits and stabilizers of these actions. Creswell and Plano Clark (2018) described this design as

> a common approach in which two parallel strands of data collected and analyzed independently and are only brought together during the interpretation. The researcher uses the two types of data to examine facets of the same phenomenon, and the two sets of independent results are then synthesized or compared during the discussion" (p. 73).

Rather than accessing the quantitative data by turning qualitative themes into numerical descriptive statistics as in the transformative design type, the quantitative numerical data in a parallel database design type is now obtained by an instrument that was developed before data collection begins. In otherwords, the existence of the numerical data is separate from and not dependent on a qualitative thematic analysis of participant data carried out by the researcher (Creswell & Plano Clark, 2018). For part III, A new prototype survey instrument, *The Non Creative versus Creative Forms of Intuition Survey* (NCCFIS), was developed from prior theoretical work and piloted for the first time to obtain numerical quantitative data that characterized each intuition the student recorded along several graded attributes.

It had become increasingly important in part III to figure out how to make intuition quantitatively tangible in order to: provide a clearer characterization of intuition in a pure math setting, improve the viability of it as a research construct, and explore the utility of quantitative analysis techniques. For this study, the NCCFIS survey data was paired with Fuzzy C-Means Clustering. This quantitative analysis method was used to classify a collection of students' recorded intuitions according to attribute similarity. Journals were used to collect qualitative data for each of the recorded intuitions. In otherwords, both quantitative and qualitative data attached to an intuition was collected, concurrently, for every intuition recorded by the student. Results from the separate qualitative and quantitative data collection and analysis were merged to gain a richer view of the (partial) make-up of students' example-based intuitions.

5.5 Participants and settings

Introductory Abstract algebra is an advanced undergraduate course typically taken at the junior or senior level and is a primer for abstract algebra at the graduate level. Participants were recruited via purposive sampling using the condition that they had exposure to a first-semester undergraduate or graduate level abstract algebra course. Three students, Max, Jenni, and Alex, agreed to participate. They represented cases at three consecutive levels in the mathematics program at the same central US R1 level research university. This sampling could also be characterized as "maximum variation" sampling which "documents diverse variations of individuals" and "increases the likelihood that the findings will reflect differences or different perspectives" (Creswell & Poth, 2018, p. 159). Max was a fourth year undergraduate mathematics major who had completed a course in Abstract Linear Algebra. He was currently taking a first undergraduate abstract algebra course that began in late August. Data collection with Max occured during mid to late October. Jenni was a first year graduate student in the mathematics department working towards a Masters in Science (MS). She had completed a first and second semester sequence of the undergraduate abstract algebra course her junior year as an undergraduate. She did not have exposure to the graduate level abstract algebra course yet. The final participant, Alex, was a second year graduate student in the mathematics department who had passed the abstract algebra qualifying examination. He was working towards a doctorate degree. Participants at three consecutive mathematical levels illustrated a developmental progression in thinking for the same abstract algebra tasks.

They all received lecture-based instruction. The abstract algebra lectures at their university occurred either twice or three times weekly for a total of 2.5 hours plus an additional office hour. The undergraduate level courses used the open source text *Abstract Algebra Theory and Applications* by Judson (2018) and the graduate level text by Dummit and Foote (2004). The two-semester undergraduate course covered equivalence relations, the division algorithm, several classes of groups (e.g., abelian, solvable, cyclic, permutation, and matrix groups), cosets and Lagrange's theorem, isomorphisms, factor groups, direct-products, homomorphisms, group actions, rings, fields, and Galois theory. Applications to cryptography were introduced during the first semester. The graduate level course covered the same material with increased difficulty in homework exercises and additional topics such as Sylow theorems, modules, and semi-direct products. Jenni and Alex both received in-person instruction the previous years where as Max was introduced to abstract algebra online during the Covid-19 pandemic. A larger sample of participants was not available for this study. All three participants participated in part I and II.

5.6 Part I: Fluency digraphs for small order groups

Sandoval and Possanni (2016) articulated that the Theory of Registers of Representations, "can be used in research in two different ways. One use is to design teaching activities to promote the comprehension of mathematical objects through a flexible use of their different representations (as Pavlopoulou (1993) and Tanguay (2002) have done). A second way is to design activities to evaluate students' comprehension after they have finished a series of lectures, to help identify the difficulties students encountered when working with different representations" (p. 113). In order to explore methodological tools for intuition research in a pure math setting, part I of this study implemented a card sorting activity (CSA) instrument that was designed to evaluate students' representational fluency and determine what registers or *modes of production* students predominantly use for finite groups after exposure to introductory group theory instruction.

In a standard card sort activity, participants are given a starting set of interrelated items. Items may be domain-specific representations, concepts, physical objects, or word statements. A participant is then asked to sort the items into card piles. "In a card sort experiment, a sort is a partition of all cards in the stimulus set into non-overlapping subsets" or piles (Fossum & Haller, 2005, p. 140). This technique is rooted in Constructivist theory and, more specifically, George Kelly's Personal Construct Theory (PCT). From the perspective of PCT, a construct is a mental organization of how objects in the environment are alike or not and it is personal in that it is formed by an individual based on the way they see and interpret relatedness and non-relatedness (Kelly, 2020). It has been an effective tool for drawing out participants tacit knowledge - knowledge stored in memory that influences decisions, with absence of local awareness of the particular knowledge inputs that led to the decision output, and is not usually verbalized unless intentionally probed (Chervinskaya & Wasserman, 2000; Rugg & McGeorge, 2005).

In mathematics education research, card sorting has been used to extract observable aspects of learners' schema. Schema has been described as: an organization of knowledge, a hierarchical classification of concepts and connections between concepts, a network of connected components in which the components have varying levels of connectivity, a network where the vertices represent states and the edges represent a path from a present to goal state, and a collection of actions processes and objects (Skemp, 1979; Dubinsky & Mcdonald, 2001). For example, Eli, Mohr-Schroeder, and Lee (2011) used the *repeated single-criterion* card sorting technique outlined by Rugg and McGeorge (2005) to probe for aspects of pre-service teachers schemas, in particular, the types of mathematical connections these teachers make across geometry concepts. Card sorts have also been used to design teaching activities that encourage students' to make connections between graphical, algebraic, numerical, and contextual representations for functions (Hillen & Malik, 2013). The CSA Instrument in this study was designed specifically to systematically extract the vertex set, the set of sign-interpretation pairs, of each students' fluency digraphs for finite groups and associations between sign-interpretation pairs.

5.6.1 Instrumentation and data collection

Card sort activity (CSA) instrument

The CSA instrument was created in *Desmos* through the custom card sort activity builder, which allows educators to create their own card sort activities by importing images and writing text. The images for cayley graphs, objects of symmetry, multiplication tables, cycle graphs, and definitions for groups in terms of generators and relations were imported from *Group Explorer*. In addition to images imported from *Group Explorer Visualization Software*, signs of groups that consisted of generators as permutation functions in cycle notation were also included as cards. Participants were sent a url link through ZOOM chat to access the card sort activity for finite groups of order 6. The participant was asked to share their screen that displays their view of the cards and to use blank paper for scratch work. All activities were audio-video recorded.

The cards for the card sort activity were assigned a number and randomly scattered on the screen as shown in Figure 5.1. The images were given generic alphabetical labels for the elements in the group. This was done to prevent participants from making decisions based on surface level features such as seeing 'r, f' notation in a multiplication table and concluding that the table represents 'D_8' without paying attention to relationships among the elements.

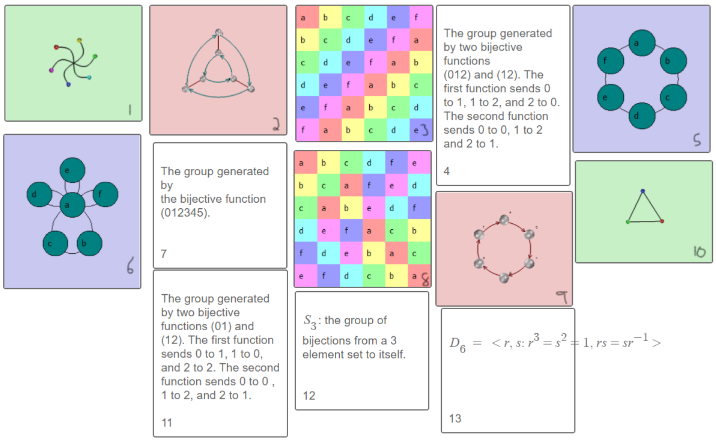


Figure 5.1: Cards for groups with order 6.

Card sorting formats may be open, closed, or both. An open card sorting format in which the participant freely makes sorts as they see fit without constraints regarding the size or number of groupings, or the "criterion" they use to make piles. "A criterion is the attribute used as the basis for a sort when using sorting techniques" (Rugg & McGeorge, 2005, p. 95). The collection of sorts that are formed partially mirrors the participant's mental organization, the patterns they are attuned to, and how they interpret these patterns. In a repeated single-criterion open format the participant chooses a pile of 2 or more cards out of a larger set that they think are similiar and assigns a single similarity or dissimilarity criterion that cards in the subset possess. This is repeated on the same set until the participant can't make any further subsets with distinct criterion (Rugg & McGeorge, 2005; Eli, Mohr-Schroeder, & Lee, 2011).

In a closed format the researcher places restrictions such as fixing criterion and then instructing the participant to form piles based on the fixed criterion. Alternatively, the researcher may fix piles and instruct the participant to provide similarity or dissimilarity relationships between the fixed piles. Aside from open, closed, or combined formats, there are many variations of card sorting techniques. The technique that researchers choose to use depends on the psychological structures or processes that they are attempting to target and the theories that these structures or processes are embedded in (Rugg & McGeorge, 2005; Fossum & Haller, 2005).

A closed "'All in one' sort" technique outlined by Rugg and McGeorge (2005) was used to collect data in the first activity session with more specialized theory-driven and goal-directed modifications to these techniques that align with the fluency digraph construct (p. 95). In a typical application of a closed *'All in one' sorts*, the respondent sorts cards one at a time into piles based on criterion fixed by the researcher (Rugg & McGeorge, 2005). In this study participants were asked to sort cards into piles that represent the same group. The participant was also instructed to make each card pile maximal. A maximal card pile for a Group G was defined to be a set of cards M where each card in M represents G and is there is no larger set of such cards M' that the participant can come up with where M is contained in M'. The sorting criterion that was fixed by the researcher was 'piles must represent the same group' and 'piles must be maximal to the participant'. Participants were also instructed to think aloud as they formed piles.

During this session the researcher deviated from a typical closed format by implementing an embedded open criterion format in which participants came up with their own reasoning criterion for why cards were added or not added to a pile as they worked towards their maximal piles. Some possible reasoning criterion for open sorts may incorporate: the number of generators, relations between generators, presence of cyclic subgroups, presence of normal subgroups, orders of subgroups, orders of elements, abelian or not, etc. The student kept a record of their piles indicating each card by its assigned number and their reasoning criterion as they worked towards their maximal piles. The data collection procedure was carried out for each participant separately.

Semi-structured interviews during CSA

The looped prompt below was used by the researcher to encourage participants to add their own sign cards and any additional interpretations. The looped prompt was used to extract as many sign-interpretation pairs as possible.

1. Researcher: Are there any additional ways you like to represent or think about a group of order (insert order) that is not given by a card on the board? (If the participant answers yes, then allow the participant to make their own card and add it to the card set or to make an additional interpretation to an already existing card.)

2. Researcher: Are there any other cards you want to add or remove from this pile?

Is this your maximal card pile? (If the participant affirms that their pile is maximal, then move on to the next activity. If the participant states that their pile is not maximal or is hesitant, move onto the next prompt.)

3. Researcher: I see that you are unsure. Take some more time to think about your piles, decide whether or not you want to make any modifications to your piles. (Once the participant is finished taking time to think more and possibly make modification repeat question 1. Continue this loop until the participant affirms that their pile is maximal or are unable to make any further modifications).

Mode of acquisition survey

After the participant confirmed that their piles were maximal data was collected on the *mode of acquisition*. The survey consisted of four statements from non-sanctioned to sanctioned semiotic representation aquisition. During this pilot the researcher displayed each sign card one at a time and asked the participant to chose the statement that best described how they acquired their thinking attached to the sign. However, this survey procedure focused on the collection of interpretations attached to the sign rather than each individual sign-interpretation pair (SR). Next time, the researcher could state a recorded sign-interpretation pair produced by the participant and ask the participant to chose the statement that best described how they aquired it. The participant may have multiple interpretations for a fixed sign. This modified survey procedure could be repeated for each stated SR individually until all observable SRs are typified according to mode in which they were aquired. In future studies, the cards could be swapped out for different cards, depending on the mathematical object of interest, registers for that mathematical object, and level of the learner. In addition, the data collection procedure on participant's fluency digraphs could be automated and made more efficient by using a computer.

5.7 Part II: Representational fluency during collapsing and adding structure tasks

For part II, data was collected on the registers student's naturally use to support their reasoning for collapsing structure (i.e., quotients) and adding structure (i.e., direct product or semi-direct product) tasks. These tasks were designed to investigate students' flexibility in converting their initial natural response in one register to an alternate register.

A methodological tactic when Duvalian semiotic theory is used is to design treatment tasks that are contained within some predominant register chosen apriori or to design conversion tasks in which the researcher fixes the order of the initial and target registers are beforehand. For example, McGee and Moore-Russo (2015) phrased a task in a way that directed the participant to go from a numerical to geometric register and created additional tasks that were phrased to direct participants to go from other apriori initial and target choices. Part II of this study experimented with a different tactic. Rather than creating several different tasks with a apriori fixed initial and target, the researcher showed stimuli for various registers and let the participant set their inital register(s) and complete their response before introducing switch prompts that directed them to alternate target registers.

5.7.1 *Instrumentation and data collection*

Collapsing and adding structure task set up with semi-structured interviews

The collapsing structure task in total consisted of four major sub-tasks: determining if the dihedral group and quaternion group of order 8 were isomorphic or not, constructing a homomorphism or quotient from the dihedral group to an image group, repeating the previous sub-task for the quaternion group, and showing an isomorphism between the image groups. The researcher asked the question"Is the quaternion group of order 8 and the dihedral group of order 8 isomorphic?" followed by "Is there some way that you can modify both D_8 and Q_8 by losing information or collapsing some structure so that the resultant modification made to D_8 is a group that is isomorphic to the resultant modification to Q_8?" Participants were also given another version of the task prompt depicted in Figure 5.2 with question marks that indicated what they needed to find. The homomorphism metaphor of losing information could invoke the loss of information of the orders in the pre-image and collapsing could invoke combining elements to form a partition. The researcher set up the collapsing structure task by arranging cards into

$$\begin{tikzcd} D_8 \arrow[d, "?"'] & Q_8 \arrow[d, "?"] \\ ?G_1 \arrow[r, leftrightarrow, "\cong"] & ?G_2 \end{tikzcd}$$

Figure 5.2: Collapsing Task Prompt.

piles as displayed in Figure 5.3 and the adding structure task by arranging cards into piles as shown below in Figure 5.4. The participant was asked to build a larger group of order 16 by combining the groups D_8 and Z_2.

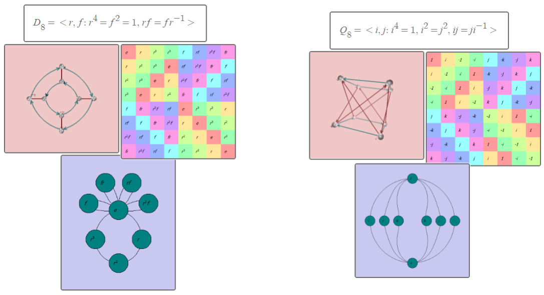


Figure 5.3: Collapsing structure activity set up.

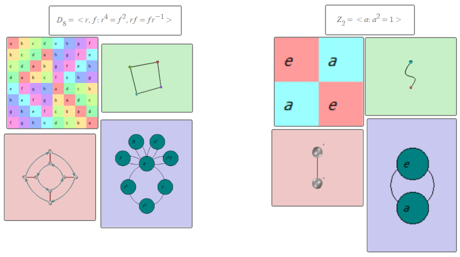


Figure 5.4: Adding structure activity set up

After the register cards were put in front of the participant, the researcher left it open to the participant to choose an initial register and noted the participant's first natural response. If the participant's natural first response consisted predominantly visual intuitive response in the collapsing activity, such as performing quotienting processes in a digraph register, the researcher followed up with the first question below to see if they could flex to a more rigorous formal response or registers. If the participant provided a predominantly formal response that involved writing down cosets or explicitly stating a formal-symbolic map from a domain group to a quotient group the researcher asked the second question below to see if they could flex to a more visual intuitive response. The following two questions were used to investigate whether or not students could flexibly go from one register or mode of representation to another:

> Question 1: Can you translate what it means to lose information or collapse structure like you did in the case of the dihedral group of order eight and Q_8 using more formal language? Can you explicitly write down what the homomorphism maps are?

> Question 2: Can you translate what it means to lose information or collapse structure like you did in the case of the dihedral group of order eight and Q_8 using the digraphs?

Similar to the collapsing activity, the adding structure activity also evaluated how students flexed from a visual intuitive responses, such as using Cayley digraphs or symmetries of geometric objects, to formal responses or vice versa. If the participant provided a predominantly formal natural response, such as using the group presentation, performing calculations, and taking the formal definition of a cross product or semi-direct products then the researcher followed up by asking students if they could use language that described visually how to combine the groups using Cayley digraphs.

Follow-up interviews were scheduled with students during study parts II and III for several reason. One reason was to obtain additional detailed aspects of the data that were relevant to the research questions. A second reason was that it was unclear what the participant was thinking or meant during an inital semi-structured interview. Follow-up interviews were needed to interpret the data more carefully. A third reason, which came up during part II, was that negative affective interference caused one of the participants to shut down.

5.7.2 *Qualitative parallel data analysis*

It is estabilished practice for mathematics education researchers to conduct side by side "parallel analysis" which is a first round of analysis from the perspective of one theory and a second round of analysis on the same data set from the perspective of another theory. These analysis are conducted separately from one another (Maracci, 2008, p. 274, Prediger, Bikner-Ahsbahs, & Arzarello, 2008). For example, Maracci (2008) analyzed data of undergraduate and graduate students reasoning of vector space theory twice, first from the perspective of tacit intuitive models and then from the perspective of process-object duality. Moreover, it is common during parallel analysis to use theories with different "grain sizes" (Halverscheid, 2008, p. 228). Theories, are often called lenses, as they dictate what is seen in the data. Grain size refers to the magnification of these lenses as either "macroscopic" or "microscopic" (Halverscheid, 2008, p. 228). Applying multiple lenses to the same set of data allows these networking theories researchers to view the data from various angles of emphasis.

For part II of this study, data collected during the collapsing structure activity was analyzed from the perspective of a four level coding framework for representational fluency developed. After becoming more comfortable with this first analytic framework, a decision was made to conduct a parallel-analysis of data collected during the adding structure activity from the perspective of two lenses: 1) the four level coding framework for representational fluency developed in this thesis and the 2) *Nested Epistemic Actions Model for Abstraction in Context* estabilished by Dreyfus, Hershkowitz and Schwarz (2015). The second lense was added after reviewing the data several times and repeatedly coming across important aspects of the data related to the conceptualization of intuition in chapter 2 that the representational fluency lens could not bring out. In particular, these aspects resembled matching associative forms of intuition, such as recognition based on perceptual cues, and intuitive representations characterized as gradually constructed mental representations that began with vague sporadic ideas, implicit assumptions, and loose sketches towards more integrated and communicable constructions.

Analytic lens 1: Four-level coding framework for representational fluency

To begin data analysis for the part II collapsing structure task, all audio files were transcribed into text. A case folder for part II was created for each participant. Each folder contained audio, video, and whiteboard screenshot files from Zoom interviews, scans of written scratch work on paper or a personal tablet. Each type of data was reviewed several times and evolved from unstructured to structured note taking. These notes included cues that students noticed or did not notice in certain registers, in what order did they visit registers, in what way did they use registers, for example a source to stimulate ideas or as a check, and conversions from natural registers that they started in to a prompted register. These notes were used to construct a chronological summary narrative for how each participant's approach to the task evolved. These narratives highlighted the participant's interpretations, strategies, and register use.

During the process of writing up, comparing, and contrasting the chronological narratives, I noticed that the three participants invoked many interpretations of what losing information, collapsing and adding structure meant to them within the context of the tasks. The set of interpretations the participants carried with them was sometimes different for different strategies that they explored. Several strategies were identified in the interview transcripts such as obtaining Z_4, obtaining $Z_2 \times Z_2$, and obtaining Z_2. The impossible strategy occured when the global interpretation, for collapsing structure or losing information, constructing a homomorphism was carried to the strategy to obtain Z_4. Participants also used multiple registers, in different ways, within the same strategy and exhibited multiple sign-interpretation pairs for the same sign. This preparatory process helped me to see and sketch the analytic coding framework in Figure 5.5. In this framework, Level 1 consists of global collapsing or adding structure interpretation themes. Level 2 lists the strategies that the learner enters. Level 3 contains the registers that the learner uses the researcher made notes of registers that were naturally entered versus registers that were entered after a switch prompt was applied. Level 4 is the most detailed level and consisted of collecting an analyzing learners' sign-interpretation pairs within registers and conversions between registers. Registers in which learner's could perform quotient map or homomoprhism processes that were of particular interest included: formal-symbolic mapping, digraph, and Cayley table registers.

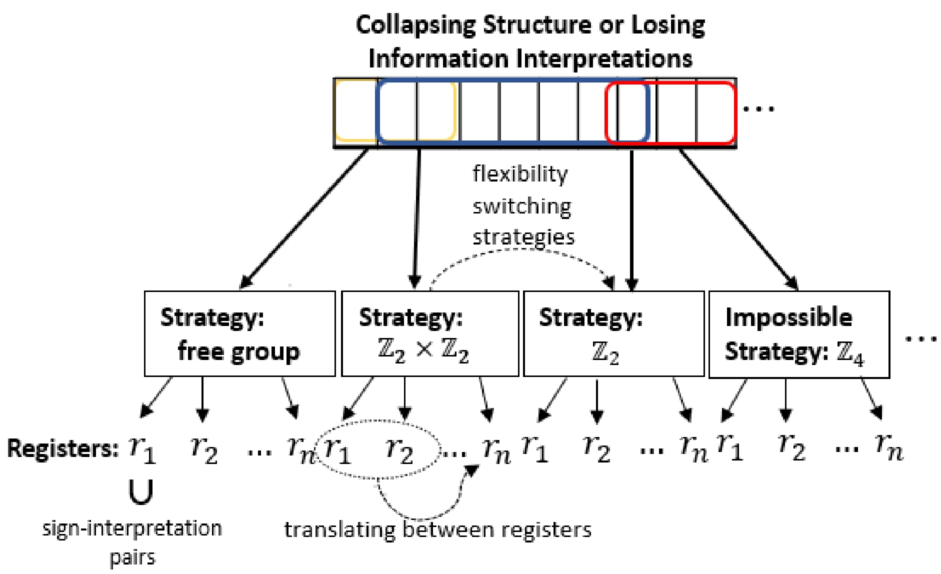


Figure 5.5: Four level analytic framework for representational fluency task.

To prepare for coding, the audio video files were imported into the Qualitative Data Analysis Software NVivo 12. The transcribed text files were then appended to the corresponding audio video files along with descriptions of their scratch work. A code book that mirrored the four level analytic coding framework in Figure 5.5 was set up in NVivo 12 manually as shown in Figure 5.6. The parent nodes: interpretation of ignoring or collapsing structure, strategy, register, and sign-interpretation pairs and conversions were selected prior to coding. Additional codes were created as sub nodes under these parent nodes through selective and open coding. Selective coding means that codes are selected prior to coding, for example the register codes Cayley tables, Cayley digraphs, group presentations, etc., were selected prior to coding. Open coding is the process of abstracting and labeling regularities or perhaps unique instances that are found in the

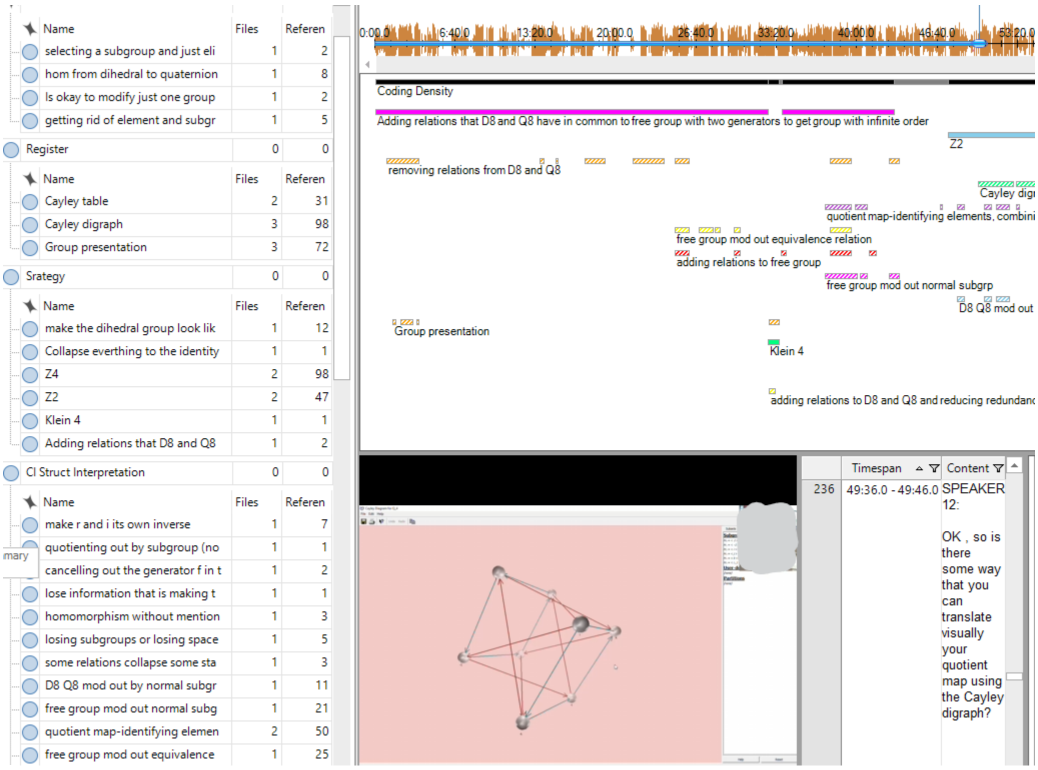


Figure 5.6: Data analysis coding framework in NVivo 12.

data. Open or combined codes are not realized until the researcher immerses themselves in the data (Creswell & Poth, 2018). The NVivo 12 coding display helped to analyze overlapping codes. After coding, a summary list of codes across all participants was transferred into table Table 5.1,Table 5.2, Table 5.3, and Table 5.4. From this coding table one can read off aspects of learners' paths in Figure 5.5. For instance, Max carried the collapsing structure or losing information interpretations with codes 1.1, 1.2, 1.2, and 1.4 to the impossible Z_4 strategy. He performed homomorphism constructions within this strategy in several registers. A specific sign-interpretation pair that Max produced in the digraph register for the impossible strategy was given by code 4.11.

Table 5.1: Code list summary for collapsing or losing structure interpretation themes.

1. Collapsing Structure or Losing Information Interpretation Themes	Cases
1.1 Restricting focus to a particular elements, subsets, subgroup and not worrying about the rest of the group as a whole	M, J
1.2 Losing subgroups or space in the subgroups	M
1.3 Get rid of what is making the two groups different and keeping what they have in common	M, J, A
1.4 Homomorphism (without mentioning quotient map)	M
1.5 Quotient out by subgroup (without awareness that the subgroup needs to be normal to form a factor group)	J
1.6 Crossing out the generator f, getting rid of relations with an f, and getting rid of words with an f	J
1.7 My initial thought was just getting rid of elements is the way to go to leave me with a subgroup and then maybe the quotient groups will help me do that and then thinking well if I quotient by a subgroup that is not just leaving me with a subgroup	J
1.8 Combining or identifying elements into sets of elements	J, A
1.9 Fixing generators, adding word relations to group presentations and reducing redundancies to get a smaller group	A
1.10 Some relations collapse some stay the same	A
1.11 Mod out by an equivalence relation, partitioning	A, J
1.12 Quotient out by normal subgroup to get group of order less than eight	A
1.13 Identifying elements by introducing a relation	M, A
1.14 Free group mod out normal subgroup	A
1.15 Free group mod out equivalence relation	A
1.16 Fixing generators and removing relations in group presentations to get larger group	A

Table 5.2: Code list summary for strategies entered.

$\rightarrow$	2. Strategy (non-trivial)	Cases
M-1.1,1.3,1.13	2.1 Make the dihedral group look the same as the quaternion group/collapsing the dihedral group to the quaternion group	M
M-1.1,1.2,1.3,1.4	2.2 Obtain Z_4 (the impossible strategy if paired with hom. interpretation)	M, J
J-1.1, 1.3, 1.5, 1.6, 1.8, 1.11	2.2 Obtain Z_4 (the impossible strategy if paired with hom. interpretation)	
J-1.5, 1.7, 1.8, 1.11	2.3 Obtain Z_2	J, A
A-1.8,1.10,1.11,1.12	2.3 Obtain Z_2	
A-1.8,1.9,1.11,1.12	2.4 Obtain $Z_2 \times Z_2$	A

A-1.3,1.10,1.14,1.15, 1.16	2.5 Adding relations that D_8 and Q_8 have in common to obtain infinite group: $F =< x, y \| x^4 = 1, xy = -yx >$	A

Table 5.3: Code list summary for registers use.

3. Registers	Cases
3.1 Cayley table	M, J
3.2 Formal-symbolic function maps	M, J, A
3.3 Cycle graph (asked to switch)	A
3.4 Cayley digraph (brief, not asked to switch)	M
3.5 Cayley digraph (asked to switch)	M, J, A
3.6 Schreier coset digraph (natural or after 3.5)	J, A
3.7 Group presentations,listing elements in the group using group presentation, without thinking about the free group	M, J
3.8 Subgroup lattices (generated on own not asked)	A
3.9 Group presentation with thinking about the free group	A

Table 5.4: Code list summary for sign-interpretation pairs.

4. Verbalized interpretations attached to signs	Cases	Register	Strategy
4.1 The dihedral group and quaternion group only differ in their products of the last four elements with each other	M	3.1	2.1
4.2 jj is i^2 and i^2j is itself	M	3.1	2.1
4.3 take the four cycle in Q4 that doesn't match with the 2 cycle in D4 and somehow make that a 2 cycle [looking at lower right quadrant]	M	3.1	2.1
4.4 last four elements are all there own inverses [in D4]	M	3.1	2.1
4.5 the diagonal with the e's[identity] on D4 and then the r^3 on one side of that element and the r on the other side of the diagonal those are like the same elements as the i^3 and in the other cayley table	M	3.1	2.1
4.6 make r, i, and j its own inverse	M, A	M-3.1, A-3.7	M-2.1, A-$Z_2 \times Z_2$

4.7 D4 and Q8 both have three subgroups of order 4 and D4 has a bunch of these subgroups of order 2. I don't know what a function would be the collapses those subgroups, but if there was some way to get those out then it would be Q8	M	3.4+sglst	2.1
4.8 elements in the old group will go into what we wanted the group properties to be in the new group so r^4 and f^4 collapse to the identity	M	3.2+3.7	Z_4
4.9 r [a D4 generator] and i [a Q8 generator] have order 4 so I can take the second generators f and j and collapse them to the identity and then map r to generator in cyclic subgroup of order 4 and map i to the generator in cyclic subgroup of order 4	M	3.2+3.7	Z_4
4.10 If you only had the two generators, you really only need to describe what it does to the two generators. Right?	M	3.2+3.7	Z_4
4.11 So take the inner green cycle of order 4 pull it out and let it hangout by itself and this is the new group. And you don't need to worry about the relationship between these guys and these guys because those guys don't exist anymore [Dihed. group]	M	3.5	Z_4
4.12 I'm seeing four four cycles and can't find the two cycle [Q8]	M	3.5	Z_4
4.13 The cycles in between the inner and outer green squares are just like more complicated. And so I selected one of the, either the top or the bottom of that and took it off by itself then I don't have to worry about what they do with the other things on the bottom, because I 'm just looking at the top	M	3.5	Z_4
4.14 They both have subgroups of size four. So if you have D_8 and you remove all the flips and just have their rotations, and then you take Q_8 and reduce it to just the generator i then you'll have i, -i, 1, -1	J	3.7	Z_4

4.15 The flips are a generator so if I get rid of it then I will get rid of all the elements with a flip in it as well so you would just be left with the rotations [Wrote out elements using group presentation and removed any elements with an f in it]	J	3.7	Z_4
4.16 Removing the flip generator and removing any relations with an f in it gives the group presentation of Z_4	J	3.7	Z_4
4.17 $< r >$ and $< i >$ are isomorphic	J	3.1	Z_4
4.18 If I physically had the [Cayley table] cards, I would cover them up and compare them	J	3.1	Z_4
4.19 I know the group generate by r is going to by isomorphic to the cyclic group generated by i, but I can't think of how to formalize that collapse... so its like Q8 quotient it out by the cyclic subgroup generated by i, that would be a proper group right? Because its the quotient of a group and it's a subgroup of it	J	3.2+3.7	Z_4
4.20 The quotient groups only have two elements. They would be isomorphic to Z_2	J	3.2+3.7	Z_2
4.21 The quotient groups are cyclic and therefore abelian	J	3.7	Z_2
4.22 I'm trying to think of a property for the quotient map to be well-defined	J	3.1+3.2+3.7	Z_2
4.23 In D_8 it's like removing the red lines because that would be the flip	J	3.6	Z_4
4.24 If you get rid of the f then you collapse these points to a point and you end up with a square. Getting rid of the generator f could collapse these lines and you would end up with Z_4	J	3.6	Z_4
4.25 I guess the top squares [referring to Q_8] that are connected by the green lines would collapse into one and then the bottom elements that are connected by the green would collapse into one. And then all of the red lines would just collapse into one red line. So again, we'd have Z_2	J	3.5→3.6	Z_2

4.26 Take out $f^2 = 1$ and $i^2 = j^2$ then, the generators i and r play the same role and f and j play the same role	A	3.9	Infgrp
4.27 Since it isn't specified y is infinite order so $< y >$ is the group **Z**	A	3.9	Infgrp
4.28 Taking out the relations $i^2 = j^2$ just means that I would still go around in a circle, but the j doesn't necessarily go around the same circle anymore, it does not have the cycle going -1, -j, 1, and back to j	A	3.3	Infgrp
4.29 Both groups [dihedral group and quaternion group] started as the free group over two elements and then you mod out by relations to get the groups	A	3.2+3.9	Infgrp
4.30 Putting more relations in is going to make more words be the same word so you're identifying more and more elements by putting in more and more relations so you are going to get fewer distinct elements. More relations means fewer elements and fewer relations means more elements	A	3.2+3.9	Infgrp
4.31 The normal subgroups dictate what the relations are because they will say whatever's in the normal subgroup is going to be effectively the identity	A	3.2	Infgrp, Z_2, $Z_2 \times Z_2$
4.32 There is still this cyclic group of order four and there is this anti-commutativity where $xy = -yx$	A	3.9	Infgrp
4.33 In $Q8$ you have this element k but if you think about it in group presentation form it is ij and if you get rid of $i^2 = j^2$ the element that was k acts like a completely different element now so $Q8$ does not sit inside of F and similarly the dihedral group does not sit inside of F, but they are quotients of it in some way	A	3.2+3.9	Infgrp
4.34 If you add $f^2 = r^2$ to the dihedral group and $i^2 = 1$ to the quaternion group I think you just the the Klein-4 group. So it just shrunk down to this really pretty little group	A	3.5→3.6	$Z_2 \times Z_2$

4.35 So since we are modding out by the subgroup generated by i, the vertices i, 1, -i, and -1 would become one element...So if i, -i, -1, and 1 are a coset and those go together, then the other elements would be the other coset	A	3.5→3.6	Z_2
If we collapsed one negative, one i and negative i into one element, then those four go to one. And then outside of that one, there would be four others leftover. So with just that one collapsing, there's still five elements. So then the other four elements that are not that new crunched one would also collapse into one element	A	3.5→3.6	Z_2
4.36 And on here, I guess the, the red arrows between one eye negative one and negative, i, those arrows would collapse because all those elements would become one element. And then the other elements, the j negative jk and negative k would collapse to be the other coset. And so the relation, like you'd ended up having, um, just one relation between those two that goes from the i coset to the j Coset and then back to the i Coset	A	3.5→3.6	Z_2
4.37 So some of these relations wouldn't collapse, they'd just become the exact same arrow, like for example, this green arrow between negative J and one that doesn't collapse, but it's going to be the exact same relation. Yeah. All those green arrows down on the bottom that go from one to J to negative one to negative J and back to one, those would turn into those don't collapse. They just become the same relation. And then same as the green ones up at the top. Those wouldn't collapse, they just become the same relation.	A	3.5→3.6	Z_2

4.38 I guess those are oriented. They wouldn't necessarily all be the exact same relation, but it make it a two way relation. So you'd have this, all the eye that got crunched down and then all the JK that got crunched down and there'd be a two-way arrow between those.	A	3.5→3.6	Z_2
4.39 I think for this one [dihedral group], the inner circle, those are going to all collapse into one element and then the outer circle would collapse into one element [dihedral group]	A	3.5→3.6	Z_2
4.40 This time the green arrows are going to be the ones where you collapse into an element and collapse into an element. Um, so that would leave the two-way red arrows, but then they would end up becoming just the same two way, one relation [dihedral group]	A	3.5→3.6	Z_2

Conflicts between collapsing or loosing structure interpretations, strategy, and sign-interpretation pairs and mathematical reality were kept track of through note taking. At this point the concept image and concept definition distinction initially proposed by Tall and Vinner (1981) felt relevant, but they did not come at things from the predominant angle of semiotics or Duvalian theory with sign. The vertical collapsing or ignoring structure interpretation, strategy, register and sign-interpretation pairs strands may include particulars from many concept images. Therefore, we say that a strand is in conflict with mathematical reality if it contains descriptions that are not consistent with formal definitions, theorems, or what an associated mathematical object actually does.

For example, an interpretation of collapsing or ignoring structure as a homomorphism, taken to the strategy to obtain the Klein-4 group, in the Schreir graph register, with the particular sign-interpretation pair that a homomorphism is like running through a quotient map and so it is like identifying vertices and collapsing them to a coset vertex is consistent with mathematical reality. Taking a the interpretation of collapsing or ignoring structure as a homomorphism, to the strategy to obtain Z_4, in the Cayley graph register, with the particular sign-interpretation pair that involves erasing vertices and edges is not consistent with what a homomorphism actually does. These inconsistent strands were often accompanied with pseudo-semiotic representations in which the signified object that was assigned to the sign-interpretation pair by the learner does not exist mathematically or is not accurately portrayed. The pseudo-semiotic phenomenon was first found in the data, it was not defined prior to data collection. The literature was revisited to try and find existing constructs that resembled the observed phenomenon and were already estabilished by other researchers that also incorporated a semiotic lens.

One partially relevant link that was found was Tsamir and Ovodenko (2013). They combined Duval's articulation of conversions and reverse conversions, Fischbein's (1993) distinction between intuitive and formal algorithmic processes, and the concept image and concept distinction made by Tall and Vinner (1981) to conduct an error analysis of learners reasoning with inflection points. From the Duvalian semiotic perspective, they focused their analysis to students difficulty with recognizing the same mathematical object across algebraic-symbolic, geometric, and verbal registers and cases in which a learner was successful in making a conversion from an initial to target register, but could not perform the reverse conversion from the previous target to initial register. There was an overlap between theories they used to frame analysis and the theories presented for this study, however they did not note the pseudo-semiotic phenomenon. Next, the search was turned to broader encompassing constructs.

A pseudo-semiotic representation could be thought of as a specialized instance that fell under the more general umbrella of pseudo-conceptual behaviors proposed by Vinner (1997), hence the added qualifier "pseudo" to the semiotic representation construct. Vinner (1997) termed "pseudo-conceptual behavior to describe a behavior which might look like conceptual behavior, but which in fact is produced by mental processes which do not characterize conceptual behavior" (p.100). Vinner admitted that this definition was not satisfactory and provided more explanation and examples of what he meant by pseudo-conceptual behavior. He explained that pseudo-conceptual behaviors or processes are "simpler, easier, and shorter than true conceptual processes" (p. 101). The more novice learner may gravitate towards pseudo-conceptual processes because they are "simpler, easier, and shorter" (p. 101). He noticed that:

> When presented with a task, they start looking for ways that will enable them to perform the task. These ways are not necessarily the way thought by the designers of the task when they decided to present it to the students. The task designers probably intended conceptual thought processes; the students came up with pseudo-conceptual thought processes...formed in a spontaneous way...and not necessarily taught to them by teachers or other

> agents. Sometimes they are the natural cognitive reactions to certain cognitive stimuli. The students use them without going through any reflective procedure, control procedure or analysis of any kind (p. 101).

Pseudo-conceptual behaviors that have been described within a semiotic frame includes gesture and speech mismatch. Those with a greater number of gesture-speech mismatches are said to be in a "discordant state" and those with very little gesture-speech mismatches are in a "concordant state" (Alibali & Goldin-Meadow, p. 468, 470, 485). Gesture and speech mismatch generalizes to the notion of conflicts in concept expression between specified registers.

At this point in the analysis of the present study, the data was combed through a final time to try to systematically identify and collect any pseudo-semiotic representations that were made by the participants for quotients or homomorphisms. While a quotient is essentially the same as a homomorphism, the distinction was made during coding to reflect the language that the learner was using. An spreadsheet was created for each participant with the columns: sign, interpretation, signified object, and register. Rows where the signified object was the same, but the registers changed indicated possible points were a conversion or pseudo-conversions, improper signification of underlying object tracked across multiple registers, might have occurred. Tapes were reviewed at those point to make a final assessment. Coding pseudo-semiotic representation was fairly straightforward, but coding became too difficult when trying to systematically track pseudo-conversions in the open format task moreover it felt like double counting pseudo-semiotic representations. Figure 5.7 gives a theoretical illustration of a pseudo-semiotic representation as a unit of analysis.

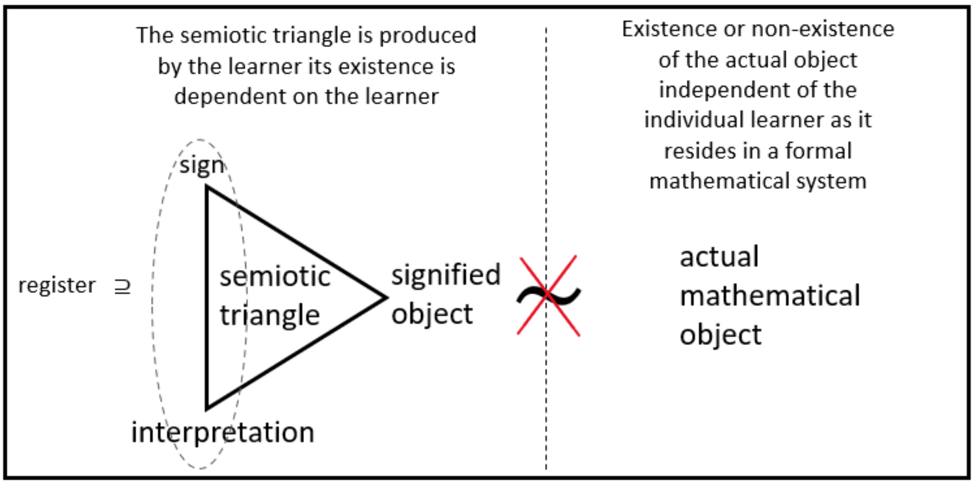


Figure 5.7: Pseudo-semiotic representation as a unit of analysis.

Going back to the definition of representational fluency, in terms of fluency digraphs covered in subsection 4.2.3, it would be important to also take pseudo-semiotic representations into account that reside within the learner's fluency digraphs. Hence, the representation fluency summation needs to be adjusted if researchers attempt to quantify learners' representational fluency within a task-based situation restricted to an object set in terms of observable semiotic representations, conversions, and pseudo-semiotic representations.

The analysis procedure using this analytic framework for representational fluency was repeated on Jenni's data from the adding structure task. Jenni was an interesting case because she started is the visual digraph register and object of symmetry register rather than the standard formal definition construction of a direct product. She ended her work by making a conjecture that she might be constructing a direct product, but would need to check it. The other learners' carried out their construction process in the reverse direction. They stated the specific groups they were constructing first in a formal definition register before building the group in a digraph register. Lens 2 was also applied to analyze how Jenni's construction evolved during the adding structure task.

Analytic lens 2: Nested Epistemic Actions for Abstraction in Context

The *Nested Epistemic Actions Model for Abstraction in Context (AiC)* is a theoretical and methodological constructivist model that was inspired by Freudenthal (1973) and Davydov's (1990) characterizations of abstraction (Dreyfus, Hershkowitz & Schwarz, 2015). Freudenthal described abstraction as a gradual process of taking previous knowledge, reorganizing it, and arriving at a new construction. Davydov described abstraction as the process of going from a vague mental image that is partial and fragmentary to a more complete, integrated and communicable whole. Dreyfus et al. (2015) contrasted Davydov's view with abstraction in the sense of going from concrete examples to an abstracted generalization. They emphasized that "It does not proceed from concrete to abstract but from an undeveloped to a developed form" and increasing levels of awareness of a new construction (p. 187). AiC is only partially observable, the AiC model provides a way to analyze abstraction through observable mental activities termed "epistemic actions" (p. 188-189).

These three epistemic actions are: "recognition" (R), "building-with" (B), and "constructing" (C). Recognition is for recognizing and bringing in to play pieces learned in the past that may be pertinent. This overlaps with the notion of matching associative forms of intuition covered in subsubsection 2.4.4.2. After recognizing such pieces students build on them, by making elaborations. After building with and extending previous knowledge the learner arrives at a new construction. The recognition of old constructs as relevant pieces can stimulate motivation to construct something new that will support goals in a particular situation. Emergence occurs when the learner experiences some partial degree of awareness of a new construct which is not fully formulated and, ”is often fragile and context dependent” (Dreyfus, Hershkowitz & Schwarz, 2015, p. 188).

Abstraction in context is described as nested meaning that the C epistemic action is dependent on the B epistemic action and the B epistemic action is dependent on the R action. Moreover, two constructions may be put together to form a third construction, so that epistemic action that led to the two constructions are embedded in the third construction. Consolidation is reached when the learner is fully aware of the generated construct and can recognize its use across multiple contexts, not just the context in which it was aquired, and build from it. Some attributes of intuition such as self-evidence, flexibility in manipulating the construct, and immediacy of the construct coming to mind in other contexts, are theorized to be signs of consolidation (Dreyfus, Hershkowitz, & Schwarz, 2015).

The AiC data analysis process developed by Dreyfus, Hershkowitz, and Schwarz (2015) was used in this study to analyze students constructing processes of direct or semi-direct products to build a larger group from two smaller groups. The AiC data analysis consists of a presumptive analysis of how students could approach the task and what they might construct followed by RBC analysis of student data. To begin the RBC analysis, data was spliced into “construction episodes” based on Dreyfus et al. (2015) and Chi’s (1997) recommendations (p. 193-194). Next, the data was coded for each construction episode in line with Dreyfus et al. (2015) coding scheme (Table 8.1, p. 197). To set up the coding sheet a column is given for the transcript line number, the participant, the transcribed data segment, and each construction labeled C_n where n is the construction number. Within each constructing column the epistemic actions recognition and building with codes are inputted that led to the specified construction. A subsequent construction may have recognition and building with codes that occurred in previous constructions.

5.8 Part III: Example-based intuitions for group actions

The two graduate students, Jenni and Alex, participated in part III of this study. To begin part III data collection a baseline questionnaire was administered. A baseline questionnaire, self-report journals, intuition excel logs, the NCCFIS with additional survey items, and a final follow-up interview were used to collect the data. The self-report journals, intuition excel logs, and NCCFIS with additional survey items were used concurrently to collect data on participant's example-based intuitions.

5.8.1 Instrumentation and data collection

Baseline questionnaire

The questionnaire consisted of 5 questions and was administered through Zoom to get a baseline of how comfortable the participant was with multiple ways of describing group actions and their knowledge of some examples. The beginning of the questionnaire introduced a typical textbook definition of a group action:

> A group action is a function $\cdot : G \times X \longrightarrow X$ that maps (g, x) to $g \cdot x$ such that $e \cdot x = x$ and $g \cdot (h \cdot x) = gh \cdot x$ for all $g \in G$ and $x \in X$.

The first question asked to translate this definition of a group action in terms of homomorphism language. The second question asked students to provide three examples of group actions. The third question gave a typical textbook definition of a group orbit, $Orb(x) = \{g \cdot x \in X : g \in G\}$, and asked the participant to describe a group orbit in their own words. The fourth question asked whether or not the definition of a group orbit felt familiar and if it reminded the participant of an object that they were more comfortable with or have worked with before. The fifth question gave a typical textbook definition of a stabilizer, $stab(x) = \{g \in G : g \cdot x = x$, and asked the participant to describe what a stabilizer is in their own words. The participant wrote down their answers on a piece of paper, tablet or whiteboard and also verbalized their responses. This was audio and video recorded. Afterwards, participants' responses were reviewed and a second session was set up with a participant if they made errors or if responses were missing. During this session, the researcher worked with participant until they were able to provide a valid interpretation of the definition of a group action using homomomorphism language and could generate some examples of group actions before moving on.

Experimentation with examples activity

An experimentation activity with examples of group actions was set up. First, each participant was asked to generate a list of classes of examples for group actions. After the participant generated their list, the researcher made sure to provide every participant with the following list of example classes: 1) G acting on itself, 2) G acting on a subgroup H, 3) The subgroup H acting on G, 3) A group G acting on the set of cosets, 4) A group G acting on a set X where the set X is not as structurally rich as a group, 5) A group G acting on something that is structurally richer than a group, namely a vector space V. The researcher also asked each participant to give an example within each class. If the participant could not provide an example of the class, then the researcher worked with the participant until they had access to an example.

Next, the researcher instructed the participant to explore patterns and relationships among the orbits and stabilizers within different example classes of group actions and across example classes. At this point, each participant went off and worked individually, and they were not confined to a particular location or time in which the work had to take place. They were given flexibility to take breaks and set their own schedule. This was done to try to reduce possible interference from negative affective factors induced by time pressure or pressure to perform in front of an audience. Each participant kept a journal of their experimentation with examples and the intuitions they arrived at. In addition, they kept a log of their intuitions in the excel spreadsheet that was formatted as shown below with an embedded survey instrument to capture qualities of their intuitions with respect to the Non-creative and Creative (NCC) attribute cluster and the additional attribute confidence in truth value.

Journals

Participants were asked to journal about their thoughts with a date and time of the journal entry. It was up to the participant to determine what work they felt was important and what they felt were the highlights of their experimentation with examples. They were also instructed to include elaborations of their example-based intuitions, what they were, how they arrived at them, and any associated sketches. The researcher provided each participant with the following combined characterization of example-based intuitions informed by definitions in the literature: ‘any patterns that you notice and guesses, hypothesis, or conjectures that come to mind, as a result of working with examples. They may seem plausible at the time you experience them and they precede the proof’. Aside from this generic characterization, students were encouraged to freely determine whether or not something that came to their mind was an example-based intuition.

Researchers have coded students verbal transcripts or written material as intuition or analytic thinking (Chiu, 1996; Bubp, 2014). For example Chiu's (1996) described her method to examine middle school students reasoning about lengths of paths: "I categorized students' utterances as indicating intuitions if they were (a) conceptual, (b) self-evident, (c) holistic, and (d) robust. In contrast, measuring each path with a ruler and comparing the numerical lengths would be coded as an analytic procedure, not an intuition" (p. 484). This type of coding only tells the story of the researchers' interpretation of student's statements as either intuition or not rather than the students interpretation or reflections about what constitutes intuition. While Chiu (1996) reports high inter-rater reliability, others have found that inter-rater reliability can be difficult to achieve and the sole use of researcher interpretations in studies on intuition has been considered a limitation (Bubp, 2014).

Intuition excel logs

Participants also kept track of their journaled intuitions in the excel spreadsheet in Figure 5.8. For each intuition that they logged in the spreadsheet, they filled out a survey that captured the qualities of their intuition in terms of: non-creative and creative attributes, and confidence in truth value. Participant's were told to log all intuitions that occurred regardless of whether or not they felt they were true or not and regardless of whether or not they were easy or difficult to express. In fact during a meeting to go over recording instructions participants asked whether or not the were supposed to record their intuitions even if they were not so sure that they were true or intuitions that would later turn out to be false. This may have indicated that the when one is unsure of something they may tend to keep it private or withhold it from others. It was important to try and capture as close to what was going on in the students mind as possible. I reiterated to the participants that it was important to log all of their intuitions even if they were unsure or not confident in them and to make note of when they felt unsure about something. If the intuitions were difficult to express the participant was instructed to do their best and make a note of it. Moreover, survey items were built in to capture these details and account for the variations in the intuiters confidence and perceived truth value of each of their intuitions.

Instructions: record a description for each intuition that comes to mind (i.e., patterns that you notice while working with examples or intuitions arrived at as guesses or conjectures that come to mind as a result of working with examples) in a separate entry box. If you have an intuition but are unable to articulate it in written form just state that you had an intuition but are having trouble expressing, express it as best as you can.	Activity: Exploring examples of group actions and relationships between orbits and stabilizers	Survey Instructions: for each of your intuitions that comes to mind fill enter an value 1-6, 1- strongly disagree to 6- strongly agree, for survey items 1-14.													
Intuition #	Description of your intuition	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1															
2															
3															
4															
...															

Figure 5.8: Intuition excel logs

Development of a new quantitative data collection instrument: the Non-creative Versus Creative Forms of Intuition Survey (NCCFIS)

The development of the NCCFIS began with an integrative literature review that involved a thematic analysis to identify and define variable attributes of mathematical intuition and an examination of existing measures for intuition. A main objective of the integrative literature review was to determine a way to represent intuition as a tangible construct for mathematics education research. Initially, a quantitative instrument was not pursued and did not seem reachable due to a lack of organization in this area. However, a key finding from the integrative literature review that made mathematical intuition a more manageable construct was the view that intuition should be represented as a class-concept and the objects, individual intuitions, may get shuffled around into different classes depending on what attributes the researcher chooses to focus on.

The coarsest classification, namely the distinction between non-creative and creative forms, was found in the psychology literature. This distinction was a reasonable place to start and opened a door towards a survey instrument that provided access to numerical data that could be analyzed using quantitative techniques. The explicit distinction between non-creative and creative forms of intuition was not found in the mathematics education literature. In fact, out of the entire literature sample only one mathematics education researcher mentioned creative intuition, a single time, without any references or description for the term. However, several scattered variable attributes were floating around the mathematics education literature that aligned with this coarse classification. For example Bruner (1960) described two different types of mathematical intuition,

> intuition is used with two rather different meanings. On the one hand, an individual is said to think intuitively when having worked for a long time on a problem, he rather suddenly achieves the solution, one for which he has

> yet to provide a formal proof. On the other hand, an individual is said to be a good intuitive mathematician if, when others come to him with questions, he can make quickly very good guesses whether something is so, or which of several approaches to a problem will prove fruitful (p. 156).

Bruner's second description "On the other hand..." could be linked to psychologists Larkin et al.'s (1980) description of a quick recognition type of intuition, "Although a sizable body of knowledge is prerequisite to expert skill, the knowledge must be indexed by large numbers of patterns that, on recognition, guide the expert in a fraction of a second to relevant parts of knowledge store" (p. 1336). Aspects of the previous descriptions were later linked to psychologists Policastro's (1995) description that contrasted intuition as "...recognition, among experiences that have similiar structure" and creative intuition that leads to "...the organization of novel structure", something new to the intuiter. While comparing and contrasting many descriptions of intuition and creating links between them, variable attributes such as time of conscious work prior to awareness of the intuition and novelty versus familiarity were noted. Constructs within the areas of creativity, semiotics, and affect started to blend as a set of attributes that separated non-creative and creative forms of intuition emerged: personal novelty, emotional intensity, ease of externalization, use of sanctioned or non-sanctioned representations to produce cues, presence of sanctioned or non-sanctioned representations contained in the intuition outcome, incubation period, and network thinking. A more thorough account for how these variables were identified and a description of them can be found in chapter 2. Next, existing self-report quantitative measures for intuition were examined.

Several general quantitative self-report survey instruments have been developed and validated for intuition within the field of Psychology. The Rational-Experiential Inventory and Multimodal Inventory (REI, REI-40; Pacini & Epstein, 1999, REIm; Norris & Epstein, 2011) is based in Cognitive Experiential Self Theory (CEST), a dual-process theory and personality theory. According to CEST there are two distinct processing systems: the rational and experiential system. The "rational system operates primarily at the conscious level and is intentional, analytic, primarily verbal, and relatively affect free" and the intuitive "experiential system is assumed to be automatic, preconscious, holistic, associationistic, primarily nonverbal, and intimately associated with affect" (Epstein, Pacini, Denes-Raj, & Heier, 1996, p. 391). As a personality theory CEST posed that people differ when it comes to how much they use the two systems when making every day decisions and whether or not they have a tendency to use one system over the other or if they balance both. A person who uses the experiential system more is said to have an intuitive thinking style and a person who uses the rational system more is said to have a rational thinking style.

The REI was the first self-report survey instrument developed to measure the two thinking styles among various populations. The REI began with two scales the Need for Cognition Scale and the Faith in Intuition Scale (Epstein, Pacini, Denes-Raj, and Heier, 1996). A revision, the REIm, included an rational-analytic thinking factor and three subscales within the experiential-intuitive thinking factor: intuition, emotionality, and imagination. Items included global statements of general preferences, beliefs, and attitudes such as "Using logic usually works well for me in figuring out problems in my life", "I tend to describe things by using images or metaphors, or creative comparisons", "I often trust my initial feelings about people", "I try to avoid situation that require thinking in depth about something", "I am not very good in solving problems that require careful logical analysis", "I have a logical mind", and "I enjoy imagining things" (Norris & Epstein, 2011, p. 1051-1052).

Another scale of similiar flavor to the REI is the Preference for Intuition and Deliberation Scale (PID). The PID measures the degree to which an individual prefers to make decisions using intuition defined as spontaneous affective states that drive decisions, "My feelings play an important role in my decisions", or to make decisions using deliberation, analytic calculations, pros and cons lists, and planning (PID; Betsch, 2008, p. 536). Betsch and Kunz (2008) used the PID to identify individuals that preferred one decision-making mode over the other, that is individuals that scored inversely on the intuition scale and deliberation scale as an initial step in a study that assessed how much value people assign to material objects when instructed to use decision-making modes that were either consistent or inconsistent with the prefered mode.

A more recent scale is the Types of Intuition Scale (TIntS; Pretz et al., 2014). It measures holistic, inferential, and affective types of intuition. Holistic type was split into two subtypes, big picture thinking and abstract thinking. Inferential type is experience or familiarity based. Affective type covers emotion based decisions. Several items from the TIntS, for example "I am a big picture person" and "It is better to break a problem into parts than to focus on the big picture", are not distinct from boundary properties of intuition found in the literature. According to both the psychology and mathematics education literature a boundary property of intuition is that it is a holistic global big picture interpretation (Fischbein, 1987; Chiu, 1996; Dane & Pratt, 2007; 2009). It seems problematic to mix boundary properties of a construct to distinguish between different types of intuition. Pretz et al. (2014) recommended that TInts could be used to confirm or reject whether or not "holistic intuition is more likely to be accurate for novices and inferential intuition for experts" (p. 465). They argued that it is important to distinguish between different types of intuitions, and showed that the relationship between occupational therapy students' diagnostic/treatment performance scores and intuition was different for different types of intuition. In particular, diagnostic and treatment performance scores were negatively correlated with affective intuition items, but positively correlated with holistic and inferential types.

Quantitative intuition scales or self-report survey instruments specific to the construct of mathematical intuition for use in mathematics education research are short in supply. Fischbein, Tirosh, and Melamed's (1981) intuitive acceptance questionnaire remains to be the only such instrument. The intuitive acceptance questionnaire is used to assess how intuitively acceptable a student perceives a provided statement or solution to be. Intuitive acceptance is defined as a combination of how obvious the statement or solution feels and how confident the student is that it is true.

Description of the NCCFIS

The Non-creative versus Creative Forms of Intuition Survey (NCCFIS) seems to be a first of its kind in the area of intuition research and extracts the (partial) local make-up of a particular cognitive object, namely an intuition. The previously discussed self-report instruments and the NCCFIS differ in: how they are administered and what they target. The administration of self-report intuition surveys from psychology take little time to complete, are filled out a single time by the participant, attempt to capture more stable thinking styles, and are detached from mathematical experimentation situations.

For example, many self-report instruments associated with intuition in psychology like the PID [Betsch, 2008] are used to target global estimations of individuals' general tendencies, thinking styles, or stable personality traits. Viewing intuitive versus analytic thinking styles as a stable personality traits within the mathematics culture seems problematic. Poincaré (1969) distinction between analytic and intuitive "minds" in mathematics has been sensationalized. He stated that, "it is impossible to study the works of the great mathematicians, or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds", the analysts were stereotyped as "logicians" and the geometers were the "intuitionalists" (p. 205-206). There was disagreement over which type of mind was better. "Doubtless Professor Klein well knows he has given here only a sketch, he has not hesitated to publish it...A logician would have rejected with horror such an conception, or rather he would not have had to reject it, because in his mind it would never have originated"

(Poincaré, 1969, p. 206). However, this is a chicken and egg statement because without intuition there would be no catalyst for a conjecture and subsequently a proof. So if such an intuition conception never originated in the analysts mind, would he be able to give a proof? And without previously proven results, formalized systems, and objects to tinker with where would we build our intuitions from? The claim that there are two distinct types of minds or tendencies in mathematics has been countered (Wilder, 1967).

Moreover, global instruments that attempt to capture stable thinking styles seems problematic and contrary to the view that intuition is fractionated, meaning that its attributes are at least dependent on the context. It is important to keep in mind that intuition is "fractionated" meaning that an individual may have strong intuition in some areas or contexts, but not in others (Kahneman & Klein, 2009, p. 522; Wilder, 1967). For example, Fields medalist Enrico Bombieri during an interview by Lipton and Regan (2013) stated, "My intuition in geometry, especially in high dimensions, is very poor...I am not sure why.... but it is very different from the study of prime numbers" (Lipton & Regan, 2013, p. 67). Bombieri wrote several conjectures related to solvable groups and one of these conjectures was quickly disproved by group theorist Colin Reid. Bombieri concluded that his "My intuition about finite groups is even worse than my geometric intuition" (p. 67).

In contrast to existing psychological measures, the NCCFIS was designed to capture local information about the make-up of each intuition reported by an individual restricted to a particular mathematical task. Even though the existing TInts instrument measures types as a global or collective valuation, without respect to a particular intuition and out of context. The TInts does not isolate and measure each intuition as a local object at the moment it occurs. Responses to the new NCCFIS survey items were dependent on a particular intuition held in the conscious mind, meaning that if the intuition held in the mind was swapped then responses to survey items would change to reflect characteristics of a currently held intuition. The journal entries provided a qualitative description of one's intuition, it tells the researcher what the content of the intuition was and the surrounding scene of experimentation with examples. The intuition excel log isolated each intuition within the journal elaborations into a succinct description. The NCCFIS provided an additional set of data on the qualities of each intuition in terms of non-creative versus creative attributes that the student recorded. The NCCFIS was repeatedly filled out by the student for every single intuition that they experienced awareness for and logged during a specified mathematical task. Rather than being a quantitative survey instrument that washed out details, it was an attached extension of the qualitative intuition journal data that provided an additional layer of information.

Attribute Targeted	Item #	**Instructions: Fill out the survey for each intuition that you recorded in your journal. Circle the number that measures how much you agree with each of the statements below.**
Personal Novelty (NCC)	1	I have never experienced this intuition before. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
Personal Novelty (NCC)*	2	This intuition feels familiar like something I have thought about or have come across before. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
Emotional Intensity (NCC)	3	The moment this intuition came to mind I had an intense and positive emotional reaction like feelings of euphoria, elation, or exhilaration. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
Emotional Intensity (NCC)*	4	The moment this intuition came to mind I had little to no emotional reaction. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
Ease of Externalization (NCC)	5	I found it difficult to externalize the intuition held in my mind in any way. I felt like I didn't have a coherent language to express it. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
Ease of Externalization (NCC)*	6	I found it easy to transfer the intuition held in my mind into an external verbal or written description. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**

Figure 5.9: Sample of items on the Non-Creative versus Creative Forms of Intuition Survey.

The NCCFIS consists of 12 items, two items for each NCC attribute variable. The NCC attributes, rather than being binary, were treated as graded attributes on a 6 point likert scale and explored whether or not intuitions could be placed on some spectrum between two extreme poles, a non-creative and creative pole. As an analogy with mathematical objects, groups can be viewed as a class-concept and a coarse classification is the distinction between abelian and non-abelian groups. But as one gets more familiar with these objects one may begin to ask how close to being abelian or non-abelian are certain groups. The extreme poles would be abelian groups and groups where the centralizer is just the identity. Groups can be mapped onto some between the two extreme poles according to the size of their centralizer or alternatively according to how large their commutator subgroup is. Similarly, we could zoom into classification that distinguishes creative versus non-creative intuitions as cognitive objects, explore how creative or non-creative they are, and ask how close is an intuition to extreme poles.

The NCCFIS was kept short in length to avoid imposing too much distraction or cognitive load on the student. Two survey items were created for for each NCC attribute. The phrasing of the first item was oriented towards the creative pole and the second oriented towards the non-creative pole. A sample of the items in the NCCFIS is provided in Figure 5.9 below. For items without asterisk a 6 was theoretically defined to be indicative of a highly creative intuition and 1 highly non-creative. This was reversed for items with an asterisk.

Conversations with experts

In addition to the integrative literature review, conversations with experts also guided the construction of the NCCFIS prior to the three-part study on abstract algebra students' representational fluency and example-based intuitions. Interviews with three mathematicians regarding their intuitions in research and teaching practice were conducted. Attributes or qualities of these mathematicians' intuitions surfaced during the interviews. These interviews influenced my construction of this survey instrument. A discussion with a Geometer about some of the survey items is provided in Table 5.5. Next, an education researcher with specialization in mathematical creativity and geometer were consulted about the NCCFIS items directly. Items of the NCCFIS were revised for incubation period based on recommendations from the creativity researcher. He emphasized that a marker of creativity was making a deliberate decision to take a break when stuck and let ideas simmer.

In summary, the NCCFIS is in the early phases of prototype development. During prototype development of a new survey instrument, the main goal is item construction, to illustrate the usability of the instrument and practically how it would work. It is expected that future discussions with others in the field and subsequent administration of the NCCFIS will raise points for revisions. Developing a survey instrument is an iterative ongoing process. At early stages, this process involves reviewing the literature, item construction and pilot testing. Item refinement, structural analysis, reliability testing, and concurrent validity testing occur at later stages (Clark & Watson, 1995). Concurrent validity for the NCCFIS may be challenging to obtain because other related instruments that have been validated are global general context measures of creativity, motivation, or intuition versus analytic thinking styles.

In contrast, the NCCFIS takes local measurements of a particular intuition object produced by an individual within a task-specific context and item values can vary among a collection of objects produced by an individual. These local measures may or may not coincide with global measures. Standard reliability testing approaches such as test-retest replicability do not fit with the NCCFIS for at least two reasons. One, if the content of an intuition outcome occurs on subsequent occasion the attributes attached to that object may be different. The NCCFIS, if eventually used to collect time-series data, would be administered to try to detect changes across time. Two, it may not be possible for the intuiter to accurately recover the original self-reported (partial) make-up of the intuition object once it has passed.

Table 5.5: Conversations with experts about mathematical intuition.

Researcher	Geometer
Kahneman and Klein's work in Psychology, it looks they are in the realm of non-creative forms of intuition. The kind that are quick, you don't think too much like a probabilistic judgement. In the Psychology field there are some papers that mentioned creative forms, but I don't think that's what Kahneman and Klein were looking at when that were looking at intuition with a focus on intuitive errors.	So you're looking for creativity in the slow thinking sense of the word.
Yeah and that's different than [next I contrasted this with non-creative forms] My intuition says, oh, pull this theorem and pull this tool when I'm thinking about this case. So it's a very quick response, your brain sort of immediately sees the pattern in the problem and gives you some sort of quick default. And so that would be like a non-creative type.	Okay. But wait, yeah how do you say, oh, this response indicates creativity or is not creative. What are you looking for?
So I use self-report survey items...It needs to come from their head of what they think that they're seeing or feeling. For example an item is "I have never experienced this intuition before, if they strongly agree, that's more indicative of a creative form. So its like this is something new it isn't a common tool that I usually pull in these situations. This something that I have never experienced before. Another one is externalization. It says I found it difficult to externalize the intuition held in my mind. And I felt like I didn't have a coherent language to express it. So I think at the really extreme end of creative intuition it like the cases where you prove things that open up new fields, the intuition sits in your mind for awhile, but it takes a long time to actually externalize them coherently and put them on paper.	Definitely in geometric group theory. That's, I mean, some of the really expressive people are exceptional writers when they can write. But, uh, that's a hard thing. You can spend a long time developing an intuition and now you need to come up with a language. And a particularly if it's very geometric, you need to come up with a language that's got enough metaphors in it that gets people on your side, way of thinking, but it also has to be precise. You need to formally give you know rigorous definitions and then prove things using them and so on. And that can take a lot of work and some people are terrible at it. And some people are very natural, very good at it.

Researcher	Geometer
So it seems like with a non-creative form, if it's something you've seen before, there is already a prefabbed language set up for you to externalize it with. That's more non-creative. If you actually have to create the language to externalize it with, then that would be on the extreme end of a creative form.	Well often creativity can mean like the aha moment. It could also be that this new environment is actually, you know, a feature of something we already know very well before making a connection between something that's very well known, or even making a weird connection between two very well known fields, but that are very far away from each other. That's very much, I don't know where that comes from, but that's very much intuition and, and some of the big people like, you know, Thurston or Terrance Tao or people like that, that is what they do. They make these connections that most people would never see in their lives.

Additional survey items: confidence in truth value

Two additional items outside the NCCFIS items for confidence in truth value of an intuition were attached to the survey for data collection purposes. This items are shown in figure 5.10. Conjectural intuitions were termed by Fischbein (1987) as, "an assumption about future events, about the course of a certain phenomenon etc. Such a conjecture is an intuition only if it is associated with a feeling of confidence" (p. 60). However, an intuiter may not always be confident that there conjectural intuitions are true. Based on interview data in Table 5.6, self-report of confidence level in the truth value and self-evidence, feelings that something is true in the an the absence of a proof, are hypothesized in this study to be variable attribute of varying degree rather than invariant properties of all intuitions.

13	I am confident that my intuition is true **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
14	I am so confident that my intuition is true that a rigorous proof of it just feels like busy work at this point. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**

Figure 5.10: Confidence in truth value survey items.

Table 5.6: Continued conversations.

Researcher	Riemannian Geometer
What are some qualities or characteristics of your intuitions?	I guess one thing that often happens when I try to draw on my intuition is that I like to say, oh, all objects of this type act in this way. Uh, I tend to make statements like that a lot in my head or in research meetings...and that leads me to conjectures as well and often times my conjectures are wrong because I'm making a broad generalization and then I have to go reconcile what I thought was a general picture with a specific thing that contradicts in a paper or a talk that I hear.
Okay. Do you usually start of with feeling that your correct or that the conjectures are possibly true?	I'm pretty cautious in terms of my conjectures even to myself so if I'm going to conjecture something to myself I probably have some kind of strong feeling about it like I've seen enough evidence or it to nice not to be true kind of feeling. Then most of the time when I have these ideas like that, there's no way I could ever answer it one way or the other. It's like just too broad. The best I'd be able to do is to be able to tackle one specific case in a particular nice situation and maybe say something about that.

5.8.2 Qualitative data analysis: Within and across case analysis

Baseline questionnaire and journal data was analyzed using a within-case and across-case analysis (Creswell & Poth, 2018). To start the within-case analysis, I read the qualitative questionnaire and journal data for each participant several times and took notes about what they did well and what they could improve upon. Next I compared their data by looking for commonalities and differences in their responses. The journaled content of the intuitions was open coded by the researcher for three additional intuition attributes: error to non-error type, unique versus common, and network thinking. Four themes emerged according to errored to non-errored type along with a few undetermined cases. The error to non-error type ranged from: 1-false intuition derived from perceived examples that were not legitimate examples, 2-false intuition with respect to some examples within the example subset that the learner was focused on, 3-true for examples in the example subset the learner was focused on but not true in general, 4-true in general beyond the example subset. A second thematic coding session grouped together intuitions with similiar content descriptions to determine whether or not the content of intuitions within the participant population were unique or common. A third coding session was done to mark whether or not the journaled content combined multiple undergraduate or graduate level courses such as linear algebra and topology. Combining multiple subject areas is an indicator of network thinking.

5.8.3 *Quantitative data analysis: Univariate tests and the Fuzzy C-Means Algorithm*

The NCCFIS was an instrument used to extract attribute values attached to a reported cognitive object, namely each learner's reported intuition. The analysis of the NCCFIS data was carried out in two stages. First, the Permuted Brunner-Munzel test, a non-parametric versions of a t-test, was used to perform univariate analysis (Neubert & Brunner, 2007). Instead of a null hypothesis in terms of equal means or other measures for central tendency, this non-parametric test detects the probability that a randomly drawn object from X will have a larger or smaller value with respect to a particular attribute than a randomly drawn object from Y. Next, the Fuzzy C-means algorithm was used to classify learners' intuition.

The Fuzzy C-Means algorithm is a clustering algorithm that has been applied across many disciplines: higher education, engineering, artificial intelligence, medicine, biology, and meteorology. Applications include: analysis of student academic performance through exams, final grades, and attributes associated with drop out, species classification, genetic analysis, segmentation of brain MRI imaging, medical diagnoses, and weather forecasting (Nayak, Naik, & Behera, 2015; Govindasamy & Velmurugan, 2018).

Let $X = \{x_1, x_2, ..., x_n\}_{i=1}^{n} \subset \mathbf{R}^{\mathbf{d}}$ be a collection of N objects characterized by d attributes. A fuzzy set was defined by Zadeh (1965) as $c_j = \{(x_i, \mu_j(x_i)) : x_i \in X\}$ where μ_j is a membership function that assigns the i^{th} object to a membership value μ_{ij} in $[0, 1]$ for the j^{th} fuzzy set or cluster. The closer the value is to 1 the greater the membership. A value of 0 means no membership and 1 is referred to as full membership. Equality, unions, intersections, complements, algebraic operations, and convexity for fuzzy sets were also defined in Zadeh (1965). Rather than a point belonging to a single set, a point can be a member of multiple sets with varying degrees of membership. Thus, the notion of a fuzzy set partition is different than that of a hard partition in classical set theory because fuzzy sets need not be disjoint (Zadeh, 1965; Bezdek, Coray, Gunderson, & Watson, 1981). Bezdek et al. (1981) defined a fuzzy partition as an element U in the vector space of real $n \times c$ matrices V_{nc} with entries μ_{ij} that satisfy three conditions:

$$\mu_{ij} \in [0, 1] \tag{5.1}$$

$$\sum_{j=1}^{c} u_{ij} = 1, \quad \forall x_i \in X \tag{5.2}$$

$$0 < \sum_{i=1}^{n} u_{ij} < n \ \ \forall \, c_j \tag{5.3}$$

The rows of U correspond to the objects, the columns correspond to fuzzy set and the entries are the membership values μ_{ij}. If for the first condition, every $\mu_{ij} \in \{0, 1\}$ then U is called a hard partition in which cluster membership is not partial. Bezdek (1981) et al. defines the set of all possible fuzzy partitions as:

$$M_{fc} = \{U \in V_{nc} | 5.1, 5.2, 5.3\} \tag{5.4}$$

The third condition (5.3) excludes the case where all membership values in a column are 0 and thus it excludes the fuzzy partition where all entries in U are 0. The FCM algorithm determines a fuzzy partition that classifies data objects by assigning objects to clusters using a membership function u_j (5.6). The membership value entries $u_j(x_i)$ for an object x_i can be interpreted as the probability that the object belongs to the cluster c_j. The higher the membership values are for the final partition matrix the closer the object x_i is to the center z_j of cluster c_j. The lower the membership values are the further the object is from the cluster center.

The goal of the algorithm is to optimize the objective function J_m (5.5) for the membership matrix $U = [u_{ij}]_{N \times c}$ and c many clusters. To do this the algorithm looks for centers called prototypes that minimizes J_m. The membership values of a reported intuition that represents the object's belongingness to various clusters and the attribute make-up of the prototypes, that were final outputs of the FCM algorithm, informed characterizations of learners' intuitions as either more creative, more non-creative, or mixed.

$$J_m(U, Z) = \sum_{i=1}^{n} \sum_{j=1}^{c} u_{ij}^{m} ||x_i - z_j||^2 \tag{5.5}$$

$$u_j(x_i) = u_{ij} = \frac{1}{\sum_{k=1}^{c} \left[\frac{||x_i - z_j||}{||x_i - z_k||} \right]^{\frac{2}{m-1}}} \tag{5.6}$$

$$z_j = \frac{\sum_{i=1}^{n} u_{ij}^m x_i}{\sum_{i=1}^{N} u_{ij}^m} \tag{5.7}$$

The steps of the FCM algorithm are:

[1] Input data set X and choose number of clusters c, weight m in the recommended interval $[1.5, 3]$ that controls the "fuzziness" of the partitions, and convergence criterion ϵ. Randomly generate an initial center vector $z^0 = \{z_1, z_2, ..., z_c\}$. Compute membership values with z^0 and (5.6) to obtain initial membership matrix U^0.

[2] Compute new centers using (5.7) and the membership values generated in the previous step.

[3] Compute membership values using (5.6) to obtain an updated membership matrix for the next iteration.

[4] Repeat [2-3] and stop when the difference in the objective function for two consecutive iterations is less than ϵ for either U or Z.

The Euclidean norm was used to define the membership function (5.8) and calculate distances between the i^{th} point and the center z_j of the j^{th} cluster. Once the algorithm has been run for several choices of $c < sqrt(N)$, $c = 2, 3, 4$ and $m = 1.5, 2$, the corresponding outputs U^ϵ, c^ϵ were evaluated using validity indices. Several types of validity indices have been proposed for fuzzy clustering. This study applied Bezdek's (1974) fuzzy partition coefficient F_c (5.8), Bezdek's (1981) partition entropy (5.11), and Campbell and Hruschka's (2006) fuzzy silhouette index (5.12).

The fuzzy silhouette index is an adaptation of the silhouette index s_i for crisp partitions, crisp also sometimes called hard is a term used for classical set theoretic partitions with non-overlapping membership. For crisp partitions, a_i in (5.11) represents the compactness of the cluster that contains object x_i as the average of within cluster distances. And b_i represents the separation between the cluster that contains object x_i and the nearest neighboring cluster. Separation is defined as the average distance between x_i and objects in the neighboring cluster. A desirable cluster is one that has a small value for a_i and large value for b_i. A value for s_i is in the interval $[-1, 1]$. A value closer to 1 means that object x_i was placed in an appropriate cluster by the FCM algorithm. For a fuzzy partition membership is not all or none. To adjust for this a fuzzy partition matrix is turned into a crisp partition matrix by changing the maximum membership value for an object x_i in the fuzzy partition matrix to a 1 and all other membership values to 0 then s_i is computed with the crisp partition matrix. Let c_p denote the cluster for which the object x_i has the largest membership value and c_q indicate the neighboring cluster with the second largest membership value. To take into account the membership values in the fuzzy partition matrix, the fuzzy silhouette index (5.11) incorporates the membership values u_{ip} and u_{iq} across all objects. A drawback of this silhouette index is that it disregards objects on the outskirts of overlapping clusters and may elevate the s_i artificially by forcing a smaller value for a_i. In summary, the optimal number of clusters occurs when the fuzzy partition coefficient and fuzzy silhouette index are maximum values across all clusterings and the entropy takes on the minimum value across all clusterings. The R package: ppclust was used for implementation.

$$F_c(U) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{c}(u_{ij})^2 \tag{5.8}$$

$$H_c(U) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{c}u_{ij}ln(u_{ij}) \tag{5.9}$$

$$FSil = \frac{\sum_{i=1}^{n}(\mu_{ip} - \mu_{iq})s_i}{\sum_{i=1}^{n}(\mu_{ip} - \mu_{iq})} \tag{5.10}$$

$$s_i = \frac{b_i - a_i}{max\{a_i, b_i\}} \tag{5.11}$$

The part III study of this thesis takes a novel approach by applying the FCM algorithm to try to better understand mathematics learners' intuitions. The (partial) make-up of a learner's intuition as a quantifiable object was defined in this thesis as a point viewed in $\mathbb{R}^{17}$, 12 Non-creative to Creative (NCC) variable values collected with the NCCFIS, 2 values for confidence in truth value, and 3 additional variable values coded by the researcher. The revised Fuzzy C-Means Clustering Algorithm (FCM) by Bezdek et al. (1981) was used to reveal the hidden underlying cluster structure of the (partial) make-up of learners' reported intuitions based on NCC attribute similarity. A compressed value for the NCC attribute make-up of each cluster center was computed using a shifted and scaled arithmetic mean score $\bar{M}_{z_j}$. The closer the $\bar{M}_{z_j}$ is to 1 the more creative the closer to 0 the more non-creative the cluster center. The FCM algorithm was also used to reveal the underlying cluster structure of learners' reported intuitions according to a subset of attributes, separate from NCC attributes, for confidence in truth value and error to non-error type. Other validity indices, such as the generalized fuzzy silhouette index (Rawashdeh & Ralescu, 2012), and modifications to the FCM algorithm as well as several other fuzzy classifiers have been proposed since Campbell and Hruschka (2006) and Bezdek (1981). An application and comparison of several indices and classifiers could be explored in future studies.

Chapter 6

Results

6.1 Introduction

Authors in a special ZDM issue on *Networking Theories* took the stance that there was a balance to strike between the use of a single theoretical lens to analyze the same set of data and the use of multiple theories. Papers in this ZDM issue applied two to four lenses on the same set of data at a time (Prediger, Bikner-Ahsbahs, & Arzarello, 2008). The use of multiple lenses has been recommended to increase understanding and objectivity. Some lenses may give a different view than others. Comparing and contrasting results obtained from a use of different theoretical lenses has offered insights or filled in details that a single lens can not account for alone. Conflicting interpretations of the same data, obtained from an analysis using different lenses, has occurred in several areas of mathematics education research (Kidron, Lenfant, Bikner-Ahsbahs, Artigue, & Dreyfus, 2008; Networking Theories Group, 2014). The application of multiple lenses in this study was done to gain a deeper understanding of learners' mathematical intuition. The representational fluency lens was chosen prior to data collection. The remaining lenses were data driven meaning that they were chosen after data collection and initial analysis to bring out other facets in the data that were related to the intuition construct.

First, this chapter presents results for the collapsing structure task, followed by the adding structure task, and ends with an analysis of learner's example-based intuitions. The collapsing, quotient map, and adding structure, direct product and semi-direct product, tasks were initially designed to investigate learners' representational fluency across various registers: formal-symbolic function maps, formal group presentation register, Cayley tables, digraphs, cycle graphs, and objects of symmetry. Representational fluency was associated with the intuition construct through a pervasive principle. The principle that the more registers one has access to and the more ways one is able to represent an object, the more informed their intuition is for that object.

For the collapsing structure task, results in terms of three lenses: representational fluency, default-interventionist, and affective are presented. To put on a fluency lens the researcher thought in terms of Duvalian semiotic theory and additional theories for the representational fluency factor that were covered in subsection 4.2.1. After saturating one's mind with semiotic theory the four level-analytic framework in section 5.7.2 was developed to analyze and organize the results. A default-interventionist perspective on intuition was given in subsection 2.4.1 and an affective perspective in section 4.5.

Next, two lenses: a theoretical *Abstraction in Context* lens with RBC analysis (Dreyfus, Hershkowitz, & Schwarz, 2015) and a fluency lens with the four-level analysis were placed on Jenni's data from the adding structure task. *Abstraction in Context* was aligned with accumulative and constructive forms of intuition in subsubsection 2.4.4.3. The RBC and four-level representational fluency analysis procedures for the adding structure task were both covered in subsection 5.7.2.

Finally, this chapter ends with results on two graduate students' example-based intuitions related to relationships between orbits and stabilizers of group actions. Their example-based intuitions were analyzed in terms of attributes that separate non-creative versus creative forms of intuition along with additional attributes covered in subsubsection 2.4.4.4.

6.2 Collapsing structure task: isomorphisms, homomorphisms, and quotients

6.2.1 Determining if D_8 and Q_8 are isomorphic

To prepare the stage for the collapsing structure task, participants first needed to come to the conclusion that the two groups D_8 and Q_8 could not be isomorphic. Various cues exist across several registers for finite groups that can be used to conclude that D_8 and Q_8 are not isomorphic. What cues are learners attuned to across the registers that help them to arrive at this conclusion? What cues are they attuned to that cause them to error? Max, Jenni, and Alex all came to the conclusion that the two groups were not isomorphic, however there were some opportunities for improvement with respect to Max and Jenni's reasoning as to why the groups were not isomorphic. Max, the fourth year undergraduate, was focused on cues contained in the group presentations, Jenni used the group cycle graphs and cayley tables, and Alex used the cayley tables. Matching associative forms of intuition in which cues in the register environment were matched to already learned fact-based formal knowledge was also theoretically relevant to the interpretation of the data. Fact-based formal knowledge in this context refers to necessary conditions that must hold if two groups are isomorphic.

Group presentation register

Max was focused on the symbolic cues for the generators and compared the group presentation for the dihedral and quaternion group. He found the order of the two generators i and j in the dihedral group presentation were not the same as the order of the two generators r and f in the quaternion group.

> Max: Okay so i^4 equals j^4 and i^2 equals j^2 so that means j^4 is also one. So I would say, no, they're not isomorphic because for D_4 one of its generators is of order 2 and Q_8, both of the generators are of order four.

While it is productive to examine the orders of the elements, it is not enough to look at the orders of the generators displayed in group presentations. Max's justification that the two groups in this task are not isomorphic, because the orders of the generators for the two groups did not match, was not valid. It follows that a false implicit assumption that Max may have used was: If the order of the generators in a group presentation for G_1 do not match with the order of the generators in the presentation for G_2 then the groups are not isomorphic. In this case, symbolic cues were not matched to fact-based formal knowledge and this caused Max to error. A counterexample to his implicit assumption would be: $Z_6 =< a : a^6 = 1 >$ and $Z_3 \times Z_2 =< (1,0),(0,1) : (1,0)^3 = 1, (0,1)^2 = 1, (1,0)(0,1) = (0,1)(1,0) >$. The order of the generators in the two group presentations do not match and the symbolic cues in the two group presentations look very different, but the the groups Z_6 and $Z_3 \times Z_2$ are isomorphic. Max did not consider other registers. He also did not dissociate his implicit assumption from the task by formulating it as a general conjecture to try and counter. Cognitive skills such as: maintaining awareness of implicit assumptions, dissociating the implicit assumption from particular examples to formulate a general conjecture, and countering, were not part of Max's thinking and needed to be trained. For the first and second year graduate level learners these skills were second nature.

Cycles register

Jenni, the first year graduate student, used the group cycles graph. It seem that the cycles representation caused confusion. Below is an excerpt from her interview.

> Jenni: I'm going to go with not isomorphic because the dihedral group has cyclic subgroups of size two and Q_8 does not have cyclic subgroups of size two.
>
> Researcher: Did you use a particular representation card to arrive at your answer?
>
> Jenni: Yeah, I used the purple one [group cycles graph].

Jenni had a relevant piece of formal knowledge activated namely that: if two groups are isomorphic with the isomorphism map ϕ then for all $x \in G$ the order of x must be equal to the order of $\phi(x)$. Her reasoning implies that the condition, x must be equal to the order of $\phi(x)$ for all x, was violated. Jenni's activation of relevant fact-based formal knowledge was accompanied with a misinterpretation of the cues in the cycle graph. The cycle graph of the quaternion group seemed to mask that $i^2 = -1$ was an element of order two.

Cayley table register

Jenni also activated a second relevant piece of fact-based formal knowledge namely that: if two groups are isomorphic then either they are both abelian or they are both non-abelian. Following her final conclusion she went back and explained that she had initially went to the Cayley tables with this fact, but could not find a match between the fact and what she was seeing in the Cayley tables.

> Jenni: My first thought was to look at the Cayley tables and see if one was abelian and the other is not abelian, but I don't think either of them are abelian.
>
> Researcher: Right, neither of them are abelian. But, what about $\{1, -1\}$ in Q_8? Is that a cyclic subgroup of order 2?

Jenni then switched back to her initial inclination of going to the Cayley tables to check and confirms that $\{1, -1\}$ is a cyclic subgroup of order 2. She then explained that she can tell the number of order 2 elements of the groups by looking along the diagonal entries of the Cayley table. She followed with a valid conclusion that the dihedral group of order 8 and Q_8 are not isomorphic because D_8 has 5 elements of order 2 and Q_8 has only one element of order 2. Similar to Jenni, Alex, the second year graduate student, used the Cayley tables. He noticed that the Cayley tables were different, no relabeling could make them look the same, and so the two groups could not be isomorphic. Max, Jenni, and Alex's responses to the collapsing structure task prompt are presented in the following section.

6.2.2 Chronological narrative for Max

Max began by looking at the Cayley tables of Q_4 and D_4 side by side [Max preferred to use the notation D_4 for the dihedral group of order 8 and Q_4 for the quaternion group of order 8]. He investigated how D_4 and Q_4 are different and how they are similiar. He narrowed his focus to the right lower quadrant of the Cayley tables, for D_4 and Q_4, outlined in Figure 6.1.

> Max: Okay. So based on the Cayley table [for Q_4], it looks like all the elements are in four cycles. So just looking at this [Cayley tables for both groups side by side], I guess it's just helpful to see, um, they [the Cayley tables] only differ in their products of the last four elements with each other [last four elements referred to for Q_4: j, ij, i^2, ji and for D_4: f, rf, r^2f, fr].

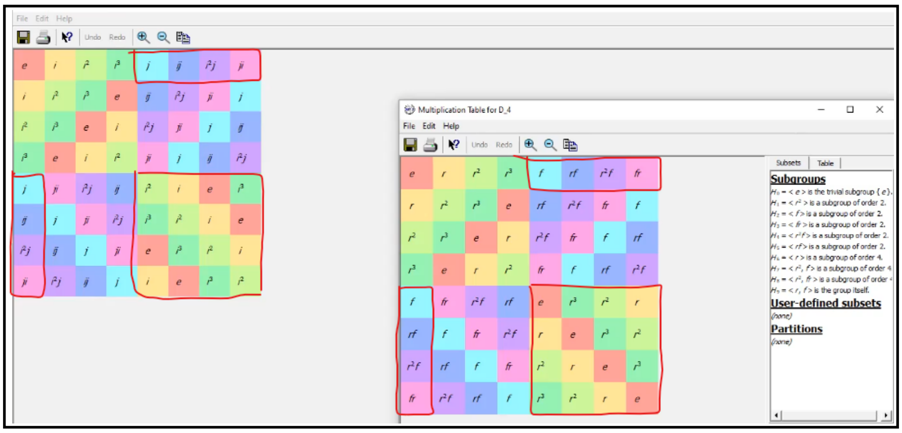


Figure 6.1: Max comparing and contrasting Cayley tables.

> Max: I guess, you can see that those last four elements [flips in D_4] are all their own inverses.

> Max: And I mean, maybe this is just a trick based on the way they're colored but the outside colors of the diagonal with e's on D_4, you have r^3 on one side of the diagonal and r on the other side of that diagonal. Um, those are like the elements

> i^3 under the same line in the other Cayley table and the i above it is swapped around. And yeah, I think if you just made each element there, its own inverse [$r = r^3$ and $i = i^3$], then the rest would follow.

I asked Max to elaborate on and explain his comment "the rest would follow".

> Max: So you can see that in D_4 f is its own inverse and so if you made j its own inverse, like you did that, then maybe this is just like a last line of thought.

Max was generating ideas sparked by perceptual cues in the Cayley tables. His initial hunch to make r its own inverse in D_4 and to make i and j its own inverse in Q_4 was consistent with the strategy of collapsing the dihedral group of order 8 and Q_4 to the Klein 4 group, $Z_2 \times Z_2$. Adding the relation to the group presentation $r^2 = e$ to the dihedral group of order 8 and reducing redundancies results in the group presentation $< r, f : r^2 = f^2 = e, rf = fr >$. The same suggested modification, make r its own inverse in the dihedral group, can be carried out predominantly in a formal-symbolic mapping register using function notation to write down a quotient map and using representative multiplication $g_1N * g_2N = g_1g_2N$ as the binary operation. Making r its own inverse is synonymous to identifying e and r^2 so that they are in the same coset $r\{e, r^2\}$. The image of the quotient map from $D_4 \longrightarrow {D_4}/{\{e, r^2\}}$ is isomorphic to the Klein 4 group. Modifying semiotic representations of Q_4 with Max's recommendation to make i and j in the quaternion group its own inverse also results in the Klein-4 group.

Alternatively if the quotient map concept with standard representative multiplication has not been aquired, one could focus on the subsets $\{e, r^2\}, \{r, r^3\}, \{f, r^2f\}$, $\{r^3f, rf\}$ in the Cayley tables with the binary operation set multiplication as illustrated in Larsen and Lockwood's (2013) guided reinvention activity to construct quotient groups. Max did not indicate that he was thinking about subsets as elements along with a binary operation. He quickly noticed some extremely pertinent cues related to the Klein-4 strategy, but he did not have the Klein-4 group in mind.

6.2.2.1 Impossible strategy to collapse the dihedral group to the quaternion group

After exploring the Cayley tables he asked for the Cayley digraphs with the subgroups list to be displayed on the screen in Figure 6.2, and looked at them briefly. Next, he compared the subgroup lists.

> Max: So yeah, I don't really know how to say it any other way, but D_4 and Q_4 both have three subgroups of order four and D_4 has a bunch of these little

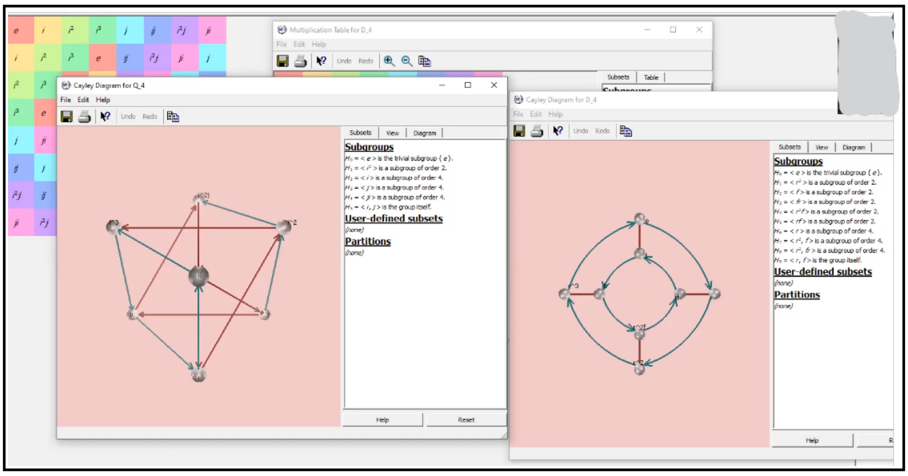


Figure 6.2: Max comparing and contrasting Cayley digraphs.

> subgroups of order two. And so I guess, I don't know what a function would be that would collapse those subgroups, but if there was some way to get those out then it would be Q_4. The groups are obviously the same size.

Max's statement, "if there was some way to get those [some subgroups of order two out of D_4] then it would be Q_4", indicated that he was viewing D_4 as a set of subsets rather than a group. It seem that Max's intuiting process was naturally activating weaker necessary conditions for two groups to be isomorphic in order to narrow the search space of possibilities. Max was focused on the group as a set of subsets in which the number and size of the subsets matter. He noticed that if he could get rid of some of the order two subgroups then the subgroup lists would look the same in terms of the number of sets and the sizes of the sets. The order four subgroups in Q_4 are all cyclic with one generator where as two of the order four subgroup in D_4 are isomorphic to the Klein-4 group. Max did not mention the binary operations of either group nor differences among the order 4 subgroups.

Trying to get a clearer picture of what Max was thinking, I asked him to take some time to think about it more. I also restated the task to try to get him to articulate and formalize some of his thoughts.

> Researcher: So you have these two groups D_4 and Q_4 [I extend both arms above my head, I shake my left hand to indicate that it is D_4 and shake my right hand to indicate that it is Q_4]. And you have some sort of mapping that tells you how you are modifying these groups [I move my left hand down to the level of my shoulders and close my index finger to my thumb to indicate an identification map from D_4 to a group and the do the same with my right fist to indicate a second map

from Q_4 to another group]. And then you need another map to show the groups you end up with are isomorphic [I move my finger back and forth horizontally at the level of my shoulders]. Max wrote on the blackboard in Figure 6.3 and attempted to write down an isomorphism map.

Max: So if we wanted them to be the same we want those group definitions to match. So r to the fourth equals one still, but then we wanted to have some extra property that r squared equals f squared and then we have the transposition thingy there at the end where r f equals f r inverse like those are the same in both groups so that can stay. And then when you start getting these products phi of r equals r, hmm, and it needs to be an isomorphism.

Researcher: So you have these two groups and some map to some sort of resulting groups, then you want to show that these two resulting groups are isomorphic.

Max: Okay. But, I don't know if it is possible, but is it possible to just modify one of them [referring to Q_4 or D_4 (quaternion group of order 8 and dihedral group of order 8)], is it okay to just do that?

Researcher: So you can focus on just modifying one of them first and then modify the other one. And then show the two modifications are isomorphic to each other.

Max: I don't know what this is going to do, but this is what I have. The new group [referring to the group being mapped to] is the same as the old group [referring to the dihedral group of order 8], except you have these like new group properties [referring to the relations]. And I guess the only way that I could figure out how to make the map was to just be very explicit and say those elements in the old group will go into like what we wanted the group properties to be in the new group. But I don't know if that makes any sense. So like r^4 and f^4 they're going to collapse to the same element.

Max kept trying to force signs of the dihedral group of order four to look like signs for the quaternion group of order four. First he tried to manipulate some symbols in the Cayley table sign of the dihedral group to make it look exactly like the Cayley table sign for the quaternion group. Next, he tried to modify the subgroup list of the dihedral group, focusing on the size of the subsets and frequency of sizes, so that it would match the subgroup list of the quaternion group with respect to the number of subsets and their sizes. Then, Max tried to manipulate the group presentation sign of the dihedral group to match the group presentation sign of the quaternion group.

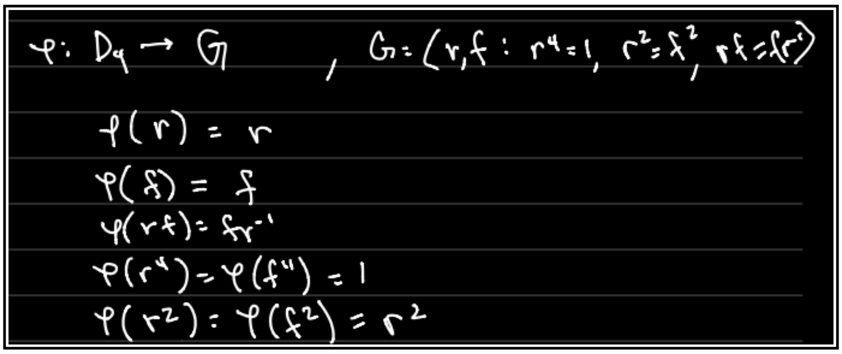


Figure 6.3: Max entering an impossible strategy in a formal-symbolic mapping register.

These modification and mapping behaviors that Max naturally engaged in were not fully consistent with homomorphisms or isomorphisms.

> Researcher: Does G have some sort of common name?
>
> Max: Well, I mean, this is literally just the statement of the quaternion and I thought, like I could skip having to do a modification on both groups and just do one straight from the other, but I guess if that's not possible, then I should try to slow down.
>
> Researcher: Yeah slow down. Let me ask a more fundamental question before we move on. Whenever I say collapsing structure or ignoring information what comes to mind mathematically?
>
> Max: I've only heard someone use the phrase collapsing in terms of like dimensionality with vector spaces. And so I guess it's something akin to that where you're just like losing a direction or just like losing some amount of freedom within the, like the new place that you're going. But I don't think I have better intuition than that at this point, I guess also, like I've done a little bit with homomorphisms so I guess how vector spaces have dimensions or whatever, um groups would have like subgroups. That's how I think about it. And so I feel like you're losing subgroups or losing space in the subgroups, but I don't know how to like, concretely make that happen.

At this point I shared my screen on Zoom during the interview and rephrased and tightened up the task using the term homomorphism since Max had some familiarity with it from class and could recall the formal definition. I drew the diagram in Figure 6.4 and stated:

Researcher: So now the task is to construct homomorphisms [pointing to ψ_1 and ψ_2] so that the these two things [pointing to the groups G_1 and G_2] are isomorphic.

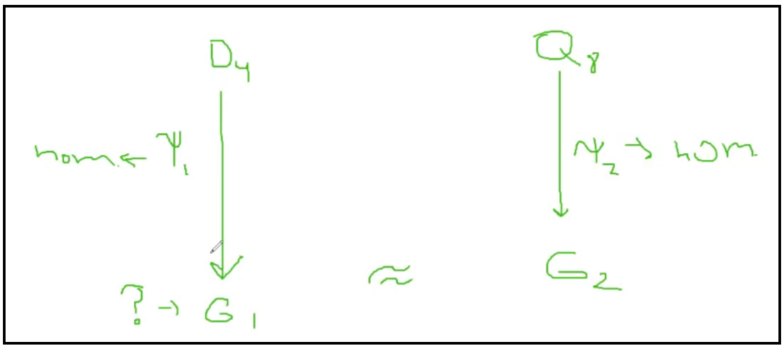


Figure 6.4: Researcher rephrasing task to construct homomorphisms.

Max: So I guess now that I've seen that, um, I can kind of explain what I was thinking. So like in my brain, the homomorphism ψ_2 was just going to be an isomorphism from Q_4 to something that, I don't know Q_4 with different labels, because Q_4, um, like it seemed to have less going on, I guess it had fewer subgroups. And so I guess I just thought that the exercise would really just be collapsing D_4 to Q_4, but I don't know if that's true, but that's kind of how I was thinking about it.

Researcher: Well is it possible? Let's see, if we take D_4 and Q_4 they're both order eight right? So if I collapse something in the dihedral group using a homomorphism that won't be isomorphic to Q_4 because now the orders are off, they would have different orders.

Max: Oh okay, so you actually do mean like losing elements, not just losing some kind of, um?

Researcher: Not exactly. So there are many ways to think about it. Um, I think maybe the best way to think about for now is that to collapse is to construct a homomorphism that is not injective, since you already mentioned homomorphism. Give yourself some serious time to think about it, just take your time.

The normative metaphor of "collapsing structure" situated in the abstract algebra was not something that Max felt comfortable with yet. The culturally normative term for the more advanced learner is just everyday language that gets thrown around and it assumed that the other person you are having a conversation with knows what you mean by the metaphor, but for Max it seemed strange within the context of this task and this made effective communication difficult.

6.2.2.2 Impossible strategy to construct a homomorphism to Z_4

After the task was rephrased, Max attempted to write down two homomorphism maps on the blackboard in Figure 6.5.

> Max: So I just rewrote the group definitions [referring to the group presentations]. I'm seeing is that they both have an element that, uh, to the fourth power is the identity, order four, and so I'm wondering if I can for both of them take the second generator which goes to a different place in each group [draw arrow to point to j] and just collapse it the identity. So you have, um, ψ_1 D_4 to G_1 and want to define it in such a way that uh $\psi_1(g) = g$ or if you have this second element here [the generator f], uh it goes to one [the identity], and then I think that would just like, you would lose the information that's making them different. And so you end up with like G_1 [writes down G]. And so yeah, I mean, I don't know what else to call that other than the cyclic subgroup of the rotations in the dihedral group.

> Max: Okay so I'm just gonna like, keep thinking about it [but immediately moves on and says...] so like moving onto the second one, um, now the homomorphism for Q_4, what we call the ψ_2...[writes down his map ψ_2 shown below going to $G_2 =< i : i^4 = 1]$. And then you can pretty clearly see that these guys are isomorphic, because they both have one generator of order four. And that's it, is that all?

> Researcher: Let me pause to think.

> Max: I probably need to be a lot more rigorous with this, but that's kind of all I can think of. I guess if I was going to write this, and if you only had the two generators, you really only need to describe what it does to the two generators. Right?

Max, had assimilated the commonly used phrase "you only need to describe what it does to the two generators" into his vocabulary, but misapplied it.

> Researcher: So if you already have a homomorphism and you know where the generators are going then you know where the rest of the elements go using the

hom property. But if your trying to build one from scratch and you just send the generators to something uh then you might not actually get a homomorphism. That is something you would have to check to see if it works or not.

Max: So it sounds kind of like a laborious thing to do, and if it is then that's fine. But when you're working with these specific elements, how can we do that in a way that isn't like super burdensome? I don't know, like, what I'm hearing is just that I'm going to have to explicitly show every product.

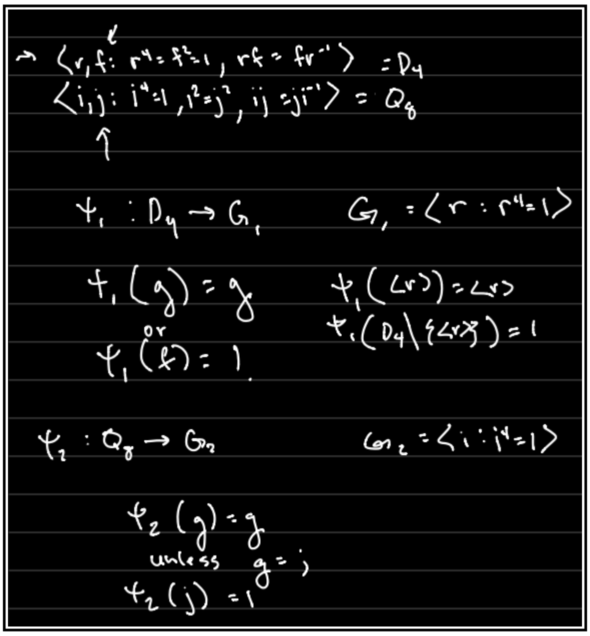


Figure 6.5: Max's Pseudo-SR for a homomorphism to Z_4 in formal mapping register.

Within the Cayley digraph register it became clearer that Max was attempting to carry out the impossible strategy due to inconsistent interpretations of a homomorphism. Recall that the impossible strategy involves a homomorphism from the dihedral group of order 8 to the cyclic group of order 4. Max restricted his focus to the cyclic subgroup of order 4 in the dihedral group and lost sight of the other elements and relations, it was if they were just erased.

Researcher: Is there some way you can describe what you are doing visually in terms of the Cayley digraphs?

Max: I can definitely try [as Max draws the picture below he explains...] uh, so at first we had like these inner green guys. Then I guess if I was going to describe what I did, I guess, like this inner piece [circles the inner green cycle] and just like pull it out and make it hang out by itself and this is a new group. And you don't need to worry about the relationship between these guys [referring to the outer green cycle] or those guys [referring to everything else], because those guys don't exist anymore.

Max referred to his drawing Figure 6.6 as representing a homomorphism from D_4 to Z_4, but drew the arrows for the two green cycles in the digraph in the same direction, so technically his digraph for D_4 was actually a digraph for $Z_4 \times Z_2$. While there exists a homomorphism from $Z_4 \times Z_2$ to Z_4 through a process of identifying elements and collapsing the red arrows to obtain four cosets of equal size, Max's process of erasing did not fit with the construction of a quotient group in a digraph register.

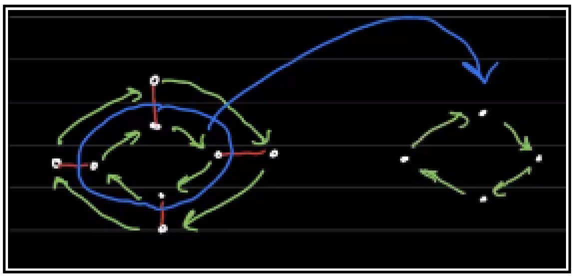


Figure 6.6: Max's Pseudo-SR for a homomorphism to Z_4 in a digraph register.

Researcher: And what about for Q_4? Can you describe what you would do visually?

Max: I'm seeing four four cycles. Is that wrong? Right?

Researcher: You should see three four cycles

Max: Okay. I guess I'm missing wherever, um the two cycle is. I can't spot it.

Researcher: The two cycle, is that what you're looking for.

Max found it difficult to see distinct cycles in the digraph register, he did not map between registers such as the cayley table, cycle graph, subgroup list or group presentations to find or check the number cyclic subgroups. There may also have been a miscommunication regarding the meaning of the term cycle that caused confusion. Max's claim that there were four four cycles was consistent with the graph theory definition of a cycle detached from thinking about subgroups. The researcher used the term cycle to refer to distinct cyclic subgroups in the digraph context. Max referred to the

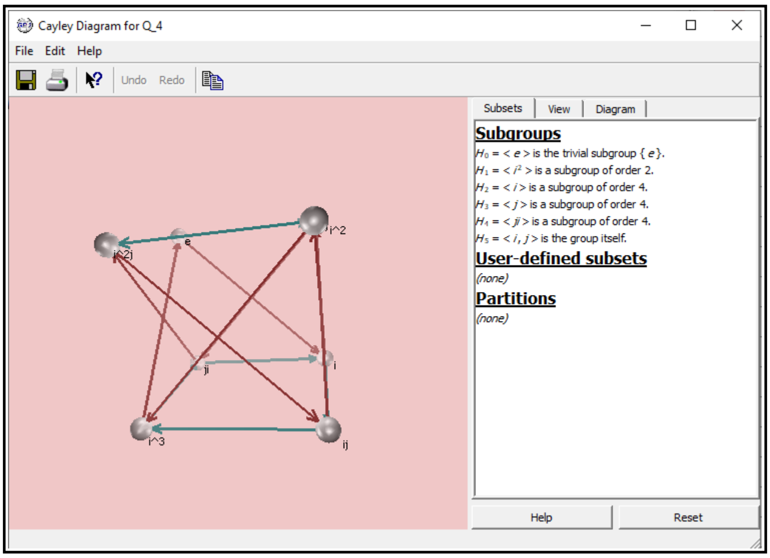


Figure 6.7: Max looking for cycles in the quaternion group.

Cayley digraph of the quaternions oriented with a four cycle on the top in Figure 6.7. The green arrows represent the generator j and red arrows are the generator i. The quaternion group is acting on itself by right action.

> Max: Okay. Well, whatever. Um, like visually you still have those two, like outer conjoining, four cycle situation happening with the green arrows. And then like, it's just the cycles in between like the inner and outer cube, or like squares, are just like more complicated. And so I guess again, I just like selected one of the, either top or bottom and took it off on its own. So like, if you straighten it out to where, like the green boxes is facing down. So on the top, the top square, there's like one four cycle between those elements. And then I just selected those, took them off by themselves. Then I don't have to worry about what they do with the other things on the bottom, because I'm just looking at the top.

What Max claimed he was doing, constructing a homomorphism, was in conflict with his thinking in the formal mapping and Cayley digraph registers. The digraph visuals were complicated and rather than connecting back to formal knowledge, binary products or homomorphisms, Max took the path of least visual resistance by making the digraph look visually simpler. He was not associating his quick visual maneuvers with a formal analog. One way to confront this conflict would be to guide Max to think about his claim in the context of the first isomorphism theorem and view a homomorphism as running through a quotient map. This view may also help to respond to Max's question of how to check if something is a homomorphism that is not burdensome. I asked Max if he had seen the mod out or quotient group notation: ${}^{G}/_{\sim}$ or ${}^{G}/_{N}$ yet. He asked “do you mean like the set without the set you would write underneath?” I asked him if he had exposure to quotient groups and quotient maps in his course work yet. He replied “No”. While Max had some exposure to homomorphisms and could recall the formal definition, he needed more space and time working with equivalence relations, normal subgroups, quotient groups, and the first isomorphism theorem in order to be equipped with tools to become aware of, confront, and resolve conflicts across various registers.

6.2.3 Chronological narrative for Jenni

Jenni’s first thought was to find subgroups within each of the groups that were isomorphic to each other. She found the cyclic subgroups $< i >$ and $< r >$ were isomorphic. Initially, she did not consider elements outside of these subgroups nor relations between elements in and outside of these subgroups. Her strategy was “just getting rid of elements”. She was looking at the group presentation and crossed out the generator f and crossed out any relations with an f in it. She also wrote down the elements in the group $e, r, r^2, r^3, f, rf, r^2f, r^3f$ and crossed out the elements with an f in it. She constructed an isomorphism map between cyclic subgroups of order four in Figure 6.8 and wrote down the Cayley tables in Figure 6.9 to double check that the subgroups were isomorphic.

> Jenni: Well they both have cyclic subgroups of size four [looking at the group presentations]. So if you take the dihedral group of order 8 and you remove all the flips and just have their rotations, and then you take Q_8 and reduce it to just the generator i until you just have i, -i, 1, and -1.

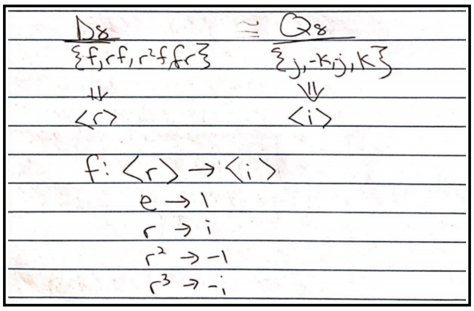


Figure 6.8: Jenni’s construction of an isomorphism for the Z_4 strategy in a formal mapping register.

> Jenni: I’m going to write down the Cayley tables for that real quick because I

> don't know if that would be...[works on paper see below]...Ya so I'm pretty sure those the same.

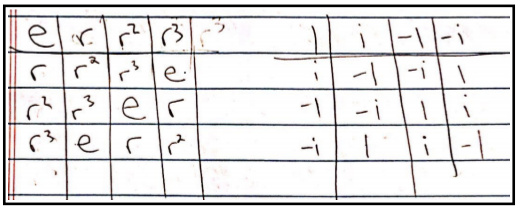


Figure 6.9: Jenni using Cayley tables to check that subgroups are isomorphic.

When asked to formalize her thinking, Jenni immediately associated the loose language of collapsing or ignoring with taking quotient maps.

> Researcher: Can you detail your thinking using more rigorous language? So when I asked you to collapse structure or ignore information you were thinking about removing the flips until you just have the rotations. What does that mean to you formally?
>
> Jenni: Formally, I guess I would think about it as like you take, you take the whole group and then you quotient out by the uh [long pause and scratches head]. Can you quotient out like uh we have two generators for each of them no its cosets. So if you know its cosets...[broken thoughts externalized to herself, stops externalizing thoughts and writes on paper].
>
> Jenni: okay I'm having trouble trying to formalize it because I think I got a little tripped up with am I, I am taking the whole group. So would I have to prove that the quotient group D_8 quotiented out by, like what I'm collapsing and reducing it to is equivalent to the cyclic group generated by r?

At this point, I summarized the task by drawing the diagram Figure 6.10 and inserted quotient map notation to fit with Jenni's interpretation of collapsing. I asked Jenni if she could prove whether or not what she was doing would be invalid or valid. Maybe this would help her to detect what would go wrong and pin down why she was stuck as she entered the impossible strategy to obtain Z_4 via quotient maps.

6.2.3.1 Impossible strategy to construct a quotient map to Z_4.

Jenni went back to working on paper at her desk quietly [she was finding it difficult to verbalize her thoughts out loud]. I conducted a follow up interview later to try and

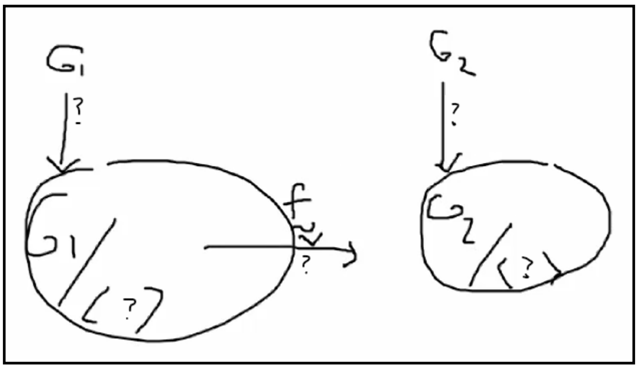


Figure 6.10: Researcher summarizing task to fit Jenni's interpretation of collapsing as quotienting.

clarify what she was wrestling with during this period of time and why she got stuck. She said she felt conflicted because her initial strategy to get rid of elements did not fit with the process of quotienting out. During this time she attempted to carry out the impossible strategy thinking in terms of formal mappings, group presentations, and Cayley to Schreier digraphs. She explained that she took her initial strategy of trying modify D_8 and Q_8 to get to Z_4 and started to write down what the cosets would be if she "quotiented out by something to get the cyclic subgroup generated by r".

Reflecting back on her work and why she was stuck, she explained that:

> Jenni: My initial thought was just getting rid of elements was the way to go to leave me with a subgroup and then maybe the quotient groups will help me do that and then thinking well they don't leave...If I like quotient by a subgroup its not just leaving me with a subgroup. And I think it took me a little bit to remember what quotient groups look like and how they relate to the non quotiented groups, to the groups [D_8 and Q_8] before they were quotiented.

While she never pinned down exactly why the impossible strategy could not work by showing that a necessary condition for a quotient map failed, she came to the realization on her own that her initial process of "getting rid of elements" and what a quotient map does felt different. She stated that "quotienting was not getting rid of elements it was combining them" and "they were combining elements into sets of elements". She showed flexibility and switched from the impossible strategy to a new strategy, namely quotienting out by order four subgroups to get Z_2.

6.2.3.2 Valid strategy to construct a quotient map to Z_2

Jenni went on to correctly write down the elements of ${}^{D_8}\!/_{< r >}$ and ${}^{Q_8}\!/_{< i >}$ and calls the elements of these factor groups "cosets". The underlying binary operation of the groups remained implicit, it was not written down or verbalized. She wrote down quotient maps, the natural projections, using symbolic notation and an explicit isomorphism map between ${}^{D_8}\!/_{< r >}$ and ${}^{Q_8}\!/_{< i >}$ as shown below. She noted that both quotient groups are of order 2 and must be isomorphic to Z_2 in Figure 6.11.

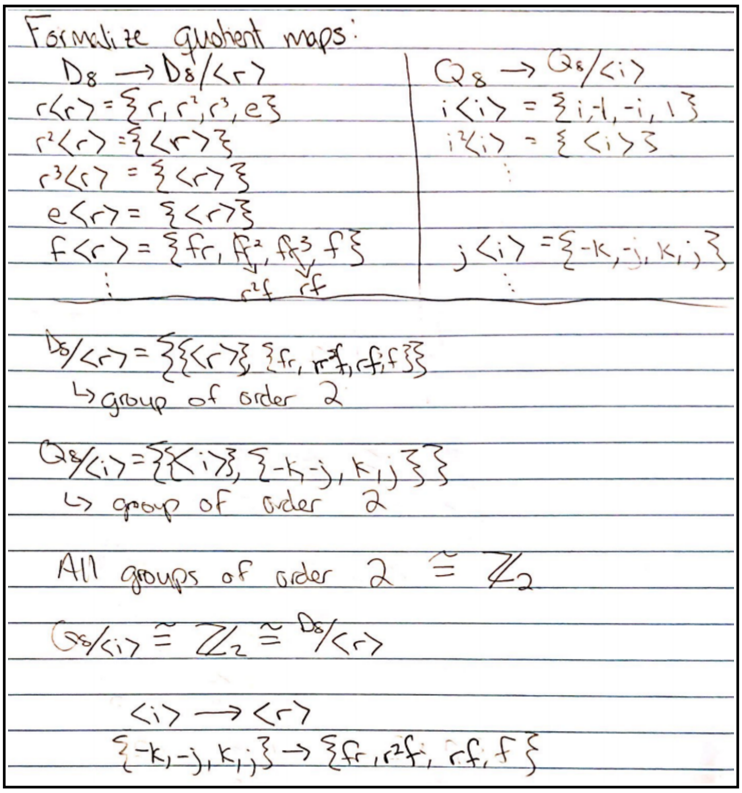


Figure 6.11: Jenni's construction of quotients in a formal mapping register for the Z_2 strategy.

Before ending the session I revisited Jenni's statement that "${}^{Q_8}\!/_{< i >}$ is a group because $< i >$ is a subgroup of Q_8" and the normality issue or equivalently the issue of well-definedness of the standard factor group binary operation. Namely, that the operation $g_1H * g_2H = g_1g_2H$ for ${}^{G}\!/_{H}$ where H is a subgroup of G is well-defined iff H is normal in G.

> Researcher: So, are there any underlying assumptions that you're making the need to be formalized further? So your quotienting out by a subgroup to get

another group. Do you need to be careful about what kind of subgroup it is? Does it need to have some special property?

Jenni: Um, I do know that the subgroups are cyclic and therefore abelian. Um I know that the order of the subgroup has to divide the order of the overall group, but I think that's kind of inherent.

Researcher: So in isolation we know that those groups that you're quotienting out by, if we just look at them in isolation, we know they are cyclic in their own right we know their abelian in their own right. What kind of relationship do they need to have with the rest of the elements in the group? Does that make sense question-wise?

Jenni: Maybe, I'm trying to think of a property for the quotient map to be well-defined. Can I see the Cayley tables to check something?

After long pause she said that she was not sure of the top of her head and it would take awhile for her to try and figure it out. We discussed that in the task it was clear to see that $D_8/_{< r >}$ and $Q_8/_{< i >}$ were both groups with the usual factor group binary operation. Jenni quickly explained they both have an identity and then an element of order two is its own identity, and associativity holds. We discussed that for larger groups that checking each of the group properties is satisfied at the elemental level may not be efficient or reasonable. Due to limitations on time I did not pursue the necessary condition of normality further.

Up to this point Jenni had used the cayley tables, group presentations and quotient maps using symbolic notation during her final collapsing structure process. Next, I investigated what her final collapsing structure process would be in terms of Cayley to Schreier digraphs. Jenni momentarily and spontaneously reverted back to her initial strategy of obtaining Z_4 by getting rid of the flips. She stated that:

Jenni: I guess I would think about it as removing the red lines. Because those would be the flips.

I reminded Jenni that what she did previously was quotient out by these subgroups, $< r >$ and $< i >$ of order 4, and ended up with Z_2.

Researcher: So hold on a second. So you were just quotienting out by the cyclic subgroup of order four. So when you quotient out by the cyclic subgroup of order four, how does that translate to something that you did to the Cayley digraph?

> Jenni: It's like collapsing the inner circle into one element and then collapsing the outer circle into one element. And then they are only connected by one line [generator].
>
> Researcher: So remember a bar or solid line is just notation for two arrows going in opposite directions. So what you are left with, what group does that represent?
>
> Jenni: That represents the, uh, quotient group, which is isomorphic to Z_2.
>
> Researcher: Okay. So going to Q_8, can you translate what you did with Q_8 by manipulating the Cayley digraph?
>
> Jenni: Yeah, I guess the top squares that are connected by the green lines would collapse into one and then the bottom elements that are connected by the green would collapse into one. And then all of the red lines would also just collapse into one red line. So again, we'd have Z_2.

Jenni transitioned from predominantly formal mappings to the digraph register seamlessly for the Z_2 strategy. She invoked her quotienting process of combining elements to form two sets of cosets, the cyclic subgroup of order four played the role of the identity. She could see that this process left her with a group and checked off the properties of inverses, associativity, and identity. However, Jenni was still missing the normality condition.

6.2.4 Chronological narrative for Alex

Alex, the second year graduate student, experienced negative affective responses at the start of the interview. After a few minutes he stated:

> Alex: I think my anxiety is kicking in, I'm just like totally [inaudible] I can't do this.

Due to this negative affective interference, the in-person interview was rescheduled to the following week. During the rescheduled interview, Alex said he preferred to use the group presentations and drew subgroup lattices. He interpreted collapsing or ignoring structure within the context of this task in two contrasting ways. One way was to obtain groups with smaller order by: adding relations to the group presentations followed by removing redundancies and taking quotient maps (quotienting out by normal subgroups). A second way was to obtain an infinite group. He went to the level of a free group on two generators and collapsed structure in this free group by adding overlapping relations of Q_8 and D_8 to it in order to obtain an infinite group. He also viewed this process as being analogous to taking out non-overlapping relations in the provided group presentation of Q_8 and D_8.

6.2.4.1 Valid strategy to obtain an infinite group

Alex explained that his solution for the later was to take the group presentation of D_4 and Q_8 and remove the relation $f^2 = 1$ and $i^2 = j^2$, that were making them different. He denoted the modification to D_4 and Q_8 as the free group on two generators with the remaining relations in Figure 6.12.

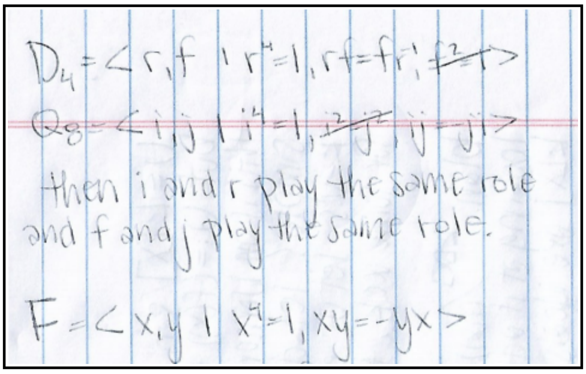


Figure 6.12: Alex's strategy to obtain an infinite group in the group presentation register.

He also described his construction of F verbally:

> Alex: So what I ended up doing was looking at F just the free group, the both don't sit in there [referring to Q_8 and D_4], but are related to quotients of F in some way. I just looked at the intersection of the quotients, so like you have the free group, and then you take some sort of normal subgroup of it and you mod it out and you get D_4 and similarly take some sort of normal subgroup and mod it out [of the free group on two generators] to get Q_8. Um, and those are two different normal subgroups that you moded out by. And so the normal subgroups dictate what the relations are. Cause that'll say whatever is in, there [referring to the normal subgroups] is effectively zero [identity]. Um so, I took the intersection of those and then moded out the free group on two generators by the intersection of the normal subgroups. [As Alex was describing his process verbally, we drew it on the white board with more explicit notation to communicate. Notation: $Q_8 = {}^{F<x,y>}/_{N_i}$, $D_4 = {}^{F<x,y>}/_{N_j}$, $F = {}^{F<x,y>}/_{N_i \cap N_j}$].

Researcher: Can you describe the resultant group F.

Alex: So one of the things since N_j intersect N_i is a smaller group that your moding out by [smaller than N_k where k=i or j] you have fewer relations in the resulting quotient group [F] so you're going to have more elements now because you're, because things in N_i intersect N_j are going to, you are saying all of these are zero [identity] now.

Researcher: So you're saying it's bigger than [${}^{F_{<x,y>}}\!/_{N_i}$ and ${}^{F_{<x,y>}}\!/_{N_j}$],

Alex: Yeah. So that would be bigger because we have a smaller section of stuff that is zero. There is still some cyclic group of order four. There is this anti commutativity property where xy equals negative yx. There's probably more stuff to it, but that's like the immediately obvious stuff.

6.2.4.2 Valid strategy to obtain Z_2 and $Z_2 \times Z_2$

After this I rephrased the task by asking him to obtain isomorphic finite groups that had order less than 8. Alex immediately carried out a valid strategy to obtain Z_2. He also associated the language "collapsing structure" to a quotient map.

Alex: So my first response is if I take the subgroup generated by i in the quaternion group, that's going to be an order four group and it's normal so we can mod it out. And then you're just going to get Z_2 and then D_4 we can mod out the rotation subgroup. You get Z_2 also. I have to double check, but that seems on the face of it, what I would think [writes down quotient maps].

Researcher: Is there some way that you can translate those quotient maps visually using the Cayley digraphs?

Alex: Um, so since we are modding out by the group generated by i we're going to want that whole subgroup to just be the same element. So like in here, the dots are the little vertices of i, negative i, one, and negative one would become one element [looking at the Cayley digraph of Q_8].

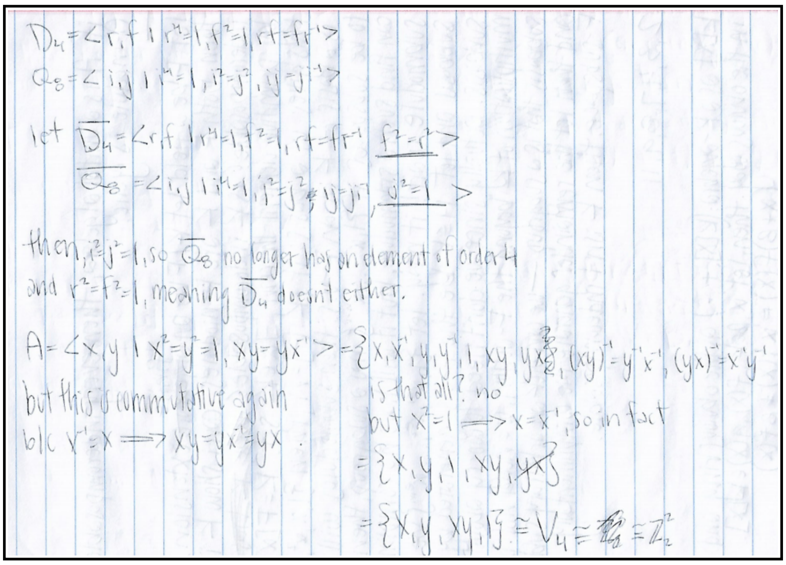
Researcher: So when you identify these vertices to one thing, that new node, what is that?

Alex: A coset.

Researcher: Okay. So what else do you do?

> Alex: So now I have five elements and the four other ones are going to be the other coset. And the red arrows between the one, i, negative one, and negative i, those arrows would collapse because all those elements would become one element. And then the other elements, the j, negative j, k, and negative k, would collapse to be the other coset. And so the relation, like you'd end up having just one relation between those two that goes from the i coset to the j coset and then back to the i coset. And some of these relations wouldn't collapse, they'd just become the exact same arrow.
>
> Researcher: And for the dihedral group what would you do?
>
> Alex: I think for this one, the inner circle, those are going to all collapse into one element and then the outer circle would collapse into one element. This time the green arrows are going to be the ones where you collapse into an element and collapse into an element. Um, so that would leave the two-way red arrows, but then they would end up becoming just the same two way, one relation.

Alex also carried out another valid strategy to obtain the Klein-4 group. To do this he started with the group presentations, added relations, and then reduced redundancies as shown in his work in Figure 6.13. He was also able to carry out this process in the Cayley digraph register. Alex never attempted to carry out the impossible strategy during the interview, moreover he never mentioned it.



$D_4 = \langle r, f \mid r^4 = 1, f^2 = 1, rf = fr^{-1} \rangle$

$Q_8 = \langle i, j \mid i^4 = 1, i^2 = j^2, ij = ji^{-1} \rangle$

let $\overline{D_4} = \langle r, f \mid r^4 = 1, f^2 = 1, rf = fr^{-1}, \underline{f^2 = r^2} \rangle$

$\overline{Q_8} = \langle i, j \mid i^4 = 1, i^2 = j^2, ij = ji^{-1}, \underline{j^2 = 1} \rangle$

then, $i^2 = j^2 = 1$, so $\overline{Q_8}$ no longer has an element of order 4 and $r^2 = f^2 = 1$, meaning $\overline{D_4}$ doesn't either.

$A = \langle x, y \mid x^2 = y^2 = 1, xy = yx^{-1} \rangle = \{x, x^{-1}, y, y^{-1}, 1, xy, yx\}, (xy)^{-1} = y^{-1}x^{-1}, (yx)^{-1} = x^{-1}y^{-1}$

is that all? no

but this is commutative again b/c $x^{-1} = x \Longrightarrow xy = yx^{-1} = yx$.

but $x^2 = 1 \Longrightarrow x = x^{-1}$, so in fact

$= \{x, y, 1, xy, \cancel{yx}\}$

$= \{x, y, xy, 1\} \cong V_4 \cong \cancel{Z_4} \cong Z_2^2$

Figure 6.13: Alex's strategy to obtain $Z_2 \times Z_2$ in formal mapping and group presentation registers.

6.2.5 Results in terms of the representational fluency lens

There was a great deal of variability among Max, Jenni, and Alex with respect to their approach to the same task, what collapsing structure or ignoring information meant to them, their register use, and the sign-interpretation pairs they produced within each of the registers.

Collapsing, losing, or ignoring structure interpretation themes

Several of Alex's and Jenni's interpretations of collapsing structure aligned with the process of constructing quotients. This included: combining or identifying elements into sets of elements, modding out by an equivalence relation, and partitioning. Max associated collapsing structure with: losing subgroups or space in the subgroups and a homomorphism. All three participants thought of losing information as losing what is making the two groups, the dihedral and quaternion group of order eight, different and keeping what they have in common (1.3). They combined this with other interpretations in different ways.

Alex, the second year graduate student, combined theme (1.3) with fixing generators and removing relations in group presentations to define a larger group (1.5). At times, the phrase "losing information" and collapsing structure took on opposite meanings for Alex as he used the Free group on two generators as a starting point. For example, he mentioned that keeping the two generators and adding relations could be thought of as collapsing the group into a smaller group and keeping the two generators and losing relations could be thought of as expanding the group. In other words, "losing or ignoring information" as getting rid of relations was expanding the initial group into a larger group because their would be fewer relations to collapse words.

In contrast, Max and Jenni, at times, combined interpretation themes in ways that were in conflict with the formal mathematical object of a homomorphism. They both mixed the interpretation (1.3) getting rid of what is making the two groups different and keeping a subgroup they have in common with a homomorphism. They carried this mix of interpretations with them to the Z_4 strategy. This mixing of more general problem-solving strategies connected to the mathematical object of a homomorphism led to the implicit assumption that a subgroup would be the image of a homomorphism. This is in conflict with the formal object of a homomorphism sometimes because the image of a homomorphism need not be a subgroup of the domain group. This thinking may have blocked them from entering into the $Z_2 \times Z_2$ strategy since $Z_2 \times Z_2$ is not isomorphic to a subgroup in the quaternion group of order eight. In the follow-up interview Jenni showed an evolving awareness of this conflict and stated: “if I quotient by a subgroup that is not just leaving me with a subgroup”.

Alex did not assume that the image of a quotient map would be a subgroup of the domain group. He also noted that in the context of fixing generators, removing relations would change how a generator interacts with other elements. Learners assign meaning to normative metaphors they encounter. This meaning seemed to depend on the strategy the participants were in and their mathematical backgrounds. For the undergraduate learner the metaphorical phrases that they hear such as “losing” or “ignoring” information may have been associated with formal objects due to temporal association and interpreted in ways that deviate from the meaning intended by an instructor or text. The more seasoned learner in this study, Alex, used these metaphors more appropriately and avoided conflicts between loose metaphorical thinking and the formal mathematical objects.

An interpretation of losing information associated with a homomorphism that did not arise in Max, Jenni, and Alex’s observable responses was the idea that you loose detailed information about individual elements orders. For example, the homomorphism from the dihedral group of order eight to the cyclic group of order two, that maps the rotations to the identity, washes out all information about the orders of the rotations. In less extreme cases some but not all information about the orders is washed out. For example, the homomorphism from the dihedral group of order eight to the Klein-4 group, that sends the rotations to order two elements, still leaves information that the order of the rotations is a multiple of two but looses information regarding precisely what the orders are.

Register use

This section covers Max, Jenni, and Alex’s natural register use which refers to registers that were used without specific prompts stated by the researcher to enter the particular registers. Max used registers as a source of ideas where as Jenni and Alex used the registers a source of ideas and to also check ideas. Max’s starting register was Cayley tables. Max predominantly used the Cayley tables, formal-symbolic function mappings, and group presentations during the task and only moved to Cayley digraphs briefly without being prompted. He rarely checked his ideas against the formal definition of a homomorphism and showed some resistance to formal verification during the interview. He become to Z_4 as a default intuitive response that he recognized in the perceptual environment and latched onto what he was familiar with. He didn’t let go of it.

Jenni predominantly used the group presentations, cayley tables, and formal-symbolic function maps. Jenni started with the group presentations to spark ideas about what the isomorphic resultant groups could be. Once she made a pick she would go to the Cayley tables to check that the resultant groups she had chosen would be isomorphic. Then she transitioned to the formal-symbolic function maps to try to construct homomorphisms from the initial domain groups to the resultant image groups. Alex predominantly used the group presentation with thinking about the free group, wrote down subgroup lattices looking for normal subgroups, and formal-symbolic quotient maps to verify and write a final answer. Alex was the only participant that referred to the free group when he used group presentations. He never used Cayley tables. Jenni briefly mentioned that she would move to the Cayley table register to investigate what property was needed to ensure that her quotient map she wrote in a formal-symbolic mapping register was well-defined. Due to time constrains, a single switch prompt to the digraph register was administered for this study.

Conversions and pseudo-semiotic representations

Quotient map conversions from registers started in to digraph register. Max did not enter a valid strategy to obtain homomorphisms with the images Z_2 or $Z_2 \times Z_2$. After giving valid quotient map responses in their natural starting registers, Jenni and Alex were prompted to switch to the Cayley digraph register. A formal definition of a Cayley digraph was given with a brief explanation of what the arrows and vertices represented, no other information was provided regarding the Schreier coset register. Both Jenni and Alex converted their formal-symbolic quotient maps to digraph registers. After the prompt to switch to a Cayley digraph register, Jenni and Alex both entered the Schreier digraph register on their own when thinking about quotient maps. Both manipulated the Cayley digraph in ways that were consistent with the formal analog of the equivalence relation: two elements are equivalent if they are in the same coset. They glued vertices together to represent elements in the domain group becoming disjoint sets of elements of equal size in the resultant quotient group for Z_2. Phrases like "collapse" and "crunched down" were used without invoking tearing or erasing behaviors. Alex made it explicit that the identity subgroup as the normal subgroup and produced the resultant quotient group for $Z_2 \times Z_2$ in the Schreier digraph register, Jenni did not arrive at the normality condition during the interview in a digraph register.

Pseudo-semiotic representations Both Jenni and Max produced pseudo-semiotic representations with respect to homomorphisms (Max) and quotient maps (Jenni). Max's

Table 6.1: Registers that participants produced quotients in prior to switch prompt.

Registers that participants naturally produced quotient in prior to switch prompt	M	J	A
Formal-symbolic quotient from start group either D_8 or Q_8 (group presentation) to Z_2 (group presentation)		$\checkmark^{D_8/\sim}$ $\checkmark^{Q_8/\sim}$	$\checkmark^{D_8/N}$ $\checkmark^{Q_8/N}$
Formal-symbolic quotient (group presentation) start group to $Z_2 \times Z_2$ (group presentation)			$\checkmark^{D_8/N}$ $\checkmark^{Q_8/N}$
....	....	....	....
Cayley tables start group to Z_2			
Cayley tables start group to $Z_2 \times Z_2$			
....	....	....	....
Cayley$\to$Schreir digraph start group to Z_2			
Cayley$\to$Schreir start group to $Z_2 \times Z_2$			

Table 6.2: Digraph register that participants converted quotient maps to after switch prompt to go to Cayley digraph.

Register that participants converted valid quotients maps to after switch prompt	J	A
Cayley$\to$Schreir digraph start group to Z_2	$\checkmark^{D_8/\sim}$ $\checkmark^{Q_8/\sim}$	$\checkmark^{D_8/N}$ $\checkmark^{Q_8/N}$
Cayley$\to$Schreir start group to $Z_2 \times Z_2$		$\checkmark^{D_8/N}$ $\checkmark^{Q_8/N}$

semiotic triangles for a homomorphism were inconsistent across multiple registers. His interpretation of collapsing and ignoring structure as a homomorphism (code 1.5) was in conflict with his map psi_1 that utilized the group presentations (maps generator r to order 4 element in Z_4 and generator f to the identity in Z_4 (code 2.2) and his focus to just the inner green cycle of rotations in the Cayley digraph (code 1.4). Following Max's instructions for his mapping ψ_1 and paying attention to just the elements we have: $\psi_1 : e \longrightarrow e, r \longrightarrow r, r^2 \longrightarrow r^2, r^3 \longrightarrow r^3, f \longrightarrow e, rf \longrightarrow r, r^2 f \longrightarrow r^2, r^3 f \longrightarrow r^3$. However, if one assumes the homomorphism condition holds and pays attention to the relations one can see that this map is not well-defined, since $r^2 = r^3 frf$ can be mapped to $\psi_1(r^2) = r^2$ and $\psi_1(r^3 frf) = \psi_1(r^3 f)\psi_1(rf) = r^3 r = e$ which leads to a contradiction. Within the digraph register, Max produced the pseudo-SR where the sign was a digraph, his interpretation was "pulling out" a subgroup and erasing everything else, and he indicated that it signified a homomorphism. Max's sign-interpretation pair was in conflict with the actual mathematical object of a homomorphism.

Similarly, Jenni's earlier interpretation of getting rid of elements by cancelling out the flip generator f and elements with an f in the group presentation register was in conflict with the signified object of a quotient map. Unlike Max, Jenni detected this conflict and realized that getting rid of elements with an f in it was not the same as the process of taking a quotient map which she reiterated "combines" elements. Even though Jenni affirmed that her earlier process of cancelling out elements with the flip generator in it, in the group presentation register with a list of elements, to get obtain Z_4 as a quotient group "didn't make sense", she was still trying to understand during a follow-up interview whether or not it was still possible to get Z_4 as a quotient group by another process in the digraph register. Her process in the digraph register was at least consistent with partitioning the elements of the dihedral group of order eight into cosets.

> Jenni: My initial instinct when looking at it was that like all of these connections [red lines] were like s's [notation for flip generator] and so if you just get rid of the s's you collapse these two points into a point and these two points into a point and so you end up with a square and that was along with my process to get to Z_4. What I had been thinking about was that completely getting rid of the generators s would collapse these lines and leave you with Z_4 [sketches Figure 6.14].

Jenni also referred to the this as being synonymous with combining the pairs of elements e and s, r and sr, r^2 and sr^2, r^3 and sr^3 to a vertex $\{e, s\}$, $\{e, s\}r$, $\{e, s\}r^2$, and $\{e, s\}r^3$. While Jenni's process forms a partition of D_8 into cosets, it does not form a quotient group. Jenni did not refer to the standard factor group binary operation or set multiplication. She was not attuned to the visual cues indicating that the standard factor group binary operation was not well defined. The following computation shows that the well-definedness property does not hold: $\{e, f\}r * \{e, f\}r = \{e, f\}r^2$ and $\{e, f\}r * \{e, f\}r = \{r, fr\} * \{e, f\}r = \{fr^3f, r^3f\} * \{e, f\}r = \{e, f\}r^3f * \{e, f\}r = \{e, f\}r^3fr = \{e, f\}fr^2$. With the operation as set multiplication, on Jenni's partition, it can be observed, in the Cayley table register, that the closure property does not hold since $\{r, fr\}*\{r, fr\} = \{r^2, f, fr^2, e\}$ and that $\{e, f\}$ does not behave like the identity, in fact none of the cosets behave like an identity with set multiplication. Jenni did not refer back this time to the Cayley tables to check whether or not her partition with the standard factor group binary operation or set multiplication formed a group. Pseudo-SRs were not found in Alex's data.

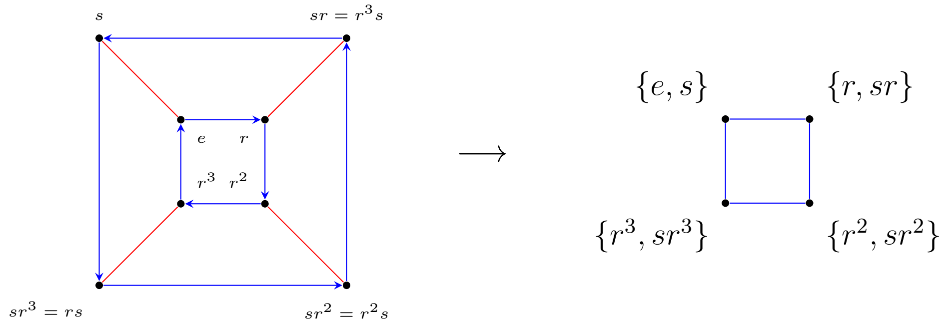


Figure 6.14: Pseudo-SR for the signified object of a quotient map from D_4 to Z_4 in a digraph register.

6.2.6 Results in terms of the default-interventionists' intuition lens

After applying the intended apriori semiotic lens to analyze data, the default-interventionist system 1 and system 2 distinction by Kahneman (2011) was used to provide additional layers. Through the new lens, it was observed that Max experienced rapid default-intuitive judgements produced by system 1. His intuitive outcomes were more indicative of matching-associative errored forms and were generated rapidly based on visual stimuli in the environment. His analytic system was not fully kicking in, eventhough he indicated that he should "probably be more rigorous". Brute force, inefficient busy work was something that Max had an aversion towards and this suppressed his analytic system 2 from carrying out binary product computations to check if his map he wrote down was a homomorphism. After the analytic system 2 was suppressed, Max's reflective system posed a beautiful question:

> Max: So it sounds like a laborious thing to do...how can we do that in a way that isn't super burdensome?

This question was relatable and a key input for Max's intuitive system to crunch on that could be leveraged by an instructor in a classroom situation to motivate the need to work towards results like the first isomorphism theorem. If one gains access to Lagrange's theorem, the first isomorphism theorem, and knows what the normal subgroups of the group are, this problem of burdensome local energy computational costs that Max is experiencing becomes an efficient cognitive energy saving strategy. This puts less stress on the analytic system in the long run. In the pure math setting of this study, signals from the analytic system 2 that one should check followed by the subsequent suppression of the analytic system 2 is not necessarily a bad thing, but highly adaptive energy saving tactic. Aversion to labor intensive computations is one reason that may drive a motivation to compress towards theorems. Unlike Max's system 2, Jenni's system 2 seemed to be on overdrive. Jenni produced pages of routine computations and she was having difficulty retrieving formal results she had learned. Alex, hardly wrote down any routine computations. His thoughts seemed to be driven more by theorems because he narrowed in on the correct strategies, Z_2 and $Z_2 \times Z_2$,quickly and without erring. It is not explicitly clear what theorems Alex was using because he did not verbalize them, his responses seemed rapid like he did not have to work very hard for them during the interview.

6.2.7 Results in terms of an affective impacts lens

Max flipped back and forth from projecting ease with a confident response using phrases like "obviously" "clearly" "the rest follows" to projecting uncertainty and insecurity with words like "sorry" and "I don't know if that makes sense, but that's what I have". Rather than using his awareness of uncertainty as a signal to take his time and comfortably investigate if his claims made sense, he would state his claim and just freeze. Despite being uncomfortable, he still wanted to try. His rapid responses with little to no time spent to check coupled with his willingness to try signaled that he could have been experiencing a need to perform, "fear of invalidity" and "premature closure" during the the interview (Fiscbein, 1987, p. 195-196). "In order to preserve the already reached equilibrium (as an effect of closure) the individual tends to avoid the situation of facing different alternatives" or strategies (Fischbein, 1987, p. 196). Max began with generating key ideas as he explored patterns in the Cayley tables of the quaternion and dihedral group. Then, he began to exhibit symptoms of premature closure when he wanted to manipulate symbols so that the cayley tables would look the exact same. However, he became so fixated on the cyclic subgroup of order 4 that he did not take time to investigate how changes like making 'r its own inverse' would induce changes to the rest of the group. This fixation persisted throughout the interview in other registers as well. The only strategy Max entered besides trying to make the quaternion group and dihedral group look the same, was the impossible strategy to obtain Z_4. Fischbein (1987) conjectured that "One tends not only to close the search for arguments, one tends to close the debate as early as possible...Very often the moment of closure takes place prematurely on the basis of incomplete information" (p. 196).

Alex also experienced affective issues during the task. I asked if he would be comfortable with sharing the impact of anxiety on his intuition. He reflected over this and shared his experiences.

> Alex: So I have been thinking about it after last Monday [initial session that was ended due to anxiety]. I think the best analogy I could think of was, if you read writers talking about writing, a lot of times they talk about having a writer and having an editor inside their head and to sit down and write, they have to put the editor somewhere else. So they can just like, make all the mistakes and like spell words incorrectly and have terrible grammar and then once they're done with that they can bring the editor in and like clean it up. And I think analogously anxiety kicks in and it's like, the editor part is like the one in control and it's the only thing that's happening. And so like the part of me that wants to just explore and see how the thing works as soon as anxiety kicks in it's just like, you don't have time for that. You got to figure this thing out right now. You have to just get to the answer and know exactly how to do it. Uh, which of course like cuts off all the conduit stuff, like actually learning how to do the thing that works and what the answer would be.
>
> Alex: It is sort of like you're so focused on the solution and the end goal, it cuts off all the paths to get to the end goal it feels like.
>
> Researcher: Yeah. I relate to that a lot.
>
> Alex: It is the worst thing.

6.3 Adding structure task: direct or semi-direct products

6.3.1 Results in terms of RBC+AiC lens

Preparation for RBC+AiC analysis: Jenni's learning history

As a preliminary step to conducting an RBC+AiC analysis, it is important that the researcher obtains background information regarding the learner's personal history and prior exposure related to the mathematical concepts under investigation (Dreyfus et al., 2015). Following the collapsing structure task I asked Jenni to describe aspects of her learning background as an undergraduate and her approach to similar tasks as an undergraduate versus her current approach to these types of tasks as a graduate student.

> Jenni: I think it took me awhile to build up intuition. I feel like what I have left is intuition and I don't have the formalization skills, that would

> take me a little while to build back up, but I think when I was first starting I wouldn't have had the intuition for it and so I would have been probably trying to find different formalizations and look at proofs that other people had written about similiar subjects or similar questions, so I think I would have gone about it [the previous task] differently [as an undergraduate].

She described an initial period of working out problems mechanically and following proofs. Later on after more exposure to group theory she began to develop intuition. Jenni also had experienced a period of not thinking about group theory for a year and forgetting. Formal definitions and proofs that she had previously learned were not quickly recoverable, these things were not fresh at her fingertip, and she had to rely on her intuition to guide her. Next, I asked her if she had been exposed to different ways of representing groups as undergraduate.

> Jenni: Yeah, I had the same professor for both the abstract algebra classes that I took and he showed a couple different ways of representing groups and then in other classes I would see other professors mention in it different ways, not usually these kinds of graphs, but different ways that they might write out the group presentations. Also reading through different textbooks and sources and stack exchange explanations and Wikipedia always has the Cayley graph on the side. I think the place I was exposed to Cayley graphs the most was Wikipedia.

Throughout all of the interview sessions Jenni was open to exploring and was relaxed. She was just comfortable with following her thoughts. Next, I transitioned into the collapsing structure task that would be analyzed through the RBC+AiC lens by presenting the following prompt.

> Researcher: Last time you were collapsing structure. This time you are going to be adding structure, well adding to one group or the other. So the task is, can you construct a larger group, of order 16, by combining the dihedral group of order 8 and the cyclic group of order 2 in some way?

Episode 1: adding a flip generator to the dihedral group in an object of symmetry register

Jenni began her construction process towards the new group by focusing on the generators of both groups and thinking about what it might mean to add a third generator that behaves like the generator for Z_2 to the dihedral group. She recognized that the two generators in the dihedral group corresponded to a flip and rotation in the square as its object of symmetry. She built with the object of symmetry for the dihedral group by adding an additional axis of symmetry for the third generator in Figure 6.15.

> Jenni: In the dihedral group we have rotations which gives the cyclic of order four and a flip which is a generator with order two. Um, so I'm thinking of just adding another one, so like a square with a different axis of symmetry, and then thinking of Z_2 as a different type of flip I guess.

Jenni maintained the construction process of adding a flip generator, left the object of symmetry register and entered the digraph register. She continued to explore what it would look like to add a third generator, of order two, to the dihedral group.

Episode 2: adding a flip generator to link two copies of the dihedral group in digraph register

Jenni recognized what the generators are in the digraph context and drew the digraph for the dihedral group. She built from this and the need for a group of order sixteen and drew two copies of the dihedral group connected with a third type of line for the additional flip generator.

> Jenni: So the way I'm thinking about it is that each type of line represents a generator so these one's are the rotation and these ones are the reflection. So I guess I thought [long pause] my thought was that it would look like different circles.

After a long pause of just sitting quietly and thinking and drawing on scratch paper. Jenni asked if she should draw her intuition even if she did not have way to formalize it. This question and reluctance to say something that was not formally sound yet indicated that some learners may withhold their intuitions from public view unless their is a norm that is okay to share their initial intuitions before they have time to check if they are right or wrong. Once it was estabilished that it was okay to share these early start-up intuitions, She stated:

> Jenni: So my idea was basically just that there is probably another flip involved. So yeah if this side is going this way, then the concentric rings arrows would go the other direction I think. So I will draw the new ones as this kind of line. My intuition is also that for this flip there should be another type of line [broken speech] connected between those nodes and then,

Building from her observation that the digraph should be connected, that you can get from and element to another by a word list of generators, Jenni drew eight new lines to connect the two copies of the dihedral group. These new lines, short-hand for two directed arrows in opposite directions, represented the new flip generator that behaved like the generator in Z_2. She drew the digraph of the newly constructed group in Figure 6.15. In the middle of her sentence, following the cliff hanger "and then,...",

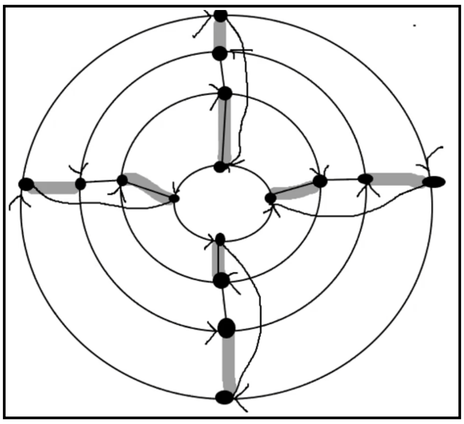


Figure 6.15: Jenni's construction of the new group in a digraph register.

a new thought interrupted her and she spontaneously switched back to the object of symmetry register.

> Jenni: Um the, the other graph should instead of a square be a cube.

It was as if Jenni's subconscious mind was listening and working on the construction of the new group in the object of symmetry register in the background while she had been explicitly building the new group in the digraph register.

Episode 3: going from a square to a cube

Jenni's initial pieces in episode 1 were re-organized and elaborated on. She spoke in a stream of consciousness with free associations as she drew a cube before prompting herself with more coherently formulated guiding questions.

> Jenni: So I guess that is how I would think about changing it with this graph adding another group to combine it adds another dimension we for sure have to have 16 new elements and I think you could do the rotations the same rotations but now you also have the flips that might, like with the cube before you were either rotating it this way or flipping and rotating it would be similiar to like rotating it,

> no it wouldn't, but you could with the square you could either rotate it around the inner circle or flip it across an axis of symmetry and then rotate it [draws a cube]. Combining Z_2 is going to act like another symmetry.
>
> Jenni: So I am starting with what I'm doing is I'm looking at, if we added another type of flip and called it, like, using Z_2 as another type of flip, what would that look like? And so I am drawing out like a cube and then seeing the types of rotations that we can do in one dimension, but then flips can happen in two different dimensions. And so that is what I'm looking at right now.

The flip generator coming from the dihedral group were like planes running in a vertical direction and the new generator would cut the cube in a different way. The rotation generator would rotate 90 degrees through the center vertical axis. Jenni thought about two different situations as either adding a slant axis of symmetry for the new flip generator or adding a horizontal axis of symmetry in Figure 6.16. She began to view the new group she was constructing as a subgroup that lived inside the symmetries of a cube. She knew that what she was constructing was not the entire symmetry group of the cube because she was only including rotations around one rotational axis .

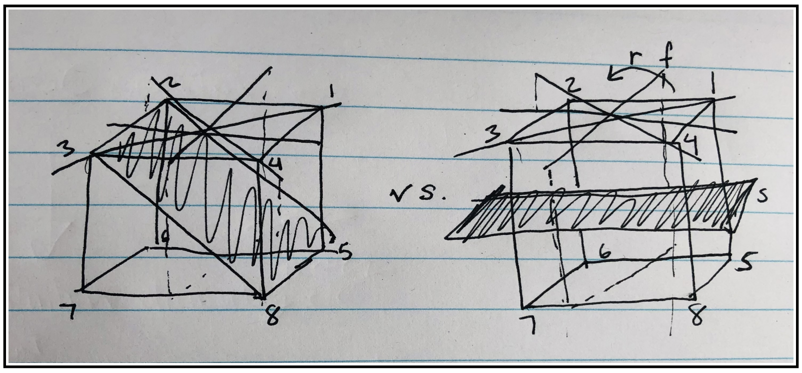


Figure 6.16: Jenni's construction of the new group in an object of symmetry register.

Episode 4: moving towards a formal construction

As Jenni tried to formalize her thoughts and write down what the relations between the generators would be in each case she paused and went back to the group presentation of the dihedral group.

> Jenni: An now I'm getting caught up in confusion about how rotations and flips work together. Because I was trying to visualize just in the square one because

> in D_8 I was trying to think if r cubed f is the same thing as f r. Yeah. I think something that is tripping me up in the presentation of the dihedral group is that r f is equal to f r cubed. And so now I'm going to try to redo the Cayley table adding on the different types of interactions that adding another symmetry you would have.

Towards the end of the session she wrote down a conjecture, with a question mark, that the group she was constructing would be isomorphic to the direct product of the dihedral group of order eight and cyclic group of order two, $D_4 \times Z_2$. Jenni was still cautious about her conjecture because it would take more time to figure out what the interactions between the generators would be whether or not the combinations of those generators matched up with Cayley table for the direct product. When asked what the most difficult part of the task was, Jenni described difficulty with expressing thoughts and moving towards formalization as well as what attempts to formalize do to her intuition.

> Jenni: I think what's difficult for me is when I'm trying to express my thoughts I think inherently there needs to be a justification. So it's hard for me to just express intuition without trying to justify it. And then that goes down the route of, oh, I should formalize it. And then if I can't I feel like it kind of blocks the intuition a little bit because I'm like, well, if I can't formalize it, then I might be wrong. And so then I want to stop and think about it again a little more so that I'm more sure of it. So I think that trying to express the thoughts sometimes does hinder my intuition. And so that makes it harder for me to talk about it and talk about the ideas as I go.
>
> Researcher: So what happens to your initial ideas and intuitions?
>
> Jenni: Um, they get fuzzier. I think for me I start questioning them and like I'm going through a formulation and think why did I start doing that? Why did I think that is going to work kind of thing. And so unless I can think of the formalization pretty quickly, then I start questioning my intuition a lot more.

Even though Jenni ended the session as more of a skeptic of her intuitions, her drawings and intuitions in the digraph register were consistent with the construction of $D_4 \times Z_2$. During the interview, Jenni expressed that she was still pretty far from having a solution. More work needed to be done to figure out how the additional flip and generators from the dihedral group could be related in order to form a new group of order 16. Her work with the cube with an added axis of symmetry was a key, it could lead in a direction towards the direct product or in the direction towards the semi-direct product with further guidance.

Representational fluency

Jenni's overall interpretation of adding structure in this context was adding a new generator that interacted with the two generators of the dihedral group and adding a reflection. The strategies that Jenni entered was to obtain the direct-product, she did not explicitly enter the strategy to obtain the semi-direct product and never mentioned the semi-direct product in any of the registers. Jenni naturally went to the digraph register rapidly followed by the object of symmetry register on her own without any switch prompts from the researcher. This was in contrast to her natural tendency without prompt to work predominantly in group presentation, formal-symbolic mapping, and Cayley table registers during the collapsing structure task.

While working in the digraph and object of symmetry register to construct the new group she was also using the group presentations in parallel. She made a partial conversion from the digraph register to the object of symmetry register by drawing the third generator from Z_2 as an additional reflection for the signified object $D_4 \times Z_2$. The conversion was considered partial because she never explicitly distinguished whether the added generator as a plane of reflection should be slanted or horizontal. In order to be a valid semiotic representation for $D_4 \times Z_2$ the added generator must be the horizontal plane of reflection.

6.4 Performance on the baseline questionnaire: group action definitions and examples

6.4.1 Definition of a group action using homomorphism language

On the baseline definition and generating example questionnaire, Jenni correctly and confidently recalled the definition of a group homomorphism, but struggled to translate the provided definition of a group action in terms of homomorphism language. Jenni stated: "Homomorphism language might just mean notation". She proceeded by mimicking the syntax of the definition of a homomorphism and insert syntax from the provided definition of a group action as indicated in the red outlined box in figure 6.17. She incorrectly assumed that $G \times X$ was equipped with a group binary operation and that X was equipped with a group binary operation. She also made a logical error and assumed that f was a homomorphism and at the same time maintained rigidity with the syntax $G \times X \to X$ rather than noticing the appropriate map $\psi : G \to Sym(X)$ where each $g \in G$ is assigned to a map $\tau_g : X \to X$ that sends x to $g \cdot x$. Moreover, she did not mention key pieces that are needed to make this conversion in one direction going from the definition of a group action to a homomorphism from $\psi : G \to Sym(X)$.

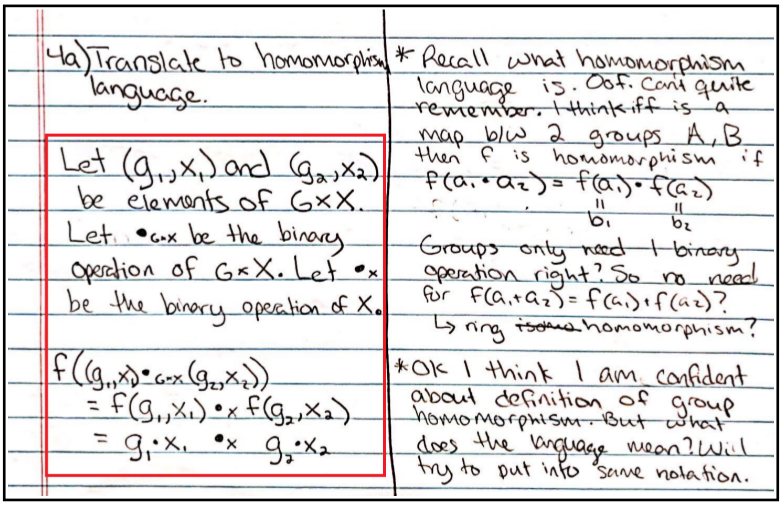


Figure 6.17: Jenni's translation of a group action in terms of homomorphism language.

During the earlier Part I of the study she seemed very comfortable with viewing elements of groups as permutation functions in the Cayley table register. She invoked Cayley's theorem while working with Cayley tables and objects of symmetry. However, when she was looking at the provided definition of a group action the perspective of viewing a group as an isomorphic copy of a subgroup of a $Sym(X)$ was not activated. She did not associate the provided definition of a group action with viewing elements in the group as a permutation functions during this questionnaire.

Alex began by writting down $\phi : G \times X \to X$ and $\phi(g, x)\phi($ and stopped because he felt something wasn't right. After emphasizing that X was just a set, he took a few more minutes to think. Next, he wrote $\phi : G \to \{mapsX \to X\}$.

> Researcher: Can you elaborate more on why your maps from X to X form a group?
>
> Alex: So this is taking element of g and mapping them into maps that go from X to X...So the maps from X to X, I'm thinking the group action is composition. So we're going to take any map that goes X to X and then we can compose those

> maps and it will still be a map that goes from X to X and the identity map would be the identity function.
>
> Researcher: So the binary operation is composition, the identity, what about
>
> Alex: Oh they would have to be bijective though to have inverses because if you don't have something surjective uh [stops abruptly], I mean that wasn't injective then your not going to have an inverse. So ya the inverses would just be the inverse map where you do something then undo it.

He went back to the provided definition of the group action and translated the identity action property using his framing of a group action as a homomorphism from the group G to the group of bijective functions from X to X and wrote down his response in Figure 6.18:

> Alex: So it has e acts on x by just giving you x so that would just send the identity in G to the identity map on X.

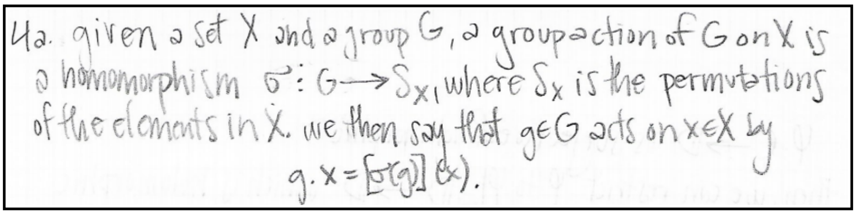

4a. given a set X and a group G, a group action of G on X is a homomorphism $\sigma : G \to S_X$, where S_X is the permutations of the elements in X. we then say that $g \in G$ acts on $x \in X$ by $g.x = [\sigma(g)](x)$.

Figure 6.18: Alex's translation of a group action in terms of homomorphism language.

6.4.2 Generating examples of group actions

Jenni wrote down three example attempts for group actions in Figure 6.19. For her first example, Jenni denoted G to be the group $\mathbf{Z_2} = (\{z : z \in \mathbf{Z}\}, +_{mod2})$ and X as the group $\mathbf{Z_6}$. She defined the group action by assigning an explicit map that took elements from the direct product group $\mathbf{Z_2} \times \mathbf{Z_6}$ to elements in $\mathbf{Z_6}$. She mapped elements of the form $(\bar{0}, x)$ to x in order to satisfy the group action property $e_G \cdot x = x$ for all $x \in X$. She repeated this same process for the equivalence class $\bar{1}$ by mapping $(\bar{1}, x)$ to x. The map (g, x) to x for any $g \in G$ naturally satisfies the second property of a group action, $g \cdot (h \cdot x) = (g + h) \cdot x$. Such a map, called the trivial action, works for any choice of G and X, but Jenni made a guess that such a map $\mathbf{Z_a} \times \mathbf{Z_b}$ to elements in $\mathbf{Z_b}$ is an action if the constraint $a|b$ holds. For the trivial action this constraint is irrelevant. She later corrected this and stated:

> Jenni: Check first intuition that $\mathbf{Z_a} \times \mathbf{Z_b}$ to elements in $\mathbf{Z_b}$ is an action if the constraint $a|b$ holds. Look at if $a \nmid b$

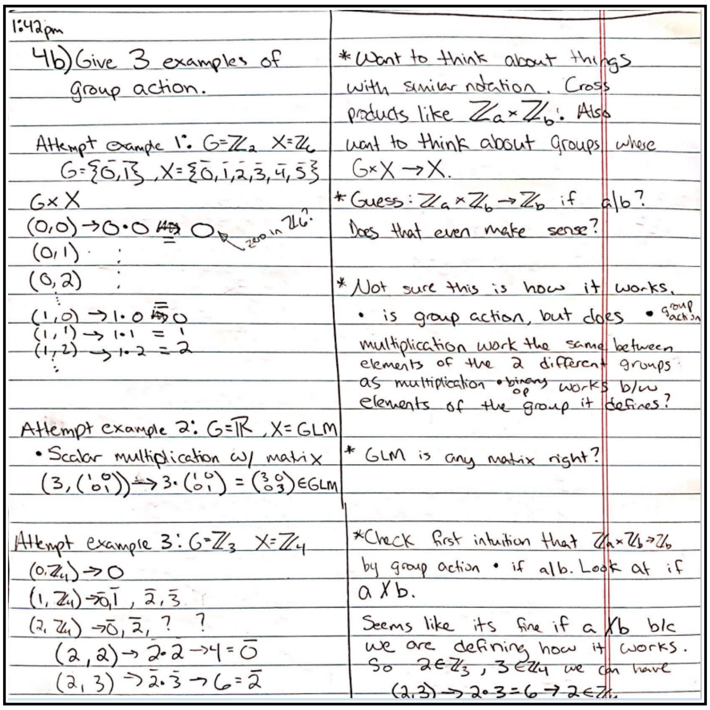
She made the following conclusion,

> Jenni: Seems like its fine if $a \nmid b$ because we are defining how it works.

However the example she used to justify this conclusion, example attempt 3, was not a valid example of a group action. Her map $(\bar{0}, \mathbf{Z_4}) \to \bar{0}$ did not satisfy the necessary property that $e_G \cdot x = x$. She incorrectly mapped $(\bar{0}_G, x)$ to $\bar{0}_X$ for every $x \in X$ and did not see the identity element in e_G as being analogous to the identity function $\tau_{e_G} : X \longrightarrow X$ in $Sym(X)$ that sends each element in X to itself.

For her second example, Jenni went back to her idea of leveraging the scalar operation of a vector space and viewing it as the action operation $\cdot : \mathbf{R} \times M_{2\times2}(\mathbf{R}) \to M_{2\times2}(\mathbf{R})$. This is a valid example of a group action and is very close to the idea of the multiplicative group $\mathbf{R}$ acting on itself. At this point her valid example was missing details and not made rigorous yet meaning that she has not provided justification that the properties of a group action hold.

For her third example attempt, Jenni picked the group G to be $\mathbf{Z_3}$ with usual addition and X to be the group $\mathbf{Z_4}$ with usual addition. She wrote down an element to element map. At first, she viewed the group action operation $\cdot$ as being synonymous with the usual integer multiplication mod 4. She mapped $(2, 3)$ to $2 \cdot 3 = 6$ mod 4 = 2. This would ensure that action operation $\cdot : G \times X \to X$ would be closed. Her third example did not satisfy the other group action properties. Jenni questioned whether or not the two groups binary operations were related to action operation $\cdot$, the action operation. She questioned how a group's binary operation and the action operation may or may not be related. The provided standard definition of a group action was not very illuminating and hid the answers to Jenni's questions.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4b) Give 3 examples of group action.

Attempt example 1: $G = \mathbb{Z}_2$ $X = \mathbb{Z}_6$

$G = \{\bar{0}, \bar{1}\}$, $X = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

$G \times X$

$(0,0) \to 0 \cdot 0 = 0$ zero in $\mathbb{Z}_6$?

$(0,1)$ ⋮

$(0,2)$ ⋮

$(1,0) \to 1 \cdot 0 = 0$

$(1,1) \to 1 \cdot 1 = 1$

$(1,2) \to 1 \cdot 2 = 2$

⋮

Attempt example 2: $G = \mathbb{R}$, $X = GLM$

• Scalar multiplication w/ matrix

$(3, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \to 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \in GLM$

Attempt example 3: $G = \mathbb{Z}_3$ $X = \mathbb{Z}_4$

$(0, \mathbb{Z}_4) \to 0$

$(1, \mathbb{Z}_4) \to \bar{0}, \bar{1}, \bar{2}, \bar{3}$

$(2, \mathbb{Z}_4) \to \bar{0}, \bar{2}, ?\ ?$

$(2,2) \to \bar{2} \cdot \bar{2} \to 4 = \bar{0}$

$(2,3) \to \bar{2} \cdot \bar{3} \to 6 = \bar{2}$

* Want to think about things with similar notation. Cross products like $\mathbb{Z}_a \times \mathbb{Z}_b$. Also want to think about groups where $G \times X \to X$.

* Guess: $\mathbb{Z}_a \times \mathbb{Z}_b \to \mathbb{Z}_b$ if $a|b$? Does that even make sense?

* Not sure this is how it works.

• is group action, but does multiplication work the same between elements of the 2 different groups as multiplication • binary op works b/w elements of the group it defines? • group action

* GLM is any matrix right?

*Check first intuition that $\mathbb{Z}_a \times \mathbb{Z}_b \to \mathbb{Z}_b$ by group action • if $a|b$. Look at if $a \nmid b$.

Seems like its fine if $a \nmid b$ b/c we are defining how it works. So $2 \in \mathbb{Z}_3$, $3 \in \mathbb{Z}_4$ we can have $(2,3) \to 2 \cdot 3 = 6 \to 2 \in \mathbb{Z}_4$.

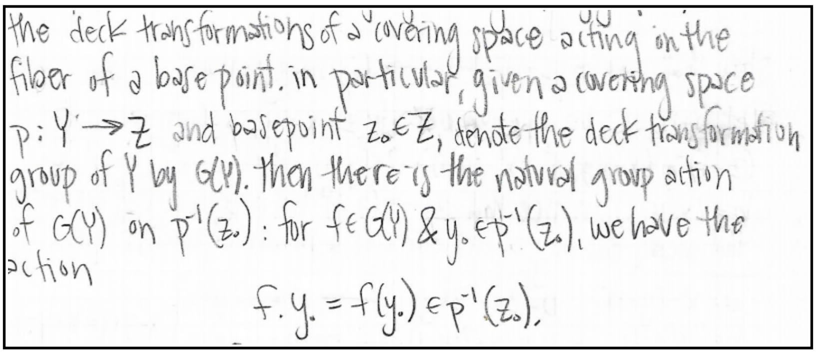
Figure 6.19: Jenni generating examples of group actions.

Alex was able to generate several examples of group actions. He started off with the most basic example of a group action where G acts on itself by left multiplication. Next, I asked him to give classes of examples according to different choices for the set X.

> Researcher: Can you give different classes of examples for group actions?
>
> Alex: What do you mean, can you explain it a little bit more.
>
> Researcher: Ya so one class of group actions you can think about is the group G acting on itself. So that would be a homomorphism G into the symmetric group of G. And if you think about examples in the class you might think about things like left action, right action, conjugation. Another class of examples would be G acting on H or H acting on G, H subgroup of G.
>
> Alex: So going along with this G acting on quotients of G [$^{G}\!/_{H}$].
>
> Researcher: Okay so that is another class of examples. So what are some other ones your thinking about?
>
> Alex: Well I had though of modules but that is ring actions, so we could restrict to thinking about just the additive group action.

> Researcher: So that would be another class of group actions, that is a good one. Another one to explore would be G acting on a vector space V [or equivalently an $FG - module$, i.e., FG acting on the vector space V]. Are you familiar with groups acting on vector spaces?
>
> Alex: Um, groups acting on [pause]. No maybe lets go through this. [I used the white board and explained that you can think of a group acting on a vector space as a homomorphism from G into the general linear group $GL(V)$ of linear transformations from V to V. I illustrated constructive thinking to build from a group acting on itself to a group acting on a vector space].
>
> Researcher: So we sort of have this basic thing, a set X has not structural properties there is nothing to it, and then we sort of increase the amount of structure to a group that has a richer structure, and then add more, richer, structure up until we get to a vector space that is super rich.
>
> Alex: Okay I see, it is similar to 4 [G acting on a module]. So your pushing G into linear maps [Alex went on to give the example of the dihedral group of order 8 acting on R^2 by mapping the generator r to the rotation matrix and the other generator s to the reflection matrix without further guidance].

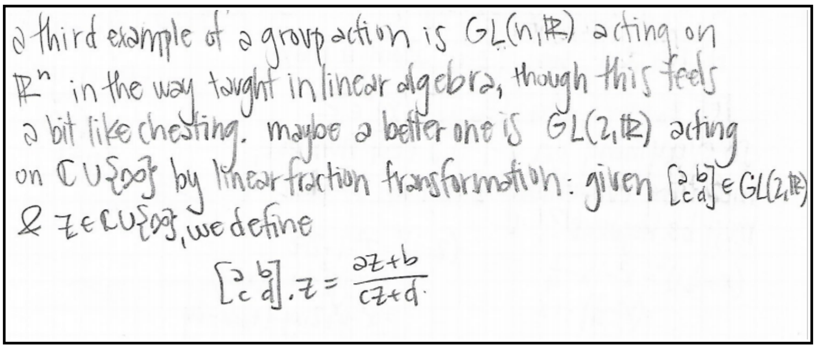
Unlike Jenni, who stayed predominantly in formal-symbolic mapping register, Alex went to a more geometric object of symmetry register to describe some of his examples. Alex could see instances of group actions across several subject areas beyond an introductory abstract algebra course. He wrote down a valid example of a group action as shown in Figure 6.20 from topology and a few more valid examples in Figure 6.21 from linear algebra:



the deck transformations of a covering space acting on the fiber of a base point. in particular, given a covering space $p: Y \to Z$ and basepoint $z_0 \in Z$, denote the deck transformation group of Y by $G(Y)$. then there is the natural group action of $G(Y)$ on $p^{-1}(z_0)$: for $f \in G(Y)$ & $y_0 \in p^{-1}(z_0)$, we have the action

$$f.y_0 = f(y_0) \in p^{-1}(z_0).$$

Figure 6.20: Alex's example of group action from topology.


a third example of a group action is $GL(n,\mathbb{R})$ acting on $\mathbb{R}^n$ in the way taught in linear algebra, though this feels a bit like cheating. maybe a better one is $GL(2,\mathbb{R})$ acting on $\mathbb{C}\cup\{\infty\}$ by linear fraction transformation: given $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{R})$ & $z \in \mathbb{C}\cup\{\infty\}$, we define

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} . z = \frac{az+b}{cz+d}.$$

Figure 6.21: Alex's example of group action leveraging linear algebra.

6.4.3 *Describing group orbits and stabilizers*

Jenni made drawings in Figure 6.22 and began to think about elements of the group g as functions again that take an elements in X to elements in X. She said that these drawings helped her to think about orbits. Taking a semiotic perspective, these drawings served as an auxiliary semiotic representation for Jenni between the initial provided definition $orb(x) = \{g \cdot x \in X : g \in G\}$ and the target definition using the homomorphism frame and viewing elements of G as functions, and the entire group G as a subgroup of a symmetric group:

> Jenni: An element $k \in X$ is in the $orb(x)$ if $\exists$ a $g \in G$ such that $g \cdot x = k$.

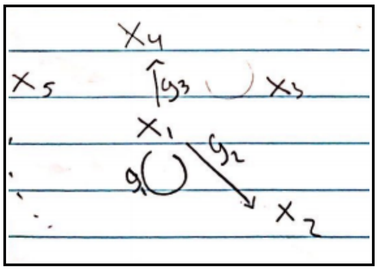


Figure 6.22: Jenni's doodles that helped her think about an orbit.

She also mentioned that an orbit reminded her of a coset. I asked Jenni to explain why an orbit had reminded her of a coset. She stated:

> Jenni: The word coset pops into my head. But to be honest the word coset pops into my head every time I think of anything algebraic. I think I just like the word coset.

Jenni's simple-associative feelings of orbits feeling familiar to cosets was revisited at the end of this study to see if this association turned into something more robust.

Jenni correctly situated the stabilizer as being contained in G and evolved into an analogy that related elements in the $stab(x)$ and identity maps on x.

> Jenni: Stabilizer is a subset of G instead of a subset of X.

> Jenni: It is all the g values that act like identity on x?

Following this, Jenni used invalid examples of group actions to make claims about stabilizers. She made these claims after writing down each of the elements in the stabilizers, as shown in figure 6.23, for each of the three examples she had previously constructed.

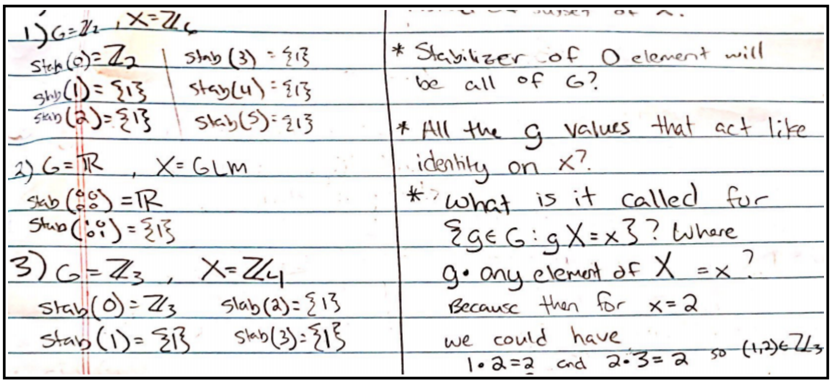


Figure 6.23: Jenni's description of stabilizers.

Referring to her first and third example in Figure 6.19, which were both invalid, Jenni made the statement:

> Jenni: Multiple elements in X can have the same stabilizer.

At face value her statement is correct, but her reasons for making this statement were drawn from invalid examples of group actions. An efficient example that could be used to help Jenni properly justify her claim, in the objects of symmetry register, is the dihedral group acting on the square of four vertices. The vertices that lie on the same reflection line have the same stabilizer. However, object of symmetry examples such as this that she was familiar with were not activated in her mind. She was not connecting the provided symbolic definition of a group action to visual processes such as rotating or reflecting objects of symmetries that she had engaged in during earlier tasks.

Jenni also made a cautious generalization question to explore later on for stabilizers that was based on patterns she noticed in her invalid examples. At this point it was not a claim she felt confident about:

> Jenni: Stabilizer of the 0 element [identity in G] will be all of G?

The answer to Jenni's question is no, not it in general. Firstly, X need not have an identity element if X is just a set. Secondly, if $X = G$ has an identity element e then with G acting on G, where $\cdot$ is the same as the binary operation in G, $stab(e) = \{g \in G : g \cdot e = e\}$ which implies that $stab(e) = \{e\}$. One possible source of Jenni's confusion could have stemmed from a focus on symbolic notation of the group action property $e \cdot x = x$ for all $x \in X$ and swapping G and X in the map $G \times X \to X$. A second source of Jenni's incorrect, but cautious generalization could also have stemmed from valid experimentation with the example where the multiplicative group of real numbers $\mathbf{R}^{\times}$ with the binary operation $\times$ and identity 1 acts on the vector space $M_{m\times n}$ over the field $\mathbf{R}$ with the scalar action $r \cdot [x] := [r \times x]$. Due to the vector space structure we have $r \cdot [0]_{m\times n} = r \cdot ([0]_{m\times n} + [0]_{m\times n}) \implies r \cdot [0]_{m\times n} = [0]_{m\times n}$ for all $r \in \mathbf{R}^{\times}$ and this shows that $stab\{0\} = \{r \in \mathbf{R}^{\times} : r \cdot [0]_{m\times n} = [0]_{m\times n}\} = \mathbf{R}^{\times}$. In this case, the stabilizer of the additive identity $[0]_{m\times n}$ in the object being acted on is all of the group $R^{\times}$.

Using the homomorphism from G to $Sym(X)$ frame, Alex correctly described the orbit and stabilizer for a fixed $x \in X$:

> Alex: [referring to orbit] It's going to be a subset in X, like the union of images of all the g's. So if I take g and okay, then the orbit of x for some fixed x is going to be the union of all the different elements and this is going to be a subset of X. So if your thinking about the elements as bijective functions on X the orbits going to be all the elements in the image of all of those. So its just like the collective image of all the g's.

> Alex:[referring to stabilizer] It's going to be all the g that when they restrict to this given x is the identity map. [Alex expressed that it was difficult to describe this in words. He used formal notation to write down what he was thinking on the whiteboard in Figure 6.24.]

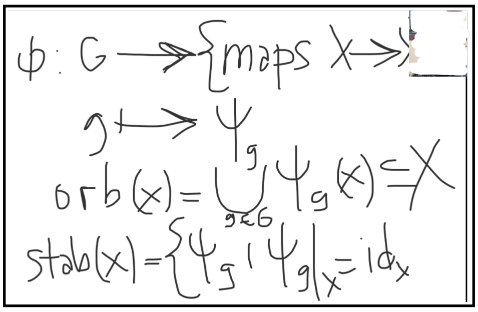


Figure 6.24: Alex describing orbits and stabilizers.

While Alex successfully translated the definitions of orbits and stabilizers into the homomorphism frame into a symmetric group, neither of the participants indicated that the equivalence relation involving orbits under the action partition the object X being acted upon. The collection of orbits as a partition of X, was not at the forefront of their minds yet.

6.4.4 Intervening session with Jenni group actions in a homomorphism frame and examples of group actions

Previously, during the baseline questionnaire, Jenni did not convert the provided definition of a group action to the homomorphism ψ from G to $Sym(X)$ frame, where $Sym(X)$ is the group of bijections from the set X to X. She generated two valid example out of three attempts. Before moving on to the experimentation with examples phase, I guided Jenni to view a group action within the homomorphism frame. First, we checked that given a group action $\cdot$ the collection $\{\tau_g : x \in X \to \tau_g(x) := g \cdot x \in X : g \in G\}$ was a subset and then a subgroup of $Sym(X)$. Next, we checked that ψ would be a homomorphism. She followed up, without further guidance, by correctly translating the initial provided definitions of a stabilizer and orbit to the definition of a stabilizer and orbit within the homomorphism frame as shown Figure 6.25. Pointing to Jenni's work on the board, I informally summarized that picking an element $x \in X$ we can track all the places that G takes x to. All the places that G takes x to is called the $orb(x)$. And that the $stab(x)$ was a subset of elements in G that fix x and leave it where it is. Since $stab(x)$ is a subgroup of G, ψ restricted to $stab(x)$ would be a trivial action.

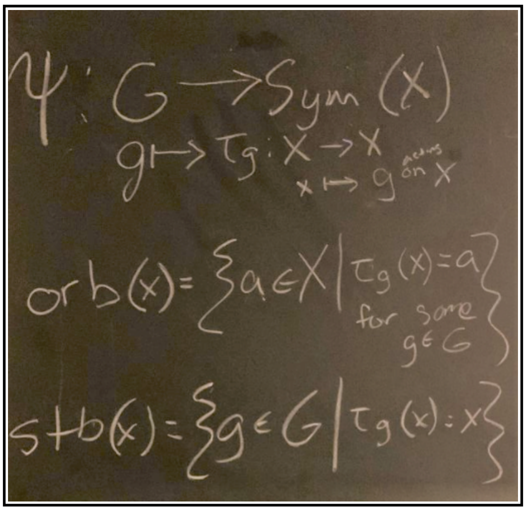


Figure 6.25: Jenni's description of orbits and stabilizers within the homomorphism frame for group actions.

The IOI principle of building from a learners ideas and questions was used. Drawing on her knowledge of homomorphisms, we worked through a few examples and I highlighted that the binary operation in the group G would behave like function composition in the symmetric group and that we could think of each element $g \in G$ as a bijection $\tau_g \in Sym(X)$. Next, her earlier questions regarding how a group's binary operation and the action operation may or may not be related were revisited. Jenni's questions and her switch to a homomorphism frame of mind led into a discussion about various examples of group actions in the table below. Examples were discriminated in Table 6.3 according to how the action operation and group operation may be related or not related. We discussed how Cayley's theorem was a consequence of a group G acting on itself with the action operation being the same as the binary operation of G.

It was emphasized that an interesting type of group action to think about and follow with was the subgroup H acting on G. Jenni connected back to her earlier example of Z_2 acting on Z_6, as a case when a subgroup H acts on G. Columns for what the orbits, stabilizers, and related theorems were not discussed with Jenni and were left open to explore. Jenni had logged some of her intuitions prior to this session and filled out the NCCFIS for each intuition logged. That is the total sample of Jenni's logged intuitions could be split into intuitions that were arrived at prior to translating the provided group action definition into the homomorphism frame and post translation into a homomorphism frame. The difference in the pre and post-translation samples along NCC attributes, error-type, and confidence in truth value for Jenni will be discussed in the next section.

Table 6.3: Examples of group actions and their consequences.

Action case	Example	Orbits	Stablizers	Related theorem
$G \curvearrowright G$	left or right action	$orb(x) = G$ for every $x \in G \Rightarrow$ one distinct orbit	$stab(x) = \{e\}$ for every $x \in G$	Cayley's Thm
$H \curvearrowright G$, $H < G$	left or right action	cosets that are all the same size $\lvert H \rvert$	size of the index $\lvert G : H \rvert$	Lagrange's Thm
$G \curvearrowright G$	conjugation	orbits are the conjugacy classes may or may not be the same size	$stab(x)$ are elements in G that commute with x	Class equation
$G \curvearrowright X$, X is a set	G permuting the set of vertices X of $G's$ object of symmetry	orbits may or may not be the same size	$\lvert stab(x) \rvert = \frac{\lvert G \rvert}{\lvert orb(x) \rvert}$	Orbit-stabilizer Thm

6.5 The make-up of learners' example-based intuitions: orbits and stabilizers of group actions

6.5.1 Descriptive summary statistics and univariate tests

Jenni reported a total of 11 example-based intuitions and Alex reported a total of 6. There was no variability within the collection of Alex's reported intuitions with respect to emotional intensity, incubation period*, and non-sanctioned cue use. He had little to no emotional reactions in response to the arrival of an intuition. He did not experience strong positive emotional reactions, which are theorized to coincide with more creative forms of intuition following periods of incubation. He strongly agreed that all of his reported intuitions were arrived at while working on the task for less than 15 minutes. Another invariant among all of Alex's reported intuitions was that he strongly disagreed with the statement below for non-sanctioned cue use:

> During this task I made sketches or drawing of the way I like to think about groups/related concepts that feels unique to me rather than something I picked up from others.

On the other hand, Jenni had more frequent instances in which she had strong positive emotional reactions when she experienced her reported intuitions and out of all the variable attributes, values for incubation had the highest standard deviation from the mean and maximal range. Overall, the make-up of Alex's reported intuitions restricted to a single attribute at at time appeared more uniform than Jenni's.

An independent univariate Welch t-test was conducted to compare the means of Jenni and Alex's reported intuition samples for each NCC attributes listed in Table 6.4 and Table 6.5. There was a significant difference in the means for both emotional intensity item scores; $t_3(10) = 4.28, p_3 < 0.01$ and $t_4(7.14) = -11, p_4 < 0.001$, for the non-sanctioned cue item score; $t_7(10) = 7.12, p_7 < 0.001$, and for both incubation item scores; $t_{11}(10.92) = 2.80, p_{11} < 0.01$ and $t_{12}(10) = -2.76, p_{12} < 0.05$. For each of these attributes, the means were on the creative end for Jenni's sample and on the non-creative end for Max. There was no significant difference between the two learners' samples for the remaining attributes; $p > 0.05$. While the Welch t-test does not assume equal variance it does assume normal distributions for each sample, however the data in this study was not normally distributed. A Mann-Whitney U test is a substitute to the Welch t-test that does not assume normal distributions and does not require equal variance. However, the Mann-Whitney U was not appropriate because there were several tied scores which violated the requirement that the variables be continuous.

As an alternative the permuted Brunner-Munzel test was conducted. This test is a better fit for sets of data with arbitrary distributions, unequal variances, small sample sizes, and discrete variables. Significance was found for both emotional intensity attributes and the non-sanctioned cue use attribute on one and two-sided tests. The null hypothesis that Jenni and Alex's self-reported scores were stochastically equal, $P(X > Y) = P(X < Y)$, for emotional intensity was rejected, $p_3 < 0.01, \hat{p_3} = 0.0909$. It is significantly more likely that an randomly sampled intuition object from Jenni is mixed with greater intensity positive emotional reactions than a randomly sampled intuition object from Alex's sample.

The make-up of intuition objects reported by Jenni and Alex were not stochastically equal for non-sanctioned cue use, $p_7 < 0.001, \hat{p_7} = 0.0454$. A randomly sampled object from Jenni tended to have a stronger agreement score for non-sanctioned cue use than a randomly sampled object from Alex. The p-value was not quite small enough to be significant for the incubation attribute to reject stochastic equality, $p_{11} = 0.075, \hat{p_{11}} = 0.2727$. A randomly sampled intuition from Jenni tends to have stronger agreement values for the incubation item getting stuck and intentionally taking breaks than an object from Alex. Alex did not have points of being stuck followed by intentional breaks to let his ideas simmer. Despite Alex having more non-creative markers for non-sanctioned cue use, incubation and emotional intensity, some of his intuitions had facets of Network thinking. For example, he recovered instances of group actions related to the first fundamental group that he had been studying in algebraic topology. There were no statistically significant differences between Jenni's pre-session objects and post-session objects based on the univariate permuted Brunner-Munzel test for each NCC attribute. In otherwords, there was not a significant tendency of higher or lower attribute values between pre and post object samples.

Table 6.4: Descriptive statistics for Jenni's 11 logged intuitions for each attribute in Total.

Attribute		n	mean	sd	medin	min	max	range	skew	krtss	se
Self-Report NCCFIS Items			1-6 scale								
Personal Novelty	1	11	2.09	1.04	2	1	4	3	0.32	-1.40	0.31
Personal Novelty*	2	11	4.45	1.37	4	3	6	3	0.09	-1.94	0.41
Emotional Intensity+	3	11	4.45	1.13	4	2	6	4	-0.47	-0.30	0.34
Emotional IntensityN*	4	11	1.64	0.50	2	1	2	1	-0.49	-1.91	0.15
Ease of Externalization	5	11	2.45	1.92	1	1	6	5	0.64	-1.39	0.58
Ease of Externalization*	6	11	4.09	2.17	5	1	6	5	-0.37	-1.81	0.65
Non-Sanct. Rep. Cues	7	11	4.73	1.74	5	1	6	5	-1.08	-0.39	0.52
Sanct. Rep. Cues*	8	11	2.00	1.41	1	1	5	4	0.96	-0.64	0.43
Non-Sanct. Outcome	9	11	3.09	1.51	3	1	6	5	0.49	-1.08	0.46
Sanct. Outcome*	10	11	3.73	1.56	5	1	5	4	-0.46	-1.61	0.47
Incubation	11	11	3.36	2.54	2	1	6	5	0.13	-2.12	0.77
Incubation*	12	11	3.18	2.18	3	1	6	5	0.26	-1.83	0.66
Self-Report Conf. in T Value			1-6 scale								
Confident intuition is T	13	11	4.18	1.60	5	2	6	4	-0.13	-1.82	0.48
Proof feels like busy work	14	11	2.82	1.72	3	1	5	4	0.15	-1.82	0.52
Coded by Researcher			1-4 scale								
Error$\rightarrow$Non-Error Type	15	11	3.27	1.01	4	1	4	3	-1.02	-0.28	0.30
Coded by Researcher			0-No 1-Yes								
Network Thinking	16	11	0.09			0	1	1			

Note- NCCFIS: For odd items 1 indicates non-creative extreme and 6 creative theoretical extreme with even items* scale reversed; Error Non-Error Type: 1-errored for worked examples, 2-errored for worked examples and self-corrected, 3-true for worked examples, but not in general, 4-true in general; see The Non-creative Versus Creative Forms of Intuition Survey with Confidence in Truth Items for item statements

Table 6.5: Descriptive statistics for Alex's 6 logged intuitions for each attribute in Total.

Attribute		n	mean	sd	medin	min	max	range	skew	krtss	se
Self-Report NCCFIS Items			1-6 scale								
Personal Novelty	1	6	1.83	1.60	1	1	5	4	1.19	-0.41	0.65
Personal Novelty*	2	6	5.00	2.00	6	1	6	5	-1.25	-0.29	0.82
Emotional Intensity	3	6	3.00	0.00	3	3	3	0	Na	Na	0.00
Emotional Intensity*	4	6	5.67	0.82	6	4	6	2	-1.36	-0.08	0.33
Ease of Externalization	5	6	2.50	0.84	3	1	3	2	-0.85	-1.17	0.34
Ease of Externalization*	6	6	4.50	0.84	4	4	6	2	0.85	-1.17	0.34
Non-Sanct. Rep. Cues	7	6	1.00	0.00	1	1	1	0	Na	Na	0.00
Sanct. Rep. Cues*	8	6	2.17	0.41	2	2	3	1	1.36	-0.08	0.17
Non-Sanct. Outcome	9	6	3.17	1.60	3	1	5	4	0.02	-1.82	0.65
Sanct. Outcome*	10	6	3.17	1.17	3	2	5	3	0.37	-1.62	0.48
Incubation	11	6	1.17	0.41	1	1	2	1	1.36	-0.08	0.17
Incubation*	12	5	5.00	0.00	5	5	5	0	Na	Na	0.00
Self-Report Conf. in T Value			1-6 scale								
Confident intuition is T	13	6	5.33	0.52	5	5	6	1	0.54	-1.96	0.21
Proof feels like busy work	14	6	3.25	0.50	3	3	4	1	0.75	-1.69	0.25
Coded by Researcher			1-4 scale								
Error→Non-Error Type	15	6	3.83		4	3	4	1			
Coded by Researcher			0-No 1-Yes								
Network Thinking	16	6	0.5			0	1	1			

Note- NCCFIS: For odd items 1 indicates non-creative extreme and 6 creative extreme with even items* scale reversed; Error Non-Error Type: 1-errored for worked examples, 2-errored for worked examples and self-corrected, 3-true for worked examples, but not in general, 4-true in general; see The Non-creative Versus Creative Forms of Intuition Survey with Confidence in Truth Items for item statements

6.5.2 *Trends looking across all cluster models*

Looking across all objects and all fuzzy partitions in Table 6.7 it was observed that five of Alex's reported intuitions had high membership values $u_{ij} > 0.88$ to the more non-creative clusters for two, three, and four partition models. It could be inferred that Alex was predominantly in more non-creative intuiting states during the task. An intuiting state is a fuzzy set in a partition that is identified by the FCM algorithm. Intuiting states are characterized by intuiting process and outcome attributes. Different intuiting states are well-separated states that are characterized by prototypes with different intuiting process and outcome attribute values.

Intuition No. 15 was an outlier case among Alex's intuitions with unclear membership. Upon inspection of the raw NCCFIS survey scores it was found that this reported intuition was high in the first personal novelty, a marked value of 5 on a 6-likert scale. It was an intuition that Alex had not experienced before. Alex journaled about the content for No. 15 as written below.

> Alex: An orbit feels the most intuitive when G is cyclic. As G moves through its cycle, x moves through its orbit and the circular motion of both is synchronic.

The novelty of No. 15 was in stark contrast to the (partial) make-up for the rest of Alex's reported intuitions which all had low personal novelty and high familiarity attribute scores indicating that the content of these intuitions were already acquired consolidated bits in memory that were recovered.

Along with personal novelty as a creative indicator, the notion of incubation with the specified context of an administered task and incubation since exposure to a concept needed to be examined. While there was no incubation period associated with No. 15 within the time span of the administered task, to investigate orbits and stabilizers, this may have been something that Alex's non-conscious processes may have been working on since he had exposure to an abstract algebra course the prior year or his more evolved schematic network at the time of the task enabled a rapid association. His investigation of examples of cyclic groups and contrasting it with non-cyclic examples on this task provided enough of a stimulus to bring the highly novel, to Alex, No. 15 intuition to his awareness. Personal novelty and incubation period attributes within the span of an administered task need not move in the same direction. Moreover, the converse to the statement: if there was an incubation period within the span of an administered task then the intuition following the period will be novel need not be true. Likewise, the statement as written in the straight forward direction need not be true. A counter-example to it was found in Jenni's data. For Jenni, it had been awhile since she had thought about abstract algebra and so it took her more time and breaks before she was able to recover already learned information. Historically, documentation of cases in which incubation periods are followed by highly novel intuitions occur when mathematicians make ground breaking findings, after many years working on a problem, and these would be considered more extreme cases of creative forms (Hadamard, 1954).

Table 6.6: Cluster validity indices.

Weight Exp. (m)	No. of Clust. (c) (c)	Partition Coef. (F_c)	Partition Entropy (H_c)	F Silhouette (FSil)
1.5	2	0.82665	0.30035	0.50671
1.5	3	0.81204	0.35754	0.53076
1.5	4	0.78151	0.43756	0.50357
2	2	0.62689	0.55550	0.50355
2	3	0.60792	0.69768	0.56922
2	4	0.56629	0.83246	0.68679

Table 6.7: Three NCC fuzzy partitions and cluster membership values for Jenni's (No. 1-11) and Alex's (No. 12-17) example-based intuitions obtain from FCM Algorithm.

Intui. Obj. No.	2-Clust 1 c=2	2-Clust 2	3-Clust 1 c=3	3-Clust 2	3-Clust 3	4-Clust 1 c=4	4-Clust 2	4-Clust 3	4-Clust 4
1	0.8267	0.1732	0.1755	0.0915	0.7330	0.1178	0.6233	0.0710	0.1879
2	0.0883	0.9117	0.0772	0.8804	0.04238	0.2526	0.0313	0.6519	0.0641
3	0.8931	0.1069	0.0059	0.0028	0.9912	0.0013	0.9962	0.0006	0.0019
4	0.8931	0.1069	0.0059	0.0028	0.9912	0.0013	0.9962	0.0006	0.0019
5	0.8931	0.1069	0.0059	0.0028	0.9912	0.0013	0.9962	0.0006	0.0019
6	0.0281	0.9718	0.0175	0.9415	0.0409	0.9407	0.0105	0.0437	0.0051
7	0.3092	0.6908	0.2876	0.5531	0.1593	0.0542	0.0326	0.8525	0.0607
8	0.0301	0.9699	0.0147	0.9553	0.0301	0.7626	0.0328	0.1866	0.0179
9	0.6487	0.3513	0.0867	0.1206	0.7927	0.1983	0.6098	0.0941	0.0978
10	0.1287	0.8713	0.0593	0.6878	0.2529	0.9447	0.0295	0.0176	0.0082
11	0.0535	0.9465	0.0355	0.9317	0.0328	0.1347	0.0202	0.8220	0.0232
12	0.9333	0.0667	0.9839	0.0058	0.0102	0.0040	0.0099	0.0113	0.9747
13	0.9720	0.0280	0.9697	0.0068	0.0236	0.0042	0.0184	0.0071	0.9704
14	0.9764	0.0236	0.9957	0.0011	0.0032	0.0006	0.0022	0.0012	0.9961
15	0.8790	0.1210	0.7149	0.0765	0.2086	0.0673	0.1749	0.0877	0.6701
16	0.9333	0.0667	0.9839	0.0058	0.0102	0.0040	0.0099	0.0113	0.9747
17	0.9781	0.0219	0.8840	0.0220	0.0940	0.0164	0.0769	0.0230	0.8837
Parameters	$m = 1.5$	$\epsilon = 0.00001$							NCC attributes 1-12

Table 6.8: Three NCC fuzzy partitions and cluster prototype arithmetic mean scores on a scale from 0 extremely non-creative to 1 extremely creative.

Intui. Obj. No.	2-Clust1 c=2	2-Clust2	3-Clust1 c=3	3-Clust2	3-Clust3	4-Clust1 c=4	4-Clust2	4-Clust3	4-Clust4
M	0.22009	0.62829	0.29840	0.64638	0.36352	0.63344	0.34446	0.65037	0.28634
%J	45.45%	54.55 %	0%	54.55%	45.45%	27.27%	45.45%	27.27%	0%
%A	100 %	0 %	100%	0%	0%	0%	0%	0%	100%
Parameters	$m = 1.5$	$\epsilon = 0.00001$				%-max membership		NCC attributes 1-12	

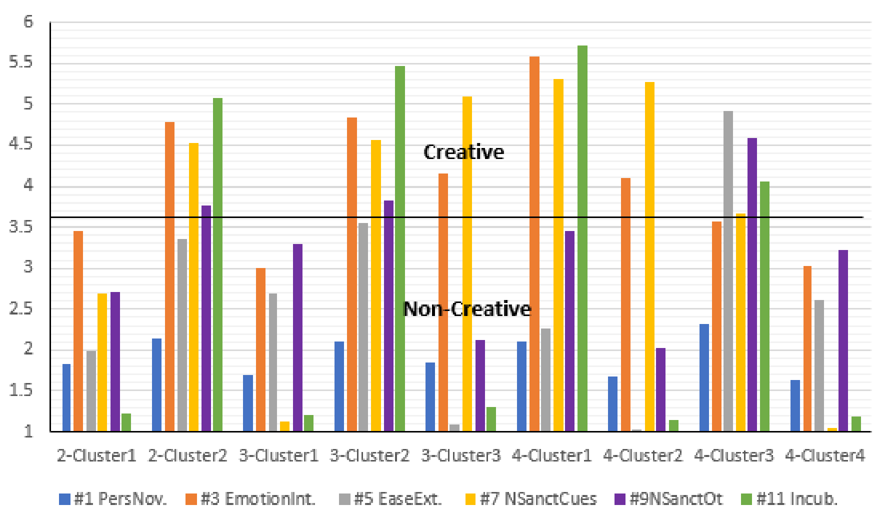


Figure 6.26: Prototypes (cluster centers) for partitions in Table 6.7.

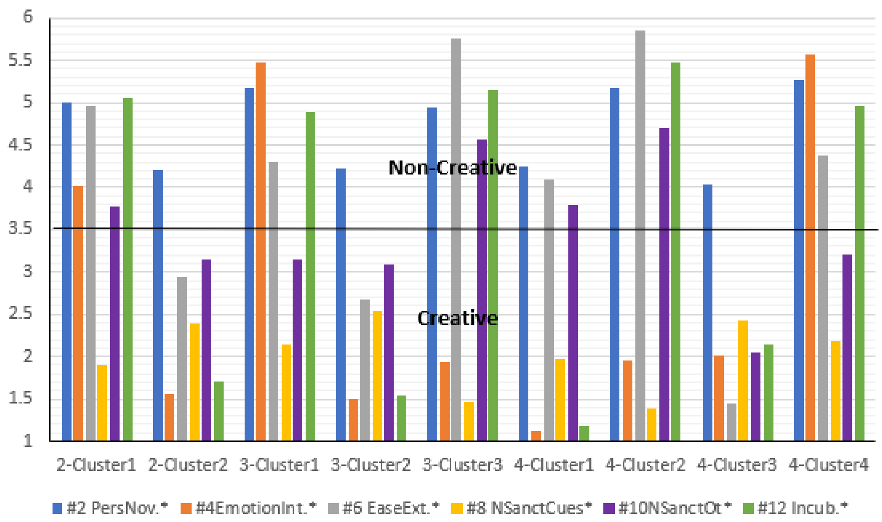


Figure 6.27: Prototypes (cluster centers) for partitions in Table 6.7 with reverse items.

6.5.3 Unique versus common intuition outcomes

Overall, common intuition outcomes reported in the intuition logs and journals were rare and unique, non-overlapping, outcomes were more prevalent. This section will focus on the small intersection of intuition outcomes with some content from journals that overlapped. The intuitions with content overlap were related to the orbit-stabilizer theorem. Jenni logged that she was seeing the pattern $|orb(x)||stab(x)| = |G|$ and that size of the orbits and stabilizers were like mirrors of each other, if one went up the other went down. Similarly Alex logged that "the cardinality of the group can put restraint on the orbit size". While both arrived at intuition outcomes with some overlapping content, the examples they were exploring within the duration of the task and intuiting processes that led to this overlapping outcome were very different.

Jenni investigated the orbits and stabilizers of the dihedral group of order eight acting on the vector space $\mathbf{R^2}$. She wrote down the example in Figure 6.28 and made θ a 90 degree rotation. Since $\mathbf{R^2}$ was infinite she decided to act on the set $\{< a, a >, < -a, a >, < -a, -a >, and < a, -a > |a \in \mathbf{R}_{>0}\}$.

Next, she drew several diagrams in Figure 6.29 and Figure 6.30 to get a feel for where the transformation matrices τ_g move vectors in R^2. Her process of gaining familiarity with the particular action example involved moving between multiple representational registers. From these cues she noticed patterns which led to new questions that she posed to herself which opened up a new line of subsequent investigations with other examples. She also tried to find counter-examples to the patterns she was noticing to try to figure out if she was in a special case. In her work leading up to the pattern

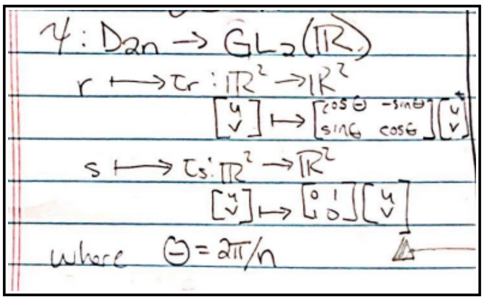


Figure 6.28: Jenni exploring examples of group actions.

stated by the orbit-stabilizer theorem she noticed that,

> Jenni: the number of times each element in an orbit appears was equal to the size of the stabilizer.

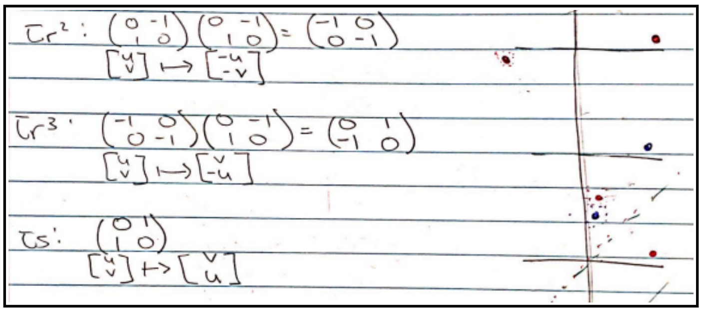


Figure 6.29: Jenni's journal entries to gain familiarity with orbits and stabilizers spatially.

Alex's examples the he experimented with had trivial stabilizers with only the identity element. Alex was aware that the the examples he chose may have not been the most interesting or informative. He followed this with the deck transformation example in Figure 6.20 which he felt was more interesting and was on his mind from his studies. He journaled about what he noticed about orbits in Figure 6.31.

This particular example helped him to see that the "cardinality of the group can put some restraint on the orbit size". He also preferred the deck transformation example because it "was sufficiently general to not allow exact calculation". He journaled that his reaction to the task to investigate relationships between orbits and stabilizers was

> Alex: "Oh, [expletive], now I have to actually calculate something", which in my experience is often harder than understanding the underlying concepts. But in actual fact, I managed to pick some fairly well-behaved actions, so this part wasn't as difficult as I anticipated, especially since the hardest example, that of the deck transformations, was sufficiently general to not allow an exact calculation.

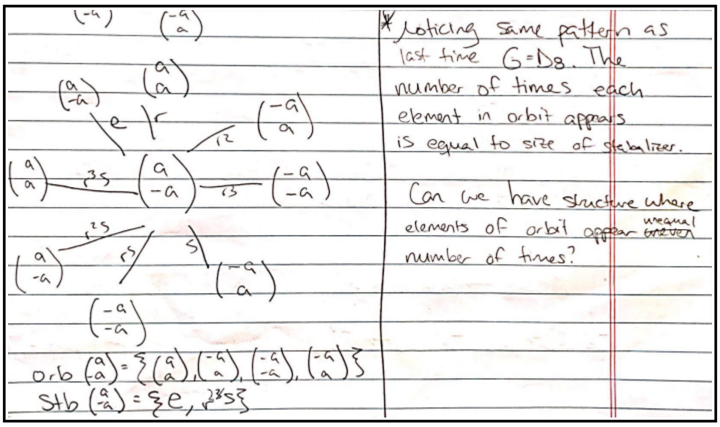


Figure 6.30: Jenni's journal entries to detect patterns among orbits and stabilizers.

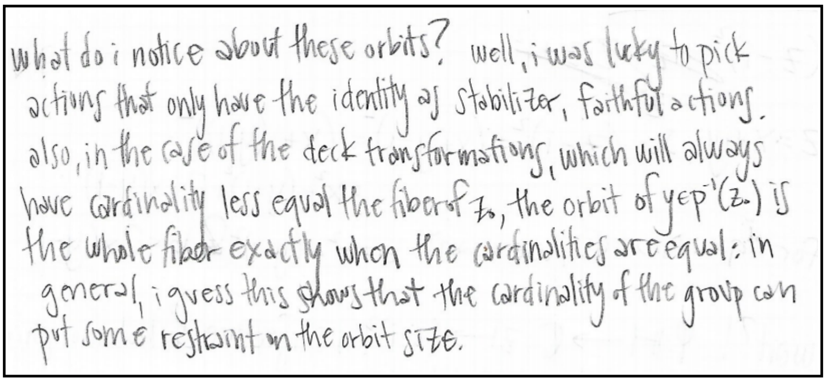


Figure 6.31: Alex's journal entry about orbits from the examples he investigated.

This was evidence of an aversion towards having to carry out calculations which Alex found was "often harder than understanding the underlying concepts". He also wrote that "once familiar with group actions they seem to crop up everywhere so coming up with other examples felt like more of a restricting process than a proof of existence".

Following this statement that group actions was something he was already very familiar with, he associated the orbit-stabilizer with more abstract notions and moved away from thinking about orbits and stabilizers within concrete example cases and computations, which he found to be too restrictive on his thinking.

> Alex: an orbit feels analogous to the image of a group homomorphism, since both in some way preserve the group operation of G. This analogy is further bolstered by the orbit-stabilizer theorem, which feels much like the first isomorphism theorem-we can even say $G/stab(x) \simeq orb(x)$ as G-sets (No. 16) and the stabilizer is analogous to the kernel of a homomorphism (No. 14).

The statement that the "stabilizer is analogous to the the kernel" holds true with the context of his investigated examples of faithful actions where the kernel is exactly the identity. However, in general one needs to be careful with this analogy the stabilizer may not be exactly the same as the kernel. The stabilizer need not be a normal subgroup or the coset of a normal subgroup and the kernel of the homomorphism from a group into the symmetric group may be smaller than the $stab(x)$ for a fixed $x \in X$. In fact in general, the kernel is $\{g \in G : g \cdot x = x, \forall x \in X\} = \bigcap_{x \in X} stab(x)$. Alex was careful to clarify that $G/stab(x)$ and $orb(x)$ were isomorphic as G-sets which involves additional terminology and definitions beyond an introductory undergraduate course. Alex elaborated on a few more details of his rough analogy between the orbit-stabilizer theorem and first isomorphism theorem in the intuition excel log.

> Alex: In a non mathematical precise way, these two feel reminiscent of one another, in that they both enact a process of quotienting the group by a subgroup to get something that in some sense is equal to the image of the map (No. 16).

Based on the FCM results in Table 6.7 and Table 6.9 the journaled intuition objects with overlapping content for the orbit-stabilizer theorem that were previously described, namely intuition object No. 10 (Jenni) and No. 16-17 (Alex), were drastically different in terms of their NCC attributes. Jenni's object had the highest membership to the more creative cluster 4-Clust1; $u = 0.9447$, $M = 0.633$. This cluster was characterized by high levels of positive emotional intensity, non-sanctioned cue use, and incubation, but was low for personal novelty. Even though the orbit-stabilizer theorem was something Jenni had some familiarity with and some outcome attributes for this intuition were on the non-creative end, the outcome attribute of positive emotional intensity and the process attributes were on the high creative end. It was like a creative recovery, leveraging a new example, of a pattern she had come across before. Her positive emotional reaction to recovered pattern was not a reaction associated with it being new to her. Instead, it was a reaction to finding a different route to seeing the old pattern. Alex's No. 16 and No. 17 object had the highest membership value to non-creative clusters 3-Clust1 and 2-Clust1; $u = 0.9839$, $M = 0.298$ and $u = 0.9781$, $M = 0.220$. These clusters were characterized by low levels of personal novelty and extremely low levels of incubation. In addition, No.17 was extremely low for non-sanctioned cue use.

6.5.4 A transition from a creative to non-creative intuiting state

Fix a mathematical context, in this case orbit-stabilizer relationships for group actions. Given a collection of reported intuition objects at t_1 and a collection at t_2 for a fixed context we can intuitively define an intuiting state change as a change in density from one cluster type to another where the distances between the centers of clusters with high density for the partition U_{t_1} and partition U_{t_2} is sufficiently large. While this potentially quantifiable definition is far from formalization, qualitative evidence of an intuiting state change was found in the journal data. Reflecting on his survey responses, Alex made a note in his journal:

> Alex: Looking now at my responses to the survey questions, I realize that though I've written fairly coherent intuitions above, I know that there were many more nonverbal thought processes at work, because some of these ideas I don't think of exactly as I was taught them-I had to come up with my own visualization in my head about how the work that I've used enough to allow them to come out immediately in verbal form. Getting back to the sub verbal or pre verbal thought processes is difficult with something that is relatively familiar like group actions.

Alex described that he had engaged in more creative type intuiting processes, marked by the attributes non-sanctioned cue generation and difficulty to verbally externalize, when he learned about group actions. Overtime what was indicative of creative intuiting processes, namely generating personal visualizations and pre-verbal thought processes, transitioned to non-creative processing, indicated by ease of externalization and familiarity, with respect to a fixed mathematical concept. Now the educator may ask what stimulus can I apply to encourage a learner that is in a non-creative state to cycle back into a creative intuiting state for a concept that they already feel very comfortable with? How does one create an environment that stimulates the learner to add a new layer or register?

6.5.5 Additional attributes: confidence in truth and error to non-error type

The FCM algorithm exposed the four clusters for confidence in truth value and error to non-error type in Table 6.9. A problematic cluster that was not present was that characterized by high confidence in the truth value and completely errored both within the context of worked examples and in general. Jenni's evaluative error-detecting intuition, feelings of rightness or wrongness, was working properly. When a completely errored intuition occurred her confidence levels in the truth value of it dropped dramatically as illustrated in Figure 6.32 for cluster 4. On the other hand, Alex's intuition objects had high membership values to Cluster 1 and 2 marked by moderate to high confidence in truth value. Contrasting this with membership values for Cluster 3, it was concluded that Jenni was more conservative and cautious in trusting her intuitions even when her intuitions were on the right track. For example, when she began to relate cosets to the generalization of cosets, namely orbits, Jenni journaled that:

> Jenni: Going back to the idea of cosets, I think a coset is like if $H \subset X$ then partition X into right cosets by $H \cdot x$. This is different because G might not be a subset of X. But all $G \cdot x \in X$ so maybe it is the same? (No. 2)

Table 6.9: Partitions for confidence in truth value and error to non-error type obtained from the Fuzzy C-Means Algorithm.

Obj. No.	Clust1	Clust2	Clust3	Clust4
1	0.001162	0.000410	0.010959	0.987469
2	0.000136	0.000023	0.999636	0.000206
3	0.002812	0.997008	0.000106	0.000074
4	0.001908	0.000401	0.981343	0.016347
5	0.002812	0.997008	0.000106	0.000074
6	0.000451	0.000202	0.003683	0.995664
7	0.000136	0.000023	0.999636	0.000206
8	0.918636	0.079325	0.001487	0.000552
9	0.017654	0.981680	0.000436	0.000230
10	0.999553	0.000342	0.000087	0.000019
11	0.999553	0.000342	0.000087	0.000019
12	0.999553	0.000342	0.000087	0.000019
13	0.228857	0.768962	0.001554	0.000626
14	0.046084	0.952602	0.000839	0.000475
15	0.918636	0.079325	0.001487	0.000552
16	0.999553	0.000342	0.000087	0.000019
17	0.999553	0.000342	0.000087	0.000019

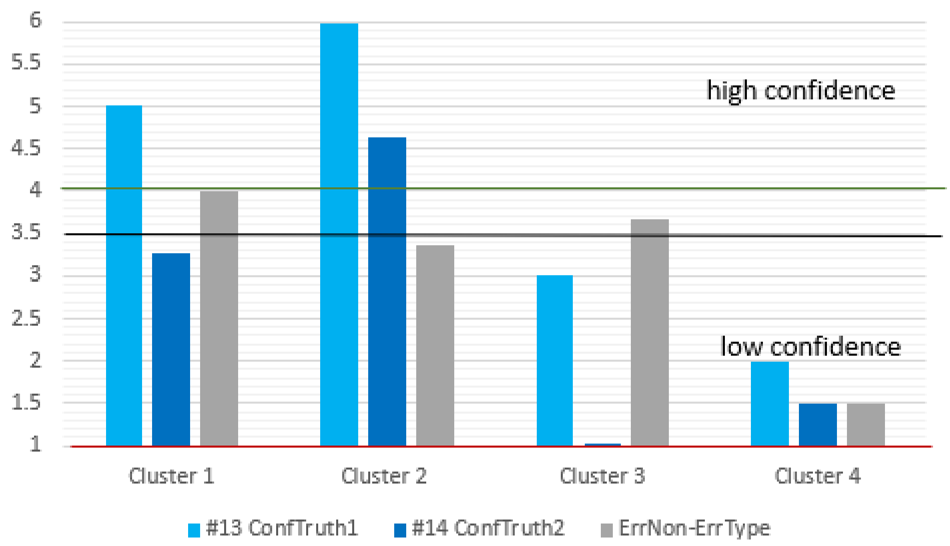


Figure 6.32: Prototypes (cluster centers) for fuzzy partitions in Table 6.9.

6.5.6 *Revisiting Jenni's associative feelings that an orbit felt like a coset: associations between Lagrange's theorem and the Orbit-stabilizer theorem*

After completing part III of the study and playing with examples of group actions, I asked Jenni again about why she thought orbits felt like cosets. She gave both a temporal and conceptual association.

> Jenni: I might have just thought that because I was trying to think back to the class and I think I had those at the same time. So I don't know if they reminded me of each other structurally or just because I was trying to think back to Abstract Algebra, that was in the same semester, the words are connected because I learned them around the same time.

> Jenni: I think it was more like when I was thinking about how quotient groups work is really when I was thinking about cosets, because it's like just partitioning the set, and then thinking about orbits, is that just like a coset, are there going to be disjoint partitions as far as like are the orbits going to be like everything is one's thing orbit but not in the other and then I pretty quickly figured out that orbits and cosets are not similiar.

Jenni felt they were not similiar because in the case of Lagrange's theorem the subgroup H acting on G partitions G, but was confused because the case G acting on the set X did not partition G. I explained that what was being partitioned was the set being acted upon and we revisited what the usual induced relation and equivalence classes for the actions, H acting G and G acting on the set X, would be. Following this, she identified her point of confusion in making a connection between Lagrange's theorem and the orbit-stabilizer theorem.

> Jenni: I think I did not, I was still thinking about, orbits not partitioning G was throwing me off. I don't think that had completely clicked.

While Alex had already consolidated several abstract associations between orbits, stabilizers and the cardinality of a group, Jenni was still forming these types of associations. Working with specific example cases helped her to see that orbits partition, what was being partitioned in the case of Lagrange's theorem was clear, but shifting to the generalization of orbits partitioning the set being acted upon was something that she was still in the process of getting straight in her mind.

Chapter 7

Discussion

This chapter highlights key findings from two separate studies conducted in this thesis, addresses research questions, and provides a discussion that weaves in related literature. The first study that will be discussed investigated three leaners' approaches to the collapsing structure task and representational fluency related to homomorphisms. Multiple registers (Cayley tables, group presentations with formal-symbolic mappings, digraphs, and objects of symmetry) were built into the task. This provided multiple insights into learners' difficulties and strengths that isolated successes or failures in a single register might mask. A semiotic lens that incorporated a new four-level analytic framework for fluency was used to analyze the data. The analytic framework captured learners' interpretations of what losing or collapsing structure meant to them, valid and invalid solution strategies, registers they entered, pseudo-semiotic triangles that they produced, and conversions across registers. Aside from the more dominant semiotic lens, two additional intuition-associated lenses, affective impacts and default-interventionist, were used during analysis.

This study addressed the following research questions:

1. What are learners' difficulties and strengths with quotients or homomorphisms, detected through a semiotic lens during the collapsing structure task?

   1a. What are learners' interpretation of the losing or collapsing structure?

   1b. Do learners produce semiotic representations that are inconsistent with quotient maps or homomorphism across multiple registers? If yes, which registers?

   1c. Did learners make quotient map conversions from registers that they started in to a prompted register for valid Z_2 or $Z_2 \times Z_2$ strategies?

2. What are learners' difficulties and strengths on the quotient task detected through the two additional intuition-associated lenses?

The next study was the example-based intuition for group actions study. It consisted of a baseline questionnaire that evaluated whether or not two graduate learners could generate multiple examples of group actions, and if they could translate the definition of a group action in terms of homomorphism language. After the baseline questionnaire, an experimentation paradigm was set up for learners to explore orbit and stabilizer relationships for different group action examples. The non-creative versus creative forms of intuition attributes along with additional attributes were used to classify the (parital) make-up of their reported intuitions. Both quantitative and qualitative data attached to each reported intuition were obtained.

The following research questions were addressed:

3a. Is it possible to quantitatively characterize the (partial) make-up of learners' example-based intuitions during a group actions task?

3b. What is the (partial) quantitative make-up of learners' reported intuitions related to orbit-stabilizer relationships induced by group actions?

3c. Is it possible to measure intuitive feelings of rightness or wrongness? Are learners' feelings of rightness and wrongness working properly?

3d. What is the qualitative content of learners' intuitions related to orbit-stabilizer relationships?

7.1 Homomorphisms or quotients through a semiotic lens

The transition to understanding quotients and the fundamental theorem of homomorphisms at least involves understanding: how to partition a set?', how to partition a set into subsets so that it is guaranteed that the subsets are of equal size?', and 'how to partition a group into subsets so that the subsets viewed as elements with some binary operation form a new group?' Many aspects of this transition have been a documented challenge for undergraduates (Dubinsky, et al., 1994; Mena-Lorca & Parraguez, 2016). The collapsing structure task shed additional light on these challenges which were pervasive across multiple registers for Max. The necessary normality condition to partition a group to obtain a new group was not fully recovered by Jenni, the first year graduate learner who had completed a year long undergraduate course, but did not have exposure to a graduate-level abstract algebra course yet.

Max associated the collapsing or losing structure prompt with a homomorphism and could comfortably recall the formal definition of a homomorphism. However, his constructions of a homomorphism using the group presentation with formal-symbolic mapping register and digraph register were invalid. The digraph register offered additional insights into Max's difficulty with the homomorphism concept. Rather than combining elements into sets of equal size and viewing these sets as elements, Max wanted to erase vertices and edges in the digraph. Max did not mention the terms cosets, partitions, or equivalence relations during the task. The link between these more fundamental underlying concepts and the homomorphism concept was missing.

The incorporation of the digraph register helped to narrow in on the source of difficulties associated with the homomorphism concept. For example, the incorporation of the digraph register made it apparent that Jenni's difficulties related to a homomorphism were not the same as Max's. Jenni related the term collapsing structure with the quotienting process of modding out by an equivalence relation and viewed sets of elements as elements in the image. She showed a strong understanding of cosets. She consistently partitioned groups into equal size cosets in more formal-symbolic and digraph registers. She got hung up with how to make a partition of a group into a group and struggled with the necessary well-definedness condition or equivalently the normality condition in both invalid, Z_4 and valid strategies Z_2.

A common finding for Max and Jenni was that neither of them entered the $Z_2 \times Z_2$ strategy. A speculated reason for this was that general problem solving strategies such as keeping what the two starting groups had in common and getting rid of what they don't applied to symbolic cues may have prevented them from seeing an alternative solution path by reinforcing a false generalization that the image of a homomorphism would be a subgroup of the domain group. While this type of thinking could lead to the Z_2 solution it does not work for the $Z_2 \times Z_2$ solution. More studies are needed to determine if a lack of seeing the $Z_2 \times Z_2$ strategy is common among undergraduates or new graduate learners and what other reasons may be causing the learner to miss this strategy.

The second year graduate student, Alex, produced valid semiotic representations within the formal-symbolic and digraph registers for a quotient. He used strong necessary conditions such as normality to quickly construct quotients. He was the only participant to produce multiple valid solutions, namely to construct quotient groups that are isomorphic to Z_2 and $Z_2 \times Z_2$, and no invalid solutions to the task.

What are learners' interpretation of losing information or collapsing structure?
Instructors use of conceptual metaphors showed up in the integrative literature review and interviews with experts as a method for transferring or facilitating intuition, and as a potential tactic for "fostering an understanding of isomorphism and homomorphism" (Rupnow, 2019, p. 299). Rupnow (2021) cataloged and compared the conceptual metaphors that instructors used during classroom instruction and interviews. She emphasized the need for future research to investigate learners' use the language their instructors use and intends "to compare students' individual scores on achievement measures to their usage of metaphors" (Rupnow, 2019, p. 299). Based on results from the current study, I also think it is important that learners' metaphor use is examined at the micro-level of semiotic and pseudo-semiotic triangles (i.e., sign, interpretation, and signified object unit of analysis).

For example, a semiotic analysis during the collapsing structure task designed in this study showed that for a new to abstract algebra learner, like Max, the meaning of metaphors and common informal phrases picked up from instruction may not be clear. On several occasions conceptual metaphors for a homomorphism and common cultural phrases were misinterpreted by Max. This was one possible cause of invalid homomorphism constructions. Max associated the words collapsing structure and homomorphism, but his homomorphism constructions did not reflect the intended meaning of the "collapsing metaphor" as defined in (Melhuish, Lew, Hicks, & Kandasamy, 2020). He asked if collapsing meant "losing elements?" His sign-interpretation pairs for a homomorphism that involved erasing and tearing did not actually resemble a homomorphism or collapsing to cosets. Another concern was that the statement of the task prompt used the metaphorical language associated with a homomorphism "collapsing structure" and "losing information" at the same time. This may have exacerbated Max's confusion. This suggests that the use of multiple metaphors too soon may be overwhelming to new learners and lead to confusion. It is important for educators to be extremely mindful of the metaphorical language they use in the classroom with undergraduates and not to assume that the more metaphors the better.

There was also evidence that Max tried to use the "sameness metaphor", as defined in Rupnow (2021) for same properties. He implicitly used the notion of sameness for an isomorphism, but did not always use valid sameness properties. For instance, he correctly used the notion of sameness as a relabeling in the Cayley table register, but incorrectly assumed that two groups could not be isomorphic because the generators in the group presentations did not have the same orders. Not being able to match up the generators orders violated Max's criteria for a isomorphism. The more advanced graduate student, Jenni however was more comfortable with the phrase collapsing and sameness metaphors. She thought of collapsing as quotienting and combining elements to form cosets and sameness as re-labeling Cayley tables. It seems that the correct meanings for the collapsing, losing structure, and sameness metaphors develop after a sizable compression of introductory abstract algebra experiences and they should be introduced carefully.

Do learners produce representations that are inconsistent with a quotient map or homomorphism across several registers? Which registers?

An integration of Duvalian semiotic theory, Peirce's semiotic triangle, and Vinner's pseudo-conceptual behavior allowed the articulation pseudo-semiotic representations. Pseudo-semiotic representation were defined in this thesis as semiotic triangle produced by the learner that either did not exist in the mathematical system or was not consistent with the actual mathematical object that they attempted to signify. According to Duvalian semiotics the more registers one can throw the same mathematical object in, the better understanding they have for that object (Duval, 2017). Likewise the more pseudo-semiotic representation produced for an object as instance of a concept, in this case the quotient concept, the less fluency and understanding.

In this study pseudo-semiotic representations were produced by Jenni for a quotient map and Max for a homomorphism. This pseudo phenomenon was observed most frequently in formal-symbolic mapping and digraph registers. Max came into the digraph register with a preconceived solution that the homomorphic image would be Z_4 and manipulated the digraph to fit his preconceived solution rather than manipulating in ways that were consistent with the actual object of a homomorphim. He was not coordinating his quick visual manipulations, such as erasing and pulling out the subgroup Z_4 in the digraph register with the formal analog of a homomorphism. Max's pseudo productions in a digraph register were further from than actual mathematical object of a homomorphism than Jenni's. Jenni's pseudo-productions were not as far or concerning because they were at least consistent with a partition into cosets. Both Max and Jenni only attempted to construct homomorphism where the image was a subgroup of the domain group, this may have been one reason why they did not enter the $Z_2 \times Z_2$ strategy. Trying to force the image of a map to be a subgroup or trying to get rid of elements or subgroups co-occurred with pseudo-semiotic representations and impossible strategies.

Did learners make quotient map conversions from registers that they natural started in to a prompted register for valid strategies?

Both graduate students went to the formal-symbolic mapping register to construct quotient maps. Neither of the graduate students entered the digraph register prior to a switch prompt for valid strategies. After a prompt that asked them to make a conversion from the formal-symbolic register to the digraph register, both were successful. Jenni was successful to the extent of modding out by an equivalence relation, but had difficulty with the normality condition. Alex was fully successful to the extent of modding out by a normal subgroup. There was no tearing or erasing in any of Jenni and Alex's digraph responses.

Overall, the results in section 6.2.5, suggested that Max had little to no fluency for homomorphism constructions across the registers. Max's homomorphism constructions were directed by perceptual stimuli in the signs, formal definitions of a homomorphism. These constructions were not directed by more powerful formal results or the concept of equivalence relations. Max's fluency difficulties across multiple registers were consistent with findings for undergraduates fluency in other related subject areas such as linear algebra (Sandoval & Possani, 2016). It seems to be a developmental norm for learners to have low fluency in subject areas where they are just getting started. Jenni had moderate fluency, and Alex had strong fluency with respect to the registers they naturally entered and the probed digraph register.

While the two graduate students were fluent in several registers, there was no natural indication that they or Max would use Cayley tables in ways that were consistent with forming a quotient group with set multiplication, as illustrated in Larsen and Lockwood's TAAFU sequence (2013). But just because learners do not naturally go to certain registers on their own to successfully construct a quotient map does not mean that they are are not able to. This was illustrated by the incorporation of a digraph switch prompt. A possible future modification to this study would be to incorporate an additional switch prompt to the Cayley table register if time allows. A few suggestions from an expert were to replace the cycle graph cards with the object of symmetry card on the collapsing structure task. The expert explained that in the object of symmetry register, normality can be seen as a symmetry within a symmetry.

7.2 Homomorphisms or quotients through other intuition lenses

According to the default-interventionist perspective, system 1 generates automatic intuitive thoughts. These thoughts come to mind as reactions to cues in a problem solving environment. The analytic system 2 either approves the intuitive thought as reasonable or signals the need to check or reject it (Kahneman, 2011). The stimuli in Group Explorer cards seemed to drive Max's thoughts with little control from formal knowledge, such as definitions and theorems. He would always start with pairs of cards in the same register, one for the dihedral group and one for the quaternion group, look for perceptual similarities to make the cards look the same, get an idea of what to do based on these perceptual cues, and give his response. At times he would maintain awareness that he wasn't sure, but was lost when trying to use his analytic system to check or verify his response. One example of this occurred when he was comparing and contrasting subgroup list for the dihedral and quaternion group. He verbalized the cues in the subgroup list that he was paying attention to and said:

> Max: I don't know what a function would be that would collapse those subgroups [referring to subgroup list for dihedral group of order eight], but if there was some way to get those out then it would be Q_8.

A second example of this occurred in the group presentation register where symbolic cues for the cyclic subgroup of order four stood out in both the group presentation for the dihedral group and quaternion group. A third example of this occurred in the digraph register when he produced a pseudo-semiotic representation for a homomorphism from the quaternion group to the cyclic group of order four. In this example Max took a path of least visual resistance. He found the cycles to be the least complicated stimuli and so he focused on the green cycle on the top in Figure 6.7.

> Max: ...the inner and outer cube, or like squares, are just like more complicated...So on the top, the top square, there's like one four cycle between those elements. And then I just selected those, took them off by themselves. Then I don't have to worry about what they do with the other things on the bottom, because I'm just looking at the top.

Once he saw two isomorphic subgroups, namely the cyclic group of order four, he felt he had found the solution to the task and assumed the corresponding homomorphisms that he needed would exist.

> Max: And then you can pretty clearly see that these guys are isomorphic, because they both have one generator of order four. And that's it, is that all?

Jenni also was initially consumed with cues for cyclic subgroups of order four. She cancelled out elements until she was left with just these subgroups, however later on she recovered from these default reactions on her own. System two kicked in with key aspects of a quotient and she explained

> Jenni: My initial though was just getting rid of elements is the way to go to leave me with a subgroup and then maybe the quotient groups will help me do that and then thinking well if I quotient by a subgroup that is not just leaving me with a subgroup...quotients don't get rid of elements they combine them.

In contrast to Jenni, Max did not recover from default intuitive reactions on his own. However, Jenni had an advantage. She was able to recover because she had an accurate association to a quotient map, knowledge that Max had not learned yet.

After giving default or reactive responses to stimuli in the cards Max would stop and make statements like "I probably need to be a lot more rigorous with this, but that's kind of all I can think of". He knew he needed to verify his responses through some kind of formal argument, but did not know how. He often looked to the researcher for confirmation to see if he was right or not. When he became aware of a way to check one of his proposed homomorphism maps by brute force binary product computations he loathed at the idea of performing these "burdensome" computations and suppressed the analytic system further.

Max's reason for not pursuing analytic thinking was more sophisticated and distinct from others findings. For example, Weber and Alcock (2009) found some learners often felt extremely satisfied with a solution formed in an intuitive system, had strong contentment and feelings of self-evidence. Due to an overwhelming contentment they had the right answer, these learners did not have a need to use the analytic system and construct a proof. Max's suppression of the analytic system was perhaps initially due to this often cited self-evidence principle, but not entirely. He eventually exhibited a need to use analytic system 2 thinking and was given one way to check, but still chose not to use analytic thinking at that moment because he was looking for a checking method that was more efficient. While Max's aversion to computation could be viewed as a negative it could also be viewed as a positive that could motivate the use of computational software and work towards more powerful formal results like the first isomorphism theorem. Max was not the only one who showed aversion to computation. The oldest graduate student also showed aversion to computation on the group actions task.

An educational implication stated by Leron and Hazzan (2006) is to:

> ...train people to be aware of the way S1 and S2 operate, and to include this awareness in their problem-solving toolbox...If analyzing typical S1/S2

> pitfalls became an inherent part of students' problem solving sessions, they might become more successful problem solvers and decision makers (p. 123-124).

In addition to system 1 errors and suppression of system 2, several affective factors that have been theorized to negatively impact intuition were also observed for Max and Alex. Max showed signs of constructs defined by Fischbein (1987) such as fear of invalidity and premature closure.

> ..."need for closure" was defined in terms of a desire for "an answer on a given topic, any answer,...compared to confusion and ambiguity. Such need was referred to as "nonspecific" and was contrasted with needs for "specific closure", that is, for particular (e.g., ego-protective or enhancing) answers to one's questions...The need for closure may also be aroused when the judgmental task appears intrinsically dull and unattractive to the individual. Under such circumstances, closure may serve as a means of escaping an unpleasant (hence, a subjectively costly) activity (Webster & Krulanski, 1994, p. 1049).

In Max's case, it was not clear if premature closure was nonspecific or specific and what the underlying reasons for closure were.

The most seasoned learner Alex, the second year graduate student, had the highest level of quotient fluency, never entered an invalid strategy, and had the most severe affective reaction. He experienced anxiety that completely blocked his intuitions during the collapsing task and he recalled having similiar anxiety episodes during oral examinations. Alex expressed that an overemphasis on getting the right solution:

> cuts of all the conduit stuff, the actually learning how to do the thing that works...it cuts of all paths to get to the goal.

7.3 The make-up of learners' example-based intuitions

Is it possible to quantitatively characterize the (partial) make-up of learners' example-based intuitions during a group action task?

Over a decade ago psychologists Dane and Pratt (2009) concluded that

> ...there is little consensus regarding how intuitions are captured methodologically...we encourage researchers to continue to craft new measures and methods for capturing intuition (p. 33)

Today, this lack of consensus continues with no direction from mathematics education researchers since Fischbein and his colleague's contributions, which ended with his passing in 1998. Fischbein, Tirosh, and Melamed (1981) asked "Is it possible to measure the intuitive acceptance of a mathematical statement?" They defined intuitive acceptance as a combined geometric mean score of two attributes: level of confidence and degree of obviousness. They used yes or no questions with a hierarchical analysis, that revealed a Guttman scale, to measure confidence and obviousness. Motivated by Fischbein et al. (1981), I also worked on an 'is it possible to measure some aspects of intuition' question, but paid attention to a different set of attributes, specifically those that separated non-creative versus creative form, confidence in truth, and error type. The distinction between non-creative and creative forms of intuition was framed by psychologists Policastro (2015) and Dane and Pratt (2007, 2009). Determining which attributes to focus on and theoretically framing them for use in mathematics education research was a significant challenge. In addition, I developed a new prototype instrument that aligned with some of these attributes to collect quantitative data. An analysis using the Fuzzy C-Means algorithm was different from Fischbein et al.'s yes or no questionnaire with hierarchical analysis.

What is the quantitative make-up of learners' reported intuitions related to orbit-stabilizer relationships induced by group actions?

A goal of the new approach, taken in this thesis, was to illustrate the possibility of quantitatively characterizing and classifying intuitions in a pure math setting. This approach uncovered the (partial) make-up of two graduate learners' intuitions that were reported during a task to explore orbit-stabilizer relationships. Univariate tests revealed significant differences between Jenni and Alex's intuitions at the level of single isolated attributes for: positive emotional intensity and non-sanctioned cue use, and near significance for incubation. Jenni was more likely to have strong positive emotional reactions that accompanied her reported intuitions, non-sanctioned cue use, and intentionally taking breaks to incubate. The FCM prototypes revealed variations for how non-creative or creative intuition objects were and that pairs of attributes associated with creativity need not move together from object to object. A Fuzzy C-Means analysis confirmed that 54.55 % of Jenni's reported intuitions belonged to more creative clusters, $M > 0.628$, and 45.45% belonged to more non-creative clusters, $M < 0.364$, across all partitions. All of Alex's intuitions belonged to the most non-creative clusters, $M < 0.299$, across all partitions.

There could be many reasons why Alex's intuitions were more non-creative.

First of all, a limitation is that network thinking was not incorporated into the FCM analysis. Alex showed several instances of network thinking. Another reason, is that Alex may have viewed the task as being too routine and went into a mode of recovering familiar knowledge he associated with the task. However, I purposefully chose this task because it could be made routine or non-routine depending on what examples the learner wanted to explore and what questions they wanted to ask. Affective factors such as lack of motivation, lack of enjoyment, lack of time, and other priorities could have also contributed to a higher frequency of non-creative forms of intuition. Based on personal experiences and interviews, one key that seems to unlock extreme creative forms of intuition is having passion in a particular subject area that drives curiousity. Tasks that are not related to a learner's passions could also be cause for seeing a higher frequency of non-creative forms in a local setting. Taking the the same set of Alex and Jenni's reported intuition objects a second set of attributes separate were focused on to investigate intuitive feelings of rightness or wrongness.

Is it possible to measure intuitive feelings of rightness or wrongness? Are learners' feelings of rightness or wrongness working properly?

Intuitive feelings of rightness and wrongness have been characterized as "nonemotional cognitive feelings, such as feelings of knowing...they can influence whether one continues a search or aborts it" and "they may be rather 'vague' and not easily noticed or focused upon, but can influence one's actions, including action that are part of the proving process" (Selden et al., 2010, p.202). Prior to proof, a reported intuition object is evaluated by another class of intuitions, namely intuitive feelings of rightness or wrongness. Intuitive feelings of rightness or wrongness fall under the classification of simple associative forms. Many types of intuitions interact with one another (Glöckner & Witteman, 2010; Dane & Pratt, 2009).

Intuitive feelings of rightness or wrongness were inferred, in this thesis, through two attributes confidence in truth value and error to non-error type. After recording an intuition learner's reported how confident they were that they were that their intuition was true on a likert scale and the researcher coded error to non-error type. A Fuzzy C-Means classification of Jenni and Alex's intuitions in terms of these attributes can be found in subsection 6.5.5. Problematic clusters marked by bipolar attribute values for the first confidence item and error type. The second item for confidence seemed to be more of indication of being cautious.

Taking this notion of problematic clusters, an example of a problematic cluster would be one characterized by high confidence in truth value and low scores for error to non-error type. This particular problematic cluster did not show up in the FCM analysis of Jenni and Alex's intuitions. Moreover, completely errored intuitions, as defined in subsection 5.8.2, were not a cause for concern because they were accompanied with low levels of confidence. In fact, completely errored intuitions coincided with extremely low confidence in truth values in cluster 4. This was seen as a positive sign that her evaluative feelings of wrongness were working properly. Cluster 3 indicated some potential issues in Jenni's feelings of rightness where she had low confidence values, but high levels of correctness for 27% of her reported intuitions. All of Alex's intuitions belonged to clusters with high confidence levels and high levels of correctness. Overall, the results suggested that Alex's feelings of rightness were working properly. Jenni's feelings of rightness were not running as smoothly, this could have also just been a natural co-occurrence with creative forms that are newer or less familiar to the learner. However, going back to the data she did report intuitions that had high membership to cluster 3 and high membership to more non-creative clusters. Overall, a measure for feelings of rightness or wrongness using confidence and error-type values would be a starting point, but the picture is more complicated than just looking at confidence levels and error type. It would also be important to consider how these attributes cross with non-creative and creative clusters.

The quantitative portion in this study improved objectivity during analysis and made research on the intuition construct more manageable and coherent. While it is not possible fully capture the intuition phenomenon due to an enormous amount of attributes and actions running around, I argue that this mixed approach allowed for discussions that would not have been possible with qualitative means alone. Likewise, the qualitative portion attached to each object provided examples connected to the quantitative data. The qualitative data for Jenni's reported object No. 2 that belonged to the potentially concerning cluster will be discussed further under the next research question.

What is the qualitative content of learners' intuitions related to orbit-stabilizer relationships?

The journal results provided several examples of the qualitative content of learners' intuitions. The discussion will focus on Jenni's objects because they provided the most insight into how learners might be linking Cayleys, Lagrange's and the Orbit-stabilizer theorem within a group actions frame, following a first year undergraduate course. Jenni drew her intuition from concrete examples of transitive an non-transitive group actions. Alex drew his intuitions from transitive group actions where the identity was trivial or equivalently an action that admits a single orbit. He felt that specific examples were too restrictive and he preferred to draw his intuition form the more abstract deck transformation example in algebraic topology. Jenni's intuition journals and follow-up interviews also highlighted a gradual transition from viewing a group as a set of elements with a binary operation to a set of functions with composition, a fundamental idea underneath Cayley's theorem.

Next, she reported intuitions relevant to Lagrange's and the orbit-stabilizer theorem. An example of a key creative intuition related to her transition between Lagrange's and the Orbit-stabilizer theorem was:

> Jenni: Going back to the idea of cosets, I think a coset is like if $H \subset X$ then partition X into right cosets by $H \cdot x$. This is different because G might not be a subset of X. But all $G \cdot x \in X$ so maybe it is the same? (No. 2)

> Jenni:...I was thinking about cosets, because it's like just partitioning the set, and then thinking about orbits, is that just like a coset, are there going to be disjoint partitions as far as like are the orbits going to be like everything is one's things orbit but not in the other...

These linking intuitions from Cayley's to Lagrange's to the Orbit-stabilizer theorem were fragile. She showed low confidence in intuition No. 2 and concluded with the statement that "orbits and cosets are not similiar". Her reason for why they weren't similiar was that orbits don't partition G and cosets do. This linking intuition No. 2 with higher membership to more creative clusters was fragile and learners. It is linking intuitions like these that need to be tended to with guidance from the instructor, however such intuitions are likely not observable to instructors during a standard lecture. Moreover, figuring out how to incorporate, strengthen, and develop or re-direct these linking intuitions during instruction would be a challenge.

Chapter 8

Concluding Remarks and Future Directions

8.1 Concluding Remarks

A goal of instruction is to improve learners' fluency across representation registers and facilitate intuition (Duval, 2006; 2017; Fischbein, 1987, Burton, 1999), and the significant goal of evaluative research on student thinking is to inform instructional design (Gravemeijer, 2004; Sandoval & Possani, 2016). The evaluative type case studies, designed in this thesis, provided ways to gather information on learners' representational fluency and example-based intuitions for group theory topics in a systematic way. A combination of several existing theoretical lenses, a new analytic framework for fluency, and new instruments were useful in bringing out different facets of the data. In this chapter, the contributions are listed, followed by limitations, and plans for future directions. These Future plans specify starting points for how to extend qualitative case studies under the intuition umbrella to larger scale mixed-method studies. Overall, this thesis represents an attempt to maintain a grip on the multi-faceted intuition construct.

8.1.1 Contributions

The contributions of this thesis were both theoretical and methodological. The first theoretical strand included:

(a) an integration of multiple research areas through an extensive integrative literature review,

(b) developing a theoretical framework for intuition associated factors,

(c) an organization of attributes associated with non-creative versus creative forms of intuition,

(d) an organization of additional attributes and viewing intuition as a cognitive object with various attributes attached to it.

The methodological contribution built from this theoretical contribution consisted of:

(e) designing a prototype survey instrument to practically extract the attribute values of learner's reported intuition objects (NCCFIS),

(f) characterizing the 'partial-make up of intuition' as an object with qualitative and quantitative data values attached to it, and

(g) applying the Fuzzy C-Means Algorithm by Bezdek et al. (1981) to classify abstract algebra learners' reported intuition objects based on attribute similarity.

To the best of my knowledge the NCCFIS instrument seems to be a first of its kind in the area of intuition research. Furthermore, to the best of my knowledge, an application of the FCM algorithm to classify different intuitions has never been done. It may or may not have been a strange coincidence that I used fuzzy partitions to analyze learners' reported intuitions related to hard partitions induced by group actions.

A second contribution strand included theoretical work in the area of semiotics that led to:

(h) the derivation of a new construct termed a fluency digraph that crossed modes of semiotic production modes of aquisition, and modality. The fluency digraph was later aligned with the practical methodological data collection tool of card-sorting,

(i) two representational fluency tasks, the collapsing and adding structure tasks were designed with multiple solutions, registers built in, and interview protocols, and

(j) An analytic representational fluency framework.

Additional contributions included:

(k) insightful results on abstract algebra learners' interpretations of metaphorical language and representational fluency related to the homomorphism concept,

(l) examples of learners' semiotic and pseudo-semiotic representations of a homomorphism, or quotient, and direct-product across several registers, and

(m) their intuitions related to group actions afforded by novel approaches to data collection and analysis.

8.1.2 Limitations

The ongoing limitations that occured in fluency studies, that also continue to be a challenge for semiotic researchers, is the ability to rigorously separate single register use or multimodal use in which several registers are being used at the same time but are not all observable (Sandoval & Possani, 2016). Knowing which mix of registers are being used seems to be exacerbated when analyzing data from more advanced learners who tend to use multiple registers simultaneously. Another difficulty during analysis was that many conversions may be occurring within the total move from one register to another for a specified object. One helpful tactic was to fix and keep focus on one mathematical object at a time (e.g., a specific quotient map) and track it across observable registers.

The current analytic framework for representational fluency, purposeful sampling, instrumentation with multiple registers and analysis used in this study highlighted the presence of different fluency levels for the homomorphism concept. While this approach brought out several criteria for different levels of fluency, the discrimination of different levels was limited by inferences based on qualitative analysis. More work is needed to define a rigorous discrimination between these levels in order to extend this qualitative demonstration to a viable larger scale mixed study.

In reference to the example-based intuition study, the new non-creative versus creative forms of intuition (NCCFIS) survey and accompanied analysis needs to undergo critical reviews and revisions. At this time the NCCFIS is based on an extensive literature review and a few interviews with mathematicians. More expert consultations with mathematicians, education researchers, and quantitative psychologists are needed to revise the instrument to satisfy content validity. At this time, it is unclear how to obtain validity, beyond theoretical grounding and content validity, for the self-report NCCFIS instrument.

While the self-report instrument, used in this thesis, may be considered a limitation, alternatives such as a researcher trying to infer non-observable aspects of learners' intuitions has also been problematic (Bubp, 2014). In this thesis, I have done my best to provide some middle ground by incorporating attribute values that are reported by the participant, attributes inferred by the researcher, and qualitative journal data interpreted by the researcher. Lastly, the quantitative portion of this study was limited by a small sample size. Overall, the tools developed in this thesis that were intended to partially capture and examine example-based intuitions are still in their infancy. Plans to address some of these limitations are discussed in the next section along with future directions.

8.2 Future Directions

A diverse supply of studies with many design types and researchers with different strengths are needed. Moreover, the availability of some study designs may be contingent on others. To start, a field may conduct many evaluative case studies to identify learner's difficulties and strengths followed by grounded theory to obtain a an extensive catalog of difficulties and strengths that learners encounter within a particular subject area, especially when theory from the literature is not sufficient or in some cases does not exist. These types of studies inform qualitative small scale teaching experiments that lead to local instructional theories. In addition, integrative reviews and phenomenological studies are needed to gain insight into a cultural phenomenon or to clarify constructs. Building from this, quantitative instruments become available. In general, an incredible and diverse amount of work is necessary before full blown efficacy studies with quasi-experimental designs are within reach.

After the work put forth in this thesis, we now return to the motivating hypothesis stated in the introduction. Efficacy studies under an intuition focused research program for abstract algebra education involves many hurdles to overcome. While the case studies conducted in this thesis do not lend themselves to generalizable results. However, they do set a potential foundation for carrying out larger scale future studies. This section looks towards the next steps and provides specific starting points where: 1) case study designs could be extended to mixed-method case designs and 2) pockets where quasi-experimental designs could be conducted under the umbrella of intuition associated factors in abstract algebra.

Abstract algebra education researchers continue to find and document that the homomorphism concept is difficult for undergraduates. It is an instructional challenge because a deepened understanding of a homomorphism, beyond the formal definition, relies on a solid understanding of the quotient group concept, the first homomorphism theorem, and first isomorphism theorem for sets and groups (Dubinsky et al., 1994; Asiala et al., 1997; Melhuish et al., 2020; Rupnow, 2021). More efficacy oriented studies are needed that compare groups of learners that receive different instructional sequences for these entangled concepts. It is of interest to determine whether or not leaners who receive Inquiry-Oriented instructional sequences perform better on other quotient tasks than participants that do not receive this sequence (Larsen, Johnson, & Bartlo, 2013). Additional efficacy type studies that compare different instructional sequences for quotient group concept is one avenue for future research.

A goal of experimental instruction is to guide undergraduate learners to develop higher levels of fluency compared to learners that receive more standard lecture based forms of instruction. It is recommended that fluency activities first allow learners to become comfortable within single registers before they move on to activities that train conversions across registers (Sandoval & Possani, 2016). In addition to the instructional sequences that predominantly intersect the Cayley table or formal-symbolic registers, there is a need to design and implement more activities that encourage fluency for the object of symmetry, beyond triangle and square, as in Bergman and French (2019), matrices as linear invertible maps, and digraph registers. Finally, experimental instruction should guide learners to confront and reduce the occurrence of pseudo-semiotic representations. The collapsing structure task, data collection protocol, and the semiotic analysis, developed in this thesis, could serve as useful tools for future studies that aim to evaluate the effectiveness of different instructional sequences on fluency outcomes for the quotient concept.

One avenue for future research is to modify and extend the qualitative case study design for the collapsing structure task, developed in this thesis, to a mixed-case design. A modification, along with adding switch prompts, would be to create a structured interview format that adds questions for what collapsing structure or losing information means to the learner and how it relates to a quotient map and how it relates to a homorphism. Next the task could be tightened by instructing participants to "Is there some way that you can modify both D_8 and Q_8 by constructing a quotient map so that the images are isomorphic with register options". The loosening or tightening of the prompt may be researcher preference. The tightened prompt would certainly be easier to handle during data analysis with a larger number of participants. To transform qualitative transcripts to quantitative data, create a fluency attribute rubric for the task, input the coded attributes into the FCM algorithm. Use the FCM algorithm to define more rigorous cut-offs for quotient fluency levels (i.e., little to none, moderate, and strong). The case study with the three consecutive level of learners helped me to see relevant attributes that could be used to distinguish different fluency levels within the context of the collapsing structure task. For example, some observable attributes were: the number of valid and invalid strategies that learner's entered, the number of registers they naturally entered for a quotient without prompt for valid strategies, the number of pseudo-semiotic representation that they produced for a quotient, the number of quotient map conversions they made across natural and prompted registers, and whether or not they were constructing quotients and making conversions at the level of an equivalence relation without normality or if they also incorporated normality. After kinks in the mixed-case design are worked out, it could be extended again to a large scale quasi design that evaluates learners' quotient fluency that receive different types of instruction characterized by the Inquiry Oriented Instructional Measure.

A second avenue for future research was motivated by an interesting piece of data found in this thesis that aligned with Lakatosian and Duvalian influences. From a Lakatosian perspective it is important to guide the learner to: build from weaker conditions and test what happens under these conditions and make progress towards stronger necessary conditions, maintaining awareness of implicit assumptions, and dissociating implicit assumptions from specific examples to formulate a general conjecture, and trying to counter the conjecture. From a Duvalian perspective in order to be a successful mathematics learner one has to develop the following skills: dissociating object from its representation or container, seeing multiple instances of an object across many registers, performing conversions, and maintaining awareness of pseudo-semiotic representations. The piece of data that is relevant to these perspectives occurred when Max was trying to decide if the dihedral group of order eight was isomorphic to the quaternion group of order eight. Max came to the conclusion that:

> So I would say, no, they're not isomorphic because for D_4 on of its generators is of order 2 and Q_8 both of the generators are of order four.

As a more seasoned mathematics learner, my brain automatically and quickly processed this piece of data using the following steps:

> step 1: detection of an implicit assumption that does not feel right.
>
> step 2: dissociate the implicit assumption from the examples and formulate it as a generalized logic statement -'if the order of the generators in the group presentation do not match then the two groups are not isomorphic'.
>
> step 3: try to find a counter-example to the generalized logic statement
>
> step 4: brainstorm, Z_6 comes to mind, I can think of two different group presentations for Z_6 where the orders of the generators in one presentation and the orders of the generators in the other do not match up.

step 5: state counter-example following a logical script - 'the orders of the generators in $< a : a^6 = 1 >$ and $< a, b : a^3 = b^2 = 1, ab = ba >$ do not match, but the two groups given by these presentations are isomorphic'.

Max did not perform this maneuver and after reflecting on it I realized how difficult such a maneuver, that I take for granted and do automatically, could be for an undergraduate learner. First feelings of rightness and wrongness, also referred to as error detecting intuition, must be working properly. Without this signal that something did not feel quite right, I would have never had awareness of a faulty implicit assumption and so I would have just gone on about my day. Second, it required dissociating an implicit assumption from a statement about examples and generalizing. Dissociating and generalizing are documented to be difficult moves in their own right for undergraduates. Third, a sound logical system has already been hardwired in my brain, meaning I automatically knew what the logical syntatic script should be and it this script is so second nature that I don't even consciously think about it. Fourth, you have to think of another example object and be able to see the same object in multiple representations. Taking in the total complexity of this maneuver and the fact that Max was often unsure during the task, but did not know how to check was a light bulb moment. Steps 1-5 mixes several core cognitive skills (e.g. dissociating object from object being represented, generalization, shifting between registers, etc.) some of which have been documented to be extremely challenging for undergraduates even in isolation. More research is needed to prepare scaffolded instructional materials that train these complex checking and error-correcting maneuvers.

A third avenue for future research is concerned with the example-based intuitions. The example-based intuition mixed design developed in this thesis illustrated ways for drawing out learners' intuitions, classifying them in terms of non-creative and creative forms attributes, and classifying them in terms of confidence in truth and error type. This has implications for characterizing learners' overall intuiting states based on reported objects (i.e., a mix of both non-creative and creative, or pre-dominantly non-creative), detecting whether or not learners' intuitive feelings of rightness or wrongness were working properly, and detecting positive and problematic densities of attribute combinations within a population. Moreover, the example-based intuition instrumentation and analysis can be used as a template meaning that the mathematical group actions content of baseline questionnaire and the experimentation task could be swapped with a baseline and experimentation task for mathematical concepts of the researchers choice. This flexible template accounts for the fact that intuition is fractionated and certain environments may be better than others at facilitating intuition for certain concepts.

At this time, recommendations for the example-based intuition factor are to conduct phenomenological studies and expert reviews to improve the NCCFIS survey. A more immediate use of the example-based intuition set up would be to repeat the group actions study in this thesis for a larger number of learners across consecutive mathematical levels and take a mixed grounded theory approach. This could help to create a catalog of intuitions for orbit-stabilizer relationships induced by group actions and inform the creation of instructional texts that incorporate the perspective of the undergraduate, first year graduate, second year graduate, an so on that adds layers to the same concepts as the text advances. Another more immediate use of the example-based intuition set up would be again to repeat the group actions study in this thesis, but this time swap the experimentation task with a Polya's necklace problem.

Chapter 2 Integrative Literature Review References

[1] Adiredja, A. (2018). Building on misconceptions and students' intuitions in advanced mathematics. In T. Bartell (Ed.). *Toward equity and social justice in mathematics education* (pp. 59-76). Cham: Springer.

[2] Alcock, L., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*(2), 111-129.

[3] Alcock, L., & Simpson, A. (2004). Convergence of Sequences and Series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 57*(1), 1-32.

[4] Antonini, S. (2019). Intuitive acceptance of proof by contradiction. *ZDM Mathematics Education, 51*, 793-806.

[5] Antonini, S., & Mariotti, M. (2008). Indirect proof: what is specific to this way of proving? *ZDM Mathematics Education, 40*(3), 401-412.

[6] Aricha-Metzer, I., & Zaslavsky, O. (2019). The nature of students' productive and non-productive example-use for proving. *The Journal of Mathematical Behavior, 53*, 304-322.

[7] Arzarello, F. (2006b). Semiosis as a multimodal process. *Revista Latinoamericana de Investigacioón en Matemática Educativa, 9*, 267-299.

[8] Attridge, N., & Inglis, M. (2015). Increasing cognitive inhibition with a difficult prior task: implications for mathematical thinking. *ZDM Mathematics Education, 47*, 723-734.

[9] Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1997). A framework for research and curriculum development in undergraduate mathematics education. *MAA Notes*, 37-54.

[10] Bastick, T. (1982). *Intuition: How we think and act*. Chichester: J. Wiley.

[11] Beth, E., & Piaget, J. (1966). *Mathematical epistemology and psychology*. Dordrecht: Reidel.

[12] Betsch, T., & Glöckner, A. (2010). Intuition in judgment and decision making: extensive thinking without effort. *Psychological Inquiry, 21*(4), 279-294.

[13] Bills, L., & Rowland, T. (1999). Examples, generalisation and proof. *Research in Mathematics Education, 1*(1), 103-116.

[14] The Networking Theories Group (2014). Bikner-Ahsbahs, A. & Prediger, S. (Eds.). *Networking of theories as a research practice in mathematics education.* Cham: Springer.

[15] Boell, S., & Cecez-Kecmanovic, D. (2014). A hermeneutic approach for conducting literature reviews and literature searches. *Communications of the Association for Information Systems, 34*(1), 12.

[16] Boero, P., Garuti, R., & Mariotti, M. (1996). Some dynamic mental processes underlying producing and proving conjectures. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education PME*(Vol.2, pp. 121-128). Spain: University of Valencia.

[17] Brouwer, L. (1913). *Intuitionism and formalism. na.*

[18] Brown, L., & Reid, D. (2006). Embodied cognition: somatic markers, purposes and emotional orientations. *Educational Studies in Mathematics, 63*(2), 179-192.

[19] Bruner, J. (1960). *The process of education*. Cambridge: Harvard University Press.

[20] Bubp, K. (2014). *To prove or disprove: The use of intuition and analysis by undergraduate students to decide on the truth value of mathematical statements and construct proofs and counterexamples* (Doctoral dissertation). Retrieved from http://etd.ohiolink.edu/

[21] Burton, L. (1999). Why is intuition so important to mathematicians but missing from mathematics education? *For the Learning of Mathematics, 19*(3), 27-32.

[22] Cangelosi, R., Madrid, S., Cooper, S., Olson, J., & Hartter, B. (2013). The negative sign and exponential expressions: unveiling students' persistent errors and misconceptions. *The Journal of Mathematical Behavior, 32*, 69-82.

[23] Carlson, M., & Bloom, I. (2005). The cyclic nature of problem solving: an emergent multidimensional problem-solving framework. *Educational studies in Mathematics, 58*(1), 45-75.

[24] Carter, N. (2009). Visual Group Theory. The Mathematical Association of America.

[25] Cayley, A. (1854). VII. On the theory of groups, as depending on the symbolic equation $\theta n = 1$. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 7*(42), 40-47.

[26] Chiu, M. (1996). Exploring the origins, uses, and interactions of student intuitions: comparing the lengths of paths. *Journal for Research in Mathematics Education, 27*(4), 478-504.

[27] Clement, J. (2008). *Creative Model Constructions in Scientists and Students*. New York: Springer.

[28] Cobb, P. (1986). Context, goals, beliefs, and learning mathematics. *For the Learning of Mathematics, 6*(2), 2-9.

[29] Cooper, H. (1982). Scientific guidelines for conducting integrative research reviews. *Review of educational research, 52*(2), 291-302.

[30] Cowan, N., Chen, Z., & Rouder, J. (2004). Constant capacity in an immediate serial-recall task: a logical sequel to Miller (1956). *Psychological science, 15*(9), 634-640.

[31] Crandall, B., & Getchell-Reiter, K. (1993). Critical decision method: a technique for eliciting concrete assessment indicators from the intuition of NICU nurses. *Advances in Nursing Science, 16*(1), 42-51.

[32] Creswell, J.W., & Creswell, J.D. (2018). *Research design: Qualitative, quantitative, and mixed methods approaches*(Fifth ed.). Thousand Oaks: SAGE Publications.

[33] Creswell, J.W., & Poth, C. (2018). *Qualitative Inquiry and Research Design*(Fourth ed.). Thousand Oaks: SAGE Publications.

[34] Czocher, J., & Weber, K. (2020). Proof as a cluster category. *Journal for Research in Mathematics Education, 51*(1), 50-74.

[35] Damasio, A. (1996). The somatic marker hypothesis and the possible functions of the prefrontal cortex. *Philosophical Transactions of the Royal Society of London. Series B: Biological Sciences, 351*(1346), 1413-1420.

[36] Dane, E., & Pratt, M. (2007). Exploring intuition and its role in managerial decision making. *The Academy of Management Review, 32*(1), 33-54.

[37] Dane, E., & Pratt, M. (2009). Conceptualizing and measuring intuition: A review of recent trends. *International Review of Industrial and Organizational Psychology, 24*(1), 1-40.

[38] Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics, 93*(1), 19-33.

[39] Davis, P. (1993). Visual theorems. *Educational Studies in Mathematics, 24*, 333-344.

[40] Davis, P., Hersh, R., & Marchisotto, E. (2012). *The mathematical experience: Study edition*. New York: Springer. doi: 10.1007/978-0-8176-8295-8

[41] Dawkins, P. (2015). Explication as a lens for the formalization of mathematical theory through guided reinvention. *The Journal of Mathematical Behavior, 37*, 63-82.

[42] De Groot, A. (2014). *Thought and Choice in Chess (Vol. 4)*. Walter de Gruyter GmbH & Co KG. (Original work published 1946).

[43] diSessa, A. (2004). Metarepresentation: native competence and targets for instruction. *Cognition and Instruction, 22*(3), 293-331.

[44] diSessa, A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: meta-representational expertise in children. *The Journal of Mathematical Behavior, 10*(2), 117-160.

[45] Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2001). Abstraction in Context II: the case of peer interaction. *Cognitive Science Quarterly, 1*, 307-368.

[46] Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: an ICMI study*(pp. 275-282). Dordrecht: Kluwer.

[47] Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *The Journal of Mathematical Behavior, 23*(3), 271-300.

[48] Dummett, M. (2000). *Elements of Intuitionism*(Vol. 39). Oxford University Press.

[49] Duval, R. (2000a). Basic issues for research in mathematics education. In T. Nakashara and M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education (PME), 1*, Hiroshima, 55-69.

[50] Duval, R. (2017). *Understanding the mathematical way of thinking-The registers of semiotic representations*. Cham: Springer. doi: 10.1007/978-3-319-56910-9

[51] Edwards, L. (1998). Odds and evens: mathematical reasoning and informal proof among high school students. *Journal of Mathematical Behavior, 17*(4), 489-504.

[52] Enyedy, N. (2005). Inventing mapping: Creating cultural forms to solve collective problems. *Cognition and instruction, 23*(4), 427-466.

[53] Epstein, S. (1994). Integration of the cognitive and the psychodynamic unconscious. *American psychologist, 49*(8), 709-724.

[54] Epstein, S. (2010). Demystifying intuition: what it is, what it does, and how it does it. *Psychological Inquiry, 21*(4), 295-312.

[55] Eubanks, D., Murphy, S., & Mumford, M. (2010). Intuition as an influence on creative problem-solving: the effects of intuition, positive affect, and training. *Creativity Research Journal, 22*(2), 170-184.

[56] Evans, J. (2006). The heuristic-analytic theory of reasoning: extension and evaluation. *Psychonomic Bulletin & Review, 13*(3), 378-395.

[57] Evans, J. (2019). *Hypothetical thinking: Dual processes in reasoning and judgement*. Psychology Press.

[58] Ferrer, E., & Nesselroade, J. R. (2003). Modeling affective processes in dyadic relations via dynamic factor analysis. *Emotion, 3*(4), 344-360.

[59] Fischbein, E. (1975). *The Intuitive Sources of Probabilistic Thinking in Children*. Dordrecht: Reidel Publishing.

[60] Fischbein, E. (1982). Intuition and proof. *For the Learning of Mathematics, 3*(2), 9-24.

[61] Fischbein, E. (1990). Intuition and information processing in mathematical activity. *International Journal of Educational Research, 14*(1), 31-50.

[62] Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139-162.

[63] Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics, 38*(1/3), 11-50.

[64] Fischbein, E. (2002). *Intuition in science and mathematics: An educational approach* (Vol. 5). New York: Kluwer Academic Publishers. (Original work published 1987)

[65] Fischbein, E., & Grossman, A. (1997). Schemata and intuitions in combinatorial reasoning. *Educational Studies in Mathematics, 34*(1), 27-47.

[66] Fischbein, E., Jehiam, R., & Cohen, D. (1995). The concept of irrational numbers in high-school students and prospective teachers. *Educational Studies in Mathematics, 29*(1), 29-44.

[67] Fischbein, E., Tirosh, D., & Hess, P. (1979). The intuition of infinity. *Educational Studies in Mathematics, 10*(1), 3-40.

[68] Fischbein, E., Tirosh, D., & Melamed, U. (1981). Is it possible to measure the intuitive acceptance of a mathematical statement? *Educational Studies in Mathematics, 12*(4), 491-512.

[69] Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

[70] Frey, K. (2002). Die Projektmethod [Project method]. IPN Universität Kiel.

[71] Fujita, T., Jones, K., & Yamamoto, S. (2004). Geometical intuition and the learning and teaching of geometry. *Paper presented to topic study group 10 at the 10th International Congress on Mathematical Education, Copenhagen, Denmark*.

[72] Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology, 22*(3-4), 455-479.

[73] Glöckner, A., & Witteman, C. (Eds.). (2010). *Foundations for tracing intuition: Challenges and methods*. New York: Psychology Press.

[74] Godino, J., Batanero, C., & Font, V. (2007). The onto semiotic approach to research in mathematics education. *ZDM Mathematics Education, 39*(1-2), 127-135.

[75] Goldenberg, P., & Mason, J. (2008). Shedding light on and with examples spaces. *Educational Studies in Mathematics, 69*(2), 183-194.

[76] Goldin, G. (1998). Representational systems, learning, and problem solving in mathematics. *The Journal of Mathematical Behavior, 17*(2), 137-165.

[77] Google Talks (2011). (2011, Nov. 10). Thinking, Fast and Slow with Daniel Kahneman [Video]. YouTube. https://www.youtube.com/watch?v=CjVQJdIrDJ0.

[78] Gore, J., & Sadler-Smith, E. (2011). Unpacking intuition: a process and outcome framework. *Review of General Psychology, 15*(4), 304-316.

[79] Gilhooly, K. (2016). Incubation and intuition in creative problem solving. *Frontiers in Psychology, 7*, 1076.

[80] Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: a calculus course as an example. *Educational Studies in mathematics, 39*(1-3), 111-129.

[81] Gray, E., Pinto, M., Pitta, D., & Tall, D. (1999). Kowledge construction and diverging thinking in elementary and advanced mathematics. *Educational Studies in Mathematics, 38*(1-3), 111-133.

[82] Griffiths, M. (2013). Intuiting the fundamental theorem of arithmetic. *Educational Studies in Mathematics, 82*(1), 75-96.

[83] Grosholz, E. (1978). *Unification and growth of mathematical knowledge* (Doctoral dissertation, No. 7915821). Retrieved from ProQuest Dissertations and Theses Global.

[84] Hadamard, J. (1954).*An essay on the psychology of invention in the mathematical field*. New York: Dover.

[85] Hammond, K. (1996). *Human judgment and social policy: irreducible uncertainty, inevitable error, unavoidable injustice*. New York, NY: Oxford University Press.

[86] Hancock, B. (2019). From qualification to consensus: the role of multimodal uncertainty in collective argumentation regarding complex integration. *Journal of Mathematical Behavior, 55*, 100700.

[87] Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6-13.

[88] Hanna, G., de Bruyn, Y., Sidoli, N., & Lomas, D. (2004). *ZDM Mathematics Education, 36*(3), 82-90.

[89] Hanna, G., & Jahnke, H. (2002). Arguments from physics in mathematical proofs: an educational perspective. *For the Learning of Mathematics, 22*(3), 338-345.

[90] Harel, G., & Sowder, L. (1996). Classifying processes of proving. In L. Puig & A. Guitierrez (Eds.), *Proceedings of the 20th Meeting of the International Group for the Psychology of Mathematics Education, 3* (pp. 59-65). Spain: University of Valencia.

[91] Hart, L., Smith, S., Swars, S., & Smith, M. (2009). An examination of research methods in mathematics education (1995-2005). *Research Methods in Mathematics Education, 3*(1), 26-41.

[92] Hegg, M., Papadopoulos, D., Katz, B., & Fukawa-Connelly, T. (2018). Preservice teacher proficiency with transformations-based congruence proofs after a college proof-based geometry class. *Journal of Mathematical Behavior, 51*, 56-70.

[93] Hélie, S., & Sun, R. (2010). Incubation, insight, and creative problem solving: a unified theory and a connectionist model. *Psychological Review, 117*(3), 994-1024.

[94] Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in context: epistemic actions. *Journal for Research in Mathematics Education*, 1995-222.

[95] Heyting, A. (1996). *Intuitionism: an introduction.*(2nd rev. ed., Studies in logic and the foundations of mathematics). Amsterdam: North-Holland Publishing Company.

[96] Hintzman, D. (1986). Schema abstraction in multiple-trace memory model. *Psychological Review, 93*(4), 411-428.

[97] Hogarth, R. (2010). Intuition: A challenge for psychological research on decision making. *Psychological Inquiry Special Issue on Intuition, 21*(4), 338-353.

[98] Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking*. New York: Basic Books.

[99] Jeshion, R. (2001). Frege's notions of self-evidence. *Mind, 110*(440), 937-976.

[100] Kahneman, D. (2003). Maps of bounded rationality: Psychology for Behavioral Economics. *The American Economic Review, 93*(5), 1449-1475.

[101] Kahneman, D. (2003). A perspective on judgement and choice: mapping bounded rationality. *American Psychologist, 58*(9), 697-720.

[102] Kahneman, D. (2011). *Thinking, fast and slow*. New York: Farrar, Straus and Giroux.

[103] Kahneman, D., & Frederick (2002). Representativeness revisited: attribute substitution in intuitive judgment. In T. Gilovich, D. Griffin, & De. Kahneman (Eds.), *Heuristics and biases: The psychology of intuitive judgment*. Cambridge: University Press.

[104] Kahneman, D., & Klein, G. (2009). Conditions for intuitive expertise: A failure to disagree. *American Psychologist, 64*(6), 515-526.

[105] Kaisari, M., & Patronis, T. (2010). So we decided to call "straight line"(...): mathematics students' interaction and negotiation of meaning in constructing a model of elliptic geometry. *Educational Studies in Mathematics, 75*(3), 253-269.

[106] Keene, K., Hall, W., & Duca, A. (2014). Sequence limits in calculus: using design research and building on intuition to support instruction. *ZDM Mathematics Education, 46*, 561-574.

[107] Kidron, I., & Dreyfus, T. (2010). Justification enlightenment and combining constructions of knowledge. *Educational Studies in Mathematics, 74*(1), 75-93.

[108] Kidron, I. (2011). Tacit models, treasured intuitions and the discrete-continuous interplay. *Educational Studies in Mathematics, 78*(1), 109-126.

[109] Kidron, I., & Dreyfus, T. (2014). Proof image. *Educational Studies in Mathematics, 87*, 297-321.

[110] Klein, G. (1993). A recognition-primed decision (RPD) model of rapid decision making. *Decision Making in Action: Models and Methods, 5*(4), 138-147.

[111] Kortenkamp, U., Monaghan, J., & Trouche, L. (2016). Johnathan M. Borwein (1951-2016): exploring, experiencing and experimenting in mathematics-an inspiring journey in mathematics. *Educational Studies in Mathematics, 93*(2), 131-136.

[112] Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom two episodes and related theoretical abductions. *Journal of Mathematical Behavior, 26*, 60-82.

[113] Kuster, G., Johnson, E., Rupnow, R., & Wilhelm, A. (2019). The Inquiry-Oriented Instructional Measure. *International Journal of Research in Undergraduate Mathematics Education, 5,* 183-204.

[114] Lakatos, I. (2015). *Proofs and refutations: The logic of mathematical discovery*. Cambridge University Press. (Original work published 1976)

[115] Larkin, J., McDermott, J., Simon, D.P., & Simon, H. A. (1980). Expert and novice performance in solving physics problems. *Science, 208*(4450), 1335-1342.

[116] Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in abstract algebra. *The Journal of Mathematical Behavior, 32,*(4), 693-711.

[117] Lem, S. (2015). The intuitiveness of the law of large numbers. *ZDM Mathematics Education, 47*, 783-392.

[118] Leron, U., & Hazzan, O. (2009). Intuitive vs analytical thinking: four perspectives. *Educational Studies in Mathematics, 71*(3), 263-278.

[119] Leung, I. (2008). Teaching and learning of inclusive and transitive properties among quadrilaterals by deductive reasoning with the aid of SmartBoard. *ZDM Mathematics Education, 40*, 1007-1021.

[120] Lockwood, E., Johnson, E., & Larsen, S. (2013). Developing instructor support materials for an inquiry-oriented curriculum. *The journal of Mathematical Behavior, 32*(4), 776-790.

[121] Lubin, A., Simon, G., Houdé, O., & Neys, W. D. (2015). Inhibition, conflict detection, and number conservation. *ZDM Mathematics Education, 47*, 793-800.

[122] Malaspina, U., & Font, V. (2009). Optimizing intuition. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*(Vol. 4, pp. 81-88).

[123] Malaspina, U., & Font, V. (2010). The role of intuition in the solving of optimization problems. *Educational Studies in Mathematics, 75*(1), 107-130.

[124] Mariotti, M., & Pedemonte, B. (2019). Intuition and proof in the solution of conjecturing problems'. *ZDM Mathematics Education, 51*, 759-777.

[125] Martínez, S., Guíñez, F., Zamora, R., Bustos, S., & Rodríguez, B. (2020). On the instructional model of a blended learning program for developing mathematical knowledge for teaching. *ZDM Mathematics Education, 52*, 877-891.

[126] Mason, J., & Watson, A. (2001). Getting students to create boundary examples. *MSOR Connections, 1*(1), 9-11.

[127] Melhuish, K. (2018). Three conceptual replication studies in group theory. *Journal for Research in Mathematics Education, 49*(1), 9-38.

[128] Michener, E. (1978). Understanding understanding mathematics. *Cognitive Science, 2*(4), 361-383.

[129] Miller, A. (1992). Scientific creativity: a comparative study of Henri Poincaré and Albert Einstein. *Creativity Research Journal, 5*(4), 385-418.

[130] Mills, M. A framework for example usage in proof presentations. *The Journal of Mathematical Behavior, 33*, 106-118.

[131] Noddings, N., & Shore, P. (1984). *Awakening in the inner eye: Intuition in education*. New York, NY: Teachers College Press.

[132] QSR International Pty Ltd. (2018). NVivo 12, https://www.qsrinternational.com/nvivo-qualitative-data-analysis-software/home

[133] Núñez, R., Edwards, L., & Matos, J. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational studies in mathematics, 39*(1), 45-65.

[134] Otte, M. (1990). Intuition and formalism in mathematical proof. *Interchange, 21*(1), 59-64.

[135] Papert, S. (1991). Situating constructionism. In I. Harel & P. Seymour (Eds.), *Constructionism*. Norwood, NJ: Ablex.

[136] Pearce, J. Similarity and discrimination: a selective review and a connectionist model. *Psychological Review, 101*(4), 587-607.

[137] Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics, 66*, 23-41.

[138] Peirce, C. (1991). Peirce on signs: writings on semiotic. UNC Press Books.

[139] Piaget, J. (1970). *Structuralism* (C. Maschler Translator). New York: Basic. (Original work published 1968)

[140] Pinto, A. (2019). Variability in the formal and informal content instructors convey in lectures. *Journal of Mathematical Behavior, 54*, 100680.

[141] Poincaré, H. (1914). *Science and Method. Translated by Francis Maitland with a pref. by Bertrand Russel.* England: T. Nelson.

[142] Poincaré, H. (1969). Intuition and logic in mathematics. *The Mathematics Teacher, 62*(3), 205-212.

[143] Policastro, E. (1995). Creative intuition: An integrative review. *Creativity Research Journal, 8*(2), 99-113.

[144] Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM The International Journal on Mathematics Education, 40*(2), 165-178.

[145] Przenioslo, M. (2004). Images of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics, 55*, 103-132.

[146] Raftopoulos, A. (2002). The spatial intuition of number and the number line. *Mediterranean Journal for Research in Mathematics Education, 1*(2), 17-36.

[147] Raidl, M., & Lubart, T. (2001). An empirical study of intuition and creativity. *Imagination, Cognition, and Personality, 20*(3), 217-230.

[148] Raman, M. (2002) Coordinating informal and formal aspects of mathematics: student behavior and textbook messages. *The Journal of Mathematical Behavior, 21*(2), 135-150.

[149] Rasmussen, C. (2001). New directions in differential equation: a framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*, 55-87.

[150] Rasmussen, C., Kwon, O., Allen, K., Marrongelle, K. & Burtch, M. (2006). Capitalizing on advances in mathematics and K-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. *Asia Pacific Education Review, 7*(1), 85-93.

[151] Reber, A. (1989). Implicit learning and tacit knowledge. *Journal of Experimental Psychology: General 118*(3), 219-235.

[152] Ross, A. A., & Onwuegbuzie, A. J. (2012). Prevalence of mixed methods research in mathematics education. *The Mathematics Educator, 22*(1), 84-113.

[153] Ross, A. A., & Onwuegbuzie, A. J. (2014). Complexity of quantitative analyses used in mixed research articles published in a flagship mathematics education journal. *International Journal of Multiple Research Approaches, 8*(1), 63-73.

[154] Rowland, T. (2002). Generic proofs in number theory. In S. Campbell & R. Zaskis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 157-183). New Jersey: Ablex Publishing Corporation.

[155] Russell, C. (2005). An overview of the integrative research review. *Progress in transplantation, 15*(1), 8-13.

[156] Sadler-Smith, E. (2015). Wallas' four-stage model of the creative process: more than meets the eye? *Creativity Research Journal, 27*(4), 342-352.

[157] Sadler-Smith, E. (2016). What happens when you intuit?: understanding human resource practitioners' subjective experience of intuition through a novel linguistic method. *Human Relations, 69*(5), 1069-1093.

[158] Savic, M. (2015). The incubation effect: how mathematicians recover from proving impasses. *Journal of Mathematical Behavior, 39*, 67-78.

[159] Savic, M., Gulden, K., Gail, T., El Turkey, H., & Naccarato, E. (2017). Formative assessment of creativity in undergraduate mathematics: using a creativity-in-progress rubric (CPR) on proving. In R. Leikin & B. Sriraman (Eds.), *Creativity and Giftedness: Interdisciplinary Perspectives from Mathematics Education*, 23-46. Cham: Springer.

[160] Schneider, M. (1992). On learning the rate of instantaneous change. *Educational Studies in Mathematics, 23*(4), 317-350.

[161] Selden, A., & Lim, K. Continuing discussion of mathematical habits of mind. *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Columbus, Ohio*.

[162] Selden, A., & Selden, J. (2013). Proof and problem solving at the university level. *The Mathematics Enthusiast, 10*(1), 303-334.

[163] Selden, J., & Selden, A. (2007). *Teaching proving by coordinating aspects of proofs with students' abilities.* Technical Report. No. 2007-2. Online Submission.

[164] Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioural schemas and the proving process. *International Journal of Mathematical Education in Science and Technology, 41*(2), 199-215.

[165] Selden, J., Selden, A., & McKee, K. (2008). The role of nonemotional cognitive feelings in constructing proofs. *Proceedings of the 11th Annual conference on Research in Undergraduate Mathematics Education*.

[166] Semadeni, Z. (2008). Deep intuition as a level in the development of the concept image. *Educational Studies in Mathematics, 68,* 1-17.

[167] Simon, H. (1986). The information processing explanation of gestalt phenomena. *Computers in Human Behavior, 2*, 241-255.

[168] Simon, H. (1992). What is an explanation of behavior? *Psychological Science 3*(3), 150-161.

[169] Sinclair, M. (2010). Misconceptions about intuition. *Psychological Inquiry, 21*, 378-386.

[170] Sinclair, M., & Ashkanasy, N. (2005). Intuition: myth or a decision-making tool? *Management learning, 36*(3), 353-370.

[171] Sirotic, N., Zazkis, R., & Zazkis, A. (2007). Irrational numbers: the gap between formal and intuitive knowledge. *Educational Studies in Mathematics, 65*(1), 49-76.

[172] Skemp, R. (1979). *Intelligence, learning, and action: A foundation for theory and practice in education.* Chichester: Wiley.

[173] Smith, E., & DeCoster (1999). Associative and rule-based processing: A connectionist interpretation of dual-process models. In S. Chaiken and Y. Trope (Eds.), *Dual-process theories in social psychology* (pp. 323-336). The Guilford Press.

[174] Sriraman, B. (2009). The characteristics of mathematical creativity. *ZDM Mathematics Education, 41*, 13-27.

[175] Sriraman, B., Haavold, P., & Lee, K. (2013). Mathematical creativity and giftedness: a commentary on and review of theory, new operational views, and ways forward. *ZDM The International Journal on Mathematics Education, 45*, 215-225.

[176] Stanovich, K., West, R., & Toplak, M. (2014). Rationality, intelligence, and the defining features of Type 1 and Type 2 processing. *Dual-process theories of the social mind, 80-91*

[177] Star, J. R., & Pollack, C. (2015). Inhibitory control and mathematics learning: definitional and operational considerations. *ZDM Mathematics Education, 47*, 859-863.

[178] Stewart, S., Thompson, C., & Brady, N. (2017). Navigating through the mathematical world: Uncovering a geometer's thought processes through his handouts and teaching journals.*The 10th Congress of the European Society for Research in Mathematics Education (CERME 10)*. Dublin, Ireland. ¡hal-01941311¿

[179] Stout, G. (1934). Self-evidence and matter of fact. *Philosophy, 9*(36), 389-404.

[180] Tall, D. (1999). The Cognitive Development of Proof: Is Mathematical Proof For All or For Some? In Z. Usiskin (Ed.), Developments in School Mathematics Education Around the World, Vol. 4. Reston, Virginia: NCTM, 117-136.

[181] Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

[182] Tanguay, D., & Venant, F. (2016). The semiotic and conceptual genesis of angle. *ZDM The International Journal on Mathematics Education, 48*, 875-894.

[183] Thomas, M. (2015). Inhibiting intuitive thinking in mathematics education. *ZDM Mathematics Education, 47*, 865-876.

[184] Thurston, W. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society, 30*(2), 161-177.

[185] Tirosh, D., Fischbein, E., Graeber, A., & Wilson, J. (1998). Prospective elementary teachers' conceptions of rational numbers. Retrieved from http://jwilson.coe.uga.edu/Texts.Folder/Tirosh/Pros.El.Tchrs.html

[186] Torraco, R. (2005). Writing integrative literature reviews: guidelines and examples. *Human resource development review, 4*(3), 356-367.

[187] Torraco, R. (2016). Writing integrative literature reviews: Using the past and present to explore the future. *Human resource development review, 15*(4), 404-428.

[188] Toulmin, S. E. (2003). *The uses of argument(updated edition of the 1958 book)*. Cambridge: Cambridge University Press.

[189] Tsamir, P., & Ovodenko, R. (2013). University students' grasp of inflection points. *Educational Studies in Mathematics, 83*(3), 409-427.

[190] Tversky, A., & Kahneman, D. (1981). The framing of decisions and the psychology of choice. *Science, New Series, 211*(4481), 453-458.

[191] Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2013). Educated adults are still affected by intuitions about the effect of arithmetical operations: evidence from a reaction-time study. *Educational Studies in Mathematics, 82*(2), 323-330.

[192] Vergnaud, G. (1994). Multiplicative conceptual field: what and why? In G. Harel & J. Confrey(Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41-59). Albany: State University of New York Press.

[193] Wallas, G. (1926). *The Art of Thought*. New York: Harcourt, Brace and Company.

[194] Watson, J., & Callingham, R. (2013). Likelihood and sample size: the understandings of students and their teachers. *The Journal of Mathematical Behavior, 32*, 660-672.

[195] Watson, A., & Mason (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, New Jersey: Lawrence Erlbaum Associates.

[196] Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *Problems, Resources, and Issues in Mathematics Undergraduate Studies, 22*(8), 577-599.

[197] Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*, 209-234.

[198] Weber, K., & Alcock, L. (2009). Proof in advanced mathematics classes: semantic and syntactic reasoning in the representation system of proof. *In Teaching and learning proof across the grades: A K-16 perspective* (pp. 323-338). Routledge Taylor & Francis Group.

[199] Webster, D.M., & Krglanski, A.W. (1994). Individual differences in need for cognitive closure. *Journal of Personality and Social Psychology, 67*(6), 1049-1062.

[200] Wilder, R. (1967). The role of intuition. *Science, New Series, 156*(3775), 605-610.

[201] Wilensky, U. (1995). Paradox, programming, and learning probability: a case study in a connected mathematics framework. *Journal of Mathematical Behavior, 14*, 253-280.

[202] Wilkie, K. (2019). Investigating secondary students' generalization, graphing, and construction of figural patterns for making sense of quadratic functions. *Journal of Mathematical Behavior, 54*, 100689.

[203] Williams, S., & Leatham, K. (2017). Journal quality in mathematics education. *Journal for Research in Mathematics Education, 48*(4), 369-396.

[204] Wittmann, E (1981). The complementary roles of intuitive and reflective thinking in mathematics teaching. *Educational Studies in Mathematics, 12*(3), 389-397.

[205] Young, J., Denton, W., & Mitchell, U. (1911). *Lectures on fundamental concepts of algebra and geometry*. Norwood: Macmillan Company and Norwood Press.

[206] Zagorianakos, A., & Shvarts, A. (2015). The role of intuition in the process of objectification of mathematical phenomena from a Husserlian perspective: a case study. *Educational Studies in Mathematics, 88*(1), 137-157.

[207] Zazkis, R., Liljedahl, P., & Chernoff, E. (2008). The role of examples in forming and refuting generalizations. *ZDM The International Journal on Mathematics Education, 40*(1), 131-141.

Chapter 3 Abstract Algebra Education Research References

[1] Agustyaningrum, N., Sari, R. N., Abadi, A. M., & Mahmudi, A. (2021). Dominant factors that cause students' difficulties in learning abstract algebra: A case study at a university in Indonesia. *International Journal of Instruction, 14*(1), 847-866.

[2] Asiala, M., Dubinsky, E., Mathews, D. M., Morics, S., & Oktac, A. (1997). Development of students' understanding of cosets, normality, and quotient groups. *Journal of Mathematical Behavior, 16*(3), 241-309.

[3] Bergman, A.M., & French T.A. (2019). Developing an active approach to chemistry-based group theory. In M.H. Towns, K. Bain, & J.G. Rodriguez (Eds.), *It's Just Math: REsearch on Students' Understanding of Chemistry and Mathematics (pp. 213-237).* American Chemical Society.

[4] Bergman, A.M. (2020). *Identifying a starting point for the guided reinvention of the classification of chemically important symmetry groups.* (Doctoral Dissertation). Retrieved from https://pdxscholar.library.pdx.edu/cgi/viewcontent.cgi?article=6550&context=open_access_etds

[5] Cook, J.P. (2012). *A guided reinvention of ring, integral domain, and field.* (Doctoral Dissertation). Retrieved from https://shareok.org/bitstream/handle/11244/319010/Cook_ou_0169D_10846.pdf?sequence=1

[6] Cook, J.P. (2018). An investigation of an undergraduate student's reasoning and the zero-product property. *The Journal of Mathematical Behavior*, 49, 95-115.

[7] Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational studies in Mathematics, 27*(3), 267-305.

[8] Fukawa-Connelly, T., Johnson, E., & Keller, R. (2016). Can math education research improve the teaching of abstract algebra. *Notices of the American Mathematical Society, 63*, 276-281.

[9] Gravemeijer, K. (2004). Local instruction theories as a means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*(2), 105-128.

[10] Hazzan, O., & Leron, U. (1996). Students' use and misuse of mathematical theorems: The case of Lagrange's theorem. *For the Learning of Mathematics, 16*(1), 23-26.

[11] Ioannou, M., & Nardi, E. (2009). Engagement, abstraction and visualisation: Cognitive and emotional aspects of year 2 mathematics undergraduates' learning experience in Abstract Algebra. *Proceedings of the British Society for Research into Learning Mathematics, 29*(2), 35-40.

[12] Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education, 51*(4), 504-516.

[13] Johnson, E., Caughman, J., Fredericks, J., & Gibson, L. (2013). Implementing inquiry-oriented curriculum: From the mathematicians' perspective. *The Journal of Mathematical Behavior, 32*(4), 743-760.

[14] Larsen, S. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *Journal of Mathematical Behavior, 32*(4), 712-725.

[15] Larsen, S., & Lockwood, E. (2013). A local instructional theory for the guided reinvention of the quotient group concept. *The Journal of Mathematical Behavior, 32*, 726-742.

[16] Lockwood, E., Johnson, E., & Larsen, S. (2013). Developing instructor support materials for an inquiry-oriented curriculum. *The journal of Mathematical Behavior, 32*(4), 776-790.

[17] Melhuish, K. (2018). Three conceptual replication studies in group theory. *Journal for Research in Mathematics Education, 49*(1), 9-38.

[18] Melhuish, K., Lew, K., Hicks, M., & Kandasamy, S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior, 60*, 100806.

[19] Mena-Lorca, A., & Parraguez, A. M. M. (2016). Mental constructions for the group isomorphism theorem. *Mathematics Education, 11*(2), 377-393.

[20] Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric' images and multi-level abstractions in group theory. *Educational Studies in Mathematics, 43*(2), 169-189.

[21] Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: an overview on achievements and challenges. *ZDM Mathematics Education, 47*(6), 877-891.

[22] Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *Journal of Mathematical Behavior, 62*, 100867.

[23] Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational studies in mathematics, 48*(1), 101-119.

Remaining References

[1] Alibali, M., & Goldin-Meadow (1993). Gesture-speech mismatch and mechanisms of learning: what the hands reveal about a child's state of mind. *Cognitive psychology, 25*, 468-523.

[2] Dubinsky, E., & McDonald, M.A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton, M. Artigue, U. Kirchgr´aber, J. Hillel, M. Niss, and A. Schoenfeld (eds) The teaching and learning of mathematics at university level (pp. 275-282). New ICMI Study Series, vol 7. Dordrecht: Springer.

[3] Betsch, C. (2008). Chronic preferences for intuition and deliberation in decision making: Lessons learned about intuition from an individual differences approach. *Intuition in judgement and decision making*, 231-248.

[4] Betsch, C., & Kunz, J.J. (2008). Individual strategy preferences and decisional fit. *Journal of Behavioral Decision Making, 21*(5), 532-555.

[5] Bezdek, J.C. (1974). Cluster validity with Fuzzy sets. *Journal of Cybernetics, 3*,58-73.

[6] Bezdek, J.C., Coray, C., Gunderson, R., & Watson, J. (1981). Detection and characterization of cluster substructure. I. Linear structure: Fuzzy c-lines*. *Society for Industrial and Applied Mathematics Journal 40*(2), 339-357.

[7] Bezdek, J., Ehrlich, R., & Full, W. (1984). FCM: The fuzzy c-means clustering algorithm. *Computers & Geosciences, 10*(2-3), 191-203.

[8] Neubert, K., & Brunner, E. (2007). A studentized permutation test for the non-parametric Behrens-Fisher problem. *Computational Statistics and Data Analysis, 51*, 5192-5204.

[9] Campello, R.J., & Hruschka, E.R. (2006). A fuzzy extension of the silhouette width criterion for cluster analysis. *Fuzzy Sets and Systems, 157*(21), 2858-2875.

[10] Capra, M. G. (2005). Factor analysis of card sort data: an alternative to hierarchical cluster analysis. In *Proceedings of the Human Factors and Ergonomics Society Annual Meeting, 49*(5), 691-695. Los Angeles: SAGE Publications.

[11] Chervinskaya, K. R., & Wasserman, E. L. (2000). Some methodological aspects of tacit knowledge elicitation. *Journal of Experimental & Theoretical Artificial Intelligence, 12*(1), 43-55.

[12] Chi, M. (1997). Quantifying qualitative analyses of verbal data: A practical guide. *The Journal of the Learning Sciences, 6*(3), 271-315.

[13] Creswell, J.W., & Plano Clark, V.L. (2018). *Designing and conducting mixed methods research*(Third ed.). Thousand Oaks: SAGE Publications.

[14] Cayley, A. (1878). Desiderata and suggestions: No. 2. The theory of groups: graphical representation. *American Journal of Mathematics, 1*(2), 174-176.

[15] Davydov, V. (1990). *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula. Soviet Studies in Mathematics Education* (Vol. 2). National Council of Teachers of Mathematics, Reston, VA.

[16] Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2015). The nested epistemic actions model for Abstraction in Context: Theory as methodological tool and methodological tool as theory. In A. Bikner-Ahsbahs, C. Knipping, and N. Presmeg (eds.), Approaches to qualitative research in mathematics education (pp. 185–217). Dordrecht, The Netherlands: Springer.

[17] Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103-131.

[18] Dummit, D.S., & Foote, R. M. (2004). *Abstract Algebra, 3rd ed.,* Hoboken: John Wiley.

[19] Eli, J.A., Mohr-Schroeder, M.J., & Lee, C.W. (2011). Exploring mathematical connections of prospective middle grades teachers through card-sorting tasks. *Mathematics Education Research Journal, 23*(3), 297-319.

[20] Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior, 48*, 152-167.

[21] Epstein, S., Pacini, R., Denes-Raj, V., & Heier, H. (1996). Individual differences in intuitive-experiential and analytical-rational thinking styles. *Journal of personality and social psychology, 71*(2), 390-405.

[22] Fonger, N. L. (2019). Meaningfulness in representational fluency: An analytic lens for students' creations, interpretations, and connections. *The Journal of Mathematical Behavior, 54,* 100678.

[23] Fossum, T., & Haller, S. (2005). Measuring card sort orthogonality. *Expert Systems, 22*(3), 139-146.

[24] Goldin, G.A. (2002). Affect, meta-affect, and mathematical belief structures. *In Beliefs: A hidden variable in mathematics education?*(pp. 59-72). Dordrecht: Springer.

[25] Govindasamy, K., & Velmurugan, T. (2018). Analysis of student academic performance using clustering techniques. *International Journal of Pure and Applied Mathematics, 119*(15), 309-323.

[26] Grootenboer, P., & Marshman, M. (2016). The affective domain, mathematics, and mathematics education. In *Mathematics, affect, and learning*(pp. 13-33). Singapore: Springer.

[27] Guetterman, T. (2017). What distinguishes a novice from an expert mixed methods researcher? *Qual & Quant, 51*, 377-398.

[28] Halverscheid, S. (2008). Building a local conceptual framework for epistemic actions in a modelling environment with experiments. *ZDM, 40*(2), 225-234.

[29] Hillen, A.F., & Malik, L. (2013). Sorting out ideas about function. *The mathematics Teacher, 106*(7), 526-533.

[30] Johansen, M. W., & Misfeldt, M. (2018). Material representations in mathematical research practice. *Synthese,*1-21.

[31] Judson, T. W. (2018). *Abstract algebra: theory and applications*. http://abstract.pugetsound.edu/

[32] Kelly, G. A. (2020). *The Psychology of Personal Constructs*. Routledge.

[33] Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M., & Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *ZDM, 40*(2), 247-264.

[34] Lajos, J., & Stewart, S. (2020). A tour of cognitive transformations of semiotic representations in advanced mathematical thinking. *Proc. 23rd Conference of the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education*, Boston, MA.

[35] Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics, 67*(3), 205-216.

[36] Leech, N.L., & Onwuegbuzie, A.J. (2009). A typology of mixed methods research designs. *Quality & Quantity, 43*(2), 265-275.

[37] Lipton, R. J., & Regan, K. W. (2013). Enrico Bombieri: On Intuition. In People, Problems, and Proofs (pp. 65-69). Springer-Verlag: Berlin, Heidelberg.

[38] Maracci, M. (2008). Combining different theoretical perspectives for analyzing students' difficulties in vector spaces theory. *ZDM, 40*(2), 265-276.

[39] McGee, D. L., & Moore-Russo, D. (2015). Impact of explicit presentation of slopes in three dimensions on students' understanding of derivatives in multivariable calculus. *International Journal of Science and Mathematics Education (13)*(2), 357-384.

[40] Merriam, S.B., & Tisdell, E.J. (2015). *Qualitative research: A guide to design and implementation*. San Francisco: John Wiley & Sons Inc.

[41] Nayak, J., Naik, B., & Behera, H. (2015). Fuzzy C-means (FCM) clustering algorithm: a decade of review from 2000 to 2014. *Computational intelligence in data mining volume 2*, 133-149.

[42] Norris, P., & Epstein, S. (2011). An experiential thinking style: Its facets and relations with objective and subjective criterion measures. *Journal of personality, 79*(5), 1043-1080.

[43] Pacini, R., & Epstein, S. (1999). The relation of rational and experiential information processing styles to personality, basic beliefs, and the ratio-bias phenomenon. *Journal of personality and social psychology, 76*(6), 972.

[44] Peirce, C. S. (1992). *The essential Peirce: selected philosophical writings* (Vol. 2). Bloomington: Indiana University Press.

[45] Piatetski-Shapiro, I. (1983). Complex Representations of GL(2,K) for Finite Fields K (Vol. 16). American Mathematical Society.

[46] Plano Clark, V. (2019). Meaningful integration within mixed methods studies: Identifying why, what, when, and how. *Contemporary Educational Psychology, 57*, 106-111.

[47] Poincaré, H. (1969). Classics in mathematics education: Intuition and logic in mathematics. *The Mathematics Teacher, 62*(3), 205-212.

[48] Pretz, J. E., Brookings, J. B., Carlson, L. A., Humbert, T. K., Roy, M., Jones, M., & Memmert, D. (2014). Development and validation of a new measure of intuition: The types of intuition scale. *Journal of Behavioral Decision Making, 27*(5), 454-467.

[49] Rawashdeh, M., & Ralescu, A. Crisp and fuzzy cluster validity: generalized intra-inter silhouette index. In *Proceedings of the IEEE Annual Meeting North American Fuzzy Information Processing Society, 2012*, 11-18.

[50] Rugg, G., & McGeorge, P. (2005). The sorting techniques: a tutorial paper on card sorts, picture sorts and item sorts. *Expert Systems, 22*(3), 94-107.

[51] Tao, T. (2010). Cayley graphs and the geometry of groups. https://terrytao.wordpress.com/2010/07/10/cayley-graphs-and-the-geometry-of-groups/

[52] Tashakkori, A., & Creswell, J.W. (2007). The new era of mixed methods. *Journal of Mixed Methods Research, 1*(1), 3-7.

[53] Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics, 34*(2), 97-129.

[54] Whaley, A. L., & Longoria, R. A. (2009). Preparing card sort data for multidimensional scaling analysis in social psychological research: a methodological approach. *The Journal of Social Psychology, 149*(1), 105-115.

[55] Zadeh, L. (1965). Fuzzy sets. *Information and Control, 8*, 338-353.

[56] Sandoval, I., & Possani, E. (2016). An analysis of different representations for vectors and planes in $\mathbb{R}^3$. *Educational Studies in Mathematics, 92*(1), 109-127.

[57] Serre, J. (1977). Linear Representations of Finite Groups, Graduate texts in Mathematics, 42. New York: Springer-Verlag.

[58] Skemp, R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher, 26*(3), 9-15.

[59] Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Eds.), *Advanced Mathematical Thinking*(pp. 25-41). Dordrecht: Springer.

[60] Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational studies in mathematics, 56*(2), 209-234.

Appendices

A. 1 Semi-structured interviews

A. 1.1 Part I: Card-sort task-based interview protocol

(Read consent to the participant. If consent is given to audio and video record, then move on with the study and make sure the record feature is on in Zoom. If consent is not given to audio and video record, then terminate study.)

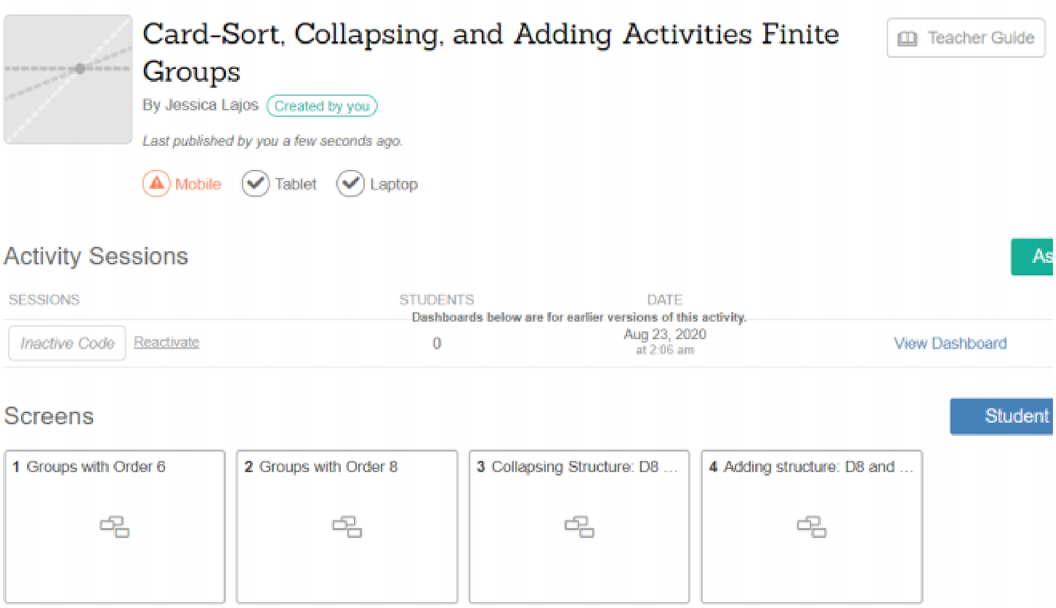
Researcher: "In your Zoom chat you will see a link, click on it."

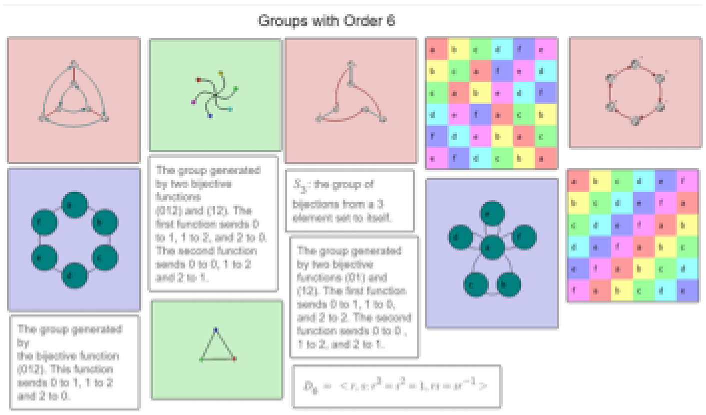
(Sends card-sort activity link through zoom chat and participant clicks on the link to see card sort activities as shown below. The card sort-activities are listed in the following order: (1) groups of order 6, (2) groups of order 8, (3) collapsing structure, and (4) adding structure.)

Card-sort activity link:
https://teacher.desmos.com/activitybuilder/custom/5f41f38e151b792d94ebc6ff)

Researcher: "Begin by selecting card-sorting activity (1) titled groups of order 6. A set of cards will now be displayed on your screen. Each card that you see represents some finite group of order 6. Take about 10 minutes to look at and get acquainted with these cards."
(Ask participant to share the screen that displays their view of the card-sort activity. The cards will be randomly scattered on the screen for each activity as shown below.)

Card images are from Carter, N. (2006). Group Explorer, Version 2.0 [computer software]. Waltham, MA.
Researcher: "What are some of your first thoughts about the cards?"





Researcher: Now I want you to form piles of cards that represent the same group. Create a maximal card pile for each group you see. A maximal card pile for a group G is a set of cards M where each card in M represents G and is the largest possible pile that you can make for G at this time.

(Participants are given time to create card piles and instructed to think aloud as they form the piles)

(Researcher asks the following questions for each card pile that the participant makes)

1. ***Researcher***: “First, tell me about each of the cards in your pile separately? What

information can be extracted from each of the cards?”

2. ***Researcher***: “Why do these cards go in the same pile? How are they related? Explain your reasoning.”

3. ***Researcher***: “Are there any additional ways you like to represent or think about a group of order (insert order) that is not given by a card on the board?” (If the participant answers yes, then allow the participant to make their own card and add it to the card set)

4. ***Researcher***: “Are there any other cards you want to add or remove from this pile? Is this your maximal card pile? Explain.”
   (If the participant affirms that their pile is maximal, then move on to the next activity.)
   (If the participant states that their pile is not maximal or is hesitant, move on to question 4.)

5. ***Researcher***: “I see that you are unsure. Take some more time to think about your piles, decide whether or not you want to make any modifications to your piles.”
   (Once the participant is finished taking time to think more and possibly make modifications repeat question 3. Continue this loop until the participant affirms that their pile is maximal or are unable to make any further modifications.)

Follow up with the mode of aquisition survey for each card-collection of interpretations. (End this interview session and schedule a follow up session with the participant for activity (3) and (4) the following week.)

A. 1.2 Part II: Collapsing and Adding structure task-based interview protocol

(Read oral consent to the participant. If consent is given to audio and video record, then move on with the study and make sure the record feature is on in Zoom. If consent is not given to audio and video record then terminate study.)

1. ***Researcher***: “In your Zoom chat you will see a link, click on it.” (Sends card-sort activity link through zoom chat and participant clicks on the link to see card sort activities. Also instruct them to pull up group explorer so that they can manipulate the cards dynamically)

   Collapsing structure task-based interview protocol

Activity link:
https://teacher.desmos.com/activitybuilder/custom/5f41f38e151b792d94ebc6ff)

2. ***Researcher***: "Click on the link and now select activity (3) titled Collapsing Structure: D4 (dihedral group of order 8) and Q8."
   (The researcher will set up activity by arranging cards into piles as shown below.)

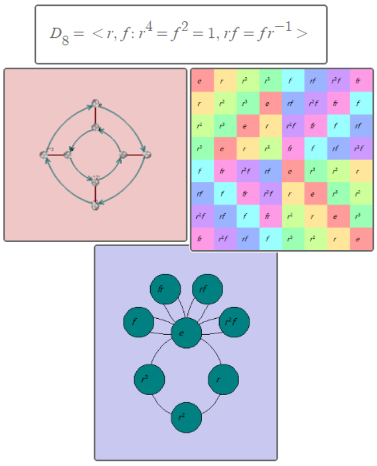
3. ***Researcher***: "Are the groups D4 [dihedral group of order eight] and Q8 isomorphic or not? How can you tell?" (If a wrong answer is given try to guide the student to recognize the conflict. After they provide an answer reiterate that D4 and Q8 are not isomorphic before moving on.)

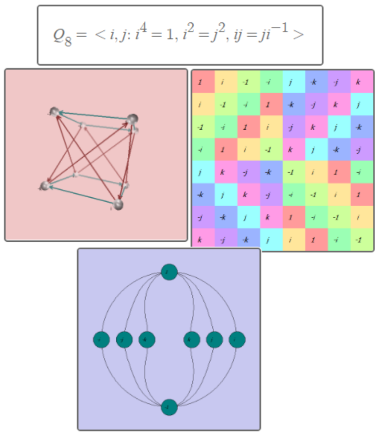
4. ***Researcher***: "Is there some way that you can modify both D4 by losing information or collapsing some structure and modify Q8 by losing information or collapsing some structure so that the resultant modification to D4 is a group that is isomorphic to the resultant modification to Q8?"
   (Remind the participant to think aloud and give them space to think. Tell them to use a piece of notebook paper if they need to work something out.)

5. ***Researcher***: "Describe what losing information or collapsing structure means to you in the context of this abstract algebra task?" (Tighten task using a mapping diagram and insert homomorphism or quotient map depending on what aligns with their response to the previous question.) (If participant provides a predominant response using a Cayley digraph such as collapsing parts of the digraph then follow up with switch prompt question 1. If the participant provides a predominantly formal response, such as using the group presentation, using quotient diagrams, and using language like homomorphism, quotient, and lists of cosets go to switch prompt question 2 and 3. If the participant provides a nice mixed response that illustrates their flexible use in language, then move on to question 4 and activity 4.)

Switch register prompts for collapsing structure task:

1. ***Researcher***: "Can you translate what it means to lose information or collapse structure like you did in the case of Dihedral group of order eight and Q8 using more formal language? What are the explicit quotient maps? What is the explicit isomorphism map? Where you looking at any of the cards to formulate an answer





to the previous question? Which cards? Are there any underlying assumptions that you are using that need to be formalized?"

2. ***Researcher***: "Can you translate what it means to lose information or collapse structure like you did in the case of the Dihedral group of order eight and Q8 using more visual intuitive language? Were you looking at any of the cards to formulate an answer to the previous question? Which cards?"

3. ***Researcher***: "Can you translate what it means to collapse structure like you did in the case of the dihedral group of order eight and Q8 using the Cayley digraphs?"

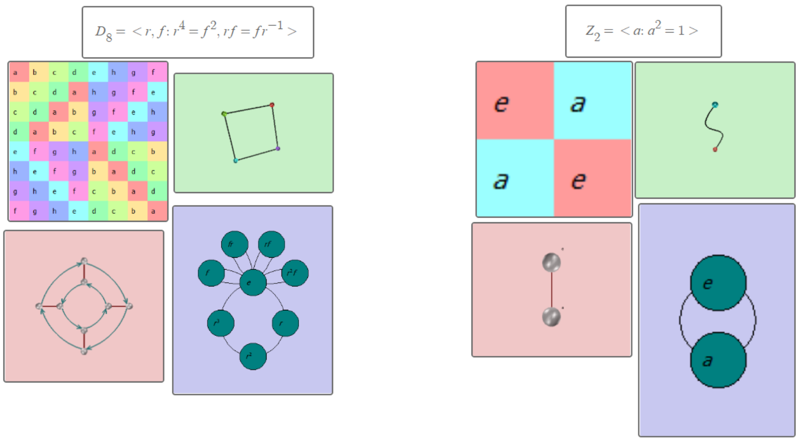
4. ***Researcher***: "What was the most difficult part of this task?"

(Instruct the participant to take a picture of their work and solutions and email it to the Principal Investigator. Have the participant take a break and move on to activity 4, the adding structure task, by hitting the next button.)
Adding structure task-based interview protocol (The researcher will set up activity 4 by arranging cards into piles as shown below.)

1. ***Researcher***:"Last time you were collapsing structure, in this task you will be adding or combining structure. Can you build a larger group by combining the dihedral group of order eight and Z2?"

   (Remind the participant to think aloud and give them space to think. Tell them to use a piece of notebook paper if they need to work something out.)

   (If participant provides a predominantly visual intuitive response, such as combining both Cayley digraphs in a way that gives a visual cross product then follow up with question 1 below. If the participant provides a response, such as using



the group presentation, multiplication table and taking the formal cross product without the use of other representations go to question 2 below. If the participant provides a nice mixed response that illustrates their flexible use in language, then end the study and thank the participant for their time.)

Switch register prompts for adding structure task:

1. ***Researcher***: “Can you translate what it means to combine the groups using more formal language?” “Were you looking at any of the cards to formulate an answer to the previous question? Which cards/representations are you thinking about?” “Is there a common name for the group you are trying to construct?”

2. ***Researcher***: “Can you translate what it means to combine the groups using language that describes what is visually going on?” Were you looking at any of the cards to formulate an answer to the previous question? Which cards?”

3. ***Researcher***: “Can you translate what it means to add structure using the Cayley digraphs?” “Does the group you are constructing have a common name?”

4. ***Researcher***: “What was the most difficult part of this task?”

(If no progress made on D4 x Z2 (3 generators) on the adding structure task try Z2 x Z4 (2 generators))

(Instruct the participant to take a picture of work, scratch work, and solutions and email it to the Principal Investigator. Thank the participant for their time and conclude part II of the study.)

A. 1.3 Part III: Baseline questionnaire for group actions

Definition: A group action is a function $\cdot : G \times X \to X$ that maps (g, x) to $g \cdot x$ such that $e \cdot x = x$ and $g \cdot (h \cdot x) = gh \cdot x$ for all $g, h \in G$ and $x \in X$.

Definition: A group orbit for an element $x \in X$ is defined as $Orb(x) = \{g \cdot \in X : g \in G\}$.

Definition: A stabilizer for an $x \in X$ is defined as $stab(x) = \{g \in G : g \cdot x\} = x$.

(1) Translate the definition of a group action above in terms of homomorphism language.

(2) Give three examples of a group action.

(3) Describe a stabilizer in your own words.

(4) Describe a group orbit in your own words.

(5) Does the definition of a group orbit feel familiar to you? Does it remind you of an object that you are more comfortable with or have worked with before?

A. 2 Surveys

A. 2.1 Modes of Semiotic Representation Acquisition Survey

Mode of Acquisition	**Statement**
Sanctioned	1. I was directly taught or told how to represent or think about (insert group) this way be my instructor 2. I learned how to represent or think about (insert group) this way through social interactions with my peers, activities in class or outside of class with little direction from an instructor 3. I have never seen this way of representing (insert group) before, but I figured out what this card is telling me about the group during this study by spending time playing with it and looking at things from various angles.
Non-sanctioned	4. I created this representation or way of thinking about (insert group) prior to this study on my own outside of class by spending a lot of time playing with and looking at (insert group) from many different angles. 5. Other

A. 2.2 The Non-creative Versus Creative Forms of Intuition Survey with Confidence in Truth Items

	Instructions: Fill out the survey for each intuition that you recorded in your journal. Circle the number that measures how much you agree with each of the statements below.
1	I have never experienced this intuition before. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
2	This intuition feels familiar like something I have thought about or have come across before. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
3	The moment this intuition came to mind I had an intense and positive emotional reaction like feelings of euphoria, elation, or exhilaration. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
4	The moment this intuition came to mind I had little to no emotional reaction. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
5	I found it difficult to externalize the intuition held in my mind. I felt like I did not have a coherent language to express it. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
6	I found it easy to transfer the intuition held in my mind into an external verbal or written description. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
7	During this task I made sketches or drawings of the way I like to think about groups/related concepts that feels unique to me rather than something I picked up from others. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
8	I relied solely on the ways I have been taught by instructor or textbook to view groups/related concepts while working on this task. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
9	I had to use every day non-mathematical language or make drawings of what was in my mind in order to initially express my intuition. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
10	I mostly used standard mathematical terms and symbols that I was taught in class or textbooks to initially express my intuition. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
11	I got to a point of being stuck so I intentionally took a break from the task to let ideas simmer in my mind before I arrived at this intuition. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
12	I experienced this intuition about a problem while working for less than 15 minutes and without taking any breaks. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**

13	I am confident that my intuition is true **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**
14	I am so confident that my intuition is true that a rigorous proof of it just feels like busy work at this point. **Strongly Disagree 1 2 3 4 5 6 Strongly Agree**

A. 3 Univariate and Fuzzy C-Means Statistics

A. 3.1 Welch t test

```
Welch Two Sample t-test (significant results)
(two-sided and confid.level=0.95)
x-Jenni, y-Alex
#3 Emotional intensity
data:  x and y
t = 4.2762, df = 10, p-value = 0.001621
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
  0.6966425 2.2124484
sample estimates:
  mean of x mean of y
4.454545  3.000000

#4 Emotional intentsity

data:  x and y
t = -11, df = 7.1446, p-value = 9.793e-06
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
  -4.893169 -3.167438
sample estimates:
  mean of x mean of y
1.636364  5.666667

#7 Non-Sanct. rep cues
data:  x and y
t = 7.1157, df = 10, p-value = 3.234e-05
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
  2.560144 4.894401
sample estimates:
  mean of x mean of y
4.727273  1.000000

#11 Incubation
data:  x and y
t = 2.8025, df = 10.92, p-value = 0.01731
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
  0.4700011 3.9239383
sample estimates:
  mean of x mean of y
3.363636  1.166667

#12 Incubation*
data:  x and y
t = -2.7629, df = 10, p-value = 0.02003
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
  -3.2844561 -0.3519076
```

```
sample estimates:
  mean of x mean of y
3.181818  5.000000
```

A. 3.2 Permuted Brunner Munzel Test

```
Brunner Munzel Permutation Test (significant results)
>brunnermunzel.permutation.test(x,y,
alternative = c("two.sided"), force=TRUE)

x-Jenni, y-Alex
#3 Emotional intensity
data:  x and y
p-value = 0.009454
sample estimates:
  P(X<Y)+.5*P(X=Y)
0.09090909

#4 Emotional intentsity
data:  x and y
p-value = 8.08e-05
sample estimates:
  P(X<Y)+.5*P(X=Y)
1

#7 Non-Sanct. rep cues
data:  x and y
p-value = 0.0003232
sample estimates:
  P(X<Y)+.5*P(X=Y)
0.04545455

#11 Incubation period
data:  x and y
p-value = 0.07523
sample estimates:
  P(X<Y)+.5*P(X=Y)
0.2727273

> brunnermunzel.permutation.test(x,y,
alternative=c("greater"), force=TRUE)

x-Jenni, y-Alex
#3 Emotional intensity
data:  x and y
p-value = 0.006626
sample estimates:
P(X<Y)+.5*P(X=Y)
      0.09090909

#7 Non-Sanct. rep cues
data:  x and y
p-value = 8.08e-05
sample estimates:
```

```
P(X<Y)+.5*P(X=Y)
      0.04545455

#11 Incubation

data:  x and y
p-value = 0.05769
sample estimates:
P(X<Y)+.5*P(X=Y)
       0.2727273

brunnermunzel.permutation.test(x,y,
alternative=c("less"), force=TRUE)
 x-Jenni, y-Alex

#4 Emotional Intensity
data:  x and y
p-value = 8.08e-05
sample estimates:
P(X<Y)+.5*P(X=Y)
 1
```

A. 3.3 Clusters with validity indices NCC attributes

```
No. Clusters = 2, Weighting Exponent m=2
>library('fcm')
>install.packages("ppclust")
>install.packages("fclust")
>JAdataNCC <- read_excel
("C:/Users/Jessi Lajos/Downloads/JAdataNCC.xls")
>res.fcm <- fcm(JAdataNCC, centers=2)
>as.data.frame(res.fcm$u)[1:17,]
 Cluster 1 Cluster 2
1  0.5862537 0.4137463
2  0.3052886 0.6947114
3  0.6161819 0.3838181
4  0.6161819 0.3838181
5  0.6161819 0.3838181
6  0.1646148 0.8353852
7  0.4309691 0.5690309
8  0.1861474 0.8138526
9  0.4330003 0.5669997
10 0.2337922 0.7662078
11 0.2539677 0.7460323
12 0.8139089 0.1860911
13 0.8613089 0.1386911
14 0.8870995 0.1129005
15 0.6952607 0.3047393
16 0.8139089 0.1860911
17 0.8542094 0.1457906
>summary(res.fcm)\s
Number of data objects:  17

Number of clusters:  2

Crisp clustering vector:
 [1] 1 2 1 1 1 2 2 2 2 2 2 1 1 1 1 1 1

Initial cluster prototypes:
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI*
Cluster 1       4           3           4           1
Cluster 2       2           5           6           1
          #5 EaseExt #6 EaseExt* #7 NonSanctCues #8 SanctCues*
Cluster 1       1           6           2           3
Cluster 2       1           6           5           1
          #9 NonSanctOut #10 SanctOut* #11 IncubP #12 IncubP*
Cluster 1       1           5           2           5
Cluster 2       3           5           6           1

Final cluster prototypes:
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI*
Cluster 1  1.765483   5.071278    3.361064       4.315312
Cluster 2  2.145535   4.275801    4.711482       1.708640
          #5 EaseExt #6EaseExt* #7 NonSanctCues #8 SanctCues*
Cluster 1 2.296533    4.637798    2.312194       2.002540
Cluster 2 2.838854    3.551115    4.590355       2.204996
          #9 NSanctOt #10 SanctOt* #11 IncubP #12 IncubP*
```

```
Cluster 1 2.896175    3.585264    1.394381       4.877264
Cluster 2 3.456567    3.404698    4.391023       2.308405

Distance between the final cluster prototypes
          Cluster 1
Cluster 2  32.02712

Difference between the initial and final cluster prototypes
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI*
Cluster 1 -2.2345170  2.0712783   -0.6389355    3.3153117
Cluster 2  0.1455347 -0.7241992   -1.2885184    0.7086398
          #5 EaseExt #6EaseExt* #7 NonSanctCues #8 SanctCues*
Cluster 1 1.296533  -1.362202      0.3121944    -0.9974599
Cluster 2 1.838854  -2.448885     -0.4096449     1.2049959
           #9 NSanctOt #10 SanctOt* #11 IncubP #12 IncubP*
Cluster 1 1.8961745 -1.414736     -0.605619     -0.1227364
Cluster 2 0.4565668 -1.595302     -1.608977      1.3084054

Root Mean Squared Deviations (RMSD): 5.101523
Mean Absolute Deviation (MAD): 180.0373

Membership degrees matrix (top and bottom 5 rows):
  Cluster 1 Cluster 2
1 0.5862537 0.4137463
2 0.3052886 0.6947114
3 0.6161819 0.3838181
4 0.6161819 0.3838181
5 0.6161819 0.3838181
...
   Cluster 1 Cluster 2
13 0.8613089 0.1386911
14 0.8870995 0.1129005
15 0.6952607 0.3047393
16 0.8139089 0.1860911
17 0.8542094 0.1457906

Descriptive statistics for the membership degrees by clusters
     Size    Min     Q1      Mean      Median        Q3       Max
Cls1 10 0.5862537 0.6161819 0.7360496 0.7545848 0.8441343 0.8870995
Cls2  7 0.5669997 0.6318712 0.7131743 0.7460323 0.7900302 0.8353852

Dunn's Fuzziness Coefficients:
dunn_coeff normalized
 0.6268877  0.2537753

Within cluster sum of squares by cluster:
       1        2
199.9000 171.1429
(between_SS / total_SS =  26.22%)

>res.fcmvalidity <- ppclust2(res.fcm, "fclust")

>S <- SIL.F(res.fcmvalidity$Xca, res.fcmvalidity$U, alpha=1)
>paste("Fuzzy Silhouette Index: ",S)
```

```
[1] "Fuzzy Silhouette Index:  0.503551319867993"

>H <- PE(res.fcmvalidity$U)
>paste("Partition Entropy: ",H)
[1] "Partition Entropy:  0.555500049246217"

>F <- PC(res.fcmvalidity$U)
>paste("Partition Coefficient", F)
[1]"Partition Coefficient 0.626887673705641"

-----------------------------------------------------------------
No. Clusters = 3, Weighting Exponent m=2, Fuzzy C-Means Algorithm
>res.fcm <- fcm(JAdataNCC, centers=3)
> as.data.frame(res.fcm$u)[1:17,]
    Cluster 1  Cluster 2  Cluster 3
1  0.23090427 0.28413601 0.48495972
2  0.65695811 0.20473749 0.13830440
3  0.02613597 0.03571561 0.93814842
4  0.02613597 0.03571561 0.93814842
5  0.02613597 0.03571561 0.93814842
6  0.74895614 0.10404533 0.14699852
7  0.43746008 0.33229443 0.23024549
8  0.77145894 0.09798747 0.13055359
9  0.28200148 0.21497613 0.50302239
10 0.55442278 0.15311493 0.29246229
11 0.69538566 0.15935982 0.14525451
12 0.05929375 0.86439828 0.07630798
13 0.06938575 0.80564514 0.12496910
14 0.02817288 0.92541201 0.04641511
15 0.19767162 0.52279657 0.27953181
16 0.05929375 0.86439828 0.07630798
17 0.12151491 0.66095704 0.21752805
> summary(res.fcm)
Summary for 'res.fcm'

Number of data objects:  17

Number of clusters:  3

Crisp clustering vector:
 [1] 3 1 3 3 3 1 1 1 3 1 1 2 2 2 2 2 2

Initial cluster prototypes:
           #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1       1          6             4            2            5
Cluster 2       5          1             3            6            1
Cluster 3       3          3             5            2            1
            #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1       1          5             2            4            2
Cluster 2       6          1             2            3            3
Cluster 3       5          5             2            3            3
          #11 Incubation Period #12 Incubation Period*
Cluster 1                     6                      3
Cluster 2                     1                      5
```

```
Cluster 3                   1                     3

Final cluster prototypes:
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1   2.154471   4.160795     4.863720     1.537990        3.371506
Cluster 2   1.544829   5.333987     3.073310     5.407282        2.786039
Cluster 3   1.525570   5.348187     4.069294     2.047270        1.167312
          #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1   2.852427   4.572239      2.538154     3.680901       3.177512
Cluster 2   4.180987   1.224863      2.126536     3.421249       3.157124
Cluster 3   5.737880   5.349752      1.306877     2.150997       4.665085
          #11 Incubation Period #12 Incubation Period*
Cluster 1             5.315996              1.652016
Cluster 2             1.264258              4.799672
Cluster 3             1.315513              5.388418

Distance between the final cluster prototypes
          Cluster 1 Cluster 2
Cluster 2  59.79938
Cluster 3  52.51882  39.24972

Difference between the initial and final cluster prototypes
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1   1.154471  -1.839205     0.86372049  -0.46201019   -1.6284940
Cluster 2  -3.455171   4.333987     0.07330983  -0.59271844    1.7860389
Cluster 3  -1.474430   2.348187    -0.93070626   0.04726973    0.1673118
          #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1   1.8524275 -0.4277610    0.5381541     -0.3190993   1.1775121
Cluster 2  -1.8190133  0.2248630    0.1265359      0.4212489   0.1571242
Cluster 3   0.7378804  0.3497518   -0.6931234     -0.8490031   1.6650852
          #11 Incubation Period #12 Incubation Period*
Cluster 1           -0.6840043            -1.3479839
Cluster 2            0.2642585            -0.2003277
Cluster 3            0.3155128             2.3884180

Root Mean Squared Deviations (RMSD): 4.932512
Mean Absolute Deviation (MAD): 150.8645

Membership degrees matrix (top and bottom 5 rows):
   Cluster 1  Cluster 2 Cluster 3
1 0.23090427 0.28413601 0.4849597
2 0.65695811 0.20473749 0.1383044
3 0.02613597 0.03571561 0.9381484
4 0.02613597 0.03571561 0.9381484
5 0.02613597 0.03571561 0.9381484
...
    Cluster 1 Cluster 2  Cluster 3
13 0.06938575 0.8056451 0.12496910
14 0.02817288 0.9254120 0.04641511
15 0.19767162 0.5227966 0.27953181
16 0.05929375 0.8643983 0.07630798
17 0.12151491 0.6609570 0.21752805

Descriptive statistics for the membership degrees by clusters
```

```
        Size  Min        Q1        Mean      Median      Q3       Max
Cluster 1 6 0.4374601 0.5800566 0.6441070 0.6761719 0.7355635 0.7714589
Cluster 2 6 0.5227966 0.6971291 0.7739346 0.8350217 0.8643983 0.9254120
Cluster 3 5 0.4849597 0.5030224 0.7604855 0.9381484 0.9381484 0.9381484

Dunn’s Fuzziness Coefficients:
dunn_coeff normalized
 0.6079195  0.4118792

Within cluster sum of squares by cluster:
       1        2        3
140.3333  64.5000  49.2000
(between_SS / total_SS =  53.18%)


> res.fcmvalidity <- ppclust2(res.fcm, "fclust")
> S <- SIL.F(res.fcmvalidity$Xca, res.fcmvalidity$U, alpha=1)
> paste("Fuzzy Silhouette Index: ",S)
[1] "Fuzzy Silhouette Index:  0.569220623213061"

> H<- PE(res.fcmvalidity$U)
> paste("Partition Entropy: ",H)
[1] "Partition Entropy:  0.697681887166419"

> F <- PC(res.fcmvalidity$U)
> paste("Partition Coefficient", F)
[1] "Partition Coefficient 0.60791945531254"
-----------------------------------------------------------------
No. Clusters = 4, Weighting Exponent m=2, Fuzzy C-Means Algorithm
> res.fcm <- fcm(JAdataNCC, centers=4)
> as.data.frame(res.fcm$u)[1:17,]
     Cluster 1  Cluster 2   Cluster 3   Cluster 4
1  0.112739096 0.17625541 0.099406408 0.611599086
2  0.155570832 0.09767078 0.505187294 0.241571091
3  0.003572314 0.98812735 0.002707738 0.005592597
4  0.003572314 0.98812735 0.002707738 0.005592597
5  0.003572314 0.98812735 0.002707738 0.005592597
6  0.083842410 0.11061203 0.646600877 0.158944683
7  0.250720484 0.15862737 0.323138085 0.267514058
8  0.072302676 0.09176985 0.701688042 0.134239429
9  0.097859260 0.18841905 0.130937860 0.582783831
10 0.113929301 0.20646980 0.428834146 0.250766750
11 0.124504534 0.10572482 0.636542628 0.133228021
12 0.870199714 0.03977316 0.032184320 0.057842805
13 0.707004223 0.10303471 0.059741623 0.130219440
14 0.906564973 0.03106656 0.019788017 0.042580454
15 0.286864215 0.15278171 0.117205930 0.443148147
16 0.870199714 0.03977316 0.032184320 0.057842805
17 0.398878006 0.14729814 0.092693698 0.361130153
> summary(res.fcm)
Summary for ’res.fcm’

Number of data objects:  17
```

```
Number of clusters:  4

Crisp clustering vector:
 [1] 4 3 2 2 2 3 3 3 4 3 3 1 1 1 4 1 1

Initial cluster prototypes:
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1     2            5           3            4            2
Cluster 2     3            3           2            2            6
Cluster 3     2            4           5            1            4
Cluster 4     1            6           4            2            5
          #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1     5            1           3            1            3
Cluster 2     1            4           1            6            1
Cluster 3     2            6           4            2            5
Cluster 4     1            5           2            4            2
          #11 Incubation Period #12 Incubation Period*
Cluster 1                  2                       5
Cluster 2                  1                       2
Cluster 3                  6                       1
Cluster 4                  6                       3

Final cluster prototypes:
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1   1.230786   5.709981    3.032044      5.703612        2.952812
Cluster 2   1.130173   5.833741    4.016774      2.032038        1.090950
Cluster 3   2.013847   4.306647    4.904378      1.473528        3.519775
Cluster 4   3.336107   3.126555    4.073434      2.506090        1.655022
          #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1   4.028123   1.128574     2.053278     3.750554        3.068387
Cluster 2   5.884758   5.807086     1.080170     2.075191        4.885769
Cluster 3   2.650801   4.813074     2.554907     3.669115        3.180550
Cluster 4   4.997198   2.884048     2.421079     2.477755        3.570505
          #11 Incubation Period #12 Incubation Period*
Cluster 1              1.153829             4.864779
Cluster 2              1.149392             5.780179
Cluster 3              5.592253             1.575667
Cluster 4              2.072703             3.797263

Distance between the final cluster prototypes
          Cluster 1 Cluster 2 Cluster 3
Cluster 2  51.17233
Cluster 3  70.56450  66.59855
Cluster 4  32.11151  30.54751  36.51371

Difference between the initial and final cluster prototypes
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1 -0.76921425  0.7099814   0.03204421    1.70361225     0.9528121
Cluster 2 -1.86982735  2.8337409   2.01677352    0.03203786    -4.9090500
Cluster 3  0.01384701  0.3066469  -0.09562205    0.47352764    -0.4802248
Cluster 4  2.33610717 -2.8734453   0.07343398    0.50608971    -3.3449778
         #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1 -0.9718773   0.1285735  -0.9467219     2.750554       0.06838685
Cluster 2  4.8847584   1.8070857   0.0801704    -3.924809       3.88576945
```

```
Cluster 3  0.6508006  -1.1869258  -1.4450933     1.669115     -1.81944963
Cluster 4  3.9971982  -2.1159525   0.4210793    -1.522245      1.57050532
          #11 Incubation Period #12 Incubation Period*
Cluster 1            -0.8461713             -0.1352211
Cluster 2             0.1493921              3.7801787
Cluster 3            -0.4077472              0.5756668
Cluster 4            -3.9272972              0.7972627

Root Mean Squared Deviations (RMSD): 7.14883
Mean Absolute Deviation (MAD): 218.3971

Membership degrees matrix (top and bottom 5 rows):
    Cluster 1  Cluster 2   Cluster 3   Cluster 4
1 0.112739096 0.17625541 0.099406408 0.611599086
2 0.155570832 0.09767078 0.505187294 0.241571091
3 0.003572314 0.98812735 0.002707738 0.005592597
4 0.003572314 0.98812735 0.002707738 0.005592597
5 0.003572314 0.98812735 0.002707738 0.005592597
...
   Cluster 1  Cluster 2  Cluster 3  Cluster 4
13 0.7070042 0.10303471 0.05974162 0.13021944
14 0.9065650 0.03106656 0.01978802 0.04258045
15 0.2868642 0.15278171 0.11720593 0.44314815
16 0.8701997 0.03977316 0.03218432 0.05784280
17 0.3988780 0.14729814 0.09269370 0.36113015

Descriptive statistics for the membership degrees by clusters
          Siz   Min        Q1      Mean    Median      Q3     Max
Cluster 1 5 0.3988780 0.7070042 0.7505693 0.8701997 0.8701997 0.9065650
Cluster 2 3 0.9881274 0.9881274 0.9881274 0.9881274 0.9881274 0.9881274
Cluster 3 6 0.3231381 0.4479224 0.5403318 0.5708650 0.6440863 0.7016880
Cluster 4 3 0.4431481 0.5129660 0.5458437 0.5827838 0.5971915 0.6115991

Dunn's Fuzziness Coefficients:
dunn_coeff normalized
 0.5662883  0.4217178

Within cluster sum of squares by cluster:
        1         2         3         4
 27.60000   0.00000 140.33333  39.33333
(between_SS / total_SS =  60.94%)


> res.fcmvalidity <- ppclust2(res.fcm, "fclust")
> S<- SIL.F(res.fcmvalidity$Xca, res.fcmvalidity$U, alpha=1)
> paste("Fuzzy Silhouette Index: ",S)
[1] "Fuzzy Silhouette Index:  0.686785780488479"

> H <- PE(res.fcmvalidity$U)
> paste("Partition Entropy: ",H)
[1] "Partition Entropy:  0.832463433402978"

> F <- PC(res.fcmvalidity$U)
> paste("Partition Coefficient", F)
```

```
[1] "Partition Coefficient 0.566288346935951"
------------------------------------------------------------------
No. Clusters = 2, Weighting Exponent m=1.5, Fuzzy C-Means Algorithm
> res.fcm <- fcm(JAdataNCC, centers=2, m=1.5)
> as.data.frame(res.fcm$u)[1:17,]
    Cluster 1  Cluster 2
1  0.82671520 0.17328480
2  0.08827339 0.91172661
3  0.89308290 0.10691710
4  0.89308290 0.10691710
5  0.89308290 0.10691710
6  0.02817517 0.97182483
7  0.30918419 0.69081581
8  0.03009909 0.96990091
9  0.64869828 0.35130172
10 0.12870720 0.87129280
11 0.05345039 0.94654961
12 0.93329972 0.06670028
13 0.97199942 0.02800058
14 0.97638541 0.02361459
15 0.87899093 0.12100907
16 0.93329972 0.06670028
17 0.97814617 0.02185383
> summary(res.fcm)
Summary for 'res.fcm'

Number of data objects:  17

Number of clusters:  2

Crisp clustering vector:
 [1] 1 2 1 1 1 2 2 2 1 2 2 1 1 1 1 1 1

Initial cluster prototypes:
           #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1   1           6             4              2            5
Cluster 2   2           4             6              1            2
           #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1   1           5             2              4            2
Cluster 2   4           6             1              5            2
          #11 Incubation Period #12 Incubation Period*
Cluster 1                     6                      3
Cluster 2                     6                      1

Final cluster prototypes:
         #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1 1.835837    5.002060    3.457181       4.015388    1.983068
Cluster 2 2.153616    4.198053    4.779840       1.559293    3.349939
         #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1 4.957394    2.699692     1.901300      2.711874    3.770262
Cluster 2 2.946994    4.531566     2.385778      3.769650    3.151335
          #11 Incubation Period #12 Incubation Period*
Cluster 1              1.228623               5.064348
Cluster 2              5.072474               1.718308
```

```
Distance between the final cluster prototypes
          Cluster 1
Cluster 2   45.5029

Difference between the initial and final cluster prototypes
         #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1 0.8358371 -0.9979400  -0.5428185    2.0153879      -3.016932
Cluster 2 0.1536164  0.1980528  -1.2201602    0.5592929       1.349939
         #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1 3.957394   -2.300308  -0.09869984  -1.288126        1.770262
Cluster 2 -1.053006  -1.468434   1.38577765  -1.230350        1.151335
        #11 Incubation Period #12 Incubation Period*
Cluster 1            -4.7713772             2.0643484
Cluster 2            -0.9275262             0.7183075

Root Mean Squared Deviations (RMSD): 6.364883
Mean Absolute Deviation (MAD): 210.4514

Membership degrees matrix (top and bottom 5 rows):
   Cluster 1 Cluster 2
1 0.82671520 0.1732848
2 0.08827339 0.9117266
3 0.89308290 0.1069171
4 0.89308290 0.1069171
5 0.89308290 0.1069171
...
   Cluster 1  Cluster 2
13 0.9719994 0.02800058
14 0.9763854 0.02361459
15 0.8789909 0.12100907
16 0.9332997 0.06670028
17 0.9781462 0.02185383

Descriptive statistics for the membership degrees by clusters
          Size  Min        Q1      Mean    Median        Q3       Max
Cluster 1 11 0.6486983 0.8860369 0.8933440 0.8930829 0.9526496 0.9781462
Cluster 2  6 0.6908158 0.8814012 0.8936851 0.9291381 0.9640631 0.9718248

Dunn's Fuzziness Coefficients:
dunn_coeff normalized
 0.8266468  0.6532937

Within cluster sum of squares by cluster:
       1        2
223.2727 140.3333
(between_SS / total_SS =  32.7%)


> res.fcmvalidity <- ppclust2(res.fcm, "fclust")
> S <- SIL.F(res.fcmvalidity$Xca, res.fcmvalidity$U, alpha=1)
> paste("Fuzzy Silhouette Index: ",S)
[1] "Fuzzy Silhouette Index:  0.506710125936752"
```

```
> H <- PE(res.fcmvalidity$U)
> paste("Partition Entropy: ",H)
[1] "Partition Entropy:  0.300346449416796"

> F <- PC(res.fcmvalidity$U)
> paste("Partition Coefficient", F)
[1] "Partition Coefficient 0.826646834868445"

-------------------------------------------------------------------
No. Clusters = 3, Weighting Exponent m=1.5, Fuzzy C-Means Algorithm

> res.fcm <- fcm(JAdataNCC, centers=3, m=1.5)
> as.data.frame(res.fcm$u)[1:17,]
     Cluster 1   Cluster 2   Cluster 3
1  0.175498477 0.091547458 0.732954065
2  0.077242056 0.880375662 0.042382283
3  0.005914397 0.002838779 0.991246823
4  0.005914397 0.002838779 0.991246823
5  0.005914397 0.002838779 0.991246823
6  0.017530783 0.941535105 0.040934112
7  0.287648141 0.553059962 0.159291897
8  0.014660263 0.955256494 0.030083243
9  0.086693784 0.120565646 0.792740570
10 0.059306535 0.687820376 0.252873089
11 0.035489546 0.931702407 0.032808046
12 0.983941095 0.005823918 0.010234987
13 0.969654686 0.006755634 0.023589680
14 0.995664966 0.001105158 0.003229875
15 0.714905667 0.076521321 0.208573012
16 0.983941095 0.005823918 0.010234987
17 0.884023201 0.022015525 0.093961274
> summary(res.fcm)
Summary for 'res.fcm'

Number of data objects:  17

Number of clusters:  3

Crisp clustering vector:
 [1] 3 2 3 3 3 2 2 2 3 2 2 1 1 1 1 1 1

Initial cluster prototypes:
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1   1           6           3             6                 3
Cluster 2   2           4           6             1                 2
Cluster 3   1           6           4             2                 1
          #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1   4           1           2             5                 2
Cluster 2   4           6           1             5                 2
Cluster 3   6           6           1             2                 5
          #11 Incubation Period #12 Incubation Period*
Cluster 1                     1                      5
Cluster 2                     6                      1
Cluster 3                     1                      6
```

```
Final cluster prototypes:
         #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1 1.690159    5.166187    3.013326      5.474945      2.685543
Cluster 2 2.106843    4.216292    4.843200      1.495029      3.548808
Cluster 3 1.859220    4.949760    4.157999      1.934086      1.093114
         #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1 4.304208    1.133088     2.141955     3.289546      3.144712
Cluster 2 2.673934    4.558191     2.533426     3.815361      3.092299
Cluster 3 5.748795    5.101979     1.468690     2.126391      4.573413
          #11 Incubation Period #12 Incubation Period*
Cluster 1             1.201896               4.878329
Cluster 2             5.461281               1.539839
Cluster 3             1.308634               5.161300

Distance between the final cluster prototypes
          Cluster 1 Cluster 2
Cluster 2  65.11881
Cluster 3  38.23701  53.58192

Difference between the initial and final cluster prototypes
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1 0.6901592   -0.8338131  0.01332645    -0.52505470   -0.31445654
Cluster 2 0.1068428    0.2162921 -1.15680030     0.49502946    1.54880786
Cluster 3 0.8592197   -1.0502398  0.15799896    -0.06591382    0.09311418
          #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1 0.3042079    0.1330875  0.1419546     -1.710454       1.1447122
Cluster 2 -1.3260661  -1.4418087  1.5334261     -1.184639       1.0922985
Cluster 3 -0.2512045  -0.8980206  0.4686897      0.126391      -0.4265871
          #11 Incubation Period #12 Incubation Period*
Cluster 1            0.2018959            -0.1216712
Cluster 2           -0.5387185             0.5398387
Cluster 3            0.3086342            -0.8387001

Root Mean Squared Deviations (RMSD): 2.78929
Mean Absolute Deviation (MAD): 91.4403

Membership degrees matrix (top and bottom 5 rows):
    Cluster 1   Cluster 2  Cluster 3
1 0.175498477 0.091547458 0.73295407
2 0.077242056 0.880375662 0.04238228
3 0.005914397 0.002838779 0.99124682
4 0.005914397 0.002838779 0.99124682
5 0.005914397 0.002838779 0.99124682
...
   Cluster 1   Cluster 2   Cluster 3
13 0.9696547 0.006755634 0.023589680
14 0.9956650 0.001105158 0.003229875
15 0.7149057 0.076521321 0.208573012
16 0.9839411 0.005823918 0.010234987
17 0.8840232 0.022015525 0.093961274

Descriptive statistics for the membership degrees by clusters
          Size       Min        Q1      Mean    Median        Q3       Max
```

```
Cluster 1    6 0.7149057 0.9054311 0.9220218 0.9767979 0.9839411 0.9956650
Cluster 2    6 0.5530600 0.7359592 0.8249583 0.9060390 0.9390769 0.9552565
Cluster 3    5 0.7329541 0.7927406 0.8998870 0.9912468 0.9912468 0.9912468

Dunn's Fuzziness Coefficients:
dunn_coeff normalized
 0.8120368  0.7180552

Within cluster sum of squares by cluster:
       1        2        3
 64.5000 140.3333  49.2000
(between_SS / total_SS =  54.14%)


> res.fcmvalidity <- ppclust2(res.fcm, "fclust")

> S<- SIL.F(res.fcmvalidity$Xca, res.fcmvalidity$U, alpha=1)
> paste("Fuzzy Silhouette Index: ",S)
[1] "Fuzzy Silhouette Index:  0.530761123800606"

> H <- PE(res.fcmvalidity$U)
> paste("Partition Entropy: ",H)
[1] "Partition Entropy:  0.357536457651714"

> F <- PC(res.fcmvalidity$U)
> paste("Partition Coefficient", F)
[1] "Partition Coefficient 0.812036801279259"

-------------------------------------------------------------------
No. Clusters = 4, Weighting Exponent m=1.5, Fuzzy C-Means Algorithm
> res.fcm <- fcm(JAdataNCC, centers=4, m=1.5)
> as.data.frame(res.fcm$u)[1:17,]
      Cluster 1   Cluster 2    Cluster 3   Cluster 4
1  0.1177650423 0.623317788 0.0709679731 0.187949197
2  0.2526238757 0.031304621 0.6519354122 0.064136091
3  0.0012597705 0.996201613 0.0006354544 0.001903162
4  0.0012597705 0.996201613 0.0006354544 0.001903162
5  0.0012597705 0.996201613 0.0006354544 0.001903162
6  0.9406793545 0.010516108 0.0436561043 0.005148433
7  0.0542050165 0.032581212 0.8525128474 0.060700924
8  0.7626315666 0.032828300 0.1866069170 0.017933217
9  0.1982729889 0.609833076 0.0940632380 0.097830697
10 0.9446679007 0.029545372 0.0176186351 0.008168092
11 0.1346597437 0.020189333 0.8219880772 0.023162846
12 0.0039877256 0.009941518 0.0113256192 0.974745138
13 0.0041674031 0.018375731 0.0070693553 0.970387510
14 0.0005671558 0.002192161 0.0011708058 0.996069877
15 0.0672929468 0.174882504 0.0877077363 0.670116813
16 0.0039877256 0.009941518 0.0113256192 0.974745138
17 0.0164268371 0.076890197 0.0229675030 0.883715463
> summary(res.fcm)
Summary for 'res.fcm'

Number of data objects:  17
```

```
Number of clusters:  4

Crisp clustering vector:
 [1] 2 3 2 2 2 1 3 1 2 1 3 4 4 4 4 4 4

Initial cluster prototypes:
          #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1   2           4           6             1               2
Cluster 2   4           3           4             1               1
Cluster 3   1           6           4             2               5
Cluster 4   1           6           3             6               3
          #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1   4           6           1             5               2
Cluster 2   6           2           3             1               5
Cluster 3   1           5           2             4               2
Cluster 4   4           1           2             5               2
          #11 Incubation Period #12 Incubation Period*
Cluster 1                    6                      1
Cluster 2                    2                      5
Cluster 3                    6                      3
Cluster 4                    1                      5

Final cluster prototypes:
         #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1 2.109620  4.247036      5.586304      1.132274      2.254348
Cluster 2 1.681337  5.179381      4.095479      1.963025      1.026220
Cluster 3 2.318122  4.028364      3.575102      2.005408      4.924602
Cluster 4 1.628453  5.258101      3.031845      5.565420      2.614437
         #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1 4.095487  5.301739      1.969014      3.454553      3.798127
Cluster 2 5.853176  5.267239      1.398282      2.022232      4.707149
Cluster 3 1.452304  3.660045      2.432122      4.582186      2.047118
Cluster 4 4.375731  1.051974      2.175674      3.209411      3.206704
        #11 Incubation Period #12 Incubation Period*
Cluster 1              5.729604               1.187582
Cluster 2              1.151102               5.475020
Cluster 3              4.066094               2.138962
Cluster 4              1.190321               4.964498

Distance between the final cluster prototypes
          Cluster 1 Cluster 2 Cluster 3
Cluster 2  51.11276
Cluster 3  29.93487  73.47548
Cluster 4  81.02328  41.11854  55.19068

Difference between the initial and final cluster prototypes
         #1 PersNov #2 PersNov* #3 EmotionalI #4 EmotionalI* #5 EaseExt
Cluster 1  0.1096203  0.2470364   -0.41369635   0.132274196   0.25434755
Cluster 2 -2.3186634  2.1793813    0.09547873   0.963025169   0.02622019
Cluster 3  1.3181222 -1.9716357   -0.42489824   0.005407781  -0.07539766
Cluster 4  0.6284532 -0.7418990    0.03184516  -0.434579835  -0.38556294
         #6EaseExt* #7 NonSanctCues #8 SanctCues* #9 NSanctOt #10 SanctOt*
Cluster 1  0.09548712 -0.69826115  0.9690142   -1.5454468     1.79812740
```

```
Cluster 2 -0.14682381  3.26723949 -1.6017175    1.0222320    -0.29285087
Cluster 3  0.45230363 -1.33995530  0.4321221    0.5821861     0.04711788
Cluster 4  0.37573095  0.05197395  0.1756736   -1.7905891     1.20670401
          #11 Incubation Period #12 Incubation Period*
Cluster 1            -0.2703962            0.18758203
Cluster 2            -0.8488982            0.47501995
Cluster 3            -1.9339059           -0.86103784
Cluster 4             0.1903215           -0.03550165

Root Mean Squared Deviations (RMSD): 3.635687
Mean Absolute Deviation (MAD): 106.3553

Membership degrees matrix (top and bottom 5 rows):
    Cluster 1  Cluster 2   Cluster 3   Cluster 4
1 0.117765042 0.62331779 0.070967973 0.187949197
2 0.252623876 0.03130462 0.651935412 0.064136091
3 0.001259771 0.99620161 0.000635454 0.001903162
4 0.001259771 0.99620161 0.000635454 0.001903162
5 0.001259771 0.99620161 0.000635454 0.001903162
...
     Cluster 1   Cluster 2   Cluster 3 Cluster 4
13 0.004167403 0.018375731 0.007069355 0.9703875
14 0.000567156 0.002192161 0.001170806 0.9960699
15 0.067292947 0.174882504 0.087707736 0.6701168
16 0.003987726 0.009941518 0.011325619 0.9747451
17 0.016426837 0.076890197 0.022967503 0.8837155

Descriptive statistics for the membership degrees by clusters
        Size  Min        Q1      Mean     Median        Q3       Max
Cluster 1 3 0.7626316 0.8516555 0.8826596 0.9406794 0.9426736 0.9446679
Cluster 2 5 0.6098331 0.6233178 0.8443511 0.9962016 0.9962016 0.9962016
Cluster 3 3 0.6519354 0.7369617 0.7754788 0.8219881 0.8372505 0.8525128
Cluster 4 6 0.6701168 0.9053835 0.9116300 0.9725663 0.9747451 0.9960699

Dunn's Fuzziness Coefficients:
dunn_coeff normalized
   0.78151    0.70868

Within cluster sum of squares by cluster:
       1        2        3        4
31.33333 49.20000 56.66667 64.50000
(between_SS / total_SS =  62.94%)


> res.fcmvalidity <- ppclust2(res.fcm, "fclust")
> S <- SIL.F(res.fcmvalidity$Xca, res.fcmvalidity$U, alpha=1)
> paste("Fuzzy Silhouette Index: ",S)
[1] "Fuzzy Silhouette Index:  0.503573648360626"

> H <- PE(res.fcmvalidity$U)
> paste("Partition Entropy: ",H)
[1] "Partition Entropy:  0.437564938382639"

> F <- PC(res.fcmvalidity$U)
```

```
> paste("Partition Coefficient", F)
[1] "Partition Coefficient 0.78150997269221"
```

A. 3.4 Clusters with validity indices additional attributes

```
-------------------------------------------------------------------
No. Clusters = 2, Weighting Exponent m=1.5, Fuzzy C-Means Algorithm
> res.fcm <- fcm(JAAddattdata, centers=2, m=1.5)
> as.data.frame(res.fcm$u)[1:17,]
      Cluster 1    Cluster 2
1  0.0081954795 0.9918045205
2  0.0135390210 0.9864609790
3  0.9924595887 0.0075404113
4  0.0002695265 0.9997304735
5  0.9924595887 0.0075404113
6  0.0176808004 0.9823191996
7  0.0135390210 0.9864609790
8  0.9996918254 0.0003081746
9  0.9958583860 0.0041416140
10 0.9920750537 0.0079249463
11 0.9920750537 0.0079249463
12 0.9920750537 0.0079249463
13 0.9995682511 0.0004317489
14 0.9976988226 0.0023011774
15 0.9996918254 0.0003081746
16 0.9920750537 0.0079249463
17 0.9920750537 0.0079249463
> summary(res.fcm)
Summary for 'res.fcm'

Number of data objects:  17

Number of clusters:  2

Crisp clustering vector:
 [1] 2 2 1 2 1 2 2 1 1 1 1 1 1 1 1 1 1

Initial cluster prototypes:
          #13 ConfTruth1 #14 ConfTruth2 ErrNon-ErrType
Cluster 1       5              3             4
Cluster 2       3              1             4

Final cluster prototypes:
          #13 ConfTruth1 #14 ConfTruth2 ErrNon-ErrType
Cluster 1   5.415637        3.833170      3.749645
Cluster 2   2.604285        1.203319      2.802750

Distance between the final cluster prototypes
          Cluster 1
Cluster 2  15.71643

Difference between the initial and final cluster prototypes
          #13 ConfTruth1 #14 ConfTruth2 ErrNon-ErrType
```

```
Cluster 1 0.4156365       0.8331699       -0.2503553
Cluster 2-0.3957151       0.2033186       -1.1972500

Root Mean Squared Deviations (RMSD): 1.131579
Mean Absolute Deviation (MAD): 4.943168

Membership degrees matrix (top and bottom 5 rows):
    Cluster 1   Cluster 2
1 0.008195479 0.991804521
2 0.013539021 0.986460979
3 0.992459589 0.007540411
4 0.000269526 0.999730474
5 0.992459589 0.007540411
...
   Cluster 1   Cluster 2
13 0.9995683 0.000431749
14 0.9976988 0.002301177
15 0.9996918 0.000308175
16 0.9920751 0.007924946
17 0.9920751 0.007924946

Descriptive statistics for the membership degrees by clusters
        Size    Min      Q1    Mean    Median        Q3       Max
Cluster1 12 0.9920751 0.9920751 0.9948170 0.9924596 0.9981662 0.9996918
Cluster2 5 0.9823192 0.9864610 0.9893552 0.9864610 0.9918045 0.9997305

Dunn’s Fuzziness Coefficients:
dunn_coeff normalized
 0.9865620  0.9731239

Within cluster sum of squares by cluster:
       1        2
12.83333  8.80000
(between_SS / total_SS =  71.94%)

> res.fcmvalidity <- ppclust2(res.fcm, "fclust")
> S <- SIL.F(res.fcmvalidity$Xca, res.fcmvalidity$U, alpha=1)
> paste("Fuzzy Silhouette Index: ",S)
[1] "Fuzzy Silhouette Index:  0.835952725302842"
> H <- PE(res.fcmvalidity$U)
> paste("Partition Entropy: ",H)
[1] "Partition Entropy:  0.0385105767753424"
> F <- PC(res.fcmvalidity$U)
> paste("Partition Coefficient", F)
[1] "Partition Coefficient 0.986561968406358"
-------------------------------------------------------------------
No. Clusters = 3, Weighting Exponent m=1.5, Fuzzy C-Means Algorithm
> res.fcm <- fcm(JAAddattdata, centers=3, m=1.5)
> as.data.frame(res.fcm$u)[1:17,]
      Cluster 1    Cluster 2    Cluster 3
1  9.835682e-01 0.0042716572 0.0121601168
2  9.637757e-01 0.0051507199 0.0310735502
3  1.043916e-04 0.9970366422 0.0028589661
4  9.993165e-01 0.0001178697 0.0005656602
```

```
5  1.043916e-04 0.9970366422 0.0028589661
6  9.658348e-01 0.0105265606 0.0236386160
7  9.637757e-01 0.0051507199 0.0310735502
8  1.167329e-03 0.0829404855 0.9158921850
9  3.668311e-04 0.9822014598 0.0174317091
10 4.719521e-05 0.0003118199 0.9996409849
11 4.719521e-05 0.0003118199 0.9996409849
12 4.719521e-05 0.0003118199 0.9996409849
13 1.167872e-03 0.7776272205 0.2212049075
14 7.472240e-04 0.9543915648 0.0448612112
15 1.167329e-03 0.0829404855 0.9158921850
16 4.719521e-05 0.0003118199 0.9996409849
17 4.719521e-05 0.0003118199 0.9996409849
> summary(res.fcm)
Summary for 'res.fcm'

Number of data objects:  17

Number of clusters:  3

Crisp clustering vector:
 [1] 1 1 2 1 2 1 1 3 2 3 3 3 2 2 3 3 3

Initial cluster prototypes:
          #13 ConfTruth1 #14 ConfTruth2 ErrNon-ErrType
Cluster 1   3                  1                  4
Cluster 2   6                  5                  4
Cluster 3   6                  4                  3

Final cluster prototypes:
          #13 ConfTruth1 #14 ConfTruth2 ErrNon-ErrType
Cluster 1   2.600449      1.202636       2.796226
Cluster 2   5.988028      4.638631       3.368104
Cluster 3   5.011520      3.267530       3.996599

Distance between the final cluster prototypes
          Cluster 1 Cluster 2
Cluster 2 23.608803
Cluster 3 11.517947  3.228494

Difference between the initial and final cluster prototypes
          #13 ConfTruth1 #14 ConfTruth2 ErrNon-ErrType
Cluster 1  -0.39955136     0.2026362    -1.2037739
Cluster 2  -0.01197156    -0.3613687    -0.6318963
Cluster 3  -0.98848028    -0.7324696     0.9965992

Root Mean Squared Deviations (RMSD): 1.249882
Mean Absolute Deviation (MAD): 5.528747

Membership degrees matrix (top and bottom 5 rows):
    Cluster 1   Cluster 2   Cluster 3
1 0.983568226 0.004271657 0.012160117
2 0.963775730 0.005150720 0.031073550
3 0.000104392 0.997036642 0.002858966
```

```
4 0.999316470 0.000117870 0.000565660
5 0.000104392 0.997036642 0.002858966
...
     Cluster 1  Cluster 2  Cluster 3
13 0.001167872 0.77762722 0.22120491
14 0.000747224 0.95439156 0.04486121
15 0.001167329 0.08294048 0.91589219
16 0.000047195 0.00031182 0.99964098
17 0.000047195 0.00031182 0.99964098

Descriptive statistics for the membership degrees by clusters
       Size    Min        Q1      Mean    Median        Q3       Max
Cluster1 5 0.9637757 0.9637757 0.9752542 0.9658348 0.9835682 0.9993165
Cluster2 5 0.7776272 0.9543916 0.9416587 0.9822015 0.9970366 0.9970366
Cluster3 7 0.9158922 0.9577666 0.9757128 0.9996410 0.9996410 0.9996410

Dunn's Fuzziness Coefficients:
dunn_coeff normalized
 0.9392371  0.9088557

Within cluster sum of squares by cluster:
       1        2        3
8.800000 2.400000 1.428571
(between_SS / total_SS =  83.75%)

> res.fcmvalidity <- ppclust2(res.fcm, "fclust")
> S <- SIL.F(res.fcmvalidity$Xca, res.fcmvalidity$U, alpha=1)
> paste("Fuzzy Silhouette Index: ",S)
[1] "Fuzzy Silhouette Index:  0.733746614524144"
> H <- PE(res.fcmvalidity$U)
> paste("Partition Entropy: ",H)
[1] "Partition Entropy:  0.122095053608784"
> F <- PC(res.fcmvalidity$U)
> paste("Partition Coefficient", F)
[1] "Partition Coefficient 0.939237111330002"
-------------------------------------------------------------------
No. Clusters = 4, Weighting Exponent m=1.5, Fuzzy C-Means Algorithm
> res.fcm <- fcm(JAAddattdata, centers=4, m=1.5)
> as.data.frame(res.fcm$u)[1:17,]
      Cluster 1    Cluster 2    Cluster 3    Cluster 4
1  0.0011616747 4.103401e-04 1.095911e-02 9.874689e-01
2  0.0001355106 2.264168e-05 9.996361e-01 2.057141e-04
3  0.0028120258 9.970083e-01 1.059983e-04 7.369762e-05
4  0.0019083483 4.007458e-04 9.813434e-01 1.634749e-02
5  0.0028120258 9.970083e-01 1.059983e-04 7.369762e-05
6  0.0004506388 2.015655e-04 3.683408e-03 9.956644e-01
7  0.0001355106 2.264168e-05 9.996361e-01 2.057141e-04
8  0.9186362453 7.932471e-02 1.487358e-03 5.516881e-04
9  0.0176543439 9.816796e-01 4.357028e-04 2.303993e-04
10 0.9995533393 3.416037e-04 8.651617e-05 1.854082e-05
11 0.9995533393 3.416037e-04 8.651617e-05 1.854082e-05
12 0.9995533393 3.416037e-04 8.651617e-05 1.854082e-05
13 0.2288573449 7.689622e-01 1.553962e-03 6.264739e-04
14 0.0460841291 9.526021e-01 8.386896e-04 4.751053e-04
```

```
15 0.9186362453 7.932471e-02 1.487358e-03 5.516881e-04
16 0.9995533393 3.416037e-04 8.651617e-05 1.854082e-05
17 0.9995533393 3.416037e-04 8.651617e-05 1.854082e-05
> summary(res.fcm)
Summary for 'res.fcm'

Number of data objects:  17

Number of clusters:  4

Crisp clustering vector:
 [1] 4 3 2 3 2 4 3 1 2 1 1 1 2 2 1 1 1

Initial cluster prototypes:
          #13 ConfTruth1 #14 ConfTruth2 ErrNon-ErrType
Cluster 1      5              3              4
Cluster 2      6              4              4
Cluster 3      3              1              4
Cluster 4      3              1              3

Final cluster prototypes:
          #13 ConfTruth1 #14 ConfTruth2 ErrNon-ErrType
Cluster 1   5.017690           3.274047      3.998491
Cluster 2   5.990291           4.642518      3.366756
Cluster 3   2.999716           1.000606      3.671962
Cluster 4   2.001163           1.496448      1.498564

Distance between the final cluster prototypes
          Cluster 1 Cluster 2 Cluster 3
Cluster 2  3.217754
Cluster 3  9.347371 22.300209
Cluster 4 18.508929 29.301043  5.966632

Difference between the initial and final cluster prototypes
          #13 ConfTruth1 #14 ConfTruth2 ErrNon-ErrType
Cluster 1 0.01768953   0.274047122       -0.001509134
Cluster 2 -0.009709335 0.642518104       -0.633243571
Cluster 3 -0.000283535 0.000605935       -0.328037500
Cluster 4 -0.998837490 0.496448246       -1.501436335

Root Mean Squared Deviations (RMSD): 1.060117
Mean Absolute Deviation (MAD): 3.678274

Membership degrees matrix (top and bottom 5 rows):
    Cluster 1   Cluster 2   Cluster 3   Cluster 4
1 0.001161675 0.000410340 0.010959107 0.987468878
2 0.000135511 0.000022642 0.999636134 0.000205714
3 0.002812026 0.997008278 0.000105998 0.000073698
4 0.001908348 0.000400746 0.981343415 0.016347491
5 0.002812026 0.997008278 0.000105998 0.000073698
...
    Cluster 1   Cluster 2   Cluster 3   Cluster 4
13 0.22885734 0.768962220 0.001553962 0.000626474
14 0.04608413 0.952602076 0.000838690 0.000475105
```

```
15 0.91863624 0.079324709 0.001487358 0.000551688
16 0.99955334 0.000341604 0.000086516 0.000018541
17 0.99955334 0.000341604 0.000086516 0.000018541

Descriptive statistics for the membership degrees by clusters
        Size    Min        Q1      Mean    Median        Q3       Max
Cluster1  7 0.9186362 0.9590948 0.9764342 0.9995533 0.9995533 0.9995533
Cluster2  5 0.7689622 0.9526021 0.9394521 0.9816796 0.9970083 0.9970083
Cluster3  3 0.9813434 0.9904898 0.9935386 0.9996361 0.9996361 0.9996361
Cluster4  2 0.9874689 0.9895178 0.9915666 0.9915666 0.9936155 0.9956644

Dunn's Fuzziness Coefficients:
dunn_coeff normalized
 0.9488041  0.9317387

Within cluster sum of squares by cluster:
        1         2         3         4
1.4285714 2.4000000 0.6666667 1.0000000
(between_SS / total_SS =  92.95%)


> res.fcmvalidity <- ppclust2(res.fcm, "fclust")
> S <- SIL.F(res.fcmvalidity$Xca, res.fcmvalidity$U, alpha=1)
> paste("Fuzzy Silhouette Index: ",S)
[1] "Fuzzy Silhouette Index:  0.776537449518855"
> H <- PE(res.fcmvalidity$U)
> paste("Partition Entropy: ",H)
[1] "Partition Entropy:  0.100381480715855"
> F <- PC(res.fcmvalidity$U)
> paste("Partition Coefficient", F)
[1] "Partition Coefficient 0.948804054380558"
```