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Rendezvous with Multiple, Intermittent Leaders

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***Abstract*— In this paper we study bipartite, first order-networks where the nodes take on leader or follower roles. In particular, we let the leaders' positions be static and assume that they are only intermittently visible to the followers. This is an assumption that is inspired by the way female silkworm moths only intermittently release pheromones to be detected by the males. The main result in this paper states that if the followers execute the linear agreement protocol, they will converge to the convex hull spanned by the leaders (may they be visible or not).**

I. INTRODUCTION

The research on multi-agent robotics and decentralized, networked control has drawn significant inspiration from interaction-rules in social animals and insects [1], [2], [3]. In particular, the widely used nearest-neighbor-based interaction rules, used for example for formation control (e.g. [5], [6]), consensus (e.g. [7], [8]), and coverage control [9], [10], has a direct biological counterpart, as pointed out in [1]. In this paper, we follow this line of inquiry by seeing if we can understand how leader-follower systems behave if the leaders are only intermittently visible to the followers. This model comes from a particular swarming phenomena observed in the silkworm moth *Bombyx Mori*.

In fact, silkworm moths are known to swarm in tight geometrical configurations, such as vertical cylindrical structures. This is caused by the females' intermittent releasing of a pheromone - *bombykol* - to attract male moths. This pheromone in essence makes the females act as attractors to the males, but the intermittent nature of the release produces an inherently switched system. Moreover, the spatial distribution of the females imply that the males are attracted to a general area rather than to a particular point, which is what is believed to cause their characteristic swarming geometry. (See for example [11], [12], [13], [14])

Based on this discussion, what we will do in this paper is to investigate a first-order network model in which stationary leaders (the female moths) are only intermittently visible to the followers (the males). This corresponds to applying a switched control input of varying dimension (since the number of visible females may be changing) to the system. And, our main result is that, asymptotically, the males will end up in the convex hull spanned by *all* the females. For the case in which the females are always visible and no edges appear or disappear between males, this is already known, and what is thus new is that we produce a hybrid version of this result, using tools from hybrid stability theory. It should be noted that a similar question was pursued in [16] but there it was shown that the followers end up in a larger set (ellipsoid) that contained the convex hull of the leaders. In this paper, we thus make this result much more tight by showing that this set can in fact be shrunk down to the convex hull itself.

The outline of this paper is as follows: We next establish some of the basic notation that will be used in the paper. We then, in Section II, recall the switched version of LaSalle's invariance principle, followed by a discussion of the underlying network model in Section III and the static case, in Section IV. The main result for switched systems is given in Section V, followed by a simulation study in Section VI.

Notation: We let $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{R}_+$ denote the natural numbers, the non-negative integer numbers, and the positive real numbers, respectively. Given the sets M, M_1 and M_2 such that $M \subset M_1 \times M_2$, we denote $\pi_1(M)$ (respectively $\pi_2(M)$) the projection of M on M_1 (respectively M_2), i.e. $\pi_1((m_1,m_2)) = m_1$ and $\pi_2((m_1,m_2)) = m_2$. We denote by $\mathbf{1}_d$, $d \in \mathbb{N}$, the vector of dimension d with all entries equal to 1 (e.g. $\mathbf{1}_2 = [1\ 1]^T$).

II. A LASALLE'S INVARIANCE PRINCIPLE FOR SWITCHED SYSTEMS

In this section, we recall a LaSalle's invariance principle for switched systems proved in [15] that will be useful to prove our main result. For the sake of clarity, we will not use the most general assumptions used in the paper, but we will impose stronger assumptions that are verified by our problem formulation.

Given a parametrized family of locally Lipschitz vector fields $\{f_\gamma : \mathbb{R}^n \to \mathbb{R}^n \mid \gamma \in \Gamma\}$, where Γ is a finite index set, we consider the switched system

$$\dot{x} = f(x, \sigma), \tag{1}$$

where $\sigma : \mathbb{R}_+ \to \Gamma$ is a piecewise constant (continuous from the right) switching signal, and where we somtimes use the notation $f(x,\gamma) = f_\gamma(x)$.

Let $\mathscr{S}$ be the set of all switching signals. A pair $(x(\cdot), \sigma(\cdot))$ is a *trajectory* of (1) if and only if $\sigma(\cdot) \in \mathscr{S}$ and $x : [0,T) \to \mathbb{R}^n$, $0 < T \leq +\infty$, is a piecewise differential

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solution to $\dot{x} = f_{\sigma(t)}(x)$, $t \in [0,T)$. Note that T is, in general, a function of $x(0)$ and $\sigma(\cdot)$ (so that we should write $T(x(0),\sigma(\cdot))$.

In the following we will consider switching signals that have positive average dwell time, i.e. signals for which the number of discontinuities in any open interval is bounded above by the length of the interval normalized by an "average dwell-time" plus a "chatter bound".

More formally, we say that a switching signal σ has an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ if the number of its switching times in any open interval $(\tau_1, \tau_2) \subset \mathbb{R}_+$ is bounded by $N_0 + (\tau_2 - \tau_1)/\tau_D$. We denote by $\mathscr{S}_a[\tau_D, N_0]$ the set of all switching signals with average dwell-time τ_D and chatter bound N_0, and by $\mathscr{T}_a[\tau_D, N_0]$ the subclass of all trajectories of (1) corresponding to some $\sigma \in \mathscr{S}_a[\tau_D, N_0]$. Also, we let

$$\mathscr{S}_a = \cup_{\tau_D > 0, N_0 \in \mathbb{N}} \mathscr{S}_a[\tau_D, N_0],$$

and consequently, we let $\mathscr{T}_a$ br the corresponding subclass of trajectories.

In order to deal with a LaSalle's Invariance Principle it is useful, following [15], to introduce the following subclasses of trajectories.

Definition 2.1 (Class of trajectories $\mathscr{T}_V$): Let $V : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. $\mathscr{T}_V$ is the class of trajectories $(x(\cdot), \sigma(\cdot)) \in \mathscr{T}$ which verify the conditions:

i) $x(t) \in \Omega$ for all $t \in [0,T)$;
ii) for any pair of times $t, t' \in [0,T)$ such that $t \leq t'$ and $\sigma(t) = \sigma(t')$, then $V(x(t), \sigma(t)) \geq V(x(t'), \sigma(t'))$.

$\mathscr{T}_V^*$ is the subfamily of $(x(\cdot), \sigma(\cdot)) \in \mathscr{T}_V$ verifying $V(x(t), \sigma(t)) = V(x(t'), \sigma(t'))$ for $\sigma(t) = \sigma(t')$. □

Then, we introduce a suitable notion of a weakly-invariant set:

Definition 2.2 (Weakly invariant set): Given a family $\mathscr{T}'$ of trajectories of (1), a non-empty subset $M \subset \mathbb{R}^n \times \Gamma$ is said to be weakly-invariant with respect to $\mathscr{T}'$ if, for each $(\xi, \gamma) \in M$, there is a trajectory $(x(\cdot), \sigma(\cdot)) \in \mathscr{T}'$ such that $x(0) = \xi$, $\sigma(0) = \gamma$ and $(x(t), \sigma(t)) \in M$ for all $t \in [0,T)$. □

We are now ready to state (a slightly modified version of) the LaSalle's Invariance Principle proved in [15] (Theorem 2.4).

Theorem 2.1 (LaSalle's IP for switched systems, [15]): Let $V : \Omega \times \Gamma \to \mathbb{R}$, with Ω an open subset of $\mathbb{R}^n$, be continuous. Suppose that $(x(\cdot), \sigma(\cdot))$ is a trajectory belonging to $\mathscr{T}_V \cap \mathscr{T}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and $N_0 \in \mathbb{N}$, such that for some compact subset $B \subset \Omega$, $x(t) \in B$ for all $t \geq 0$. Let $M \subset \mathbb{R}^n \times \Gamma$ be the largest weakly invariant set with respect to $\mathscr{T}_V^* \cap \mathscr{T}_a[\tau_D, N_0]$ contained in $\Omega \times \Gamma$. Then $x(t)$ converges to $\pi_1(M)$ as $t \to \infty$. □

III. Network Model

In this section, we introduce a mathematical model that is based on the model in [16], and that describes the swarming behavior encountered among the silkworm moths. Informally, we consider a network with agents of two sorts: leaders (representing the female moths) and followers (representing the males). Leaders and followers are both described as first order integrators, but they apply different control laws. In this paper we assume the leaders to be stationary, that is, their control input is identically zero. Also, we assume they may be active or inactive, equivalently visible or invisible to the followers. The followers apply a Laplacian based averaging control law. They communicate among themselves and with active leaders according to a fixed, undirected communication graph. In other words, the follower subgraph graph is a fixed graph whereas, at each time instant, the edge with a leader is present if and only if that leader is active at that time instant.

More formally, we consider a network of agents labeled by a set of identifiers $I = \{1, \ldots, n\}$, $n \in \mathbb{N}$, such that the labels $\{1, \ldots, n^f\}$, $n^f \in \mathbb{N}$, correspond to the followers and the remaining ones to the leaders. The agents leave in the state space $\mathbb{R}^d$, $d \in \mathbb{N}$, and obey a first order, continuous time dynamics. We assume that the dynamics along each dimension can be decoupled so that along each direction, it is given by $\dot{x}_i = u_i$, for all $i \in \{1, \ldots, n\}$, where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are respectively the state and the input of agent i. The agents communicate according to a communication edge map $t \mapsto E(t)$, $t \in \mathbb{R}_+$, defined as follows: an edge (i, j) belongs to $E(t)$ if and only if agents i and j can communicate at time t. We assume E to be piecewise constant and defined according to the following set up. We let $G_a = (I, E_a)$ be a time invariant undirected connected graph describing the communication among leaders and followers when *all* the leaders are active (hence the subscript a). The communication edge set $E(t)$ at instant t is a subset of E_a obtained by dropping the edges $(i, j) \in E_a$ such that i is a follower and j is a nonactive leader. We denote by $G(t) = (I, E(t)) \subset G_a$ the graph at instant t.

The followers subgraph is assumed to be fixed and connected, and is denoted by $G^f = (\{1, \ldots, n^f\}, E^f)$. Also, we let N_i^f be the static set of followers communicating with follower i and N_i^l the total set of leaders that (may) communicate with follower i.

In order to model the fact that a leader j is visible or invisible, we define the switching signal $T : \mathbb{R}_+ \to \{0, 1\}$ as

$$T_j(t) = \begin{cases} 1 & \text{if leader } j \text{ is active} \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

Before introducing the network dynamics, we state the standing assumptions that we will use in the paper.

Standing Assumptions (SA)

i) Each follower communicates with all active leaders. That is, for any $t \in \mathbb{R}_+$ and any active leader $j_a \in \{n^f + 1, \ldots, n\}$, then $(i, j_a) \in E(t)$ for all $i \in \{1, \ldots, n^f\}$.
ii) For any $t \in \mathbb{R}_+$, there exists at least one active leader.
iii) For each leader $j \in \{n^f + 1, \ldots, n\}$ the switching signal $T_j(\cdot)$ has positive average dwell time, that is, $T_j(\cdot) \in \mathscr{S}_a$.

Due to Standing Assumption i), we have that N_i^l does not depend on agent i, therefore we will denote it by N^l.

We are now ready to introduce the network dynamics, and we let the dynamics of the followers is given by

$$\dot{x}_i^f(t) = -\sum_{j\in N_i^f}(x_i^f(t)-x_j^f(t)) - \sum_{j\in N^l} T_j(t)(x_i^f(t)-x_j^l(t)) \quad i\in\{1,\ldots,n^f\}. \tag{3}$$

The leaders are stationary, i.e. their dynamics is simply

$$\dot{x}_i^l(t) = 0,\ i\in\{n^f+1,\ldots,n\}. \tag{4}$$

And, although leaders do not apply a Laplacian based control law (as the follower do), it is useful to consider the dynamics obtained if all the agents (both leaders and followers) applied such control law. Indeed, the dynamics would be

$$\dot{x}(t) = -L(t)x(t), \tag{5}$$

where $x=[x_1,\ldots,x_n]^T$ and $L(t)$ is the Laplacian of the graph $G(t)$ at instant t. If we partition the Laplacian with respect to leaders and followers as

$$L(t) = \begin{bmatrix} L^f(t) & l^{fl}(t) \\ l^{lf}(t) & L^l(t) \end{bmatrix} \tag{6}$$

the followers dynamics becomes

$$\dot{x}^f(t) = -L^f(t)x^f(t) - l^{fl}x^l(t). \tag{7}$$

In particular, due to Standing Assumption i), we may rewrite this

$$\dot{x}^f(t) = -L^f(t)x^f(t) + \mathbf{1}_{n^f}\mathbf{1}_{n^l}^T x^l(t). \tag{8}$$

It is worth noting that $L^f(t)$ is not the follower's Laplacian, but depends on active leaders (this is why it is time dependent). It can be written in terms of the Laplacian of the follower's subgraph, L_0^f, as

$$L^f(t) = L_0^f + n_a^l(t) I_{n^f} \tag{9}$$

where $n_a^l(t)\in\mathbb{N}$ is the number of active leaders at time t, and I_{n^f} is the identity matrix of dimension n^f.

In order to emphasize the switched nature of the followers dynamics, we will rewrite it as

$$\dot{x}^f(t) = -L_{\sigma(t)}^f x^f(t) + \mathbf{1}_{n^f}\mathbf{1}_{n^l}^T x_{\sigma(t)}^l, \tag{10}$$

where $\sigma:[0,T)\to\Gamma=\{0,1\}^{n^l}$ is a suitable (piecewise constant) switching signal such that

$$L_{\sigma(t)}^f = L_0^f + \sum_{j\in N^l} T_j(t) I_{n^f}$$

and $x_{\sigma(t)}^l = [x_1^l T_1(t)\ldots x_{n^l}^l T_{n^l}(t)]^T$. Consistently with Standing Assumption iii), $\sigma(\cdot)$ belongs to $\mathscr{S}_a$ (the set of switching signals with positive average dwell-time and chatter bound).

IV. Problem Statement and Static Case

Before we can state and prove the main result, i.e. that the followers end up in the convex hull spanned by the static and only intermittently visible leaders, we first need to investigate and recall what happens when the leaders are all visible all the time. This is the topic under consideration in this section, i.e. we investigate what happens under static network topologies.

One standard way in which the graph Laplacian can be obtained is from the product of incidence matrices,

$$L = \mathscr{I}^\sigma \mathscr{I}^{\sigma T},$$

where we have assumed that the network is *static* (L does not depend on t), and where $\sigma: E\to\{-1,1\}$ is any arbitrary orientation assignment to the edges of the graph (essentially turning it from an undirected to a directed graph), $\mathscr{I}\in\mathbb{R}^{n\times m}$ is the incidence matrix, where m is the number of edges in the graph.

By writing the incidence matrix as $\mathscr{I}^T = [\mathscr{I}^{f^T}, \mathscr{I}^{l^T}]^T$, where $\mathscr{I}^f\in\mathbb{R}^{n^f\times m}$ and $\mathscr{I}^l\in\mathbb{R}^{(n-n^f)\times m}$ (note here we have dropped the explicit dependence on σ since it does not matter what σ is[1]), we get

$$L^f = \mathscr{I}^f\mathscr{I}^{f^T},\ L^l = \mathscr{I}^l\mathscr{I}^{l^T} \text{ and } l^{fl} = \mathscr{I}^f\mathscr{I}^{l^T}. \tag{11}$$

We know that $L\succeq 0$. In addition, if the graph is connected, we have that $\mathrm{null}(L)=\mathrm{span}\{\mathbf{1}_n\}$. And, since

$$x^{f^T}L^f x^f = [x^{f^T}\ 0]L\begin{bmatrix} x^f \\ 0 \end{bmatrix}$$

and $[x^{f^T}\ 0]^T\notin\mathrm{null}(L)$, we have that

$$[x^{f^T}\ 0]L\begin{bmatrix} x^f \\ 0 \end{bmatrix} > 0\ \forall\ x^f\neq 0.$$

As such, we have the following result (see for example [5] for another version of this proof)

Lemma 4.1: If the graph is connected, then L^f is positive definite,

This lemma allows us to state the following lemma (also avialable in [5])

Lemma 4.2: Given fixed leader positions x^l, the quasi-static equilibrium point[2] is

$$x^f = -L^{f^{-1}}L^{fl}x_l, \tag{12}$$

which is globally asymptotically stable.

As a result of this, we have that if the leaders are stationary (located at x^l), the followers will asymptotically approach the equilibrium point

$$x^{f^e} = -L^{f^{-1}}L^{fl}x^l.$$

[1]Even though $\mathscr{I}^\sigma$ depends on σ, L does *not*.

[2]A process is called quasi-static when it follows a succession of equilibrium states. In such a process, a sufficiently slow transition of a thermodynamic system from one equilibrium state to another occurs such that at every moment in time the state of the system is close to an equilibrium state.

Now, since x_f^e is an equilibrium, we must have that

$$\dot{x}_i^e = 0 = -\sum_{j\in N(i)} (x_i^e - x_j^e)$$

for all follower agents. (Here we have used the notation that if agent j is a leader, x_j^e is the static position of that leader, even though it strictly speaking is not an equilibrium point but a static input.) What this means is that

$$x_i^e = \frac{1}{|N(i)|}\sum_{j\in N(i)} x_j^e.$$

In other words, the equilibrium point x_i^e for follower agent i lies in the convex hull spanned by agent i's neighbors - may they be leaders of followers.

Now, if every follower ends up in the convex hull spanned by its neighbors, and the only agents who do not need to satisfy this are the leaders, every follower will end up in the convex hull spanned by the leaders. We denote this convex hull by Ω_L and we recall the following key lemma from [4] (formulated in a slightly different way)

Lemma 4.3: Given a connected, static network topology with multiple static leaders, the followers will asymptotically end up in the convex hull spanned by the leaders, i.e.

$$x_i^e \in \Omega_L,\ i = 1,\ldots,n^f.$$

So far, in this section, the results are previously known. What we would like to do in this paper is to extend Lemma **??** to hold also for intermittently visible leaders. In other words, we would like to show that even though only a non-empty subset of the leaders is visible at any given time, the followers will still converge asymptotically to the convex hull spanned by *all* leaders. It should be noted that a similar question was pursued in [16] but there it was shown that the followers end up in a larger set (ellipsoid) that contains Ω_L. In this paper, we thus make this result much more tight by showing that this set can in fact be shrunk down to Ω_L itself. And for that, we need to combine the tools from the switched LaSalle's invariance principle, with the geometrical arguments presented in this section.

V. An Invariant Set for Networks with Intermittent Leaders

In this section, we prove the main result of the paper, i.e. that the followers (running the Laplacian based averaging law), driven by the intermittently visible, stationary leaders, asymptotically converge to the convex hull of the all leaders.

Lemma 5.1: Let $\mathscr{N}$ be a leader-follower network as in Section III, with stationary, intermittently visible leaders, and followers dynamics as in (10). Suppose that the Standing Assumptions (SA) hold. Then, for any $x^f(0) \in \mathbb{R}^n$, and any $\sigma \in \mathscr{S}_a[\tau_D, N_0]$, $\tau_D > 0$ and $N_0 \in \mathbb{N}$, there exists a compact set $B \subset \mathbb{R}^n$ such that $x(t) \in B$ for all $t \geq 0$.

Proof: Let x^l be the state of all the leaders. Consider the change of variable $\delta = x^f - \mathbf{1}_{n^f}\mathbf{1}_{n^l}^T x^l/n^l$. Informally, δ is the distance of the follower states from the centroid of *all* leaders. The followers' dynamics, written with respect to δ, is

$$\dot{\delta} = -L^f_{\sigma(t)}\delta - L^f_{\sigma(t)}\mathbf{1}_{n^f}\mathbf{1}_{n^l}^T x^l/n^l + \mathbf{1}_{n^f}\mathbf{1}_{n^l}^T x^l_{\sigma(t)} \qquad (13)$$

Using the fact that $L^f_{\sigma(t)} = L^f_0 + n^l_a(t) I_{n^f}$ and $L^f_0\mathbf{1}_{n^f} = 0$, we may rewrite the δ dynamics as

$$\begin{aligned}\dot{\delta} &= -L^f_{\sigma(t)}\delta - n^l_a(t)/n^l\mathbf{1}_{n^f}\mathbf{1}_{n^l}^T x^l + \mathbf{1}_{n^f}\mathbf{1}_{n^l}^T x^l_{\sigma(t)} \\ &= -L^f_{\sigma(t)}\delta + \mathbf{1}_{n^f}(\bar{x}^l_{\sigma(t)} - \bar{x}^l))n^l_a(t),\end{aligned} \qquad (14)$$

where $\bar{x}^l_{\sigma(t)}$ is the centroid of the *active* leaders, and $\bar{x}^l$ is the centroid of *all* leaders (active and inactive). Since the set of leaders is finite and they are stationary, the distance $|\bar{x}^l_{\sigma(t)} - \bar{x}^l|$ is bounded (e.g. by the maximum distance between each leader and the centroid of all the leaders). Also, $n^l_a(t)$ is bounded by n^l.

Remembering that $-L^f_{\sigma(t)}$ is a *uniformly* negative definite, exponentially stable matrix, we have a uniformly exponentially stable, linear, time-varying system driven by a bounded input. This implies that $\delta(t)$, and therefore $x^f(t)$, $t \in \mathbb{R}_+$, is bounded. [3] ■

We are now ready to state the main result:

Theorem 5.1: Let $\mathscr{N}$ be a leader-follower network as in Section III, with stationary, intermittently visble leaders, and follower dynamics as in (10). Let Ω_L be the convex hull of all the leaders and suppose that the Standing Assumptions (SA) hold. Then the set $\Omega_L \times \Gamma$ is (weakly) invariant with respect to the followers' trajectories, and the followers asymptotically converge to Ω_L. In other words, for any $\varepsilon > 0$, there exists $\bar{t} > 0$ such that dist

$$\left(x^f_j(t), \Omega_L\right) < \varepsilon$$

for all $t \geq \bar{t}$.

Proof: We prove the result in two steps. In the first step, we show that (along each dimension) the set $\Omega_F \times \Gamma$, with $\Omega_F = \{(x^f_1,\ldots,x^f_{n^f}) \in \mathbb{R}^{n^f} \mid x^f_1 = \ldots = x^f_{n^f}\}$, is (weakly) invariant with respect to the followers' trajectories, and that the followers asymptotically converge to Ω_F. In the second step, we prove that the average of the positions (in $\mathbb{R}^d$) of the followers converges to the convex hull of all the leaders.

In order to prove the first part, consider the continuous function $V_1(x^f,\gamma) = 1/2(x^f)^T L^f_0 x^f$. The derivative along the followers trajectories is, for all $(x^f,\gamma) \in \mathbb{R}^{n^f} \times \Gamma$,

$$\begin{aligned}\frac{\partial V_1}{\partial x^f}(x^f,\gamma) f(x^f,\gamma) &= -(x^f)^T L^f_0 L^f_\gamma(x^f) + (x^f)^T L^f_0 \mathbf{1}_{n^f}\mathbf{1}_{n^l}^T x^l_\gamma \\ &= -(x^f)^T L^f_0 L^f_\gamma(x^f).\end{aligned} \qquad (15)$$

For any γ such that at least one leader is active (Standing Assumption ii)), L^f_γ is a positive definite matrix. Also, L^f_0 is a positive semidefinite matrix such that $L^f_0\mathbf{1}_{n^f} = 0$.

Let $\mathscr{T}^f_a[\tau_D, N_0]$, for given $\tau_D > 0$ and $N_0 \in \mathbb{N}$, be the class of followers trajectories $(x^f(\cdot),\sigma(\cdot))$ such that $\sigma(\cdot) \in \mathscr{S}_a[\tau_D, N_0]$. We denote $\mathscr{T}^f_{V_1}$ the subclass of followers trajectories $(x^f(\cdot),\sigma(\cdot)) \in \mathscr{T}_a[\tau_D, N_0]$ such that for any $t \leq t'$ such that $\sigma(t) = \sigma(t')$, then $V_1(x^f(t),\sigma(t)) \geq V_1(x^f(t'),\sigma(t'))$. From equation (15) it follows that $\mathscr{T}_{V_1} = \mathscr{T}_a[\tau_D, N_0]$.

[3] To explicitly show it one could choose, for example, the Lyapunov function $V(\delta) = 1/2\delta^T\delta$.

If we denote $\mathscr{T}^*_{V_1} = \{(x^f(\cdot),\sigma(\cdot)) \in \mathscr{T}_a[\tau_D,N_0] \mid x^f_1(t) = \ldots = x^{n^f}(t)$ for all $t \in \mathbb{R}_+\}$, using Theorem 2.1, we have that the set $\Omega_F \times \Gamma$ is the largest (weakly) invariant set for the followers trajectories $\mathscr{T}^*_{V_1}$ and that $x^f(t)$ converges to Ω_F as $t \to \infty$ for all $(x^f(\cdot),\sigma(\cdot)) \in \mathscr{T}_a[\tau_D,N_0]$, and the first part of the proof follows.

In the second step, we prove that the average of the followers converges to the convex-hull. Let $\bar{x}^f \in \mathbb{R}^d$, $\bar{x}^l \in \mathbb{R}^d$, and $\bar{x}^l_\gamma \in \mathbb{R}^d$ denote, respectively, the average of the followers, of all the leaders and of the active (for a given γ) leaders.

Consider the function $V_2(\bar{x}^f,\gamma) = \text{dist}(\bar{x}^f,\Omega_l)$, i.e. the distance between the average position of the followers and the convex hull of the leaders. Clearly, V is a continuous function. In order to invoke (again) Theorem 2.1, we need to show that for any $t \le t'$ such that $\sigma(t) = \sigma(t')$, $V_2(\bar{x}^f(t),\sigma(t)) \ge V_2(\bar{x}^f(t'),\sigma(t'))$. In other words, we need to show that for any $t \le t'$ such that $\sigma(t) = \sigma(t')$, $\text{dist}(\bar{x}^f(t),\Omega_L) \le \text{dist}(\bar{x}^f(t'),\Omega_L)$. The dynamics of $\bar{x}^f$ between two switching intervals is

$$\begin{aligned}\dot{\bar{x}}^f(t) &= -n^l_\sigma(t)(\bar{x}^f(t) - \bar{x}^l_{\sigma(t)}) \\ &= -n^l_\gamma(\bar{x}^f(t) - \bar{x}^l_\gamma),\end{aligned} \tag{16}$$

where n^l_γ is the number of active leaders between two switching intervals and $\bar{x}^l_\gamma \in \mathbb{R}^d$ their average position. This means that $\bar{x}^f(t)$ converges monotonically to $\bar{x}^l_\gamma$ and therefore for any $t \le t'$ such that $\sigma(t) = \sigma(t') = \gamma$, $\bar{x}^f(t')$ may be written as a linear combination of $\bar{x}^f(t)$ and $\bar{x}^l_\gamma$. That is, we may write

$$\bar{x}^f(t') = \lambda\bar{x}^f(t) + (1-\lambda)\bar{x}^l_\gamma.$$

Now, using the fact that $\text{dist}(\bar{x}^l_\gamma,\Omega_L) = 0$ and that $\text{dist}(\cdot,\Omega_L)$ is a convex function we have

$$\begin{aligned}\text{dist}(\bar{x}^f(t'),\Omega_L) &\le \lambda\text{dist}(\bar{x}^f(t),\Omega_L) + (1-\lambda)\text{dist}(\bar{x}^l_\gamma,\Omega_L) \\ &\le \text{dist}(\bar{x}^f(t),\Omega_L).\end{aligned}$$

Therefore we may apply Theorem 2.1 to the switching system describing the dynamics of $\bar{x}^f$. The set $\Omega_L \times \Gamma$ is the largest (weakly) invariant set for the class of trajectories $\bar{\mathscr{T}}^*_{V_2} = \{(x^f(\cdot),\sigma(\cdot)) \in \bar{\mathscr{T}}_a[\tau_D,N_0] \mid x^f_1(t) = \ldots = x^{n^f}(t)$ for all $t \in \mathbb{R}_+\}$, and the theorem follows. ■

VI. Simulations

In this section we provide a simulation illustrating the analysis performed in the paper. We simulate a leader-follower network scenario with 30 followers (dots) and 4 leaders (squares), as shown in Figure 1. Leaders in the network that are visible to (all) the followers are selected at random and are shown with a ring around them. The simulation illustrates the fact that the followers in the network converge to locations inside the convex hull spanned by all the leaders.

VII. Conclusions

In this paper we showed that it is possible to characterize the set to which a collection of follower agents converge to as the convex hull spanned by the leader agents. This is the case even if only a non-empty subset of the leader agents are visible to the followers at each instant of time. As a result, a tighter result is obtained as compared to that in [16], where the region of attraction was found to be a subset of the convex hull of the leaders. The main result in this paper relies on recent advances in the switched LaSalle invariance principle, and it can help explain the swarming behaviors observed in the silkworm moth, where the male moths are attracted to the female moths that only intermittently release pheromones that can be detected by the males.

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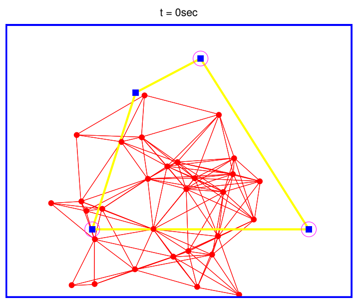
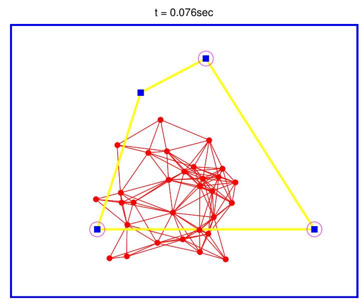
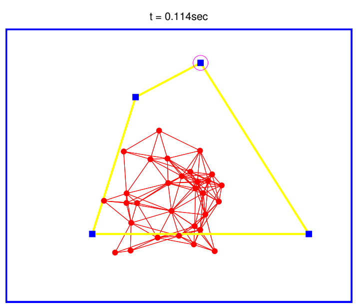


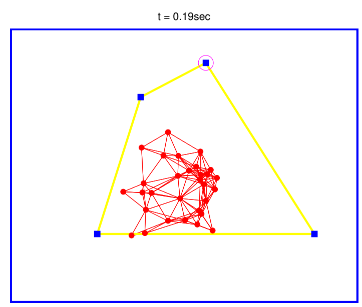
(a) Followers are influenced by three leaders.



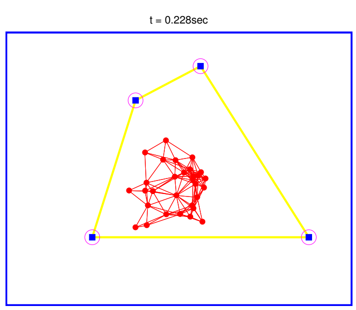
(b) Followers are influenced by three leaders.



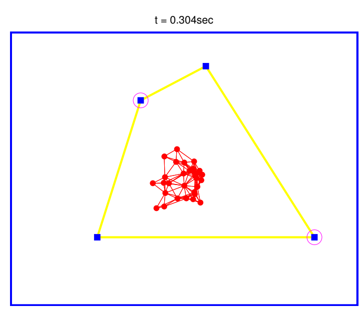
(c) Followers are influenced by one leader.



(d) Followers are influenced by one leader.



(e) Followers are influenced by four leaders.



(f) Followers are influenced by two leaders.

Fig. 1. Simulation of followers (dots) converging to the convex hull spanned by all leaders (squares). The convex hull is shown by the line segments connecting the leaders, while lines between followers denote edges. A leader with a ring around indicates that it is visible by all followers.