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A CATEGORY FOR EXTENSIVE-FORM GAMES

Peter A. Streufert
Economics Department
Western University

Abstract. This paper introduces **Gm**, which is a category for extensive-form games. It also provides some applications.

The category's objects are games, which are understood to be sets of nodes which have been endowed with edges, information sets, actions, players, and utility functions. Its arrows are functions from source nodes to target nodes that preserve the additional structure. For instance, a game's information-set collection is newly regarded as a topological basis for the game's decision-node set, and thus a morphism's continuity serves to preserve information sets. Given these definitions, a game monomorphism is characterized by the property of not mapping two source runs (plays) to the same target run. Further, a game isomorphism is characterized as a bijection whose restriction to decision nodes is a homeomorphism, whose induced player transformation is injective, and which strictly preserves the ordinal content of the utility functions.

The category is then applied to some game-theoretic concepts beyond the definition of a game. A Selten subgame is characterized as a special kind of categorical subgame, and game isomorphisms are shown to preserve strategy sets, Nash equilibria, Selten subgames, subgame-perfect equilibria, perfect-information, and no-absentmindedness. Further, it is shown that the full subcategory for distinguished-action sequence games is essentially wide in the category of all games, and that the full subcategory of action-set games is essentially wide in the full subcategory for games with no-absentmindedness.

1. Introduction

1.1. A foundational question

Extensive-form games can be specified in many different styles, as shown in Figures 1.1–1.4 below. Game theorists informally understand that any substantial concept in one style should have the same meaning in any other style, and that any substantial result in one style should also hold in any another style.

This informal understanding might be formally developed. In particular, when a concept or result is translated from one style to another, how would we define the sense in which the translation itself was correct or incorrect? A good answer to this question promises to help us identify and manipulate the substance of game-theoretic concepts and results. This paper is part of a larger agenda which aims to answer this and related questions by means of category theory.

Date: May 24, 2021. *Keywords:* Continuously labeled tree, Selten subgame, Nash equilibrium, subgame-perfect equilibrium, distinguished-action game, sequence game, action-set game. *Classifications:* MSC 91A70, JEL C73. *Contact information:* pstreuf@uwo.ca, 519-661-2111x85384, Economics Department, University of Western Ontario, London, Ontario, N6A 5C2, Canada.

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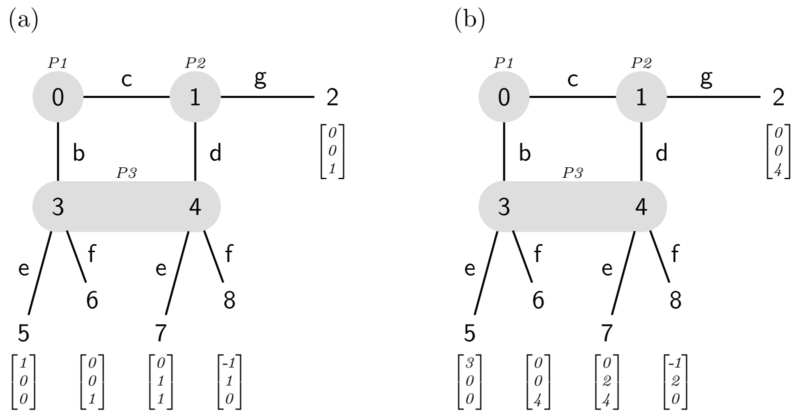

Figure 1.1. Two **Gm** games. In both games, player *P3* selects action e or action f without knowing whether she is at node 3 or node 4. In the context of this paper, the utilities of the two games have the same meaning (see note 2).

1.2. Examples of equivalences

Figures 1.1–1.4 depict seven games, which game theorists regard as "equivalent" in an informal sense. The equivalences between these games illustrate three "dimensions" in which games can be transformed without changing anything substantial. First, utilities can be altered, as in Figure 1.1(a) versus Figure 1.1(b). Second, nodes and actions can be specified in various "styles", as in Figures 1.1(a), 1.2(a,b), and 1.3(a,b). Third, information sets might or might not have distinguished actions, as in Figure 1.1(a) versus Figure 1.4. All three dimensions will be formalized by this paper's isomorphisms, and thus the examples here provide background for the paper's summary in Section 1.3 below.

To be more specific, Figure 1.1 depicts two games, both resembling Selten 1975, Figure 1. For readers less acquainted with game theory, it may be helpful to accompany these games with a story.[1] Note that the two games have the same configuration of nodes, edges, actions, information sets, and players. In both games, nodes are numbers, edges are pairs of nodes shown by line segments, actions are letters, player *P1* moves at the node 0 in the information set {0}, player *P2* moves at the node 1 in the information set {1}, and player *P3* moves at the nodes 3 and 4 in the information set {3,4}.

The two games differ in that they have different utilities. For example, consider the run {0,3,5}, that is, the run (or "play") through nodes 0, 3, and 5. In game (a),

[1]To tell a story matching Figure 1.1's games, suppose a student (called player *P1*) must decide between the bad action of not doing her homework (called b) and the correct action of doing her homework (called c). Knowing that the homework has been finished (node 1), a dog (player *P2*) must decide between the dumb action of eating the homework (d) and the good action of taking a nap (g). Finally, without knowing whether the student chose bad (node 3) or the student chose correct and the dog chose dumb (node 4), the teacher (player *P3*) must decide between excusing the student (e) and failing the student (f). The student most prefers being excused without doing the homework (run {0,3,5}), and least prefers failing after doing the homework (run {0,1,4,8}). The dog likes eating homework (runs {0,1,4,7} and {0,1,4,8}). The teacher does not want to excuse a badly behaving student (run {0,3,5}) or to fail a correctly behaving student (run {0,1,4,8}).

player $P1$ gets utility 1 from this run (shown as the first entry in the vector beneath node 5). Meanwhile in game (b), player $P1$ gets utility 3 from this run. Nonetheless, in the context of this paper, the utilities in the two games have the same meaning in the sense that they have the same ordinal content.[2] For example, in both games, player $P1$ most desires run {0,3,5}, least desires run {0,1,4,8}, and regards the remaining three runs as equally desirable.

In addition to the different ways of assigning utilities, there are different styles in which to specify nodes and actions. The following paragraphs arrange these styles into five broad groups. Streufert 2019, Section 7, explains how each of these five groups has its own advantages and disadvantages. Note that elsewhere in the literature, nodes are sometimes called "vertices", "states", or "histories", and actions are sometimes called "labels", "alternatives", "choices", or "programs".

Group 1. Some styles specify nodes and actions abstractly without restriction. Examples from economics include the styles in Selten 1975 and Myerson 1991, Section 2.1. Examples from computer science and/or logic include the "labeled transition system" style in Blackburn, de Rijke, and Venema 2001, page 3 (and elsewhere); the style in Shoham and Leyton-Brown 2009, page 125; and the "epistemic process graph" style in van Benthem 2014, page 70.[3] A final example is the "**Gm**" style in this paper. Since this group specifies nodes and actions abstractly without restriction, this group can be understood to subsume the other four groups. Relatedly, this paper's **Gm** games include virtually all extensive-form games.[4]

Group 2. Other styles specify nodes as sequences of actions. Examples from economics include the styles in Harris 1985 and Osborne and Rubinstein 1994, page 200. Examples from computer science and/or logic include the "protocol" style in Parikh and Ramanujam 1985, the "history-based multi-agent structure" style in Pacuit 2007, the "sequence-form representation" style in Shoham and Leyton-Brown 2009, page 129, the "logical game" style in Hodges 2013, Section 2, and the "epistemic forest model" style in van Benthem 2014, page 130. A final example is the "sequence" style in this paper (see Figure 1.2(a)).

Group 3. Some new styles specify nodes as sets of actions. These are the "choice-set" style in Streufert 2019, and the closely related "action-set" style in this paper (see Figure 1.2(b)). **Gm** subcategories corresponding to Groups 2 and 3 will be developed in Section 5 below. These subcategories will be called **SGm** and **AGm**.

[2]The utilities in the two games would not have the same meaning if mixed strategies were allowed and the specified utilities were used to construct expected utilities. Expected utilities attach more meaning to the specified utilities, and this additional meaning can be embodied by a **Gm** subcategory that admits only affine utility transformations. The construction of this subcategory is left for future research.

[3]Some labeled transition systems and process graphs have recursive transitions. These do not support extensive-form games because extensive-form games require trees. Similarly, stochastic games, such as those in Mertens 2002, are not extensive-form games because they have recursive transitions.

[4]Some aspects of this claim should be clarified. [a] Although infinite runs are allowed, a **Gm** game is discrete in the sense that each node has a finite number of predecessors. This excludes non-discrete extensive-form games such as those in Dockner, Jørgensen, Long, and Sorger 2000, and Alós-Ferrer and Ritzberger 2016, Chapter 5. [b] A **Gm** game assumes that exactly one player moves at each information set. Accordingly, simultaneous moves by several players are to be specified by several information sets, as discussed in Osborne and Rubinstein, 1994, page 202. [c] A **Gm** game specifies player preferences over runs by means of utility functions. Alternatively, preferences could be specified by binary relations, as in Osborne and Rubinstein 1994, page 201.

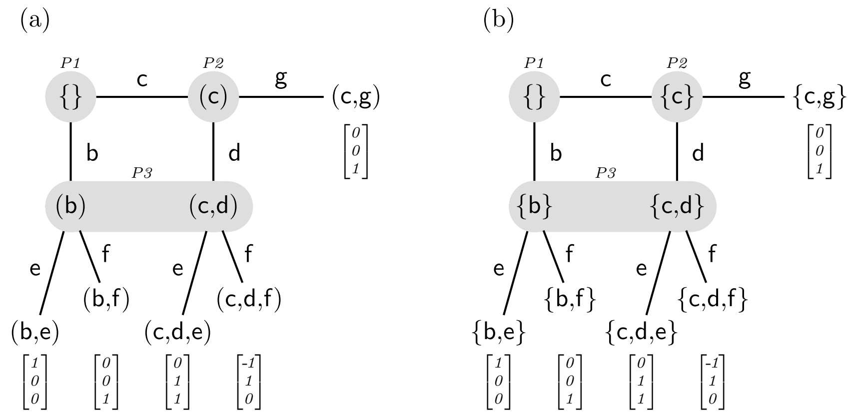

FIGURE 1.2. (a) A sequence game. (b) An action-set game. Section 5 develops subcategories for these kinds of games (**SGm** and **AGm**).

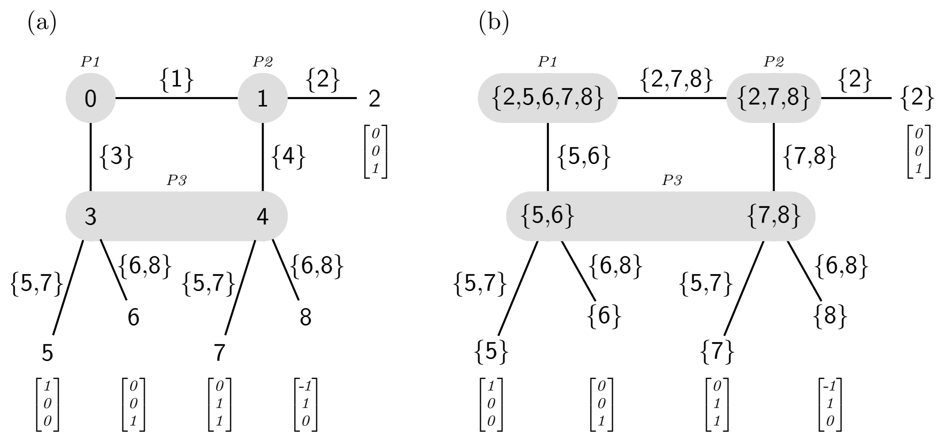

FIGURE 1.3. (a) A node-set game. (b) An outcome-set game. Although such games are **Gm** games, this paper does not define subcategories for these two kinds of games.

Two more groups remain. *Group 4.* Some styles specify actions as sets of nodes. An example is the "simple" style in Alós-Ferrer and Ritzberger 2016, Section 6.3 (see Figure 1.3(a)). *Group 5.* Other styles express both nodes and actions as sets of outcomes. These include the style in von Neumann and Morgenstern 1944, Section 10, and the style in Alós-Ferrer and Ritzberger 2016, Section 6.2 (see Figure 1.3(b)). Although Figure 1.3's games are **Gm** games, the construction of **Gm** subcategories for Groups 4 and 5 is left for future research (see note 5).

Finally, this five-group catalog of styles needs another dimension. In Figure 1.1(a), the actions of each information set are distinguished from the actions of any other information set. In contrast, in Figure 1.4, the actions of one information set are not distinguished from the actions of the other information sets. **Gm** games allow undistinguished actions, and relatedly, Figure 1.4's game is a **Gm** game.

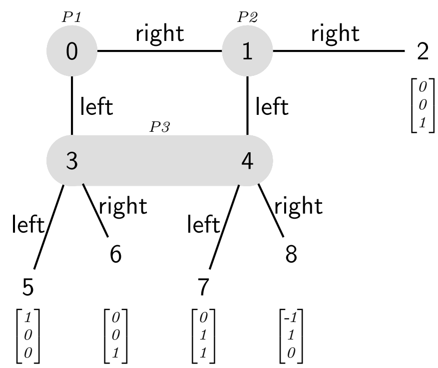

Figure 1.4. A **Gm** game with undistinguished actions.

1.3. Summary

This paper [a] develops a category for games and [b] applies the category to some game-theoretic concepts beyond the definition of a game. For [a], Section 2 defines the category, Section 3.1 characterizes its monomorphisms, and Section 4.1 characterizes its isomorphisms. For [b], Section 3.2 considers Selten subgames, Sections 4.2 and 4.3 consider two equilibrium concepts, and Section 5 considers some properties of games. The results for [a] are called "propositions", and the results for [b] are called "theorems" (the results in the appendices are called "lemmas").

[a] The category itself. To be more specific, Section 2 defines a game to be a set of nodes which is accompanied by a large amount of additional structure. That structure specifies (i) edges which link the nodes to create a tree, (ii) information sets which partition the decision nodes, (iii) actions which label the edges, (iv) an assignment of players to decision nodes which specifies who moves where, and (v) for each player, a utility value for each run of the game. The information-set collection is newly regarded as a basis for a topology on the decision-node set. Then a continuity of the labeling imposes the restriction that the same feasible actions are available at any two nodes in one information set (condition [C4]). Analogously, the continuity of the player assignment imposes the restriction that the same player moves at any two nodes in one information set (condition [G2]). The remaining conditions in the definition of a game are relatively familiar.

Then Section 2 defines a game morphism to be a function from source nodes to target nodes, which preserves all the additional structure. To preserve this structure, a morphism is defined to satisfy six conditions. The first condition ([cE]) requires that edges are preserved. The second condition ([cI]) is the morphism's continuity, which ensures that information sets are preserved. Third, any morphism determines an action transformation at each source decision node. These action transformations are required to vary continuously across decision nodes so that actions are transformed identically at any two source nodes in one information set (condition [cL]). Further, a morphism is required to be compatible with a player transformation which preserves moves (condition [gM]). Yet further, a morphism, with its induced player transformation, is required to weakly preserve the ordinal content of the players' utility functions (condition [gU]). Finally, to make such utility comparisons possible, a morphism is required to preserve the end node of each finite source run (condition [gZ]).

These games and game morphisms become the objects and arrows of the category **Gm**. Proposition 2.4 shows that **Gm** is well-defined. Proposition 2.5 shows that there is a forgetful functor to **CLT**, which is the category of continuously labeled trees (CLTs). CLTs have nodes, edges, information sets, and actions, but do not have players and utilities. Next, Proposition 3.1 shows that a game morphism is a monomorphism iff its induced run transformation is injective. Finally, Proposition 4.1 shows that a game morphism is an isomorphism iff (i) it is bijective, (ii) its restriction to decision nodes is a homeomorphism, (iii) its induced player transformation is injective, and (iv) the ordinal content of the utility functions is strictly preserved. Examples illustrate the independent roles of these four conditions.

[b] Applications. The remainder of the paper's results concern applications of **Gm** to game-theoretic concepts beyond the definition of a game. To begin, recall Selten 1975 defines a kind of subgame which is standard in the game-theory literature. Meanwhile, Section 3.2 defines a categorical subgame to be the source of a monomorphism. Theorem 3.2 characterizes Selten subgames as a special type of categorical subgame in which (i) the subgame's nodes are a particular node and all its successors in the original game, (ii) the subgame's information sets are in the original game, (iii) the subgame's actions and players are as in the original game, and (iv) the subgame's utility functions are essentially restrictions of the original utility functions. Examples explore each of these four conditions.

Section 4.2 derives, from each game, its set of grand strategies, its sets of individual player strategies, and its set of Nash equilibria. Theorems 4.3, 4.4, and 4.5 show that a game isomorphism preserves these sets. The manner in which a game isomorphism preserves strategies and equilibria is somewhat complicated because it must accommodate the action, player, and run transformations that are implied by the isomorphism. In the same vein, Theorems 4.6 and 4.7 show how a game isomorphism preserves Selten subgames and the subgame-perfect equilibria from Selten 1975.

Section 5 develops some full subcategories of **Gm**. The first subcategory is **DGm**, which is for "distinguished-action" games in which the actions of each information set are distinguished from the actions of every other information set. **DGm** contains all six games in Figures 1.1–1.3, but not the game in Figure 1.4. The second subcategory is **SGm**, which is for "sequence" games in which the nodes are sequences of past actions, as in Figure 1.2(a). Theorems 5.1 and 5.2 show that neither subcategory is restrictive in the sense that both (and also their intersection **DSGm**) are essentially wide in **Gm**.

Next Section 5 considers the standard game-theoretic concepts of no-absentmindedness and perfect-information. Theorem 5.3 shows that both are invariant to game isomorphisms. Finally, the section develops the subcategory **AGm**, which is for "action-set" games in which the nodes are sets of past actions, as in Figure 1.2(b). Theorem 5.5 shows that this is very mildly restrictive in the sense that **AGm** is essentially wide in the **Gm** subcategory for games with no-absentmindedness.

1.4. Motivation

Section 1.1 proposed an agenda of translating concepts and results across different styles of games. This section places the present paper within that agenda and within the literature. The first four paragraphs are four steps in the same direction.

Translating Games. A necessary first step is to formally define a notion of equivalence between individual games. This paper does so via **Gm** isomorphisms. For example, the seven games in Figures 1.1–1.4 are isomorphic, in accord with the intuition of game theorists.

Translating "Styles". Theorems 5.1, 5.2, and 5.5 imply categorical equivalences between **Gm** subcategories. These unify and extend earlier ad hoc equivalences in the game-theory literature.[5] In terms of Sections 1.1 and 1.2, the theorems systematically translate between the general "style", the sequence "style", the action-set "style", the distinguished-action "style", and the distinguished-action sequence "style". All proofs are constructive. Specifically, Lemma D.7 takes an arbitrary game and constructs a distinguished-action sequence isomorph, and Lemma D.13 takes an arbitrary game with no-absentmindedness and constructs an action-set isomorph.

Translating Concepts. Theorems 4.5, 4.6, 4.7, and 5.3 show that the concepts of Nash equilibrium, Selten subgame, subgame-perfect equilibrium, no-absentmindedness, and perfect-information are preserved by **Gm** isomorphisms. Thus it is shown how these five concepts can be easily translated from a game to its isomorphs, regardless of "style". (Since the "style" concepts themselves do vary with **Gm** isomorphisms, they have relatively little "substance" in the sense of Section 1.1.)

Translating Results. The above success in translating concepts suggests a future agenda for translating results. Such translations promise conceptual benefits. Foremost in the author's mind is the formal synthesis of results and questions from the many disciplines and subdisciplines which are each studying some version of game theory. There seems much to gain because there is so much diversity. In addition, the author has been made aware of another benefit, namely, that categorical translations between games may allow for syntactic translations between the logical languages that are interpreted in those games. This would accord with the correspondence theory of van Benthem 2001, and Conradie, Ghilardi, and Palmigiano 2014.

Categorical Perspective. In a different direction, it appears that a categorical perspective can sometimes benefit game theory in unexpected ways. For example, consider this paper's novel use of topology to understand a game's collection of information sets. This happened as parallels between **Gm** and **Top** emerged, and the idea seems to measurably clarify the definition of a game. Other examples may appear in the future.

Related Literature. This paper is preceded by Streufert 2018, 2020a and 2020b. These papers develop, respectively, the categories of node-and-choice preforms, node-and-choice forms, and node-and-choice games. The present paper develops

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[5] Theorems 5.2 and 5.5 unify and extend the ad hoc equivalences by Kline and Luckraz 2016 and Streufert 2019. Similar categorical results promise to extend the ad hoc equivalences in Alós-Ferrer and Ritzberger 2016, Section 6.3. This future research will concern the two games in Figure 1.3.

an entirely different formulation which is considerably more transparent. In addition, the present paper is more general in the sense that it does not assume distinguished actions (the other category names use the term "choice" as opposed to "action" because that term is historically correlated with assuming distinguished actions).

A complementary literature studies how finite extensive-form games can be regarded as compositions of fragments called "open games". Within this literature, the papers with nontrivial information sets are Bolt, Hedges, and Zahn 2019, and the very recent work of Capucci, Ghani, Ledent, and Nordvall Forsberg 2021. These papers define open games as morphisms, to be composed within a category. In contrast, the present paper defines games as objects, to be compared within a category. Correspondingly, the compositions there have no counterpart here, and the general morphisms here have no counterpart there.

Incidentally, categories with games as objects have also been constructed for some other kinds of games.[6] Both Lapitsky 1999 and Jiménez 2014 define categories for simultaneous-move games. Machover and Terrington 2014 defines a category for some specialized cooperative games in political science. Finally, Abramsky, Jagadeesan, and Malacaria 2000, Hyland and Ong 2000, McCusker 2000, and Honsell, Lenisa, and Redamalla 2012 develop specialized categories for some extensive-form games in computer science.

1.5. Organization

The paper is organized as described in Section 1.3. Briefly, Section 2 defines the category **Gm**. Section 3 characterizes monomorphisms and Selten subgames. Section 4 characterizes isomorphisms and shows how they preserve Nash equilibria and subgame-perfect equilibria. Finally, Section 5 develops subcategories for distinguished-action games, sequence games, action-set games, games with no-absentmindedness, and games with perfect-information. Appendices A, B, C, and D support Sections 2, 3, 4, and 5, respectively.

2. Definition

This section defines the category **Gm** in three steps. In brief, Section 2.1 reviews oriented trees, Section 2.2 introduces continuously labeled trees (CLTs), and Section 2.3 builds games on CLTs.

2.1. Out-trees

The next two paragraphs define nontrivial out-trees, which are rooted oriented trees with possibly unbounded depth, possibly unbounded and uncountable degree, and at least two nodes. Thereafter are some derivative concepts.

As in Diestel 2010, Chapter 1, an *unoriented graph* is a pair $(X, \mathcal{E})$ such that X is a set and $\mathcal{E}$ is a collection of two-element subsets of X. The elements of X are called *nodes*, and the elements of $\mathcal{E}$ are called *edges*. A *path linking* x_0 *and* x_ℓ is an unoriented graph $(\bar{X}, \bar{\mathcal{E}})$ of the form $\bar{X} = \{x_0, x_1, x_2, \ldots\ x_\ell\}$ and $\bar{\mathcal{E}} = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots\ \{x_{\ell-1}, x_\ell\}\}$ in which distinct i and j satisfy $x_i \neq x_j$.

[6] In addition, other papers study equivalences between games without using category theory. These include the references in note 5, as well as McKinsey 1950, Thompson 1952, Dalkey 1953, Kohlberg and Mertens 1986, Bonanno 1992, Elmes and Reny 1994, and van Benthem 2014 (pages 43–51).

Out-tree (X, E)	
X	set of nodes x
E	set of (oriented) edges xy
r	↳ root node
$\preccurlyeq$	↳ weak tree order
$\prec$	↳ strict tree order
P	↳ strict predecessor correspondence
p	↳ immediate predecessor function
W	↳ set of decision nodes x
$X \smallsetminus W$	↳ set of end nodes x
$\mathcal{Z}$	↳ collection of runs Z
$\mathcal{Z}_{\mathsf{ft}}$	↳ collection of finite runs Z
$\mathcal{Z}_{\mathsf{inft}}$	↳ collection of infinite runs Z

CLT (Continuously Labeled Tree) $\Theta = (X, E, \mathcal{H}, \lambda)$ satisfying [C1]–[C4]	
$\mathcal{H}$	collection of information sets H
λ	labeling function
A	↳ set of actions a
F	↳ feasibility correspondence
n	↳ next-node function
S	↳ set of grand strategies s (Section 4)
o	↳ outcome function (Section 4)

CLT morphism $\theta = [\Theta, \Theta', \tau]$ satisfying [cE], [cI], [cL]	
τ	node transformation
α	↳ action transformation

Game $\Gamma = (X, E, \mathcal{H}, \lambda, \mu, U)$ satisfying [G1]–[G3]	
μ	move-assigning function
I	↳ set of players i
U	profile $\langle U_i \rangle_{i \in I}$ listing a utility function U_i for each player i
W_i	↳ player i's set of decision nodes x (Section 4)
S_i	↳ set of player-i strategies s_i (Section 4)

Game morphism $\gamma = [\Gamma, \Gamma', \tau]$ satisfying [gZ], [gM], [gU]	
ζ	↳ run transformation
ι	↳ player transformation

Table 2.1. Terms defined in Sections 2 and 4. Out-trees, CLTs, CLT morphisms, games, and game morphisms are implicitly accompanied by their components and derivatives (↳).

Further, one graph $(X^o, \mathcal{E}^o)$ is said to be *in* another graph $(X, \mathcal{E})$ iff $X^o \subseteq X$ and $\mathcal{E}^o \subseteq \mathcal{E}$. An *unoriented tree* is an unoriented graph $(X, \mathcal{E})$ in which every two elements of X are linked by exactly one path in $(X, \mathcal{E})$.

As in Bang-Jensen and Gutin 2009, Chapter 1, an *oriented graph* is a pair (X, E) such that X is a set and E is a collection of ordered pairs from X such that $(\forall x{\in}X, y{\in}X)\ (x, y) \in E \Rightarrow (y, x) \notin E$ [this implies $(\forall x{\in}X)\ (x, x) \notin E$]. Denote the edges of an oriented graph by xy rather than (x, y). It is easily seen that each oriented graph (X, E) determines an unoriented graph $(X, \mathcal{E})$ by means of $E \ni xy \mapsto \{x, y\} \in \mathcal{E}$. An *oriented tree* is an oriented graph whose unoriented graph is an unoriented tree. Further, an *out-tree* is an oriented tree (X, E) which has a node

$r \in X$ such that $X \setminus \{r\} = \pi_2 E$, where $\pi_2 E$ is the projection of E onto its second coordinate. Call r the *root node*. Finally, an out-tree (X, E) is *nontrivial* iff $E \neq \varnothing$.

A *path from* x_0 *to* x_ℓ is an oriented graph $(\bar{X}, \bar{E})$ of the form $\bar{X} = \{x_0, x_1, x_2, ... x_\ell\}$ and $\bar{E} = \{x_0 x_1, x_1 x_2, ... x_{\ell-1} x_\ell\}$ in which distinct i and j satisfy $x_i \neq x_j$. Similarly, *infinite path from* x_0 is an oriented graph $(\hat{X}, \hat{\mathcal{E}})$ of the form $\hat{X} = \{x_0, x_1, x_2, ...\}$ and $\hat{E} = \{x_0 x_1, x_1 x_2, ...\}$ in which distinct i and j satisfy $x_i \neq x_j$. If a path is in an out-tree, the path's indices and edge set are redundant. In particular, if $(\dot{X}, \dot{E})$ is a path in an out-tree (X, E), then $\dot{E} = \{\, xy{\in}E \mid \{x, y\}{\subseteq}\dot{X} \,\}$. Accordingly, the paths in an out-tree will be identified with their node sets.

Consider a nontrivial out-tree (X, E). Let $\preccurlyeq$ be the binary relation on X defined by $x \preccurlyeq y$ iff there is a path from x to y. Call $\preccurlyeq$ the *weak tree order*. Next let $\prec$ be the binary relation on X defined by $x \prec y$ iff $x \preccurlyeq y$ and $x \neq y$ (this excludes trivial one-node paths). Call $\prec$ the *strict tree order*. Further, define the correspondence[7] $P{:}X{\rightrightarrows}W$ by $P(y) = \{\, x{\in}X \mid x{\prec}y \,\}$. Call P the *strict predecessor correspondence*. It is easily shown that $(\forall y{\in}X)$ $P(y){\cup}\{y\}$ is (the node set of) the path from r to y. Incidentally, $P(r){\cup}\{r\}$ is the trivial path $\{r\}$.

Further, define $W = \pi_1 E$, where $\pi_1 E$ is the projection of E onto its first coordinate. Call W the set of *decision nodes*. It is easily shown that E determines the surjective function p with domain $X \setminus \{r\}$, codomain W, and graph $\{(y, x) | xy{\in}E\}$. Call p the *(immediate) predecessor function*. It is easily shown that $(\forall y{\in}X{\setminus}\{r\})$ $p(y) \in P(y)$.

Call $X \setminus W$ the set of *end nodes*, and let $\mathcal{Z}_{\mathsf{ft}}$ be the collection of all paths from r to an end node. In addition, let $\mathcal{Z}_{\mathsf{inft}}$ be the collection of all infinite paths from r. Finally, let $\mathcal{Z} = \mathcal{Z}_{\mathsf{ft}} {\cup} \mathcal{Z}_{\mathsf{inft}}$, and call $\mathcal{Z}$ the set of *runs* (elsewhere "plays"). Nontriviality implies $\mathcal{Z}$ is nonempty. Possibly $\mathcal{Z} = \mathcal{Z}_{\mathsf{ft}}$, possibly $\mathcal{Z} = \mathcal{Z}_{\mathsf{inft}}$, and possibly both $\mathcal{Z}_{\mathsf{ft}}$ and $\mathcal{Z}_{\mathsf{inft}}$ are nonempty.

2.2. Continuously labeled trees (CLTs)

Consider a nontrivial out-tree (X, E). Let $\mathcal{H}$ be a partition of W, and call its elements *information sets*. One can regard $\mathcal{H}$ as the basis of a topology for W. Call this topology the *information topology*. This paper uses the information topology for W and the discrete topology everywhere else.

Let λ be a surjective function with domain E. Call λ the *labeling function*,[8] and say that $\lambda(xy)$ *labels* the edge xy. Next define A to be the codomain of λ.[9] Although A could be called the set of "labels", A will instead be called the set of *actions*. Finally, it will be assumed that λ is *deterministic* (Blackburn, de Rijke, and Venema 2001, page 3) in the sense that for any two edges of the form xy_1 and xy_2, $\lambda(xy_1) = \lambda(xy_2)$ implies $y_1 = y_2$.

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[7]To be clear, a *correspondence* $F{:}X{\rightrightarrows}Y$ is taken to be a triple (X, Y, F^{gr}) such that $F^{\mathsf{gr}} \subseteq X{\times}Y$, and a *function* $f{:}X{\rightarrow}Y$ is taken to be a triple (X, Y, f^{gr}) such that $f^{\mathsf{gr}} \subseteq X{\times}Y$ and $(\forall x{\in}X)(\exists! y{\in}Y)$ $(x, y) \in f^{\mathsf{gr}}$ (the three components of a correspondence or function are its domain, codomain, and graph).

[8]Since there is a bijection between E and $X \setminus \{r\}$, a labeling function λ on E could be alternatively specified by a function on $X \setminus \{r\}$.

[9]To be clear, recall from note 7 that a function is regarded as a triple. First, $\lambda = (\lambda^{\mathsf{dom}}, \lambda^{\mathsf{cod}}, \lambda^{\mathsf{gr}})$ is assumed to be a function such that $\lambda^{\mathsf{dom}} = E$ and $\lambda^{\mathsf{cod}} = \pi_2 \lambda^{\mathsf{gr}}$, where $\pi_2 \lambda^{\mathsf{gr}}$ is the projection of λ^{gr} onto its second coordinate. Second, A is defined to be λ^{cod}. Since A is derived from λ, A will not appear in the tuple defining a CLT. Accordingly, Table 2.1 lists λ as a component of a CLT, and A as a derivative of a CLT.

From λ derive the correspondence $F{:}W \rightrightarrows A$ by

$$(\forall x{\in}W)\ F(x) = \{\, a{\in}A \,|\, (\exists y{\in}X{\smallsetminus}\{r\})\, \lambda(xy){=}a \,\}.$$

Thus $F(x)$ is the set of actions that label the edges leaving x. Call F the *feasibility correspondence*, and call $F(x)$ the set of actions that are *feasible* at x.

If λ is deterministic, there is a function $n{:}F^{\mathsf{gr}} \to X{\smallsetminus}\{r\}$ which takes each[10] $(x,a) \in F^{\mathsf{gr}}$ to the unique $y \in X{\smallsetminus}\{r\}$ such that $\lambda(xy) = a$ (Lemma A.1(b)). Call n the *next-node function*. Notice that if the feasible action $a \in F(x)$ is chosen at node x, then the next node will be $n(x,a)$.

From a different perspective, $F = \langle F(x) \rangle_{x \in W}$ can be regarded as a set-valued function from W. It will be assumed that F is continuous as a function of W, endowed with the information topology, into the collection of subsets of A, endowed with the discrete topology.[11] This is equivalent to assuming that two nodes in one information set have the same feasible set.

A *continuously labeled tree (CLT)* is a tuple $\Theta = (X, E, \mathcal{H}, \lambda)$ such that

[C1] (X, E) is a nontrivial out-tree,

[C2] $\mathcal{H}$ is a partition of W,

[C3] λ is a deterministic surjective function from E, and

[C4] $\langle F(x) \rangle_{x \in W}$ is continuous from W.

Figures 2.1 and 2.2 provide four examples (temporarily ignore the dashed arrows suggesting morphisms, and the remarks about morphisms in the captions). A tree diagram with underlined root node, shaded information sets, and labeled edges unambiguously specifies all the components of a CLT.

The ensuing paragraphs will define CLT morphisms. This paragraph begins by fixing some general notation. For $f{:}X \to Y$ and $A \subseteq X$, let $\bar{f}(A) = \{\, f(x) \,|\, x{\in}A \,\}$. Also, for $f{:}X \to Y$, $A \subseteq X$, and $B \subseteq Y$, let $f|_{A,B}$ be the function with domain A, codomain B, and the graph of $f|_A$. This construction is well-defined iff $\bar{f}(A) \subseteq B$.

Consider a source CLT Θ and a target CLT Θ'. A *node transformation* is a function of the form $\tau{:}X \to X'$. It will be assumed that τ preserves edges in the sense that $(\forall xy{\in}E)\ \tau(x)\tau(y) \in E'$. Edge preservation easily leads to several observations about the out-trees (X, E) and (X', E'). These are included far below in Proposition 2.1(a–e). Among them is $\bar{\tau}(W) \subseteq W'$, which implies that $\tau|_{W,W'}$ is well-defined (recall the general notation of the previous paragraph).

It will also be assumed that $\tau|_{W,W'}$ is continuous. This is equivalent to saying that the inverse image of every target information set is the union of a collection of source information sets. For another perspective, recall that W and W' are endowed with partition topologies. Thus continuity is equivalent to saying that the image of each source information set is included in a target information set (Lemma A.2). In other words, continuity says that source information sets cannot be "split".

For example, consider Figure 2.2. There $\tau|_{W,W'} = \mathsf{id}_W$ is discontinuous because the inverse image $\{\, x{\in}W \,|\, \tau(x){\in}\{\mathsf{1}\} \,\} = \{\mathsf{1}\}$ of the target information set $\{\mathsf{1}\}$ fails to be open in the source information topology. Equivalently, $\tau|_{W,W'} = \mathsf{id}_W$ is

[10] F^{gr} is the graph of the correspondence F, as in note 7.

[11] This is equivalent to assuming that $F{:}W \rightrightarrows A$ is lower semicontinuous in the sense of Berge 1963, page 109, when its domain is endowed with the information topology, and when its codomain is endowed with the discrete topology.

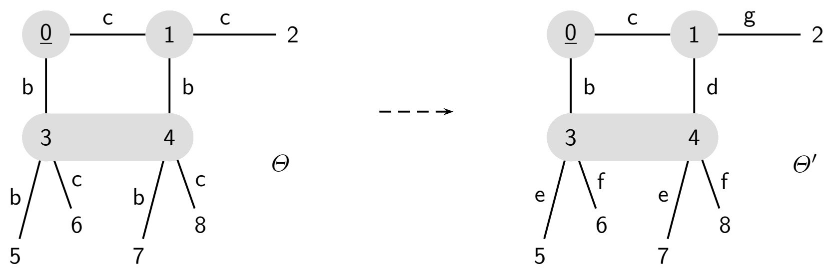

FIGURE 2.1. Θ and Θ' are CLTs. Figure 1.1's games are built on Θ', and game theorists would informally regard Θ and Θ' as equivalent. $[\Theta, \Theta', \mathsf{id}_X]$ is a morphism (where id_X is an identity in **Set**).

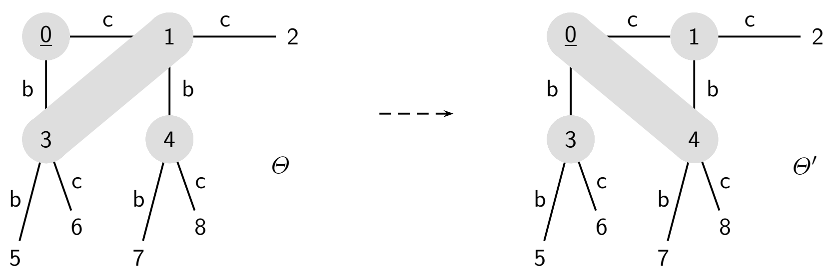

FIGURE 2.2. Θ and Θ' are CLTs. $[\Theta, \Theta', \mathsf{id}_X]$ is not a morphism because $\tau|_{W,W'} = \mathsf{id}_W$ is not continuous.

discontinuous because the image $\bar{\tau}(\{1,3\}) = \{1,3\}$ of the source information set $\{1,3\}$ is not included in a target information set. In other words, $\tau|_{W,W'} = \mathsf{id}_W$ is discontinuous because it "splits" the source information set $\{1,3\}$. In contrast, consider Figure 2.1. There $\tau|_{W,W'} = \mathsf{id}_W$ is continuous since no source information sets are split. The same can be said about Figure 4.1 far below.

To interpret continuity in the context of game theory, remember that continuity prevents source information sets from being split. Thus, since large information sets correspond to less information, continuity corresponds to preserving a lack of information.

From the tuple $[\Theta, \Theta', \tau]$, derive $\alpha = \langle \alpha_x{:}F(x){\to}F'(\tau(x)) \rangle_{x \in W}$ at each $x \in W$ by

$$(\forall a{\in}F(x))\ \alpha_x(a) = \lambda'(\,\tau(x)\,\tau(n(x,a))\,).$$

Call α the tuple's *action transformation*. At each $x \in W$, the function α_x sends source actions feasible at x to target actions feasible at the image of x. In particular, consider a source action a which is feasible at x. This a is sent to the target action that labels the target edge from the image of x to the image of the next source node determined by x and a. This target action belongs to $F'(\tau(x))$ by the definition of F'.

As a whole, $\alpha = \langle \alpha_x \rangle_{x \in W}$ is a function-valued function with domain W. It will be assumed that α is continuous as a function from W, endowed with the information topology, into the set of functions, endowed with the discrete topology. This is equivalent to assuming that $(\forall H{\in}\mathcal{H}, x_1{\in}H, x_2{\in}H)\ \alpha_{x_1} = \alpha_{x_2}$. In words, two nodes in one information set transform actions in the same way.

To explore this carefully, suppose $\{x_1, x_2\} \subseteq H \in \mathcal{H}$ and consider the functions $\alpha_{x_1}{:}F(x_1){\to}F'(\tau(x_1))$ and $\alpha_{x_2}{:}F(x_2){\to}F'(\tau(x_2))$. [C4] for Θ implies $F(x_1) = F(x_2)$. Further, if $\tau|_{W,W'}$ is continuous, there is $H' \in \mathcal{H}'$ such that $\{\tau(x_1), \tau(x_2)\} \subseteq H'$, and thus [C4] for Θ' implies $F'(\tau(x_1)) = F'(\tau(x_2))$. Hence, if $\tau|_{W,W'}$ is continuous, the domains and codomains of α_{x_1} and α_{x_2} coincide, and thus the equality $\alpha_{x_1} = \alpha_{x_2}$ reduces to $(\forall a{\in}F(x_1){=}F(x_2))\ \alpha_{x_1}(a) = \alpha_{x_2}(a)$.

For example, consider Figure 2.3. Here

$$\alpha_3(\mathsf{b}) = \lambda'(\,\tau(3)\,\tau(n(3,\mathsf{b}))\,) = \lambda'(\,\tau(3)\,\tau(5)\,) = \lambda'(\,3\,5\,) = \mathsf{e} \text{ and}$$
$$\alpha_4(\mathsf{b}) = \lambda'(\,\tau(4)\,\tau(n(4,\mathsf{b}))\,) = \lambda'(\,\tau(4)\,\tau(7)\,) = \lambda'(\,4\,7\,) = \mathsf{f}.$$

Since these differ, $\alpha_3 \neq \alpha_4$. Thus since 3 and 4 belong to the same information set in Θ, α is discontinuous. In contrast, Figure 2.1 provides an example of a continuous α. There $\alpha_3(\mathsf{b}) = \alpha_4(\mathsf{b}) = \mathsf{e}$ and $\alpha_3(\mathsf{c}) = \alpha_4(\mathsf{c}) = \mathsf{f}$. Thus (by the previous paragraph), $\alpha_3 = \alpha_4$. So, since $\{3,4\} \subseteq W$ is the only nonsingleton information set in Θ, α is continuous from W.

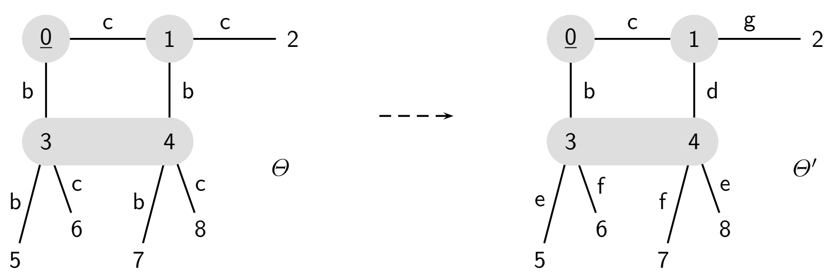

Figure 2.3. Θ and Θ' are CLTs. $[\Theta, \Theta', \mathsf{id}_X]$ is not a morphism because its $\alpha = \langle \alpha_x \rangle_{x \in W}$ is not continuous from W.

Note that the continuity of α from W allows "local" action changes in the sense that actions can be changed differently at different information sets. For example, return to Figure 2.1, where $\alpha_0(\mathsf{b}) = \mathsf{b}$ and $\alpha_3(\mathsf{b}) = \mathsf{e}$. This is consistent with the continuity of α because 0 and 3 are in different information sets.

A CLT morphism is a node transformation which preserves edges, information sets, and labels, but possibly not the root. More precisely, a *CLT morphism* is a tuple $\theta = [\Theta, \Theta', \tau]$ such that Θ and Θ' are CLTs, $\tau{:}X{\to}X'$,

[cE] $\quad (\forall xy{\in}E)\ \tau(x)\tau(y) \in E'$,

[cI] $\quad \tau|_{W,W'}$ is continuous, and

[cL] $\quad \langle \alpha_x \rangle_{x \in W}$ is continuous from W

(where $\langle \alpha_x \rangle_{x \in W}$ is derived from $[\Theta, \Theta', \tau]$). Call [cE] *edge preservation*, call [cI] *information-set preservation*, and call [cL] *label preservation.* Note that the source root r may or may not be taken to the target root r'.

Proposition 2.1. *Suppose that $[\Theta, \Theta', \tau]$ is a morphism. Then the following hold.*

(a) $(\forall x{\in}X, y{\in}X)\ x \preccurlyeq y \Rightarrow \tau(x) \preccurlyeq' \tau(y)$.

(b) $(\forall x{\in}X, y{\in}X)\ x \prec y \Rightarrow \tau(x) \prec' \tau(y)$.

(c) $(\forall y{\in}X)\ \bar{\tau}(P(y)) \subseteq P'(\tau(y))$.[12]

[12] Recall $\bar{f}(A)$ was defined as $\{\, f(x) \,|\, x{\in}A \,\}$ shortly after the definition of a CLT.

(d) $(\forall y{\in}X{\setminus}\{r\})\ \tau(p(y)) = p'(\tau(y))$.
(e) $\bar{\tau}(W) \subseteq W'$.[12]
(f) $(\forall xy{\in}E)\ \alpha_x(\lambda(xy)) = \lambda'(\tau(x)\tau(y))$.
(g) $(\forall x{\in}W, a{\in}F(x))\ \tau(n(x,a)) = n'(\tau(x), \alpha_x(a))$. *(Proof A.3.)*

This paragraph defines the category **CLT**, which is called the *category of CLTs (continuously labeled trees)*. Let an object be a CLT Θ. Let an arrow be a CLT morphism $[\Theta, \Theta', \tau]$. Let source, target, identity, and composition be

$$[\Theta, \Theta', \tau]^{\mathsf{src}} = \Theta,\ [\Theta, \Theta', \tau]^{\mathsf{trg}} = \Theta',\ \mathsf{id}_\Theta = [\Theta, \Theta, \mathsf{id}_X],\ \text{and}$$
$$[\Theta', \Theta'', \tau']\circ[\Theta, \Theta', \tau] = [\Theta, \Theta'', \tau'\circ\tau]$$

(where id_X is an identity in **Set**).

Proposition 2.2. *(a)* **CLT** *is well-defined. (b) If* $[\Theta, \Theta', \tau]$ *and* $[\Theta', \Theta'', \tau']$ *are morphisms, then the action transformation of* $[\Theta', \Theta'', \tau']\circ[\Theta, \Theta', \tau]$ *is* $\langle \alpha'_{\tau(x)}\circ\alpha_x\rangle_{x\in W}$. *(Proof A.4.)*

2.3. Games

Let μ be a surjective function with domain W. Next define I to be the codomain of μ.[13] Call μ the *move-assigning function*, call a member of I a *player*, and say that player $\mu(x)$ *moves* at node x. It will be assumed that μ is continuous. This means that, for each $i \in I$, the inverse image $\{\, x{\in}W \mid \mu(x){=}i \,\}$ is open in the information topology. This is equivalent to saying that, for each $i \in I$, $\{\, x{\in}W \mid \mu(x){=}i \,\}$ is the union of a collection of information sets. Equivalently, the nodes in each information set are assigned to the same player.

For each $i \in I$, consider a function $U_i{:}\mathcal{Z}{\to}\mathbb{R}$. Call U_i the *utility function* of player i (elsewhere the "payoff function" of player i). The intended meaning of U_i is that, for all $Z_1 \in \mathcal{Z}$ and $Z_2 \in \mathcal{Z}$, $U_i(Z_1) \geq U_i(Z_2)$ iff player i weakly prefers the run Z_1 over the run Z_2. This meaning is called the "ordinal preference content" of U_i.

An *(extensive-form) game* is a tuple $(X, E, \mathcal{H}, \lambda, \mu, U)$ such that

[G1] $(X, E, \mathcal{H}, \lambda)$ is a CLT,

[G2] μ is a continuous surjective function from W, and

[G3] $U = \langle U_i{:}\mathcal{Z}{\to}\mathbb{R}\rangle_{i\in I}$.

Figures 1.1–1.4 provide seven examples. Note that a CLT diagram with movers and utility vectors unambiguously specifies almost all the components of a game. The only thing that may need to be specified externally is the order in which the utility vectors list the players.

Game morphisms will be built on certain CLT morphisms. Say that a CLT morphism $[\Theta, \Theta', \tau]$ *preserves ends* iff $\tau(X{\setminus}W) \subseteq X'{\setminus}W'$. Figure 2.4 shows a morphism which does not preserve ends, because the end node 2 is mapped to the decision node 12. Accordingly, the run $\{0, 2\}$ is mapped to the non-run path $\{10, 12\}$. Arguably, the source utility assigned to the source run $\{0, 2\}$ should imply nothing about the target utilities assigned to the target runs $\{10, 12, 13\}$ and $\{10, 12, 14\}$ because the connection between that source run and those two

[13] This derivation of I from μ is very similar to the derivation of A from λ. In particular, note 9 would apply with I, μ, W, and "game" replacing, respectively, A, λ, E, and "CLT".

target runs is so nebulous. For this reason, game morphisms are not built on CLT morphisms that fail to preserve ends.

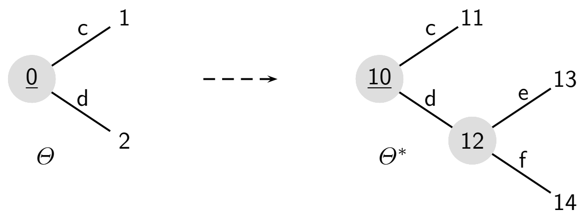

Figure 2.4. The morphism $[\Theta, \Theta^*, \tau]$ defined by $\tau(\mathsf{x}){=}\mathsf{x}{+}10$. $[\Theta, \Theta^*, \tau]$ is not end-preserving.

In contrast, Figure 2.5 shows an end-preserving morphism. Here it seems reasonable that a player's preference between the source runs $\{0,1\}$ and $\{0,2\}$ could be preserved in a preference between the target runs $\{50,10,11\}$ and $\{50,10,12\}$. Note that the images $\{10,11\}$ and $\{10,12\}$ of the two source runs are preceded by the target node 50, and that 50 is not the image of any source node.

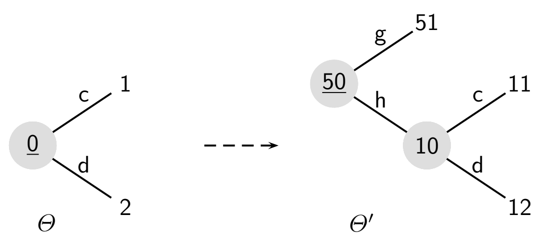

Figure 2.5. The morphism $[\Theta, \Theta', \tau]$ defined by $\tau(\mathsf{x}){=}\mathsf{x}{+}10$. $[\Theta, \Theta', \tau]$ is end-preserving.

From an end-preserving $[\Theta, \Theta', \tau]$, derive the function $\zeta{:}\mathcal{Z}{\to}\mathcal{Z}'$ by

$$\zeta(Z) = P'{\circ}\tau(r){\cup}\bar{\tau}(Z).$$

Call ζ the *run transformation*. Proposition 2.3 shows that ζ is well-defined. For example, in Figure 2.5,

$$\zeta(\{0,2\}) = P'{\circ}\tau(r) \cup \bar{\tau}(\{0,2\}) = \{50\} \cup \{10,12\} = \{50,10,12\},$$

where $P'{\circ}\tau(r) = P'{\circ}\tau(0) = P'(10) = \{50\}$.

Proposition 2.3. *If a morphism $[\Theta, \Theta', \tau]$ is end-preserving, then its run transformation ζ is well-defined. (Proof A.6.)*

To go further, consider two games Γ and Γ' (with their Θ and Θ'), and suppose that $[\Theta, \Theta', \tau]$ is an end-preserving morphism. It will be assumed that there exists a function $\iota{:}I{\to}I'$ such that $\iota{\circ}\mu = \mu'{\circ}\tau|_{W,W'}$. This implies $(\forall x{\in}W)\ \iota(\mu(x)) = \mu'(\tau(x))$. This says that at each $x \in W$, the source player with the move at x is sent to the target player with the move at the image of x. Call ι the *player transformation* of $[\Gamma, \Gamma', \tau]$. There can be no more than one player transformation, because for each $i \in I$, the definition of I and [G2] for Γ implies there is an $x \in W$ such that $i = \mu(x)$, and thus $\iota(i)$ is determined as $\mu'(\tau(x))$.

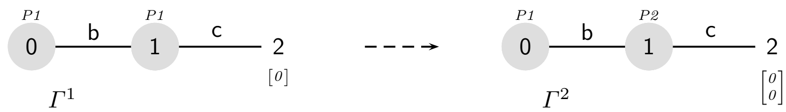

Figure 2.6. $[\Theta^1, \Theta^2, \mathsf{id}_{X^1}]$ is an end-preserving CLT morphism. But $[\Gamma^1, \Gamma^2, \mathsf{id}_{X^1}]$ is not a game morphism because there is no player transformation ι.

At the same time, the existence of a player transformation is a restrictive assumption. For example, consider Figure 2.6. There the tuple $[\Theta^1, \Theta^2, \mathsf{id}_X]$ is an end-preserving CLT morphism. But the equation $\iota \circ \mu = \mu' \circ \tau|_{W^1,W^2}$ is equivalent to $\iota \circ \mu^1 = \mu^2 \circ \mathsf{id}_{W^1}$, which implies $(\forall x {\in} \{0,1\})\ \iota(\mu^1(x)) = \mu^2(x)$, which is equivalent to the two equations $\iota(\textit{P1}) = \iota \circ \mu^1(0) = \mu^2(0) = \textit{P1}$ and $\iota(\textit{P1}) = \iota \circ \mu^1(1) = \mu^2(1) = \textit{P2}$. Since it is impossible to satisfy both these equations, no player transformation ι exists.

A game morphism is a node transformation which preserves edges, information sets, labels, ends, movers, and the ordinal preference content of the utility functions. More specifically, a *game morphism* is a tuple $\gamma = [\Gamma, \Gamma', \tau]$ such that Γ and Γ' are games,

[gZ] $\quad [\Theta, \Theta', \tau]$ is an end-preserving CLT morphism,

and there exists $\iota{:}I {\to} I'$ such that

[gM] $\quad \iota \circ \mu = \mu' \circ \tau|_{W,W'}$ and

[gU] $\quad (\forall i{\in}I, Z_1{\in}\mathcal{Z}, Z_2{\in}\mathcal{Z})\ U_i(Z_1) \geq U_i(Z_2)$ implies $U'_{\iota(i)} \circ \zeta(Z_1) \geq U'_{\iota(i)} \circ \zeta(Z_2)$

(where ζ is derived from $[\Theta, \Theta', \tau]$). Call [gM] *move preservation*, and call [gU] *utility preservation*.

This paragraph defines the category **Gm**, which is called the *category of extensive-form games*. Let an object be a game Γ. Let an arrow be a game morphism $[\Gamma, \Gamma', \tau]$. Let source, target, identity, and composition be

$$[\Gamma, \Gamma', \tau]^{\mathsf{src}} = \Gamma,\ [\Gamma, \Gamma', \tau]^{\mathsf{trg}} = \Gamma',\ \mathsf{id}_\Gamma = [\Gamma, \Gamma, \mathsf{id}_X],\ \text{and}$$
$$[\Gamma', \Gamma'', \tau'] \circ [\Gamma, \Gamma', \tau] = [\Gamma, \Gamma'', \tau' \circ \tau].$$

Proposition 2.4. *(a)* **Gm** *is well-defined. (b) Suppose* $[\Gamma, \Gamma', \tau]$ *and* $[\Gamma', \Gamma'', \tau']$ *are morphisms. Then the run transformation of* $[\Gamma', \Gamma'', \tau'] \circ [\Gamma, \Gamma', \tau]$ *is* $\zeta' \circ \zeta$, *and its player transformation is* $\iota' \circ \iota$. *(Proof A.8.)*

Proposition 2.5. *Define* F *from* **Gm** *to* **CLT** *by*

$$\mathsf{F}_0 : (X, E, \mathcal{H}, \lambda, \mu, U) \mapsto (X, E, \mathcal{H}, \lambda)\ \textit{and}$$
$$\mathsf{F}_1 : [\Gamma, \Gamma', \tau] \mapsto [\mathsf{F}_0(\Gamma), \mathsf{F}_0(\Gamma'), \tau].$$

Then F *is a well-defined functor. (Proof A.10.)*

3. Monomorphisms

3.1. Characterizing monomorphisms

The source of a **CLT** monomorphism is called a *categorical subCLT* of the target. Similarly, the source of a **Gm** monomorphism is called a *categorical subgame* of the target. A wide variety of examples appear in Figures 3.1, 3.2, 3.3, 3.5, and 3.6 (temporarily ignore their captions' remarks about Selten subCLTs and Selten subgames). Further, the morphisms in Figures 2.1 and 4.1 also happen to be monic.

Proposition 3.1.
(a) A **CLT** *morphism* $[\Theta, \Theta', \tau]$ *is monic iff* τ *is injective.*
(b) A **Gm** *morphism* $[\Gamma, \Gamma', \tau]$ *is monic iff its* ζ *is injective. (Proof B.5.)*

Proposition 3.1(a) characterizes CLT monomorphisms by the injectivity of τ. The injectivity of τ implies the injectivity of each action transformation α_x (Lemma B.6). Figure 3.1 provides an example of a CLT monomorphism.

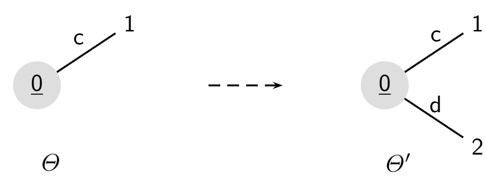

FIGURE 3.1. The monomorphism $[\Theta, \Theta', \mathsf{inc}_{X,X'}]$.[14] Θ is a categorical subCLT of Θ', but not a Selten subCLT.

Proposition 3.1(b) characterizes a game monomorphism by the injectivity of its run transformation ζ. By inspection, the injectivity of ζ is weaker than the injectivity of τ, and the next paragraph will show by example that it is strictly weaker. The salient difference between parts (a) and (b) of the proposition is that game morphisms are built on CLT morphisms that are end-preserving (condition [gZ]).

For example, consider Figure 3.2. Here ζ injectively maps the two source runs to the two target runs. Yet τ is not injective because $\tau(41) = \tau(42) = 40$. Thus Proposition 3.1 implies $[\Gamma, \Gamma', \tau]$ is monic but $[\Theta, \Theta', \tau]$ is not.[15] To develop intuition for $[\Gamma, \Gamma', \tau]$ being monic, let γ denote $[\Gamma, \Gamma', \tau]$ and consider the problem of specifying distinct $\gamma^1 = [\Gamma^*, \Gamma, \tau^1]$ and $\gamma^2 = [\Gamma^*, \Gamma, \tau^2]$ such that $\gamma \circ \gamma^1 = \gamma \circ \gamma^2$. The non-injectivity of τ suggests specifying an $x^* \in X^*$ such that $\tau^1(x^*) = 41$ and $\tau^2(x^*) = 42$. But since γ^1 and γ^2 must be built on end-preserving CLT morphisms, and since 41 and 42 are not end nodes, x^* cannot be an end node. This suggests specifying a successor y^* of x^*. But then [cE] for γ^1 implies $\tau^1(y^*) = 81$, and [cE] for γ^2 implies $\tau^2(y^*) = 82$. These equations and the definition of τ imply $\tau \circ \tau^1(y^*) = 81$ and $\tau \circ \tau^2(y^*) = 82$, which contradict $\gamma \circ \gamma^1 = \gamma \circ \gamma^2$.

Game monomorphisms can also have non-injective action and player transformations. Figure 3.2's example has a non-injective action transformation because the definition of α implies $\alpha_0(\mathsf{b}) = \alpha_0(\mathsf{d}) = \mathsf{f}$. Meanwhile, Figure 3.3's example has a non-injective player transformation because [gM] implies both $\iota(\mathit{P1}) = \iota(\mu(0)) = \mu'(\tau(0)) = \mathit{P1}$ and $\iota(\mathit{P2}) = \iota(\mu(1)) = \mu'(\tau(1)) = \mathit{P1}$.

[14]Let $\mathsf{inc}_{C,D}$ be the inclusion function from C to D.
[15]This is consistent with Proposition 2.5's forgetful functor because a functor need not take a monomorphism to a monomorphism.

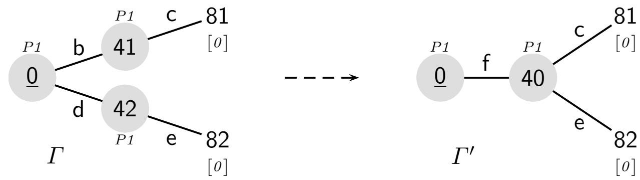

FIGURE 3.2. The monomorphism $[\Gamma,\Gamma',\tau]$ defined by $\tau(x)=\mathsf{40}$ for $x\in\{\mathsf{41},\mathsf{42}\}$ and $\tau(x)=x$ otherwise. Its $[\Theta,\Theta',\tau]$ is not monic. Γ is a categorical subgame of Γ', but not a Selten subgame.

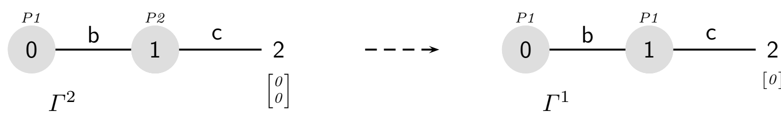

FIGURE 3.3. The monomorphism $[\Gamma^2,\Gamma^1,\mathsf{id}_{X^2}]$. Its ι is not injective. Γ^2 is a categorical subgame of Γ^1, but not a Selten subgame.

3.2. Selten subgames

Selten 1975 defines a different concept of subgame.[16] Consider a CLT Θ'. Then for each $r\in W'$, define the *Selten tuple* $\Theta=(X,E,\mathcal{H},\lambda)$ by

$$X=\{\,y'{\in}X'\,|\,r{\succcurlyeq}'y'\,\},$$
$$E=\{\,x'y'{\in}E'\,|\,\{x',y'\}{\subseteq}X\,\},$$
$$\mathcal{H}=\{\,H'{\in}\mathcal{H}'\,|\,H'{\subseteq}X\,\},\text{ and}$$
$$\lambda=\lambda'|_{E,\overline{\lambda'}(E)}.$$

This is a CLT iff [$*$] $(\forall H'{\in}\mathcal{H}')$ $H'\subseteq X$ or $H'\subseteq X'\smallsetminus X$ (Lemma B.10). If the tuple is a CLT, call it the *Selten subCLT of* Θ' *at* r. Otherwise, the Selten subCLT of Θ' at r does not exist. For example, consider Θ' in Figure 3.4. The Selten subCLT at $\mathsf{24}$ has three nodes, and the Selten subCLT at $\mathsf{0}$ is Θ' itself. In contrast, there is no Selten subCLT at any other node. For example, the figure's Θ illustrates the Selten tuple below $\mathsf{11}$. This is not a CLT because $\mathsf{11}$ is not in an information set, in violation of [C2]. Relatedly, condition [$*$] is violated because $\{\mathsf{11},\mathsf{12}\}\in\mathcal{H}'$, $\mathsf{11}\in X$, and $\mathsf{12}\notin X$.

Finally, consider a game Γ' and an $r\in W'$. If the Selten subCLT of Θ' at r exists, the *Selten subgame of* Γ' *at* r is the game $\Gamma=(X,E,\mathcal{H},\lambda,\mu,U)$ constructed by letting $\Theta=(X,E,\mathcal{H},\lambda)$ be the Selten subCLT of Θ' at r, and by defining μ and $U=\langle U_i{:}\mathcal{Z}{\rightarrow}\mathbb{R}\rangle_{i\in I}$ by

$$\mu=\mu'|_{W,\overline{\mu'}(W)}\text{ and}$$
$$(\forall i{\in}I,Z{\in}\mathcal{Z})\;U_i(Z)=U_i'(P'(r){\cup}Z)$$

(Lemma B.11 verifies that Γ is well-defined). Because Γ determines r, the statement that "Γ is a Selten subgame of Γ'" is equivalent to the statement that "Γ is the Selten subgame of Γ' at r". The parallel statement holds for CLTs.

[16]The construction here accords with the standard definition in Selten 1975 and Myerson 1991, page 184, except that the standard definition admits trivial one-node subgames at each of a game's end nodes. In this paper, all games have at least two nodes because of the nontriviality in [C1].

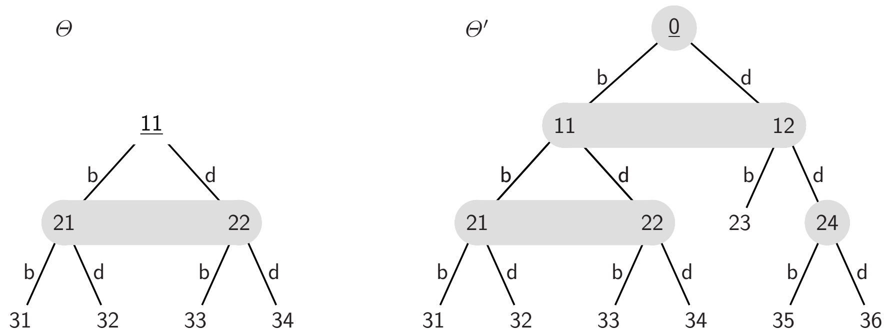

Figure 3.4. The tuple Θ is not a CLT because 11 is not in an information set. The CLT Θ' does not have a Selten subCLT at 11.

Theorem 3.2.

(a) Θ is a Selten subCLT of Θ' iff (i) Θ is a categorical subCLT of Θ' via $[\Theta, \Theta', \mathsf{inc}_{X,X'}]$ with $\alpha = \langle \mathsf{id}_{F(x)} \rangle_{x \in W}$, (ii) $X = \{\, y' {\in} X' \,|\, r {\succcurlyeq}' y' \,\}$, and (iii) $\mathcal{H} \subseteq \mathcal{H}'$.

(b) Γ is a Selten subgame of Γ' iff (i∗) Γ is a categorical subgame of Γ' via $[\Gamma, \Gamma', \mathsf{inc}_{X,X'}]$ with $\alpha = \langle \mathsf{id}_{F(x)} \rangle_{x \in W}$ and $\iota = \mathsf{inc}_{I,I'}$, (ii) $X = \{\, y' {\in} X' \,|\, r {\succcurlyeq}' y' \,\}$, (iii) $\mathcal{H} \subseteq \mathcal{H}'$, and (iv) $U = \langle U'_i {\circ} \zeta \rangle_{i \in I}$. (Proof B.13.)

Theorem 3.2(a) shows that a Selten subCLT is a special kind of categorical subCLT. Restriction (ii) is identical to the Selten definition of X. It rules out the categorical subCLTs in Figures 3.1 and 3.2. Meanwhile, the action transformation in (i) implies that the subgame's actions are the same as in the game itself. This rules out the categorical subCLT implicit in Figure 2.1 long ago.

Restriction (iii) is relatively subtle. To explore it, recall that $\mathcal{H}$ generates a topology for W. This topology can be compared with the subspace topology for W that is implied by the topology for W' generated by $\mathcal{H}'$ (restriction (ii) implies $W \subseteq W'$). By inspection, restriction (iii) implies that $\mathcal{H}$ generates the subspace topology. In fact, (iii) is strictly stronger.

In this paragraph, the first example's $\mathcal{H}$ does not generate the subspace topology, and the second example violates (iii) even though its $\mathcal{H}$ generates the subspace topology. First, consider Figure 4.1 far below. There Θ and Θ' satisfy (i) and (ii), but they violate (iii) because $\{3\} \in \mathcal{H} \smallsetminus \mathcal{H}'$. Note $\mathcal{H}$ generates a topology other than the subspace topology (the former topology is strictly finer than the latter). Second, consider Figure 3.5. Here Θ and Θ' satisfy (i) and (ii), but they violate (iii) because $\{11\} \in \mathcal{H} \smallsetminus \mathcal{H}'$. In contrast to the first example, this example's $\mathcal{H}$ generates the subspace topology. This violation of (iii) accords with Figure 3.4 and the earlier observation that this Θ' does not have a Selten subCLT at 11.

Next, Theorem 3.2(b) shows that a Selten subgame is a special kind of categorical subgame. All the restrictions of the preceding paragraph are imposed. In addition, restriction (i∗) stipulates that $\iota = \mathsf{inc}_{I,I'}$, which implies that ι is injective. This rules out the categorical subgame in Figure 3.3. Finally, restriction (iv) implies that the ordinal content of the utility function is strictly preserved in the sense that [gU]'s conditional holds in both directions. This rules out the categorical subgame in Figure 3.6.

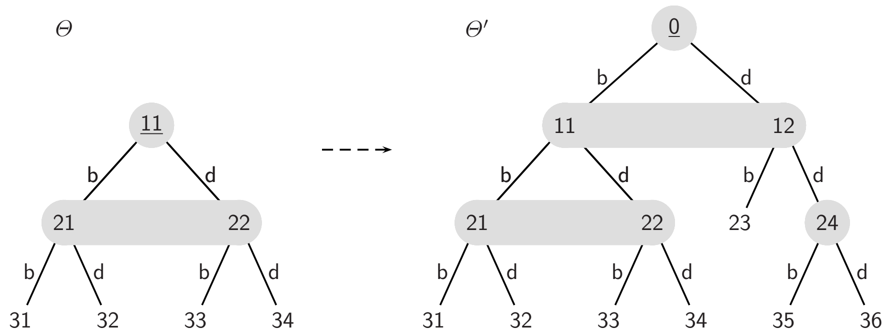

FIGURE 3.5. The monomorphism $[\Theta, \Theta', \mathsf{inc}_{X,X'}]$. Θ is a categorical subCLT of Θ', but not a Selten subCLT. Θ' also appears in Figure 3.4.

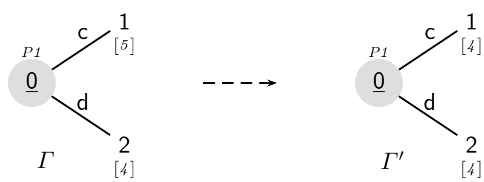

FIGURE 3.6. The monomorphism $[\Gamma, \Gamma', \mathsf{id}_X]$. Γ is a categorical subgame of Γ', but not a Selten subgame.

4. Isomorphisms

4.1. Characterizing isomorphisms

The following theorem uses the term "homeomorphism" from topology. If $[\Theta, \Theta', \tau]$ is a morphism and τ is bijective, then $\tau|_{W,W'}$ is a homeomorphism iff $(\tau|_{W,W'})^{-1}$ is continuous. For example, in Figure 4.1, $\tau|_{W,W'} = \mathsf{id}_W$ is not a homeomorphism because $(\tau|_{W,W'})^{-1} = \mathsf{id}_{W'}$ is not continuous. In particular, $(\tau|_{W,W'})^{-1} = \mathsf{id}_{W'}$ splits the information set $\{3, 4\}$. For another perspective, recall that W and W' are endowed with partition topologies. Thus, if $[\Theta, \Theta', \tau]$ is a morphism and τ is bijective, $\tau|_{W,W'}$ is a homeomorphism iff $\mathcal{H} \ni H \mapsto \bar{\tau}(H) \in \mathcal{H}'$ is a bijection. For example, in Figure 4.1, $\tau|_{W,W'} = \mathsf{id}_W$ is not a homeomorphism because it does not define a bijection between $\mathcal{H}$ and $\mathcal{H}'$.

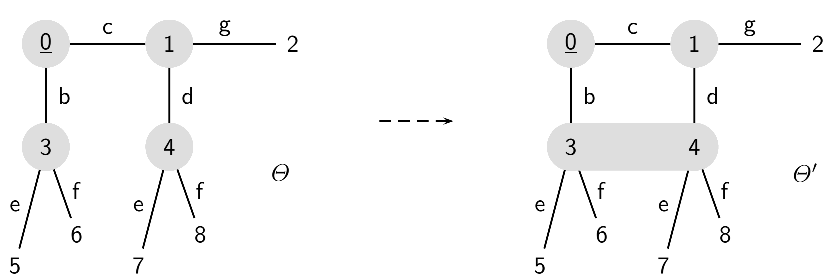

FIGURE 4.1. The morphism $[\Theta, \Theta', \mathsf{id}_X]$ is not an isomorphism because $\tau|_{W,W'} = \mathsf{id}_W$ is not a homeomorphism.

Proposition 4.1.

(a) A **CLT** *morphism* $[\Theta,\Theta',\tau]$ *is an isomorphism iff* τ *is bijective and* $\tau|_{W,W'}$ *is a homeomorphism. Further, if* $[\Theta,\Theta',\tau]$ *is an isomorphism, then each* α_x *in* $\langle\alpha_x\rangle_{x\in W}$ *is bijective, the inverse of* $[\Theta,\Theta',\tau]$ *is* $[\Theta',\Theta,\tau^{-1}]$*, and the inverse's action transformation is* $\langle\alpha^{-1}_{\tau^{-1}(x')}\rangle_{x'\in W'}$.

(b) A **Gm** *morphism* $[\Gamma,\Gamma',\tau]$ *is an isomorphism iff* τ *is bijective,* $\tau|_{W,W'}$ *is a homeomorphism,* ι *is injective, and* $(\forall i{\in}I, Z_1{\in}\mathcal{Z}, Z_2{\in}\mathcal{Z})\ U_i(Z_1) \geq U_i(Z_2) \Leftrightarrow U'_{\iota(i)}{\circ}\zeta(Z_1) \geq U'_{\iota(i)}{\circ}\zeta(Z_2)$.[17] *Further, if* $[\Gamma,\Gamma',\tau]$ *is an isomorphism, then its* ζ *is the bijection* $\bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$*, its* ι *is bijective, its inverse is* $[\Gamma',\Gamma,\tau^{-1}]$*, the inverse's run transformation is* ζ^{-1}*, and the inverse's player transformation is* ι^{-1}.[18] *(Proof C.4.)*

To explore Proposition 4.1(a)'s characterization, this paragraph relaxes each of its two conditions independently. In Figure 3.1 (long ago), τ is not a bijection but $\tau|_{W,W'} = \mathsf{id}_{\{0\}}$ is a homeomorphism. Here the tuple $[\Theta',\Theta,\tau^{-1}]$ cannot be constructed because τ^{-1} does not exist. In Figure 4.1, τ is a bijection but $\tau|_{W,W'}$ is not a homeomorphism. Here the tuple $[\Theta',\Theta,\tau^{-1}] = [\Theta',\Theta,\mathsf{id}_{X'}]$ is not a morphism because $\mathsf{id}_{X'}$ is not continuous in violation of [cI].

To explore Proposition 4.1(b)'s characterization, this paragraph relaxes its third and fourth conditions independently (its first two conditions are equivalent to $[\Theta,\Theta',\tau]$ being an isomorphism by part (a)). In Figure 3.3, $[\Theta,\Theta',\tau] = [\Theta^2,\Theta^1,\tau]$ is an isomorphism, ι is not injective, and the biconditional holds. Here the tuple $[\Gamma',\Gamma,\tau^{-1}] = [\Gamma^1,\Gamma^2,\mathsf{id}_{X^1}]$ is not a morphism because it does not have a player transformation in violation of [gM] (Figure 2.6 illustrated this in a different context). In Figure 3.6, $[\Theta,\Theta',\tau]$ is an isomorphism, ι is injective, and the biconditional is violated. Here the tuple $[\Gamma',\Gamma,\tau^{-1}] = [\Gamma',\Gamma,\mathsf{id}_{X'}]$ is not a morphism because it violates [gU].

To build further intuition for Proposition 4.1(a), note that the bijectivity of τ is powerful. In particular, Lemma C.1(a) shows that a bijective τ implies a bijective edge transformation in the sense of Proposition 4.2(a). This result relies on the special properties of trees, and plays a central role in proving the reverse direction of Proposition 4.1(a). Proposition 4.2 itself collects some implications of CLT isomorphisms which do not appear in Proposition 4.1. It is comparable to Proposition 2.1.

Proposition 4.2. *Suppose* $[\Theta,\Theta',\tau]$ *is an isomorphism. Then the following hold.*

(a) $E \ni xy \mapsto \tau(x)\tau(y) \in E'$ *is a bijection.*
(b) $(\forall x{\in}X, y{\in}X)\ x \preccurlyeq y \Leftrightarrow \tau(x) \preccurlyeq' \tau(y)$.
(c) $(\forall x{\in}X, y{\in}X)\ x \prec y \Leftrightarrow \tau(x) \prec' \tau(y)$.
(d) $\tau(r) = r'$.
(e) $(\forall y{\in}X)\ \bar{\tau}(P(y)) = P'(\tau(y))$.
(f) $\tau|_{X\smallsetminus W, X'\smallsetminus W'}$ *is a bijection. (Proof C.5.)*

---

[17]In this sentence, ζ can be replaced by $\bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$ because of the following. Part (a), τ being bijective, and $\tau|_{W,W'}$ being a homomorphism together imply that $[\Theta,\Theta',\tau]$ is an isomorphism, which by Lemma C.2(b) implies $\zeta = \bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$.

[18]The three conclusions of part (a)'s second sentence could be added to part (b)'s second sentence. These additional conclusions follow immediately from part (a) and Proposition 2.5's forgetful functor.

4.2. Nash equilibrium

Consider a CLT Θ. Let S be the set of continuous $s{:}W{\to}A$ such that $(\forall x{\in}W)$ $s(x) \in F(x)$. Call a member of S a *grand strategy.* Continuity is equivalent to s being constant across each information set.

At each decision node $x \in W$, a grand strategy $s \in S$ determines the action $s(x)$, and thus the node's successor $n(x, s(x))$. As a result, a grand strategy $s \in S$ determines the run $Z \in \mathcal{Z}$ consisting of r, $x_1{=}n(r, s(r))$, $x_2{=}n(x_1, s(x_1))$, and so on, either indefinitely or until an end node occurs. To put this another way, for each grand strategy $s \in S$, there is exactly one run $Z \in \mathcal{Z}$ such that $(\forall x{\in}Z{\cap}W)$ $n(x, s(x)) \in Z$. This defines a function o from S into $\mathcal{Z}$. Call o the *outcome function.*

Theorem 4.3. *Suppose* $[\Theta, \Theta', \tau]$ *is an isomorphism. Then the following hold.*

(a) $S \ni s \mapsto \langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x' \in W'} \in S'$ *is a bijection. Its inverse is* $S \ni \langle\, \alpha_x^{-1}(s'(\tau(x)))\,\rangle_{x \in W} \leftarrow\!\shortmid s' \in S'$.
(b) $(\forall s{\in}S)$ $\zeta{\circ}o(s) = o'(\,\langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x' \in W'}\,)$. *(Proof C.6.)*

Consider a game Γ. Then consider a player $i \in I$, and let W_i be the inverse image $\{\, x{\in}W \,|\, \mu(x){=}i \,\}$. Thus W_i is the set of nodes at which player i moves. Endow W_i with the subspace topology from W. The continuity of μ implies that this topology is generated by $\{\, H{\subseteq}\mathcal{H} \,|\, H{\subseteq}W_i \,\}$, and that this collection partitions W_i (Lemma C.7). Finally, let S_i be the set of continuous functions $s_i{:}W_i{\to}A$ such that $(\forall x{\in}W_i)$ $s_i(x) \in F(x)$. In light of the second-previous sentence, continuity requires that each s_i is constant across each information set H that is a subset of W_i. Call S_i the set of *player-i strategies.*

Theorem 4.4. *Suppose* $[\Gamma, \Gamma', \tau]$ *is an isomorphism. Then the following hold.*

(a) $(\forall i{\in}I)$ $\tau|_{W_i, W'_{\iota(i)}}$ *is a homeomorphism.*
(b) $(\forall i{\in}I)$ $S_i \ni s_i \mapsto \langle\, \alpha_{\tau^{-1}(x')}(s_i(\tau^{-1}(x')))\,\rangle_{x' \in W'_{\iota(i)}} \in S'_{\iota(i)}$ *is a bijection. Its inverse is* $S_i \ni \langle\, \alpha_x^{-1}(s'_{\iota(i)}(\tau(x)))\,\rangle_{x \in W_i} \leftarrow\!\shortmid s'_{\iota(i)} \in S'_{\iota(i)}$. *(Proof C.8.)*

There is a straightforward bijection between the set S of grand strategies and the set $\Pi_{i \in I} S_i$ of player-strategy profiles. In particular, $S \ni s \mapsto \langle s|_{W_i} \rangle_{i \in I} \in \Pi_{i \in I} S_i$ is a bijection. Its inverse is $S \ni \cup_{i \in I} s_i \leftarrow\!\shortmid \langle s_i \rangle_{i \in I} \in \Pi_{i \in I} S_i$, where $\cup_{i \in I} s_i$ is the function whose domain is $W = \cup_{i \in I} W_i$, whose codomain is A, and whose graph is $\cup_{i \in I} s_i^{\mathsf{gr}}$.

In light of the preceding bijection, identify a grand strategy $s \in S$ with a player-strategy profile $\langle s_i \rangle_{i \in I} \in \Pi_{i \in I} S_i$. Relatedly, for each $i \in I$, identify an $s \in S$ with a pair (s_i, s_{-i}), where $s_i \in S_i$, and where s_{-i} is understood as both the grand-strategy restriction $s|_{W \smallsetminus W_i}$ and the player-strategy subprofile $\langle s_j \rangle_{j \in I \smallsetminus \{i\}} \in \Pi_{j \in I \smallsetminus \{i\}} S_j$. In this fashion s_{-i} specifies the strategies of player i's opponents.

A *Nash equilibrium* is an $s \in S$ such that $(\forall i{\in}I, s_i^+{\in}S_i)$ $U_i{\circ}o(s) \geq U_i{\circ}o(s_i^+, s_{-i})$. Thus a Nash equilibrium is a strategy profile in which each player's strategy is a best response to their opponents' strategies. Let $S_{\mathsf{NE}} \subseteq S$ denote the set of Nash equilibria. The following theorem shows how isomorphisms preserve Nash equilibria.

Theorem 4.5. *Suppose* $[\mathit{\Gamma},\mathit{\Gamma}',\tau]$ *is an isomorphism. Then*

$$S_{\mathsf{NE}} \ni s \mapsto \langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x'\in W'} \in S'_{\mathsf{NE}}$$

is a bijection. Its inverse is $S_{\mathsf{NE}} \ni \langle\, \alpha_x^{-1}(s'(\tau(x)))\,\rangle_{x\in W} \leftarrow\!\shortmid s' \in S'_{\mathsf{NE}}$. *(Proof C.9.)*

4.3. Subgame-perfect equilibrium

The following theorem shows how game isomorphisms preserve Selten subgames. In the remainder of this section, games with small-letter superscripts are Selten subgames of the games with the corresponding capital-letter superscripts.

Theorem 4.6. *Suppose* $[\mathit{\Gamma}^{\mathsf{D}},\mathit{\Gamma}^{\mathsf{E}},\tau]$ *is an isomorphism. Then the following hold.*

(a) Let R^{D} *be the set of nodes at which* $\mathit{\Gamma}^{\mathsf{D}}$ *has a Selten subgame. Similarly let* R^{E} *be the set of nodes at which* $\mathit{\Gamma}^{\mathsf{E}}$ *has a Selten subgame. Then* $\tau|_{R^{\mathsf{D}},R^{\mathsf{E}}}$ *is a bijection.*

(b) Take $r^{\mathsf{d}} \in R^{\mathsf{D}}$*, let* $\mathit{\Gamma}^{\mathsf{d}}$ *be the Selten subgame of* $\mathit{\Gamma}^{\mathsf{D}}$ *at* r^{d}*, and let* $\mathit{\Gamma}^{\mathsf{e}}$ *be the Selten subgame of* $\mathit{\Gamma}^{\mathsf{E}}$ *at* $\tau(r^{\mathsf{d}})$*. Then* $[\mathit{\Gamma}^{\mathsf{d}},\mathit{\Gamma}^{\mathsf{e}},\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$ *is an isomorphism. Its action transformation is* $\langle \alpha_{x^{\mathsf{d}}}\rangle_{x^{\mathsf{d}}\in W^{\mathsf{d}}}$*, its run transformation is* $\zeta|_{\mathcal{Z}^{\mathsf{d}},\mathcal{Z}^{\mathsf{e}}}$*, and its player transformation is* $\iota|_{I^{\mathsf{d}},I^{\mathsf{e}}}$*. (Proof C.10.)*

As in Selten 1975, a *subgame-perfect equilibrium* in a game $\mathit{\Gamma}^{\mathsf{B}}$ is an $s^{\mathsf{B}} \in S^{\mathsf{B}}$ such that, for each Selten subgame $\mathit{\Gamma}^{\mathsf{b}}$ of $\mathit{\Gamma}^{\mathsf{B}}$, the restriction $\langle\, s^{\mathsf{B}}(x^{\mathsf{b}})\,\rangle_{x^{\mathsf{b}}\in W^{\mathsf{b}}}$ is a Nash equilibrium in $\mathit{\Gamma}^{\mathsf{b}}$. Let $S^{\mathsf{B}}_{\mathsf{SPE}} \subseteq S^{\mathsf{B}}$ denote the set of subgame-perfect equilibria in $\mathit{\Gamma}^{\mathsf{B}}$.[19] The following theorem shows how isomorphisms preserve subgame-perfect equilibria.

Theorem 4.7. *Suppose* $[\mathit{\Gamma}^{\mathsf{D}},\mathit{\Gamma}^{\mathsf{E}},\tau]$ *is an isomorphism. Then*

$$S^{\mathsf{D}}_{\mathsf{SPE}} \ni s^{\mathsf{D}} \mapsto \langle\, \alpha_{\tau^{-1}(x^{\mathsf{E}})}(s^{\mathsf{D}}(\tau^{-1}(x^{\mathsf{E}})))\,\rangle_{x^{\mathsf{E}}\in W^{\mathsf{E}}} \in S^{\mathsf{E}}_{\mathsf{SPE}}$$

is a bijection. Its inverse is $S^{\mathsf{D}}_{\mathsf{SPE}} \ni \langle\, \alpha_{x^{\mathsf{D}}}^{-1}(s^{\mathsf{E}}(\tau(x^{\mathsf{D}})))\,\rangle_{x^{\mathsf{D}}\in W^{\mathsf{D}}} \leftarrow\!\shortmid s^{\mathsf{E}} \in S^{\mathsf{E}}_{\mathsf{SPE}}$. *(Proof C.11.)*

5. Full Subcategories

5.1. Distinguished-action games

A CLT $\mathit{\Theta}$ is said to have *distinguished actions* iff $(\forall a{\in}A)\ \{\, x{\in}W \mid a{\in}F(x)\,\} \in \mathcal{H}$. This strengthens [C4], which states that each inverse image $\{\, x{\in}W \mid a{\in}F(x)\,\}$ is the union of a subcollection of $\mathcal{H}$. It requires that each action is associated with exactly one information set. Thus the actions of each information set are "distinguished" from the actions of all the other information sets. For example, in Figure 2.1, $\mathit{\Theta}'$ has distinguished actions and $\mathit{\Theta}$ does not. For a well-known example from the literature, note that Kreps and Wilson 1982 assumes distinguished actions. Say that a game has *distinguished actions* iff its CLT does, and let **DGm** be the full subcategory of **Gm** for distinguished-action games.[20]

Consider a subcategory of an arbitrary category. As usual, the subcategory is said to be essentially wide in the category iff each of the category's objects

[19] Since a game is a Selten subgame of itself, each subgame-perfect equilibrium is a Nash equilibrium. Hence $S^{\mathsf{B}}_{\mathsf{SPE}} \subseteq S^{\mathsf{B}}_{\mathsf{NE}}$.

[20] Streufert 2020b defines the category **NCG**. It is conjectured that **DGm** is isomorphic to the subcategory of **NCG** for end-preserving morphisms.

is isomorphic to an object of the subcategory. Since every essentially wide full subcategory is equivalent to the category itself, the following theorem implies that **DGm** and **Gm** are equivalent. Similar equivalences are implied by Theorems 5.2 and 5.5 below, and by Corollaries 5.4 and 5.6 below.

Theorem 5.1. **DGm** *is an essentially wide full subcategory of* **Gm**. *(Lemma D.3.*[21]*)*

5.2. Sequence games

Let a *(finite) sequence* be a function with domain $\{1, 2, ... \ell\}$ for some nonnegative integer ℓ. To be clear, the empty sequence $\{\}$ is admitted by $\ell = 0$, and nonempty sequences will be written in the form $x = (x_1, x_2, ... x_\ell)$. Call ℓ the *length* of a sequence x, and denote it hereafter as $|x|$. Further, for any sequence x and any $m \in \{0, 1, ... |x|\}$, let ${}_1x_m$ denote the *initial segment* $(x_1, x_2, ... x_m)$. Thus for any sequence x, ${}_1x_0$ is the empty sequence $\{\}$.

A CLT Θ is said to *use sequences* iff

$$X \text{ is a collection of (finite) sequences,}$$
$$E = \{\, {}_1y_{|y|-1}\, y \mid y \in X \setminus \{\{\}\}\,\}, \text{ and}$$
$$(\forall\, {}_1y_{|y|-1}\, y \in E)\ \lambda({}_1y_{|y|-1}\, y) = y_{|y|}.$$

Note that a sequence CLT satisfies $r = \{\}$. Similar structures appear widely in the literature, as discussed in Section 1.2 in connection with "Group 2" and Figure 1.2(a).

A *sequence game* is a game whose CLT uses sequences. Let **SGm** be the full subcategory of **Gm** for sequence games. Further, let **DSGm** be the full subcategory of **Gm** for distinguished-action sequence games. The following implies the equivalence of **SGm**, **DSGm**, and **Gm**.

Theorem 5.2. **SGm** *and* **DSGm** *are essentially wide full subcategories of* **Gm**. *(Proof D.8.)*

5.3. Two invariant properties

A CLT Θ is said to have *no-absentmindedness* iff $(\nexists H \in \mathcal{H}, x \in H, y \in H)\ x \prec y$, and further, a game is said to have no-absentmindedness iff its CLT does (the term "no-absentmindedness" is from Piccione and Rubinstein 1997). If violated, some information set H would contain both a node and one of its predecessors, which would mean that the decision maker at that information set would not know whether they had made the same decision previously. Such "absentmindedness" is routinely ruled out by assumption, as in Kuhn 1953. In the present paper, Figure 2.2's Θ' is the only example that violates no-absentmindedness.

A CLT Θ is said to have *perfect-information* iff $(\forall H \in \mathcal{H})\ |H| = 1$, and a game is said to have perfect-information iff its CLT does. In other words, perfect-information is equivalent to the information topology being discrete. For example, consider Figure 4.1. There, Θ satisfies perfect-information, and Θ' violates

[21] Lemma D.3 shows that each game has a distinguished-action isomorph. Its proof uses Lemma D.1. That lemma begins with (i) an arbitrary game and (ii) arbitrary node, action, and player transformations which are bijective. It then constructs an isomorph of the game by means of the transformations. This general construction is also used to prove Theorems 5.2 and 5.5, and could be useful beyond this paper.

perfect-information. Although perfect-information is quite restrictive, there are many interesting perfect-information games, such as chess.

Consider an arbitrary category, and a property which is defined for its objects. The property is said to be *(isomorphically) invariant* iff, for each object, the object satisfies the property iff all its isomorphs satisfy the property. Equivalently, the property is invariant iff, for each object, the object satisfies the property only if all its isomorphs satisfy the property.

Theorem 5.3.
(a) No-absentmindedness is (isomorphically) invariant in $\mathbf{Gm}$.
(b) Perfect-information is (isomorphically) invariant in $\mathbf{Gm}$. *(Proof D.9.)*

Let $\mathbf{Gm_{\tilde{a}}}$ be the full $\mathbf{Gm}$ subcategory for no-absentminded games. As noted, Figure 2.2's Θ' violates no-absentmindedness, and hence, the one-player-named-*P1* zero-utility game[22] built on Θ' is not an object of $\mathbf{Gm_{\tilde{a}}}$. Further, since no-absentmindedness is invariant by Theorem 5.3(a), this game is not isomorphic to an object of $\mathbf{Gm_{\tilde{a}}}$. Thus $\mathbf{Gm_{\tilde{a}}}$ is not essentially wide in $\mathbf{Gm}$.

Let $\mathbf{Gm_p}$ be the full $\mathbf{Gm}$ subcategory for perfect-information games. By inspection, perfect-information implies no-absentmindedness. Thus $\mathbf{Gm_p}$ is a full subcategory of $\mathbf{Gm_{\tilde{a}}}$. Further, Figure 4.1's Θ' satisfies no-absentmindedness but not perfect-information, and hence, the one-player-named-*P1* zero-utility game[22] built on Θ' is an object of $\mathbf{Gm_{\tilde{a}}}$ but not an object of $\mathbf{Gm_p}$. Thus, since perfect-information is invariant by Theorem 5.3(b), $\mathbf{Gm_p}$ is not essentially wide in $\mathbf{Gm_{\tilde{a}}}$.

In a similar fashion, define $\mathbf{DGm_{\tilde{a}}}$ and $\mathbf{DGm_p}$, $\mathbf{SGm_{\tilde{a}}}$ and $\mathbf{SGm_p}$, and $\mathbf{DSGm_{\tilde{a}}}$ and $\mathbf{DSGm_p}$. For example, $\mathbf{DSGm_p}$ is the full $\mathbf{Gm}$ subcategory for distinguished-action sequence games with perfect-information. (Subscripts are being used for invariant properties.)

Corollary 5.4.
(a) $\mathbf{DGm_{\tilde{a}}}$, $\mathbf{SGm_{\tilde{a}}}$ *and* $\mathbf{DSGm_{\tilde{a}}}$ *are essentially wide full subcategories of* $\mathbf{Gm_{\tilde{a}}}$.
(b) $\mathbf{DGm_p}$, $\mathbf{SGm_p}$ *and* $\mathbf{DSGm_p}$ *are essentially wide full subcategories of* $\mathbf{Gm_p}$. *(Proof D.10.)*

5.4. Action-set games

A distinguished-action CLT Θ is said to *use action sets* iff

$$X \text{ is a collection of finite sets containing } \{\},$$
$$E = \{\, xy{\in}X^2 \mid x{\subseteq}y,\ |y{\setminus}x|{=}1 \,\}, \text{ and}$$
$$(\forall xy{\in}E)\ \{\lambda(xy)\} = y{\setminus}x.$$

Note that an action-set CLT satisfies $r = \{\}$. Figure 1.2(b) provides an example, and similar structures appear in Streufert 2015 and 2019. Let an *action-set* game be a game whose CLT uses action sets.[23] Further, let $\mathbf{AGm}$ be the full $\mathbf{Gm}$

[22]The "one-player-named-*P1* zero-utility game" built on a CLT Θ is the game constructed by augmenting Θ with (μ, U), where $\mu{:}W{\to}\{P1\}$ is defined by $\mu(x) = P1$, and where $U = \langle U_i{:}\mathcal{Z}{\to}\mathbb{R}\rangle_{i\in\{P1\}}$ is defined by $U_{P1}(Z) = 0$. For example, both games in Figure 3.2 are one-player-named-*P1* zero-utility games.

[23]Section 5.2's games are called "sequence games" rather than "action-sequence games". In a similar vein, it is tempting to call Section 5.4's games "set games" rather than "action-set games". But this abbreviation would be confusing because there are other kinds of "set games", such as those shown in Figure 1.3.

subcategory for action-set games. The following theorem implies the equivalence of **AGm** and $\mathbf{Gm_{\tilde{a}}}$. This reveals that using action sets is only slightly restrictive in the sense that using action sets rules out absentmindedness but essentially nothing else.

Theorem 5.5. **AGm** *is an essentially wide full subcategory of* $\mathbf{Gm_{\tilde{a}}}$. *(Proof D.15.)*

Let $\mathbf{AGm_p}$ be the full **Gm** subcategory for action-set games with perfect information. The following implies the equivalence of $\mathbf{AGm_p}$ and $\mathbf{Gm_p}$.

Corollary 5.6. $\mathbf{AGm_p}$ *is an essentially wide full subcategory of* $\mathbf{Gm_p}$. *(Proof D.16.)*

APPENDIX A. FOR DEFINITION

Lemma A.1. *Suppose that* (X, E) *is a nontrivial out-tree, and that* $\lambda{:}E{\to}A$ *is surjective and deterministic. Then the following hold.*

(a) $\cup_{x\in W} F(x) = A$.

(b) For each $(x, a) \in F^{\mathsf{gr}}$ *there is a unique* $y \in X\smallsetminus\{r\}$ *such that* $\lambda(xy) = a$.

(c) Define $n{:}F^{\mathsf{gr}}{\to}X\smallsetminus\{r\}$ *by part (b), and consider* $x_o \in W$. *Then*

$$F(x_o) \ni a \mapsto n(x_o, a) \in \{y_+|x_o y_+{\in}E\}$$

is a bijection. Its inverse is $F(x_o) \ni \lambda(x_o y) \leftarrow\!\shortmid y \in \{y_+|x_o y_+{\in}E\}$.

Proof. (a). For $\cup_{x\in W} F(x) \subseteq A$, note that each $F(x) \subseteq A$ by the definition of F. For the reverse inclusion, take $a \in A$. Then the surjectivity of λ implies there is [1] $xy \in E$ such that [2] $\lambda(xy) = a$. [1] and the definition of W imply $x \in W$. [2] and the definition of F imply $a \in F(x)$.

(b). Take $(x, a) \in F^{\mathsf{gr}}$. Then by the definition of F, there is $y \in X\smallsetminus\{r\}$ such that $\lambda(xy) = a$. Further, if $y_* \in X\smallsetminus\{r\}$ satisfies $\lambda(xy_*) = a$, then the determinism of λ implies $y = y_*$.

(c). The forward function is well-defined by part (b), and the reverse function is well-defined since the domain of λ is E. Thus it suffices to show that (1) the forward followed by the reverse is $\mathsf{id}_{F(x_o)}$, and that (2) the reverse followed by the forward is $\mathsf{id}_{\{y_+|x_o y_+\in E\}}$.

For (1), take $a \in F(x_o)$. By definition, $n(x_o, a)$ is the unique y_* such that $\lambda(x_o y_*) = a$. Thus (by substituting for y_*), $\lambda(x_o n(x_o, a)) = a$.

For (2), take y such that $x_o y \in E$. Then the definition of $F(x_o)$ implies $\lambda(x_o y) \in F(x_o)$, which easily implies $(x_o, \lambda(x_o y)) \in F^{\mathsf{gr}}$. Thus the definition of n implies that (i) $n(x_o, \lambda(x_o y)) = y_*$ where (ii) y_* is the unique solution to $\lambda(x_o y_*) = \lambda(x_o y)$. To conclude, (ii) implies $y_* = y$ and thus (i) implies $n(x_o, \lambda(x_o y)) = y$. □

Lemma A.2. *Suppose* Θ *and* Θ' *are CLTs and* $\bar{\tau}(W) \subseteq W'$. *Then* $\tau|_{W,W'}$ *is continuous iff* $(\forall H{\in}\mathcal{H})(\exists H'{\in}\mathcal{H}')\ \bar{\tau}(H) \subseteq H'$.

Proof. By general topology, $\tau|_{W,W'}$ is continuous iff

(1) $\quad (\forall x{\in}W, H'{\in}\mathcal{H}')\ \tau(x) \in H'$ implies $(\exists H{\in}\mathcal{H})\ x \in H$ and $\bar{\tau}(H) \subseteq H'$.

Thus it suffices to show (1) iff (2) $(\forall H{\in}\mathcal{H})(\exists H'{\in}\mathcal{H}')\ \bar{\tau}(H) \subseteq H'$.

For the forward direction, assume (1) and take [a] $H \in \mathcal{H}$. Then [C2] for Θ implies there is [b] $x \in W$ such that [c] $x \in H$. Further $\bar{\tau}(W) \subseteq W'$ and [C2] for Θ' imply there is [d] $H' \in \mathcal{H}'$ such that [e] $\tau(x) \in H'$. [b], [d], [e] and (1) imply there is [f] $H_o \in \mathcal{H}$ such that [g] $x \in H_o$ and [h] $\bar{\tau}(H_o) \subseteq H'$. [a] and [c], [f] and [g], and [C2] for Θ imply $H = H_o$. Thus [h] implies $\bar{\tau}(H) \subseteq H'$.

For the reverse direction, assume (2), and take [a] $x \in W$ and [b] $H' \in \mathcal{H}'$ such that [c] $\tau(x) \in H'$. [a] and [C2] for Θ imply there is [d] $H \in \mathcal{H}$ such that [e] $x \in H$. It suffices to show $\bar{\tau}(H) \subseteq H'$. [d] and (2) imply there is [f] $H'_* \in \mathcal{H}'$ such that [g] $\bar{\tau}(H) \subseteq H'_*$. Easily, [e] implies $\tau(x) \in \bar{\tau}(H)$, which by [g] implies [h] $\tau(x) \in H'_*$. [b] and [c], and [f] and [h], and [C2] for Θ' imply $H' = H'_*$. Thus [g] implies $\bar{\tau}(H) \subseteq H'$. □

Proof A.3 (for Proposition 2.1)**.** *(a)–(e).* These follow immediately from [cE].

(f). Take $xy \in E$. In steps, the definition of F implies $\lambda(xy) \in F(x)$, which by the definition of α_x implies $\alpha_x(\lambda(xy)) = \lambda'(\,\tau(x)\,\tau(n(x,\lambda(xy)))\,)$, which by applying Lemma A.1(c) at $x_o = x$ reduces to $\alpha_x(\lambda(xy)) = \lambda'(\,\tau(x)\,\tau(y)\,)$.

(g). Take $x \in W$ and $a \in F(x)$. Then the definition of α_x implies [1] $\alpha_x(a) = \lambda'(\,\tau(x)\,\tau(n(x,a))\,)$. In addition, the definition of α_x implies [2] $\alpha_x(a) \in F'(\tau(x))$. By [2], the definition of n' implies that $n'(\tau(x),\alpha_x(a))$ is the unique y' such that $\lambda'(\,\tau(x)\,y'\,) = \alpha_x(a)$. By [1], that y' is $\tau(n(x,a))$. □

Proof A.4 (for Proposition 2.2)**.** By inspection, the identities are well-defined, the identity laws hold, and associativity holds. Thus the proposition's part (a) follows from Claim 2. Part (b) follows from Claim 1.

Claim 1: Suppose $[\Theta,\Theta',\tau]$ *and* $[\Theta',\Theta'',\tau']$ *are morphisms. Then the action transformation of the tuple* $[\Theta,\Theta'',\tau'{\circ}\tau]$ *is* $\langle \alpha'_{\tau(x)}{\circ}\alpha_x\rangle_{x\in W}$. To see this, let $\langle \alpha^*_x{:}F(x){\to}F''(\tau'{\circ}\tau(x))\rangle_{x\in W}$ be the action transformation derived from the tuple $[\Theta,\Theta'',\tau'{\circ}\tau]$. Fix $x \in W$. Then $\alpha'_{\tau(x)}{\circ}\alpha_x{:}F(x){\to}F''(\tau'{\circ}\tau(x))$ is well-defined since $\alpha_x{:}F(x){\to}F'(\tau(x))$ by construction, and since $\alpha'_{\tau(x)}{:}F'(\tau(x)){\to}F''(\tau'{\circ}\tau(x))$ by the construction of $\langle \alpha'_{x'}{:}F'(x'){\to}F''(\tau'(x'))\rangle_{x'\in W'}$ at $x' = \tau(x)$. Thus it remains to show that $(\forall a{\in}F(x))\ \alpha^*_x(a) = \alpha'_{\tau(x)}(\alpha_x(a))$. Toward that end, take $a \in F(x)$ and note

$$\begin{aligned}\alpha^*_x(a) &= \lambda''(\ \tau'{\circ}\tau(x)\ \tau'{\circ}\tau(n(x,a))\ ) \\ &= \lambda''(\ \tau'(\tau(x))\ \tau'(n'(\tau(x),\alpha_x(a)))\ ) \\ &= \alpha'_{\tau(x)}(\alpha_x(a)),\end{aligned}$$

where the first equality holds by the definition of α^*_x, the second holds by Proposition 2.1(g) applied to $[\Theta,\Theta',\tau]$, and the third holds by the definition of $\alpha'_{\tau(x)}$.

Claim 2: Composition is well-defined. Let $\theta = [\Theta,\Theta',\tau]$ and $\theta' = [\Theta',\Theta'',\tau']$ be morphisms. It suffices to show that the tuple $[\Theta,\Theta'',\tau'{\circ}\tau]$ satisfies [cE], [cI], and [cL]. In this context, [cE] is

$$(\forall xy{\in}E)\ \tau'{\circ}\tau(x)\,\tau'{\circ}\tau(y) \in E''.$$

Take $xy \in E$. Then [cE] for θ implies $\tau(x)\tau(y) \in E'$, which by [cE] for θ' implies $\tau'{\circ}\tau(x)\,\tau'{\circ}\tau(y) \in E''$. Further, [cI] is

$$(\tau'{\circ}\tau)|_{W,W''} \text{ is continuous.}$$

Proposition 2.1(e) implies $(\tau'{\circ}\tau)|_{W,W''} = \tau'|_{W',W''} \circ \tau|_{W,W'}$. This composition is continuous since $\tau|_{W,W'}$ is continuous by [cI] for θ, and since $\tau'|_{W',W''}$ is continuous by [cI] for θ'. Finally, Claim 1 implies that [cL] is

$$\langle \alpha'_{\tau(x)}{\circ}\alpha_x\rangle_{x\in W} \text{ is continuous from } W.$$

Suppose x_1 and x_2 are in the same member of $\mathcal{H}$. By [cL] for θ, this assumption implies [1] $\alpha_{x_1} = \alpha_{x_2}$. By [cI] for θ and Lemma A.2, the assumption also implies $\tau(x_1)$ and $\tau(x_2)$ are in the same member of $\mathcal{H}'$, which by [cL] for θ' implies [2] $\alpha'_{\tau(x_1)} = \alpha'_{\tau(x_2)}$. [1] and [2] imply $\alpha'_{\tau(x_1)}\circ\alpha_{x_1} = \alpha'_{\tau(x_2)}\circ\alpha_{x_2}$. □

Lemma A.5. *Suppose Θ is a CLT. Then the action transformation of id_Θ is $\langle \mathsf{id}_{F(x)}\rangle_{x\in W}$.*

Proof. Let $\langle\alpha_x\rangle_{x\in W}$ be the action transformation of id_Θ. It suffices to show $\langle\alpha_x\rangle_{x\in W} = \langle \mathsf{id}_{F(x)}\rangle_{x\in W}$. Take $x \in W$. Since $F'(\tau(x))$ reduces to $F(x)$, the definition of α_x implies $\alpha_x{:}F(x){\to}F(x)$. Thus it remains to show that $(\forall a{\in}F(x))$ $\alpha_x(a) = a$. Toward that end, take $a \in F(x)$. In steps, $\alpha_x(a)$ by definition is $\lambda'(\,\tau(x)\,\tau(n(x,a))\,)$, which reduces to $\lambda'(\,x\,n(x,a)\,)$, which reduces to $\lambda(\,x\,n(x,a)\,)$, which by Lemma A.1(c) is a. □

Proof A.6 (for Proposition 2.3)**.** Take $Z \in \mathcal{Z}$. Before beginning, note that the definition of P' implies [1] $P'{\circ}\tau(r){\cup}\{\tau(r)\}$ is the path from r' to $\tau(r)$.

On the one hand, suppose $Z \in \mathcal{Z}_{\mathsf{ft}}$. Then there is a [2] $y \in X{\smallsetminus}W$ such that [3] Z is the path from r to y. [2] and end-preservation imply [4] $\tau(y) \in X'{\smallsetminus}W'$. Meanwhile, [3] and [cE] imply $\bar{\tau}(Z)$ is the path from $\tau(r)$ to $\tau(y)$, which by [1] implies that the concatenation $P'{\circ}\tau(r){\cup}\bar{\tau}(Z)$ is the path from r' to $\tau(y)$. Thus [4] implies $P'{\circ}\tau(r){\cup}\bar{\tau}(Z) \in \mathcal{Z}'_{\mathsf{ft}}$.

On the other hand, take $Z \in \mathcal{Z}_{\mathsf{inft}}$. Then Z is an infinite path from r. Thus [cE] implies $\bar{\tau}(Z)$ is an infinite path from $\tau(r)$, which by [1] implies that the concatenation $P'{\circ}\tau(r){\cup}\bar{\tau}(Z)$ is an infinite path from r'. Hence $P'{\circ}\tau(r){\cup}\bar{\tau}(Z) \in \mathcal{Z}'_{\mathsf{inft}}$. □

Lemma A.7. *Suppose $[\Theta,\Theta',\tau]$ and $[\Theta',\Theta'',\tau']$ are end-preserving morphisms. Then $[\Theta',\Theta'',\tau']{\circ}[\Theta,\Theta',\tau]$ is end-preserving and its run transformation is $\zeta'{\circ}\zeta$.*

Proof. The lemma follows from Claims 1 and 4.

Claim 1: $[\Theta',\Theta'',\tau']{\circ}[\Theta,\Theta',\tau]$ is end-preserving. Take $x \in X{\smallsetminus}W$. Since $[\Theta,\Theta',\tau]$ is end-preserving, $\tau(x) \in X'{\smallsetminus}W'$. Thus since $[\Theta',\Theta'',\tau']$ is end-preserving, $\tau'{\circ}\tau(x) \in X''{\smallsetminus}W''$.

Claim 2: $P''{\circ}\tau'(r') \cup \overline{\tau'}(P'{\circ}\tau(r){\cup}\{\tau(r)\}) = P''{\circ}\tau'{\circ}\tau(r){\cup}\{\tau'{\circ}\tau(r)\}$.

First, the definition of P'' implies

$$P''{\circ}\tau'(r'){\cup}\{\tau'(r')\} \text{ is the path from } r'' \text{ to } \tau'(r'),$$

Second, the definition of P' implies $P'{\circ}\tau(r){\cup}\{\tau(r)\}$ is the path in (X',E') from r' to $\tau(r)$, which by [cE] for $[\Theta',\Theta'',\tau']$ implies

$$\overline{\tau'}(P'{\circ}\tau(r){\cup}\{\tau(r)\}) \text{ is the path from } \tau'(r') \text{ to } \tau'{\circ}\tau(r).$$

The above imply that the concatenation $P''{\circ}\tau'(r') \cup \overline{\tau'}(P'{\circ}\tau(r){\cup}\{\tau(r)\})$ is the path from r'' to $\tau'{\circ}\tau(r)$. Yet by the definition of P'', $P''{\circ}\tau'{\circ}\tau(r){\cup}\{\tau'{\circ}\tau(r)\}$ is the path from r'' to $\tau'{\circ}\tau(r)$. Thus the two are equal.

Claim 3: $(\forall Z{\in}\mathcal{Z})$ $P''{\circ}\tau'(r') \cup \overline{\tau'}(P'{\circ}\tau(r){\cup}\bar{\tau}(Z)) = P''{\circ}\tau'{\circ}\tau(r) \cup \overline{\tau'{\circ}\tau}(Z)$.

Take $Z \in \mathcal{Z}$. It will be argued that

$$\begin{aligned} & P''{\circ}\tau'(r') \cup \overline{\tau'}(\ P'{\circ}\tau(r){\cup}\bar{\tau}(Z)\ ) \\ = {} & P''{\circ}\tau'(r') \cup \overline{\tau'}(\ P'{\circ}\tau(r){\cup}\{\tau(r)\} \cup \bar{\tau}(Z)\ ) \end{aligned}$$

$$= P''{\circ}\tau'(r') \cup \overline{\tau'}(P'{\circ}\tau(r){\cup}\{\tau(r)\}) \cup \overline{\tau'{\circ}\tau}(Z)$$
$$= P''{\circ}\tau'{\circ}\tau(r){\cup}\{\tau'{\circ}\tau(r)\} \cup \overline{\tau'{\circ}\tau}(Z)$$
$$= P''{\circ}\tau'{\circ}\tau(r) \cup \overline{\tau'{\circ}\tau}(Z).$$

The first holds because $r \in Z$, the second by manipulation, the third by Claim 2, and the fourth because $r \in Z$.

Claim 4: The run transformation of $[\Theta', \Theta'', \tau']{\circ}[\Theta, \Theta', \tau]$ *is* $\zeta'{\circ}\zeta$. Since the morphisms are end-preserving, $\zeta{:}\mathcal{Z}{\to}\mathcal{Z}'$ and $\zeta'{:}\mathcal{Z}'{\to}\mathcal{Z}''$ are well-defined. Thus $\zeta'{\circ}\zeta{:}\mathcal{Z}{\to}\mathcal{Z}''$ is well-defined. Meanwhile, Claim 1 implies that the run transformation of $[\Theta', \Theta'', \tau']{\circ}[\Theta, \Theta', \tau]$ is well-defined. By definition, it has domain $\mathcal{Z}$ and codomain $\mathcal{Z}''$. Thus is suffices to show $(\forall Z{\in}\mathcal{Z})\ \zeta'(\zeta(Z)) = P''{\circ}\tau'{\circ}\tau(r) \cup \overline{\tau'{\circ}\tau}(Z)$. This is equivalent to Claim 3. □

Proof A.8 (for Proposition 2.4)**.** By inspection, the identities are well-defined, the identity laws hold, and associativity holds. Thus the proposition's part (a) follows from Claim 3. Part (b) follows from Claims 1 and 2.

Claim 1: Suppose $\gamma = [\Gamma, \Gamma', \tau]$ *and* $\gamma' = [\Gamma', \Gamma'', \tau']$ *are morphisms. Then the run transformation of the tuple* $[\Gamma, \Gamma'', \tau'{\circ}\tau]$ *is* $\zeta'{\circ}\zeta'$. [gZ] for γ implies $[\Theta, \Theta', \tau]$ is end-preserving. Similarly [gZ] for γ' implies $[\Theta', \Theta'', \tau']$ is end-preserving. Thus the claim follows from Lemma A.7.

Claim 2: Suppose $\gamma = [\Gamma, \Gamma', \tau]$ *and* $\gamma' = [\Gamma', \Gamma'', \tau']$ *are morphisms. Then the player transformation of the tuple* $[\Gamma, \Gamma'', \tau'{\circ}\tau]$ *is* $\iota'{\circ}\iota$. It suffices to show that $\iota'{\circ}\iota{\circ}\mu = \mu''{\circ}(\tau'{\circ}\tau)|_{W,W''}$. In steps, $\iota'{\circ}\iota{\circ}\mu$ by [gM] for γ equals $\iota'{\circ}\mu'{\circ}\tau|_{W,W'}$, which by [gM] for γ' equals $\mu''{\circ}\tau'|_{W',W''}{\circ}\tau|_{W,W'}$, which by inspection equals $\mu''{\circ}(\tau'{\circ}\tau)|_{W,W''}$.

Claim 3: Composition is well-defined. Suppose $\gamma = [\Gamma, \Gamma', \tau]$ and $\gamma' = [\Gamma', \Gamma'', \tau']$ are morphisms. It suffices to show that the tuple $[\Gamma, \Gamma'', \tau'{\circ}\tau]$ satisfies [gZ], [gM], and [gU]. First, [gZ] follows from [gZ] for γ, [gZ] for γ', and Lemma A.7. Second, [gM] holds since $\iota'{\circ}\iota$ is the player transformation of $[\Gamma, \Gamma'', \tau'{\circ}\tau]$ by Claim 2. Finally, Claims 1 and 2 imply [gU] is equivalent to

$$(\forall i{\in}I, Z_1{\in}\mathcal{Z}, Z_2{\in}\mathcal{Z})$$
$$U_i(Z_1) \geq U_i(Z_2) \text{ implies } U''_{\iota'\circ\iota(i)}{\circ}\zeta'{\circ}\zeta(Z_1) \geq U''_{\iota'\circ\iota(i)}{\circ}\zeta'{\circ}\zeta(Z_2).$$

To show this conditional holds, take $i \in I$, $Z_1 \in \mathcal{Z}$, and $Z_2 \in \mathcal{Z}$. Assume $U_i(Z_1) \geq U_i(Z_2)$. Then [gU] for γ implies $U'_{\iota(i)}{\circ}\zeta(Z_1)) \geq U'_{\iota(i)}{\circ}\zeta(Z_2)$, which by [gU] for γ' implies $U''_{\iota'\circ\iota(i)}{\circ}\zeta'{\circ}\zeta(Z_1) \geq U''_{\iota'\circ\iota(i)}{\circ}\zeta'{\circ}\zeta(Z_2)$. □

Lemma A.9. *Suppose* Γ *is a game. Then* id_Γ*'s run transformation is* $\mathsf{id}_\mathcal{Z}$*, and its player transformation is* id_I*.*

Proof. Let $\zeta{:}\mathcal{Z}{\to}\mathcal{Z}$ be the run transformation of id_Θ. It suffices to show $(\forall Z{\in}\mathcal{Z})$ $\zeta(Z) = Z$. Take $Z \in \mathcal{Z}$. Then $\zeta(Z)$ by definition is $P(r){\cup}Z$, which by inspection is Z.

Let $\iota{:}I{\to}I$ be the player transformation of id_Γ. It suffices to show $(\forall i{\in}I)\ \iota(i) = i$. Take $i \in I$. By the definition of I and [G2], there is $x \in W$ such that $\mu(x) = i$. In steps, $\iota(i)$ by the definition of x is equal to $\iota(\mu(x))$, which by [gM] for id_Γ is equal to $\mu'(\tau(x))$, which reduces to $\mu(\tau(x))$, which reduces to $\mu(x)$, which by the definition of x is i. □

Proof A.10 (for Proposition 2.5)**.** By [G1], F_0 maps any game to a CLT. By [gZ], F_1 maps any game morphism to a CLT morphism. By inspection, F preserves source and target. Thus the proposition follows from Claims 1 and 2.

Claim 1: F *preserves identity.* In steps, $\mathsf{F}_1(\mathsf{id}_\Gamma)$ by the definition of id in **Gm** is equal to $\mathsf{F}_1([\Gamma,\Gamma,\mathsf{id}_X])$, which by the definition of F_1 is equal to $[\mathsf{F}_0(\Gamma),\mathsf{F}_0(\Gamma),\mathsf{id}_X]$, which by the definition of id in **CLT** is equal to $\mathsf{id}_{\mathsf{F}_0(\Gamma)}$.

Claim 2: F *preserves composition.* In steps, $\mathsf{F}_1([\Gamma',\Gamma'',\tau']\circ[\Gamma,\Gamma',\tau])$ by the definition of $\circ$ in **Gm** is equal to $\mathsf{F}_1([\Gamma,\Gamma'',\tau'\circ\tau])$, which by the definition of F_1 is equal to $[\mathsf{F}_0(\Gamma),\mathsf{F}_0(\Gamma''),\tau'\circ\tau]$, which by the definition of $\circ$ in **CLT** is equal to $[\mathsf{F}_0(\Gamma'),\mathsf{F}_0(\Gamma''),\tau']\circ[\mathsf{F}_0(\Gamma),\mathsf{F}_0(\Gamma'),\tau]$, which by two applications of the definition of F_1 is equal to $\mathsf{F}_1([\Gamma',\Gamma'',\tau'])\circ\mathsf{F}_1([\Gamma,\Gamma',\tau])$. □

APPENDIX B. FOR MONOMORPHISMS

Lemma B.1. *Suppose that* (X,E) *and* (X',E') *are out-trees, that* $\tau{:}X{\to}X'$ *satisfies* $(\forall xy{\in}E)\ \tau(x)\tau(y)\in E'$*, and that* $\hat{X}$ *is a path in* (X,E)*. Then* $\bar{\tau}(\hat{X})$ *is a path in* (X',E') *and* $\tau|_{\hat{X},\bar{\tau}(\hat{X})}$ *is a graph isomorphism.*[24]

Proof. Let $\hat{E}$ be the edge set of $\hat{X}$. The lemma is proved after the following claims.

Claim 1: $\tau|_{\hat{X},\bar{\tau}(\hat{X})}$ *is a bijection.* It suffices to show that $\tau|_{\hat{X},\bar{\tau}(\hat{X})}$ is injective. Toward that end, take distinct x_1 and x_2 in $\hat{X}$. Because $\hat{X}$ is a path, either $x_1\prec x_2$ or $x_1\succ x_2$. Assume the former without loss of generality. Then the lemma's assumption on τ implies $\tau(x_1)\prec\tau(x_2)$, which implies $\tau(x_1)\neq\tau(x_2)$.

Claim 2: Suppose $\hat{X}$ *is finite. Then (a)* $\bar{\tau}(\hat{X})$ *is a path in* (X',E')*, and (b) the edge set of* $\bar{\tau}(\hat{X})$ *is* $\{\,\tau(x)\tau(y)\,|\,xy{\in}\hat{E}\,\}$.

Since $\hat{X}$ is a finite path with edge set $\hat{E}$, there is an $\ell\geq 0$ and a bijection $\phi{:}\{0,1,...\,\ell\}{\to}\hat{X}$ such that $\{\phi_0\phi_1,\phi_1\phi_2,...\,\phi_{\ell-1}\phi_\ell\}=\hat{E}$. Define $\psi=\tau|_{\hat{X},\bar{\tau}(\hat{X})}{\circ}\phi$. The bijectivity of ϕ and Claim 1 imply $\psi{:}\{0,1,...\,\ell\}{\to}\bar{\tau}(\hat{X})$ is a well-defined bijection. Thus the pair consisting of $\bar{\tau}(\hat{X})$ and $\{\psi_0\psi_1,\psi_1\psi_2,...\,\psi_{\ell-1}\psi_\ell\}$ is a path.

It suffices to show this path is in (X',E'), and $\{\psi_0\psi_1,\psi_1\psi_2,...\,\psi_{\ell-1}\psi_\ell\}=\{\,\tau(x)\tau(y)\,|\,xy{\in}\hat{E}\,\}$. For the equality, $\{\psi_0\psi_1,\psi_1\psi_2,...\,\psi_{\ell-1}\psi_\ell\}$ by the definition of ψ is equal to $\{\,\tau(x)\tau(y)\mid xy{\in}\{\phi_0\phi_1,\phi_1\phi_2,...\,\phi_{\ell-1}\phi_\ell\}\,\}$, which by the definition of ϕ is equal to $\{\,\tau(x)\tau(y)\,|\,xy{\in}\hat{E}\,\}$. Further, this equality and the lemma's assumption on τ imply $\{\psi_0\psi_1,\psi_1\psi_2,...\,\psi_{\ell-1}\psi_\ell\}\subseteq E'$. This and the fact that $\bar{\tau}(\hat{X})\subseteq X'$ imply that the path is in (X',E').

Claim 3: Suppose $\hat{X}$ *is infinite. Then (a)* $\bar{\tau}(\hat{X})$ *is a path in* (X',E')*, and (b) the edge set of* $\bar{\tau}(\hat{X})$ *is* $\{\,\tau(x)\tau(y)\,|\,xy{\in}\hat{E}\,\}$.

Since $\hat{X}$ is an infinite path in the out-tree (X,E), $\hat{X}$ is an infinite path from some node. Thus there is a bijection $\phi{:}\{0,1,...\,\}{\to}\hat{X}$ such that $\{\phi_0\phi_1,\phi_1\phi_2,...\,\}=\hat{E}$. Define $\psi=\tau|_{\hat{X},\bar{\tau}(\hat{X})}{\circ}\phi$. The bijectivity of ϕ and Claim 1 imply $\psi{:}\{0,1,...\,\}{\to}\bar{\tau}(\hat{X})$

[24]Bang-Jensen and Gutin 2009, page 7, defines graph isomorphisms for directed pseudographs. To specialize that definition to the present context, suppose X^1 and X^2 are two paths in out-trees. Then $\psi{:}X^1{\to}X^2$ is a graph isomorphism iff (a) ψ is a bijection and (b) the edge set of X^2 is $\{\,\psi(x)\psi(y)\,|\,xy{\in}E^1\,\}$, where E^1 is the edge set of X^1.

is a well-defined bijection. Thus the pair consisting of $\bar{\tau}(\hat{X})$ and $\{\psi_0\psi_1, \psi_1\psi_2, ...\}$ is a path.

It remains to be shown that this path is in (X', E') and that $\{\psi_0\psi_1, \psi_1\psi_2, ...\} = \{\, \tau(x)\tau(y) \,|\, xy{\in}\hat{E} \,\}$. For the equality, $\{\psi_0\psi_1, \psi_1\psi_2, ...\}$ by the definition of ψ is equal to $\{\, \tau(x)\tau(y) \mid xy{\in}\{\phi_0\phi_1, \phi_1\phi_2, ...\} \,\}$, which by the definition of ϕ is equal to $\{\, \tau(x)\tau(y) \,|\, xy{\in}\hat{E} \,\}$. Further, this equality and the lemma's assumption on τ imply $\{\psi_0\psi_1, \psi_1\psi_2, ...\} \subseteq E'$. This and the fact that $\bar{\tau}(\hat{X}) \subseteq X'$ imply that the path is in (X', E').

Conclusion. Claims 2(a) and 3(a) imply $\bar{\tau}(\hat{X})$ is a path in (X', E'). Thus it suffices to show that $\tau|_{\hat{X}, \bar{\tau}(\hat{X})}$ is a graph isomorphism. Claim 1 shows it is a bijection. Thus it suffices to show that $\{\, \tau(x)\tau(y) \,|\, xy{\in}\hat{E} \,\}$ is equal to the edge set of $\bar{\tau}(\hat{X})$. This follows from Claims 2(b) and 3(b). □

Lemma B.2. *Suppose $[\Theta, \Theta', \tau]$ is a monomorphism. Then τ is injective.*

Proof. To show the contrapositive, suppose τ is not injective. Then there are $x_1 \in X$ and $x_2 \in X$ such that [a] $x_1 \neq x_2$ and [b] $\tau(x_1) = \tau(x_2)$.

This paragraph concerns the predecessors of x_1 and x_2. The contrapositive of Proposition 2.1(b) and [b] imply not $x_1 \prec x_2$, which by [a] implies [c] $x_1 \in X {\smallsetminus} \{r\}$, which by Proposition 2.1(d) implies [d] $\tau(p(x_1)) = p'(\tau(x_1))$. Similarly, the contrapositive of Proposition 2.1(b) and [b] imply not $x_2 \prec x_1$, which by [a] implies [e] $x_2 \in X {\smallsetminus} \{r\}$, which by Proposition 2.1(d) implies [f] $\tau(p(x_2)) = p'(\tau(x_2))$. Finally, [b], [d], and [f] imply [g] $\tau(p(x_1)) = \tau(p(x_2))$.

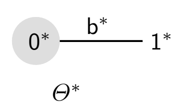

Figure B.1. The Θ^* in Lemma B.2's proof.

Now define the tuple $\theta^1 = [\Theta^*, \Theta, \tau^1]$ by taking Θ^* from Figure B.1,[25] and by setting $\tau^1(0^*) = p(x_1)$ and $\tau^1(1^*) = x_1$. To show θ^1 is a morphism, note that Θ^* is a CLT by inspection, that Θ is a CLT by assumption, and that $\tau^1{:}X^* {\to} X$ by the definition of x_1 and [c]. [cE] holds because $E^* = \{\, 0^*1^* \,\}$ and because $\tau^1(0^*)\tau^1(1^*) = p(x_1)\, x_1 \in E$. [cI] and [cL] hold because $\mathcal{H}^*$ generates the discrete topology.

Similarly define the tuple $\theta^2 = [\Theta^*, \Theta, \tau^2]$ by taking Θ^* from Figure B.1 and by setting $\tau^2(0^*) = p(x_2)$ and $\tau^2(1^*) = x_2$. By reasoning like the previous paragraph, θ^2 is a morphism (replace τ^1 with τ^2, x_1 with x_2, and [c] with [e]). Note [a] implies $\tau^1(1^*) \neq \tau^2(1^*)$, which implies $\theta^1 \neq \theta^2$.

Thus it remains to show $\theta{\circ}\theta^1 = \theta{\circ}\theta^2$. By definition, this is equivalent to showing $[\Theta^*, \Theta', \tau{\circ}\tau^1] = [\Theta^*, \Theta', \tau{\circ}\tau^2]$, which is equivalent to showing $\tau{\circ}\tau^1(0^*) = \tau{\circ}\tau^2(0^*)$ and $\tau{\circ}\tau^1(1^*) = \tau{\circ}\tau^2(1^*)$. For the first, note that the left-hand side by definition is $\tau(p(x_1))$, which by [g] equals $\tau(p(x_2))$, which by definition is the right-hand side. For the second, note that the left-hand side by definition is $\tau(x_1)$, which by [b] equals $\tau(x_2)$, which by definition is right-hand side. □

[25] Θ^* must have at least two nodes because of [C1]'s assumption of nontriviality. Otherwise the proof could be simpler.

Lemma B.3. *Suppose $\gamma = [\mathit{\Gamma}, \mathit{\Gamma}', \tau]$ is a morphism. Then if ζ is injective, γ is monic.*

Proof. To prove the contrapositive, suppose γ is not monic. Then there are γ^1 and γ^2 such that $\gamma^1 \neq \gamma^2$ and $\gamma \circ \gamma^1 = \gamma \circ \gamma^2$. The equality implies that γ^1 and γ^2 have a common source. Denote that source $\dot{\mathit{\Gamma}}$. Thus $\tau^1{:}\dot{X} \to X$ and $\tau^2{:}\dot{X} \to X$ satisfy $\tau^1 \neq \tau^2$ and [a] $\tau \circ \tau^1 = \tau \circ \tau^2$. The inequality implies there is $\dot{x} \in \dot{X}$ such that [b] $\tau^1(\dot{x}) \neq \tau^2(\dot{x})$. Further, since any node is in at least one run, there is $\dot{Z} \in \dot{\mathcal{Z}}$ such that [c] $\dot{x} \in \dot{Z}$. Note [a] and two applications of Proposition 2.4(b) imply $\zeta(\zeta^1(\dot{Z})) = \zeta(\zeta^2(\dot{Z}))$. Thus Claim 2 below implies ζ is not injective. This completes the proof.

Claim 1: $\tau^1(\dot{x}) \not\prec \tau^2(\dot{x})$ and $\tau^1(\dot{x}) \not\succ \tau^2(\dot{x})$. By symmetry, it suffices to show the first. If $\tau^1(\dot{x}) \prec \tau^2(\dot{x})$, then Proposition 2.1(b) for γ implies $\tau \circ \tau^1(\dot{x}) \prec \tau \circ \tau^2(\dot{x})$, which contradicts [a].

Claim 2: $\zeta^1(\dot{Z}) \neq \zeta^2(\dot{Z})$. To see this, suppose $\zeta^1(\dot{Z}) = \zeta^2(\dot{Z})$. Then the definition of ζ implies $P \circ \tau^1(\dot{r}) \cup \overline{\tau^1}(\dot{Z}) = P \circ \tau^2(\dot{r}) \cup \overline{\tau^2}(\dot{Z})$, which by [c] implies $\tau^1(\dot{x}) \in P \circ \tau^2(\dot{r}) \cup \overline{\tau^2}(\dot{Z})$. On the one hand, suppose $\tau^1(\dot{x}) \in P \circ \tau^2(\dot{r})$. Then $\tau^1(\dot{x}) \prec \tau^2(\dot{r})$. Meanwhile, $\dot{r} \,\dot{\preccurlyeq}\, \dot{x}$ and Proposition 2.1(a) for γ^2 imply $\tau^2(\dot{r}) \preccurlyeq \tau^2(\dot{x})$. The last two sentences imply $\tau^1(\dot{x}) \prec \tau^2(\dot{x})$, which contradicts Claim 1. On the other hand, suppose $\tau^1(\dot{x}) \in \overline{\tau^2}(\dot{Z})$. Note [c] implies $\tau^2(\dot{x}) \in \overline{\tau^2}(\dot{Z})$. Further, since $\dot{Z}$ is a path, [cE] for τ^2 implies $\overline{\tau^2}(\dot{Z})$ is a path. Thus $\tau^1(\dot{x})$ and $\tau^2(\dot{x})$ belong to the same path, which implies the satisfaction of $\tau^1(\dot{x}) \prec \tau^2(\dot{x})$ or $\tau^1(\dot{x}) = \tau^2(\dot{x})$ or $\tau^1(\dot{x}) \succ \tau^2(\dot{x})$. This contradicts the combination of [b] and Claim 1. □

Lemma B.4. *Suppose $\gamma = [\mathit{\Gamma}, \mathit{\Gamma}', \tau]$ is a morphism. Then if γ is monic, ζ is injective.*

Proof. To prove the contrapositive, suppose ζ is not injective. Then there are [∗1] $X^1 \in \mathcal{Z}$ and [∗2] $X^2 \in \mathcal{Z}$ such that [∗3] $X^1 \neq X^2$ and [∗4] $\zeta(X^1) = \zeta(X^2)$ (in other contexts X^1 and X^2 would be denoted Z_1 and Z_2). Note [∗1] and [∗2] imply X^1 and X^2 are paths. Let E^1 and E^2 be their respective edge sets.

Define the tuple $\mathit{\Gamma}^1 = (X^1, E^1, \mathcal{H}^1, \lambda^1, \mu^1, U^1)$ by augmenting (X^1, E^1) with the following. First define $W^1 = \pi_1 E^1$ and $\mathcal{H}^1 = \{\{x^1\} | x^1 {\in} W^1\}$. Second define $\lambda^1{:}E^1 \to \{\mathsf{b}\}$ by $(\forall x^1 y^1 {\in} E^1)\ \lambda^1(x^1 y^1) = \mathsf{b}$. Third define $\mu^1 = \mathsf{id}_{W^1}$ (this unusual construction will identify the player set I^1 with the decision-node set W^1, as shown in Claim 4(a) below). Finally define $U^1 = \langle U^1_{i^1}{:}\{X^1\} \to \mathbb{R} \rangle_{i^1 \in W^1}$ by $(\forall i^1 {\in} W^1)$ $U^1_{i^1}(X^1) = 0$. By inspection, the tuple $\mathit{\Gamma}^1$ is well-defined (Claim 4(b) will show it is a game).

Define the tuple $\gamma^1 = [\mathit{\Gamma}^1, \mathit{\Gamma}, \mathsf{inc}_{X^1,X}]$. This tuple is well-defined since $X^1 \subseteq X$ by [∗1] (Claim 6 will show the tuple is a morphism). Further, define $\delta = (\tau|_{X^2,\bar{\tau}(X^2)})^{-1} \tau|_{X^1,\bar{\tau}(X^1)}$. Claim 2(c) will show δ is well-defined. Given δ, define the tuple $\gamma^2 = [\mathit{\Gamma}^1, \mathit{\Gamma}, \mathsf{inc}_{X^2,X} \circ \delta]$. This tuple is well-defined since $X^2 \subseteq X$ by [∗2] (Claim 8 will show the tuple is a morphism).

After a series of claims, the proof will conclude by showing that γ is not monic.

Claim 1: $\bar{\tau}(X^1) = \bar{\tau}(X^2)$. The definition of ζ and [∗4] imply [∗] $P' \circ \tau(r) \cup \bar{\tau}(X^1) = P' \circ \tau(r) \cup \bar{\tau}(X^1)$. Since $(\forall x {\in} X)$ $r \preccurlyeq x$, Proposition 2.1(a) implies $(\forall x{\in}X)\ \tau(r) \preccurlyeq' \tau(x)$. Thus $X^1 \subseteq X$ implies $P'{\circ}\tau(r)$ and $\bar{\tau}(X^1)$ are disjoint. Similarly, $X^2 \subseteq X$ implies $P'{\circ}\tau(r)$ and $\bar{\tau}(X^2)$ are disjoint. Thus [$*$] implies the claim.

Claim 2: (a) $\tau|_{X^1,\bar{\tau}(X^1)}$ *is a graph isomorphism from the path* X^1 *to the path* $\bar{\tau}(X^1)$*. (b)* $\tau|_{X^2,\bar{\tau}(X^2)}$ *is a graph isomorphism from the path* X^2 *to the path* $\bar{\tau}(X^2)$*. (c)* δ *is a (well-defined) graph isomorphism from the path* X^1 *to the path* X^2. Part (a) follows from Lemma B.1 at $\hat{X} = X^1$. Part (b) follows from Lemma B.1 at $\hat{X} = X^2$. Part (c) follows from parts (a) and (b) and Claim 1.

Claim 3: (a) $(X^1, E^1, \mathcal{H}^1, \lambda^1)$ *is a CLT. (b)* $\mathcal{Z}^1 = \{X^1\}$. For part (a), it suffices to show that the tuple satisfies [C1]–[C4]. Part (b) will be shown en route.

Consider [C1]. By definition, (X^1, E^1) is a path. This implies (X^1, E^1) is an out-tree. It also implies $\mathcal{Z}^1 = \{X^1\}$, which proves (b). Further [$*$1] and the nontriviality in [C1] for Γ imply that X^1 has at least two nodes. Thus (X^1, E^1) is a nontrivial out-tree, which proves [C1].

[C2] follows immediately from the definition of $\mathcal{H}^1$. For [C3], recall X^1 is a path, which implies each element of X^1 has exactly one immediate successor, which vacuously implies λ^1 is deterministic. Further, λ^1 is surjective by inspection. Finally, [C4] holds vacuously because $\mathcal{H}^1$ generates the discrete topology.

Claim 4: (a) $I^1 = W^1$*. (b)* Γ^1 *is a game.* For (a), recall I^1 is the codomain of μ^1 by general definition. Thus the definition $\mu^1 = \mathsf{id}_{W^1}$ implies $I^1 = W^1$. For (b), it suffices to show that the tuple satisfies [G1]–[G3]. [G1] holds by Claim 3. For [G2], μ^1 is continuous because $\mathcal{H}^1$ generates the discrete topology for its domain. Further, μ^1 is surjective by inspection. Finally for [G3], Claim 3(b) and the definition of U^1 implies $U^1 = \langle U^1_{i^1}{:}\mathcal{Z}{\rightarrow}\mathbb{R}\rangle_{i^1 \in W^1}$, which by part (a) implies $U^1 = \langle U^1_{i^1}{:}\mathcal{Z}{\rightarrow}\mathbb{R}\rangle_{i^1 \in I^1}$.

Claim 5: $[\Theta^1, \Theta, \mathsf{inc}_{X^1,X}]$ *is a morphism.* Θ^1 is a CLT by Claim 3 and Θ is a CLT by assumption. Thus it suffices to show that the tuple satisfies [cE], [cI], and [cL]. For [cE], the definition of E^1 implies that every edge in E^1 is also in E. [cI] and [cL] hold vacuously because $\mathcal{H}^1$ generates the discrete topology.

Claim 6: γ^1 *is a morphism.* Γ^1 is a game by Claim 4(b) and Γ is a game by assumption. Define $\iota^1 = \mu{\circ}\mathsf{inc}_{W^1,W}$, and note $\iota^1{:}I^1{\rightarrow}I$ by Claim 4(a). Thus it suffices to show that the tuple γ^1 satisfies [gZ], [gM] with ι^1, and [gU] with ι^1. For [gZ], note $[\Theta^1, \Theta, \mathsf{inc}_{X^1,X}]$ is a morphism by Claim 5. To show that it is end-preserving, recall X^1 is a path. If X^1 is infinite, Θ^1 has no end nodes, so end nodes are preserved vacuously. If X^1 is finite, the only end node in Θ^1 is the end of X^1, which by [$*$1] is also an end in Θ. For [gM], note

$$\iota^1{\circ}\mu^1 = \iota^1{\circ}\mathsf{id}_{W^1} = \mu{\circ}\mathsf{inc}_{W^1,W}{\circ}\mathsf{id}_{W^1} = \mu{\circ}\mathsf{inc}_{W^1,W},$$

where the first equality holds by the definition of μ^1, the second holds by the definition of ι^1, and the third holds by inspection. The entire equality is [gM] with ι^1 by general definition. Finally, [gU] with ι^1 holds vacuously because Claim 3(b) shows $\mathcal{Z}^1$ is the singleton $\{X^1\}$.

Claim 7: $[\Theta^1, \Theta, \mathsf{inc}_{X^2,X}{\circ}\delta]$ *is a morphism.* Θ^1 is a CLT by Claim 3, Θ is a CLT by assumption, and $\mathsf{inc}_{X^2,X}{\circ}\delta : X^1{\rightarrow}X$ by Claim 2(c). Thus it suffices to show that the tuple satisfies [cE], [cI], and [cL]. For [cE], take $x^1y^1 \in E^1$. Then Claim 2(c) implies $\delta(x^1)\delta(y^1) \in E^2$, which by the definition of E^2 implies $\delta(x^1)\delta(y^1) \in E$. [cI] and [cL] hold vacuously because $\mathcal{H}^1$ generates the discrete topology.

Claim 8: γ^2 is a morphism. $\mathit{\Gamma}^1$ is a game by Claim 4(b), $\mathit{\Gamma}$ is a game by assumption, and $\mathsf{inc}_{X^2,X}\circ\delta : X^1 \to X$ by Claim 2(c). Define $\iota^2 = \mu\circ(\mathsf{inc}_{X^2,X}\circ\delta)|_{W^1,W}$, and note $\iota^2{:}I^1\to I$ by Claim 4(a). Thus it suffices to show that the tuple γ^2 satisfies [gZ], [gM] with ι^2, and [gU] with ι^2. For [gZ], note $[\mathit{\Theta}^1, \mathit{\Theta}, \mathsf{inc}_{X^2,X}\circ\delta]$ is a morphism by Claim 7. To show that it is end-preserving, recall X^1 is a path. If X^1 is infinite, $\mathit{\Theta}^1$ has no end nodes, so end nodes are preserved vacuously. If X^1 is finite, the only end node in $\mathit{\Theta}^1$ is the end of X^1. Call this end node x^1. Lemma 2(c) implies $\delta(x^1)$ is the end of X^2. Thus [∗2] implies $\delta(x^1)$ is an end node in $\mathit{\Theta}$. For [gM], note

$$\iota^2\circ\mu^1 = \iota^2\circ\mathsf{id}_{W^1} = \mu\circ(\mathsf{inc}_{X^2,X}\circ\delta)|_{W^1,W}\circ\mathsf{id}_{W^1} = \mu\circ(\mathsf{inc}_{X^2,X}\circ\delta)|_{W^1,W},$$

where the first equality holds by the definition of μ^1, the second holds by the definition of ι^2, and the third holds by inspection. The entire equality is [gM] with ι^2 by general definition. Finally, [gU] with ι^2 holds vacuously because Claim 3(b) shows $\mathcal{Z}^1$ is the singleton $\{X^1\}$.

Claim 9: $\gamma^1 \neq \gamma^2$. It suffices to show $\mathsf{inc}_{X^1,X} \neq \mathsf{inc}_{X^2,X}\circ\delta$. Thus it suffices to show there is $x^1 \in X^1$ such that $x^1 \neq \delta(x^1)$. Mechanically, [∗3] implies that $X^1\smallsetminus X^2$ or $X^2\smallsetminus X^1$ is nonempty. In the first contingency, take [a] $x^1 \in X^1\smallsetminus X^2$. Then Claim 2(c) implies $\delta(x^1) \in X^2$, which by [a] implies $\delta(x^1) \neq x^1$. In the second contingency, take $x^2 \in X^2\smallsetminus X^1$. Thus [b] $x^2 \in X^2$ and [c] $x^2 \notin X^1$. First, [b] and Claim 2(c) imply there is [d] $x^1 \in X^1$ such that [e] $\delta(x^1) = x^2$. Second, [c] and [e] imply $\delta(x^1) \notin X^1$, which by [d] implies $\delta(x^1) \neq x^1$.

Claim 10: $\gamma\circ\gamma^1 = \gamma\circ\gamma^2$. Claims 6 and 8 imply γ^1 and γ^2 are morphisms. Thus, since γ^1 and γ^2 share the target $\mathit{\Gamma}$, the compositions are well-defined. Thus, since γ^1 and γ^2 share the source $\mathit{\Gamma}^1$, it suffices to show $\tau\circ\mathsf{inc}_{X^1,X} = \tau\circ\mathsf{inc}_{X^2,X}\circ\delta$. This equality is equivalent to each of the following by the definition of δ, by Claim 1, and by Claim 2(a):

$$\tau\circ\mathsf{inc}_{X^1,X} = \tau\circ\mathsf{inc}_{X^2,X}\circ(\tau|_{X^2,\bar\tau(X^2)})^{-1}\circ\tau|_{X^1,\bar\tau(X^1)}\,,$$
$$\tau\circ\mathsf{inc}_{X^1,X} = \tau\circ\mathsf{inc}_{X^2,X}\circ(\tau|_{X^2,\bar\tau(X^1)})^{-1}\circ\tau|_{X^1,\bar\tau(X^1)}\,,\ \text{and}$$
$$\tau\circ\mathsf{inc}_{X^1,X}\circ(\tau|_{X^1,\bar\tau(X^1)})^{-1} = \tau\circ\mathsf{inc}_{X^2,X}\circ(\tau|_{X^2,\bar\tau(X^1)})^{-1}.$$

The last holds by inspection.

Conclusion. Claims 9 and 10 imply that γ is not monic. □

Proof B.5 (for Proposition 3.1)**.** *(a).* By inspection, the injectivity of τ is sufficient. Lemma B.2 shows it is necessary. *(b).* Lemma B.3 shows the injectivity of ζ is sufficient. Lemma B.4 shows it is necessary. □

Lemma B.6. *Suppose $[\mathit{\Theta}, \mathit{\Theta}', \tau]$ is a morphism and τ is injective. Then $(\forall x{\in}W)$ α_x is injective.*

Proof. Take $x \in W$. It suffices to show that $(\forall a_1{\in}F(x), a_2{\in}F(x))$ $\alpha_x(a_1) = \alpha_x(a_2) \Rightarrow a_1 = a_2$. Toward that end, suppose $a_1 \in F(x)$ and $a_2 \in F(x)$ are such that $\alpha_x(a_1) = \alpha_x(a_2)$. Then the definition of α_x implies $\lambda'(\,\tau(x)\,\tau(n(x,a_1))\,) = \lambda'(\,\tau(x)\,\tau(n(x,a_2))\,)$. Thus the determinism of λ' from [C3] for $\mathit{\Theta}'$ implies $\tau(n(x,a_1)) = \tau(n(x,a_x))$. Thus the injectivity of τ implies $n(x,a_1) = n(x,a_2)$. Thus Lemma A.1(c) at $x_o = x$ implies $a_1 = a_2$. □

Lemma B.7. *Suppose (X',E') is a nontrivial out-tree and $r \in W'$. Construct (X,E) by the Selten definitions. Then the following hold.*

(a) (X,E) is a nontrivial out-tree.

(b) $W = W' \cap X$.

(c) $W' \smallsetminus W \subseteq X' \smallsetminus X$.

(d) $X \smallsetminus W \subseteq X' \smallsetminus W'$.

Proof. (a). Since (X',E') is an out-tree, the definitions of X and E immediately imply (X,E) is a out-tree. Further, the assumption $r \in W'$ implies E is nonempty, which implies (X,E) is nontrivial.

(b). For the forward direction, note $W \subseteq X$ by definition. Further, in steps, the definition of E implies $E \subseteq E'$, which implies $\pi_1 E \subseteq \pi_1 E'$, which by definition implies $W \subseteq W'$. Conversely, take $x' \in W' \cap X$. Then [1] $x' \in W'$ and [2] $x' \in X$. Note [1] implies there is $y' \in X'$ such that [3] $x'y' \in E'$. Further, in steps, [2] by the definition of X implies $r \preccurlyeq' x'$, which by [3] implies $r \preccurlyeq' y'$, which by the definition of X implies $y' \in X$. This, [2], [3], and the definition of E imply $x'y' \in E$, which implies $x' \in W$.

(c). In steps, $W' \smallsetminus W$ by part (b) is equal to $W' \smallsetminus (W' \cap X)$, which is equal to $W' \smallsetminus X$, which is a subset of $X' \smallsetminus X$.

(d). In steps, $X \smallsetminus W$ by part (b) is equal to $X \smallsetminus (W' \cap X)$, which is equal to $X \smallsetminus W'$, which is a subset of $X' \smallsetminus W'$. □

Lemma B.8. *Suppose (X',E') is a nontrivial out-tree, $\mathcal{H}'$ partitions W', and $r \in W'$. Construct $(X,E,\mathcal{H})$ by the Selten definitions and assume [∗] $(\forall H' {\in} \mathcal{H}')$ $H' {\subseteq} X$ or $H' {\subseteq} X' \smallsetminus X$. Then the following hold.*

(a) $\mathcal{H}$ generates the subspace topology for W that is implied by the topology for W' generated by $\mathcal{H}'$.

(b) $\mathcal{H}$ is a partition of W.

Proof. (a). Since $W \subseteq W'$ by Lemma B.7(b), $\{\, H' {\cap} W {\neq} \varnothing \mid H' {\in} \mathcal{H}' \,\}$ generates the subspace topology for W. In steps, this collection by Lemma B.7(b) is equal to $\{\, H' {\cap} W' {\cap} X {\neq} \varnothing \mid H' {\in} \mathcal{H}' \,\}$, which by $\mathcal{H}'$ partitioning W' is equal to $\{\, H' {\cap} X {\neq} \varnothing \mid H' {\in} \mathcal{H}' \,\}$, which by [∗] is equal to $\{\, H' {\in} \mathcal{H}' \mid H' {\subseteq} X \,\}$, which by Selten definition is equal to $\mathcal{H}$.

(b). Part (a) implies $\cup \mathcal{H} = W$. Thus it remains to show that $\mathcal{H}$ is pairwise disjoint and that each element of $\mathcal{H}$ is nonempty. These facts hold because $\mathcal{H} \subseteq \mathcal{H}'$ by the Selten definition of $\mathcal{H}$, and because $\mathcal{H}'$ is a partition by assumption. □

Lemma B.9. *Suppose (X',E') is a nontrivial out-tree, $\lambda'{:}E' {\to} A'$ is deterministic and surjective, and $r \in W'$. Construct (X,E,λ) by the Selten definitions. Then the following hold.*

(a) $(\forall x {\in} W)$ $F(x) = F'(x)$.

(b) $(\forall (x,a) {\in} F^{\mathsf{gr}})$ $n(x,a) = n'(x,a)$.

Proof. (a). Take $x \in W$. This implies [1] $x \in X$, which by the definition of X implies [2] $r \preccurlyeq' x$. In steps, $a \in F(x)$ by the construction of F is equivalent to $(\exists y {\in} X)$ $xy {\in} E$ and $a {=} \lambda(xy)$, which by the definition of λ is equivalent to $(\exists y {\in} X)$ $xy {\in} E$ and $a {=} \lambda'(xy)$, which by [1] and the definition of E is equivalent to $(\exists y {\in} X)$ $xy {\in} E'$ and $a {=} \lambda'(xy)$, which by the definition of X is equivalent to [3] $(\exists y' {\in} X')$ $r {\preccurlyeq'} y'$ and $xy' {\in} E'$ and $a {=} \lambda'(xy')$. Within [3], $r {\preccurlyeq'} y'$ is redundant because it is implied by [2] and $xy' {\in} E'$. Thus [3] is equivalent to $(\exists y' {\in} X')$ $xy' {\in} E'$ and $a {=} \lambda'(xy')$, which by the construction of F' is equivalent to $a \in F'(x)$.

(b). Take $(x,a) \in F^{\mathsf{gr}}$. Then part (a) implies $(x,a) \in F'^{\mathsf{gr}}$, which implies $n'(x,a)$ is well-defined. In steps, the definition of n implies $a = \lambda(\, x\, n(x,a)\,)$, which by the Selten definition of λ' implies $a = \lambda'(\, x\, n(x,a)\,)$, which by the definition of n' implies $n'(x,a) = n(x,a)$. □

Lemma B.10. *Suppose that Θ' is a CLT and construct Θ by the Selten definitions at $r \in W'$. Then Θ is a CLT iff [$*$] $(\forall H' {\in} \mathcal{H}')\ H' \subseteq X$ or $H' \subseteq X' \smallsetminus X$.*

Proof. This follows from Claims 1 and 2.

Claim 1: Suppose Θ is a CLT. Then [$*$] holds. To prove the contrapositive, suppose [$*$] is false. Then there is [a] $H' \in \mathcal{H}'$ such that both $H' {\cap} X$ and $H' {\smallsetminus} X$ are nonempty. The nonemptiness of $H' {\smallsetminus} X$ and the Selten definition of $\mathcal{H}$ imply that [b] $H' \notin \mathcal{H}$. The nonemptiness of $H' {\cap} X$ implies there is [c] $x \in X$ such that [d] $x \in H'$.

Because $\mathcal{H}'$ is a partition, and because $\mathcal{H} \subseteq \mathcal{H}'$ by the Selten definition of $\mathcal{H}$, the sets $\cup\mathcal{H}$ and $\cup(\mathcal{H}' {\smallsetminus} \mathcal{H})$ are disjoint. Further, [a] and [b] imply $H' \in \mathcal{H}' {\smallsetminus} \mathcal{H}$, which by [d] implies $x \in \cup(\mathcal{H}' {\smallsetminus} \mathcal{H})$, which by the previous sentence implies [e] $x \notin \cup\mathcal{H}$. Yet [a], [d], and [C2] for Θ' imply $x \in W'$, which by [c] implies $x \in W' {\cap} X$, which by Lemma B.7(b) implies $x \in W$. This and [e] contradict [C2] for Θ.

Claim 2: Suppose [$*$] holds. Then Θ is a CLT. It suffices to show that Θ satisfies [C1]–[C4]. [C1] follows from Lemma B.7(a). [C2] follows from Lemma B.8(b).

For [C3], recall $\lambda = \lambda'|_{E,\overline{\lambda'}(E)}$ by the Selten definition. Thus λ is surjective by definition. Further, λ is deterministic because (i) λ' is deterministic by [C3] for Θ' and (ii) a restriction of a deterministic labeling function is deterministic.

For [C4], it suffices to show that F is constant over each $H \in \mathcal{H}$. To see this, take $H \in \mathcal{H}$. Then the Selten definition of $\mathcal{H}$ implies $H \in \mathcal{H}'$, which by [C4] for Θ' implies F' is constant over H, which by Lemma B.9(a) implies F is constant over H. □

Lemma B.11. *Suppose that Γ' is a game and that the Selten subCLT at $r \in W'$ exists. Then the Selten subgame of Γ' at r is well-defined.*

Proof. It suffices to show that Γ satisfies [G1]–[G3]. [G1] holds by assumption.

For [G2], recall $\mu = \mu'|_{W,\overline{\mu'}(W)}$ by the Selten definition. Thus μ is surjective by definition. Further, μ is continuous because (i) μ' is continuous by [G2] for Γ', (ii) a restriction of a continuous function is continuous from the subspace topology by general topology, and (iii) $\mathcal{H}$ generates the subspace topology by Lemma B.8(a).

For [G3], it suffices to show $(\forall i {\in} I, Z {\in} \mathcal{Z})\ U_i(Z) \in \mathbb{R}$. Toward that end, take $i \in I$ and $Z \in \mathcal{Z}$. The definition of P' implies $P'(r) {\cup} \{r\}$ is the path from r' to r, and the Selten definition of X implies Z is either an infinite path from r or a finite path from r to an end node of (X', E'). Thus the concatenation $P'(r) {\cup} Z$ is either an infinite path from r' or a finite path from r' to an end node of (X', E'). Thus $P'(r) {\cup} Z \in \mathcal{Z}'$, which by [G3] for Γ' implies $U'_i(P'(r) {\cup} Z) \in \mathbb{R}$, which by the Selten definition of U_i implies $U_i(Z) \in \mathbb{R}$. □

Lemma B.12.

(a) Θ is a Selten subCLT of Θ' iff (i) $[\Theta, \Theta', \mathsf{inc}_{X,X'}]$ is a morphism with $\alpha = \langle \mathsf{id}_{F(x)} \rangle_{x \in W}$, (ii) $X = \{\, y' {\in} X' \,|\, r {\succcurlyeq}' y' \,\}$, and (iii) $\mathcal{H} \subseteq \mathcal{H}'$.

(b) Γ is a Selten subgame of Γ' iff (i$*$) $[\Gamma, \Gamma', \mathsf{inc}_{X,X'}]$ is a morphism with $\alpha = \langle \mathsf{id}_{F(x)} \rangle_{x \in W}$ and $\iota = \mathsf{inc}_{I,I'}$, (ii) $X = \{\, y' {\in} X' \,|\, r {\succcurlyeq}' y' \,\}$, (iii) $\mathcal{H} \subseteq \mathcal{H}'$, and (iv) $U = \langle U'_i {\circ} \zeta \rangle_{i \in I}$.

Proof. Part (a) holds by Claims 1 and 3. Part (b) holds by Claims 2 and 4.

Claim 1: Suppose Θ is a Selten subCLT of Θ'. Then $[\Theta,\Theta']$ satisfies (i)–(iii). The Selten definitions of X and $\mathcal{H}$ implies (ii) and (iii).

Derive α from the tuple $[\Theta,\Theta',\mathsf{inc}_{X,X'}]$. To see that $\alpha = \langle \mathsf{id}_{F(x)}\rangle_{x\in W}$, take $x \in W$. By the definition of α_x, $\alpha_x{:}F(x){\rightarrow}F'(\tau(x))$. Note $F'(\tau(x))$ reduces to $F'(x)$, which by Lemma B.9(a) is equal to $F(x)$. Thus it remains to show that $(\forall a{\in}F(x))\ \alpha_x(a) = a$. Toward that end, take $a \in F(x)$. In steps, $\alpha_x(a)$ by definition is equal to $\lambda'(\,\tau(x)\,\tau(n(x,a))\,)$, which reduces to $\lambda'(\,x\,n(x,a)\,)$, which by the Selten definition of λ is equal to $\lambda(\,x\,n(x,a)\,)$, which by Lemma A.1(c) is a.

Finally, to see (i), it suffices to show that the tuple $[\Theta,\Theta',\mathsf{inc}_{X,X'}]$ satisfies [cE], [cI], and [cL]. First, [cE] reduces to $(\forall xy{\in}E)\ xy \in E'$, which holds by the Selten definition of E. Second, [cI] reduces to the continuity of $\mathsf{inc}_{W,W'}$, which holds by Lemma B.8(a). Finally, [cL] by the previous paragraph reduces to the continuity of $\langle \mathsf{id}_{F(x)}\rangle_{x\in W}$ from W, which is equivalent to the continuity of $\langle F(x)\rangle_{x\in W}$ from W, which is [C4] for Θ.

Claim 2: Suppose Γ is a Selten subgame of Γ'. Then $[\Gamma,\Gamma']$ satisfies (i∗), (ii), (iii), and (iv). Claim 1 implies (ii) and (iii). Thus it suffices to prove (i∗) and (iv).

Note [a] $I \subseteq I'$ because I by general definition is the codomain of μ, which by Selten definition is the codomain of $\mu'|_{W,\overline{\mu'(W)}}$, which is a subset of the codomain of μ', which by general definition is I'. Note [a] implies $\mathsf{inc}_{I,I'}$ is well-defined. Thus, for (i∗), it suffices to show that [gZ] holds with $\alpha = \langle \mathsf{id}_{F(x)}\rangle_{x\in W}$, that [gM] holds with $\iota = \mathsf{inc}_{I,I'}$, and that [gU] holds with $\iota = \mathsf{inc}_{I,I'}$. These three statements and (iv) are proved in the following four paragraphs.

First, consider [gZ] with $\alpha = \langle \mathsf{id}_{F(x)}\rangle_{x\in W}$. Claim 1 implies (i), which states that $[\Theta,\Theta',\mathsf{inc}_{X,X'}]$ is a morphism with $\alpha = \langle \mathsf{id}_{F(x)}\rangle_{x\in W}$. In addition, Lemma B.7(d) implies $[\Theta,\Theta',\mathsf{inc}_{X,X'}]$ is end-preserving.

Second, consider [gM] with $\iota = \mathsf{inc}_{I,I'}$. This reduces to $(\forall x{\in}W)\ \mu(x) = \mu'(x)$, which follows from the Selten definition of μ.

Third, consider (iv). Take $i \in I$. Since $U_i{:}\mathcal{Z}{\rightarrow}\mathbb{R}$ by [G3] for Γ, it suffices to show (1) $U_i'{\circ}\zeta{:}\mathcal{Z}{\rightarrow}\mathbb{R}$ is well-defined and (2) $(\forall Z{\in}\mathcal{Z})\ U_i(Z) = U_i'(\zeta(Z))$. For (1), note [a] implies $i \in I'$, which by [G3] for Γ' implies $U_i'{:}\mathcal{Z}'{\rightarrow}\mathbb{R}$. Thus it suffices that $\zeta{:}\mathcal{Z}{\rightarrow}\mathcal{Z}'$ by general definition. For (2), take $Z \in \mathcal{Z}$. Then $U_i(Z)$ by Selten definition is $U_i'(P(r){\cup}Z)$, which by the general definition of ζ is $U_i'(\zeta(Z))$.

Fourth, consider [gU] with $\iota = \mathsf{inc}_{I,I'}$. This reduces to $(\forall i{\in}I, Z_1{\in}\mathcal{Z}, Z_2{\in}\mathcal{Z})$ $U_i(Z_1) \geq U_i(Z_2) \Rightarrow U_i'{\circ}\zeta(Z_1)) \geq U_i'{\circ}\zeta(Z_2)$, which follows immediately from (iv).

Claim 3: Suppose $[\Theta,\Theta']$ satisfies (i)–(iii). Then Θ is a Selten subCLT of Θ'. Assumption (i) and Proposition 2.1(e) imply $W \subseteq W'$, which implies $r \in W'$. Further, (i) implies Θ is the source of a morphism, which easily implies Θ is a CLT. Thus it suffices to show that Θ satisfies the Selten definitions for X, E, $\mathcal{H}$, and λ. The Selten definition for X is the same as (ii), which is assumed.

The Selten definition for E is $E = \{\, x'y'{\in}E' \mid \{x',y'\}{\subseteq}X \,\}$. To show this, start with the observation that (i) and [cE] imply [a] $E \subseteq E'$. Next, for the forward direction, take $xy \in E$. Then [a] implies $xy \in E'$, and [C1] for Θ easily implies $\{x,y\} \subseteq X$. Then, for the reverse direction, take [b] $x'y' \in E'$ such that [c] $\{x',y'\} \subseteq X$. By [C1] for Θ, $(X,\mathcal{E})$ is an unoriented tree, where $\mathcal{E} = \{\, \{x,y\} \,|\, xy{\in}E \,\}$. Similarly by [C1] for Θ', $(X',\mathcal{E}')$ is an unoriented tree, where $\mathcal{E}' = \{\, \{x',y'\} \,|\, x'y'{\in}E' \,\}$. [a] implies [d] $\mathcal{E} \subseteq \mathcal{E}'$. Further, by [c] and the definition of a tree, the nodes x' and y' are linked by a path $(X^*,\mathcal{E}^*)$ in $(X,\mathcal{E})$. Thus by [d], x' and y' are linked by the path $(X^*,\mathcal{E}^*)$ in $(X',\mathcal{E}')$. Meanwhile, [b] implies that x' and y' are also linked by the one-edge path $(\{x',y'\},\{\{x',y'\}\})$ in $(X',\mathcal{E}')$. Since any two nodes in a tree a linked by a unique path, $(X^*,\mathcal{E}^*) = (\{x',y'\},\{\{x',y'\}\})$. Thus since $(X^*,\mathcal{E}^*)$ was defined to be a path in $(X,\mathcal{E})$, $(\{x',y'\},\{\{x',y'\}\})$ is a path in $(X,\mathcal{E})$. Hence $\{x',y'\} \in \mathcal{E}$. Thus the definition of $\mathcal{E}$ implies $x'y' \in E$ or $y'x' \in E$. Thus it suffices to show $y'x' \notin E$. To see this, note [b] and the definition of an oriented tree imply $y'x' \notin E'$, which by [a] implies $y'x' \notin E$.

The Selten definition for $\mathcal{H}$ is $\mathcal{H} = \{\, H'{\in}\mathcal{H}' \mid H'{\subseteq}X \,\}$. To show the forward direction, take $H \in \mathcal{H}$. Then (iii) implies $H \in \mathcal{H}'$. Further, [C2] for Θ implies $H \subseteq W$, which implies $H \subseteq X$. Conversely, take [e] $H' \in \mathcal{H}'$ such that [f] $H' \subseteq X$. In steps, [e] by [C2] for Θ' implies $H' \subseteq W'$, which by [f] implies $H' \subseteq X{\cap}W'$, which by Lemma B.7(b) and the preceding two paragraphs implies $H' \subseteq W$, which by [C2] for Θ implies [g] $H' \subseteq \cup\mathcal{H}$. Meanwhile, since $\mathcal{H} \subseteq \mathcal{H}'$ by (iii), and since both $\mathcal{H}$ and $\mathcal{H}'$ are partitions by [C2] for Θ and [C2] for Θ', $\mathcal{H}' = \mathcal{H} \cup \{H'{\in}\mathcal{H}' | H'{\subseteq}(\cup\mathcal{H}'){\setminus}(\cup\mathcal{H})\}$. Thus [e] and [g] imply $H' \in \mathcal{H}$.

The Selten definition for λ is $\lambda = \lambda'|_{E,\overline{\lambda'}(E)}$. To show this, first note that the domain of λ is E by [C3] for Θ. Thus, the domain of λ equals the domain of $\lambda'|_{E,\overline{\lambda'}(E)}$. Further, to show that the graph of λ equals the graph of $\lambda'|_{E,\overline{\lambda'}(E)}$, it suffices to show that $(\forall xy{\in}E)\ \lambda(xy) = \lambda'(xy)$. Toward that end, take $xy \in E$. In steps, $\lambda(xy)$ by the definition of F is equal to $\mathsf{id}_{F(x)}(\lambda(xy))$, which by (i)'s action-transformation statement is equal to $\alpha_x(\lambda(xy))$, which by (i) and Proposition 2.1(f) is equal to $\lambda'(\tau(x)\tau(y))$, which by (i) reduces to $\lambda'(xy)$. Finally, λ is surjective by [C3] for Θ and $\lambda'|_{E,\overline{\lambda'}(E)}$ is surjective by construction. Thus, since their graphs are equal, the codomain of λ is equal to the codomain of $\lambda'|_{E,\overline{\lambda'}(E)}$.

Claim 4: Suppose $[\Gamma,\Gamma']$ satisfies (i∗), (ii), (iii), and (iv). Then Γ is a Selten subgame of Γ'. Note (i∗) and [gZ] imply (i). Thus (ii), (iii), and Claim 3 imply that Θ is a Selten subCLT of Θ'. Hence it remains to show that Γ satisfies the Selten definitions for μ and U.

The Selten definition for μ is $\mu = \mu'|_{W,\overline{\mu'}(W)}$. To show this, first note that the domain of μ is W by [G2] for Γ. Thus, the domain of μ equals the domain of $\mu'|_{W,\overline{\mu'}(W)}$. Further, (i∗) and [gM] imply $(\forall x{\in}W)\ \mu(x) = \mu'(x)$. Thus the graph of μ equals the graph of $\mu'|_{W,\overline{\mu'}(W)}$. Finally, μ is surjective by [G2] for Γ, and $\mu'|_{W,\overline{\mu'}(W)}$ is surjective by construction. Thus, since their graphs are equal, the codomain of μ is equal to the codomain of $\mu'|_{W,\overline{\mu'}(W)}$.

The Selten definition for U is (a) $U = \langle U_i{:}\mathcal{Z}{\to}\mathbb{R}\rangle_{i\in I}$ and (b) $(\forall i{\in}I, Z{\in}\mathcal{Z})$ $U_i(Z) = U_i'(P'(r){\cup}Z)$. To show (a), take $i \in I$. Then (iv) implies $U_i = U_i'{\circ}\zeta$. The domain of ζ is $\mathcal{Z}$ by the definition of ζ, and the codomain of U_i' is $\mathbb{R}$ by [G3] for Γ'. To show (b), take $i \in I$ and $Z \in \mathcal{Z}$. Then $U_i(Z)$ by (iv) is equal to $U_i'(\zeta(Z))$, which by the definition of ζ is equal to $U_i'(P'(r){\cup}Z)$. □

Proof B.13 (for Theorem 3.2)**.** *(a).* This is equivalent to Lemma B.12(a). In particular, the theorem's (i) implies the lemma's (i) by inspection. The converse holds because $\mathsf{inc}_{X,X'}$ is an injective τ, which by Proposition 3.1(a) implies $[\Theta,\Theta',\tau]$ is monic. Finally by inspection, the theorem's (ii) and (iii) are the same as the lemma's (ii) and (iii).

(b). This is equivalent to Lemma B.12(b). In particular, the theorem's (i∗) implies the lemma's (i∗). The converse holds because $\mathsf{inc}_{X,X'}$ is an injective τ,

which implies an injective ζ, which by Proposition 3.1(b) implies that $[\varGamma, \varGamma', \tau]$ is monic. Finally by inspection, the theorem's (ii)–(iv) are the same as the lemma's (ii)–(iv). □

Appendix C. For Isomorphisms

Lemma C.1. *Suppose that (X, E) and (X', E') are nontrivial out-trees, that $\tau{:}X{\to}X'$ is a bijection, and that $\{\, \tau(x)\tau(y) \,|\, xy{\in}E \,\} \subseteq E'$. Then the following hold.*

(a) $E \ni xy \mapsto \tau(x)\tau(y) \in E'$ *is a bijection.*[26]

(b) $(\forall x{\in}X, y{\in}X)\ x \preccurlyeq y \Leftrightarrow \tau(x) \preccurlyeq' \tau(y)$.

(c) $(\forall x{\in}X, y{\in}X)\ x \prec y \Leftrightarrow \tau(x) \prec' \tau(y)$.

(d) $\tau(r) = r'$.

(e) $(\forall y{\in}X)\ \bar{\tau}(P(y)) = P'(\tau(y))$.

(f) $\tau|_{W,W'}$ *is a bijection.*

(g) $\tau|_{X\smallsetminus W, X'\smallsetminus W'}$ *is a bijection.*

(h) $\bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$ *is a bijection.*

Proof. For part (a), it suffices to show that $\{\, \tau(x)\tau(y) \,|\, xy{\in}E \,\} \supseteq E'$.

To begin, define $\mathcal{E} = \{\, \{x, y\} \,|\, xy{\in}E \,\}$, $\mathcal{E}' = \{\, \{x', y'\} \,|\, x'y'{\in}E' \,\}$, and

$$\mathcal{E}^\tau = \{\, \{\tau(x), \tau(y)\} \,|\, xy{\in}E \,\}.$$

The assumption $\{\, \tau(x)\tau(y) \,|\, xy{\in}E \,\} \subseteq E'$ implies [1] $\mathcal{E}^\tau \subseteq \mathcal{E}'$. Also, because (X', E') is an out-tree, [2] $(X', \mathcal{E}')$ is an unoriented tree. Also, because (X, E) is an out-tree, $(X, \mathcal{E})$ is an unoriented tree, which by the bijectivity of τ implies [3] $(X', \mathcal{E}^\tau)$ is an unoriented tree. Together, [1]–[3] state that $(X', \mathcal{E}^\tau)$ is an unoriented tree within the unoriented tree $(X', \mathcal{E}')$. This implies [4] $\mathcal{E}^\tau = \mathcal{E}'$ because one cannot add an edge to an unoriented tree without also adding a node (Diestel 2012, Theorem 1.5.1).

Now take [5] $x'y' \in E'$. Then $\{x', y'\} \in \mathcal{E}'$, which by [4] implies $\{x', y'\} \in \mathcal{E}^\tau$, which by definition implies there is [6] $xy \in E$ such that [7] $\{\tau(x), \tau(y)\} = \{x', y'\}$. Further, [5] implies $x' \neq y'$, which by [7] implies [8] $\tau(x)\tau(y) = x'y'$ or [9] $\tau(x)\tau(y) = y'x'$. [6] and [8] would complete the proof. Thus it suffices to show that [9] is impossible.

Toward that end, suppose [9]. Then [6] and the assumption $\{\, \tau(x)\tau(y) \,|\, xy{\in}E \,\} \subseteq E'$ imply $\tau(x)\tau(y) \in E'$, which by [9] implies $y'x' \in E'$, which by [5] contradicts (X', E') being an out-tree.

Parts (b)–(h) follow by inspection from part (a). □

Lemma C.2. *Suppose $[\varTheta, \varTheta', \tau]$ is a morphism and τ is bijective. Then the following hold.*

(a) $(\forall x{\in}W)\ \alpha_x$ *is bijective.*

(b) $[\varTheta, \varTheta', \tau]$ *is end-preserving and* $\zeta = \bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$.

Proof. (a). Take $x \in W$. Since Lemma B.6 shows α_x is injective, it suffices to show that α_x is surjective. Thus, since $\alpha_x{:}F(x){\to}F'(\tau(x))$ by definition, it suffices to show $\overline{\alpha_x}(F(x)) \supseteq F'(\tau(x))$. Toward that end, take $a' \in F'(\tau(x))$. By the definition of F', there is $y' \in X'$ such that $a' = \lambda'(\, \tau(x)\, y'\,)$. Thus by the bijectivity of τ, there is $y \in X$ such that $a' = \lambda'(\, \tau(x)\, \tau(y)\,)$. Thus Proposition 2.1(f) implies

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[26] This is equivalent to (X, E) and (X', E') being graph-isomorphic.

[1] $a' = \alpha_x(\lambda(xy))$. Meanwhile, the definition of F implies $\lambda(xy) \in F(x)$. So [1] implies $a' \in \overline{\alpha_x}(F(x))$.

(b). Lemma C.1(g) implies $[\Theta,\Theta',\tau]$ is end-preserving. Recall $\zeta{:}\mathcal{Z}{\to}\mathcal{Z}'$ is defined by $\zeta(Z) = P'{\circ}\tau(r){\cup}\bar{\tau}(Z)$. Lemma C.1(d) implies $P'{\circ}\tau(r) = \varnothing$. Thus $\zeta = \bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$. □

Lemma C.3. *Suppose $[\Gamma,\Gamma',\tau]$ is a morphism and $\tau|_{W,W'}$ is bijective. Then ι is surjective.*

Proof. Since $\iota{:}I{\to}I'$ by the definition of a morphism, it suffices to show that $\bar{\iota}(I) \supseteq I'$. Toward that end, take $i' \in I'$. By the definition of I' and [G2] for Γ', there is $x' \in W'$ such that [1] $\mu'(x') = i'$. By the bijectivity of $\tau|_{W,W'}$ there is $x \in W$ such that [2] $\tau(x) = x'$. Since $\mu(x) \in I$ by the definition of I, it suffices to show that $\iota(\mu(x)) = i'$. In steps, $\iota(\mu(x))$ by [gM] for $[\Gamma,\Gamma',\tau]$ is equal to $\mu'(\tau(x))$, which by [2] is equal to $\mu'(x')$, which by [1] is equal to i'. □

Proof C.4 (for Proposition 4.1)**.** (a) follows from Claims 1 and 3. (b) follows from Claims 2 and 4.

Claim 1: Suppose $[\Theta,\Theta',\tau]$ is an isomorphism. Then the following hold.
(a) τ is a bijection.
(b) The inverse is $[\Theta',\Theta,\tau^{-1}]$.
(c) $\tau|_{W,W'}$ is a homeomorphism.
(d) $(\forall x{\in}W)$ α_x is a bijection.
(e) The inverse's action transformation is $\langle\alpha^{-1}_{\tau^{-1}(x')}\rangle_{x'\in W'}$.

(a) and (b) hold by inspection. For (c), note $\tau|_{W,W'}$ is a bijection by (a) and Lemma C.1(f). Then, $\tau|_{W,W'}$ is continuous by [cI] for $[\Theta,\Theta',\tau]$, and $(\tau|_{W,W'})^{-1}$ is continuous by (b) and [cI] for $[\Theta',\Theta,\tau^{-1}]$.

For (d) and (e), let $\langle\alpha^*_{x'}\rangle_{x'\in W'}$ be the action transformation of $[\Theta',\Theta,\tau^{-1}]$. Since $\tau|_{W,W'}$ is bijective by (c), it suffices for both (d) and (e) to show $\langle\alpha^*_{x'}\rangle_{x'\in W'} = \langle\alpha^{-1}_{\tau^{-1}(x')}\rangle_{x'\in W'}$. First note (b) implies $[\Theta',\Theta,\tau^{-1}]{\circ}[\Theta,\Theta',\tau] = \mathsf{id}_\Theta$, which by Proposition 2.2(b) and Lemma A.5 implies [1] $\langle\alpha^*_{\tau(x)}{\circ}\alpha_x\rangle_{x\in W} = \langle\mathsf{id}_{F(x)}\rangle_{x\in W}$. Second note (b) implies $[\Theta,\Theta',\tau]{\circ}[\Theta',\Theta,\tau^{-1}] = \mathsf{id}_{\Theta'}$, which by Proposition 2.2(b) and Lemma A.5 implies [2] $\langle\alpha_{\tau^{-1}(x')}{\circ}\alpha^*_{x'}\rangle_{x'\in W'} = \langle\mathsf{id}_{F'(x')}\rangle_{x'\in W'}$. It suffices to show $(\forall x'{\in}W')$ $\alpha^*_{x'} = \alpha^{-1}_{\tau^{-1}(x')}$. Toward that end, take $x' \in W'$. By definition, $\alpha^*_{x'}{:}F'(x'){\to}F(\tau^{-1}(x'))$, and by the definition of $\langle\alpha_x{:}F(x){\to}F'(\tau(x))\rangle_{x\in W}$ at $x = \tau^{-1}(x')$, $\alpha_{\tau^{-1}(x')}{:}F(\tau^{-1}(x')){\to}F'(x')$. Further, [2] implies $\alpha_{\tau^{-1}(x')}{\circ}\alpha^*_{x'} = \mathsf{id}_{F'(x')}$, and [1] at $x = \tau^{-1}(x')$ implies $\alpha^*_{x'}{\circ}\alpha_{\tau^{-1}(x')} = \mathsf{id}_{F(\tau^{-1}(x'))}$. Hence $\alpha^*_{x'} = \alpha^{-1}_{\tau^{-1}(x')}$.

Claim 2: Suppose $[\Gamma,\Gamma',\tau]$ is an isomorphism. Then the following hold.
(a) τ is a bijection.
(b) The inverse is $[\Gamma',\Gamma,\tau^{-1}]$.
(c) $\tau|_{W,W'}$ is a homeomorphism.
(d) ζ is the bijection $\bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$.
(e) The inverse's run transformation is ζ^{-1}.
(f) ι is a bijection.
(g) The inverse's player transformation is ι^{-1}.
(h) $(\forall i{\in}I, Z_1{\in}\mathcal{Z}, Z_2{\in}\mathcal{Z})$ $U_i(Z_1) \geq U_i(Z_2) \Leftrightarrow U'_{\iota(i)}{\circ}\zeta(Z_1) \geq U'_{\iota(i)}{\circ}\zeta(Z_2)$.

(a) and (b) hold by inspection. (c) holds by Claim 1(c) (and Proposition 2.5's forgetful functor).

For (d) and (e), note (a) and Lemma C.2(b) imply that [1] $\zeta = \bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$. Symmetrically, since (b) implies $[\Theta',\Theta,\tau^{-1}]$ is a morphism, Lemma C.2(b) implies that [2] $[\Theta',\Theta,\tau^{-1}]$'s run transformation is $\overline{\tau^{-1}}|_{\mathcal{Z}',\mathcal{Z}}$. In addition, (a) implies that [3] $\bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$ and $\overline{\tau^{-1}}|_{\mathcal{Z}',\mathcal{Z}}$ are inverses. [1] and [3] imply (d). Further, [1] and [3] imply ζ and $\overline{\tau^{-1}}|_{\mathcal{Z}',\mathcal{Z}}$ are inverses, which by [2] implies ζ^{-1} is $[\Theta',\Theta,\tau^{-1}]$'s run transformation, which by (b) implies (e).

For (f) and (g), let ι^* be the player transformation of $[\Gamma',\Gamma,\tau^{-1}]$. It suffices for (f) and (g) to show $\iota^* = \iota^{-1}$. (b) implies $[\Theta',\Theta,\tau^{-1}]\circ[\Theta,\Theta',\tau] = \mathsf{id}_\Theta$, which by Proposition 2.4(b) and Lemma A.9 implies [1] $\iota^*\circ\iota = \mathsf{id}_I$. Similarly, (b) implies $[\Theta,\Theta',\tau]\circ[\Theta',\Theta,\tau^{-1}] = \mathsf{id}_{\Theta'}$, which by Proposition 2.4(b) and Lemma A.9 implies [2] $\iota\circ\iota^* = \mathsf{id}_{I'}$. Since $\iota{:}I\to I'$ and $\iota^*{:}I'\to I$, [1] and [2] imply $\iota^* = \iota^{-1}$.

For (h), take $i \in I$, $Z_1 \in \mathcal{Z}$, and $Z_2 \in \mathcal{Z}$. First assume $U_i(Z_1) \geq U_i(Z_2)$. Then [gU] for $[\Gamma,\Gamma',\tau]$ implies $U'_{\iota(i)}\circ\zeta(Z_1) \geq U'_{\iota(i)}\circ\zeta(Z_2)$. Conversely, suppose $U'_{\iota(i)}\circ\zeta(Z_1) \geq U'_{\iota(i)}\circ\zeta(Z_2)$. Then (g), (e), and [gU] for $[\Gamma',\Gamma,\tau^{-1}]$ at $i' = \iota(i)$, $Z'_1 = \zeta(Z_1)$, and $Z'_2 = \zeta(Z_2)$ imply

$$U_{\iota^{-1}\circ\iota(i)}\circ\zeta^{-1}\circ\zeta(Z_1) \geq U_{\iota^{-1}\circ\iota(i)}\circ\zeta^{-1}\circ\zeta(Z_2).$$

This reduces to $U_i(Z_1) \geq U_i(Z_2)$.

Claim 3: Suppose $[\Theta,\Theta',\tau]$ is a morphism, τ is a bijection, and $\tau|_{W,W'}$ is a homeomorphism. Then the inverse of $[\Theta,\Theta',\tau]$ is $[\Theta',\Theta,\tau^{-1}]$.

By inspection, it suffices to show that $[\Theta',\Theta,\tau^{-1}]$ is a morphism. Thus it suffices to show that $[\Theta',\Theta,\tau^{-1}]$ satisfies [cE], [cI], and [cL]. [cE] for $[\Theta',\Theta,\tau^{-1}]$ is $(\forall x'y'{\in}E')\ \tau^{-1}(x')\tau^{-1}(y') \in E$, which follows from Lemma C.1(a). [cI] for $[\Theta',\Theta,\tau^{-1}]$ is the continuity of $\tau^{-1}|_{W',W}$, which follows from $\tau|_{W,W'}$ being a homeomorphism.

To show $\langle\alpha^{-1}_{\tau^{-1}(x')}\rangle_{x'\in W'}$ is the action transformation of $[\Theta',\Theta,\tau^{-1}]$, it suffices by definition to show

$$(\forall x'{\in}W', a'{\in}F'(x'))\ \alpha^{-1}_{\tau^{-1}(x')}(a') = \lambda(\ \tau^{-1}(x')\ \tau^{-1}(n'(x',a'))\ ).$$

Thus, since $\tau|_{W,W'}$ is a bijection, it suffices to show

$$(\forall x{\in}W, a'{\in}F'(\tau(x)))\ \alpha_x^{-1}(a') = \lambda(\ x\ \tau^{-1}(n'(\tau(x),a'))\ ).$$

Thus, since each $\alpha_x{:}F(x)\to F'(\tau(x))$ is a bijection by Lemma C.2(a), it suffices to show

$$(\forall x{\in}W, a{\in}F(x))\ a = \lambda(\ x\ \tau^{-1}(n'(\tau(x),\alpha_x(a)))\ ).$$

Take $x \in W$ and $a \in F(x)$. In steps, a by Lemma A.1(c) is equal to $\lambda(\,x\,n(x,a)\,)$, which is equal to $\lambda(\,x\,\tau^{-1}(\tau(n(x,a)))\,)$, which by Proposition 2.1(g) is equal to $\lambda(\,x\,\tau^{-1}(n'(\tau(x),\alpha_x(a)))\,)$.

By the previous paragraph, [cL] is equivalent to the continuity of $\langle\alpha^{-1}_{\tau^{-1}(x')}\rangle_{x'\in W'}$ from W'. Since $\tau|_{W,W'}$ is a homeomorphism, this is equivalent to the continuity of $\langle\alpha_x^{-1}\rangle_{x\in W}$ from W, which is equivalent to the continuity of $\langle\alpha_x\rangle_{x\in W}$ from W, which is implied by [cL] for $[\Theta,\Theta',\tau]$.

Claim 4: Suppose $[\Gamma,\Gamma',\tau]$ is a morphism, τ is bijective, $\tau|_{W,W'}$ is a homeomorphism, ι is injective, and $(\forall i{\in}I, Z_1{\in}\mathcal{Z}, Z_2{\in}\mathcal{Z})\ U_i(Z_1) \geq U_i(Z_2) \Leftrightarrow U'_{\iota(i)}\circ\zeta(Z_1) \geq U'_{\iota(i)}\circ\zeta(Z_2)$. Then $[\Gamma,\Gamma',\tau]$ is an isomorphism.

It suffices to show that $[\Gamma', \Gamma, \tau^{-1}]$ is a morphism. Toward that end, note that the injectivity of ι and Lemma C.3 imply that ι is bijective. Hence it suffices to show that $[\Gamma', \Gamma, \tau^{-1}]$ satisfies [gZ], [gM] with ι^{-1}, and [gU] with ι^{-1}.

First, [gZ] for $[\Gamma', \Gamma, \tau^{-1}]$ is that $[\Theta', \Theta, \tau^{-1}]$ is an end-preserving morphism. $[\Theta', \Theta, \tau^{-1}]$ is a morphism by Claim 3, and it is end-preserving by Lemma C.1(g).

Second, [gM] for $[\Gamma', \Gamma, \tau^{-1}]$ with ι^{-1} is $\iota^{-1}\circ\mu' = \mu\circ\tau^{-1}|_{W',W}$. In steps, [gM] for $[\Gamma, \Gamma', \tau]$ is $\iota\circ\mu = \mu'\circ\tau|_{W,W'}$, which by the bijectivity of ι implies $\mu = \iota^{-1}\circ\mu'\circ\tau|_{W,W'}$, which by the bijectivity of $\tau|_{W,W'}$ implies $\mu\circ\tau^{-1}|_{W',W} = \iota^{-1}\circ\mu'$.

Finally, consider [gU] for $[\Gamma', \Gamma, \tau^{-1}]$ with ι^{-1}. Lemma C.2(b) implies [a] $\zeta = \bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$. Thus ζ is bijective. Since ι is also bijective, the claim's last assumption can be rewritten as

$$(\forall i'{\in}I', Z_1'{\in}\mathcal{Z}', Z_2'{\in}\mathcal{Z}')$$
$$U_{\iota^{-1}(i')}\circ\zeta^{-1}(Z_1') \geq U_{\iota^{-1}(i')}\circ\zeta^{-1}(Z_2') \Leftrightarrow U_{i'}'(Z_1') \geq U_{i'}'(Z_2').$$

Thus it suffices to show that the reverse direction of this biconditional is [gU] for $[\Gamma', \Gamma, \tau^{-1}]$ with ι^{-1}. For this, it suffices to show that ζ^{-1} is the run transformation of $[\Gamma', \Gamma, \tau^{-1}]$. Lemma C.2(b) implies that the run transformation of $[\Theta', \Theta, \tau^{-1}]$ is equal to $\overline{\tau^{-1}}|_{\mathcal{Z}',\mathcal{Z}}$, which by [a] is equal to ζ^{-1}. □

Proof C.5 (for Proposition 4.2)**.** By the forward direction of Proposition 4.1(a), τ is bijective. Thus Lemma C.1(a–e,g) imply (a–e,f) here. (Lemma C.1(f,h) has been incorporated into the statement of Proposition 4.1.) □

Proof C.6 (for Theorem 4.3)**.** *(a).* To show that $S \ni s \mapsto \langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x'\in W'} \in S'$ is well-defined, take $s \in S$. By the definition of S', it suffices to show that

(1) $$\langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x'\in W'} \text{ is continuous from } W'\text{, and}$$

(2) $$(\forall x'{\in}W')\ \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x'))) \in F'(x').$$

For (2), take $x' \in W'$. The forward direction of Proposition 4.1(b) implies $\tau|_{W,W'}$ is bijective, which implies $\tau^{-1}(x') \in W$, which by the definition of S implies $s(\tau^{-1}(x')) \in F(\tau^{-1}(x'))$. Thus it suffices to show $\alpha_{\tau^{-1}(x')}{:}F(\tau^{-1}(x')){\to}F'(x')$. This holds since the definition of $\alpha_{\tau^{-1}(x')}$ implies $\alpha_{\tau^{-1}(x')}{:}F(\tau^{-1}(x')){\to}F'(\tau(\tau^{-1}(x')))$, and since $F'(\tau(\tau(^{-1}(x')))) = F'(x')$. For (1), it suffices to show that $(\forall H'{\in}\mathcal{H}')\ \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))$ is constant over $x' \in H'$. Take $H' \in \mathcal{H}'$. The forward direction of Proposition 4.1(a) implies $\tau|_{W,W'}$ is a homeomorphism, which implies [a] $\tau^{-1}(H') \in \mathcal{H}$. Since s is continuous by the definition of S, [a] implies that [b] the source action $s(\tau^{-1}(x'))$ is constant over $x' \in H'$. Further, [a] and [cL] for $[\Theta, \Theta', \tau]$ imply that [c] the function $\alpha_{\tau^{-1}(x')}$ is constant over $x' \in H'$. Finally, [b] and [c] imply that the target action $\alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))$ is constant over $x' \in H'$.

Conversely, to show that $S \ni \langle\, \alpha_x^{-1}(s'(\tau(x)))\,\rangle_{x\in W} \leftarrow\!\shortmid s' \in S'$ is well-defined, first note that each α_x^{-1} exists because each α_x is bijective by Proposition 4.1(a). Next take $s' \in S'$. By the definition of S, it suffices to show that

(1) $$\langle\, \alpha_x^{-1}(s'(\tau(x)))\,\rangle_{x\in W} \text{ is continuous from } W\text{, and}$$

(2) $$(\forall x{\in}W)\ \alpha_x^{-1}(s'(\tau(x))) \in F(x).$$

For (2), take $x \in W$. Since $\tau(x) \in W'$ by Proposition 2.1(e), the definition of S' implies $s'(\tau(x)) \in F'(\tau(x))$. Since $\alpha_x^{-1}{:}F'(\tau(x)){\to}F(x)$ by the definition of α_x, this implies $\alpha_x^{-1}(s'(\tau(x))) \in F(x)$. For (1), it suffices to show that $(\forall H{\in}\mathcal{H})$ $\alpha_x^{-1}(s'(\tau(x)))$ is constant over $x \in H$. Take $H \in \mathcal{H}$. The forward direction of Proposition 4.1(a) implies $\tau|_{W,W'}$ is a homeomorphism, which implies [a] $\tau(H) \in \mathcal{H}'$. Since s' is continuous by the definition of S', [a] implies that [b] the target action $s'{\circ}\tau(x)$ is constant over $x \in H$. Further, [cL] for $[\Theta, \Theta', \tau]$ implies that [c] the function α_x^{-1} is constant over $x \in H$. Finally, [b] and [c] imply that the source action $\alpha_x^{-1}(s'(\tau(x)))$ is constant over $x \in H$.

To show that the first function followed by the second function is id_S, take $s \in S$. Then

$$\begin{aligned}&\langle\, \alpha_x^{-1}(\ \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))|_{x'=\tau(x)}\ )\,\rangle_{x\in W}\\ &= \langle\, \alpha_x^{-1}(\alpha_x(s(x)))\,\rangle_{x\in W} = \langle\, s(x)\,\rangle_{x\in W} = s.\end{aligned}$$

Further, to show that the second function followed by the first function is $\mathsf{id}_{S'}$, take $s' \in S'$. Then

$$\begin{aligned}&\langle\, \alpha_{\tau^{-1}(x')}(\ \alpha_x^{-1}(s'(\tau(x)))|_{x=\tau^{-1}(x')}\ )\,\rangle_{x'\in W'}\\ &= \langle\, \alpha_{\tau^{-1}(x')}(\ \alpha_{\tau^{-1}(x')}^{-1}(s'(x')))\,\rangle_{x'\in W'} = \langle\, s'(x')\,\rangle_{x'\in W'} = s'.\end{aligned}$$

(b). Take $s \in S$. By the definition of o, [1] $o(s) \in \mathcal{Z}$ and [2] $(\forall x{\in}W{\cap}o(s))$ $n(x, s(x)) \in o(s)$. Recall Proposition 4.1(b) shows $\zeta = \bar\tau|_{\mathcal{Z},\mathcal{Z}'}$. Thus [1] implies [3] $\zeta(o(s)) = \bar\tau(o(s))$ and [4] $\zeta(o(s)) \in \mathcal{Z}'$. Meanwhile, [2] implies

$$\begin{gathered}(\forall x{\in}W{\cap}o(s))\ \tau(n(x,s(x))) \in \bar\tau(o(s)),\\ (\forall x{\in}W{\cap}o(s))\ n'(\,\tau(x), \alpha_x(s(x))\,) \in \bar\tau(o(s)),\\ (\forall x{\in}W{\cap}o(s))\ n'(\,\tau(x), \alpha_{\tau^{-1}(\tau(x))}(s(\tau^{-1}(\tau(x))))\,) \in \bar\tau(o(s)),\\ (\forall x'{\in}\bar\tau(W{\cap}o(s)))\ n'(\,x', \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,) \in \bar\tau(o(s)),\\ (\forall x'{\in}W'{\cap}\bar\tau(o(s)))\ n'(\,x', \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,) \in \bar\tau(o(s)),\ \text{and}\\ (\forall x'{\in}W'{\cap}\zeta(o(s)))\ n'(\,x', \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,) \in \zeta(o(s)),\end{gathered}$$

where the second statement follows from the first by Proposition 2.1(g), where the fifth follows from the fourth because (i) τ is bijective and (ii) $\bar\tau(W) = W'$ by Proposition 4.1(a), and where the sixth follows from the fifth by [3]. Finally, the sixth statement, [4], and the definition of o' imply $\zeta(o(s)) = o'(\,\langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x'\in W'}\,)$. □

Lemma C.7. *Suppose Γ is a game and $i \in I$. Define $\mathcal{H}_i = \{\, H{\in}\mathcal{H} \,|\, H{\subseteq}W_i \,\}$ Then the following hold.*

(a) $\mathcal{H}_i$ generates the subspace topology for W_i that is induced from the topology for W generated by $\mathcal{H}$.

(b) $\mathcal{H}_i$ is a partition of W_i.

Proof. (a). Since $\mathcal{H}$ generates the topology for W, $\{\, H{\cap}W_i{\neq}\varnothing \,|\, H{\in}\mathcal{H} \,\}$ generates the subspace topology for W_i. Since $W_i = \{\, x{\in}W \,|\, \mu(x){=}i \,\}$ by definition, and since μ is continuous by [G2], W_i is the union of a subcollection of $\mathcal{H}$. Thus since $\mathcal{H}$ is a partition, each $H \in \mathcal{H}$ is either a subset of W_i or disjoint from W_i. Hence $\{\, H{\cap}W_i{\neq}\varnothing \,|\, H{\in}\mathcal{H} \,\} = \{\, H{\subseteq}W_i \,|\, H{\in}\mathcal{H} \,\}$, which equals $\mathcal{H}_i$ by the lemma's definition.

(b). Part (a) implies $\cup\mathcal{H}_i = W_i$. Thus it remains to show that $\mathcal{H}_i$ is a pairwise disjoint collection of nonempty sets. These facts hold because $\mathcal{H}_i \subseteq \mathcal{H}$ and because $\mathcal{H}$ is a partition. □

Proof C.8 (for Theorem 4.4). *(a).* Take $i \in I$.

To show that $\tau|_{W_i,W'_{\iota(i)}}$ is bijective, note that Proposition 4.1(b) implies τ is bijective. Thus it suffices to show that $\bar{\tau}(W_i) = W'_{\iota(i)}$. For the forward direction, suppose [a] $x \in W_i$. Note [gM] for $[\Gamma,\Gamma',\tau]$ implies [b] $\iota(\mu(x)) = \mu'(\tau(x))$. Then in steps, [a] and the definition of W_i imply $\mu(x) = i$, which by [b] implies $\iota(i) = \mu'(\tau(x))$, which by the definition of $W'_{\iota(i)}$ implies $\tau(x) \in W'_{\iota(i)}$. Conversely, suppose [c] $x' \in W'_{\iota(i)}$. Note Proposition 4.1(b) implies that the player transformation of $[\Gamma',\Gamma,\tau^{-1}]$ is ι^{-1}. Thus [gM] for $[\Gamma',\Gamma,\tau^{-1}]$ implies [d] $\iota^{-1}(\mu'(x')) = \mu(\tau^{-1}(x'))$. Then in steps, [c] and the definition of $W'_{\iota(i)}$ imply $\mu'(x') = \iota(i)$, which by [d] implies $i = \mu(\tau^{-1}(x'))$, which by the definition of W_i implies $\tau^{-1}(x') \in W_i$, which implies $x' \in \bar{\tau}(W_i)$.

To show that $\tau|_{W_i,W'_{\iota(i)}}$ is a homeomorphism, note that Proposition 4.1(b) implies $\tau|_{W,W'}$ is a homeomorphism. The definitions of W_i and $W'_{\iota(i)}$ imply $\tau|_{W_i,W'_{\iota(i)}}$ is a restriction of $\tau|_{W,W'}$, and the previous paragraph shows $\tau|_{W_i,W'_{\iota(i)}}$ is a bijection. Hence $\tau|_{W_i,W'_{\iota(i)}}$ is a homeomorphism (as stated in the text, W_i and $W'_{\iota(i)}$ are endowed with the subspace topologies).

(b). Take $i \in I$. To show that $S_i \ni s_i \mapsto \langle\, \alpha_{\tau^{-1}(x')}(s_i(\tau^{-1}(x')))\,\rangle_{x' \in W'_{\iota(i)}} \in S'_{\iota(i)}$ is well-defined, take $s_i \in S_i$. By the definition of $S'_{\iota(i)}$, it suffices to show that

(1) $$\langle\, \alpha_{\tau^{-1}(x')}(s_i(\tau^{-1}(x')))\,\rangle_{x' \in W'_{\iota(i)}} \text{ is continuous from } W'_{\iota(i)}, \text{ and}$$

(2) $$(\forall x' {\in} W'_{\iota(i)})\ \alpha_{\tau^{-1}(x')}(s_i(\tau^{-1}(x'))) \in F'(x').$$

For (2), take $x' \in W'_{\iota(i)}$. Part (a) implies $\tau^{-1}(x') \in W_i$, which by the definition of S_i implies $s_i(\tau^{-1}(x')) \in F(\tau^{-1}(x'))$. Thus it suffices to show $\alpha_{\tau^{-1}(x')}{:}F(\tau^{-1}(x')) {\rightarrow} F'(x')$. This holds since the definition of $\alpha_{\tau^{-1}(x')}$ implies $\alpha_{\tau^{-1}(x')}{:}F(\tau^{-1}(x')) {\rightarrow} F'(\tau(\tau^{-1}(x')))$, and since $F'(\tau(\tau^{-1}(x'))) = F'(x')$. For (1), note Lemma C.7(a) implies that the topology for $W'_{\iota(i)}$ is generated by $\{\, H' {\in} \mathcal{H}' \mid H' {\subseteq} W'_{\iota(i)} \,\}$. Thus it suffices to show, for each $H' \in \mathcal{H}'$ such that $H' \subseteq W'_{\iota(i)}$, that the action $\alpha_{\tau^{-1}(x')}(s_i(\tau^{-1}(x')))$ is constant over $x' \in H'$. Take such an H'. Part (a) and Lemma C.7(a) imply [a] $\tau^{-1}(H') \in \mathcal{H}$ and [b] $\tau^{-1}(H') \subseteq W_i$. Since s_i is continuous by the definition of S_i, [a] and [b] imply that [c] the source action $s_i(\tau^{-1}(x'))$ is constant over $x' \in H'$. Further, [a] and [cL] for $[\Theta,\Theta',\tau]$ imply that [d] the function $\alpha_{\tau^{-1}(x')}$ is constant over $x' \in H'$. Finally, [c] and [d] imply that the target action $\alpha_{\tau^{-1}(x')}(s_i(\tau^{-1}(x')))$ is constant over $x' \in H'$.

Conversely, to show that $S_i \ni \langle\, \alpha_x^{-1}(s'_{\iota(i)}(\tau(x)))\,\rangle_{x \in W_i} \leftarrow\!\shortmid s'_{\iota(i)} \in S'_{\iota(i)}$ is well-defined, first note that each α_x^{-1} is well-defined by Proposition 4.1(a). Next take $s'_{\iota(i)} \in S'_{\iota(i)}$. By the definition of S_i, it suffices to show that

(1) $$\langle\, \alpha_x^{-1}(s'_{\iota(i)}(\tau(x)))\,\rangle_{x \in W_i} \text{ is continuous from } W_i, \text{ and}$$

(2) $$(\forall x {\in} W_i)\ \alpha_x^{-1}(s'_{\iota(i)}(\tau(x))) \in F(x).$$

For (2), take $x \in W_i$. Part (a) implies $\tau(x) \in W'_{\iota(i)}$, which by the definition of $S'_{\iota(i)}$ implies $s'_{\iota(i)}(\tau(x)) \in F'(\tau(x))$. Since $\alpha_x^{-1}{:}F'(\tau(x)) {\rightarrow} F(x)$ by the definition of α_x, this implies $\alpha_x^{-1}(s'_{\iota(i)}(\tau(x))) \in F(x)$. For (1), note that Lemma C.7(a) implies that the topology for W_i is generated by $\{\, H {\in} \mathcal{H} \mid H {\subseteq} W_i \,\}$. Thus it suffices to show, for each $H \in \mathcal{H}$ such that $H \subseteq W_i$, that $\alpha_x^{-1}(s'_{\iota(i)}(\tau(x)))$ is constant for $x \in H$. Take such an H. Part (a) and Lemma C.7(a) imply [a] $\tau(H) \in \mathcal{H}'$ and [b] $\tau(H) \subseteq W'_{\iota(i)}$. Since $s'_{\iota(i)}$ is continuous by the definition of $S'_{\iota(i)}$, [a] and [b] imply that [c] the target action $s'_{\iota(i)}(\tau(x))$ is constant over $x \in H$. Further, [cL] for $[\Theta, \Theta', \tau]$ implies that [d] the function α_x^{-1} is constant over $x \in H$. Finally, [c] and [d] imply that the source action $\alpha_x^{-1}(s'_{\iota(i)}(\tau(x)))$ is constant over $x \in H$.

To show that the first function followed by the second function is id_{S_i}, take $s_i \in S_i$. Then

$$\begin{aligned} &\langle\, \alpha_x^{-1}(\ \alpha_{\tau^{-1}(x')}(s_i(\tau^{-1}(x')))|_{x'=\tau(x)}\ )\,\rangle_{x \in W_i} \\ &= \langle\, \alpha_x^{-1}(\alpha_x(s_i(x)))\,\rangle_{x \in W_i} = \langle\, s_i(x)\,\rangle_{x \in W_i} = s_i. \end{aligned}$$

Further, to show that the second function followed by the first function is $\mathsf{id}_{S'_{\iota(i)}}$, take $s'_{\iota(i)} \in S'_{\iota(i)}$. Then

$$\begin{aligned} &\langle\, \alpha_{\tau^{-1}(x')}(\ \alpha_x^{-1}(s'_{\iota(i)}(\tau(x)))|_{x=\tau^{-1}(x')}\ )\,\rangle_{x' \in W'_{\iota(i)}} \\ &= \langle\, \alpha_{\tau^{-1}(x')}(\ \alpha_{\tau^{-1}(x')}^{-1}(s'_{\iota(i)}(x')))\,\rangle_{x' \in W'_{\iota(i)}} = \langle\, s'_{\iota(i)}(x')\,\rangle_{x' \in W'_{\iota(i)}} = s'_{\iota(i)}. \end{aligned}$$

□

Proof C.9 (for Theorem 4.5)**.** Because of Theorem 4.3(a), it suffices to show that $(\forall s{\in}S)$ s is a Nash equilibrium in Γ if and only if $\langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{s' \in W'}$ is a Nash equilibrium in Γ'. Toward that end, take $s \in S$. It suffices to argue that

$$s \text{ is a Nash equilibrium in } \Gamma$$

$$\Leftrightarrow\ (\forall i{\in}I, s_i^+{\in}S_i)\ U_i{\circ}o(s) \geq U_i{\circ}o(s_i^+, s_{-i}) \tag{1}$$

$$\Leftrightarrow\ (\forall i{\in}I, s_i^+{\in}S_i)\ U'_{\iota(i)}{\circ}\zeta{\circ}o(s) \geq U'_{\iota(i)}{\circ}\zeta{\circ}o(s_i^+, s_{-i}) \tag{2}$$

$$\begin{aligned} \Leftrightarrow\ &(\forall i{\in}I, s_i^+{\in}S_i)\ U'_{\iota(i)}{\circ}o'(\ \langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x' \in W'}\ ) \geq \\ &U'_{\iota(i)}{\circ}o'(\ \langle\, \alpha_{\tau^{-1}(x')}((s_i^+, s_{-i})(\tau^{-1}(x')))\,\rangle_{x' \in W'}\ ) \end{aligned} \tag{3}$$

$$\begin{aligned} \Leftrightarrow\ &(\forall i{\in}I, s_i^+{\in}S_i)\ U'_{\iota(i)}{\circ}o'(\ \langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x' \in W'}\ ) \geq \\ &U'_{\iota(i)}{\circ}o'(\ \langle\, \alpha_{\tau^{-1}(x')}(s_i^+(\tau^{-1}(x')))\,\rangle_{x' \in W'_{\iota(i)}},\ \langle\, \alpha_{\tau^{-1}(x')}(s_{-i}(\tau^{-1}(x')))\,\rangle_{x' \in W' \smallsetminus W'_{\iota(i)}}\ ) \end{aligned} \tag{4}$$

$$\begin{aligned} \Leftrightarrow\ &(\forall i{\in}I, s_{\iota(i)}^+{\in}S'_{\iota(i)})\ U'_{\iota(i)}{\circ}o'(\ \langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x' \in W'}\ ) \geq \\ &U'_{\iota(i)}{\circ}o'(\ s_{\iota(i)}^+,\ \langle\, \alpha_{\tau^{-1}(x')}(s_{-i}(\tau^{-1}(x')))\,\rangle_{x' \in W' \smallsetminus W'_{\iota(i)}}\ ) \end{aligned} \tag{5}$$

$$\begin{aligned} \Leftrightarrow\ &(\forall i'{\in}I', s_{i'}^+{\in}S'_{i'})\ U'_{i'}{\circ}o'(\ \langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x' \in W'}\ ) \geq \\ &U'_{i'}{\circ}o'(\ s_{i'}^+,\ \langle\, \alpha_{\tau^{-1}(x')}(s_{-i}(\tau^{-1}(x')))\,\rangle_{x' \in W' \smallsetminus W'_{i'}}\ ) \end{aligned} \tag{6}$$

$$\Leftrightarrow\ \langle\, \alpha_{\tau^{-1}(x')}(s(\tau^{-1}(x')))\,\rangle_{x' \in W'} \text{ is a Nash equilibrium in } \Gamma'. \tag{7}$$

Equivalence (1) holds by definition, (2) holds by the biconditional in Proposition 4.1(b), and (3) holds by two applications of Theorem 4.3(b). Equivalence (4) holds because $\tau|_{W_i, W'_{\iota(i)}}$ is bijective by Theorem 4.4(a), because the domain of s_i^+ is W_i, and because the domain of s_{-i} is $W \smallsetminus W_i$. Finally, equivalence (5) holds by Theorem 4.4(b), (6) holds because ι is bijective by Proposition 4.1(b), and (7) holds by definition. □

Proof C.10 (for Theorem 4.6)**.** *(a).* Proposition 4.1(a) implies τ is a bijection. Thus it suffices to show that $\bar{\tau}(R^{\mathsf{D}}) = R^{\mathsf{E}}$. This is equivalent to the combination of $\bar{\tau}(R^{\mathsf{D}}) \subseteq R^{\mathsf{E}}$ and $R^{\mathsf{D}} \supseteq \overline{\tau^{-1}}(R^{\mathsf{E}})$. Since the two are symmetric it suffices to show the first.

Toward that end, take $r^{\mathsf{d}} \in R^{\mathsf{D}}$. Then [a] $r^{\mathsf{d}} \in W^{\mathsf{D}}$ and [b] Θ^{d} is a CLT, where Θ^{d} is constructed from Θ^{D} by the Selten definitions at r^{d}. By Lemma B.10, [b] implies [c] $(\forall H^{\mathsf{D}} {\in} \mathcal{H}^{\mathsf{D}})\ H^{\mathsf{D}} \subseteq X^{\mathsf{d}}$ or $H^{\mathsf{D}} \subseteq X^{\mathsf{D}} \smallsetminus X^{\mathsf{d}}$. Since $\tau|_{W^{\mathsf{D}},W^{\mathsf{E}}}$ is a bijection by Proposition 4.1(a), [a] implies $\tau(r^{\mathsf{d}}) \in W^{\mathsf{E}}$. Thus we may construct Θ^{e} from Θ^{E} by applying the Selten definitions at $\tau(r^{\mathsf{d}})$. It suffices to show that Θ^{e} is a CLT. Thus by Lemma B.10, it suffices to show that $(\forall H^{\mathsf{E}} {\in} \mathcal{H}^{\mathsf{E}})\ H^{\mathsf{E}} \subseteq X^{\mathsf{e}}$ or $H^{\mathsf{E}} \subseteq X^{\mathsf{E}} \smallsetminus X^{\mathsf{e}}$.

Toward that end, take $H^{\mathsf{E}} \in \mathcal{H}^{\mathsf{E}}$. Since $\tau|_{W^{\mathsf{D}},W^{\mathsf{E}}}$ is a homeomorphism by Proposition 4.1(a), $\overline{\tau^{-1}}(H^{\mathsf{E}}) \in \mathcal{H}^{\mathsf{D}}$. Thus [c] implies $\overline{\tau^{-1}}(H^{\mathsf{E}}) \subseteq X^{\mathsf{d}}$ or $\overline{\tau^{-1}}(H^{\mathsf{E}}) \subseteq X^{\mathsf{D}} \smallsetminus X^{\mathsf{d}}$. Thus the bijectivity of τ implies $H^{\mathsf{E}} \subseteq \bar{\tau}(X^{\mathsf{d}})$ or $H^{\mathsf{E}} \subseteq X^{\mathsf{E}} \smallsetminus \bar{\tau}(X^{\mathsf{d}})$. Thus it suffices to show that $\bar{\tau}(X^{\mathsf{d}}) = X^{\mathsf{e}}$. This is equivalent to the combination of $\bar{\tau}(X^{\mathsf{d}}) \subseteq X^{\mathsf{e}}$ and $X^{\mathsf{d}} \supseteq \overline{\tau^{-1}}(X^{\mathsf{e}})$. For the first, take $x^{\mathsf{d}} \in X^{\mathsf{d}}$. Then the Selten definition of X^{d} implies $r^{\mathsf{d}} \succcurlyeq^{\mathsf{D}} x^{\mathsf{d}}$, which by Proposition 4.2(b) for $[\Theta^{\mathsf{D}}, \Theta^{\mathsf{E}}, \tau]$ implies $\tau(r^{\mathsf{d}}) \succcurlyeq^{\mathsf{E}} \tau(x^{\mathsf{d}})$, which by the Selten definition of X^{e} implies $\tau(x^{\mathsf{d}}) \in X^{\mathsf{e}}$. For the second, take $x^{\mathsf{e}} \in X^{\mathsf{e}}$. Then the Selten definition of X^{e} implies $\tau(r^{\mathsf{d}}) \succcurlyeq^{\mathsf{E}} x^{\mathsf{e}}$, which by Proposition 4.2(b) for $[\Theta^{\mathsf{E}}, \Theta^{\mathsf{D}}, \tau^{-1}]$ implies $\tau^{-1}(\tau(r^{\mathsf{d}})) \succcurlyeq^{\mathsf{D}} \tau^{-1}(x^{\mathsf{e}})$, which implies $r^{\mathsf{d}} \succcurlyeq^{\mathsf{D}} \tau^{-1}(x^{\mathsf{e}})$, which by the Selten definition of X^{d} implies $\tau^{-1}(x^{\mathsf{e}}) \in X^{\mathsf{d}}$.

(b). Part (b) follows from Claims 5, 8, 9(b), and 12.

Claim 1: $\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}$ is a bijection. Proposition 4.1(a) implies τ is a bijection. Thus it suffices to show $\bar{\tau}(X^{\mathsf{d}}) = X^{\mathsf{e}}$. This is equivalent to the combination of $\bar{\tau}(X^{\mathsf{d}}) \subseteq X^{\mathsf{e}}$ and $X^{\mathsf{d}} \supseteq \overline{\tau^{-1}}(X^{\mathsf{e}})$. For the first, take $x^{\mathsf{d}} \in X^{\mathsf{d}}$. Then the Selten definition of X^{d} implies $r^{\mathsf{d}} \succcurlyeq^{\mathsf{D}} x^{\mathsf{d}}$, which by Proposition 4.2(b) for $[\Theta^{\mathsf{D}}, \Theta^{\mathsf{E}}, \tau]$ implies $\tau(r^{\mathsf{d}}) \succcurlyeq^{\mathsf{E}} \tau(x^{\mathsf{d}})$, which by the Selten definition of X^{e} implies $\tau(x^{\mathsf{d}}) \in X^{\mathsf{e}}$. For the second, take $x^{\mathsf{e}} \in X^{\mathsf{e}}$. Then the Selten definition of X^{e} implies $\tau(r^{\mathsf{d}}) \succcurlyeq^{\mathsf{E}} x^{\mathsf{e}}$, which by Proposition 4.2(b) for $[\Theta^{\mathsf{E}}, \Theta^{\mathsf{D}}, \tau^{-1}]$ implies $\tau^{-1}(\tau(r^{\mathsf{d}})) \succcurlyeq^{\mathsf{D}} \tau^{-1}(x^{\mathsf{e}})$, which implies $r^{\mathsf{d}} \succcurlyeq^{\mathsf{D}} \tau^{-1}(x^{\mathsf{e}})$, which by the Selten definition of X^{d} implies $\tau^{-1}(x^{\mathsf{e}}) \in X^{\mathsf{d}}$.

Claim 2: $E^{\mathsf{d}} \ni x^{\mathsf{d}}y^{\mathsf{d}} \mapsto \tau(x^{\mathsf{d}})\tau(y^{\mathsf{d}}) \in E^{\mathsf{e}}$ is a bijection. Because of Claim 1, it suffices to show that $(\forall x^{\mathsf{d}}y^{\mathsf{d}} {\in} E^{\mathsf{d}})\ \tau(x^{\mathsf{d}})\tau(x^{\mathsf{d}}) \in E^{\mathsf{e}}$ and that $(\forall x^{\mathsf{e}}y^{\mathsf{e}} {\in} E^{\mathsf{e}})$ $\tau^{-1}(x^{\mathsf{e}})\tau^{-1}(y^{\mathsf{e}}) \in E^{\mathsf{d}}$. For the first, take [1] $x^{\mathsf{d}}y^{\mathsf{d}} \in E^{\mathsf{d}}$. Note [1] implies $\{x^{\mathsf{d}}, y^{\mathsf{d}}\} \in X^{\mathsf{d}}$, which by Claim 1 implies [2] $\{\tau(x^{\mathsf{d}}), \tau(y^{\mathsf{d}})\} \in X^{\mathsf{e}}$. Further, [1] and the Selten definition of E^{d} imply $x^{\mathsf{d}}y^{\mathsf{d}} \in E^{\mathsf{D}}$, which by [cE] for $[\Theta^{\mathsf{D}}, \Theta^{\mathsf{E}}, \tau]$ implies $\tau(x^{\mathsf{d}})\tau(y^{\mathsf{d}}) \in E^{\mathsf{E}}$, which by [2] and the Selten definition of E^{e} implies $\tau(x^{\mathsf{d}})\tau(y^{\mathsf{d}}) \in E^{\mathsf{e}}$. For the second, a similar argument can be made. In particular, switch d with e, D with E, and τ with τ^{-1}.

Claim 3: $\tau|_{W^{\mathsf{d}},W^{\mathsf{e}}}$ is a bijection. Because of Claim 1, it suffices to show that $\bar{\tau}(W^{\mathsf{d}}) = W^{\mathsf{e}}$. This is the combination of $\bar{\tau}(W^{\mathsf{d}}) \subseteq W^{\mathsf{e}}$ and $W^{\mathsf{d}} \supseteq \overline{\tau^{-1}}(W^{\mathsf{e}})$. For the first, take $x^{\mathsf{d}} \in W^{\mathsf{d}}$. Then the definition of W^{d} implies there is $y^{\mathsf{d}} \in X^{\mathsf{d}}$ such that $x^{\mathsf{d}}y^{\mathsf{d}} \in E^{\mathsf{d}}$, which by Claim 2 implies $\tau(x^{\mathsf{d}})\tau(y^{\mathsf{d}}) \in E^{\mathsf{e}}$, which by the definition of W^{e} implies $\tau(x^{\mathsf{d}}) \in W^{\mathsf{e}}$. For the second, a similar argument can be made. In particular, switch d with e, D with E, and τ with τ^{-1}.

Claim 4: $\tau|_{W^{\mathsf{d}},W^{\mathsf{e}}}$ is a homeomorphism. Because of Claim 3, it suffices to show that both $\tau|_{W^{\mathsf{d}},W^{\mathsf{e}}}$ and $\tau^{-1}|_{W^{\mathsf{e}},W^{\mathsf{d}}}$ are continuous. For the first, take [1] $H^{\mathsf{e}} \in \mathcal{H}^{\mathsf{e}}$.

Note [1] implies $H^{\mathsf{e}} \subseteq W^{\mathsf{e}}$, which by Claim 3 implies [2] $\overline{\tau^{-1}}(H^{\mathsf{e}}) \subseteq W^{\mathsf{d}}$. Meanwhile, [1] and the Selten definition of $\mathcal{H}^{\mathsf{e}}$ implies $H^{\mathsf{e}} \in \mathcal{H}^{\mathsf{E}}$. Thus, since $\tau|_{W^C,W^D}$ is a homeomorphism by Proposition 4.1(a), [3] $\overline{\tau^{-1}}(H^{\mathsf{e}}) \in \mathcal{H}^{\mathsf{D}}$. By the Selten definition of $\mathcal{H}^{\mathsf{e}}$, [2] and [3] imply $\overline{\tau^{-1}}(H^{\mathsf{e}}) \in \mathcal{H}^{\mathsf{d}}$. For the second, a similar argument can be made. In particular, switch d with e, D with E, and τ with τ^{-1}.

Claim 5: $[\varTheta^{\mathsf{d}},\varTheta^{\mathsf{e}},\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$*'s action transformation is* $\langle\alpha_{x^{\mathsf{d}}}\rangle_{x^{\mathsf{d}}\in W^{\mathsf{d}}}$. Let $\langle\alpha^*_{x^{\mathsf{d}}}\rangle_{x^{\mathsf{d}}\in W^{\mathsf{d}}}$ denote $[\varTheta^{\mathsf{d}},\varTheta^{\mathsf{e}},\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$'s action transformation. It suffices to show that $(\forall x^{\mathsf{d}}{\in}W^{\mathsf{d}})\ \alpha^*_{x^{\mathsf{d}}} = \alpha_{x^{\mathsf{d}}}$. Toward that end, take [a] $x^{\mathsf{d}} \in W^{\mathsf{d}}$. First consider the domains and codomains of the two functions. By definition, $\alpha^*_{x^{\mathsf{d}}}{:}F^{\mathsf{d}}(x^{\mathsf{d}}){\to}F^{\mathsf{e}}(\tau(x^{\mathsf{d}}))$ and $\alpha_{x^{\mathsf{d}}}{:}F^{\mathsf{D}}(x^{\mathsf{d}}){\to}F^{\mathsf{E}}(\tau(x^{\mathsf{d}}))$. Note that [a], with Lemma B.9(a) applied at $\varTheta = \varTheta^{\mathsf{d}}$ and $\varTheta' = \varTheta^{\mathsf{D}}$, implies $F^{\mathsf{d}}(x^{\mathsf{d}}) = F^{\mathsf{D}}(x^{\mathsf{d}})$. Similarly, [a] and Claim 4 imply $\tau(x^{\mathsf{d}}) \in W^{\mathsf{e}}$, which by Lemma B.9(a) applied at $\varTheta = \varTheta^{\mathsf{e}}$ and $\varTheta' = \varTheta^{\mathsf{E}}$ implies $F^{\mathsf{e}}(\tau(x^{\mathsf{d}})) = F^{\mathsf{E}}(\tau(x^{\mathsf{d}}))$. Thus it suffices to show that $(\forall a^{\mathsf{d}}{\in}F^{\mathsf{d}}(x^{\mathsf{d}}))\ \alpha^*_{x^{\mathsf{d}}}(a^{\mathsf{d}}) = \alpha_{x^{\mathsf{d}}}(a^{\mathsf{d}})$. Toward that end, take $a^{\mathsf{d}} \in F^{\mathsf{d}}(x^{\mathsf{d}})$. In steps, $\alpha^*_{x^{\mathsf{d}}}(a^{\mathsf{d}})$ by definition is equal to $\lambda^{\mathsf{e}}(\,\tau(x^{\mathsf{d}})\,\tau(n^{\mathsf{d}}(x^{\mathsf{d}},a^{\mathsf{d}}))\,)$, which by Claim 2 and the Selten definition of λ^{e} is equal to $\lambda^{\mathsf{E}}(\,\tau(x^{\mathsf{d}})\,\tau(n^{\mathsf{d}}(x^{\mathsf{d}},a^{\mathsf{d}}))\,)$, which by Lemma B.9(b) at $\varTheta = \varTheta^{\mathsf{d}}$ and $\varTheta' = \varTheta^{\mathsf{D}}$ is equal to $\lambda^{\mathsf{E}}(\,\tau(x^{\mathsf{d}})\,\tau(n^{\mathsf{D}}(x^{\mathsf{d}},a^{\mathsf{d}}))\,)$, which by definition is equal to $\alpha_{x^{\mathsf{d}}}(a^{\mathsf{d}})$.

Claim 6: $[\varTheta^{\mathsf{d}},\varTheta^{\mathsf{e}},\tau|_{X^{\mathsf{d}},x^{\mathsf{e}}}]$ *is a morphism.* It suffices to show that the tuple satisfies [cE], [cI], and [cL]. [cE] follows from Claim 2. [cI] follows from Claim 4. For [cL], note that [cL] for $[\varTheta^{\mathsf{D}},\varTheta^{\mathsf{E}},\tau]$ implies $\langle\alpha_x\rangle_{x\in W^{\mathsf{D}}}$ is continuous from W^{D}. Thus the restriction $\langle\alpha_x\rangle_{x\in W^{\mathsf{d}}}$ is continuous from W^{d} endowed with the subspace topology. Thus Lemma B.8(a) at $\varTheta = \varTheta^{\mathsf{d}}$ and $\varTheta = \varTheta^{\mathsf{D}}$ implies that $\langle\alpha_x\rangle_{x\in W^{\mathsf{d}}}$ is continuous from W^{d} with the topology generated by $\mathcal{H}^{\mathsf{d}}$. Claim 5 showed $\langle\alpha_x\rangle_{x\in W^{\mathsf{d}}}$ is the action transformation of $[\varTheta^{\mathsf{d}},\varTheta^{\mathsf{e}},\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$.

Claim 7: $[\varTheta^{\mathsf{d}},\varTheta^{\mathsf{e}},\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$ *is an end-preserving isomorphism.* The tuple is a morphism by Claim 6. It is end-preserving because Claims 1 and 3 imply the stronger result that $\bar{\tau}(X^{\mathsf{d}}{\smallsetminus}W^{\mathsf{d}}) = X^{\mathsf{e}}{\smallsetminus}W^{\mathsf{e}}$. Finally, it is an isomorphism by Claim 1, Claim 4, and Proposition 4.1(a).

Claim 8: $[\varGamma^{\mathsf{d}},\varGamma^{\mathsf{e}},\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$*'s run transformation is* $\zeta|_{\mathcal{Z}^{\mathsf{d}},\mathcal{Z}^{\mathsf{e}}}$. Let ζ^* denote $[\varGamma^{\mathsf{d}},\varGamma^{\mathsf{e}},\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$'s run transformation. Claim 7 and Proposition 2.3 imply ζ^* is well-defined. By definition, [a] $\zeta^*{:}\mathcal{Z}^{\mathsf{d}}{\to}\mathcal{Z}^{\mathsf{e}}$ and [b] $(\forall Z^{\mathsf{d}}{\in}\mathcal{Z}^{\mathsf{d}})\ \zeta^*(Z^{\mathsf{d}}) = P^{\mathsf{e}}{\circ}\tau(r^{\mathsf{d}}){\cup}\bar{\tau}(Z^{\mathsf{d}})$. Because of [a], it suffices to show that $(\forall Z^{\mathsf{d}}{\in}\mathcal{Z}^{\mathsf{d}})\ \zeta^*(Z^{\mathsf{d}}) = \zeta(Z^{\mathsf{d}})$. Toward that end, take $Z^{\mathsf{d}} \in \mathcal{Z}^{\mathsf{d}}$. Then $\zeta^*(Z^{\mathsf{d}})$ by [b] is equal to $P^{\mathsf{e}}{\circ}\tau(r^{\mathsf{d}}){\cup}\bar{\tau}(Z^{\mathsf{d}})$, which by the theorem's definition of $\varGamma^{\mathsf{e}}$ is equal to $P^{\mathsf{e}}(r^{\mathsf{e}}){\cup}\bar{\tau}(Z^{\mathsf{d}})$, which reduces to $\bar{\tau}(Z^{\mathsf{d}})$, which by Proposition 4.1(b) is equal to $\zeta(Z^{\mathsf{d}})$.

Claim 9: (a) $\iota|_{I^{\mathsf{d}},I^{\mathsf{e}}}{\circ}\mu^{\mathsf{d}} = \mu^{\mathsf{e}}{\circ}\tau|_{W^{\mathsf{d}},W^{\mathsf{e}}}$. *(b)* $[\varGamma^{\mathsf{d}},\varGamma^{\mathsf{e}},\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$*'s player transformation is* $\iota|_{I^{\mathsf{d}},I^{\mathsf{e}}}$. Parts (a) and (b) are equivalent by the definition of a player transformation.

Consider (a). The left-hand composition is well-defined since I^{d} is the codomain of μ^{d} by definition. The right-hand composition is well-defined since W^{e} is the domain of μ^{e} by [G2] for $\varGamma^{\mathsf{e}}$. The domains of the compositions coincide since W^{d} is the domain of μ^{d} by [G2] for $\varGamma^{\mathsf{d}}$. The codomains coincide since I^{e} is the codomain of μ^{e} by definition. Thus it suffices to show that $(\forall x{\in}W^{\mathsf{d}})\ \iota(\mu^{\mathsf{d}}(x)) = \mu^{\mathsf{e}}(\tau(x))$. Toward that end, take [1] $x^{\mathsf{d}} \in W^{\mathsf{d}}$. Then Claim 3 implies [2] $\tau(x^{\mathsf{d}}) \in W^{\mathsf{e}}$. In steps, $\iota(\mu^{\mathsf{d}}(x^{\mathsf{d}}))$ by [1] and the Selten definition of μ^{d} is equal to $\iota(\mu^{\mathsf{D}}(x^{\mathsf{d}}))$, which by [gM] for $[\mathit{\Gamma}^{\mathsf{D}},\mathit{\Gamma}^{\mathsf{E}},\tau]$ is equal to $\mu^{\mathsf{E}}(\tau(x^{\mathsf{d}}))$, which by [2] and the Selten definition of μ^{e} is equal to $\mu^{\mathsf{e}}(\tau(x^{\mathsf{d}}))$.

Claim 10: $\iota|_{I^{\mathsf{d}},I^{\mathsf{e}}}$ *is injective.* Proposition 4.1(b) for $[\mathit{\Gamma}^{\mathsf{D}},\mathit{\Gamma}^{\mathsf{E}},\tau]$ implies ι is injective. Thus the restriction $\iota|_{I^{\mathsf{d}},I^{\mathsf{e}}}$ is injective.

Claim 11: $(\forall i{\in}I, Z_1^{\mathsf{d}}{\in}\mathcal{Z}^{\mathsf{d}}, Z_2^{\mathsf{d}}{\in}\mathcal{Z}^{\mathsf{d}})\;\; U_i^{\mathsf{d}}(Z_1^{\mathsf{d}}) \;\geq\; U_i^{\mathsf{d}}(Z_2^{\mathsf{d}}) \;\;\Leftrightarrow\;\; U_{\iota(i)}^{\mathsf{e}}(\zeta(Z_1^{\mathsf{d}})) \;\geq\; U_{\iota(i)}^{\mathsf{e}}(\zeta(Z_2^{\mathsf{d}}))$. Take $i \in I$, $Z_1^{\mathsf{d}} \in \mathcal{Z}^{\mathsf{d}}$, and $Z_2^{\mathsf{d}} \in \mathcal{Z}^{\mathsf{d}}$. It suffices to show

$$U_i^{\mathsf{d}}(Z_1^{\mathsf{d}}) \geq U_i^{\mathsf{d}}(Z_2^{\mathsf{d}})$$

$$\text{(a)}\quad \Leftrightarrow U_i^{\mathsf{D}}(P^{\mathsf{D}}(r^{\mathsf{d}}){\cup}Z_1^{\mathsf{d}}) \geq U_i^{\mathsf{D}}(P^{\mathsf{D}}(r^{\mathsf{d}}){\cup}Z_2^{\mathsf{d}})$$

$$\text{(b)}\quad \Leftrightarrow U_{\iota(i)}^{\mathsf{E}}(\zeta(P^{\mathsf{D}}(r^{\mathsf{d}}){\cup}Z_1^{\mathsf{d}})) \geq U_{\iota(i)}^{\mathsf{E}}(\zeta(P^{\mathsf{D}}(r^{\mathsf{d}}){\cup}Z_2^{\mathsf{d}}))$$

$$\text{(c)}\quad \Leftrightarrow U_{\iota(i)}^{\mathsf{E}}(\bar{\tau}(P^{\mathsf{D}}(r^{\mathsf{d}}){\cup}Z_1^{\mathsf{d}})) \geq U_{\iota(i)}^{\mathsf{E}}(\bar{\tau}(P^{\mathsf{D}}(r^{\mathsf{d}}){\cup}Z_2^{\mathsf{d}}))$$

$$\text{(d)}\quad \Leftrightarrow U_{\iota(i)}^{\mathsf{E}}(\bar{\tau}(P^{\mathsf{D}}(r^{\mathsf{d}})){\cup}\bar{\tau}(Z_1^{\mathsf{d}})) \geq U_{\iota(i)}^{\mathsf{E}}(\bar{\tau}(P^{\mathsf{D}}(r^{\mathsf{d}})){\cup}\bar{\tau}(Z_2^{\mathsf{d}}))$$

$$\text{(e)}\quad \Leftrightarrow U_{\iota(i)}^{\mathsf{E}}(\bar{\tau}(P^{\mathsf{D}}(r^{\mathsf{d}})){\cup}\zeta(Z_1^{\mathsf{d}})) \geq U_{\iota(i)}^{\mathsf{E}}(\bar{\tau}(P^{\mathsf{D}}(r^{\mathsf{d}})){\cup}\zeta(Z_2^{\mathsf{d}}))$$

$$\text{(f)}\quad \Leftrightarrow U_{\iota(i)}^{\mathsf{E}}(P^{\mathsf{E}}(\tau(r^{\mathsf{d}})){\cup}\zeta(Z_1^{\mathsf{d}})) \geq U_{\iota(i)}^{\mathsf{E}}(P^{\mathsf{E}}(\tau(r^{\mathsf{d}})){\cup}\zeta(Z_2^{\mathsf{d}}))$$

$$\text{(g)}\quad \Leftrightarrow U_{\iota(i)}^{\mathsf{e}}(\zeta(Z_1^{\mathsf{d}})) \geq U_{\iota(i)}^{\mathsf{e}}(\zeta(Z_2^{\mathsf{d}})).$$

Equivalence (a) holds by the Selten definition of U_i^{d}. Equivalence (b) holds by the biconditional from Proposition 4.1(b) for $[\mathit{\Gamma}^{\mathsf{D}},\mathit{\Gamma}^{\mathsf{E}},\tau]$. Equivalences (c) and (e) hold by the equation $\zeta = \bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$ from Proposition 4.1(b) for $[\mathit{\Gamma}^{\mathsf{D}},\mathit{\Gamma}^{\mathsf{E}},\tau]$. Equivalence (f) holds by Proposition 4.2(e) for $[\mathit{\Gamma}^{\mathsf{D}},\mathit{\Gamma}^{\mathsf{E}},\tau]$. Finally, equivalence (g) follows from [1] the Selten definition of $U_{\iota(i)}^{\mathsf{e}}$ and [2] the fact that $\tau(r^{\mathsf{d}})$ is the root node of $\mathit{\Gamma}^{\mathsf{e}}$ by construction.

Claim 12: $[\mathit{\Gamma}^{\mathsf{d}},\mathit{\Gamma}^{\mathsf{e}},\tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$ *is an isomorphism.* The tuple is a morphism because [gZ] follows from Claim 7, [gM] follows from Claim 9(a), and [gU] follows from Claim 8 and the forward direction of Claim 11. Thus the tuple is an isomorphism by Proposition 4.1(b) with Claim 1, Claim 4, the combination of Claims 9(b) and 10, and the combination of Claims 8 and 11. □

Proof C.11 (for Theorem 4.7)**.** Because of Theorem 4.3(a), it suffices to show that $(\forall s^{\mathsf{D}}{\in}S^{\mathsf{D}})$ s^{D} is a subgame-perfect equilibrium in $\mathit{\Gamma}^{\mathsf{D}}$ if and only if $\langle\, \alpha_{\tau^{-1}(x^{\mathsf{E}})}(s^{\mathsf{D}}(\tau^{-1}(x^{\mathsf{E}})))\,\rangle_{x^{\mathsf{E}}\in W^{\mathsf{E}}}$ is a subgame-perfect equilibrium in $\mathit{\Gamma}^{\mathsf{E}}$.

Toward that end, take $s^{\mathsf{D}} \in S^{\mathsf{D}}$. Let R^{D} be the set of nodes at which $\mathit{\Gamma}^{\mathsf{D}}$ has a Selten subgame, and let R^{E} be the set of nodes at which $\mathit{\Gamma}^{\mathsf{E}}$ has a Selten subgame. It suffices to argue that

$$s^{\mathsf{D}} \text{ is a subgame-perfect equilibrium in } \mathit{\Gamma}^{\mathsf{D}}$$

$$\Leftrightarrow (\forall r^{\mathsf{d}}{\in}R^{\mathsf{D}})\; \langle\, s^{\mathsf{D}}(x^{\mathsf{d}})\,\rangle_{x^{\mathsf{d}}\in W^{\mathsf{d}}} \text{ is a Nash equilibrium in } \mathit{\Gamma}^{\mathsf{d}} \text{ where } \mathit{\Gamma}^{\mathsf{d}} \text{ is the Selten subgame of } \mathit{\Gamma}^{\mathsf{D}} \text{ at } r^{\mathsf{d}}$$

$$\Leftrightarrow (\forall r^{\mathsf{d}}{\in}R^{\mathsf{D}})\; \langle\, \alpha_{\tau^{-1}(x^{\mathsf{e}})}(s^{\mathsf{D}}(\tau^{-1}(x^{\mathsf{e}})))\,\rangle_{x^{\mathsf{e}}\in W^{\mathsf{e}}} \text{ is a Nash equilibrium in } \mathit{\Gamma}^{\mathsf{e}} \text{ where } \mathit{\Gamma}^{\mathsf{e}} \text{ is the Selten subgame of } \mathit{\Gamma}^{\mathsf{E}} \text{ at } \tau(r^{\mathsf{d}})$$

$$\Leftrightarrow (\forall r^{\mathsf{e}}{\in}R^{\mathsf{E}})\; \langle\, \alpha_{\tau^{-1}(x^{\mathsf{e}})}(s^{\mathsf{D}}(\tau^{-1}(x^{\mathsf{e}})))\,\rangle_{x^{\mathsf{e}}\in W^{\mathsf{e}}} \text{ is a Nash equilibrium in } \mathit{\Gamma}^{\mathsf{e}} \text{ where } \mathit{\Gamma}^{\mathsf{e}} \text{ is the Selten subgame of } \mathit{\Gamma}^{\mathsf{E}} \text{ at } r^{\mathsf{e}}$$

$$\Leftrightarrow \langle\, \alpha_{\tau^{-1}(x^{\mathsf{E}})}(s^{\mathsf{D}}(\tau^{-1}(x^{\mathsf{E}})))\,\rangle_{x^{\mathsf{E}}\in W^{\mathsf{E}}} \text{ is a subgame-perfect equilibrium in } \mathit{\Gamma}^{\mathsf{E}}.$$

The first equivalence holds by the definition of subgame-perfect equilibrium. To show the second equivalence, take $r^{\mathsf{d}} \in R^{\mathsf{D}}$. Then Theorem 4.6 implies that $[\mathit{\Gamma}^{\mathsf{d}}, \mathit{\Gamma}^{\mathsf{e}}, \tau|_{X^{\mathsf{d}},X^{\mathsf{e}}}]$ is an isomorphism. Thus Theorem 4.5 implies that $\langle\, s^{\mathsf{D}}(x^{\mathsf{d}})\,\rangle_{x^{\mathsf{d}}\in W^{\mathsf{d}}}$ is a Nash equilibrium in $\mathit{\Gamma}^{\mathsf{d}}$ iff $\langle\, \alpha_{\tau^{-1}(x^{\mathsf{e}})}(s^{\mathsf{E}}(\tau^{-1}(x^{\mathsf{e}})))\,\rangle_{x^{\mathsf{e}}\in W^{\mathsf{e}}}$ is a Nash equilibrium in $\mathit{\Gamma}^{\mathsf{e}}$. The third equivalence holds because $\tau|_{R^{\mathsf{D}},R^{\mathsf{E}}}$ is a bijection by Theorem 4.6(a). The fourth equivalence holds by the definition of subgame-perfect equilibrium. □

Appendix D. For Full Subcategories

Lemma D.1.

(a) Suppose $\mathit{\Theta}$ is a CLT, $\tau{:}X{\to}X'$ is a bijection, $\langle \alpha^*_x{:}F(x){\to}A^*_x\rangle_{x\in W}$ is continuous from W, and each α^*_x is a bijection.[27] *Construct the tuple $\mathit{\Theta}' = (X', E', \mathcal{H}', \lambda')$ by combining X' with*

$$E' = \{\, \tau(x)\tau(y) \mid xy{\in}E \,\},$$
$$\mathcal{H}' = \{\, \bar{\tau}(H) \mid H{\in}\mathcal{H} \,\}, \text{ and}$$
$$\lambda'{:}E'{\to}\cup_{x\in W} A^*_x \text{ defined by } \lambda'(x'y') = \alpha^*_{\tau^{-1}(x')}(\lambda(\tau^{-1}(x')\tau^{-1}(y'))).$$

Then $\mathit{\Theta}'$ is a CLT and $[\mathit{\Theta}, \mathit{\Theta}', \tau]$ is an isomorphism. Further, $\cup_{x\in W} A^*_x = A'$, each A^*_x equals $F'(\tau(x))$, and $\langle a^*_x\rangle_{x\in W}$ is the action transformation of $[\mathit{\Theta}, \mathit{\Theta}', \tau]$.

(b) In addition, suppose μ and U are such that $\mathit{\Gamma} = (X, E, \mathcal{H}, \lambda, \mu, U)$ is a game, and suppose $\iota^*{:}I{\to}I^*$ is a bijection.[28] *Construct the tuple $\mathit{\Gamma}' = (X', E', \mathcal{H}', \lambda', \mu', U')$ by combining $\mathit{\Theta}'$ with*

$$\mu' = \iota^*{\circ}\mu{\circ}(\tau|_{W,W'})^{-1} \text{ and}$$
$$U' = \langle U'_{i'}\rangle_{i'\in I^*} \text{ defined by } U'_{i'} = U_{\iota^{*-1}(i')}{\circ}(\bar{\tau}|_{\mathcal{Z},\mathcal{Z}'})^{-1}.$$

Then $\mathit{\Gamma}'$ is a game and $[\mathit{\Gamma}, \mathit{\Gamma}', \tau]$ is an isomorphism. Further, $I^* = I'$ and ι^* is the player transformation of $[\mathit{\Gamma}, \mathit{\Gamma}', \tau]$.

Proof. As the next two paragraphs explain, both parts (a) and (b) are implied by the following sequence of claims.

For (a), Claims 1(a), 3(a), and 4(a) show the well-definition of E', $\mathcal{H}'$, and λ'. Claims 9 and 12 imply $\mathit{\Theta}'$ is a CLT and $[\mathit{\Theta}, \mathit{\Theta}', \tau]$ is an isomorphism. The remaining conclusions follow from Claims 4(b), 10, and 11.

For (b), Claim 13(a,c) shows the well-definition of μ' and U'. Claims 14 and 17 imply $\mathit{\Gamma}'$ is a game and $[\mathit{\Gamma}, \mathit{\Gamma}', \tau]$ is an isomorphism. The remaining conclusions follow from Claims 13(b) and 16(b).

Claim 1: (a) E' is well-defined. (b) $E \ni xy \mapsto \tau(x)\tau(y) \in E'$ is a bijection. (c) (X', E') is a nontrivial out-tree. (a) holds by inspection. (b) follows from the lemma's definition of E' and the assumed bijectivity of τ. (c) follows from [C1] for $\mathit{\Theta}$, the bijectivity of τ, and (b).

[27] Part (a) uses $\langle \alpha^*_x{:}F(x){\to}A^*_x\rangle_{x\in W}$ to construct a CLT $\mathit{\Theta}'$ and a morphism $[\mathit{\Theta}, \mathit{\Theta}', \tau]$. By the text's general definitions, $\mathit{\Theta}'$ determines its action set A' (as the codomain of λ'), and $[\mathit{\Gamma}, \mathit{\Gamma}', \tau]$ determines its action transformation $\langle \alpha_x\rangle_{x\in W}$. The superscript * distinguishes the assumed entities from the derived entities. Part (a)'s final sentence shows how they relate.

[28] Part (b) uses $\iota^*{:}I{\to}I^*$ to construct a game $\mathit{\Gamma}'$ and a morphism $[\mathit{\Gamma}, \mathit{\Gamma}', \tau]$. By the text's general definitions, $\mathit{\Gamma}'$ determines its player set I' (as the codomain of μ'), and $[\mathit{\Gamma}, \mathit{\Gamma}', \tau]$ determines its player transformation ι. The superscript * distinguishes the assumed entities from the derived entities. Part (b)'s final sentence shows how they relate.

Claim 2: (a) $\tau(r)=r'$. (b) $\tau|_{W,W'}$ is a bijection. (c) $\tau|_{X\smallsetminus W,X'\smallsetminus W'}$ is a bijection. (d) $\bar{\tau}|_{\mathcal{Z},\mathcal{Z}'}$ is a bijection. The assumptions of Lemma C.1 are implied by [C1] for Θ, Claim 1(c), the assumed bijectivity of τ, and the lemma's definition of E'. Thus Lemma C.1(d) implies (a), and Lemma C.1(f–h) imply (b–d).

Claim 3: (a) $\mathcal{H}'$ is well-defined. (b) $\mathcal{H}'$ partitions W'. (c) $\tau|_{W,W'}$ is a homeomorphism. (a) holds by inspection. Further, [C2] for Θ implies $\mathcal{H}$ partitions W. Thus Claim 2(b) and the lemma's definition of $\mathcal{H}'$ imply (b) and (c).

Claim 4: (a) λ' is well-defined. (b) $A'=\cup_{x\in W}A^*_x$. For (a), take $x'y'\in E'$. Then the lemma's definition of E' implies [1] $\tau^{-1}(x')\tau^{-1}(y')\in E$. [1] and the general definition of W imply [2] $\tau^{-1}(x')\in W$. Also, [1] and the general definition of F imply $\lambda(\tau^{-1}(x')\tau^{-1}(y'))\in F(\tau^{-1}(x'))$, which by the lemma's definition of $\alpha^*_{\tau^{-1}(x')}$ implies $\alpha^*_{\tau^{-1}(x')}(\lambda(\tau^{-1}(x')\tau^{-1}(y')))\in A^*_{\tau^{-1}(x')}$. This and [2] imply $\alpha^*_{\tau^{-1}(x')}(\lambda(\tau^{-1}(x')\tau^{-1}(y')))\in\cup_{x\in W}A^*_x$. For (b), A' by general definition is the codomain of λ', which by the lemma's definition of λ' is $\cup_{x\in W}A^*_x$.

Claim 5: λ' is surjective. By the definition of λ', it suffices to show $\overline{\lambda'}(E')=\cup_{x\in W}A^*_x$. The forward inclusion follows from the well-definition of λ', which was shown in Claim 4(a). For the reverse inclusion, take $x\in W$ and $a'\in A^*_x$. Thus, since $\alpha^*_x{:}F(x)\to A^*_x$ is surjective by assumption, there is [1] $a\in F(x)$ such that [2] $\alpha^*_x(a)=a'$. Further, [1] and the general definition of F imply there is $y\in X$ such that [3] $xy\in E$ and [4] $\lambda(xy)=a$. To conclude, [3] and the lemma's definition of E' imply $\tau(x)\tau(y)\in E'$. Thus it suffices to show $\lambda'(\tau(x)\tau(y))=a'$. In steps, $\lambda'(\tau(x)\tau(y))$ by the lemma's definition of λ' is equal to $\alpha^*_{\tau^{-1}(\tau(x))}(\lambda(\tau^{-1}(\tau(x))\tau^{-1}(\tau(y))))$, which reduces to $\alpha^*_x(\lambda(xy))$, which by [4] is $\alpha^*_x(a)$, which by [2] is a'.

Claim 6: λ' is deterministic. Suppose $x'y'_1\in E'$ and $x'y'_2\in E'$ satisfy $\lambda'(x'y'_1)=\lambda'(x'y'_2)$. Then the lemma's definition of λ' implies $\alpha^*_{\tau^{-1}(x')}\lambda(\tau^{-1}(x')\tau^{-1}(y'_1))$ $=\alpha^*_{\tau^{-1}(x')}\lambda(\tau^{-1}(x')\tau^{-1}(y'_2))$. Thus the assumed bijectivity of $\alpha^*_{\tau^{-1}(x')}$ implies $\lambda(\tau^{-1}(x')\tau^{-1}(y'_1))=\lambda(\tau^{-1}(x')\tau^{-1}(y'_2))$. Thus, since λ is deterministic by [C3] for Θ, $\tau^{-1}(y'_1)=\tau^{-1}(y'_2)$. Hence $y'_1=y'_2$.

Claim 7: $(\forall x'{\in}W')$ $F'(x')=\overline{\alpha^*_{\tau^{-1}(x')}}(F(\tau^{-1}(x')))$. To prove this, take $x'\in W'$. For the forward inclusion, take $a'\in F'(x')$. Then the general definition of F' implies there is $y'\in X'$ such that $a'=\lambda'(x'y')$. Thus the lemma's definition of λ' implies $a'=\alpha^*_{\tau^{-1}(x')}(\lambda(\tau^{-1}(x')\tau^{-1}(y')))$, which by the general definition of F implies $a'\in\overline{\alpha^*_{\tau^{-1}(x')}}(F(\tau^{-1}(x')))$.

For the reverse inclusion, take $a'\in\overline{\alpha^*_{\tau^{-1}(x')}}(F(\tau^{-1}(x')))$. Then there is [1] $a\in F(\tau^{-1}(x'))$ such that [2] $a'=\alpha^*_{\tau^{-1}(x')}(a)$. Further, [1], the general definition of F, and the bijectivity of τ together imply there is $y'\in X'$ such that $a=\lambda(\tau^{-1}(x')\tau^{-1}(y'))$. Thus [2] implies $a'=\alpha^*_{\tau^{-1}(x')}(\lambda(\tau^{-1}(x')\tau^{-1}(y')))$, which by the lemma's definition of λ' implies $a'=\lambda'(x'y')$, which by the general definition of F' implies $a'\in F'(x')$.

Claim 8: $\langle F'(x')\rangle_{x'\in W'}$ is continuous from W'. It suffices to show that $\langle F'(x')\rangle_{x'\in W'}$ is constant over each $H'\in\mathcal{H}'$. Thus by Claim 7, it suffices to show that $\langle\,\overline{\alpha^*_{\tau^{-1}(x')}}(F(\tau^{-1}(x')))\,\rangle_{x'\in W'}$ is constant over each $H'\in\mathcal{H}'$. Toward that end, take $H'\in\mathcal{H}'$. Then Claim 3(c) implies [a] $\tau^{-1}(H')\in\mathcal{H}$. Note [C4] for Θ and [a] imply that [b] the set $F(\tau^{-1}(x'))$ is constant for $x' \in H'$. Meanwhile, the assumed continuity of $\langle\alpha^*_x\rangle_{x\in W}$ from W and [a] imply that [c] the function $\alpha^*_{\tau^{-1}(x')}$ is constant for $x' \in H'$. Together, [b] and [c] imply that the image $\overline{\alpha^*_{\tau^{-1}(x')}}(F(\tau^{-1}(x')))$ is constant for $x' \in H'$.

Claim 9: Θ' is a CLT. It suffices to prove that the tuple Θ' satisfies [C1]–[C4]. [C1] holds by Claim 1(c). [C2] holds by Claim 3(b). [C3] holds by Claims 5 and 6. Finally, [C4] holds by Claim 8.

Claim 10: $(\forall x{\in}W)$ $A^*_x = F'(\tau(x))$. To prove this, take $x \in W$. Claim 2(b) implies $\tau(x) \in W'$. Thus Claim 7 at $x' = \tau(x)$ implies $F'(\tau(x)) = \overline{\alpha^*_x}(F(x))$. The right-hand side is equal to A^*_x by the assumed surjectivity of $\alpha^*_x{:}F(x){\to}A^*_x$.

Claim 11: The action transformation of the tuple $[\Theta,\Theta',\tau]$ is $\langle\alpha^*_x\rangle_{x\in W}$. Let $\langle\alpha_x\rangle_{x\in W}$ denote the action transformation of $[\Theta,\Theta',\tau]$. It suffices to show that $(\forall x{\in}W)$ $\alpha_x = \alpha^*_x$. Toward that end, take $x \in W$. By general definition, $\alpha_x{:}F(x){\to}F'(\tau(x))$ and [1] $\alpha_x(a) = \lambda'(\,\tau(x)\,\tau(n(x,a))\,)$. Meanwhile by the lemma's assumption, $\alpha^*_x{:}F(x){\to}A^*_x$. Thus the domain of α_x equals the domain of α^*_x. Further, Claim 10 implies the codomain of α_x equals the codomain of α^*_x. Thus it remains to show that $(\forall a{\in}F(x))$ $\alpha_x(a) = \alpha^*_x(a)$.

Toward that end, take $a \in F(x)$. The general definition of n implies $x\,n(x,a) \in E$. Thus the lemma's definition of E' implies $\tau(x)\,\tau(n(x,a)) \in E'$. Thus the lemma's definition of λ' implies $\lambda'(\,\tau(x)\,\tau(n(x,a))\,) = \alpha^*_{\tau^{-1}(\tau(x))}(\lambda(\,\tau^{-1}(\tau(x))\,\tau^{-1}(\tau(n(x,a)))\,))$. This equality reduces to $\lambda'(\,\tau(x)\,\tau(n(x,a))\,) = \alpha^*_x(\lambda(x,n(x,a)))$, which by Lemma A.1(c) reduces to $\lambda'(\,\tau(x)\,\tau(n(x,a))\,) = \alpha^*_x(a)$. This and [1] imply $\alpha_x(a) = \alpha^*_x(a)$.

Claim 12: $[\Theta,\Theta',\tau]$ is an end-preserving isomorphism. The tuple is a morphism because [cE] holds by the lemma's definition of E', [cI] holds by Claim 3(c), and [cL] holds by Claim 11 and the lemma's assumption that $\langle\alpha^*_x\rangle_{x\in W}$ is continuous from W. It is end-preserving by Claim 2(c). Finally, it is an isomorphism by Proposition 4.1(a), the bijectivity of τ, and Claim 3(c).

Claim 13: (a) μ' is well-defined. (b) $I' = I^*$. (c) U' is well-defined. (a) The inverse $(\tau|_{W,W'})^{-1}$ is well-defined by Claim 2(b). (b) In steps, I' by general definition is the codomain of μ', which by the lemma's definition of μ' is the codomain of ι^*, which by assumption is I^*. (c) The inverse $\iota^{*\,-1}$ is well-defined by assumption, and the inverse $(\bar\tau|_{\mathcal{Z},\mathcal{Z}'})^{-1}$ is well-defined by Claim 2(d).

Claim 14: Γ' is a game. It suffices to show that the tuple Γ' satisfies [G1]–[G3]. [G1] holds by Claim 9. For [G2], note that (i) $\mu' = \iota^*{\circ}\mu{\circ}(\tau|_{W,W'})^{-1}$ by the lemma's definition, that (ii) $\tau|_{W,W'}$ is a homeomorphism by Claim 3(c), that (iii) $\mu{:}W{\to}I$ is surjective and continuous by [G2] for Γ and the definition of I, and that (iv) $\iota^*{:}I{\to}I^*$ is a homeomorphism by its assumed bijectivity and the fact that I and I^* are endowed with the discrete topology. Thus $\mu'{:}W'{\to}I^*$ is surjective and continuous. Finally for [G3], take $i' \in I'$. Since $i' \in I^*$ by Claim 13(b), the lemma's definition of U' implies $U'_{i'} = U_{\iota^{*-1}(i')}{\circ}(\bar\tau|_{\mathcal{Z},\mathcal{Z}'})^{-1}$. Further, $U_{\iota^{*-1}(i')}{:}\mathcal{Z}{\to}\mathbb{R}$ by [G3] for Γ. Thus $U'_{i'}{:}\mathcal{Z}'{\to}\mathbb{R}$.

Claim 15: (a) $[\Theta,\Theta',\tau]$'s run transformation is $\bar\tau|_{\mathcal{Z},\mathcal{Z}'}$. (b) $(\forall i{\in}I)$ $U_i = U'_{\iota^*(i)}{\circ}\bar\tau|_{\mathcal{Z},\mathcal{Z}'}$. For (a), let ζ denote $[\Theta,\Theta',\tau]$'s run transformation. Claim 12 and Proposition 2.3 imply ζ is well-defined. By definition, [a] $\zeta{:}\mathcal{Z}{\to}\mathcal{Z}'$ and [b] $(\forall Z{\in}\mathcal{Z})$ $\zeta(Z) = P'{\circ}\tau(r){\cup}\bar\tau(Z)$. By [a], it suffices to show that $(\forall Z{\in}\mathcal{Z})$ $\zeta(Z) = \bar\tau(Z)$. Thus by [b], it suffices to show that $P'\circ\tau(r) = \varnothing$. This follows from Claim 2(a). For (b), note that the lemma's definition of U' implies $(\forall i'{\in}I^*)\ U'_{i'}\circ\bar{\tau}|_{\mathcal{Z},\mathcal{Z}'} = U_{\iota^{*-1}(i')}$. Thus the assumed bijectivity of ι^* implies $(\forall i{\in}I)\ U'_{\iota^*(i)}\circ\bar{\tau}|_{\mathcal{Z},\mathcal{Z}'} = U_i$.

Claim 16: (a) $[\Gamma,\Gamma',\tau]$ is a morphism. (b) Its player transformation is ι^*. Γ is a game by assumption and Γ' is a game by Claim 14. Further, since $\iota^*{:}I{\to}I^*$ by assumption, Claim 13(b) implies $\iota^*{:}I{\to}I'$. Thus for (a), it suffices to show that the tuple $[\Gamma,\Gamma',\tau]$ satisfies [gZ], [gM] with ι^*, and [gU] with ι^*. In the process, (b) will also be proved. [gZ] follows from Claim 12. For [gM] with ι^*, the lemma's definition of μ' states $\mu' = \iota^*\circ\mu\circ(\tau|_{W,W'})^{-1}$, which is equivalent to $\mu'\circ\tau|_{W,W'} = \iota^*\circ\mu$. This also proves (b) because [gM] is equivalent to the definition of the player transformation. Finally, for [gU] with ι^* follows from Claim 15(a,b).

Claim 17: $[\Gamma,\Gamma',\tau]$ is an isomorphism. Since $[\Gamma,\Gamma',\tau]$ is a morphism by Claim 16(a), it suffices to show that $[\Gamma,\Gamma',\tau]$ satisfies the four conditions of Proposition 4.1(b). First, τ is bijective by assumption. Second, $\tau|_{W,W'}$ is a homeomorphism by Claim 3(c). Third, the player transformation is injective by Claim 16(b) and the assumed injectivity of ι^*. Fourth, the utility biconditional follows from Claim 15(a,b). □

Lemma D.2. *A CLT Θ has distinguished-actions iff $(\forall H_1{\in}\mathcal{H}, H_2{\in}\mathcal{H}, x_1{\in}H_1, x_2{\in}H_2)$ $H_1 \neq H_2$ implies $F(x_1)\cap F(x_2) = \varnothing$.*

Proof. For the forward direction, suppose Θ has distinguished-actions. Then suppose [a] $x_1 \in H_1 \in \mathcal{H}$ and [b] $x_2 \in H_2 \in \mathcal{H}$. To prove the contrapositive of the conditional, suppose $a \in F(x_1)\cap F(x_2)$. Then both x_1 and x_2 belong to $\{\, x{\in}W \mid a{\in}F(x)\,\}$, which by distinguished-actions is a member of $\mathcal{H}$. Thus, using x_1 only, [a] and [C2] imply $H_1 = \{\, x{\in}W \mid a{\in}F(x)\,\}$. By similar reasoning, using x_2 only, [b] and [C2] imply $H_2 = \{\, x{\in}W \mid a{\in}F(x)\,\}$. Therefore $H_1 = H_2$.

For the reverse direction, assume the conditional holds. By the definition of distinguished-actions, it suffices to prove $(\forall a{\in}A)\ \{\, x{\in}W \mid a{\in}F(x)\,\} \in \mathcal{H}$. Take $a \in A$. [C4] implies there is $\mathcal{H}_* \subseteq \mathcal{H}$ such that $\{\, x{\in}W \mid a{\in}F(x)\,\} = \cup\mathcal{H}_*$. Thus it suffices to show that $\mathcal{H}_*$ is a singleton. To do so, suppose there were distinct H_1 and H_2 in $\mathcal{H}_*$. Then [C2] implies there are $x_1 \in H_1$ and $x_2 \in H_2$, and thus the assumed conditional implies [c] $F(x_1)\cap F(x_2) = \varnothing$. But, $x_1 \in H_1 \in \mathcal{H}_*$ and the definition of $\mathcal{H}_*$ implies $a \in F(x_1)$. Similarly, $x_2 \in H_2 \in \mathcal{H}_*$ and the definition of $\mathcal{H}_*$ implies $a \in F(x_2)$. Thus $a \in F(x_1)\cap F(x_2)$ in contradiction to [c]. □

Lemma D.3. *Each game is isomorphic to an distinguished-action game.*

Proof. Take a game Γ. A distinguished-action isomorph will be constructed via Lemma D.1. For that purpose, define $\tau = \mathsf{id}_X$ and $\iota^* = \mathsf{id}_I$. To define $\langle\alpha^*_x\rangle_{x\in W}$, take an $x \in W$. By [C2], let H_x be the member of $\mathcal{H}$ that contains x. Then define $A^*_x = \{H_x\}{\times}F(x)$, and define $\alpha_x{:}F(x){\to}A^*_x$ by $\alpha_x(a) = (H_x, a)$.

To see that τ, $\langle\alpha^*_x\rangle_{x\in W}$, and ι^* satisfy Lemma D.1's assumptions, first note that τ, each α^*_x, and ι^* are bijective by inspection. Thus it suffices to show that $\langle\alpha^*_x\rangle_{x\in W}$ is continuous from W. In other words, it suffices to show that $\langle\alpha^*_x\rangle_{x\in W}$ is constant over each $H \in \mathcal{H}$. Toward that end, take $H_o \in \mathcal{H}$. The domain of each α_x is $F(x)$, which is constant over $x \in H_o$ by [C4]. Thus since each α^*_x is surjective, it suffices to show that $(\forall a{\in}F(x))\ \alpha^*_x(a)$ is constant over $x \in H_o$. Take $a \in F(x)$. By definition $\alpha^*_x(a) = (H_x, a)$, which is constant at (H_o, a) over $x \in H_o$.

Lemma D.1 constructs a game Γ' such that $[\Gamma,\Gamma',\tau]$ is an isomorphism. To show that Γ' has distinguished-actions, it suffices by Lemma D.2 to show that $(\forall H_1'{\in}\mathcal{H}', H_2'{\in}\mathcal{H}', x_1'{\in}H_1', x_2'{\in}H_2')$ $H_1' \neq H_2'$ implies $F'(x_1'){\cap}F'(x_2') = \varnothing$. Toward that end, suppose $x_1' \in H_1' \in \mathcal{H}'$ and $x_2' \in H_2' \in \mathcal{H}'$ and $H_1' \neq H_2'$. Because $\tau|_{W,W'}$ is a homeomorphism by Proposition 4.1(a) applied to $[\Gamma,\Gamma',\tau]$, [1] $\tau^{-1}(x_1') \in \overline{\tau^{-1}}(H_1') \in \mathcal{H}$ and [2] $\tau^{-1}(x_2') \in \overline{\tau^{-1}}(H_2') \in \mathcal{H}$ and [3] $\overline{\tau^{-1}}(H_1') \neq \overline{\tau^{-1}}(H_2')$. The definition of $\langle A^*_x\rangle_{x\in W}$ and [1] imply $A^*_{\tau^{-1}(x_1')} = \{\overline{\tau^{-1}}(H_1')\}{\times}F(\tau^{-1}(x_1'))$. Similarly [2] implies $A^*_{\tau^{-1}(x_2')} = \{\overline{\tau^{-1}}(H_2')\}{\times}F(\tau^{-1}(x_2'))$. Thus [3] implies $A^*_{\tau^{-1}(x_1')}$ and $A^*_{\tau^{-1}(x_2')}$ are disjoint. Further, Lemma D.1(a)'s last sentence implies $A^*_{\tau^{-1}(x_1')}$ equals $F'(\tau(\tau^{-1}(x_1')))$, which reduces to $F'(x_1')$. Similarly $A^*_{\tau^{-1}(x_2')}$ equals $F'(x_2')$. The last three sentences imply $F'(x_1')$ and $F'(x_2')$ are disjoint. □

Lemma D.4 (corollary of Lemma D.1)**.**
(a) Suppose Θ is a CLT and $\tau{:}X{\to}X'$ is bijective. Construct the tuple $\Theta' = (X',E',\mathcal{H}',\lambda')$ by combining X' with $E' = \{\,\tau(x)\tau(y)\,|\,xy{\in}E\,\}$, $\mathcal{H}' = \{\,\bar\tau(H)\,|\,H{\in}\mathcal{H}\,\}$, and $\lambda'{:}E'{\to}A$ defined by $\lambda'(x'y') = \lambda(\tau^{-1}(x')\tau^{-1}(y'))$. Then Θ' is a CLT and $[\Theta,\Theta',\tau]$ is an isomorphism. Further, $A = A'$ and $\langle \mathsf{id}_{F(x)}\rangle_{x\in W}$ is the action transformation of $[\Theta,\Theta',\tau]$.

(b) In addition, suppose μ and U are such that $\Gamma = (X,E,\mathcal{H},\lambda,\mu,U)$ is a game. Construct the tuple $\Gamma' = (X',E',\mathcal{H}',\lambda',\mu',U')$ by combining Θ' with $\mu' = \mu{\circ}(\tau|_{W,W'})^{-1}$ and $U' = \langle U_i'\rangle_{i\in I}$ defined by $U_i' = U_i{\circ}(\bar\tau|_{\mathcal{Z},\mathcal{Z}'})^{-1}$. Then Γ' is a game and $[\Gamma,\Gamma',\tau]$ is an isomorphism. Further, $I = I'$ and id_I is the player transformation of $[\Gamma,\Gamma',\tau]$.

Proof. Consider Lemma D.1 in the special case that $\langle\alpha^*_x\rangle_{x\in W} = \langle\mathsf{id}_{F(x)}\rangle_{x\in W}$, $\langle A^*_x\rangle_{x\in W} = \langle F(x)\rangle_{x\in W}$, $\iota^* = \mathsf{id}_I$, and $I^* = I$. To see that Lemma D.1's assumptions follow from the assumptions here, note that $\langle\mathsf{id}_{F(x)}\rangle_{x\in W}$ is continuous from W because $\langle F(x)\rangle_{x\in W}$ is continuous from W by [C4] for Θ. The remainder of Lemma D.1's assumptions follow by inspection. To see that Lemma D.1's conclusions imply the conclusions here, note that $A' = A$ because, in steps, A' by Lemma D.1(a)'s last sentence is equal to $\cup_{x\in W}A^*_x$, which by this proof's definition of $\langle A^*_x\rangle_{x\in W}$ is equal to $\cup_{x\in W}F(x)$, which by Lemma A.1(a) is equal to A. The remainder of the conclusions here follow by inspection. □

Lemma D.5.[29] *Suppose Θ is a CLT. Let $k(x)$ denote the height of node $x \in X$, define $q{:}X{\smallsetminus}\{r\}{\to}A$ by $q(y) = \lambda(p(y)y)$, and let $\dot X = \{\,\langle q{\circ}p^{k(x)-\ell}(x)\rangle_{\ell=1}^{k(x)}\,|\,x{\in}X\,\}$. Then*

$$X \ni x \;\mapsto\; \langle q{\circ}p^{k(x)-\ell}(x)\rangle_{\ell=1}^{k(x)} \in \dot X$$

is a bijection. Its inverse is

$$X \ni n(n(...n(n(r,\dot x_1),\,\dot x_2),\dots,\,\dot x_{|\dot x|-1})\,,\,\dot x_{|\dot x|}) \;\;\leftarrow\!\shortmid\;\; \dot x \in \dot X$$

(to be clear, $X \ni r \leftarrow\!\shortmid \{\} \in \dot X$).

Proof. Let τ be the function from X to $\dot X$. It is well-defined by inspection and the definition of $\dot X$. Conversely, let β be the function from $\dot X$ to X. It is well-defined by Claim 3 and the definition of $\dot X$. The lemma follows from Claims 3 and 4.

[29] This lemma adapts and extends Streufert 2020a, Lemma B.2.

Claim 1: $(\forall y{\in}X{\smallsetminus}\{r\})\ n[p(y),q(y)] = y$. To show this, take $y \in X{\smallsetminus}\{r\}$. In steps, $n[p(y),q(y)]$ by the definition of q is equal to $n[p(y),\lambda(p(y)y)]$, which by Lemma A.1(c) is equal to y.

Claim 2: (a) $(\forall x{\in}W, a{\in}F(x))\, p(n[x,a]) = x$. *(b)* $(\forall x{\in}W, a{\in}F(x))\, q(n[x,a]) = a$. For (a), take $x \in W$ and $a \in F(x)$. Then the definition of n implies $x\,n[x,a] \in E$, which by the definition of p implies $p(n[x,a]) = x$. For (b), take $x \in W$ and $a \in F(x)$. In steps, $q(n[x,a])$ by the definition of q is equal to $\lambda(\,p(n[x,a])\,n[x,a]\,)$, which by part (a) is equal to $\lambda(\,x\,n[x,a]\,)$, which by Lemma A.1(c) is equal to a.

Claim 3: $\beta{\circ}\tau$ *is the identity function on* X. Take $x \in X$. First, suppose $k(x) = 0$. In steps, $\beta{\circ}\tau(x)$ by the definition of τ equals $\beta(\{\})$, which by the definition of β equals r, which by $k(x) = 0$ equals x. Second, suppose $k(x) = 1$. In steps, $\beta{\circ}\tau(x)$ by the definition of τ equals $\beta(\langle q(x)\rangle)$, which by the definition of β equals $n[r,q(x)]$, which by $k(x) = 1$ equals $n[p(x),q(x)]$, which by Claim 1 equals x. Third and finally, suppose $k(x) \geq 2$. It will be argued that

$$\begin{aligned}
\beta{\circ}\tau(x) &= \beta(\ \langle q{\circ}p^{k(x)-\ell}(x)\rangle_{\ell=1}^{k(x)}\ )\\
&= n[n[...n[n[r, q{\circ}p^{k(x)-1}(x)],\ q{\circ}p^{k(x)-2}(x)]\,...\,,\ q{\circ}p(x)],\ q(x)]\\
&= n[n[...n[n[p^{k(x)}(x), q{\circ}p^{k(x)-1}(x)],\ q{\circ}p^{k(x)-2}(x)]\,...\,,\ q{\circ}p(x)], q(x)]\\
&= n[n[...n[n[p{\circ}p^{k(x)-1}(x), q{\circ}p^{k(x)-1}(x)],\ q{\circ}p^{k(x)-2}(x)]\,...\,,\ q{\circ}p(x)], q(x)]\\
&= n[n[...n[p^{k(x)-1}(x), q{\circ}p^{k(x)-2}]\,...\,,\ q{\circ}p(x)], q(x)]\\
&\ \ \cdots\\
&= n[p(x),q(x)] = x.
\end{aligned}$$

The first equality holds by the definition of τ, the second by the definition of β, and the third by the definition of k. The fourth and fifth equalities hold by a rearrangement and Claim 1. The sixth equality holds by $k(x){-}2$ similar applications of Claim 1, and the final equality holds by a final application of Claim 1.

Claim 4: $\tau{\circ}\beta$ *is the identity function on* $\dot{X}$. Take $\dot{x} \in \dot{X}$. First, suppose $\dot{x} = \{\}$. In steps, $\tau{\circ}\beta(\{\})$ by the definition of β is equal to $\tau(r)$, which by the definition of τ is equal to $\{\}$. Second, suppose $\dot{x} \neq \{\}$. Then it suffices to show that $(\forall \ell{\in}\{1,2,...\,|\dot{x}|\})$ $(\tau{\circ}\beta(\dot{x}))_\ell = \dot{x}_\ell$. Toward that end, take $\ell \in \{1,2,...\,|\dot{x}|\}$.

First suppose $\ell < |\dot{x}|$. It will be argued that

$$\begin{aligned}
(\tau{\circ}\beta(\dot{x}))_\ell &= q(p^{k(\beta(\dot{x}))-\ell}(\beta(\dot{x})))\\
&= q(p^{|\dot{x}|-\ell}(\beta(\dot{x})))\\
&= q(p^{|\dot{x}|-\ell}(n[n[...n[n[r,\dot{x}_1],\ \dot{x}_2]\,...\,,\ \dot{x}_{|\dot{x}|-1}],\ \dot{x}_{|\dot{x}|}]))\\
&= q(p^{|\dot{x}|-\ell-1}(n[n[...n[n[r,\dot{x}_1],\ \dot{x}_2]\,...\,,\ \dot{x}_{|\dot{x}|-2}],\ \dot{x}_{|\dot{x}|-1}]))\\
&\ \ \cdots\\
&= q(p^{|\dot{x}|-\ell-(|\dot{x}|-\ell)}(n[n[...n[n[r,\dot{x}_1],\ \dot{x}_2],...\,\dot{x}_{|\dot{x}|-(|\dot{x}|-\ell)-1}],\ \dot{x}_{|\dot{x}|-(|\dot{x}|-\ell)}]))\\
&= q(n[n[...n[n[r,\dot{x}_1],\ \dot{x}_2]\,...\,\dot{x}_{\ell-1}],\ \dot{x}_\ell])\\
&= \dot{x}_\ell.
\end{aligned}$$

The first equality holds by the definition of τ, the second holds because $k(\beta(\dot{x})) = |\dot{x}|$ by the definitions of k and β, and the third holds by the definition of β. The fourth holds by Claim 2(a), and the fifth holds by $|\dot{x}|{-}\ell{-}1$ similar applications of

Claim 2(a). The sixth is a rearrangement, and the seventh holds by one application of Claim 2(b).

Second suppose $\ell = |\dot{x}|$. It will be argued that

$$(\tau\circ\beta(\dot{x}))_{|\dot{x}|} = q(p^{k(\beta(\dot{x}))-|\dot{x}|}(\beta(\dot{x}))) = q(p^{|\dot{x}|-|\dot{x}|}(\beta(\dot{x}))) = q(\beta(\dot{x}))$$
$$= q(n[n[...n[n[r,\dot{x}_1],\, \dot{x}_2], \dots,\, \dot{x}_{|\dot{x}|-1}],\, \dot{x}_{|\dot{x}|}]) = \dot{x}_{|\dot{x}|},$$

where the first equality holds by the definition of τ, the second holds because $k(\beta(\dot{x})) = |\dot{x}|$ by the definitions of k and β, the fourth holds by the definition of β, and the fifth by one application of Claim 2(b). □

Lemma D.6. *Suppose $\mathit{\Gamma}$ is a game. Then there is an isomorphism $[\mathit{\Gamma},\dot{\mathit{\Gamma}},\tau]$ whose $\dot{\mathit{\Gamma}}$ is a sequence game. Further, $A = \dot{A}$ and the isomorphism's action transformation is $\langle \mathsf{id}_{F(x)}\rangle_{x\in W}$.*

Proof. First use Lemma D.5 to construct a bijection $\tau{:}X{\rightarrow}\dot{X}$. Second use τ and Lemma D.4 to construct a game $\dot{\mathit{\Gamma}}$ such that $[\mathit{\Gamma},\dot{\mathit{\Gamma}},\tau]$ is an isomorphism. That lemma also shows that $A = \dot{A}$ and that $[\mathit{\Gamma},\dot{\mathit{\Gamma}},\tau]$'s action transformation is $\langle \mathsf{id}_{F(x)}\rangle_{x\in W}$. Thus it suffices to show that $\dot{\mathit{\Gamma}}$ is a sequence game. By inspection $\dot{X}$ consists of (finite) sequences. Thus the lemma is proved by Claims 2 and 3 below.

Claim 1: $(\forall y{\in}X{\setminus}\{r\})\ \tau(p(y)) = {}_1[\tau(y)]_{|\tau(y)|-1}$. To show this, take $y \in X{\setminus}\{r\}$. It suffices to show

$$\tau(p(y)) = \langle q\circ p^{k(p(y))-\ell}(p(y))\rangle_{\ell=1}^{k(p(y))}$$
$$= \langle q\circ p^{k(y)-1-\ell}(p(y))\rangle_{\ell=1}^{k(y)-1} = \langle q\circ p^{k(y)-\ell}(y)\rangle_{\ell=1}^{k(y)-1}$$
$$= {}_1[\langle q\circ p^{k(y)-\ell}(y)\rangle_{\ell=1}^{k(y)}]_{k(y)-1} = {}_1[\tau(y)]_{k(y)-1} = {}_1[\tau(y)]_{|\tau(y)|-1}.$$

The first equality holds by Lemma D.5's definition of τ, the second because $k(p(y)) = k(y){-}1$, the third and fourth by rearrangement, the fifth by the definition of τ, and the sixth because $k(y) = |\tau(y)|$ by the definition of τ.

Claim 2: $\dot{E} = \{\, {}_1\dot{y}_{|\dot{y}|-1}\,\dot{y} \mid \dot{y}{\in}\dot{X}{\setminus}\{\{\}\}\,\}$. It suffices to show

$$\dot{E} = \{\, \tau(x)\,\tau(y) \mid xy{\in}E \,\} = \{\, \tau(p(y))\,\tau(y) \mid y{\in}X{\setminus}\{r\}\,\}$$
$$= \{\, {}_1[\tau(y)]_{|\tau(y)|-1}\,\tau(y) \mid y{\in}X{\setminus}\{r\}\,\} = \{\, {}_1[\dot{y}]_{|\dot{y}|-1}\,\dot{y} \mid \dot{y}{\in}\dot{X}{\setminus}\{\{\}\}\,\}.$$

The first equality holds by Lemma D.4's definition of $\dot{E}$, the second by the general definition of p, the third by Claim 1, and the last by the bijectivity of τ together with the fact that $\tau(r) = \{\}$ by the definition of τ.

Claim 3: $(\forall\, {}_1\dot{y}_{|\dot{y}|-1}\,\dot{y} \in \dot{E})\ \dot{\lambda}(\,{}_1\dot{y}_{|\dot{y}|-1}\,\dot{y}\,) = \dot{y}_{|\dot{y}|}$. To prove this, take ${}_1\dot{y}_{|\dot{y}|-1}\,\dot{y} \in \dot{E}$. Note that two applications of the Lemma D.5's last sentence imply

$$\tau^{-1}(\dot{y}) = n(n(...n(n(r,\dot{y}_1),\, \dot{y}_2), \dots,\, \dot{y}_{|\dot{y}|-1})\,, \dot{y}_{|\dot{y}|})$$
$$= n(\tau^{-1}({}_1\dot{y}_{|\dot{y}|-1}), \dot{y}_{|\dot{y}|}).$$

It suffices to show

$$\dot{\lambda}(\ {}_1\dot{y}_{|\dot{y}|-1}\ \dot{y}\ ) = \lambda(\ \tau^{-1}({}_1\dot{y}_{|\dot{y}|-1})\ \tau^{-1}(\dot{y})\ )$$
$$= \lambda(\ \tau^{-1}({}_1\dot{y}_{|\dot{y}|-1})\ n(\tau^{-1}({}_1\dot{y}_{|\dot{y}|-1}), \dot{y}_{|\dot{y}|})\ ) = \dot{y}_{|\dot{y}|},$$

where the first equality holds by Lemma D.4's definition of $\dot{\lambda}$, the second equality holds by the previous sentence, and the third equality holds by Lemma A.1(c) at $x_o = \tau^{-1}({}_1\dot{y}_{|\dot{y}|-1})$. □

Lemma D.7. *Each game is isomorphic to a distinguished-action sequence game.*

Proof. Consider a game $\mathit{\Gamma}^*$. Lemma D.3 implies that $\mathit{\Gamma}^*$ is isomorphic to an distinguished-action game $\mathit{\Gamma}$. Thus it suffices to show that $\mathit{\Gamma}$ is isomorphic to an distinguished-action sequence game. Lemma D.6 implies there is an isomorphism $[\mathit{\Gamma}, \dot{\mathit{\Gamma}}, \tau]$ such that $\dot{\mathit{\Gamma}}$ is a sequence game. Thus it suffices to show that $\dot{\mathit{\Gamma}}$ has distinguished actions. In other words, it suffices to show that $(\forall \dot{a} {\in} \dot{A})$ $\{\, \dot{x}{\in}\dot{W} \mid \dot{a}{\in}\dot{F}(\dot{x}) \,\} \in \dot{\mathcal{H}}$.

For that purpose, take $\dot{a} \in \dot{A}$. Lemma D.6 shows $A = \dot{A}$, which implies $\dot{a} \in A$. Thus since $\mathit{\Gamma}$ has distinguished actions, $\{\, x{\in}W \mid \dot{a}{\in}F(x) \,\} \in \mathcal{H}$. Since $[\mathit{\Gamma}, \dot{\mathit{\Gamma}}, \tau]$ is an isomorphism, Proposition 4.1 implies $\tau|_{W,\dot{W}}$ is a homeomorphism. Thus $\bar{\tau}(\{\, x{\in}W \mid \dot{a}{\in}F(x) \,\}) \in \dot{\mathcal{H}}$. Hence it suffices to show that $\bar{\tau}(\{\, x{\in}W \mid \dot{a}{\in}F(x) \,\}) = \{\, \dot{x}{\in}\dot{W} \mid \dot{a}{\in}\dot{F}(\dot{x}) \,\}$.

By general definition, $[\mathit{\Gamma}, \dot{\mathit{\Gamma}}, \tau]$'s action transformation has the form $\langle \alpha_x{:}F(x){\to}\dot{F}(\tau(x)) \rangle_{x \in W}$. Meanwhile, Lemma D.6 shows $[\mathit{\Gamma}, \dot{\mathit{\Gamma}}, \tau]$'s action transformation is $\langle \mathsf{id}_{F(x)} \rangle_{x \in W}$. Thus $(\forall x{\in}W)$ $\dot{F}(\tau(x)) = F(x)$. Finally, in steps, $\bar{\tau}(\{\, x{\in}W \mid \dot{a}{\in}F(x) \,\})$ by the previous sentence is equal to $\bar{\tau}(\{\, x{\in}W \mid \dot{a}{\in}\dot{F}(\tau(x)) \,\})$, which by rearrangement is equal to $\{\, \tau(x){\in}\bar{\tau}(W) \mid \dot{a}{\in}\dot{F}(\tau(x)) \,\})$, which by the bijectivity of $\tau|_{W,\dot{W}}$ from Proposition 4.1 is equal to $\{\, \dot{x}{\in}\dot{W} \mid \dot{a}{\in}\dot{F}(\dot{x}) \,\}$. □

Proof D.8 (for Theorem 5.2)**.** Lemma D.7 suffices for the entire theorem. □

Proof D.9 (for Theorem 5.3)**.** *(a).* Suppose $[\mathit{\Theta}, \mathit{\Theta}', \tau]$ is an isomorphism. It suffices to show that $\mathit{\Theta}$ violates no-absentmindedness only if $\mathit{\Theta}'$ violates no-absentmindedness. Toward that end, suppose $\mathit{\Theta}$ violates no-absentmindedness. Then there is [1] $H \in \mathcal{H}$, [2] $x \in H$, and [3] $y \in H$ such that [4] $x \prec y$. Proposition 4.1(a) implies $\tau|_{W,W'}$ is a homeomorphism and thus [1] implies $\bar{\tau}(H) \in \mathcal{H}'$. [2] and [3] imply $\tau(x) \in \bar{\tau}(H)$ and $\tau(y) \in \bar{\tau}(H)$. Finally, Proposition 4.2(c) and [4] imply $\tau(x) \prec' \tau(y)$. The conclusions of the three previous sentences imply $\mathit{\Theta}'$ violates no-absentmindedness.

(b). Suppose $[\mathit{\Theta}, \mathit{\Theta}', \tau]$ is an isomorphism. Proposition 4.1(a) implies $\tau|_{W,W'}$ is a homeomorphism, which implies that $\mathcal{H}$ is discrete iff $\mathcal{H}'$ is discrete, which by the definition of perfect-information implies $\mathit{\Theta}$ has perfect-information iff $\mathcal{H}'$ has perfect-information. □

Proof D.10 (for Corollary 5.4)**.** *(a).* It suffices for all of part (a) to show that each game with no-absentmindedness is isomorphic to an distinguished-action sequence game with no-absentmindedness. Toward that end, suppose $\mathit{\Gamma}$ is a game with no-absentmindedness. Then Lemma D.7 implies that $\mathit{\Gamma}$ is isomorphic to an distinguished-action sequence game. Because no-absentmindedness is invariant by Theorem 5.3(a), this isomorph has no-absentmindedness.

(b). This is proved as part (a) was proved. In particular, replace part (a) with part (b), replace no-absentmindedness with perfection-information, and replace Theorem 5.3(a) with Theorem 5.3(b). □

Lemma D.11.[30] *Suppose $\dot{\Theta}$ is a distinguished-action sequence CLT with no-absentmindedness. Let R be the function, from finite sequences to finite sets, that takes each sequence to its range.*[31] *Then the following hold.*

(a) $(\forall \dot{x}{\in}\dot{X})\ |R(\dot{x})| = |\dot{x}|$

(b) $(\forall \dot{y}{\in}\dot{X}{\smallsetminus}\{\{\}\})\ R(\dot{y}){\smallsetminus}R({}_1\dot{y}_{|\dot{y}|-1}) = \{\dot{y}_{|\dot{y}|}\}$.

(c) $(\forall \dot{x}{\in}\dot{X}, \dot{y}{\in}\dot{X})\ R(\dot{x}) \subseteq R(\dot{y}) \Rightarrow \dot{x} = {}_1\dot{y}_{|\dot{x}|}$.

(d) $R|_{\dot{X}}$ *is injective.*

Proof. (a). Take $\dot{x} \in \dot{X}$. Suppose $|R(\dot{x})| \neq |\dot{x}|$. Then since $|R(\dot{x})| > |\dot{x}|$ is inconceivable, $|R(\dot{x})| < |\dot{x}|$. Thus there are ℓ and m such that [1] $\ell < m$ and [2] $\dot{x}_\ell = \dot{x}_m$. Using sequences implies $\dot{x}_\ell \in \dot{F}({}_1\dot{x}_{\ell-1})$ and $\dot{x}_m \in \dot{F}({}_1\dot{x}_{m-1})$. Thus [2] implies that the same action is feasible from both ${}_1\dot{x}_{\ell-1}$ and ${}_1\dot{x}_{m-1}$. Hence distinguished actions implies that the two sequences are in the same information set H. Further, using sequences and [1] imply ${}_1\dot{x}_{\ell-1} \prec {}_1\dot{x}_{m-1}$. The previous two sentences contradict no-absentmindedness.

(b). By inspection, $R(\dot{y}){\smallsetminus}R({}_1\dot{y}_{|\dot{y}|-1})$ can have no more than one element. Thus it suffices to show that the set contains $\dot{y}_{|\dot{y}|}$. For this, it suffices to show that $\dot{y}_{|\dot{y}|} \notin R({}_1\dot{y}_{|\dot{y}|-1})$. Toward that end, note that the alternative $\dot{y}_{|\dot{y}|} \in R({}_1\dot{y}_{|\dot{y}|-1})$ implies $R(\dot{y}) = R({}_1\dot{y}_{|\dot{y}|-1})$, which implies $|R(\dot{y})| = |R({}_1\dot{y}_{|\dot{y}|-1})|$, which by part (a) implies $|\dot{y}| = |{}_1\dot{y}_{|\dot{y}|-1}|$, which implies the falsehood $|\dot{y}| = |\dot{y}|{-}1$.

(c). Take $\dot{x} \in \dot{X}$ and $\dot{y} \in \dot{X}$. Suppose [1] $R(\dot{x}) \subseteq R(\dot{y})$ and [2] $\dot{x} \neq {}_1\dot{y}_{|\dot{x}|}$. Part (a) and [1] imply $|\dot{x}| \leq |\dot{y}|$, and thus [2] implies there is $m \in \{1, 2, ... |\dot{x}|\}$ such that $\dot{x}_m \neq \dot{y}_m$. Let ℓ be the least such m. Then [3] ${}_1\dot{x}_{\ell-1} = {}_1\dot{y}_{\ell-1}$ and [4] $\dot{x}_\ell \neq \dot{y}_\ell$. Using sequences implies $\dot{y}_\ell \in \dot{F}({}_1\dot{y}_{\ell-1})$ and thus [3] implies [5] $\dot{y}_\ell \in \dot{F}({}_1\dot{x}_{\ell-1})$. Further, [1] and [4] imply there is an [6] $\ell_* \neq \ell$ such that $\dot{x}_{\ell_*} = \dot{y}_\ell$. Thus [5] implies $\dot{x}_{\ell_*} \in \dot{F}({}_1\dot{x}_{\ell-1})$. At the same time, using sequences implies $\dot{x}_{\ell_*} \in \dot{F}({}_1\dot{x}_{\ell_*-1})$. The previous two sentences imply that $\dot{x}_{\ell_*}$ is feasible from both ${}_1\dot{x}_{\ell-1}$ and ${}_1\dot{x}_{\ell_*-1}$. Thus distinguished actions imply that the two sequences belong to the same information set H. Further, [6] implies ${}_1\dot{x}_{\ell-1} \prec {}_1\dot{x}_{\ell_*-1}$ or ${}_1\dot{x}_{\ell_*-1} \prec {}_1\dot{x}_{\ell-1}$. Either contingency together with the second-previous sentence leads to a contradiction of no-absentmindedness.

(d). It suffices to show $(\forall \dot{x}{\in}\dot{X}, \dot{y}{\in}\dot{X})\ R(\dot{x}) = R(\dot{y}) \Rightarrow \dot{x} = \dot{y}$. Suppose $\dot{x} \in \dot{X}$ and $\dot{y} \in \dot{X}$ are such that [1] $R(\dot{x}) = R(\dot{y})$. Then $|R(\dot{x})| = |R(\dot{y})|$, so part (a) implies $|\dot{x}| = |\dot{y}|$. In steps, $\dot{x}$ by [1] and part (c) is equal to ${}_1\dot{y}_{|\dot{x}|}$, which by the previous sentence is equal to ${}_1\dot{y}_{|\dot{y}|}$, which is $\dot{y}$. □

Lemma D.12. *Each distinguished-action sequence game with no-absentmindedness is isomorphic to an action-set game.*

Proof. Consider a distinguished-action sequence game $\dot{\Gamma}$ with no-absentmindedness. Lemma D.11(d) shows that $R|_{\dot{X}}$ is injective, where R is the function, from finite sequences to finite sets, that takes each sequence to its range. Define $X = \{\, R(\dot{x}) \,|\, \dot{x}{\in}\dot{X} \,\}$. Since $R|_{\dot{X}}$ is injective, $R|_{\dot{X},X}$ is a bijection. So use $R|_{\dot{X},X}$ and Lemma D.4 to construct a game Γ such that $[\dot{\Gamma}, \Gamma, R|_{\dot{X},X}]$ is an isomorphism. Thus it suffices to show that Γ is an action-set game. This follows from Claims 1–3.

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[30]This lemma extends part of Streufert 2020a, Lemma B.4.

[31]To be clear, $R(\{\}) = \{\}$, and for any nonempty sequence $\dot{x} = (\dot{x}_1, \dot{x}_2, ... \dot{x}_m)$, $R(\dot{x}) = \{\, \dot{x}_\ell \mid \ell{\in}\{1, 2, ... m\} \,\}$.

Claim 1: X is a collection of finite sets which contains $\{\}$. By the assumption of using sequences, $\dot{X}$ is a collection of finite sequences. Thus the previous paragraph's definition of X implies X is a collection of finite sets. Further, the assumption of using sequences implies $\{\}$ is the root node of $\dot{\Gamma}$, and thus $R(\{\}) = \{\}$ is a member of X.

Claim 2: $E = \{\, xy{\in}X^2 \,|\, x{\subseteq}y,\ |y{\smallsetminus}x|{=}1 \,\}$. For the forward direction, take $xy \in E$. Then [C1] for Γ implies $xy \in X^2$. Further, Lemma D.4's definition of E implies there is [a] $\dot{x}\dot{y} \in \dot{E}$ such that [b] $R(\dot{x})R(\dot{y}) = xy$. Using sequences and [a] implies $\dot{x} = {}_1\dot{y}_{|\dot{y}|-1}$. This implies [c] $R(\dot{x}) \subseteq R(\dot{y})$. It also implies $|\dot{x}| = |{}_1\dot{y}_{|\dot{y}|-1}|$, which by Lemma D.11(a) implies $|R(\dot{x})| = |R(\dot{y})|{-}1$, which by [c] implies [d] $|R(\dot{y}){\smallsetminus}R(\dot{x})| = 1$. Finally, [b]–[d] imply $x \subseteq y$ and $|y{\smallsetminus}x| = 1$.

For the reverse direction, suppose [1] $xy \in X^2$, [2] $x \subseteq y$, and [3] $|y{\smallsetminus}x| = 1$. By [1] and the opening paragraph's definition of X, there exist $\dot{x} \in \dot{X}$ and $\dot{y} \in \dot{X}$ such that $x = R(\dot{x})$ and $y = R(\dot{y})$. Thus [2] and [3] imply [4] $R(\dot{x}) \subseteq R(\dot{y})$ and [5] $|R(\dot{y}){\smallsetminus}R(\dot{x})| = 1$. [4] and [5] imply $|R(\dot{x})| = |R(\dot{y})|{-}1$, which by Lemma D.11(a) implies $|\dot{x}| = |\dot{y}| - 1$. In steps, Lemma D.11(c) and [4] implies $\dot{x} = {}_1\dot{y}_{|\dot{x}|}$, which by the previous sentence implies $\dot{x} = {}_1\dot{y}_{|\dot{y}|-1}$, which by using sequences implies $\dot{x}\dot{y} \in \dot{E}$. Thus this paragraph's definition of $\dot{x}$ and $\dot{y}$ and Lemma D.4's definition of E imply $xy \in E$.

Claim 3: $(\forall xy{\in}E)\ \{\lambda(xy)\} = y{\smallsetminus}x$. Take $xy \in E$. Then Lemma D.4's definition of E implies there is [a] $\dot{x}\dot{y} \in \dot{E}$ such that [b] $R(\dot{x})R(\dot{y}) = xy$. Using sequences and [a] implies both [c] $\dot{x} = {}_1\dot{y}_{|\dot{y}|-1}$ and [d] $\dot{\lambda}(\dot{x}\dot{y}) = \dot{y}_{|\dot{y}|}$. Also note [e] $\lambda(xy) = \dot{\lambda}(\dot{x}\dot{y})$, because $\lambda(xy)$ by Lemma D.4(a)'s definition of λ is equal to $\dot{\lambda}((R|_{\dot{X},X})^{-1}(x)(R|_{\dot{X},X})^{-1}(y))$, which by [b] is equal to $\dot{\lambda}(\dot{x}\dot{y})$. Finally, to conclude, $y{\smallsetminus}x$ by [b] is equal to $R(\dot{y}){\smallsetminus}R(\dot{x})$, which by [c] is equal to $R(\dot{y}){\smallsetminus}R({}_1\dot{y}_{|\dot{y}|-1})$, which by Lemma D.11(b) is equal to $\{\dot{y}_{|\dot{y}|}\}$, which by [d] is equal to $\{\dot{\lambda}(\dot{x}\dot{y})\}$, which by [e] is equal to $\{\lambda(xy)\}$. □

Lemma D.13. *Each game with no-absentmindedness is isomorphic to an action-set game.*

Proof. Suppose Γ^* is a game with no-absentmindedness. Then Lemma D.7 implies that Γ^* is isomorphic to a distinguished-action sequence game $\dot{\Gamma}$. Since no-absentmindedness is invariant by Theorem 5.3(a), $\dot{\Gamma}$ has no-absentmindedness. Thus Lemma D.12 implies that $\dot{\Gamma}$ is isomorphic to an action-set game. In summary, Γ^* is isomorphic to $\dot{\Gamma}$, which is isomorphic to an action-set game. □

Lemma D.14. *Suppose Θ is an action-set CLT. Then Θ satisfies no-absentmindedness.*

Proof. Suppose Θ violates no-absentmindedness. Then there are [a] $H \in \mathcal{H}$, [b] $x \in H$, and [c] $y \in H$ such that [d] $x \prec y$. Note [d] implies there is a nontrivial path $\{x_0, x_1, ... x_{\ell-1}, x_\ell\}$ from $x_0 = x$ to $x_\ell = y$. The nontriviality of the path implies [e] $\ell \geq 1$. Thus $x_0x_1 \in E$, which implies $xx_1 \in E$, which implies $a = \lambda(xx_1)$ is well-defined.

Note $a = \lambda(xx_1)$ and the definition of using action sets implies [f] $a \in x_1$. By [e], $\ell = 1$ or $\ell > 1$. If $\ell = 1$, $x_1 = y$. If $\ell > 1$, $\{x_1x_2, x_2x_3, ... x_{\ell-1}x_\ell\}$ is a subset of E, and hence the definition of using action sets implies $x_1 \subseteq x_\ell = y$. Thus in either event, $x_1 \subseteq y$, which by [f] implies [g] $a \in y$.

Meanwhile, $a = \lambda(xx_1)$ and the general definition of F imply $a \in F(x)$. Thus [a]–[c] and [C4] imply $a \in F(y)$. Hence the general definition of F implies there is $y_+ \in X$ such that $\lambda(yy_+) = a$, which by using action sets implies $a \notin y$. This contradicts [g]. □

Proof D.15 (for Theorem 5.5)**.** Lemma D.14 implies that each action-set game is a game with no-absentmindedness. Thus **AGm** is a full subcategory of $\mathbf{Gm}_{\tilde{\mathbf{a}}}$. Hence the theorem follows from Lemma D.13. □

Proof D.16 (for Corollary 5.6)**.** Easily, $\mathbf{AGm_p}$ is a full subcategory of $\mathbf{Gm_p}$. Thus it suffices to show that every game with perfect-information is isomorphic to an action-set game with perfect-information. Toward that end, suppose Γ is a game with perfect-information. Since, perfect-information implies no-absentmindedness, Lemma D.13 implies that Γ is isomorphic to an action-set game. Since perfect-information is invariant by Theorem 5.3(b), this action-set game has perfect-information. □

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