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Teaching Mathematics: Heuristics Can and Ought to Lead the Way

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Synopsis

In contrast to problem-solving procedures that are the bricks and mortar of demonstrations in mathematics textbooks, heuristics, defined by Polya as "the study of means and methods of problem solving", are those mental actions that enable the practitioner to make progress when it is not clear how to solve problems directly. Yet, as essential as heuristic tools are, they tend not to be included in presentations in mathematics textbooks. The overarching problem can be understood in terms of students' not developing productive means for engaging problems. A few mathematics problems are included to argue for the validity, if not the priority, of the need for the incorporation of heuristics along with problem-solving procedures as standard content in mathematics textbooks.

Keywords: heuristics; language of investigation vs. language of description.

1. Introduction

In contrast to problem-solving procedures that are the bricks and mortar of demonstrations in mathematics textbooks, heuristics, defined by Polya as "the study of means and methods of problem solving" [13, Volume 1 page *vi*], are those mental actions that enable the practitioner to make progress when it is not clear how to solve problems directly. It is those mental actions that are essential for "productive struggles", which Hiebert and Grouws describe as representing the "effort to make sense of mathematics, to figure something out that is not immediately apparent" [6, page 387]. And the National Council of Teachers of Mathematics (NCTM) supports students engaging mathematics situations for investigation and exploration, and sees "struggles" as opportunities for "delving deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions" [10, page 48].

Some standard heuristic tools include *tinkering, describing, taking things apart, reasoning by analogy, trial and error*, etc. Yet, as essential as they are, these types of tools tend not to be included in presentations in mathematics textbooks. While problem solving has been said to be at the heart of mathematics (see for example [5]), and has been extensively written about by Polya [11, 12, 13] and others (see for example [9, 15]), heuristics "has not been given the focus it deserves, and so teachers of mathematics have not been educated in a systematic approach to heuristics" [13, page *viii*]. Instead mathematics textbooks and mathematics classroom focus tend to be on teaching problem-solving techniques and algorithms where "the tasks can be solved according to the provided template without any conceptual understanding of the actual problem" [7, page 21]. This pedagogical format has been recognized as problematic by the Conference Board of the Mathematical Sciences in *The Mathematical Education of Teachers II*, where the authors expressed their considerable concern (restating a statement from MET I and adding their emphasis): "For many prospective teachers, learning mathematics has meant *only* learning its procedures and, they may have been rewarded with high grades in mathematics for their fluency using procedures" [2, page 11]. Thus that particular classroom practice emphasizing procedural learning almost exclusively tends to be promulgated and students rewarded for being proficient at memorizing algorithms and other problem-solving techniques.

The overarching problem I'm trying to point toward here can be understood best in terms of students not developing productive means of engaging problems so as to make headway when it is not clear how to proceed. That is the value of having an educated intuition, where heuristic thinking is an essential part of the mathematics experience. Were heuristic elements to be incorporated on a regular basis, students would have the opportunity to gain greater insight into how the presented mathematics came to be, and in so doing become more educated and confident practitioners in the art and science of mathematical investigation.

A possible explanation for the omission of heuristic tools in the classroom practice of most teachers is that what tends to be presented in mathematics textbooks and as a consequence mathematics classrooms are demonstrations, and as such are elements of a language of presentation. The content is a representation of what has come to be known, not what was involved in coming to know. The latter requires a language of investigation, where problem-clarifying strategies (in contrast to problem-solving procedures, see [4]) help the student investigator gain direction when no explicit procedure or argument is evident. This missing language of investigation is heuristics. While the language of mathematics in textbooks involves expressions of mathematical actions that were taken — the steps of an algorithm, problem-solving procedure being applied, mathematical proofs being presented—the language of heuristics would be the language for engaging with mathematics in the absence of the finished material. The latter is in effect a meta-language, and consequently tends to be omitted from standard mathematics textbooks. From early grades through university, the problem is clear: "Finished mathematics presented in a finished form appears as purely demonstrations, consisting of proofs only. Yet mathematics in the making resembles any other human knowledge in the making ... If the learning of mathematics reflects to any degree the invention of mathematics, it must have a place for guessing, for plausible inference" [12, page ıvi]. And that readily draws upon the language of heuristics.

2. Mathematics Problems from a Heuristic Perspective

It is problem-clarifying strategies that serve to make problems simpler, which Keith Devlin has shared (in personal communication) as the way mathematics is essentially done. From this vantage point, problem-clarifying strategies ought to be given space in textbooks. For if students understand that to successfully engage with a new and challenging problem requires them to begin by making the problem simpler, they can appreciate that it is the language of heuristics that will need to be called upon. Polya's imperative to "let them learn guessing" [11] also offers real promise with students having the chance to become aware of heuristic practices. Educators' efforts over time have made clear there are many problem-clarifying strategies that deserve consideration. A few instances are included here to argue for the validity, if not the priority, of the need for their incorporation along with problem-solving procedures as standard content in mathematics textbooks.

2.1. The Harmonic Series and Tinkering

Students encounter the harmonic series in the context of determining whether particular infinite series converge or diverge. Consider the harmonic series:

$$HS = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \cdots .$$

One elegant solution approach that demonstrates the harmonic series diverges involves comparison with another infinite series all of whose corresponding terms are either equal or lesser quantities.

$$HS > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \cdots .$$

This series clearly diverges, as the 3^{rd} and 4^{th} terms sum to 1/2, the 5^{th} through 8^{th} terms sum to 1/2, the 9^{th} through 16^{th} terms sum to 1/2, and so on. More generally, the sum of term $1 + 2^{n-1}$ through the term 2^n is at least 1/2, for all integers $n \geqslant 1$.

This formulation establishes a series of infinite halves plus 1, which increases without bound. Here we have what has been referred to as a "beautiful" demonstration by Oresme that the harmonic series must diverge, as it is clearly greater than the divergent series he created. And students have an opportunity to appreciate the elegance of the demonstration.

However, from a pedagogical perspective we can ask, what does the student take from the demonstration, besides appreciating its cleverness and beauty? Do they have the opportunity to reflect on the notion that the initial impetus for proceeding as demonstrated was to create a simpler problem? Apparently the divergence/convergence of the harmonic series was not found amenable to direct demonstration. Is that mentioned in the accompanying text material? If not, is that not a significant pedagogical omission? It is indeed informative to find that an infinite series whose terms converge to zero actually diverges. But for students to more fully appreciate the art and science of doing mathematics, being informed that a direct approach has apparently not been found would seem worth including. Additionally, it would seem valuable to ask if the problem-clarifying strategies of *tinkering* and *taking things apart* that apparently lie at the base of Oresme's proof was included in the discussion along with the proof in textbooks. If these heuristic practices are omitted from the conversation, then it seems reasonable to ask what exactly the student has come to know after being presented with the demonstration, other than the fact that the harmonic series diverges. As a contemporary mathematics professor asks with regard to Oresme's 14$^{\text{th}}$ century proof, "how would one come up with the idea of grouping more and more terms together?"[1] Without consideration of the heuristics involved, "inspiration" would likely be the response, though, all in all, it is not very informative.

2.2. The Parabolic Curve and Taking Things Apart

While the problem-clarifying strategy of *taking things ap*art doesn't seem to get much if any mention in mathematics textbook presentations, it is a valuable heuristic, serving, for example, to explain the significance of each of the terms in the quadratic equations associated with projectile motion. What follows is an application of *taking things apart* to determine a formula for the height of a projectile, with time replaced by the horizontal distance travelled. The problem can be considered as means for determining the mathematical curves seen as parabolic water arcs of fountains [3].

Students asked to create such a representation discover the need to take into consideration the angle the water exits from the fountain's surface, along with rethinking how to reconstruct the equation as a function of the horizontal distance travelled, and including the initial velocity of the water. The problem requires converting the standard projectile arc formula

$$h(t) = -16t^2 + v_0 t + h_0$$

(in terms of feet travelled), to a function $h(x)$ where $h_0 = 0$, and x represents the horizontal distance travelled. The problem-clarifying strategy of *taking*

[1] This question is asked by Leo Goldmakher, who offers his own proof that the harmonic series diverges at `https://web.Williams.edu/Mathematics/lg5/harmonic.pdf`, last accessed on July 26, 2021.

things apart can help students make the problem simpler, and thus make possible the construction of the appropriate projectile equation(s).

The angle of ascent from the water's surface may be considered without reference to the gravitational factor as a means to simplify the problem. That is, with the heuristic tool of *taking things apart*, the investigation becomes one of considering the right triangle(s) associated with an exit angle, and the new representation of height and distance, where the water location leaving the surface of the fountain at angles A and D. Two right triangles provide representations toward resolution where each provides complementary information (see Figure 1.)

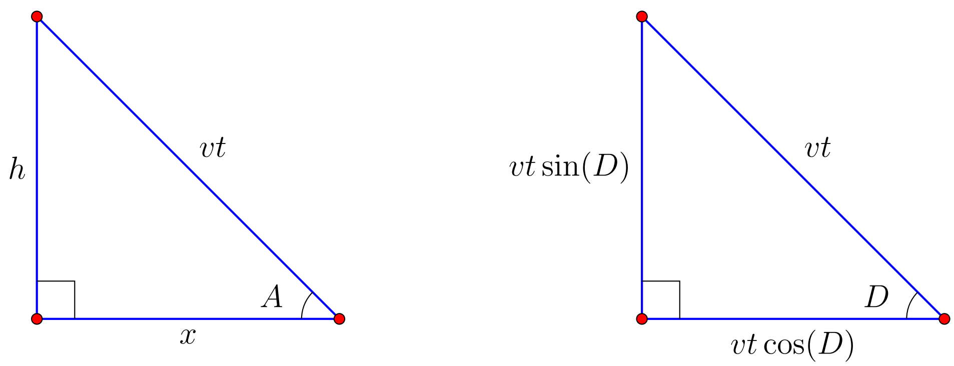

Figure 1: Water shooting up and to the left from the fountain source.

Working with the right triangle on the left in Figure 1, students can establish a new formula for the height as a function of horizontal distance travelled absent gravity's effect, arriving at $h(x) = x\tan(A)$. Then reintroducing the force of gravitational attraction, and expressing the gravity term $-16t^2$ in terms of x, which requires noting that

$$t = \frac{x}{v\cos(A)},$$

we get

$$h(x) = \text{ gravity term } + \text{ no-gravity term } = \frac{-16x^2}{(v\cos(A))^2} + x\tan A.$$

The right triangle on the right side in the figure can be used to establish parametric equations $x(t) = vt\cos(D)$ and for the vertical dimension accommodating for gravity,

$$h(t) = vt\sin(D) - 16t^2.$$

The first representation considered determines directly the height of the parabolic fountain arc as a function of the horizontal distance from the point where the water exited the fountain surface and the chosen exit angle and initial velocity. The parametric equations set provides further application, as students can now consider water jets, which mathematically speaking are degenerate parabolas (where x = 0). Looking back, students can appreciate that both solution forms came into existence by making the problem simpler via *taking the problem apart*.

2.3. The Area of a Simple Closed Curve and Trial and Error

It is often the case that mathematics students learn at some point that, when looking for the root of an algebraic equation, the method of false position is more effective than the bisection method. Yet college mathematics textbooks tend not to mention that the mental action that established the more effective method was apparently one of *trial and error*. The following investigation provides an opportunity for students to establish a variation of the formula for the average value of a function, not as a consequence of being presented with the Mean-Value Theorem, but by drawing upon *trial and error* and *reasoning by analogy*.

The impetus for the investigation could come from trying to find the dimensions of a rectangle that would be a good fit for determining the area of a simple closed curve. The initial consideration could be established with noting, for example, that the given areas of any of the Great Lakes in the United States differ considerably from the area of the rectangle defined by the maximum width and height provided as the associated dimensions for each. Seeing the circumscribed rectangle (see Figure 2) makes it clear that that rectangle provides a poor approximation to the area of the simple closed curve.

In *tinkering* with the problem, students can come up with a better approximation by fitting a trapezoidal region to the simple closed curve (see Figure 3).

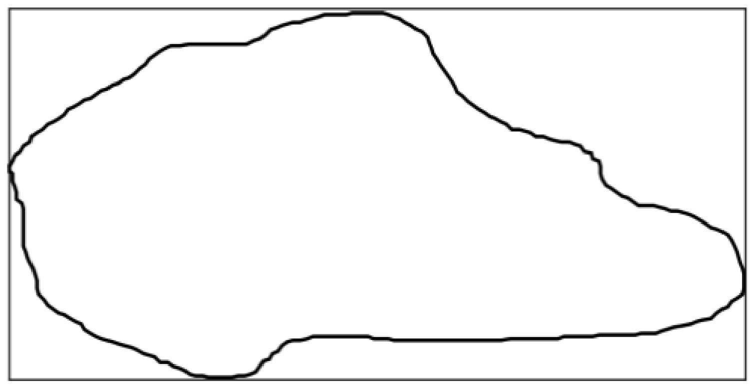

Figure 2: A simple closed curve and a bounding rectangle.

While the problem asks for a rectangle that would provide the approximation, it's clear that *tinkering* with the trapezoidal form provides a decent approximation. So *reasoning by analogy* suggests considering a rectangle associated with the formula for the area of a trapezoid.

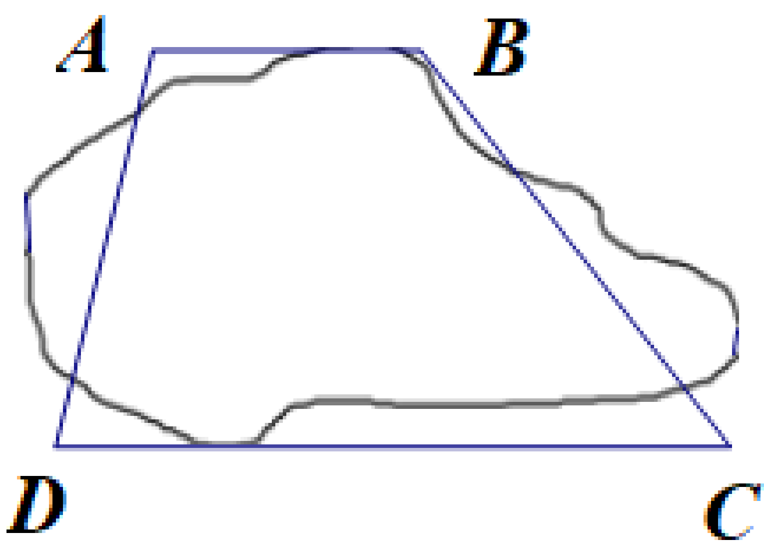

Figure 3: A simple closed curve and an approximating trapezoid.

While the formula for the area of a trapezoid is usually written as

$$\frac{h}{2}(b_1 + b_2),$$

if we rewrite it as

$$h\left(\frac{b_1 + b_2}{2}\right),$$

students can come to realize that the latter expression provides a way to rethink the area solution in terms of a rectangle. Namely, the curved region area can be approximated by finding the maximum height x times the average of the bases of the trapezoid. And this suggests a rectangle whose height would be the maximum height of the curved shape, in either the horizontal or vertical dimension, and whose width would be determined by the average of a series of parallel segments drawn perpendicular to the height direction, with the more chosen the better the approximation.

What students uncover in essence is the formula for determining the average height of a function. But instead of being introduced to the formula via the Mean-Value Theorem:

$$\bar{y} = \frac{\int_a^b f(x)\ dx}{b - a},$$

they work with the problem-clarifying strategies of *trial and er*ror and *reasoning by analogy.* In this way they come to gain a deep sense in understanding and appreciating the formula: It's quite extraordinary to be able to determine the average length of an infinite number of segments. And valuable as well to be able to find an excellent approximation to the area of a simple closed curve, including a pond in the neighborhood, or a Great Lake, by applying heuristics.

2.4. Problem Solving with Describing Carefully

Presenting visualizations is a time-honored means for demonstrating mathematical arguments (recall Bhaskara's "Behold!"), and for encouraging the activity of looking carefully mathematically as well. It is the latter consideration that makes *describing* a valuable problem-clarifying strategy as the student has to look with care and imagination to secure understanding.

While the jury is still out regarding whether visual demonstration is acceptable as mathematical proof (see, for example, the related discussion in [14]), I share below two instantiations of visualization to demonstrate its pedagogical value: engagement with a visualization requires students to visually *take things apart* and draw upon their deductive analysis skills in putting the argument back together, a most valuable quality when engaging in mathematics.

Figure 4 is Apostol's lovely visual proof by contradiction [1] that the square root of two is irrational. The reader is informed that the figure represents the *smallest* right-angled isosceles triangle with integer sides. Then a circle with radius equal in length to the shorter sides of the triangle is drawn along with the tangent to the circle where the circle cuts the hypotenuse. The *smaller* isosceles triangle can be seen to have integer sides.

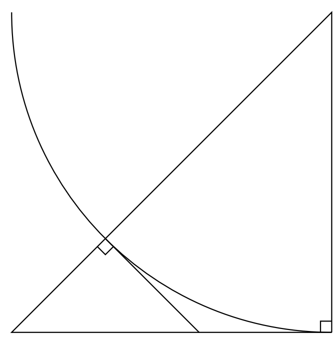

Figure 4: Apostol's visual proof by contradiction that $\sqrt{2}$ is irrational [1]. For an isoceles right triangle with integer sides, there is always a smaller such triangle with integer sides.

Another instance of *describing* to promote students' mathematical thinking is provided when they are asked to draw a representation of the first quadrant of the Cartesian coordinate plane and then asked to add the portion of the line $y = x$ that falls within the first quadrant, and as such, splits the set of all possible lines with positive slopes into two equally sized sections: the lines between the x-axis and the line $y = x$ (that is, lines with slopes $(0, 1]$), and the lines between the line $y = x$ and the y-axis (that is, lines with slopes $[1, \infty)$). Doing so, students can discover they have demonstrated the equipotence of the two sets of real numbers represented by $(0, 1]$ and $[1, \infty)$. Here too, students have an opportunity to discover the argument as a consequence of the heuristic of *describing carefully.*

3. Discussion

Mathematics textbooks, focusing primarily on demonstrations of efficiency and elegance, often leave students at a loss with respect to gaining understanding for want of a "thicker" description. I have argued here that a critical source of the difficulty is that formal demonstrations tend to be absent of problem-clarifying actions, as the presentations represent the conclusion of dedicated, thoughtful, imaginative effort, with all the scaffolding unfortunately removed. Thus students might feel discouraged as they do not see how they themselves could get to similar results. However, "[i]f students feel successful and in control, they tend to have a more positive relationship with the material they are working on" [8]. Toward supporting students in gaining a more complete understanding, and so developing greater confidence, I have suggested here that helping them think in terms of making the problem simpler and seeking some productive heuristic(s) could well provide a fruitful avenue in their problem-solving efforts and mathematics reading. I have argued that including heuristics will enable students to make headway when no particular problem-solving technique is apparent. "Innocent though this conclusion may seem, it is of fundamental importance" [16, page 28]. Hopefully the discussion presented here makes a reasonable case to think that heuristics, in the form of problem-clarifying strategies, deserve to be at the heart of mathematics textbook presentations.

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