

The List: Proverbs for Calculus

Bruce H. Pourciau
Lawrence University

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The List: Proverbs for Calculus

Cover Page Footnote

Meticulous editing by my Best Man, Dr. Selden Crary, caught numerous errors in an earlier draft. To my students, over forty years of them, I owe everything.

The List: Proverbs for Calculus

Bruce H. Pourciau

Department of Mathematics, Lawrence University, Wisconsin, USA
bruce.h.pourciau@lawrence.edu

Synopsis

Topics chosen from first-year calculus illustrate a number of “sayings” or “proverbs,” the first three, for example, being: be awed, like a child; meaning before truth; and act with intention. Many are proverbs for life as well as mathematics.

Keywords: calculus, proverbs, sayings.

0. THE LIST

After some years of teaching, I found myself repeating certain “sayings” or “proverbs,” each capturing one facet of the “mathematical way of thinking.” As the years went by, the number of these proverbs grew, and eventually I collected them in a list which I would hand out on the first day of class. They are proverbs for any mathematics class, not just calculus. Some are proverbs for life.

- | | |
|----------------------------------|---|
| 1 BE AWED, LIKE A CHILD | 9 GUESS BRAVELY AND BEAUTIFULLY |
| 2 MEANING BEFORE TRUTH | 10 UNDERSTANDING IS TRANSLATING |
| 3 ACT WITH INTENTION | 11 CELEBRATE YOUR MISTAKES |
| 4 ACCOUNT FOR ASSUMPTIONS | 12 EXPLORE THE TUG OF WAR |
| 5 GOD GIVES, WE CHOOSE | 13 TOO BEAUTIFUL TO BE FALSE |
| 6 MAKE YOUR DREAMS COME TRUE | 14 DO YOU BELIEVE IN MAGIC? |
| 7 PLAUSIBILITY BEFORE PROOF | 15 FOLLOW THE VEIL: BE MOVED BY MYSTERY |
| 8 PRINCESS DI THEN ATTICUS FINCH | 16 PHILOSOPHY MATTERS |

Though a given saying might surface several times over a semester, here we illustrate each proverb with just one calculus topic.

1. BE AWED, LIKE A CHILD

If I had influence with the good fairy... I should ask that her gift to each child in the world be a sense of wonder so indestructible that it would last throughout life. – RACHEL CARSON [5, page 44]

Ah, the first day of class. So much joy, excitement, and mystery ahead of us! Do you remember the last time you felt genuine awe? Do you remember the *first* time? I do. I was only six or seven, lying on my back in the grass at night, the stars of the Milky Way spread out across the black sky. And the next morning my dad said, “you know, the sun is a star.” Talk about having your mind blown twice in the span of one day!

What happens to us? Where does that awe go? Do we start to worry that we look silly with our mouths open? Does the awe get schooled out of us? Really, shouldn't school be *filling* us with awe? Especially here, now, in college? So many classes to choose from, so many new, sexy subjects: anthropology, classics, cognitive science, data science, economics, gender studies, linguistics, neuroscience, psychology, religious studies, philosophy. Such potential for awe there. But actually, the discipline with the greatest potential for awe is not new. It's the oldest of disciplines, the one you've all studied since you were babies: mathematics.

We all know the stereotypes — the boring high school math teacher whose monotone delivery drones the students to sleep and mathematics portrayed as deadly dry number crunching. Yet nothing, literally nothing, could be further from the truth:

- Mathematics, especially at the upper levels, exhibits a crystalline aesthetic beauty unlike any other discipline. As Bertrand Russell put it, “Mathematics, rightly viewed, possesses not only truth, but supreme beauty — a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show...” [20, page 32]

- Imagination and creativity abound: “There is an astonishing imagination,” said Voltaire, “even in the science of mathematics.... We repeat, there was far more imagination in the head of Archimedes than in that of Homer.” [7, page 126] Unconstrained by the physical universe, the freedom of invention is extraordinary, limited only by what can be imagined. A mathematician can imagine, define, and study intricate and beautiful algebraic structures, higher (even infinite) dimensional exotic spaces and geometries — all not of this world, existing only in the human mind. “There is nothing as dreamy and poetic, nothing as radical, subversive, and psychedelic, as mathematics. It is every bit as mind blowing as cosmology or physics . . . and allows more freedom of expression than poetry, art, or music Mathematics is the purest of the arts, as well as the most misunderstood.” [17, page 23]
- Unlike other disciplines, mathematics puts us in direct contact with the infinite — sets countably and uncountably infinite, sets of infinite length, area, and volume, infinite limits, infinite integrals, infinite sequences, infinite sums, infinite “polynomials,” infinite dimensional spaces — and even gives us the god-like powers to define precisely and control every one of these infinite concepts and processes! “The goal of mathematics,” writes Hermann Weyl, “is the symbolic comprehension of the infinite with human, that is finite, means.” [26, page 12]
- Using axioms, definitions, and logic of pristine clarity, the language of modern mathematics has no ambiguity. Mathematicians communicate their findings across the globe in rigorous proofs with perfect precision. There are no debates in mathematics. When a proof is extremely complex and difficult — Andrew Wiles’s proof of Fermat’s Last Theorem in 1995 was 129 pages long — there may be a pause, while the mathematical community, in seminars around the world, studies the argument looking for flaws — but in the end the final verdict is unanimous. No debate? Perfect precision? Sound like any other discipline? No.
- Modern mathematics has great power and sophistication, and yet many, many deep and exciting problems remain unsolved. Here’s just one: 5 and 7, 11 and 13, 17 and 19 are examples of “twin primes.” Is the number of twin primes finite or infinite? To this day, no one knows.

Of the seven Millennium Prize Problems set out by the Clay Mathematics Institute in 2000, only one had been solved as this writing.

- “[Nature] is written in that great book which ever is before our eyes,” wrote Galileo, “. . . but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in mathematical language.” [11, page 350] Does it surprise us that the Universe has always expressed itself in the language of mathematics? It shouldn’t. To understand the universe, we strive to illuminate its structures and patterns, and *mathematics is the language and study of structure and pattern!* In the mathematics of Einstein’s general theory of relativity, the universe is a four-dimensional spacetime “surface” whose curvature, determined by mass, varies from point to point and where light travels along “curving” paths of least length. In more modern “string theories,” spacetime may have 10, 11, or 25 dimensions! M-theory, in particular, posits extended two-dimensional membranes and five spatial dimensions that reside in a universe of eleven dimensions!
- Even at our level here, in Calculus I, the fantastically varied and seemingly uncountable applications come from almost every area you can think of: acoustics, archeology, biology, business, chemistry, computer science, ecology, economics, engineering, environmental science, government, health science, music theory, neuroscience, medicine, physics, the list goes on. I’m handing out a sample list of some several hundred applications. Let me just read a bunch of them:

predator-prey interaction	detecting art forgeries
modeling climate change	present value of an oil well
firefly synchronization	modeling whale populations
credit card payment structure	surface area of a black hole
spread of infectious diseases	predicting hurricane paths
trajectories of Martian probes	algorithms for search engines
earthquake-proof buildings	dating ancient cave paintings

pollution of marine ecosystems	stability of planetary orbits
⋮	⋮

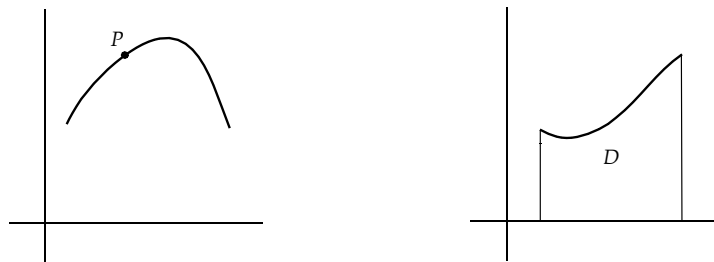
I've just read thirty or so of the roughly five hundred applications on the handout, and that five hundred represents just the tiniest of tiny slivers of what's actually out there. At the end of Calculus I, this is the kind of power you will wield!

If this course rekindles your child-like sense of awe, not just for mathematics, but for all the awesome things this college, this world, this universe, have to offer, I will be very happy.

Over the next weeks, various proverbs will arise in a natural way, each enclosing a kernel of truth concerning this course, and sometimes concerning life in general. The first proverb, BE AWED, LIKE A CHILD, may be the most fundamental.

2. MEANING BEFORE TRUTH

In this list of applications I just handed out, we have glimpsed the enormous power of calculus. Given the incredible number and variety of these applications, it would be natural to assume that calculus must be an absurdly complicated subject. Yet calculus is so beautiful in no small part because it is so *simple*. In fact, unbelievably, nearly all of its power stems from solving just two simply-stated geometric problems: *Find the slope at the point P* and *Find the area of the region D*.



Yes, Rowen? How can that be? That's exactly the question, isn't it? How can it be that all that power and variety can derive from solving these two simple problems? After all, you can't get something from nothing, right?

It just has to be that these two geometric problems, both simple to state, are not actually simple to *solve*! In fact, that's not the half of it, by which I mean: these two problems – *Find the slope at P* and *Find the area of D* – are really just the second halves of two-part problems, the first halves being, *What do we mean by the slope at P ?* and *What do we mean by the area of D ?*

And here we have one of our proverbs: MEANING BEFORE TRUTH, which means, *Do not ask whether a given statement is true until you know what it means.* If, for example, I were to claim that “The floop at point P is 1” or “The glorb of region D is 2,” it would obviously make no sense to debate the *truth* of these statements until we had agreed on the *meanings* of these statements, meanings which depend entirely on the meanings of “floop at point P ” and “glorb of region D .” And the same goes for “slope” and “area.” Of course we have clear meanings for the slope of a line and the area of a rectangle, but we need clear and precise *mathematical meanings* for the “slope at point P ” and the “area of region D ,” before we can *find* that slope or *find* that area. And it is here, in the struggle to formulate these two definitions, precisely and mathematically, where we find the true subtleties of calculus and the true source of its great power. For it is here where the subtle notion of limit gets developed and applied.

This proverb, MEANING BEFORE TRUTH, fundamental everywhere in mathematics, is fundamental in life as well. In so many of our debates, arguments, and everyday back and forth discussions, we use the same words and phrases, but with different meanings, and confused by confusions of meaning we end up talking past each other.

3. ACT WITH INTENTION

Here's a question: What does a mathematician do? Emily? She proves theorems? I like that answer! Well, apart, of course, from department meetings, faculty meetings, committee meetings, grading, lecturing, advising, supervising independent studies, going to talks, meeting with prospective students, responding to emails, and so on, yes, she proves theorems. Technically, though, a theorem is a theorem because it's already been proved. So we might say she proves conjectures, which then become theorems. And how does she come up with reasonable conjectures? She guesses!! Naturally, I

don't mean wild, out-of-the-blue, baseless guesses. That would be a silly waste of time. No, I mean guesses based on plausible reasoning applied to the evidence drawn from examples.

Once proved, a conjecture becomes a theorem. What that theorem might involve would, of course, depend on the area of mathematics it comes from — algebra, analysis, combinatorics, geometry and topology, or number theory — but no matter the area, the most likely objects of study are *functions*. Here in Calculus I and II, a branch of analysis, we investigate functions that take real numbers to real numbers. Calculus theorems can involve a single, particular function, such as —

THEOREM *The function $f : [0, 1] \rightarrow \mathbb{R}$ given by*

$$f(x) := \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

is not integrable.

THEOREM *Extend $h(x) = |x|$ from $[-1, 1]$ to \mathbb{R} by requiring $h(x+2) = h(x)$. Then the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by*

$$f(x) := \sum_{n=0}^{\infty} \frac{h(2^n x)}{2^n}$$

is continuous everywhere yet differentiable nowhere.

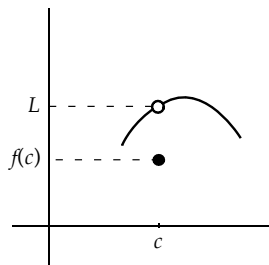
— whatever integrable, continuous, and differentiable mean.

But much more commonly, calculus theorems apply to an entire class of functions. Of course the power of such a theorem increases with both the breadth of the class and the interest and usefulness of its conclusions. Since we are unlikely to be able to say anything interesting or useful about the (too wildly diverse) class of *all* functions, it makes sense to look for a mild restriction of that class which does lead to conclusions both interesting *and* useful. One natural candidate for that restriction, expressed intuitively — that the graph of f is “unbroken” — is certainly not mild enough, for it rules out an entire swath of the elementary functions, all of which we would like to keep in the tent. Even the graph of the simplest rational function $1/x$ is “broken” (at $x = 0$). Yet notice that for every elementary function with a

broken graph, the breaks do not occur at points *of the graph*, but only at points where the function is undefined. The graph of $1/x$, for example, is actually “unbroken” at each of its points. Functions with this property are said to be *continuous*:

INTUITIVE DEFINITION We call a function f *continuous* if its graph is “unbroken” at each of its points, that is, if its graph is “unbroken” at $(x, f(x))$ for each x in the domain of f .

Let’s work toward making this rough, intuitive definition more precise. What do we mean when we say the graph of f is “unbroken” at $(c, f(c))$? Well, what do we mean when we say the graph is “broken” at $(c, f(c))$? Here’s an example of such a break:



In this figure, based on the values $f(x)$ as $x \rightarrow c$, f appears to be “intending” to do one thing at c , namely to take on the value L , but actually *at* c the function f does something entirely different:

$$L \neq f(c)$$

intention of f at $c \neq$ action of f at c

Replacing one intuition with another, we can then rewrite our rough definition of continuity:

INTUITIVE DEFINITION (REVISED) We call a function f *continuous* if
intention of f at $c =$ action of f at c

for each c in the domain of f .

Ah, but remember a core intuition from our days discussing limits: that the limit of f as $x \rightarrow c$ measures the “intention of f at c .” To refresh that memory, let’s imagine that in the minutes leading up to 11 o’clock, instead of

being here in class with you, suppose that I'm outside the president's office, pacing, thinking of asking for a raise, all the while muttering, "At 11 I'm going to knock on this door." In the seconds leading up to 11, I'm still muttering, "At 11 I'm going to knock on this door." And in the milliseconds and microseconds leading up to 11, I'm still muttering, "At 11 I'm going to knock on this door." What would you say is my *intention* at 11? Of course, yes, to knock on the president's door. Similarly, we think of $\lim_{x \rightarrow c} f(x)$ as measuring the "intention of f at c ," because it represents what f seems to be *intending* to do *at* c based, not on what f does *at* c , but rather on what f does at x 's leading up to (and leading down to) and getting arbitrarily close to c .

But back to the president's door, where my intention at 11 was to knock on the door. Now what I actually do *at* 11, my *action* at 11, could very well be different from my intention at 11: perhaps I freeze and do nothing, perhaps I dance a jig, or perhaps my action at 11 matches my intention at 11 and I knock on the door. In this last case, when action matches intention, we might say that I ACTED (CONSISTENT) WITH INTENTION or, more oddly, that I acted continuously at 11.

When action matches intention for functions, the term "continuous" is not odd, but standard:

DEFINITION We say f is *continuous* if $\lim_{x \rightarrow c} f(x) = f(c)$ for every c in the domain of f , where we use the appropriate left or right limit when the domain contains an interval with endpoint c .

We have made our intuitive definition precise. Containing the set of all elementary functions as a very thin slice, the class \mathcal{C} of continuous functions is really quite broad, and with further restrictions we obtain theorems with conclusions both interesting and useful, some fundamental examples being:

THEOREM *The continuous image of an interval is an interval.*

COROLLARY (INTERMEDIATE VALUE THEOREM) *Given a continuous function defined on an interval, every number between two values of the function must also be a value of the function.*

EXTREME VALUE THEOREM *A continuous function on a closed and bounded interval attains global extremes.*

4. ACCOUNT FOR ASSUMPTIONS

Your assumptions are your windows on the world. Scrub them off every once in a while, or the light won't come in. –ALAN ALDA [1]

At the end of our last class, we had stated, as a corollary, a rather famous theorem:

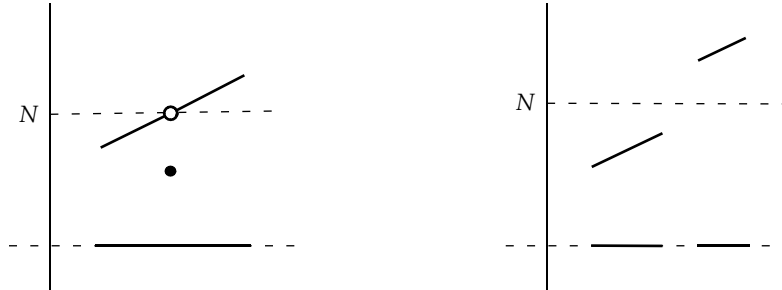
INTERMEDIATE VALUE THEOREM *Given a continuous function defined on an interval, every number between two values of the function must also be a value of the function.*

Now, how should we study such a theorem? Naturally, we should read its statement with great care, perhaps draw a figure to illustrate it, and then go through the proof line by line. If we can follow it, the proof may convince us that the theorem is true — not that most of us would ever really doubt a theorem appearing in a textbook — but more crucially, a theorem asserts a connection between the assumptions and the conclusion, and the proof lays bare the path connecting them. To understand a proof, then, is to have walked that path, taking in and making your own all the twists, turns, and scenery along the way.

But what if we can't understand the proof? What if we can follow each individual step, but cannot anticipate the next, and can't figure out why the steps were taken? Or, like the INTERMEDIATE VALUE THEOREM here, what if the theorem is asserted without proof? How should we study a theorem in such cases?

Whether a theorem is stated with a proof or without, we can always ACCOUNT FOR ASSUMPTIONS, in two opposing directions. First, the study of specific examples where all the assumptions hold can supply some intuition and insight into why those assumptions, taken together, might be *sufficient* for the conclusion. (Of course, only a rigorous proof could replace this “might be” with complete conviction.) And second, the construction of what we call *accounting counterexamples* — one for each assumption, where that assumption only fails, along with the conclusion — will account for each individual assumption by telling us why that assumption is *necessary* for the conclusion. To see, for example, that each assumption of the INTERMEDIATE VALUE THEOREM, interval and continuity, is necessary, we need two accounting counters.

Volunteers? Ezra and Eliah?



Perfect. In the left figure, the domain is an interval, but the function is not continuous, while in the right figure the function is continuous but the domain is not an interval. And in each case, the conclusion fails, for we can see that the number N lies between two values of the function, yet is not itself a value.

The assumptions of a theorem, like the two assumptions of the INTERMEDIATE VALUE THEOREM, are generally quite explicit. But not always. For example, on its face the following statement —

EXTREME VALUE THEOREM *A continuous function on a closed and bounded interval attains global extremes.*

— could be read as asserting that the conclusion holds only for some particular closed and bounded interval and some particular continuous function on that interval. Instead, mathematicians understand this statement to have hidden but implicitly understood “for every” quantifiers: *For every closed and bounded interval and for every continuous function on that interval. . . .*

Other assumptions in mathematics are hidden much more deeply. Nearly all mathematicians, for instance, assume that mathematical assertions refer to “things” — numbers, sets, functions, matrices, spaces, and the like — that stand apart from us and, as a consequence, they assume that a mathematical assertion must be either true or false, quite independently of our knowing (or being able to know) which. We might call these “background assumptions,” for there is no conscious act of accepting these assumptions, no awareness of having made them. Deeply ingrained, these background assumptions shape how we “see” mathematics — sanctioning various procedures, determining

what constitutes a legitimate proof and what makes an important line of inquiry. Later in the course, if we ever encounter some unexpected free time, we can return to this topic (which belongs to the philosophy of mathematics!) for a fuller discussion.

Outside of mathematics, each of us has a personal collection of background assumptions which shape our fears and concerns, our attitudes, arguments, and conclusions, how we see and respond to the world around us. To identify such a deeply entrenched assumption, to remove it, at least temporarily, then to see the world without it and ask, Can I account for this assumption? Is it justified? — this is a very difficult task, often unpleasant, but vital. “Your assumptions are your windows on the world,” as Alan Alda put it, “scrub them off every once in a while, or the light won’t come in.”

5. GOD GIVES, WE CHOOSE

[while tapping a balloon to keep it in the air] We’ve been working with limits for awhile now, calculating specific limits, stating and applying various limit laws, like the SQUEEZE THEOREM, but along the way we have reminded ourselves that we really can’t be *certain* about any of our limit calculations or limit laws. Why? Because the rough and intuitive definitions of limit statements which have served us well so far lack the precision required for the construction of rigorous *proofs*. It’s finally time to fix this serious problem.

Let’s begin with the simplest case: $g(x) \rightarrow +\infty$ as $x \rightarrow +\infty$. [still bouncing the balloon] You know, when I took abstract algebra in college, the professor wore white gloves every day, and no one asked him why, until, on the very last day, someone finally did. Turned out he’d become allergic to chalk! A rather unfortunate allergy for a mathematics professor. Yes, Jon? Why do I keep bouncing this balloon? Gee, I’m glad you asked. It’s a prop, since the real thing is too big to bring into class. At home, I have a magic weather balloon. It’s filled with a gas lighter than helium, lighter than hydrogen, lighter even than a vacuum!

CLAIM *When let go, my magic balloon will rise toward infinity.*

Question: how would you test my claim? Silas? What does this claim even mean? Oh, yes, the perfect counter-question! One of our proverbs, right?
MEANING BEFORE TRUTH: *Do not ask whether a statement is true until you*

know what it means. We cannot debate the truth of my claim until we know precisely what that claim means. But it is exactly that, namely the precise meaning of my claim, which we were trying to tease out with the question, How would you test my claim? Soren? I would agree: we're traversing a circle.

To get off this circle, let's agree to treat my claim — that my balloon will rise toward infinity — intuitively and apply that intuition to see if we can agree on what we should take for the precise meaning of my claim. So, back to my question: How would you test my claim? Laura? Pose some challenging heights? Excellent idea. Let the balloon go and check. Does it rise higher than that tree? Higher than that cloud? Higher than 10 miles? 100 miles? If my magic balloon were able to rise above every one of your specific challenges, you might be very impressed with my balloon, but still doubt my claim, for no matter how many specific challenging heights the balloon surpasses, there could still be some even higher challenging height that it would fail to surpass. So somehow, you would have to become convinced that given *any* arbitrary challenging height, no matter how large, my magic balloon will rise above that height.

PROVISIONAL DEFINITION OF MY CLAIM “my magic balloon will rise toward infinity”: given any challenging height, no matter how high, the balloon will rise above that height.

Let's examine this provisional definition to see if it matches our intuitive understanding of my claim. Suppose my balloon kept rising above the challenging heights, but in between those high flying times it kept sinking back down to the ground. Note that this odd behavior would be permitted by our provisional definition, but does it correspond with the intuitive meaning we assign to “rising toward infinity”? Maybe, maybe not. Do we want to rule out this odd behavior by revising our provisional definition? It's totally up to us, for we are the masters of meaning:

“When I use a word,” Humpty Dumpty said in rather a scornful tone, “it means just what I choose it to mean — neither more nor less.” “The question is,” said Alice, “whether you can make words mean so many different things.” “The question is,” said Humpty Dumpty, “which is to be master — that's all.” [4, page 57]

Shall we put it to a vote? Raise your hand if you think we should revise our provisional definition. OK, let's revise it. We need to make sure that the balloon can't keep coming back down. Said differently, given any challenging height, eventually the balloon must rise above that height and stay above that height.

DEFINITION OF MY CLAIM “my magic balloon will rise toward infinity”: given any challenging height, no matter how high, the balloon eventually will rise above that height to stay. More explicitly, given any challenging height E , there exists a time D such that for every time $t > D$, the balloon will be above E .

Now let's apply this line of reasoning to define our limit statement:

DEFINITION The assertion $g(x) \rightarrow +\infty$ as $x \rightarrow +\infty$ means: given any challenging measure $E > 0$ of bigness of $g(x)$, we can choose some responding measure $D > 0$ of bigness of x such that for every x ,

$$x > D \text{ implies } g(x) > E.$$

The wordy phrasing of this definition highlights its challenge-response character, but we can be more concise:

DEFINITION The assertion $g(x) \rightarrow +\infty$ as $x \rightarrow +\infty$ means: given any $E > 0$, we can choose some $D > 0$ such that for every x ,

$$x > D \text{ implies } g(x) > E.$$

Note that E is given first and is arbitrary, while D is chosen, second, and is definitely *not* arbitrary, for it is chosen in response to and is dependent on the given E . To stress this logic, I like to say: first, E is *given* by God, then, second, some D is *chosen* by us in response. GOD GIVES, WE CHOOSE.

Let's press PAUSE for a minute and think about what just happened. An assertion about an “infinite process,” $g(x) \rightarrow +\infty$ as $x \rightarrow +\infty$, having at first just a rough intuitive sense, has been assigned a perfectly precise mathematical meaning in completely finite terms: given any $E > 0$, we must be able to produce some $D > 0$ such that one inequality implies another inequality. What Hermann Weyl wrote so beautifully bears repeating: “The goal of mathematics is the symbolic comprehension of the infinite with human, that is finite, means.” [26, page 12]

The rough definitions we have given for other limit statements can be made precise in the same way, except that the challenge or the response may be a measure of closeness rather than bigness. For instance:

DEFINITION For numbers L and c , the assertion $g(x) \rightarrow L$ as $x \rightarrow c^+$ means: given any $\varepsilon > 0$, we can choose some $\delta > 0$ such that for every x ,

$$c < x < c + \delta \text{ implies } L - \varepsilon < g(x) < L + \varepsilon$$

Here the challenge ε is a measure of closeness to L , and the response δ is a measure of closeness to c . For the measures of closeness, the traditional ε and δ follow Cauchy ([6], also see [12, page 44]), while for the measures of bigness, there is no tradition, but the E and D follow Stein [22, page 88]. Given the pleasing big-small and Roman-Greek correspondences in the pairings E - ε and D - δ , the letters E and D really *should* be traditional.

As E is a measure of bigness and ε a measure of smallness, we sometimes see descriptive phrases like, “where E can be arbitrarily large” or “where ε can be made arbitrarily small.” Which reminds me of a story involving the English mathematicians G. H. Hardy and J. E. Littlewood, who, during the first half of the 20th century, formed the most famous and productive mathematical collaboration in history.

One Sunday morning, in Littlewood’s rooms at Cambridge, Hardy, reading over the page proofs for one of their papers, called Littlewood over, pointed to a blank spot, and asked, “Shouldn’t there be an ε there?” Littlewood nodded, then looked more closely. “What’s that?” he said. “What?” Hardy replied. Littlewood retrieved a magnifying glass from his desk, set it over the blank spot, where a tiny speck, enlarged, resolved itself into the tiniest ε either of them had ever seen. The text that followed in the page proof, “where ε can be arbitrarily small,” had apparently been taken quite literally by the printer!

Before class, because I knew I’d want to tell this funny story, I looked for the source. The closest thing I could find was the following account in the delightful little book, *Littlewood’s Miscellany*:

A minute I wrote (about 1917) for the Ballistic Office ended with the sentence ‘Thus σ should be made as small as possible.’ This did not appear in the printed minute, but P. J. Grigg said, ‘What

is that?’ A speck in a blank space at the end proved to be the tiniest σ I had ever seen (the printers must have scoured London for it). [16, page 56]

Apparently, the story I’ve just told you, set in Littlewood’s rooms, the story I’ve been telling generations of students, is mostly fiction, based on a tiny speck of truth — though even that speck, first a σ then an ε , got altered in my telling. So we have Littlewood’s σ anecdote, presumably true, and we have my ε story, not true, but more beautiful, like a Hollywood movie adaptation. Following Hermann Weyl — “My work has always tried to unite the True with the Beautiful and when I had to choose one or the other, I usually chose the Beautiful” [8]– I like my story better.

GOD GIVES, YOU CHOOSE. Always remember, your intellect is a gift from God. Others, not so graced, long for what you have been given. Choose a life and a way of living that honor this gift.

6. MAKE YOUR DREAMS COME TRUE (THEN MAKE AMENDS).

In the broad light of day mathematicians check their equations and their proofs, leaving no stone unturned in their search for rigour. But, at night, under the full moon, they dream, they float among the stars and wonder at the miracle of the heavens. They are inspired. Without dreams there is no art, no mathematics, no life. –MICHAEL ATIYAH [2, page 267]

We’ve just seen that the derivative of a sum is the sum of the derivatives. Let’s turn to products. Since $(f + g)' = f' + g'$, we might initially wonder if $(fg)' = f'g'$, but simple counterexamples say no: $(x^2)' = 2x$ but $x'x' = 1$. What then *is* the correct formula for $(fg)'$? Because the formula for $(fg)'$ surely involves f' and g' , we expect the Newton quotient for fg involves, or can be made to involve, the Newton quotients for f and g . If $p(x) = f(x)g(x)$, then

$$\frac{p(x+h) - p(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Do we see the Newton quotient for, say, f ? Yes! No! Well, sort of, but we can’t factor it out, because the factors $g(x+h)$ and $g(x)$ are different.

In our dreams, these two factors would be the same so we could pull out the Newton quotient for f :

$$\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} = \frac{f(x+h) - f(x)}{h} \cdot g(x+h)$$

Our subconscious created this dream expression from reality, from the Newton quotient for the product, by subtracting $f(x)g(x+h)$ instead of $f(x)g(x)$. This bull-in-the-china-shop behavior has broken the tie with reality — duh, it's a dream — but we can repair the broken equality with the Newton quotient and still MAKE OUR DREAMS COME TRUE by undoing (MAKING AMENDS for) the aggressive actions of our subconscious, that is, by undoing the subtraction of $f(x)g(x+h)$ and redoing the subtraction of $f(x)g(x)$:

$$\begin{aligned} \frac{p(x+h) - p(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \\ &\rightarrow f'(x)g(x) + f(x)g'(x) \end{aligned}$$

Although we did this subtract-add trick to make the Newton quotient for f appear, note that the Newton quotient for g appeared as well, without our even trying! By the way, some textbooks write this PRODUCT RULE as $(fg)' = fg' + gf'$, instead of $f'g + fg'$. The latter is better: it points us toward the correct extension for longer products — for example, $(fgh)' = f'gh + fg'h + fgh'$ — and with just a change of sign it gives us the numerator of the QUOTIENT RULE.

7. PLAUSIBILITY BEFORE PROOF

Does anyone remember what we mean by an “elementary function”? Soren? Yes, exactly right:

DEFINITION An *elementary function* is (a) a polynomial, exponential function, logarithmic function, trigonometric function, or inverse trigonometric function or (b) a finite combination of the functions in (a) using addition, subtraction, multiplication, division, and composition.

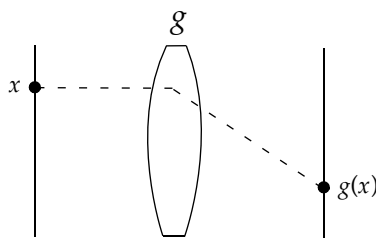
From the day we first introduced the derivative, we have wanted to get to the point where we could differentiate every elementary function. That goal

is now within reach. At this point, we can differentiate every function in category (a) and, with just one omission, all the combinations in category (b). That omission, though, is glaring: the derivative of a *composite*.

So how do we find the rule for differentiating a composite: $(f \circ g)' = ?$ We could of course just look it up in our textbook, but if we really want to find the rule ourselves, a textbook might be the last place to look for advice. Textbooks create the illusion that theorems, like colorful pieces of beach glass, just appear in front of us, already formed and smoothed, ready to be studied, proved, and applied. Textbooks mostly ignore a central part of mathematics — making conjectures.

To make a conjecture, we could begin by studying many examples, hoping to see some pattern which might hold generally. In another strategy, we could interpret, physically or geometrically, the concept or quantity being studied, apply our knowledge of how that physical or geometric quantity would behave, and then conjecture that the purely mathematical quantity might behave in the same way.

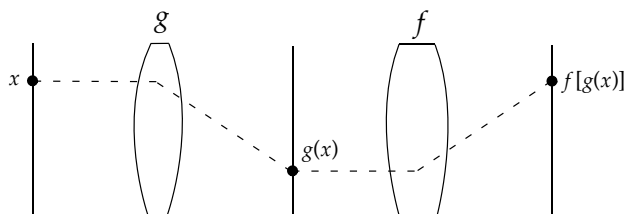
Let's use this second strategy to see if we can anticipate the rule for differentiating a composite. We begin with what we call the “optical interpretation” of a function [22, page 109]:



Leaving the point x on the slide, the ray of light, bent by the lens g , arrives at the point $g(x)$ on the screen. With this optical interpretation of g , both the Newton quotient and the derivative represent magnifications:

$$\begin{aligned} \frac{g(x+h) - g(x)}{h} &= \frac{\text{length of image}}{\text{length of original}} \\ &= \text{average magnification on } [x, x+h] \rightarrow \text{magnification at } x \end{aligned}$$

Thus the derivative $g'(x)$ measures the magnification produced by the g -lens at the point x . In the optical interpretation of a composite $f \circ g$, we would have two lenses in series:



Now if the g -lens magnifies tiny intervals at x by 2, say, and the f -lens magnifies tiny intervals at $g(x)$ by 3, then of course the composite $f \circ g$ will magnify tiny intervals at x by $3 \cdot 2 = 6$, because magnifications, produced by lenses in series, multiply! And hence, in general, we would expect that

$$\begin{aligned} (f \circ g)'(x) &= \text{magnification of } f \circ g \text{ at } x \\ &= [\text{magnification of } f \text{ at } g(x)] \cdot [\text{magnification of } g \text{ at } x] \\ &= f'[g(x)] \cdot g'(x) \end{aligned}$$

which leads to our conjectured

$$\text{COMPOSITE RULE } (f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

While this argument does not *prove* anything, it does succeed in producing a conjecture (without any smoothness conditions of course) and in making that conjecture quite plausible. Let's call it a *plausibility argument*.

Plausibility arguments can be used, as we did today, to help us make conjectures, but they can also be used when a theorem has been given to us without proof, to help us gain some intuition into the plausibility of that theorem. Even when a proof of the theorem has been given to us, a plausibility argument may still offer important insight, for a proof, laid out in a series irrefutable steps, may supply complete conviction, yet incomplete understanding. "We are not very pleased," writes Hermann Weyl, "when we are forced to accept a mathematical truth by virtue of a complicated chain of formal conclusions and computations, which we traverse blindly, link by link, feeling our way by touch. We want first an overview of the aim and of the road; we want to understand the idea of the proof, the deeper context." [25, page 177]

8. PRINCESS DI THEN ATTICUS FINCH

For the last week or more, we have separately taken up several ways of analyzing the behavior of a given function. Now it's time to put these methods together and apply them in concert. Because such concerted effort can involve quite a bit of work, it will pay us to be organized bookkeepers, applying our methods in an efficient order. PRINCESS DI THEN ATTICUS FINCH may have appeared on the list of proverbs we handed out on the first day, but that was really a sham, for it's not a proverb at all, but rather a pronunciation guide for DIADECIS (dī-AT - icus), an ancient Greek mathematician (not really) whose name just happens to spell out a logical arrangement for our work:

$$\begin{array}{l}
 f \quad \left\{ \begin{array}{l} \text{D} \quad \text{Domain} \\ \text{I} \quad \text{Intercept behavior} \\ \text{A} \quad \text{Asymptotic and boonies behavior} \end{array} \right. \\
 f' \quad \left\{ \begin{array}{l} \text{D} \quad \text{Decreasing and increasing} \\ \text{E} \quad \text{Extremes} \end{array} \right. \\
 f'' \quad \left\{ \begin{array}{l} \text{C} \quad \text{Concavity} \\ \text{I} \quad \text{Inflections} \\ \text{S} \quad \text{Sketch} \end{array} \right.
 \end{array}$$

“Boonies behavior”?? Anyone? Soren? Yes, exactly: “boonies” is short for “boondocks,” meaning a remote region, so “boonies behavior” refers to the behavior as $x \rightarrow \pm\infty$. (To give credit where credit is due, even if it's anonymous, I used to call this “far away behavior,” until one summer, in a class I was giving for high school calculus teachers, one of the participants suggested “boonies behavior,” and I've been using the alliteration ever since.)

Before we run through DIADECIS for any specific functions, let's introduce a notation convenient for investigating and recording intercept, asymptotic, and boonies behavior. Our intuition tells us that $f(x) = x^2(x-1)^3$ “behaves like” $g(x) = (x-1)^3$ as $x \rightarrow 1$, that $f(x) = \frac{1}{(x+1)^3(x+2)^4}$ “behaves like” $g(x) = \frac{-1}{(x+2)^4}$ as $x \rightarrow -2$, and that $f(x) = x^2 + x$ “behaves like” $g(x) = x^2$

as $x \rightarrow +\infty$. How can we express this “behaves like” mathematically? Note that, in each of these cases, the quotient f/g tends to a positive finite limit.

DEFINITION As $x \rightarrow c$, suppose both f and g tend to $+\infty$ (or $-\infty$ or 0). We say f and g are asymptotic as $x \rightarrow c$ and write $f \sim g$ if the quotient f/g tends to some positive finite limit. Here c may be finite or $\pm\infty$.

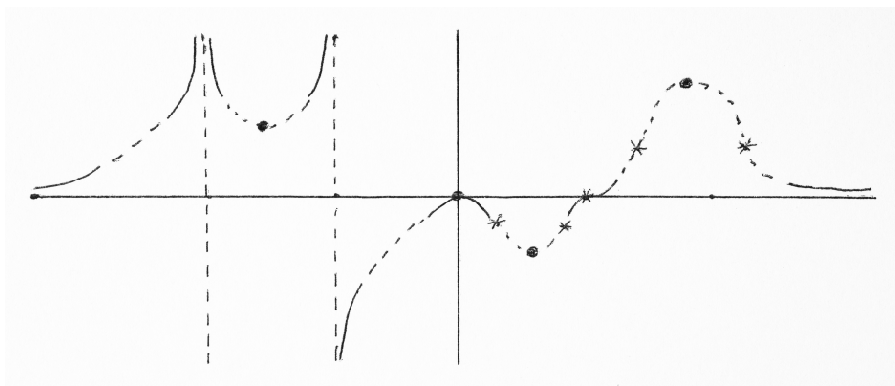
The “behaves like” behavior encoded in $f \sim g$ as $x \rightarrow c$ certainly does not imply that the *absolute* difference $f - g$ tends to zero. For example, $\frac{x^2 + x}{x^2} \rightarrow 1$ as $x \rightarrow +\infty$, yet the difference x grows arbitrarily large. But

$f \sim g$ as $x \rightarrow c$ does imply that a *relative* difference tends to zero: $\frac{f}{g} \rightarrow L$ if and only if $\frac{f - Lg}{g} \rightarrow 0$.

OK, time for some examples.

$$(1) f(x) = \frac{x^2(x-1)^3}{(x+1)^3(x+2)^4} \sim \begin{cases} (x-1)^3 & \text{as } x \rightarrow 1 \\ -x^2 & \text{as } x \rightarrow 0 \\ \frac{-1}{(x+1)^3} & \text{as } x \rightarrow -1 \\ \frac{1}{(x+2)^4} & \text{as } x \rightarrow -2 \\ \frac{1}{x^2} & \text{as } x \rightarrow \pm\infty \end{cases}$$

Let’s first use solid curves to sketch in what these asymptotic approximations tell us. Then we’ll complete the graph by drawing dotted curves to join those solid portions up in the most natural way:



Of course, we don't know the heights of the hills nor the depths of the valleys, but this qualitatively correct sketch is still good enough to let us predict the number (4 and 5) of extremes (\bullet) and inflections ($*$) as well as their (very) rough locations. Derivative calculations can now make those rough locations precise. Often, as in this example, DIA by itself provides so much information that DECIS and its derivatives just confirm and fine-tune what we already know. And that's a comfortable position to be in.

9. GUESS BRAVELY AND BEAUTIFULLY

... the feeling of mathematical beauty, of the harmony of numbers and of forms, of geometric elegance. It is a genuinely aesthetic feeling, which all mathematicians know. —HENRI POINCARÉ [18, page 59]

Last time we defined the integral as a limit of Riemann sums, made some comments on that definition, and then went through a couple of examples. Let's spend the day doing more examples. Here's the first one:

$$\int_1^2 \frac{1}{t} dt \approx \frac{1}{1 + \frac{1}{100}} \frac{1}{100} + \frac{1}{1 + \frac{2}{100}} \frac{1}{100} + \cdots + \frac{1}{1 + \frac{100}{100}} \frac{1}{100} \approx 0.693$$

Seth? That decimal reminds you of the natural log of 2? Wow, that's a good memory! And you're quite right: $\ln 2 \approx 0.693$, which means

$$\int_1^2 \frac{1}{t} dt \approx \ln 2$$

Now wouldn't it be lovely if this approximation were actually an equality?

$$\int_1^2 \frac{1}{t} dt \stackrel{?}{=} \ln 2$$

Do we imagine having that 2 appear on each side is an accident? Let's be brave and guess that we can substitute any (positive) number x for 2:

$$\int_1^x \frac{1}{t} dt \stackrel{?}{=} \ln x$$

Do you notice anything here? Marianne? Right, the derivative of $\ln x$ is $\frac{1}{x}$, so we would have

$$\frac{d}{dx} \int_1^x \frac{1}{t} dt \stackrel{?}{=} \frac{1}{x}$$

Notice we now have the same reciprocal function in the integrand and on the right hand side. What if we could replace that reciprocal function with any function at all? Wouldn't that be just beautiful?

$$\frac{d}{dx} \int_1^x f(t) dt \stackrel{?}{=} f(x)$$

Thinking about what would be beautiful and guessing bravely (and a bit wildly), we have arrived at a stunning conjecture, which (if f is continuous) turns out to be true. It's called the FIRST FUNDAMENTAL THEOREM OF CALCULUS, and we shall prove it later next week! Amazing.

Early on in mathematics, probably by first grade, we learn to associate guessing with ignorance: if you don't know how to find the answer, you might as well guess. Even in advanced mathematics, the standard textbook format — here are some theorems, here are the proofs; here are some problems, supply the solutions — would appear to associate guessing, not with ignorance perhaps, but with being ignored, since the format completely ignores the role of guessing in mathematical work. Yet mathematicians themselves associate guessing with *creativity*! George Pólya, one of the great mathematical problem solvers, put it this way:

You have to guess the mathematical theorem before you prove it: you have to guess the idea of the proof before you carry through the details. You have to combine observations and follow analogies: you have to try and try again. The result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing. [19, page *vi*]

The first part of our proverb, GUESS BRAVELY, concerns the fear of guessing instilled in us by our education. We must overcome that fear and guess bravely — not wildly of course, not out of the blue, but based on examples, educated guessing, what Pólya calls plausible reasoning. The second

part, GUESS BEAUTIFULLY, reminds us to let those guesses reflect our innate belief in the underlying architectural beauty of nature, the Universe, and mathematics.

10. UNDERSTANDING IS TRANSLATING

Would someone please remind me where we were at the end of class on Monday? Ah, yes, thank you Gaard. We had just proved the SECOND FUNDAMENTAL THEOREM OF CALCULUS (FTC II). Here's an alternate version of this wonderful result:

FTC II *If F' is continuous on $[a, b]$, then $\int_a^b F'(x) dx = F(b) - F(a)$.*

One way to increase our understanding of this theorem, or indeed any mathematical statement, would be to translate that statement into a different language. As an illustration, let's translate FTC II into three languages.

First, the magical language of Leibnizian infinitesimals: If dx represents the length of the infinitely short interval $[x, x + dx]$ at x and dy represents the corresponding infinitely small change $F(x + dx) - F(x)$ in F , then

$$\int_a^b F'(x) dx = \int_a^b \frac{dy}{dx} dx = \int_a^b dy = y(b) - y(a) = F(b) - F(a)$$

Next, the language of geometry:

$$\frac{1}{b-a} \int_a^b F'(x) dx = \frac{F(b) - F(a)}{b-a}$$

average of all the
slopes along the graph = slope of the chord

Phrased this way, FTC II seems quite surprising and pretty. And finally, the language of physics: Given the trip of a particle along a line, the "average velocity during the trip" has two natural interpretations, one industrious — the average of all the instantaneous velocities during the trip — and one lazy: the change in position over the change in time. FTC II tells us that these

two interpretations yield the same result. For if $F(t)$ is the position of the particle at time t , then

$$\frac{1}{b-a} \int_a^b F'(t) dx = \frac{F(b) - F(a)}{b-a}$$

average of all the instantaneous velocities during the trip = $\frac{\text{change in position}}{\text{change in time}}$

Our three translations have helped us to understand FTC II. Here the word “understand” suggests something distinct from and perhaps beyond just absorbing the literal meaning, something deeper, more organic, intuitive, and contextual. Any attempt to understand a given theorem in this sense will involve some sort of *translation*. Even the seemingly neutral process of studying the statement and its proof, for example, is a process of translation, from the mathematical language on the page into the mental language of our own interior mathematical landscape. Some translations help our understanding by simplifying down to the kernel of the assertion, say by translating the detailed symbolic statement into a prose sentence. Richard Feynman suggested stripping a concept or statement down to its essentials, ending with a verbal description that would make sense to a child.

In the translation of a theorem — whether into the language of geometry, physics, or simple prose — the reproduction of *exact literal* meaning gives way in order to underline the fundamental sense and intention of the original. In his essay “The Task of the Translator,” Walter Benjamin (in a literary, not mathematical context) argued that, “as regards the meaning, the language of translation can — in fact, must — let itself go, so that it gives voice to the *intentio* of the original not as reproduction but as harmony. . . .” [3, page 79]

11. CELEBRATE YOUR MISTAKES

*A man's mistakes are his portals of discovery.*¹ —JAMES JOYCE
[14, page 182]

¹A mistaken quotation: it’s apt, but not accurate. While Google attributes this saying to James Joyce, what Joyce has Stephen Dedalus claim in *Ulysses* is actually quite different: “A man of genius makes no mistakes. His errors are volitional and are the portals of discovery.” One reading: knowing full well that a given approach will fail, the man of

Where did we end up last time? Anyone remember? Emily? Ah, yes, we had just used the INTEGRAL TEST to investigate the convergence of a p -series. And here's what we found:

THEOREM *The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.*

Let's do some more integral test examples:

$$(1) \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Now before we test the convergence of this series, why don't we see if the intuition we have developed concerning the convergence of positive-term series leads us to a conjecture.

INTUITION *A positive-term series $\sum p_n$ converges when and only when the terms $p_n \rightarrow 0$ sufficiently fast.*

The question then is: do the terms $\frac{1}{n \ln n} \rightarrow 0$ sufficiently fast? Well, we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges provided $p > 1$, so it must be that the terms $\frac{1}{n^p} \rightarrow 0$ sufficiently fast. But then $\frac{1}{n^p} = \frac{1}{n \cdot n^q} \rightarrow 0$ sufficiently fast for every $q = p - 1 > 0$. Since we can make $n^q \rightarrow +\infty$ more and more slowly by choosing q closer and closer to 0, it seems that we should be able to make $n^q \rightarrow +\infty$ more slowly than $\ln n \rightarrow +\infty$ by choosing q sufficiently close to 0. For such q , $\frac{1}{n \cdot n^q}$ would $\rightarrow 0$ sufficiently fast and consequently $\frac{1}{n \ln n}$ would also $\rightarrow 0$ sufficiently fast. This intuitive line of thought leads to our

CONJECTURE $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

What's remarkable is that I still remember thinking about just this series, following this line of thought, and making this same conjecture, from when I took calculus way back in 1967!! I also remember how I felt when the

genius follows it anyway, knowing also that the particular way it fails will open "portals of discovery."

INTEGRAL TEST

$$\int_2^{+\infty} \frac{1}{x \ln x} dx = \ln(\ln x)|_2^{+\infty} = +\infty$$

proved my conjecture wrong! I felt deflated, disappointed that my intuition had led me astray. But really, looking back, I should have felt *elated*! I should have celebrated the opportunity to deepen my understanding, to sharpen my intuition by finding where my thinking had gone wrong. And it was, of course, in thinking that just because I could make $n^q \rightarrow +\infty$ more and more slowly by choosing q closer and closer to 0, that I could make $n^q \rightarrow +\infty$ more slowly than $\ln n$, when, in fact, $\ln n \rightarrow +\infty$ more slowly than *any* positive power of n :

$$\frac{\ln n}{n^q} \rightarrow 0$$

by L'Hôpital (really Johann Bernoulli) for *any* $q > 0$.

My point being, CELEBRATE YOUR MISTAKES! See them as double joys, not only signaling that our mathematical mental furniture is out of alignment but also, once we trace our error back to its origins, placing a finger on the particular chair, desk, or table that needs shifting. Without mistakes, we don't know what we don't know.

12. EXPLORE THE TUG OF WAR: CUI DOMINETUR?

The argument we gave yesterday proved the following theorem:

LEIBNIZ ALTERNATING SERIES TEST *The alternating series*

$$\sum_{n=0}^{\infty} (-1)^n p_n,$$

where $p_n > 0$, converges if $p_n \downarrow 0$.

Before we go on to look at some alternating series examples, here's a question: Which of our proverbs has come up the most this year, in Calculus I and II? Rowen? EXPLORE THE TUG OF WAR? Yes, I think you're right. And why is that? Is it the most fundamental for life? No, that honor would probably go to BE AWED, LIKE A CHILD or MEANING BEFORE TRUTH or GOD GIVES, WE CHOOSE. But EXPLORE THE TUG OF WAR is certainly a fundamental

proverb for *calculus*: because calculus rests on the limit concept, because “All interesting limits involve a tug of war,” as I’ve said so often, and because understanding its tug of war is the key to understanding a limit.

But why, I see you all wondering, am I bringing up the tug of war proverb now? Well, because an alternating series represents the final tug of war category that we’ll meet in this class, which makes this the perfect time to look back and reflect on a few of the more memorable tug of wars (tugs of war?) we’ve seen during this past year.

$$(1) \frac{1 - \cos x}{x^n} \text{ as } x \rightarrow 0^+ \quad (n = 1, 2, 3)$$

$$\frac{1 - \cos x}{x} \rightarrow 0 \quad \frac{1 - \cos x}{x^2} \rightarrow \frac{1}{2} \quad \frac{1 - \cos x}{x^3} \rightarrow +\infty$$

Here the top and bottom both tend to 0, creating a classic tug of war, with the top team pulling the fraction toward 0 while the bottom team pulls the fraction toward $+\infty$. The top team wins (dominates) when $n = 1$, the bottom team wins (dominates) when $n = 3$, and there’s a tie when $n = 2$. As $x \rightarrow 0^+$, we say $1 - \cos x$ tends to 0 faster than x , slower than x^3 , and at the same rate as x^2 .

In the full proverb, EXPLORE THE TUG OF WAR: CUI DOMINETUR?, the question “Which dominates?” has two distinct interpretations: (a) Which *team* dominates the tug of war? and (b) Which team *members* dominate their teammates? The example above illustrates just the first interpretation, while the next example illustrates both.

$$(2) \frac{2x^3 + x^2 + x}{3x^3 - x^2 + 2x} \text{ as } x \rightarrow 0 \text{ and } x \rightarrow +\infty$$

When $x \rightarrow 0$ the top team and the bottom team both tend to 0, setting up our tug of war. But as $x \rightarrow 0$, x dominates its teammates on the top team while $2x$ dominates its teammates on the bottom team, and “dividing by the dominator”

$$\frac{2x^3 + x^2 + x}{3x^3 - x^2 + 2x} = \frac{2x^2 + x + 1}{3x^2 - x + 2} \rightarrow \frac{1}{2}$$

we see that neither *team* dominates, for the tug of war ends in a tie. On the other hand, when $x \rightarrow +\infty$ the top team and the bottom team both tend toward $+\infty$, producing a different tug of war. Since, in this case, $2x^3$ dominates its teammates on the top team and $3x^3$ dominates its teammates on the bottom team, we “divide by the dominator” again

$$\frac{2x^3 + x^2 + x}{3x^3 - x^2 + 2x} = \frac{2 + \frac{1}{x} + \frac{1}{x^2}}{3 - \frac{1}{x} + \frac{2}{x^2}} \rightarrow \frac{2}{3}$$

to find once more that neither *team* dominates, as the tug of war ends in a tie.

$$(3) \left(1 + \frac{1}{n}\right)^n \text{ as } n \rightarrow \infty$$

In this famous and fundamental limit, the term $\frac{1}{n}$ pulls the expression toward 1, while, at the same time, the exponent n pulls the expression toward $+\infty$. As we know, this tug of war ends in a tie:

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e.$$

$$(4) \frac{f(c+h) - f(c)}{h} \text{ as } h \rightarrow 0$$

Obviously, every derivative derives from a tug of war, with the top and bottom both tending toward 0.

$$(5) \sum_{i=1}^n f(c_i) \Delta x_i \text{ as mesh} \rightarrow 0$$

Also every integral involves a tug of war, not with opposing teams so much as opposing forces. As the mesh tends toward 0, the *size* of each term in the Riemann sum tends toward 0, while the *number* of terms tends toward ∞ .

$$(6) \sum_{n=1}^{\infty} p_n \quad (p_n > 0)$$

Intuitively, the positive-term infinite series

$$\sum_{n=1}^{\infty} p_n := \lim_{N \rightarrow \infty} \sum_{n=1}^N p_n$$

converges if and only if the pull toward convergence produced by the terms p_n tending to 0 counteracts the pull toward $+\infty$ (and hence divergence) produced by the number N of terms tending to ∞ .

BASIC INTUITION *A positive-term series $\sum_{n=1}^{\infty} p_n$ converges if and only if its terms tend to 0 sufficiently fast.*

$$(7) \sum_{n=1}^{\infty} (-1)^{n+1} p_n \quad (p_n > 0)$$

When the series alternates, an extra force pulls toward convergence – the cancellation created by the alternating signs — and as a result, to ensure convergence of the series, the force produced by the terms tending to 0 does not need to pull as hard as it did when the terms were all positive. In fact, the terms p_n no longer need to tend toward 0 “sufficiently fast”; they just need to decrease to 0, at any rate at all, which is where we started the class:

LEIBNIZ ALTERNATING SERIES TEST *The alternating series*

$$\sum_{n=0}^{\infty} (-1)^n p_n,$$

where $p_n > 0$, converges if $p_n \downarrow 0$.

13. TOO BEAUTIFUL TO BE FALSE

My work has always tried to unite the True with the Beautiful and when I had to choose one or the other, I usually chose the Beautiful. —HERMANN WEYL [8]

As class ended on Wednesday, we had just stated a theorem on the nice behavior of power series. We can't prove it now, but you'll see a proof if you go on to the analysis course next year. Let's get this theorem back up, on the side board here, and take a look at some examples.

PLAYING WITH POWER SERIES THEOREM (PPST) *On the interior of its interval of convergence, a power series acts like a polynomial: it's infinitely differentiable and all the basic operations (\pm , \times , \div , substitution, \int , $()'$) can be done term-by-term!*

Let's play. On the interval $(-1, 1)$, we have

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

Applying the PPST, we replace x by $-x^2$ to see that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

Perhaps we should pause here. Remember that an *algebraic function* contains only (and finitely many) additions, subtractions, divisions, multiplications, rational powers, and rational roots), and a *transcendental function* is any function that's not algebraic. Thus a transcendental function "transcends" algebra. For example, \arctan is a transcendental function. In particular, then, \arctan transcends the polynomials: given any open interval, there is no polynomial which equals \arctan everywhere on that interval. But apparently, if we allow our polynomials to go on forever, to become transcendental themselves, they can catch up to \arctan , at least on the interval $(-1, 1)$: by the PPST, we can integrate the power series representation above term-by-term to find

$$\arctan x \stackrel{!}{=} x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Now we know this holds on $(-1, 1)$, but the PPST is mute about whether a convergent power series ever "acts like a polynomial" even at the boundary points of the open interval, in this case at ± 1 . Just for fun, let's see what happens when $x = 1$:

$$\frac{\pi}{4} \stackrel{?}{=} 1 - \frac{1}{3} + \frac{1}{5} - \dots \quad \text{!!!!}$$

Oh, my god, this is far TOO BEAUTIFUL TO BE FALSE! Ah, you're laughing. I know, it sounds goofy: TOO BEAUTIFUL TO BE FALSE. But I'm really quite serious.

Let's think about what just happened. We have an equality

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

which holds for every x in $(-1, 1)$, and in particular, for every x just short of the boundary at 1. It's certainly *possible* that the equation also holds *at* 1, and there's some evidence that it *might* hold at 1, given that the statement when $x = 1$ is "arbitrarily close" to statements that are true. And *at* 1 we get:

$$\frac{\pi}{4} \stackrel{?}{=} 1 - \frac{1}{3} + \frac{1}{5} + \dots$$

a stunningly beautiful statement which lies on the "boundary of an interval of true statements." Would God, the architect of the universe, really be so perverse as to make this lovely equality *false*? Personally, I don't think so. That would be like God smirking at us while singing na na na na boo boo.

Of course, we have proofs by induction, proofs by counterexample, contraposition and contradiction, but no "proofs by beauty." There are, however, proofs by Abel, the wonderful Norwegian mathematician, who died so young, at 26:

ABEL'S THEOREM *If $\sum_{n=0}^{\infty} c_n$ converges, then $s(x) := \sum_{n=0}^{\infty} c_n x^n$ is continuous on $[0, 1]$.*

Since the alternating series $1 - \frac{1}{3} + \frac{1}{5} + \dots$ converges, we then have, by continuity at $x = 1$,

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} + \dots$$

Abel assures us that God, at least this time, has not been perverse.

Recall Hermann Weyl's statement concerning beauty and truth at the beginning of this section: "My work has always tried to unite the True with the Beautiful and when I had to choose one or the other, I usually chose the Beautiful." Perhaps Weyl had in mind that choosing the Beautiful over the True in the moment often led to a deeper truth in the end.

14. DO YOU BELIEVE IN MAGIC?

I don't like magic — but I have been known to make guys disappear. —MR. T [24] [leaving behind, one assumes, the ghosts of departed quantities!]

From your reading last week in *Calculus Gems* [21] on Leibniz and the Bernoulli brothers, you may remember that Leibniz employed a marvelously productive way of thinking about calculus which involved “infinitesimals” — quantities supposedly smaller than any other quantity yet not zero. Of course, in our standard real number system, such infinitesimals do not exist, since given any real number $r > 0$, no matter how small, $r/2$ is smaller. Nevertheless, Leibniz and his followers made wonderful discoveries using these intuitive, magical, nonexistent quantities, taking mathematics on the continent well beyond the mathematics in Britain, where, due to Newton’s worries about the lack of rigor, infinitesimals were effectively banned.

Actually, Leibniz agreed with Newton that infinitesimals did not exist. In a letter written in 1706, he was quite clear about this: “Philosophically speaking, I no more believe in infinitely small quantities than in infinitely great ones . . . I consider both as fictions of the mind for succinct ways of speaking, appropriate to the calculus. . . .” [13, page 159] He was more willing than Newton, though, to let these infinitesimals direct not only his thinking but also his writing about calculus problems. It seems that Leibniz believed in the fruitfulness of his magical thinking, but not in the existence of his magical quantities. Still fruitful even today, Leibniz’s infinitesimal intuitions survive encoded in his flexible, evocative, and magical notations: where dx represents the infinitesimal length of the interval $[x, x + dx]$ at x ,

$$\frac{dy}{dx} = \text{the "quotient" } \frac{f(x + dx) - f(x)}{dx}$$

$$\int_a^b f(x) dx = \text{the "sum" of all the "products" } f(x) \cdot dx \text{ from } x = a \text{ to } x = b$$

To see an illustration of Leibnizian magical thinking, let’s rotate the graph G of f lying over the interval $[a, b]$ about the x -axis to generate a surface of revolution M . Suppose the infinitesimal portion of the graph G which lies

over the interval $[x, x + dx]$ has length ds , and suppose that portion of G , rotated about the x -axis, generates an infinitesimal strip of M having area dA . Then we would have

$$\begin{aligned} \text{length}(G) &= \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ \text{area}(M) &= \int_a^b dA = \int_a^b 2\pi f(x) ds = \int_a^b 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \end{aligned}$$

Such infinitesimal manipulations can be used, like we did here, to come up with conjectures, which then must be established rigorously by other means, in this case by careful Riemann sum arguments. But this magical Leibnizian thinking can also be used in the reverse direction, not to produce a conjecture, but to produce insight into a given expression or equation that might appear in an advanced text or a research article on ecology, say, or biology.

For example, suppose an ecology text claims, without explanation, that the following equation has been used to predict the population $p(T)$ of humpback whales along the Alaskan coastline T years from the present:

$$p(T) = p(0)s(T) + \int_0^T r(t)s(T-t) dt$$

Here $r(t)$ is the birth rate and $s(t)$ is the “survival fraction”: given any P humpback whales, $Ps(t)$ are expected to be alive t years later. Is this a plausible formula for $p(T)$? Well, thinking like Leibniz, the product $r(t) dt$ should be the number of humpbacks born during the time interval $[t, t + dt]$, and by the time T only $r(t) dt \cdot s(T-t)$ of those should still be alive. Adding up these survivors over the interval $[0, T]$, we get the integral $\int_0^T r(t)s(T-t) dt$. We then have to add to this the number of humpbacks alive at $t = 0$ who would be expected to survive until time T , namely $p(0)s(T)$. So although we may have no idea how well the given estimate for $p(T)$ actually works to predict the population of humpbacks, Leibnizian infinitesimals have at least supplied the formula with some plausibility.

15. FOLLOW THE VEIL: BE MOVED BY MYSTERY

There are two ways to live: you can live as if nothing is a miracle; you can live as if everything is a miracle. The most beautiful experience we can have is the mysterious. It is the source of all true art and science. —ALBERT EINSTEIN [9]

We're going to celebrate the last day of Calculus II by talking about complex numbers. In high school, we are told that complex numbers have the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ is "imaginary." We are then told that we can add and multiply these complex numbers normally, the way we do real numbers, except that we should always replace i^2 by -1 . Of course, this is all very mysterious: "The nature, mother of the eternal diversities, or the divine spirit . . . has invented," wrote Leibniz, "this elegant and admirable proceeding, this wonder of Analysis, prodigy of the universe of ideas, a kind of hermaphrodite between existence and non-existence, which we have named imaginary root." [15]

Let's see if we can lift the veil on these complex numbers. We all know how to add points in the plane, using vector addition, and multiply points by real numbers, using scalar multiplication. But we've never seen a *product* of points, where the product is another point: $(x, y)(u, v) = (,)$. Such a new and presumably fundamental operation, if it exists, would surely turn the plane into a lush and fruitful plain. For think about the real line. In its fertile soil grows so much beautiful mathematics — all the elementary functions, limits, calculus, analysis generally, and a huge harvest of theorems and applications — but take away the multiplication of real numbers and that dark, rich earth becomes pale, desert sand. With no multiplication of points, surely the plane must be just as barren. Yet with an appropriate product of points, together with its vector addition and scalar multiplication, we imagine the plane would become as fertile as the Amazon jungle, with elementary functions, derivatives, integrals, sequences, series, and power series, all growing wildly in a tangle of stunning theorems and applications.

But how should this "appropriate" product be defined? Perhaps we should do what mathematicians so often do in this sort of situation, where they're looking for the right way to define something: use the properties we would like that something to satisfy to help us narrow the choices down to one.

Of course, what we'd like and what's possible are often two different things. We might be overly optimistic in writing out our wish list, only to find that no definition of the product will give us every property on our list. But keeping this in mind, we certainly would like our product to be associative and commutative, it should distribute over the vector addition, there should be a neutral point, probably $(1, 0)$, for this product, and all points (except the origin) should be invertible. Oh, and we really need the following property as well: because points on the x -axis can be viewed both as points in the plane and as real numbers, they can be multiplied both as points and as real numbers, and the two products must be consistent. So however we define the product, for points on the x -axis we must have $(x, 0)(u, 0) = (xu, 0)$. In other words, our product on the plane must *extend* the product we already have for points on the x -axis (viewed as real numbers).

Concentrating on this extension property, let's think about how we multiply real numbers. Using polar notation $\langle r, \theta \rangle$, we have

$$\begin{aligned} -2 \cdot 3 &= -6 \\ \langle 2, \pi \rangle \cdot \langle 3, 0 \rangle &= \langle 6, \pi \rangle \\ &= \langle 2 \cdot 3, \pi + 0 \rangle \end{aligned}$$

So in a product of real numbers, the lengths multiply and the angles add! And of course this way of multiplying real numbers, and hence this way of multiplying points on the x -axis, extends in a totally natural way to the entire plane:

DEFINITION Given points $z = \langle r, \theta \rangle$ and $w = \langle \rho, \phi \rangle$ in the plane, the product $zw := \langle r\rho, \theta + \phi \rangle$ is called *complex multiplication*.

One can then verify that this complex multiplication satisfies every property on our wish list!

DEFINITION By the *complex plane* \mathbb{C} we mean the real plane \mathbb{R}^2 endowed with vector addition, scalar multiplication, and complex multiplication. By a *complex number* we mean any point in this complex plane.

Noting that multiplication by the point $i := (0, 1)$ produces a counterclockwise rotation by a right angle and writing x and y for $(x, 0)$ and $(y, 0)$, we can relate our new definition of complex number to the mysterious $x + iy$ notation:

$$\begin{aligned}
 (x, y) &= (x, 0) + (0, y) \\
 &= (x, 0) + (0, 1)(y, 0) \\
 &= x + iy
 \end{aligned}$$

where $i^2 = (0, 1)^2 = (-1, 0) = -1$!! The imaginary number $i = \sqrt{-1}$, which couldn't be found among the real numbers, has been found in a higher dimension: it's the point $(0, 1)$ in the complex plane!

Having unveiled one mystery, we turn to a deeper mystery. Question: How should we define e^z when $z = (x, y) = x + iy$ is a complex number? However we define e^{x+iy} , we certainly want the basic exponential property — turning sums into products — to still hold, which means e^{x+iy} should equal $e^x e^{iy}$, but this reduces defining e^{x+iy} to defining e^{iy} . Now it turns out that much of what we've learned in Calculus II about infinite series of real numbers extends quite naturally to infinite series of complex numbers. Instead, for example, of an interval of convergence, we now have a *disk* of convergence. It would make sense, then, to define e^{iy} as the sum of the exponential series:

$$\begin{aligned}
 e^{iy} &:= 1 + iy + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \cdots \\
 &= 1 + iy - \frac{1}{2!}y^2 - i\frac{1}{3!}y^3 + \cdots \\
 &= (1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \cdots) + i(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \cdots) \\
 e^{iy} &= \cos y + i \sin y !!!
 \end{aligned}$$

Here is the long hidden, but sometimes glimpsed, intimate connection between the exponential and trigonometric worlds, first discovered by Leonhard Euler! [10, Chapter VIII] And when $y = \pi$, we find:

$$e^{i\pi} + 1 = 0 !!!$$

Viewed by physicists and mathematicians as one of, if not the, most beautiful equation in the world, this relationship, Euler's equation, connects, in a stunningly simple way, the five most fundamental mathematical constants $(1, \pi, 0, i, e)$, invented (discovered?) in very different times for very different reasons. 1: shrouded in prehistory, π : Egypt 1850 BC, 0: Mesopotamia 3 BC, i : Heron of Alexandria 50 CE, e : Leibniz 1690. Euler's equation should make chills race up your mathematical spine.

Rowen? You'd rather feel chills race up your *physical* spine? Ah, yes, I totally get it: containing only *mathematical* constants, Euler's equation seems, well, abstract. Let's see if I can think of an equation that will produce the kind of chills you're after. Hm, how about this one: no less mysterious and at least as deep as Euler's equation, the following relationship involves not only the five mathematical constants above, but also, unbelievably, eight fundamental physical constants! It will leave your physics friends and faculty speechless:

$$cGh\varepsilon_0q_e\alpha\mu_0m_e(e^{i\pi} + 1) = 0 !!!$$

[laughter and groans]

Of course, instead of being silly, we really should give Rowen's reaction to Euler's equation the serious attention it deserves.

$$E = mc^2, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \Delta P \Delta x \geq \frac{\hbar}{2}, \dots$$

— the equations of physics naturally excite us, for they describe the physical world as we know it, from black holes and pulsars to bridges and turbulence to the interactions of elementary particles. In contrast, the equations of pure mathematics, such as Euler's equation, containing neither physical constants nor physical quantities, can appear sterile and empty, the fruits of a meaningless, abstract game. But *mathematics is no game; it is the language and study of abstract structures and patterns*. It certainly applies, deeply and in detail, to the "real world," for to the extent that a physical structure or pattern resembles an abstract structure or pattern, the mathematics of the latter may be applied to approximate or model the physics of the former. This is how mathematical modeling works: abstract mathematics models real world phenomena.

But in a very real (or non-real) sense, mathematics also bears on *immutable* and *eternal* truth, truths beyond and independent of the evolving and transient physical universe we happen to reside in, truths that would hold in *any* universe. And Euler's equation, $e^{i\pi} + 1 = 0$, being one such beautiful truth, makes chills race up both my mathematical *and* physical spines!

16. PHILOSOPHY MATTERS (AKA I'LL SEE IT WHEN I BELIEVE IT)

Once a mathematician has seen that his perception of the "self-evident correctness" of the law of excluded middle [bivalence] is

nothing more than the linguistic equivalent of an optical illusion, neither his practice of mathematics nor his understanding of it can ever be the same. –GABRIEL STOLZENBERG [23, page 268]

Well, we finished that topic earlier than anticipated! Hmm, what to do with the twenty minutes we have left? I know, let's talk philosophy! Ah, I can see the question in your faces: What's philosophy got to do with mathematics? After all, in philosophy the central questions have no single, agreed upon answer, while in mathematics rigorous proofs settle questions definitively. Philosophy is all about continuing debate; mathematics is all about preventing debate.

Nevertheless, philosophy has a crucial role to play in mathematics, and the philosophical choices we make can determine how mathematics is done! We may have ignored philosophical issues in this class, but they lie at the heart of what we've been doing. In fact, nearly all mathematicians follow procedures sanctified by a particular philosophical stance toward mathematical statements.

Take, for example, the procedure we use in a proof by contradiction: in order to prove a mathematical statement S , we show that the negation of S implies a contradiction. The force of this argument stems from a philosophical assumption about mathematical statements: that they are either true or false, independent of our knowing (or being able to know) which. This "bivalence" or "excluded middle" assumption is not one that mathematicians make consciously. Rather it's a background assumption, built into the way they see mathematical statements. Why? Partly because the present-tense language we use in mathematics — referring to sets, functions, sequences, spaces, and so on, as if they were pre-existing shells on a beach — creates a nearly irresistible sense of reality. Such language is then taken *literally*, as referring to "things" that stand apart from us. A mathematical statement is then quite naturally seen as a statement about these "things" and therefore, taken this way, a mathematical statement will be seen as having to be either true or false, from which then follows, in particular, the force of an argument by contradiction: if the negation of a statement S implies a contradiction, then that negation cannot be true, and hence (using bivalence) S must be true.

I’LL SEE IT WHEN I BELIEVE IT: if we *believe* in the existence of these mathematical “things” that stand apart from us, we then *see* any mathematical statement, since it refers to such “things,” as being obviously true or false independent of our knowing.

But there are other quite natural ways of viewing a mathematical statement. Such an assertion could be seen, for instance — and here I’ll read from a delightfully meticulous paper by Gabriel Stolzenberg —

as an announcement, or signal, that one is in possession of a certain piece of [mathematical] knowledge . . . knowledge that one is in a position to share; for example, by using language to specify certain procedures that are to be followed in order to attain this knowledge. From this standpoint, to inquire about some statement whether “it might be true, independent of our knowing it” is merely idle talk, devoid of substance. For there are not literally “things” as “statements,” only acts “of stating.” [23, page 245]

Nouns have become verbs: mathematics as the study of mathematical “things” has morphed into mathematics as the study of mathematical “acts.” This philosophical stance stems from the belief that the foundations of mathematics, that most rational and precise of disciplines, ought to rest, not on talk, but on knowledge, knowledge that one can share.

Under this view, when would it be correct to assert a given mathematical statement S ? “Since such a statement,” continues Stolzenberg, “is supposed to be a signal that one knows that S is true, it is correct to assert it when one does know that S is true and it is incorrect when one does not.” And again, one “knows that S is true” when “one is in possession of a certain piece of [mathematical] knowledge . . . knowledge that one is in a position to share.” Now observe that taking this (quite natural) philosophical stance *completely blunts the force of an argument by contradiction*: proving that the negation of the assertion S implies a contradiction provides certainty that no one will ever be in a position to assert the negation of S , but does not in general provide that “certain piece of [mathematical] knowledge,” that piece being a proof, required to assert correctly that S is true!

Switching from the view that a mathematical statement refers to “things” that stand apart from us (the “bivalence view”) to the view that asserting a mathematical statement signals that one possesses a proof of that statement (the “signal view”) alters not just the force of an argument by contradiction, but so much more. The meanings of existence, negation, and disjunction, what counts as a legitimate method of proof, what counts as a legitimate definition, the meanings of theorems, what counts as a theorem, what questions should be asked, what problems should be investigated — all of this changes!

PHILOSOPHY MATTERS: *The philosophical stance we take toward mathematical statements shapes the landscape of mathematics!*

Before we run out of time, we’ll look at one small part of that mathematical landscape, first from the “bivalence view” and then from the “signal view.” We’ve mentioned twin primes before: 3 and 5, 5 and 7, 11 and 13, and so on. At the present time, no one knows whether there are an infinite number of twin primes or a finite number. Now, set

$$n := \begin{cases} 1 & \text{if there are infinitely many twin primes} \\ 0 & \text{if there are finitely many twin primes} \end{cases} \quad (\star)$$

and ask: *Does this assignment (\star) define an integer?* If T stands for the assertion, “there are infinitely many twin primes,” then under the “bivalence view” of mathematical statements, T is either true or false, so that n is definitely either 1 or 0, we just don’t know which. Hence the assignment (\star) defines an integer. But under the “signal view” of mathematical statements, to assert that $n = 1$ or $n = 0$ is to signal that one is in possession of a proof that there are infinitely many twin primes or one is in possession of a proof that there are finitely many twin primes. Because, at the present time, no one has either, the assignment (\star) does *not* define an integer.

The answer to that most fundamental, gut-level mathematical question — “What is an integer?” — has been altered dramatically by a change in philosophy. PHILOSOPHY MATTERS!

Uh-oh, I see by the clock that we’ve gone into overtime. Sorry, got carried away!

But before you go, let me leave you with one final thought on this philosophy business. One of our proverbs has surfaced so often this semester that it could be seen as a theme for the course: MEANING BEFORE TRUTH. And it surfaces here as well, in a truly basic way: the *meaning* of the statement “the assignment (\star) defines an integer” must come before its *truth*.

And so too in life: *Agree on meaning before debating truth.*

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