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On the Tropicalization of Lines onto Tropical Quadrics

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Abstract

Tropical geometry uses the minimum and addition operations to consider tropical versions of the curves, surfaces, and more generally the zero set of polynomials, called varieties, that are the objects of study in classical algebraic geometry. One known result in classical geometry is that smooth quadric surfaces in three-dimensional projective space, $\mathbb{P}^3$, are doubly ruled, and those rulings form a disjoint union of conics in $\mathbb{P}^5$. We wish to see if the same result holds for smooth tropical quadrics. We use the Fundamental Theorem of Tropical Algebraic Geometry to outline an approach to studying how lines lift onto a tropical quadric, which is necessary for understanding what lines are on smooth tropical quadrics and their structure. We also provide suggestions of how computational tools can be used to implement the approach.

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Chapter 1

Introduction

Tropical geometry is a branch of algebraic geometry where algebra is done over the tropical semiring, which is defined by minimum and addition. In classical algebraic geometry, we study geometric objects that are associated with polynomials. In the tropical semiring, because the operations change, these geometrical objects also change. Therefore, one way to study algebraic geometry is to bring problems from classical alegbraic geometry and study them using the tropical semiring. Not all classical results easily lift to tropical ones, because classical tools do not always easily translate to tropical ones.

Some geometric objects one can study are the lines on a general quadric surface in $\mathbb{CP}^3$, complex projective space of dimension 3. A classical result shows that the general smooth quadric is the disjoint union of two conics, sitting inside the Grassmannian $G(2,4)$, the 2-dimensional subspaces of three dimensional space $\mathbb{P}^3$, of lines in $\mathbb{P}^3$. One may then be curious about this problem in tropical geometry. Sturmfels posed the following question, which was investigated in O'Neill (2013):

Question 1. What tropical lines, and how many, are contained on the tropicalization of a smooth quadric in $\mathbb{P}^3$? What is their structure as a subset of the tropical Grassmannian?

Vigeland (2010) demonstrated that the tropical quadric in $\mathbb{R}^3$ is doubly ruled, meaning that the quadric contains two tropical families of lines. Our technique will involve attempting to create tropical analogues of classical constructions and studying how classical lines lift through tropicalization to tropical space.

A classical quadric surface in $\mathbb{P}^3$ has two distinct rulings, so it is of interest to see if this property holds for tropical surfaces in $\mathbb{P}^3$. O'Neill

(2013) demonstrated that the image of these rulings on the tropical quadric in the tropical Grassmannian is the union of two conics, but it is unknown if these conics are disjoint. To answer this question, we provide the necessary definitions for the classical problem, their tropical analogues, and attempt to translate the classical solution into a tropical one. We also discuss the connection between tropical hypersurfaces and their polyhedral geometry. We also take a different approach to answering the question by studying where lines not on the classical quadric go after tropicalization, where the now tropical lines are in tropical projective space. We seek to see if tropicalization allows for lines not in the rulings of the classical quadric to be on the tropical quadric by studying a known smooth tropical quadric.

Chapter 2

Tropical Geometry

We begin by introducing tropical geometry, starting an introduction of the new operations, which will allow us to define tropical hypersurfaces and tropicalization.

We first define the operations we use in tropical geometry. In tropical geometry,

$$x \oplus y = \min\{x, y\}$$

and

$$x \odot y = x + y.$$

We refer to the operation $\oplus$ as being tropical addition and $\odot$ as tropical multiplication. The tropical semiring, which is a ring without additive inverses being required, is then defined as $(\mathbb{R} \cup \infty, \oplus, \odot)$, where ∞ is necessary because it is the additive identity.

Example 1. We see that for any real number x,

$$x \oplus \infty = x$$

which is why ∞ is the additive identity.

For any x, we see that

$$x \odot 0 = x,$$

so 0 is the multiplicative identity in the tropical semiring.

Using these operations, we can define monomials and polynomials.

Definition 1. A **tropical monomial** in variables $x_1, \ldots, x_n$ is a tropical product of these variables, which is of the form $x_1^{i_1} \odot x_2^{i_2} \odot \cdots \odot x_n^{i_n}$. The monomial has degree $i_1 + \cdots + i_n$.

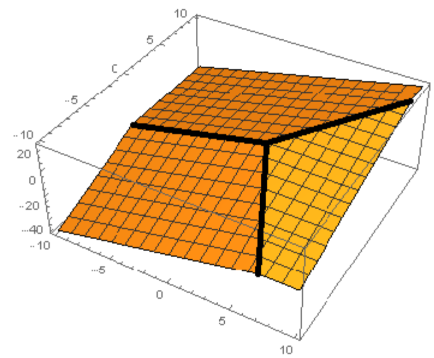

Figure 2.1 The Polynomial from Example 2, with Hypersurface in Black. Created in Mathematica.

It is important to remember that $x_1^{i_1}$ corresponds to $i_1 x_1$, because of the way tropical multiplication is defined. Monomials can be combined to form tropical polynomials.

Definition 2. A **tropical polynomial** is a finite tropical linear combination of tropical monomials. The degree is the maximum degree of the monomials.

Monomials are combined using tropical addition, but because tropical addition of two elements is the minimum of the two, a graph of a tropical polynomial is the minimum of the tropical monomials.

Example 2. Consider the tropical polynomial $x^2 \oplus x^2 y^3 \oplus 4 \odot y^2$. This polynomial has degree 5. Classically, this is the piecewise linear function $\min\{2x, 2x+3y, 4+2y\}$, illustrated in Figure 2.1.

Figure 2.1 illustrates that the graph of a tropical polynomial is the minimum of an intersection of planes. We describe the points where these planes intersect in the following way.

Definition 3. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ by a tropical polynomial. Then, the **tropical hypersurface** is all points $w \in \mathbb{R}^n$ were the minimum of f is attained at least twice within the terms in f. Each of these points is also referred to as being a corner locus of f.

In Figure 2.1, the hypersurface is marked with thick black lines, which are the locations where at least two planes intersect.

We've presented the tropicalized versions of monomials and polynomials here, and we can study their hypersurfaces. However, we are also interested in studying collections of polynomials. Rather than defining tropical analogues for these objects, we instead describe how to "tropicalize" them.

2.1 Tropicalization

Throughout this thesis, K is a field. In this section we define affine varieties and then how to tropicalize them, following notation and definitions from Maclagan and Sturmfels (2015).

Definition 4. Let I be an ideal with $I \subseteq K[x_1, ..., x_n]$. Then, the affine variety[1] $V(I)$ is defined as

$$V(I) = \{a \in \mathbb{A}^n_k : f(a) = 0 \text{ for all } f \in I\}.$$

We note $K[x_1, ..., x_n]$ is a polynomial ring, but we often use the Laurent polynomial ring, defined as $K[x_1^{\pm 1}, ..., x_n^{\pm 1}]$.

In order to "tropicalize" a variety $V(I)$, we use a special kind of map, called a valuation.

Definition 5. A valuation on field K is a map val : $K \longrightarrow \mathbb{R} \cup \infty$ such that:

1. $\text{val}(a) = \infty$ if and only if $a = 0$
2. $\text{val}(ab) = \text{val}(a) + \text{val}(b)$
3. $\text{val}(a + b) \geq \min\{\text{val}(a), \text{val}(b)\}$

The image of the valuation map is denoted Γ_{val} and is called the **value group**.

This naturally extends to val : $K^n \longrightarrow \mathbb{R}^n$, which we refer to as coordinatewise valuation.

One can observe that a valuation takes us from classical operations to the tropical operations. For example, we look at the first requirement for a map to be a valuation. In the tropical semiring, the additive identity is ∞, and in other fields, the additive identity is 0. The valuation of a is only ∞ if $a = 0$, so the valuation map is mapping the additive identity in the tropical

[1]We don't assume irreducibility.

semiring to the additive identity in K. Similarly, we see that the valuation of a product is the sum of the valuations of each factor, which corresponds to tropical multiplication being classical addition.

Example 3. One type of valuation is **p-adic valuation**, with $\mathrm{val}_p : \mathbb{Q} \to \mathbb{R}$. For fixed prime p and $q = p^k a/b$ with $a, b \in \mathbb{Z}$ and $p \nmid a, b$, we have $\mathrm{val}_p(q) = k$.

For example, $\frac{4}{7} = 2^2 \cdot \frac{1}{7}$, so $\mathrm{val}_2(\frac{4}{7}) = 2$.

We use a specific valuation map in the tropicalization process. To use this map, we must first define the Puiseux series and its corresponding field.

Definition 6. The **Puiseux series** in a field K is a generalization of power series that allows for negative and fractional exponents. We usually take $K = \mathbb{C}$. The **field of Puiseux series** over $\mathbb{C}$ is then the set of expressions of the form $c(t) = \sum_{k=k_0}^{\infty} c_k t^{k/n}$ where c_k is complex, n is a positive integer, and k_0 is an arbitrary integer. The field is denoted $\mathbb{C}\{\{t\}\}$.

Definition 7. The valuation map on $\mathbb{C}\{\{t\}\}$ is **order**, where $\mathrm{order}(c(t))$ is the smallest exponent on t in the Puiseux series $c(t)$.

With the order map, we can finally define tropicalization.

Definition 8. Let $f = \sum_{u \in \mathbb{N}^{n+1}} c_u x^u$ be a polynomial in $K[x_1, \ldots, x_n]$. The **tropicalization** of f is the function $\mathrm{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathrm{trop}(f)(w) = \min\{\mathrm{val}(c_u) + w \cdot u : c_u \neq 0\}.$$

Example 4. Let $f = (1+t)x - 2t^{-1}y + t^3x^2y \in K[x, y]$, where $K = \mathbb{C}\{\{t\}\}$. We are interested in $\mathrm{trop}(f)$.

We see that $\mathrm{ord}(1+t) = 0$, $\mathrm{ord}(-2t^{-1}) = -1$, and $\mathrm{ord}(t^3) = 3$. Therefore,

$$\mathrm{trop}(f)(x, y) = 0 \odot x \oplus -1 \odot y \oplus 3 \odot 2x \odot y.$$

However, we can tropicalize more than polynomials. Tropicalizing varieties allows us to tropicalize hypersurfaces, giving us the ability to examine conics and hyperplanes using tropical geometry.

Definition 9. Let $I \subseteq K[x_1^{\pm 1}, \cdots x_n^{\pm 1}]$, the Laurent polynomial ring, where $V(I) \subseteq (K \backslash \{0\})^n$. Then,

$$\mathcal{T}(I) = \overline{\mathrm{val}(V(I))} \subseteq \mathbb{R}^n$$

is a **tropical variety**, where $\overline{\mathrm{val}(V(I))}$ is the closure in $\mathbb{R}^n$ of the coordinatewise valuation of all points in $V(I)$.

Equivalent ways to define $\mathcal{T}(I)$ will be shown in Chapter 4.

Tropical varieties allow us to study conics and hyperplanes with tropical rather than classical algebra. However, we've only been discussing tropical geometry in affine space, but to answer our question, we need to explore projective space, both classically and then tropically.

Chapter 3

Projective Space

3.1 What is Projective Space?

Informally, some authors describe projective space as a space that views points (and in higher dimensions, lines and planes) at infinity just the same as regular points. We provide a more formal definition.

Definition 10. Given a finite-dimensional vector space V over a field K, the **projective space** $\mathbb{P}(V)$ induced by V is the set $V \backslash \{0\}/\sim$ of equivalence classes of nonzero vectors in V under the equivalence relation $\sim$ defined such that for all $u, v \in V \backslash \{0\}$,

$$u \sim v \text{ if and only if } v = \lambda u, \text{ for some } \lambda \in K \backslash \{0\}.$$

When $V = K^{n+1}$, as it is in this thesis, we note that projective coordinates are of the form $(x_0 : x_1 : x_2 : x_3 : \cdots : x_n)$. This notation highlights that the coordinates are the basis of an equivalence class, which contains all nonzero scalar multiples of the point.

We sometimes think about projective spaces $\mathbb{P}^n$ over a field K as the one dimensional subspaces of the vector space K^{n+1}. Locally, $\mathbb{P}^n$ looks like $\mathbb{A}^n$. To see this, we take a subset of points $\mathcal{U}_i \subset \mathbb{P}^n$, where these points are $(z_0 : \cdots : z_n)$ with $z_i \neq 0$. We then call $\mathcal{U}_i$ a **standard open set** of $\mathbb{P}^n$. We can then show $\mathcal{U}_i \cong \mathbb{A}^n$.

We define $\pi : \mathcal{U}_i \longrightarrow \mathbb{A}^n$, where

$$\pi(z_0 : \cdots : z_n) = (\frac{z_0}{z_i}, \ldots, \frac{\hat{z_i}}{z_i}, \ldots, \frac{z_n}{z_i})$$

and $\frac{\hat{z_i}}{z_i}$ denotes that we leave this coordinate out when looking at the coordinates in $\mathbb{A}^n$.

We then define $\phi : \mathbb{A}^n \to \mathcal{U}_i$, where

$$\phi(z_0, ..., z_n) = (z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n).$$

These maps are mutually inverse, and therefore define a bijection. Since these maps are polynomial maps, it follows that $\mathcal{U}_i$ is isomorphic to $\mathbb{A}^n$. These sets $\mathcal{U}_i$ are called the standard cover of $\mathbb{P}^n$ by affine open sets.

In tropical geometry, it is common to use $\mathbb{R}^{n+1}/\mathbb{R}(1, 1, ..., 1)$ when referring to tropical projective space $\mathbb{TP}^n$. Because of tropical operations, vectors u, v in tropical projective space are equivalent if and only if $v = (\lambda + u_1 : \cdots : \lambda + u_n)$.

3.2 Projective Varieties

We have affine varieties in affine space, and similarly we have projective varieties in projective space.

Definition 11. A **homogeneous** polynomial is a polynomial that is the sum of monomials of the same degree.

Polynomials in $K[x_0, ..., x_n]$ on the vector space K^{n+1} are not necessarily well-defined on $\mathbb{P}^n$. However, for a degree d homogeneous polynomial $f \in K[x_0, ..., x_n]$, we see that

$$f(\lambda x_0, ..., \lambda x_n) = \lambda^d \cdot f(x_0, ..., x_n)$$

and $\{(x_0, ..., x_n) | f(x_1, ..., x_n) = 0\}$ is well-defined on $\mathbb{P}^n$. Therefore, the idea of a projective variety makes sense.

Definition 12. A **projective variety** is the zero set of a collection of homogeneous polynomials.

In the previous chapter we were able to tropicalize affine varieties. We can also do this for projective varieties.

Definition 13. Let I be a homogeneous ideal in the Laurent polynomial ring $K[x_1^{\pm 1}, ..., x_n^{\pm 1}]$. A **tropical projective variety** is a subset of $\mathbb{TP}^{n-1}$ of the form

$$\mathcal{T}(I) = \overline{\text{val}(V(I))}/\mathbb{R}(1, 1, ..., 1).$$

This is similar to the definition of a tropical variety, with the main difference being that a tropical projective variety is in $\mathbb{TP}^{n-1} = \mathbb{R}^n/\mathbb{R}(1,1,1,\dots,1)$, while a tropical variety is in $\mathbb{R}^n$.

Question 1 is interested in studying quadric surfaces in projective spaces. Therefore, projective varieties are important, because we use them to represent quadric surfaces in projective space. With this projective variety, we will then show the surface it defines contains two families of lines, by which we mean two copies of $\mathbb{P}^1$, and is therefore doubly ruled. This process of embedding will be covered when we discuss the Grassmannian.

Chapter 4

The Grassmannian

We started with affine varieties, and now have defined projective space and projective varieties. In this chapter, we'll discuss some of the tools that take us from affine to projective space, as well as mappings from projective lines, $\mathbb{P}^1$, to projective spaces of higher dimensions. These tools are necessary for studying quadric surfaces in projective space, and what we hope to find tropical equivalents to when we begin working towards an answer of Question 1.

4.1 The classical Grassmannian

We begin by defining the Grassmannian. The material in this section is a summary of Chapter 11 in Hassett (2007) and Lecture 6 of Harris (1992).

Theorem 4.1. *The* ***Grassmannian****, $G(k,n)$, is a smooth projective variety of dimension $k(n-k)$. Each point corresponds to an k-dimensional linear subspace of a fixed n-dimensional vector space V.*

In other words, the Grassmannian is the set of all k-dimensional hyperplanes through the origin. In this paper, we will use $G(k,n)$ for the Grassmannian and $\mathbb{G}$ for the projective Grassmannian.

Definition 14. The **projective Grassmannian** $\mathbb{G}(k-1,n-1)$ is the set of $(k-1)$-dimensional planes in $\mathbb{P}^{n-1}$.

Theorem 4.2. *The Grassmannian and the projective Grassmannian relate in the following way:*

$$\mathbb{G}(k-1,n-1) \cong G(k,n).$$

To use the Grassmannian, we also need some exterior algebra.

Definition 15. Let $V = \{c_1e_1 + \cdots + c_ne_n\}$ be a finite dimensional vector space with basis $\{e_1, ..., e_n\}$. For each $k = 0, ..., n$, the **kth exterior power** is defined as the vector space

$$\bigwedge^k V = \{ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} c_{i_1 \ldots i_k} e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k} \}$$

with the convention that $\bigwedge^0 V = K$, where K is the field underlying V.
The **wedge product**, $\wedge$, has the following properties:

1. $v \wedge v = 0$

2. $v \wedge w = -w \wedge v$

3. $(av + w) \wedge u = av \wedge u + w \wedge u$

Proposition 4.1. *The vector space $\bigwedge^k(V)$ has dimension $\binom{n}{k}$ and a basis given by $\{e_I\}$, with I ranging over all subsets of size k of $\{1, \ldots, n\}$ and $\{e_i\}$ being the standard lexicographically ordered basis.*

Corollary 4.1. *$\bigwedge^k(V)$ is isomorphic to $\mathbb{R}^{\binom{n}{k}}$ as a vector space. As a result, $\mathbb{P}(\bigwedge^k(V)) \cong \mathbb{P}^{\binom{n}{k}-1}$.*

Now, we can define the Plücker embedding.

Definition 16. Let W be a k-dimensional plane in K^n spanned by vectors $w_1, \ldots, w_k$. Then we define a map $\Phi : G(k, n) \to \mathbb{P}(\bigwedge^k(V))$ by

$$W \mapsto [w_1 \wedge \cdots \wedge w_k]$$

The map Φ is known as the **Plücker embedding**. We see that this embedding is well-defined: Let us consider another ordered basis for W, $w_i = (\sum_j a_{ij} v_j)_{1 \leq i \leq k}$ where $A = (a_{ij})$ is an invertible matrix. Then, $v_1 \wedge \cdots \wedge v_k = \det(A)(w_1 \wedge \cdots \wedge w_k)$, and these points are equivalent in projective space.

We demonstrate the Plücker embedding through an example.

Example 5. We look at two dimensional subspaces, or planes, in $\mathbb{R}^4$, which are within $G(2,4)$. We define a plane P by the span of the vectors $(1,2,0,0)$ and $(0,0,3,4)$. Then, we have

$$\begin{aligned}(1,2,0,0) \wedge (0,0,3,4) &= (1e_1 + 2e_2) \wedge (3e_3 + 4e_4) \\ &= 3e_1 \wedge e_3 + 4e_1 \wedge e_4 + 6e_2 \wedge e_3 + 8e_2 \wedge e_4\end{aligned}$$

which corresponds to the point $(0:3:4:6:8:0) \in \mathbb{P}^5$.

It is shown in Harris (1992) that the image of the Grassmannian under this map is a variety. The embedding of the Grassmannian satisfies a set of homegeneous equations, called the Plücker relations. A complete description of them can be found in Hassett (2007).

4.2 The tropical Grassmannian

This section is rooted in the findings published in Speyer and Sturmfels (2003). We begin from the tropical geometry we've already discussed, and work up to defining the tropical Grassmannian.

We first defined tropical varieties in terms of tropicalized ideals. We provide another definition to highlight the geometric and algebraic properties of the set. We let K be the field of Puiseux series we introduced earlier, $\mathbb{C}\{\{t\}\}$, and $\mathrm{ord}(c(t))$ be the order valuation from Definition 7.

Definition 17 (Maclagan and Sturmfels (2015))**.** Consider a polynomial f and its tropicalization, $\mathrm{trop}(f)$.
Fix $w \in (\mathbb{R})^{n+1}$ and let $W = \mathrm{trop}(f)(w) = \min\{\mathrm{ord}(c_u) + w \cdot u : c_u \neq 0\}$. We also let $\overline{a}$ be the image of a in the residue field $\Bbbk = \mathbb{C}\{\{t\}\}/m$, where maximal ideal m is the set of elements with a positive valuation in $\mathbb{C}\{\{t\}\}$.
The **initial form of f with respect to w** is

$$\mathrm{in}_w(f) = \sum_{\substack{u \in \mathbb{N}^{n+1}: \\ \mathrm{ord}(c_u) + w \cdot u = W}} \overline{c_u t^{-\mathrm{ord}(c_u)} x^u} \in \Bbbk[x_0, \ldots, x_n].$$

Definition 18. Given an ideal $I \subset K[x]$, its **initial ideal** is defined to be

$$\mathrm{in}_w(I) = \left\langle \mathrm{in}_w(f) : f \in I \right\rangle \subset \Bbbk[x].$$

We then have the following theorem.

Theorem 4.3 (Theorem 2.1, Speyer and Sturmfels (2003)). *For an ideal $I \subset K[x]$, the following subsets of $\mathbb{R}^n$ coincide*:

1. *The closure of the set* $\{(ord(u_1), \ldots, ord(u_n)) : (u_1, \ldots, u_n) \in V(I)\}$;
2. *The intersection of the tropical hypersurfaces* $\mathcal{T}(trop(f))$ *where* $f \in I$;
3. *The set of all vectors* $w \in \mathbb{R}^n$ *such that* $in_w(I)$ *contains no monomial.*

This subset of $\mathbb{R}^n$ *is the tropical variety* $\mathcal{T}(I)$.

We then use this theorem to draw conclusions about the tropical Grassmannian.

First, we fix a polynomial ring in $\binom{n}{d}$ variables with integer coefficients, denoted with $\mathbb{Z}[p] = \mathbb{Z}[p_{i_1,i_2,\ldots,i_d} : 1 \le i_1 < i_2 < \cdots < i_d \le n]$. The coordinate p_{i_k} is referred to as a Plücker coordinate.

Definition 19. The **Plücker ideal** $I_{d,n}$ is the homogeneous prime ideal in $\mathbb{Z}[p]$ consisting of the algebraic relations among the $d \times d$-minors of an arbitrary $d \times n$-matrix.

We defined a projective variety earlier to be the zero locus of a collection of homogeneous polynomials. We can provide another definition of the Grassmannian using this.

Theorem 4.4. *The projective variety of* $I_{d,n}$ *is the* ***Grassmannian*** $G(d, n)$.

We can now provide the definition of the tropical Grassmannian, from Speyer and Sturmfels (2003).

Definition 20. The **tropical Grassmannian** $\mathcal{G}_{d,n}$ is the tropical variety $\mathcal{T}(I_{d,n})$ over a field K. Theorem 4.3(c) implies

$$\mathcal{G}_{d,n} = \{w \in \mathbb{R}^{\binom{n}{d}} : \text{in}_w(I_{d,n}) \text{ contains no monomial}\}.$$

To study Question 1, we are interested in the lines contained in $\mathbb{TP}^3$. We see in Speyer and Sturmfels (2003) that the tropical Grassmannian of lines in $\mathbb{TP}^3$ is the tropical projective variety $\mathcal{T}(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})$, where $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$ is the Plücker relation corresponding to $n = 4$, $d = 2$.

Such lines can be viewed as the tropicalization of classical lines, but they have also been described tropically in Jürgen et al. (2003).

A tropical line in $\mathbb{TP}^3$ in the tropical Grassmannian has the coordinates $(a_{12} : a_{13} : a_{14} : a_{23} : a_{24} : a_{34})$. These coordinates come from the tropical basis of I.

Theorem 4.5 (Theorem 2.6, Jürgen et al. (2003))**.** *Every ideal I has a finite subset $\mathcal{B}$ with the following properties*:

1. *If $w \in \mathcal{T}(I)$ then $\{in_w(b) : b \in \mathcal{B}\}$ generates the initial ideal $in_w(I)$.*
2. *If $w \notin \mathcal{T}(I)$ then $\{in_w(b) : b \in \mathcal{B}\}$ contains a monomial.*

   *The finite set $\mathcal{B}$ is a **tropical basis** of I.*

For an ideal that defines a tropical line in $\mathbb{TP}^3$, the tropical basis of I is

$$\begin{aligned}\mathcal{B} = \{&p_{12} \cdot x_2 + p_{13} \cdot x_3 + p_{14} \cdot x_4,\\ &-p_{12} \cdot x_1 + p_{23} \cdot x_3 + p_{24} \cdot x_4,\\ &-p_{13} \cdot x_1 - p_{23} \cdot x_2 + p_{34} \cdot x_4,\\ &-p_{14} \cdot x_1 - p_{24} \cdot x_2 - p_{34} \cdot x_3\}\end{aligned}$$

where the coefficients satisfy the Plücker relation above. Then, the coordinate a_{ij} is the valuation of p_{ij}, which is in the field of Puiseux series. Because the Grassmannian is a corner locus, the lines fall into these three cases:

$$\begin{aligned}&a_{14} + a_{23} = a_{13} + a_{24} \leq a_{12} + a_{34},\\ &a_{14} + a_{23} = a_{12} + a_{34} \leq a_{13} + a_{24}, \text{or}\\ &a_{13} + a_{24} = a_{12} + a_{34} \leq a_{14} + a_{23}\end{aligned}$$

which combinatorially, are the trivalent trees with four labeled leaves, shown in Speyer and Sturmfels (2003).

4.2.1 The geometry of $\mathcal{G}(2, 4)$

One result in Speyer and Sturmfels (2003) allows us to better understand the geometry of $\mathcal{G}(2, 4)$. Before stating this result, we must provide some definitions.

Definition 21. Let $S \subset \mathbb{R}^n$. Then, the **convex hull** of S, sometimes denoted $\text{conv}(S)$, is the smallest convex set containing S. If S is a finite point set, then $\text{conv}(S)$ is a **polytope**.

Definition 22. A **polyhedron** is the intersection of finitely many closed half-spaces in $\mathbb{R}^n$.

Polyhedra come together to form a complex, which must satisfy some rules.

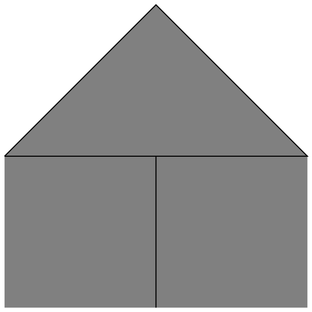

Figure 4.1 Not a polyhedral complex

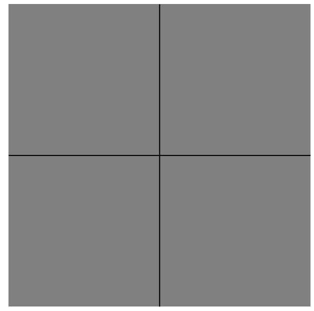

Figure 4.2 An example of a polyhedral complex

Definition 23. A **face** of a polyhedron P is the set of the form $\{x \in P : wx \leq wy \text{ for all } y \in P\}$, where w is a linear functional. This is sometimes denoted by $\text{face}_w(P)$.

Definition 24. A **polyhedral complex** is a collection of polyhedra such that

1. If P is in the collection, then so is any face of P.
2. If P and Q are both in the collection, then $P \cap Q$ is a face of both P and Q or the empty set.

The **support** of a polyhedral complex is the union of its polyhedra.

In Figure 4.1, the polyhedra do not form a complex, but Figure 4.2 is a polyhedral complex. Figure 4.1 fails second requirement for polyhedral complexes, as the intersection of one of the squares and the triangle is not a face of the triangle.

When a polyhedral complex is made up of cones from the origin, we call it a **polyhedral fan**. This leads us to the result.

Theorem 4.6 (Corollary 3.1, Speyer and Sturmfels (2003))**.** *The tropical Grassmannian $\mathcal{G}_{d,n}$ is a polyhedral fan in $\mathbb{R}^{\binom{n}{d}}$. Each of its maximal cones has the same dimension, namely $(n-d)d+1$.*

We explore this result using $\mathcal{G}(2,4)$, guided by Example 3.2 in Speyer and Sturmfels (2003) and Example 4.3.1 in Maclagan and Sturmfels (2015).

Example 6. We know that $G(2,4)$ is a hypersurface in $\mathbb{P}^5$, because of the Plücker embedding, defined by $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$, the corresponding Plücker relation. Corollary 4.6 tells us that $\mathcal{G}(2,4)$ is a polyhedral fan in $\mathbb{R}^6$, and that each of its three maximal cones has dimension 5. These cones are $\mathbb{R}^4 \times \mathbb{R}_{\leq 0}$.

From Speyer and Sturmfels (2003), the image of $\mathcal{G}(2,4)$ in $\mathbb{R}^6/\mathbb{R}(1,1,....,1)$ is a fan $\mathcal{G}'(2,4)$ of dimension 4.

We define a map

$$\phi : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{d}}$$

where n-vector $(a_1, ..., a_n)$ maps to a $\binom{n}{d}$-vector whose $(i_1, ..., i_d)$-coordinate is $a_{i_1} + \cdots + a_{i_d}$. The image of ϕ is the lineality space L, which is

$$L = \text{span}(\sum_{I:i\in I} e_I : 1 \leq i \leq n) \subseteq \mathbb{R}^{\binom{n}{d}}.$$

In this case with $n = 4$, we have

$$L = \text{span}(e_{12} + e_{13} + e_{14}, e_{12} + e_{23} + e_{24}, e_{13} + e_{23} + e_{34}, e_{14} + e_{24} + e_{34}).$$

The image of ϕ is also the common intersection of all cones in $\mathcal{G}(2,4)$, which in this example is $\mathbb{R}^4$. In terms of the lineality space, these cones are $L + \text{pos}(e_{12} + e_{34})$, $L + \text{pos}(e_{13} + e_{24})$, and $L + \text{pos}(e_{23} + e_{14})$.

We can further reduce the dimension of the tropical Grassmannian using the process outlined in Speyer and Sturmfels (2003). The image of $\mathcal{G}(2,4)$ in $\mathbb{R}^6/\mathbb{R}^4$ is a fan $\mathcal{G}''(2,4)$ of dimension 1. We identify this fan by sending e_{12}, e_{34} to $(1,0)$, e_{13}, e_{24} to $(0,1)$, and e_{14}, e_{23} to $(-1,-1)$; therefore, $\mathcal{G}''(2,4)$ is a tropical line in $\mathbb{R}^2$.

Finally, if we intersect $\mathcal{G}''(2,4)$ with a unit sphere, we get a polyhedral complex $\mathcal{G}'''(2,4)$. Each maximal face is a polytope with dimension 0, so $\mathcal{G}'''(2,4)$ is three points.

Chapter 5

The classical solution

We hope to use the methods and tools in the classical solution to aid us in answering Question 1. Therefore, the classical solution provided here could be a roadmap to how one might solve the analogous tropical problem. This solution follows the one given in O'Neill (2013), and with assistance from Harris (1992). To start, we need to characterize quadric surfaces and then prove some necessary lemmas.

Lemma 5.1. *Let $Q \subset \mathbb{P}^3$ be the nonsingular variety of a single nondegenerate homegeneous degree 2 polynomial $f(x_0, \ldots, x_3)$. Then, there exists a change of coordinates $(x_0, \ldots, x_3) \longrightarrow (z_0, \ldots, z_3)$ such that $Q = V(z_0 z_3 - z_1 z_2)$.*

Proof. The proof in Chapter 5 of O'Neill (2013) uses symmetric bilinear forms to produce a change of coordinates from (x_0, x_1, x_2, x_3) to (z_0, z_1, z_2, z_3). $\square$

To demonstrate that lines on a classical quadric are disjoint conics in the Grassmannian $G(2, 4)$, we use the following embedding.

Definition 25. The **Segre embedding**,

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}$$

takes a pair of points $((x_0 : \cdots : x_n), (y_0 : \cdots : y_m)) \in \mathbb{P}^n \times \mathbb{P}^m$ to $(x_0 y_0 : x_0 y_1 : \cdots : x_n y_m)$. The image of the map is a variety, called the **Segre variety**.

In projective space, projective lines are denoted with $\mathbb{P}^1$, and points on projective lines are of the form $(x_0 : x_1)$.

We now can prove the classical result.

Theorem 5.1 (see lecture 2 and 22 of Harris (1992))**.** *Let K be algebraically closed. Let Q be a smooth (non-degenerate) quadric surface in $\mathbb{P}^3$. Then, the set of lines contained in Q is two disjoint conics in $\mathbb{G}(2,4) \subseteq \mathbb{P}^5$. A description of the lines is given in the proof.*

Proof. This proof is presented in O'Neill (2013). We assume by Lemma 5.1, without loss of generality, that Q is the zero set of the quadratic $z_0z_3 - z_1z_2 = 0$. Q is a quadric surface in $\mathbb{P}^3$, so it is the image of the Segre embedding for $n = m = 1$. Therefore, there are two families of lines are contained in Q. With the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$,

$$\sigma((a_0 : a_1) \times \mathbb{P}^1) = \{(a_0y_0 : a_0y_1 : a_1y_0 : a_1y_1) | (y_0 : y_1) \in \mathbb{P}^1\}$$

$$\sigma(\mathbb{P}^1 \times (b_0 : b_1)) = \{(x_0b_0 : x_0b_1 : x_1b_0 : x_1b_1) | (x_0 : x_1) \in \mathbb{P}^1\}$$

These are all the lines contained in Q. By examining the inverse map of σ, given by

$$\sigma^{-1}((z_0 : z_1 : z_2 : z_3)) = \begin{cases} ((z_0 : z_2), (z_0 : z_1)) : z_0 \neq 0 \\ ((z_1 : z_3), (z_0 : z_3)) : z_1 \neq 0 \\ ((z_0 : z_2), (z_2 : z_3)) : z_2 \neq 0 \\ ((z_1 : z_3), (z_2 : z_3)) : z_3 \neq 0. \end{cases}$$

we see σ^{-1} is linear. We only consider lines in $\mathbb{P}^1 \times \mathbb{P}^1$, because if $L \subseteq Q$, then $\sigma^{-1}(L)$ is a line in $\mathbb{P}^1 \times \mathbb{P}^1$. Because these lines are in $\mathbb{P}^1 \times \mathbb{P}^1$, we can parametrize them using

$$(x : y) \mapsto ((a_0x + b_0y : a_1x + b_1y), (c_0x + d_0y : c_1x + d_1y)).$$

The image of a line in $\mathbb{P}^1 \times \mathbb{P}^1$ under σ is only a line in $\mathbb{P}^3$ if the products of x_iy_j do not have quadratic terms; however,

$$\begin{aligned} \sigma(x : y) = &(a_0c_0x^2 + (b_0c_0 + a_0d_0)xy + b_0d_0y^2 : \\ &a_0c_1x^2 + (b_0c_1 + a_0d_1)xy + b_0d_1y^2 : \\ &a_1c_0x^2 + (b_1c_0 + a_1d_0)xy + b_1d_0y^2 : \\ &a_1c_1x^2 + (b_1c_1 + a_1d_1)xy + b_1d_1y^2) \end{aligned}$$

so we must have either $a_0 = a_1 = b_0 = b_1 = 0$ or $c_0 = c_1 = d_0 = d_1$ in order to eliminate the quadratic terms. Either of these options fixes a coordinate of the lines in $\mathbb{P}^1 \times \mathbb{P}^1$, which we did when looking at the lines contained in Q. We can now use the Plücker embedding to study these lines in $\mathbb{P}^5$. Consider a line $(a_0 y_0 : a_0 y_1 : a_1 y_0 : a_1 y_1) \in \mathbb{P}^3$. Lines in $\mathbb{P}^3$ correspond to planes through the origin in K^4, and the plane corresponding to this line is spanned by vectors $(a_0, 0, a_1, 0)$ and $(0, a_0, 0, a_1)$.

We use the definition of the Plücker embedding to find what point this line corresponds to. The wedge product $(a_0, 0, a_1, 0) \wedge (0, a_0, 0, a_1)$ corresponds to the point $(a_0^2 : 0 : a_0 a_1 : -a_0 a_1 : 0 : a_1^2) \in \mathbb{P}^5$. Our other line is $(b_0 x_0 : b_1 x_0 : b_0 x_1 : b_1 x_1)$, which corresponds to the plane spanned by $(b_0, b_1, 0, 0)$ and $(0, 0, b_0, b_1)$. We see that the Plücker embedding then takes (b_0, b_1) to $(0 : b_0^2 : b_0 b_1 : -b_0 b_1 : b_1^2 : 0)$. Both of these points are parametrizations of conics, and these conics are disjoint because no line is in both families of lines, these conics are disjoint. $\square$

Chapter 6

Tropical Hypersurfaces and Polyhedral Complexes

In subsection 4.2.1, we discussed the geometry of the tropical Grassmannian $\mathcal{G}(2,4)$. Here, we further explore the connections between tropical hypersurfaces and polyhedral geometry, as well as define what it means to be a smooth quadric and look at the geometry of the tropicalization of the quadric discussed in Chapter 5. This will allow us to work with smooth quadrics in $\mathbb{P}^3_K$, where K is the field of Puiseux series, $\mathbb{C}\{\{t\}\}$.

6.1 Connecting Polyhedral Geometry to Tropical Hypersurfaces

In this section, we introduce more polyhedral geometry and provide a geometrical definition of a tropical hypersurface.

Proposition 6.1 (Maclagan and Sturmfels (2015), Proposition 3.1.6)**.** *Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial. The tropical hypersurface trop$(V(f))$ is the support of a pure Γ_{val}-rational polyhedral complex of dimension $n-1$ in $\mathbb{R}^n$. It is the $(n-1)$-skeleton of the polyhedral complex dual to a regular subdivision of the Newton polytope of $f = \sum c_u x^u$ given by the weights val(c_u) on the lattice points in Newt(f).*

In subsection 4.2.1 we defined a polyhedral complex to be a collection of polyhedra that satisfy certain conditions, and the support of a polyhedral complex to be the union of its polyhedra. A **pure** polyhedral complex is one where all maximal polyhedra have the same dimension. The image of our

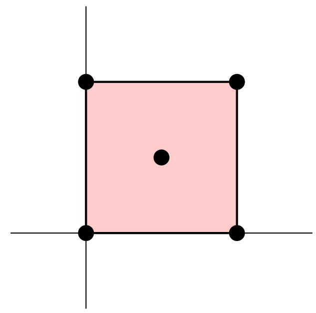

Figure 6.1 Newton Polytope of $f = x^2 + 3x^2y^2 + 3xy + 5y^2 + 1$

valuation map, order, is represented by Γ_{val}, and Γ_{val}-rational means that the points of the polyhedral complex are in Γ_{val}. The range of the order map is in $\mathbb{R}$, so the points of the polyhedral complex must therefore be real. The $(n-1)$-skeleton of a polyhedral complex is the set of all faces of the complex of dimension at most $n-1$.

To complete our understanding of the proposition, we define Newton polytopes and subdivisions, which is what we use to characterize tropical quadrics as smooth.

Definition 26. Let $S = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the Laurent polynomial ring. Given $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in S$, the **Newton polytope** of f is the polytope

$$\text{Newt}(f) = \text{conv}(u : c_u \neq 0) \subset \mathbb{R}^n.$$

Example 7. Let $f(x, y) = x^2 + 3x^2y^2 + 3xy + 5y^2 + 1$. The Newton polytope of f is then given by the convex hull of $(0,0), (2,0), (0,2), (1,1), (2,2)$ and illustrated in Figure 6.1.

In this example, the Newton polytope is two dimensional, so it is also a Newton polygon.

The final piece in understanding Proposition 6.1 is to define what it means to be a subdivision of a Newton polytope. We use a construction from Sturmfels (2007) to define a regular subdivision.

Definition 27 (Sturmfels (2007))**.** For a tropical polynomial p, we define a polytope P in $\mathbb{R}^{n+1}$ to be all $(c, u_1, \ldots, u_n)$ where $c \odot x^{u_1} \odot \ldots \odot x_n^{u_n}$ is a term of p. Then, consider the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ on the last n coordinates. The Newton polytope of p is the image of P under projection π, because the points of the Newton polytope are exactly the last n coordinates of the points on polytope P.

We define a linear functional $w' = (1, w) \in \mathbb{R}^{n+1}$, where $w \in \mathbb{R}^n$. Then, we define a **lower face** of P to be a face F of P that has the form $F = \text{face}_{w'}(P)$. The set of all lower faces of P form a pure polyhedral complex of dimension n, and the image of this complex under π is the **regular subdivision** of the Newton polytope of p.

By Proposition 6.1, the tropical hypersurface of a tropical polynomial trop(f) is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision, as described above.

6.1.1 The polyhedral geometry associated with trop$(V(x_0x_3-x_1x_2))$

As mentioned before, the valuation map we use to tropicalize polynomials is the order map from Definition 7. Consider a polynomial whose coefficients are in $\mathbb{C}$. The order of these coefficients is then always zero. This is then a special case of Proposition 6.1.

Proposition 6.2 (Maclagan and Sturmfels (2015)). *Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial whose coefficients all have valuation zero. Then the tropical hypersurface trop$(V(f))$ is the support of an $n-1$-dimensional polyhedral fan in $\mathbb{R}^n$. That fan is the $(n-1)$-skeleton of the normal fan to the Newton polytope of f.*

We first define a normal fan.

Definition 28. Let the **normal cone** of a face F of a polyhedron P be

$$N_p(F) = \{w \in \mathbb{R}^n : \text{face}_w(P) \subseteq F\}.$$

The **normal fan** of a polyhedron P is then

$$\mathcal{N}(P) = \{N_p(F) : F \text{ face of } P\}.$$

We then explore Proposition 6.2 using the polynomial $f(x_0, x_1, x_2, x_3) = x_0x_3 - x_1x_2$, which defines a smooth quadric; in fact, this is the quadric studied in Chapter 5.

Example 8. We first see that $\text{Newt}(x_0x_3-x_1x_2) = \text{conv}((1,0,0,1),(0,1,1,0)) \in \mathbb{R}^4$. The normal fan of this Newton polytope is given in the proof of Proposition 6.2 in Maclagan and Sturmfels (2015) as the complex $\Sigma_{\text{trop}(f)}$, which has maximal cells

$$\sigma = \{w \in \mathbb{R}^4 : \text{trop}(f)(w) = a + w \cdot u\}$$

where $a \odot x^u$ runs over monomials of trop(f). In this case, we know that $a = 0$.

The 3-skeleton of this complex is then all the faces of the complex at most dimension 3, and trop$(V(x_0x_3 - x_1x_2))$ is the support of this 3-dimensional skeleton.

Other polynomials that only have coefficients with valuation 0 are those that are the determinants of an $n \times n$ matrix with variables as entries. The polynomial $f(x_0, x_1, x_2, x_3) = x_0x_3 - x_1x_2$ is the determinant of the 2×2 matrix $\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix}$. It is shown in Example 3.1.11 from Maclagan and Sturmfels (2015) that the tropical hypersurface of trop$(x_0x_3 - x_1x_2)$ is a 3 dimensional fan, with 4 rays, and its maximal cones are indexed by pairs (π, π') such that $\pi^{-1} \circ \pi'$ is a cycle.

6.2 Smooth Tropical Quadrics

In Chapter 5, we studied the smooth quadric $V(x_0x_3 - x_1x_2)$, where smooth meant that the quadric was non-degenerate. For a tropical quadric, smoothness comes from the polyhedral geometry of the tropical hypersurface.

Proposition 6.3 (Maclagan and Sturmfels (2015), Proposition 4.5.1)**.** *Fix $f = \sum c_u x^u \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\Delta_{val(c_u)}$ the regular subdivision of the Newton polytope Newt(f) induced by the weights $val(c_u)$. If $\Delta_{val(c_u)}$ is unimodular, then $V(f) \subset T^n$ is a smooth hypersurface.*

Definition 29. A regular subdivision of a Newt(p) is **unimodular** if all simplices have volume $\frac{1}{n!}$.

An algorithm for determining if a tropical quadric defined by generic tropical quadratic polynomial f in $\mathbb{TP}^3$ is smooth can be found in O'Neill (2013).

6.2.1 Study of Smooth Quadric Surfaces in $\mathbb{P}^3$

To study smooth quadric surfaces Q in $\mathbb{P}^3$ and their tropicalizations, we present the characterization of these surfaces as given in Maclagan and Sturmfels (2015).

In chapter 5, we demonstrated that a quadric in $\mathbb{P}^3$ is a ruled surface; that there exactly two families of lines on Q. In chapter 5, we represented these lines in projective coordinates; now we will instead work with polynomial ideals that represent the incidence correspondence for the two rulings.

In Chapter 4, we defined the Plücker embedding, which embedded the Grassmannian $G(k,n)$ into $\mathbb{P}(\bigwedge^k(V))$. Lines in $\mathbb{P}^3$ are $G(2,4)$, so the Plücker embedding maps these lines into $\mathbb{P}^5$. The coordinates of the line L embedded into $\mathbb{P}^5$ are then referred to as Plücker coordinates. They are denoted by $\mathbf{p} = (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34})$. These coordinates satisfy the Plücker relation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$, which was introduced in Example 6.

Points in $\mathbb{P}^3$ are represented by $x = (x_1 : x_2 : x_3 : x_4)$. Then, we consider the following skew-symmetric matrix

$$\begin{pmatrix} \mathbf{p} & \mathbf{x} \\ -\mathbf{x} & 0 \end{pmatrix} = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} & x_1 \\ -p_{12} & 0 & p_{23} & p_{24} & x_2 \\ -p_{13} & -p_{23} & 0 & p_{34} & x_3 \\ -p_{14} & -p_{24} & -p_{34} & 0 & x_4 \\ -x_1 & -x_2 & -x_3 & -x_4 & 0 \end{pmatrix} \tag{6.1}$$

To define the incidence correspondence of points on lines, we need to define the Pfaffian.

Definition 30. The **Pfaffian** of a $2n \times 2n$ matrix A, $\mathrm{pf}(A)$, is defined a $\mathrm{pf}(A)^2 = \det(A)$. If A is an $n \times n$ matrix where n is odd, the Pfaffian vanishes.

The incidence correspondence of points on lines is the irreducible subvariety of $\mathbb{P}^3 \times \mathbb{P}^5$ defined by the 4×4 subpfaffians of the skew-symmetric matrix presented above.

To represent the quadric, we consider the standard quadric $V(x_1^2 + x_2^2 + x_3^2 + x_4^2)$. In chapter 5, we defined a change of coordinates to transform one quadric into another. Here, we similarly define the change of coordinates matrix

$$\mathbf{m} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}$$

Matrix m has 16 coordinates, but our quadrics are in $\mathbb{P}^3$, so we are over-parametrizing the space of quadrics by the $\mathbb{P}^{15}$. For $x = (x_1 : x_2 : x_3 : x_4)$ to lie on the quadric is equivalent to

$$\begin{aligned}(\mathbf{mx})^T(\mathbf{mx}) = {} & (m_{11}^2 + m_{21}^2 + m_{31}^2 + m_{41}^2)x_1^2 + (m_{12}^2 + m_{22}^2 + m_{32}^2 + m_{42}^2)x_2^2 \\ & + (m_{13}^2 + m_{23}^2 + m_{33}^2 + m_{43}^2)x_3^2 + (m_{14}^2 + m_{24}^2 + m_{34}^2 + m_{44}^2)x_4^2 \\ & + 2(m_{11}m_{12} + m_{21}m_{22} + m_{31}m_{32} + m_{41}m_{42})x_1x_2 \\ & + 2(m_{11}m_{13} + m_{21}m_{23} + m_{31}m_{33} + m_{41}m_{43})x_1x_3 \\ & + 2(m_{11}m_{14} + m_{21}m_{24} + m_{31}m_{34} + m_{41}m_{44})x_1x_4 \\ & + 2(m_{12}m_{13} + m_{22}m_{23} + m_{32}m_{33} + m_{42}m_{43})x_2x_3 \\ & + 2(m_{12}m_{14} + m_{22}m_{24} + m_{32}m_{34} + m_{42}m_{44})x_2x_4 \\ & + 2(m_{13}m_{14} + m_{23}m_{24} + m_{33}m_{34} + m_{43}m_{44})x_3x_4.\end{aligned} \tag{6.2}$$

With $\mathbf{p}$ being the upper left part of matrix 6.1, the conditions for a line to lie on the quadric are given by the entries of $\mathbf{pm}^T\mathbf{mp}$, and the 4×4 subpfaffians of matrix 6.1.

As previously discussed, smooth quadrics in $\mathbb{P}^3$ are doubly ruled, and is the union of two disjoint conics in $\mathbb{P}^5$. The following proposition describes these two rulings without proof.

Proposition 6.4 (Maclagan and Sturmfels (2015), Proposition 4.5.5)**.** *Let I denote the ideal generated by $\mathbf{x}^T\mathbf{m}^T\mathbf{mx}$, the entries of $\mathbf{pm}^T\mathbf{mp}$, and the subpfaffians of matrix* 6.1*. This ideal is radical and the intersection of the following two prime ideals, which represent the two rulings in $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^{15}$.*

$$I + \langle (\mathbf{m}^T\mathbf{pm})_{12} - (\mathbf{m}^T\mathbf{pm})_{34}, (\mathbf{m}^T\mathbf{pm})_{13} + (\mathbf{m}^T\mathbf{pm})_{24}, (\mathbf{m}^T\mathbf{pm})_{14} - (\mathbf{m}^T\mathbf{pm})_{23} \rangle$$
$$I + \langle (\mathbf{m}^T\mathbf{pm})_{12} + (\mathbf{m}^T\mathbf{pm})_{34}, (\mathbf{m}^T\mathbf{pm})_{13} - (\mathbf{m}^T\mathbf{pm})_{24}, (\mathbf{m}^T\mathbf{pm})_{14} + (\mathbf{m}^T\mathbf{pm})_{23} \rangle$$

To study a smooth quadric, we then represent it with matrix $\mathbf{m}$, and can then study it using the description above.

Chapter 7

Examining the Lifting of Lines onto the Tropical Quadric Surface

One approach to discussing the kinds of lines on a tropical quadric is to compare quadrics with their tropicalizations. When we consider the tropicalization of some quadric Q, it is important to explore how the lines on Q lift to trop(Q). We proceed by attempting to answer the following questions.

Question 2. Let I be some ideal in the Laurent polynomial ring $K[x_1, x_2, x_3, x_4]$, such that $Q = V(I)$ is a quadric in $\mathbb{P}^3$. Does there exist a line $\ell \subset \mathbb{P}^3$ such that $\ell \not\subset Q$ but $\text{trop}(\ell) \subseteq \text{trop}(Q)$?

In order to better explore the tropicalization of lines, we reintroduce the Fundamental Theorem of Tropical Alegbraic Geometry.

Theorem 7.1 (Fundamental Theorem of Tropical Algebraic Geometry, Maclagan and Sturmfels (2015))**.** *Let I be an ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $X = V(I)$ its variety in the algebraic torus $T^n \cong (K^*)^n$. Then the following three subsets of $\mathbb{R}^n$ coincide*:

(1) *$trop(X) = \bigcap_{f \in I} trop(V(f))$;*

(2) *the closure $\mathbb{R}^n$ of the set of all vectors $w \in \Gamma_{val}^n$ with $in_w(I) \neq \langle 1 \rangle$;*

(3) *the closure of the set of coordinatewise valuations of points in X*:

$$val(X) = \{(val(u_1), \dots, val(u_n)) : (u_1, \dots, u_n) \in X\}.$$

We will use Theorem 7.1 to prove the following theorem, which is our first step in exploring Question 2.

Theorem 7.2. *Let I be some ideal in the Laurent polynomial ring $K[x_1, x_2, x_3, x_4]$. Let $X = V(I)$ be the variety of I in the algebraic torus T^n. Let ℓ be a line contained in X. Then, the tropicalization of ℓ is contained in the tropicalization of X.*

Proof. Theorem 7.1 tells us that $\mathrm{trop}(X)$ coincides with the closure of the set of the coordinatewise valuations of points in X. Because $\ell \subseteq X$, the coordinatewise valuation of the points on ℓ is contained within this set. Therefore, $\mathrm{trop}(l) \subseteq \mathrm{trop}(X)$. □

To begin exploring Question 2, there are two approaches that can be taken: either look at the lines on a smooth tropical quadric $\mathrm{trop}(Q)$ and see which classical lines they lifted from during tropicalization, or find lines not on the classical quadric and see where the tropicalizations of those lines are, specifically, if they lift onto the tropical quadric. We start with second approach using a specific known smooth quadric detailed in Maclagan and Sturmfels (2015).

7.1 Studying the Lifting on Lines on a Specific Smooth Quadric

In section 6.2.1, we presented a representation of an arbitrary classical quadric in $\mathbb{P}^3$. Each quadric was represented with a 4×4 matrix $\mathbf{m}$. In this section, we will work with a quadric represented by

$$\mathbf{m} = \begin{pmatrix} t + t^3 & t + t^2 & i + t^3 & i + t^3 \\ 1 + t^3 & it + t^2 & i + t^2 & it + t^2 \\ it + t^3 & i + t^2 & 1 + t^2 & 1 + t^2 \\ i + t^2 & 1 + t^3 & 1 + t^3 & it + t^2 \end{pmatrix}$$

in $\mathbb{P}^3_K$, where $K = \mathbb{C}\{\{t\}\}$. The corresponding polynomial, given by equation

6.2, is then

$$\begin{aligned} p(x_1, x_2, x_3, x_4) = {} & (3t^6 + (2i+3)t^4 + 2t^3 + 2it^2)x_1^2 + (t^6 + 3t^4 + (2i+4)t^3 + 2it^2)x_2^2 \\ & + (2t^6 + 2t^4 + (2i+2)t^3 + (2i+2)t^2)x_3^2 + (t^6 + 3t^4 + 6it^3)x_4^2 \\ & + 2(4t^5 + (i+1)t^4 + (3i+1)t^3 + 3t^2 + (i-1)t + i)x_1x_2 \\ & + 2(t^6 + 3t^5 + t^4 + (4i+1)t^3 + 2t^2 + 2it + 2i)x_1x_3 \\ & + 2(t^6 + 2t^5 + (i+2)t^4 + (3i+1)t^3 + (i+1)t^2 + (3i-1)t)x_1x_4 \\ & + 2(t^6 + t^5 + 3t^4 + (i+2)t^3 + (3i+1)t^2 + (i-1)t + i + 1)x_2x_3 \\ & + 2(2t^5 + (i+3)t^4 + 2it^3 + (2i+1)t^2 + 2it + i)x_2x_4 \\ & + 2(t^6 + t^5 + (i+2)t^4 + 3it^3 + (i+3)t^2 + (i-1)t)x_3x_4 \end{aligned}$$

and the condition for $\mathbf{x}$ to lie on the quadric equals the above.

To be able to discuss the quadric more easily, we organize the coordinate vectors and coefficients in Table 7.1.

Coordinate Vector	Corresponding Coefficient
$U_1 = (2000)$	$C_{U_1} = (3t^6 + (2i+3)t^4 + 2t^3 + 2it^2)$
$U_2 = (0200)$	$C_{U_2} = (t^6 + 3t^4 + (2i+4)t^3 + 2it^2)$
$U_3 = (0020)$	$C_{U_3} = (2t^6 + 2t^4 + (2i+2)t^3 + (2i+2)t^2)$
$U_4 = (0002)$	$C_{U_4} = (t^6 + 3t^4 + 6it^3)$
$U_5 = (1100)$	$C_{U_5} = 2(4t^5 + (i+1)t^4 + (3i+1)t^3 + 3t^2 + (i-1)t + i)$
$U_6 = (1010)$	$C_{U_6} = 2(t^6 + 3t^5 + t^4 + (4i+1)t^3 + 2t^2 + 2it + 2i)$
$U_7 = (1001)$	$C_{U_7} = 2(t^6 + 2t^5 + (i+2)t^4 + (3i+1)t^3 + (i+1)t^2 + (3i-1)t)$
$U_8 = (0110)$	$C_{U_8} = 2(t^6 + t^5 + 3t^4 + (i+2)t^3 + (3i+1)t^2 + (i-1)t + i + 1)$
$U_9 = (0101)$	$C_{U_9} = 2(2t^5 + (i+3)t^4 + 2it^3 + (2i+1)t^2 + 2it + i)$
$U_{10} = (0011)$	$C_{U_{10}} = 2(t^6 + t^5 + (i+2)t^4 + 3it^3 + (i+3)t^2 + (i-1)t)$

Table 7.1 Organization of Coefficients of $p(x_1, x_2, x_3, x_4)$

Therefore, the tropicalization of $p(x_1, x_2, x_3, x_4)$ is

$$\begin{aligned} \operatorname{trop}(p) = {} & 2 \odot u_1^2 \oplus 0 \odot u_1u_2 \oplus 0 \odot u_1u_3 \oplus 1 \odot u_1u_4 \\ & \oplus 2 \odot u_2^2 \oplus 0 \odot u_2u_3 \oplus 0 \odot u_2u_4 \oplus 2 \odot u_3^2 \oplus 1 \odot u_3u_4 \oplus 3 \odot u_4^2. \end{aligned}$$

We refer to the classical quadric represented by $V(p)$ as $V(p) = Q$, and the tropical quadric represented by $\operatorname{trop}(V(p))$ as $\boldsymbol{Q} = \operatorname{trop}(V(p))$. It's shown in Maclagan and Sturmfels (2015) that $\boldsymbol{Q}$ is tropically smooth, but one can verify the smoothness of $\boldsymbol{Q}$ with the algorithm in Chapter 7 of O'Neill (2013).

For this approach, we'll pick a projective line ℓ in $\mathbb{P}^3$ and examine the points on the line. If ℓ is not on Q, then for points on ℓ, $\mathbf{x} = (x_1, x_2, x_3, x_4)$, we have $p(\mathbf{x}) \neq 0$. We will then tropicalize ℓ using the Fundamental Theorem of Tropical Geometry. Since ℓ is a line in $\mathbb{P}^3$, it is represented as the zero set of two polynomials, or $\ell = V(F_1, F_2)$. Then by the Fundamental theorem, trop(ℓ) coincides with the closure of the coordinatewise valuation of points in $V(F_1, F_2)$. Since our points on the line are in the field of Puiseux series, the points in the set trop($V(F_1, F_2)$) are coordinates in $\mathbb{R}^4/\mathbb{R}$ with rational coordinates.

We want the points in line ℓ to not be roots of the polynomial that defines the smooth quadric in $\mathbb{P}^3$, because we don't want ℓ to be on the quadric. However, if trop(ℓ) $\subseteq \boldsymbol{Q}$, then the points of trop(ℓ) should be corner loci of trop(p), where $\boldsymbol{Q} = V(\text{trop}(p))$. More specifically, we want an line ℓ such that for $V(F_1, F_2) = \ell$ and $V(p(x_1, x_2, x_3, x_4)) = Q$,

$$V(F_1, F_2) \nsubseteq V(p(x_1, x_2, x_3, x_4))$$

but

$$\overline{\{(\text{ord}(x_1(t)), \text{ord}(x_2(t))), \text{ord}(x_3(t))), \text{ord}(x_4(t))|(x_1(t), x_2(t), x_3(t), x_4(t)) \in V(F_1, F_2)\}} \\ \subseteq \overline{\{\ (\text{ord}(y_1), \text{ord}(y_2), \text{ord}(y_3), \text{ord}(y_4))|(y_1, y_2, y_3, y_4) \in Q\}}.$$

One might ask if the points in ℓ have to have the same coordinatewise valuations as the points (x_1, x_2, x_3, x_4) such that $p(x_1, x_2, x_3, x_4) = 0$ in order to be a corner locus of trop(p).
If f is a polynomial in one variable, then this must be the case.

Proposition 7.1 (Proposition 2.5, Sturmfels (2007))**.** *Let $f(x)$ be a polynomial in one variable with coefficients in the Puiseux series field K and let $u_1, \ldots, u_m$ be the roots of $f(x)$ in K. Their orders ord(u_1), ..., ord(u_m) are the roots of the tropical polynomial trop(f).*

However, Proposition 7.1 does not extend to polynomials in more than one variable. The following example demonstrates a counterexample.

Example 9. Consider the polynomial $f(x, y) = (1 + t)x - 2t^{-1}y$. We see that one root of f is $(2t^{-2}, 1 + t^{-1})$. We have trop(f) $= 1 \odot x \oplus -1 \odot y$. We see ord($2t^{-2}$) $= -2$ and ord($1 + t^{-1}$) $= -1$, so

$$\text{trop}(f)(-2, -1) = -1 \oplus -2$$

and therefore $(-2, -1)$ is not a root of trop(f).

Therefore, the points on ℓ are not forced to have the same coordinatewise valuation as the roots of p.
We then demonstrate this approach through an example.

Example 10. We consider the line defined by $V(F_1, F_2)$, where $F_1 = x_1 + x_2$ and $F_2 = x_3 + x_4$. Then, $V(F_1, F_2)$ is the line $\ell = \{(x_1 : -x_1 : x_2 : -x_2)\}$ in $\mathbb{CP}^3$. We then consider $\ell_K \in \mathbb{P}^3_K$ be the points $\ell_K = \{(x_1(t), -x_1(t), x_2(t), -x_2(t))\}$, where $x_1(t) = t^{a_1} + t^{a_2} + \cdots$ and $x_2(t) = t^{b_1} + t^{b_2} + \cdots$. We let $\operatorname{ord}(x_1(t)) = a_1$ and $\operatorname{ord}(x_2(t)) = b_1$.

We first see that ℓ is not on the quadric Q because not all points in ℓ are roots of $p(x_1, x_2, x_3, x_4)$. We see that

$$\begin{aligned}
p(x(t), -x(t), y(t), -y(t)) &= C_{U_1}x(t)^2 + C_{U_2}x(t)^2 + C_{U_3}y(t)^2 + C_{U_4}y(t)^2 \\
&\quad - C_{U_5}x(t)^2 + C_{U_6}x(t)y(t) - C_{U_7}x(t)y(t) - C_{U_8}x(t)y(t) \\
&\quad + C_{U_9}x(t)y(t) - C_{U_{10}}y(t)^2 \\
&= (C_{U_1} + C_{U_2} - C_{U_5})x(t)^2 + (C_{U_6} - C_{U_7} - C_{U_8} + C_{U_9})x(t)y(t) \\
&\quad + (C_{U_3} + C_{U_4} - C_{U_{10}})y(t)^2
\end{aligned}$$

where

$$\begin{aligned}
C_{U_1} + C_{U_2} - C_{U_5} &= 4t^6 - 8t^5 + 4t^4 + (4 - 4i)t^3 + (-6 + 4i)t^2 - (-2 + 2i)t - 2i \\
C_{U_6} - C_{U_7} - C_{U_8} + C_{U_9} &= -2t^6 + 4t^5 - 2t^4 - (4 - 4i)t^3 + (2 - 4i)t^2 + 4t + (-2 + 4i) \\
C_{U_3} + C_{U_4} - C_{U_{10}} &= t^6 - 2t^5 + (1 - 2i)t^4 + (2 + 2i)t^3 - 4t^2 + (2 - 2i)t.
\end{aligned}$$

If $(x_1(t), -x_1(t), x_2(t), -x_2(t))$ is on quadric Q, then $x_1(t)$ and $x_2(t)$ satisfy the following relation given by the quadratic formula:

$$x_1(t) = x_2(t)\frac{-(C_{U_6} - C_{U_7} - C_{U_8} + C_{U_9}) \pm \sqrt{(C_{U_6} - C_{U_7} - C_{U_8} + C_{U_9})^2 - 4(C_{U_1} + C_{U_2} - C_{U_5})(C_{U_3} + C_{U_4} - C_{U_{10}})}}{2(C_{U_1} + C_{U_2} - C_{U_5})}.$$

We then consider $\operatorname{trop}(\ell)$. Theorem 7.1 tells us that the tropical variety $\operatorname{trop}(\ell)$ is given by the closure of the coordinatewise valuation of the points in $V(F_1, F_2)$. Both a_1 and b_1 are rational numbers, so the closure of all points of the form (a_1, a_1, b_1, b_1) is the set $\{(a, a, b, b)\}$, where a and b are in $\mathbb{R}$. However, $(a, a, b, b) \in \mathbb{R}^4$, and we are looking at points on tropical quadric $\mathcal{Q}$, which is in $\mathbb{TP}^3$. Therefore, we mod by $(1, 1, 1, 1)$ to finally have that

$$\operatorname{trop}(\ell) = \{(a + c, a + c, b + c, b + c)\},$$

where $a, b, c \in \mathbb{R}$. We then plug these points into trop(p) to look at when they are on Q.

$$\begin{aligned}\text{trop}(p)(a+c, a+c, b+c, b+c) &= 2+2a+2c \oplus 2a+2c \oplus a+b+2c \oplus 1+a+b+2c \\ &\oplus 2+2a+2c \oplus a+b+2c \oplus a+b+2c \\ &\oplus 2+2b+2c \oplus 1+2b+2c \oplus 3+2b+2c \\ &= 2+2a+2c \oplus 2a+2c \oplus a+b+2c \oplus 1+a+b+2c \\ &\oplus 2+2b+2c \oplus 1+2b+2c \oplus 3+2b+2c\end{aligned}$$

but we can further simplify this by thinking about where the minimum occurs. In this example, we know $3+2b+2c > 2+2b+2c > 1+2b+2c$, $2+2a+2c > 2a+2c$, and $1+a+b+2c > a+b+2c$. Therefore, we further simplify trop(p) to be

$$\text{trop}(p)(a+c, a+c, b+c, b+c) = 2a+2c \oplus a+b+2c \oplus 1+2b+2c.$$

We then look at the corner locus of trop(p), which is where the minimum is achieved twice. This happens when $a = b$ or $a = b+1$. However, trop(ℓ) contains more points than $(a+c, a+c, a+c, a+c)$ and $(b+1+c, b+1+c, b, b)$, so not all points of trop(ℓ) are also points of $trop(Q)$. Therefore trop(ℓ) is not completely contained within Q.

Another way to evaluate if tropical lines are on the tropical quadric is to use the idea of a tropical basis, defined in Theorem 4.5. A tropical basis of an ideal I can be found using packages such as `Gfan`[1]. If a line is lifting onto the tropical quadric, all points that satisfy the linear inequalities detailed in Example 2.8 from Jürgen et al. (2003) should also be on the tropical quadric.

Example 11. Consider an ideal I, which is generated by a two dimensional space of linear equations in $K[x_1, x_2, x_3, x_4]$, and the tropical variety $\mathcal{T}(I)$, which defines a line in $\mathbb{TP}^3$. Then from Theorem 4.5, the tropical basis of ideal I is

$$\begin{aligned}\mathcal{B} = \{&p_{12} \cdot x_2 + p_{13} \cdot x_3 + p_{14} \cdot x_4, \\ &-p_{12} \cdot x_1 + p_{23} \cdot x_3 + p_{24} \cdot x_4, \\ &-p_{13} \cdot x_1 - p_{23} \cdot x_2 + p_{34} \cdot x_4, \\ &-p_{14} \cdot x_1 - p_{24} \cdot x_2 - p_{34} \cdot x_3\}\end{aligned}$$

[1]See https://users-math.au.dk/jensen/software/gfan/gfan.html

where the p_{ij} satisfy the Plücker relation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. Then, assume I is an ideal such that $\mathrm{ord}(p_{ij}) = 0$ for all p_{ij}, so each p_{ij} is a Puiseux series with a constant. Then from Example 2.8 in Jürgen et al. (2003), we have the following inequalities that describes every point $w \in \mathbb{TP}^3$ on line $\mathcal{T}(I)$:

$$
\begin{aligned}
&x_2 = x_3 \leq x_4 \text{ or } x_2 = x_4 \leq x_3 \text{ or } x_3 = x_4 \leq x_2 \\
&\text{and } x_1 = x_3 \leq x_4 \text{ or } x_1 = x_4 \leq x_3 \text{ or } x_3 = x_4 \leq x_1 \\
&\text{and } x_1 = x_2 \leq x_4 \text{ or } x_1 = x_4 \leq x_2 \text{ or } x_2 = x_4 \leq x_1 \\
&\text{and } x_1 = x_2 \leq x_3 \text{ or } x_1 = x_3 \leq x_2 \text{ or } x_2 = x_3 \leq x_1
\end{aligned}
$$

We consider a $w \in \mathbb{TP}^3$ such that $x_2 = x_3 \leq x_4$, $x_1 = x_3 \leq x_4$, $x_1 = x_2 \leq x_4$, and $x_1 = x_2 \leq x_3$. This implies that $x_1 = x_2 = x_3 \leq x_4$.

To compare points on $\mathcal{T}(I)$ to points on our smooth tropical quadric, we look at $\mathrm{trop}(p)(w)$ and simplify by removing repeated terms and identifying terms that will never be the minimum:

$$
\begin{aligned}
\mathrm{trop}(p)(w) &= 2 + 2x_1 \oplus 2x_1 \oplus 2x_1 \oplus 1 + x_1 + x_4 \oplus 2 + 2x_1 \oplus 2x_1 \\
&\qquad\qquad \oplus x_1 + x_4 \oplus 2 + 2x_1 \oplus 1 + x_1 + x_4 \oplus 3 + 2x_4 \\
&= 2x_1 \oplus x_1 + x_4 \oplus 3 + 2x_4.
\end{aligned}
$$

If $x_1 = x_4$, then this is a corner locus of $\mathrm{trop}(p)$, since $2x_1 < 3 + 2x_1$. However, we also want to look at points where $x_1 < x_4$. Then, the minimum cannot be attained twice, since $2x_1 < 3+2x_4$, $2x_1 < x_1+x_4$, and $x_1+x_4 < 3+2x_4$. So, not all points of the line $\mathcal{T}(I)$ are on the tropical quadric Q.

There could be several ideals who satisfy the conditions in the above example. Working with a line's tropical basis rather than tropicalizing $\ell = V(I)$ could provide an easier way to study lines and their points, since one can group lines by the valuations of the p_{ij} that define their bases.

Chapter 8

Conclusion and Future Work

We've outlined an approach to determining how lines lift onto a general smooth tropical surface in $\mathbb{TP}^3$, which is a necessary step towards understanding which lines are on the general smooth tropical quadric and what their structure is. In the classical problem, there are many tools one can use to study the lines on a smooth surface in $\mathbb{P}^3$, but we are still lacking many of the tropical analogues necessary to create a proof mirroring the classical one. However, our approach provides an opportunity to explore one of the questions necessary to provide an answer to Question 1.

To understand the structure of tropical lines contained in a general smooth tropical surface, we are interested in learning where those lines lift from. Tropicalization can yield interesting and maybe unexpected results: different Puiseux series can have the same valuation, and two seemingly different tropical polynomials can have the same hypersurface. Therefore, it's important to ask if a projective line not on a smooth surface can lift to the tropicalization of that smooth surface. The smooth surface has two families of lines; so if this lifting is possible, we know that lines on the smooth tropical surfaces exist that are not the tropicalizations of the families of lines. It would then be informative to compare the structure of lines that lifted from outside the smooth surface to lines that lited from inside the smooth surface.

This approach was detailed in Chapter 7, but not explored to its full potential. Next steps might be very computational, such as generating large sets of lines in $\mathbb{P}^3$, taking the valuations of the points of those lines, and plugging them into the equation that defines a smooth tropical quadric. With more time and better knowledge of the computational tools available in tropical geometry, this would have been the direction the thesis continued in, especially in the continued exploration of tropical bases. It would also be useful to explore tropical lifting literature to work towards an answer for Question 2.

To answer Question 1, it appears that many small steps must be taken in order to come to a conclusive answer. Tropical analogues to classical tools are still needed to understand how tropical projective lines and surfaces behave. Speyer and Sturmfels (2003) provides one piece by defining the tropical Grassmannian, O'Neill (2013) contains an algorithm that can easily find smooth quadrics and their combinatorial types, and Jürgen et al. (2003) defines tropical projective space and the requirements on a tropical projective line. Continuing to tropicalize the ideas necessary to build a proof, even without the explicit goal of answering Question 1, provide another necessary part of the big picture.

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