Using Functional Data Models in longevity and mortality hedging: a cross-gender study

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Presented to

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Thesis presented for the degree of
Master's in Finance

University of Sherbrooke
Sherbrooke
May 2021

Résumé

Cet essai a pour but d'étudier le risque de longevité dans la population Canadienne et de proposer des solutions de couverture adaptées à un fond de pension. Il se joint à une série de travaux dans le but de créer une ouverture vers un marché liquide et transparant de produits de longevité et/ou de mortalité.

Le risque de longevité d'un fond de pension peut être défini comme le risque que ses rentiers vivent plus longtemps que prévu. C'est un réel enjeu pour ces institutions, qui représentent une grande partie du marché financier Canadien. Un marché de produits derivés financiers qui permettent de se couvrir pour ce risque sera de mise étant donné la réalité démographique future du pays.

Cet essai a pour but de contribuer à la littérature scientifique en validant la portée possible des modèles de type *functional data approach* (FDA). Nous allons calibrer les instruments de couverture grâce aux résultats générés par ce modèle et nous allons en tester l'efficacité.

Afin d'évaluer la pertinence des moyens de couvertures, nous présenterons des forwards et des options, sur des sous-jacents comme les taux de mortalité et les taux de survie. Nous faisons une étude sur des portefeuilles unicohortes, multicohortes et sur un portefeuille de rentiers mixte. Nos résultats démontrent tous une amélioration par rapport à nos mesures de risque, soit une diminution de l'écart-type, de la valeur à risque et de la perte espérée.

Il faut noter que la majorité de cette étude a été travaillée avant la pandémie de la COVID-19 de 2020-2021. L'excès de mortalité durant la pandémie pour des âges pertinents à des fonds de pensions sera un facteur important à considérer lors de travaux futurs.

Contents

1 Acknowledgments

I want to begin this essay with a special thanks to Alain Bélanger, who has allowed me to challenge myself and learn a lot more than I could have imagined. Not only was he a great director, but he is also a great teacher. I remember when I thought that the financial field wasn't for me, and his class and this project lit a spark that got me reinvested.

Second, I would like to thank the scholarship program Mitacs and the Canadian Institute of Actuaries for the opportunity to do a research project. it is programs like this that encourage young graduates to go into the research field.

I was lucky to spend some time with the professionals at Addenda Capital, notably Christian Robert. I'm more than happy that he gets to read the thesis I started during my summer spent with his team.

I want to thank my partner, Simon, and my parents, for their encouragement and help during my last 5 years of study. A big thanks goes out to my teammates, Valérie Beaupré and Sullivan Giannini, who provided moral support through difficult times; the experience would not have been the same without them.

Finally, a word for all the professors and teachers at the University of Sherbrooke who have contributed to my knowledge throughout the bachelor's degree and the master's. To the professors in economics (David, François, Mario), those in finance (Guy, Marc-André, Jennifer), and so many others, thank you! I hope you can inspire many more.

2 Introduction

2.1 Overview

Alarming news was presented by the Institut de la Statistique du Québec (ISQ) in their last demographic report [17]. By 2022, the number of seniors aged 65 and over will surpass the number of people in the group of 0-19 years old. By 2066 the Quebec population will grow by 1.7 million people, of which 1.3 million will be elders, raising the total population of people above 65 to 2.7 million. According to the ISQ, Québec's total population in 2066 is projected to be 9.9 million, bringing the elderly close to 30% of the population. These projections, primarily lead by the overall improvement of longevity, are the essence of the longevity risk defined by Barrieu and al.[1] as the "the risk that the trend of longevity improvements significantly changes in the future". Although this risk can be applied to various fields of our society (i.e., political, medical, sociological), some of the most affected institutions remain pension funds and life insurance companies. To understand where their risk arises, let us analyze a pension plan's liabilities. The plan's subscribers are entitled to a certain amount each year until they die. Thus, if the survival rate increases beyond expectations, the pension plan is exposed to higher liabilities, potentially leading to its bankruptcy.

To control this exposition to unexpected changes in the longevity rate, pension plans can use hedging products, such as forward contracts. For example, a longevity contract pays an amount that increases as its realization of a survival rate increases to offset the unexpected increase in liabilities. This position's counterpart could be a hedge fund that diversifies its portfolio; longevity is, after all, uncorrelated with the capital markets, unlike interest rates or stock prices (Li and Luo, 2011) [21].

There has been a significant advancement in products offered, but most of the work still lies ahead. One of the important aspects of the financial products available for hedging is their reliance on the accuracy of the forecast of mortality rates.

Mortality rate forecasting has been a field of interest for quite some time. One of the first models appeared in 1825 and is due to Benjamin Gompertz. In his paper, he expressed that over the majority of the human life span, age-specific mortality rates increase exponentially, using a parametric model (Gompertz, 1825)[9]:

$$\log[\mu_x] = \log[R_0](\alpha x)$$

where μ_x is the force of mortality at age x, R_o and α are level parameters of the log equation slope. This soon became the Gompertz law or the Gompertz mortality force law. Even though the first model was created more than 200 years ago, it has been only in the past 15 years that more sophisticated models have been conceived (Booth and Tickle, 2008)[2]. The modern models shift from a deterministic point forecast to a stochastic one; hence we no longer predict a single mortality rate but rather a probability distribution (Booth and Tickle, 2008)[2].

Mortality tables give an example of a deterministic mortality forecasting model. A mortality table is a non-dynamic model that takes the mortality rate at a given year and reduces it the following year by an amount that is considered to be the general improvement in the population's longevity. This model, however, heavily relies on a subjective estimation of the improvement in longevity.

On the other hand, an example of a model that focuses on a probability distribution is the Lee-Carter model (Lee and Carter, 1992) [19]. The Lee-Carter (LC) model has since its publication become the gold standard of actuarial science [19]. It is a parsimonious model based strictly on the persistent long-term historical patterns found in the historical mortality dataset. The use of this model can be justified because there is an underlying and consistent pattern in mortality rates (Kalben, 2002)[18]. An example of a pattern is the lower mortality rate for women than men in more advanced ages. One of LC's most attractive features is its complete transparency and the fact that it does not need to analyze medical advancements or the trends in causes of death. The model in itself is easily accessible and provides better estimates than the previously discussed tables of mortality.

However, it remains a limited model that forecasts age-specific mortality rates in straight projections (Lee and Carter, 1992)[19]. The model focuses on the single most important principal component, explaining the maximum variance of the data. The model can then be expressed as :

$$\log[m_x(t)] = a_x + b_x k_t + \varepsilon_{x,t}$$

where,

$m_x(t)$ is the central mortality rate at age x on year t,computed as

$$m_{x,t} = \frac{\text{Number of deaths of age x at year t}}{\text{Total population of age x on June 30th of t}}$$

a_x is the age specific component, it can also be interpreted as the mean of the $\log[m_x(t)]$ over the years t considered

k_t represents an index of the level of mortality

b_x is the age specific sensitivity for age x to the level of mortality k_t

$\varepsilon_{x,t}$ are the residuals influences, not captured by the model

Like Gomptez's model, we see that the LC model also takes the log of mortality rates. This transformation is generally employed in modeling to ensure that the rates remain positive. One of LC model's disadvantages is that the parameter b_x is fixed over time, negatively impacting long-term accuracy.

Functional data model (FDM), can be seen as a generalization of the Lee-Carter model. Functional data analysis (FDA) models look over a series of factors that best explain the variance of the data. We compute these factors using principal component analysis. The functional data paradigm is a newer approach used in mortality, fertility, and migration data. Through this approach, FDA models are used to predict future structures of a given population (Hyndman, 2006)[15]. During a discussion with Patrice Dion from Statistique Canda, we learned that functional data models have spiked interest in the demographic field because they can get around the shortcomings of the LC model, such as the fixed parameter β_x. An extensive overview of the functional data model used in this thesis can be found in Section 3.3.

Mortality frameworks model the female and male mortality curves separately. The idea behind splitting the two curves is that male and female mortality attributes vary a lot. There is a well-known, general acceptance that mortality rates of men are higher than women's, and that appears to be true for almost any group age. For example, in the Canadian population, there is approximately a 4-year gap between the life expectancy at birth between men (79.8 years) and women (83.9 years)(Statistics Canada, 2018) [23]. Furthermore, the general population is generally slightly over 50% women. In the last 20 years, the percentage of women in the Canadian population has been slightly higher than men (between 0.5% and 0.92%). When we break it by age, we see that in more advanced ages, particularly ages pertinent to a pension plan, women are more predominant in the population pool (Statistics Canada, 2020)[24]. From Figure 1 below, we can also note the general improvement in the male mortality curve over time.

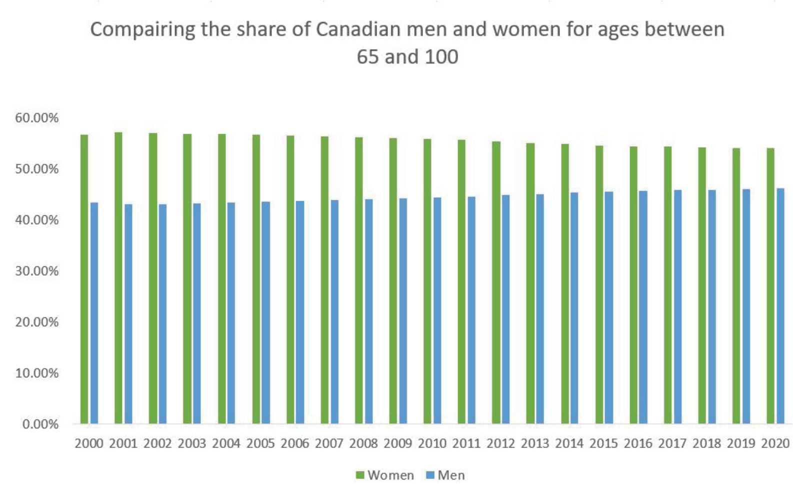

Figure 1: Distribution of men and women in the canadian poopulation, for ages 65 to 100, for the last 20 years

These differences present an additional challenge for a pension plan because the percentage of men and women in the portfolio may change their liabilities. Therefore, in this essay, we will begin to model and apply the hedging scheme to the historical data "Total Mortality Rates," which considers men and women's historical weight in the Canadian population. In a later section, we will do a cross-gender study by applying various hedges to see which ones best fit a portfolio composed of 65% women and 35% men.

2.2 Essay Motivation

This essay is meant to provide an analysis of the effectiveness of different types of hedging instruments. Functional data models are relatively new to the field, so we want to contribute to the literature by calibrating our longevity and mortality instruments using the FDA paradigm.

We will begin by modeling the Canadian mortality rates using the FDA paradigm, and then we will apply the hedging scheme to a pension plan's portfolio. We will apply our analysis to the total population to test the robustness of the results before moving to a cross-gender hedging scheme to divide the population into men and women.

The Canadian population poses different challenges because it is a relatively small sample size. The Canadian market is also well known to be an illiquid market for financial products. To prove that our results are robust, we will also apply our model and hedging scheme to the American population. The American population seems like a logical choice because of the countries' proximity and similar socio-political climate. The American population is a bigger sampling pool than Canada, with a population roughly ten times the Canadian one. The results for the American population will be available in Annex 6: Results from hedging the American Population (Section 8.6).

The importance of looking at cross-gender hedging comes from the fact that most models are only calibrated to fit one of the two genders. Male and female mortality curves do not exhibit the same patterns, and a pension plan's portfolio has inherently specific distribution for men and women. We will explore different types of cross-gender hedging schemes using longevity and mortality instruments.

3 Methodology

3.1 Data

Let us begin by looking at how mortality data is calculated. The most common approach put forward by Cairns et al [3], is to look at the central crude mortality rates. These rates are calculated by taking the number of deaths at a given age x divided by the population's mean in any given year t. By taking the population on June 30^{th}, we are assuming that deaths constantly occur during the year; therefore, the population mean should approximately be the population at the half-year mark. The mortality rate $m_{x,t}$ is therefore given by

$$m_{x,t} = \frac{\text{Number of deaths of age x at year t}}{\text{Total population of age x on June 30th of t}} \tag{1}$$

This calculation occurs before any smoothing, averaging, or adjustments (LLMA, 2012)[20]. Most mortality data presented in this thesis, unless stated otherwise, is in its crude form. The data is easily accessible through the Human Mortality Database (HMD), created jointly by the Max Planck Institute and the University of California, Berkeley [12]. HMD provides different data sets necessary to compute Equation 1, such as total deaths and total population. The information provided by HMD is split into men, women, and total population. Therefore, Equation (1) can be computed for men and women individually and the total population. As described in the introduction, we will first treat the total population curves, meaning we will consider the historical weights of men and women in the Canadian population. Further along in the thesis, we will use the data provided for men and women as separate curves for the cross-gender analysis.

From a fitting point of view, most mortality models use age-specific characteristics to predict mortality rates. Therefore, mortality rates are presented in an age-specific form, meaning that each row represents a given age throughout the entire period at study. In matrix form, we have:

$$m_{x,t} = \begin{bmatrix} m_{20,1951} & m_{20,1952} & \cdots & m_{20,2016} \\ \vdots & m_{50,1952} & \ddots & \vdots \\ m_{100,1951} & m_{100,1952} & \cdots & m_{100,2016} \end{bmatrix}$$

From a hedging point of view, we want to portray the reality of a pension plan. We will transform the matrix into a presentation that follows different cohorts through time. This presentation is also called a Lexis diagram. The matrix of mortality rates will therefore look like,

$$m_{x,t} = \begin{bmatrix} m_{20,1951} & m_{21,1952} & \cdots & m_{85,2016} \\ m_{21,1951} & m_{22,1952} & \cdots & m_{86,2016} \\ \vdots & \ddots & \vdots & \\ m_{35,1951} & m_{36,1952} & \cdots & m_{100,2016} \end{bmatrix}$$

The model is calibrated using the Canadian and American total populations, from 20 to 100 years old. We disregard the first 20 years, as they introduce unwanted volatility in the mortality curve due to infant mortality. Furthermore, the ages above 100 introduce a different type of volatility due to a smaller pool of people available. From a pension plan's perspective, the few remaining people pose a more negligible risk and no longer need hedging. We concur that ages above 65 and below 100 are pertinent to our analysis.

The period that has been chosen begins in 1951 and expands to 2016, the most recent available data from HMD. We hope that by trimming the data frame, we reduce severe outliers caused by historical events, such as WWII, while still keeping a fair sample to work with. Other significant wars have particularly touched the United States, such as the Vietnam War. We can keep this in mind if further along in the thesis if we need to explain differences between the Canadian and the American mortality curves.

3.2 Fitting the mortality rates

The approach we will be considering is a generalization of the base model of LC[19], which will take into account more than just the first principal component. We will use a functional data model (FDM) to extend the analysis to as many common factors as explain the data set's maximum variance. As first proposed by Hyndman and Ullah (2006)[15], the functional data model for forecasting fertility and mortality rates is based on the work of Ramsay and Silverman (2005)[25]. Functional data analysis (FDA) is increasingly used in modeling time series data, especially in terms of biomedical applications (Ullah and Finch, 2013)[27] . The idea behind switching from traditional methods of forecasting multi-variate time series to more complex models (such as FDM) is that the paradigm of the functional data analysis permits a more in-depth decomposition of the data. In addition, one of the well-known problems of using age-specific forecasting is the high dimensionality of the data. Techniques, such as principal component analysis, are a way to avoid the problem by reducing the dimensionality of the data and providing a more parsimonious model. The key concepts behind FDM are the choice of a smoothing technique, data reduction, and forecasting methods (Ullah and Finch, 2013)[27].

If we denote by y_t the log of the observed mortality rates, then we can express the data as

$$y_t(x_i) = f_t(x_i) + \sigma_t(x_i)\varepsilon_{t,i} \tag{2}$$

We assume that there is an underlying smoothing function $f_t(x)$ that we observe with error at discrete ages x. The error term, $\varepsilon_{t,i}$ is normally distributed with mean zero and variance of 1. The importance of using smooth data arises from the fact that age-specific mortality rates tend to have much statistical noise. It is not uncommon that the mortality rate at a given age x is higher than that of age x+1, due to the changing size of the available population. Let us take, for example, the mortality rates of the total Canadian population.

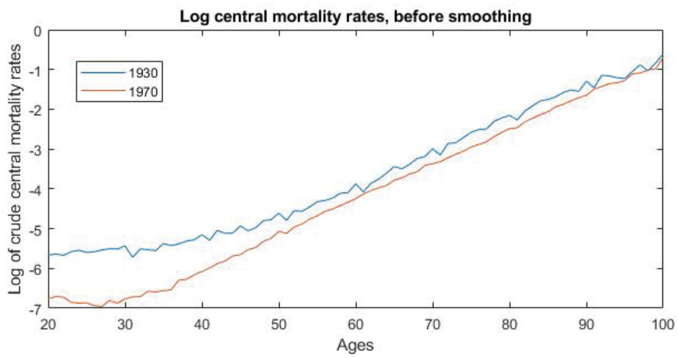

Figure 2: Two different years showing age-specific mortality rates for the Canadian total population. We can see the statistical discrepancies happening across the ages

From Figure 2, we can see that the curve in 1970 (close to modern age) is smoother than the one in 1930. That is due to the general improvement of longevity in the population. However, the curve never truly becomes smooth, and statistical discrepancies remain present in ages between 20 and 40 and further ages, between 65 and 100. Figure 3 shows the statistical noise present in ages pertinent to pension plans.

There are several reasons why these discrepancies occur. Recall that the numerator in the crude mortality rates formula is the total population in a given year. The population may change for various reason, one of which is population migration. The methodology applied in this essay calls for a smoothing of the mortality curves to avoid this noise when forecasting the mortality curves. Furthermore, some of the hedging instruments discussed in later sections heavily rely on the correlation between neighbouring ages in the mortality curve, so we do not want the correlation to be underestimated.

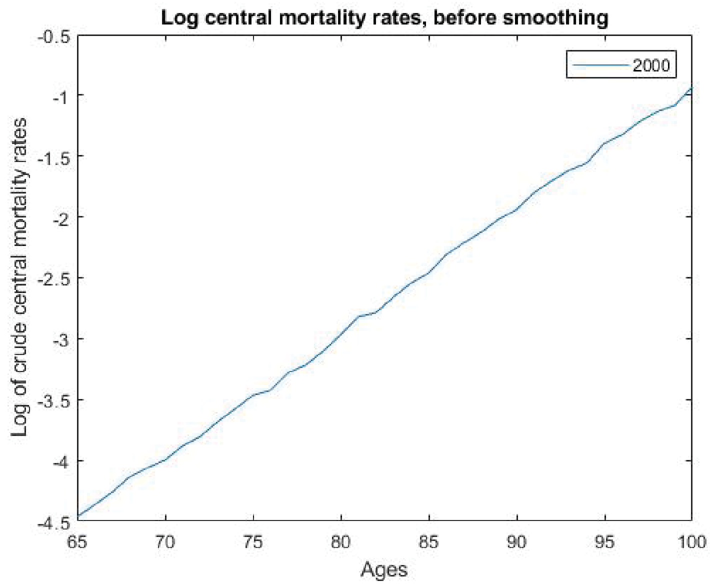

Figure 3: Log of crude mortality rates in the year 2000 for ages 65 to 100.

3.3 HU Methodology; Forecasting of Mortality Rates

The single population model of Hyndman and Ullah (HU), follows these steps:

1. Smooth the data for each age x using a nonparametric smoothing method to estimate $f_t(x)$ for $x \in [x_1, x_p]$ from $\{x_i, y_t(x_i)\}, i = 1, 2, ..., p$

2. Decompose the fitted curves via a basis function expansion using the following model:

$$f_t(x) = \mu_x + \sum_{k=1}^{K} \beta_{t,k}\phi_k(x) + \varepsilon_t(x) \tag{3}$$

   where μ_x is $\sum_{t=1}^{T} \frac{ln(m_t(x))}{T}$, $\phi_k(x)$ is an orthonormal basis and the error term $\varepsilon_t(x) \sim N(0, v(x))$

3. Fit univariate time series models to each of the coefficients $\{\beta_{t,k}\}$, $k = 1, ..., K$

4. Forecast the coefficients $\{\beta_{t,k}\}$, $k = 1, ..., K$, $fort = n+1, ..., n+h$ using the fitted time series model.

5. Use these forecasted coefficients with (2) to obtain the forecasts of $f_t(x), t = n+1, ..., n+h$

3.3.1 Non parametrical smoothing

Smoothing techniques are used to reduce statistical noise by capturing additional information present between adjacent data. Mortality rates for adjacent ages are highly correlated in neighbouring areas (LLMA,2012)[20].

Throughout the literature (Duchon, 1997[4]; Wahba,1990[28]; Gu, 2002[10]), many different models are available for smoothing data. The use of splines is an effective way of estimating models such as the one we have in Equation 2.

One way of using splines is through weighted regression B-splines (He and Ng, 1999)[11]. This technique involves constraints based on spline parameters, which may give us a certain degree of monotonicity (Wood,1994)[29] but not enough to ensure a mortality curve's logical shape.

In this thesis, we will use the smoothing technique put forward by Hyndman and Ullah. In Figure 3, we can see the statistical discrepancies in advanced ages. Using a smoothing technique, we can make sure that we remove these discrepancies while still keeping the most important aspects of the mortality curve. Penalized regression splines, as put forward by He and Hg (1999)[11] permit the addition of a monotonic smoothing constraint. This means that after a given age, the mortality curve will only be upward sloping. Because this thesis focuses on pension plan's, the smoothing restraint will begin at age 50.

To smooth the data, we define $N_t(x)$ as the total population of age x mid-year t. We can assume that the mortality rates $m_{x,t}$ are approximately binomially distributed[1] because of our main assumption that deaths are evenly distributed throughout the year. Therefore we have

$$m_{x,t} \sim \mathrm{B}(N_t, m_x(t))$$

The estimated variance is $N_t(x)m_t(x)[1 - m_t(x)]$.Via the Taylor approximation, we can obtain the variance of $y_t(x) = log[m_t(x)]$:

$$\hat{\sigma}_t^2(x) \approx [1 - m_t(x)]N_t^{-1}(x)m_t^{-1}(x) \tag{4}$$

We will define the inverse of the variances as the regression weights, and we use a weighted penalized regression splines to estimate the curve for each year (Wood,2003)[30].

We can illustrate the smoothing's overall effect with these 2 different age-specific curves.

[1]Recall that by definition $m_{x,t}$ are the central crude mortality rates

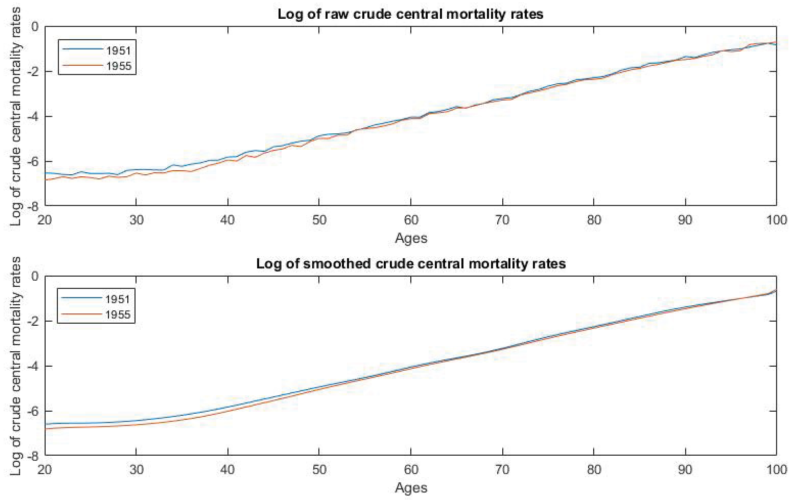

Figure 4: We can see that the smoothing has suppressed most of the statistical noise and that all the curves are monotonically increasing with age

3.3.2 Decomposition of the basis functions

Recall Equation 3,

$$f_t(x) = \mu_x + \sum_{k=1}^{K} \beta_{t,k}\phi_k(x) + \varepsilon_t(x)$$

This section will model the smoothed age-specific mortality curves obtained in the previous section via a basis function expansion. We will compute the ϕ_k using principal components, to help reduce the dimensionality of the data set. However, as per Equation 3, we need to first subtract the L1-median of the data, before adding it back once the principal components are fitted. When we are done with the principal component analysis, we will add the mean to obtain our model from the equation above. The L1-median is defined to be any point that minimizes the sum of Euclidean distances to all points in the data set (Fetter,1930)[5].

$$\hat{\mu}(x) = \underset{\theta(x)}{argmin} \sum_{t=1}^{n} \left\| \hat{f}(x) - \theta(x) \right\|_1$$

where $\|g(x)\|_1 = (\int_{-\infty}^{+\infty} \|g(x)\| dx)$. The norm of a function from the vector space of integrable functions to the nonnegative real numbers. We write the mean-adjusted-smoothed data as,

$$\hat{f}_t^*(x) = \hat{f}_t(x) - \hat{\mu}(x)$$

We suppose that each function has a basis expansion[25],

$$\hat{f}_t^*(x) = \sum_{j=1}^{m} a_{t,j}\xi_j(x)$$

where $A = (a_{t,j})$ is an $n \times m$ matrix of coefficients that arises from the computation of the penalized regression splines and $\xi(x) = [\xi_1(x), ..., \xi_m(x)]'$ is a vector of splines.

Let J denote $m \times m$ matrix, with the (i,k)th element $J_{ik} = \int \xi_i(x)\xi_k(x)dx$ from which we can find the Cholesky decomposition $J = U'U$

The basis function ϕ_k is,

$$\phi_k(x) = (U^{-1}g^k)'\xi(x)$$

If W is the diagonal matrix with diagonal values being weights $w_j = 1/\hat{\sigma}_t^2(x)$ computed from Equation 4 and g^k as the kth normalized eigenvector of $(U^{-1})'JSJ'U^{-1}$, where $S = (n-1)^{-1}A'W^2A$.

The next step is for us to find the coefficients β associated to the basis. The matrix of coefficients arises when we take the matrix $\Phi_{p\times(n-1)}$ which contains the elements from $\phi_k(x)$ and $F_{n\times p}$ which contains the elements from $f_t^*(x)$. Thus we can find the matrix $B = F\Phi$, which contains the elements $\hat{\beta}_{t,k}$. Notice that the coefficients β are time dependant. Step 3 from the HU methodology calls for fitting the historical β using a univariate times series model ARIMA. The next section (Section 3.3.3) covers step 4 of the HU methodology, using the ARIMA process to forecast the mortality curve.

To assess how much variance the basis functions can explain, we recall that an eigenvalue of a given matrix A, λ_i,is such that :

$$Ax = \lambda_i x$$

The equation can be simplified to

$$(A - \lambda_i)x = 0$$

Thus, Λ is the diagonal matrix containing all the eigenvalues,λ_i, of matrix A. We denote x the eigenvector containing the eigenvalues. By taking the sum of the diagonal of the matrix Λ,

$$\Lambda = \begin{bmatrix} \lambda_{1,i} & \dots & 0 \\ \vdots & \ddots & \\ 0 & & \lambda_{n,n} \end{bmatrix}$$

we can compute the trace of the matrix.

$$\sum_{n=1}^{K} \lambda_{i,i} = tr(\Lambda)$$

With this equation, we can compute the marginal contribution of each factor to the variance of the data by taking $\frac{\lambda_i}{tr(\Lambda)}$ for each of the factors. We usually take into account only the first factors, as the marginal contributions get insignificant quite rapidly.

3.3.3 Forecasting

Once we have established the orthonormal basis, the forecast expansion becomes:

$$\widehat{f_t(x)} = \widehat{\mu_x}, + \sum_{k=1}^{K} \widehat{\beta_{t,k}} \phi_k(x) \ + \widehat{\varepsilon_t(x)}$$

We can then proceed to forecast the $\beta_{t,k}$ for $k = 1, ..., K$ and $t = n+1, ..., n+h$. For $K \geq 1$, this becomes a multivariate time series, however since the set of basis is orthonormal, we can assume that the coefficients are uncorrelated and therefore can forecast them with a univariate ARIMA model.

When we combine Equations (2) and (3), we obtain,

$$y_t(x_i) = \mu_x + \sum_{k=1}^{K} \beta_{t,k}\phi_k(x) + \varepsilon_t(x) + \sigma_t(x_i)\varepsilon_{t,j} \tag{5}$$

We can see that there are two terms of error in Equation 5. The first $\varepsilon_t(x)$ is the difference between the splined curves and the fitted curves, while the $\sigma_t(x_i)\varepsilon_{t,j}$ is the observational error, the difference between the observed death rates and the splined curves.

By conditioning on our set of observable data $\ell = \{y_t(x_i); t = 1, ..., n; i = 1, ..., p\}$ and on ϕ, we obtain the one step ahead forecast

$$\hat{y}_{n,1}(x) = \mathrm{E}[y_{n+1}(x)|\ell, \Phi] = \hat{\mu}(x_i) + \sum_{k=1}^{K} \beta_{n,k,1}\hat{\phi}_k(x)$$

We use the ARIMA models to forecast the one-step ahead for the following year and we apply to the basis function associated with the age and its mean.

4 Model calibration

To calibrate our FDA model, we used the R package *demography* provided by Professor Hyndman[16]. We analyze the principal components selected for each of the models and populations. Following our decision mentioned in Data (Section 3.1), we will fit the models for ages 20 to 100 and from 1951 to 2016. Note that the model calibration when the whole available data is taken into account is available in Appendix 1: Results from fitting all available data.

The results presented below should be read like this: the basis interactions are age-specific, whereas their coefficients are time-specific. In line with the methodology, we forecast the coefficients for future years; hence the age-specific patterns in the data set remain the same.

4.1 Model results

Canadian Population

The Canadian population exhibits the following interactions between the coefficients and the basis.

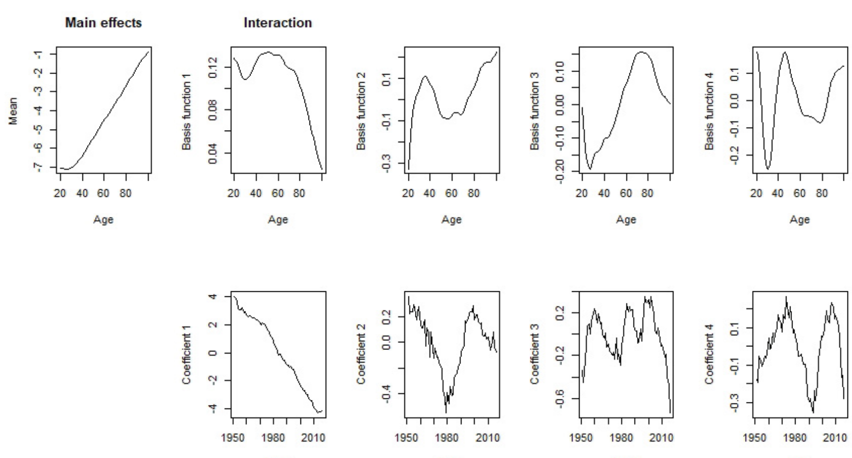

Figure 5: The four main interactions for the Canadian population explain individually 97.4%, 1.0%, 0.8% and 0.4%

We decided to keep on the four first components as they cumulatively explain 99.6% of the variance. We might be tempted to conclude the movements. However, the only pattern that seems to be recognizable is the first component explaining the general improvement of longevity. We see an important drop within the last 20 ages, which echoes that the average mortality rate has significantly dropped in the last 50 years. The first movement is generally applicable across similar nations[2]. However, some of the interactions are not as easily interpretable, and it is preferable to leave the other possible interpretations to demographic experts.

As for the ARIMA models for the coefficients, we have

Coefficient	Process
$\beta_{t,1}$	ARIMA(0,1,0)
$\beta_{t,2}$	ARIMA(0,1,3)
$\beta_{t,3}$	ARIMA(1,0,0)
$\beta_{t,4}$	ARIMA(1,0,0)

Table 1: Models for each basis coefficient for the Canadian Population Model

[2]We looked over this hypothesis in countries with slower development. While the same trend is found in the data, the drop is significantly slower. In Annex 2 FDA Modelling for Slovenia and Estonia

American Population

The American population exhibits the following interactions between the coefficients and the basis.

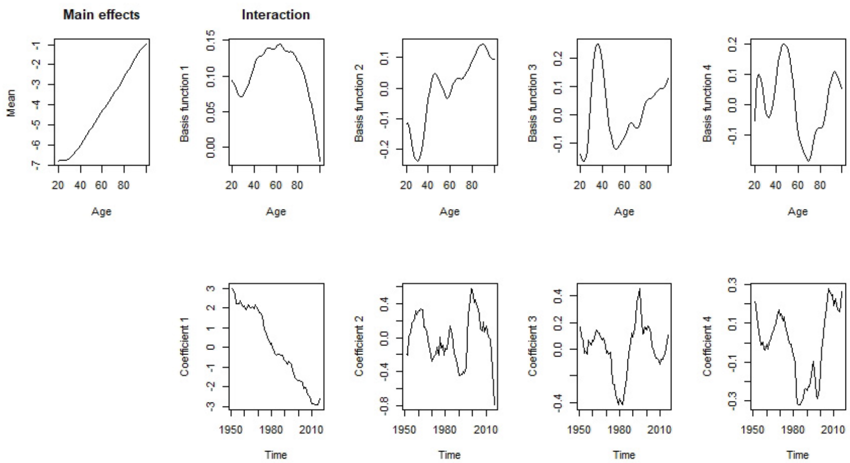

Figure 6: The four main interactions for the American population explain individually 95.2%, 2.2%, 1.0% and 0.9%

As for the ARIMA models for the coefficients, we have

Coefficient	Process
$\beta_{t,1}$	ARIMA(0,1,0)
$\beta_{t,2}$	ARIMA(0,2,2)
$\beta_{t,3}$	ARIMA(1,0,3)
$\beta_{t,4}$	ARIMA(1,0,0)

Table 2: Models for each basis coefficient for the American Population Model

What is interesting from these curves is that the first component that explains the maximum variance is a random walk with drift. The same model is used in Li-Luo[21], and Giannini 2020 [8]. Although the models are very different, we see the strong characteristics of mortality rates that the models pick up.

4.2 Survival Curves

Using the Monte-Carlo (MC) technique, we simulate 10 000 occurrences for each of the coefficients mentioned above per their ARIMA model for a period of 35 years. From this point on, we are done using mortality rates in their age-specific form for fitting. Therefore, we can now express mortality rates for age x+t at time t

$$m_x(t) = m_{x+t}(t)$$

We then take the cumulative sum for each period to obtain the cumulative survival rates. Figure 7 below shows the obtained mortality curves for the Canadian total population, computed using Equation 6.

$$S_t = \exp(-\sum_{n=1}^{t} m_x(n)) \tag{6}$$

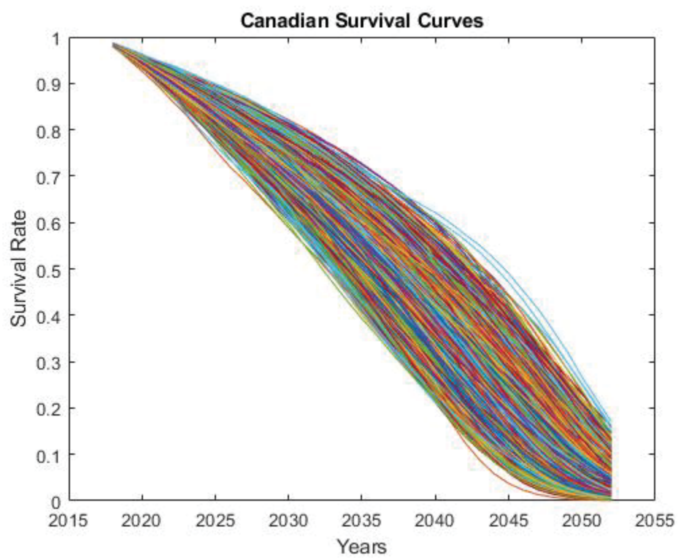

Figure 7: 10 000 simulations of cumulative survival rates for the Canadian population

The survival curves in Figure 7 were obtained by aggregating the mortality rates forecasted using the MC technique. The variance in further ages can be explained by the fact that a change in the first years will have a water-fall effect on all subsequent years. On the other hand, when multiplying mortality rates together, we perform an averaging (or smoothing). Therefore the curve will dampen near the end, and the most volatility is found in the mid-years of the cohort.

Now that we have successfully calibrated our data using a functional data model, we will apply different hedging instruments to a pension plan's portfolio.

5 Hedging scheme

Once the model is calibrated, we can apply hedging instruments and analyze their effectiveness from a pension plan's perspective. We separate them into two categories: longevity (Section 5.2) and mortality instruments (Section 5.3). We will analyze the effectiveness through comparative tables that will include the following metrics: mean, standard deviation, skewness, kurtosis, $\text{VaR}_{99\%}$ and the $\text{ES}_{99\%}$ of the unhedged and hedged distributions of final profits and losses.

5.1 Basic portfolio

We begin with a basic portfolio, which we will, at first, keep unhedged for 35 years and then hedge with survival instruments for 20 and 25 years. This closed portfolio consists of 4 000 annuitants, all aged 65 years old at the beginning of the period, and no other annuitants will be added throughout time. Because we have calibrated our models to the Canadian total population, the 4 000 annuitants will reflect men and women's natural, historical weights. We restrict the maximum age of the population to 100. We make this assumption in line with LLMA, that more often than not the mortality rates after 100 years old are not reliable considering the small pool of population. All the cash flows will be discounted at a fixed 4% risk-free rate. This last assumption allows for a simpler model, with which we can keep the focus on longevity risk. We first review the different hedging methods before applying them to our population.

5.1.1 Longevity bond

Unfortunately, because a liquid longevity/mortality market is far from sight, we have only one instrument to price longevity risk. We base our calculations on the 2004 longevity bond announced by BNP Paribas and European Investment Bank (EIB). The original bond has a maturity of 25 years and was based on the male population of England and Wales. Although we will not be using this exact bond, the calculations provided in the 2006 paper by Cairns et al. help us price an equivalent bond for a period of 35 years[3] based on the Canadian population. We define V(0) as the price of the longevity bond,

$$V(0) = \sum_{t=1}^{35} B(0,t) e^{\delta t} E_0^{BE}[e^{-\int_0^t m_x(v)dv}]$$

Where,

1. δ is a spread, or an average risk premium per annum
2. $B(0,t)$ are the discounted factors for each period
3. $E_0^{BE}[e^{-\int_0^t m_x(v)dv}]$ is the historical best estimate (BE) of the survival rate for the ith period

We will use $\delta = 0.002$, as priced by the BNP/EIB bond that is 20pbs below the standard EIB rates. Although this pricing gives us a head start, it should be noted that this particular longevity bond never found a buyer ready to take on the risk transfer. However the calculations remian useful. We can take the initial bond and adjust the mortality rates to incorporate risk. We write the value of the longevity bond using risk adjusted rates (RA) as,

$$V(0) = \sum_{t=1}^{35} B(0,t) E_0^{RA}[e^{-\int_0^t m_x(v)dv}] \tag{7}$$

5.1.2 Risk adjusted probabilities

So far, using the MC simulations, we find the best estimate (BE) mortality rates by applying a probability equal to 1/N, where N is the number of MC simulations. As first put forward by Stutzer(1996)[26], canonical valuation is a method that consists of adjusting the probability distribution of each simulation. It is nonparametric and has been proven to work well in pricing options on equity indexes and different derivative securities. The 2011 paper by Li and Ng extends the canonical valuation model on pricing mortality-linked securities [22].

We begin by generating the trajectories of mortality rates using the functional data framework. Under the basic assumption, when we generate N scenarios of mortality data, they carry each an equal probability, so the probability mass function is

$$\Pr(\omega = \omega_j) = \pi_j = \frac{1}{N},\ j = 1, 2, ..N$$

From each of the N scenarios, we can calculate the value of a longevity index $I(t)$ at $t = 1, 2, ..., 35$, based on the trajectory of simulated mortality rates. Let $I(t, w_j)$ denote the value of the longevity index at time t in the jth scenario (Li,Ng 2011)[22]. We can then derive, from the above-mentioned PNB/EIB formula,

$$v(w_j) = \sum_{t=1}^{35} B(0,t) I(t, w_j)$$

The principle of canonical valuation is based on the Kullback-Leibler information criterion (Kullback and Leibler,1951),

$$D(Q,P) = E^P\left[\frac{dQ}{dP} ln \frac{dQ}{dP}\right]$$

The Kullback-Liebler information criterion $D(Q,p)$ is defined as the information obtained by moving from measure P to measure Q. We can think of P as the real-world probability measure. Due to a lack of market prices for specific longevity products, this real-world distribution of P is the only tool we can use. Therefore, to have a fairly priced longevity product, we need to solve for the real-world probabilities and use them to obtain the risk-adjusted mortality distribution. To obtain these probabilities, we solve the Lagrange function defined by,

$$L = \sum_{j=1}^{N} \pi_j^* \ln \pi_j^* - \lambda_0 \left(\sum_{j=1}^{N} \pi_j^* - 1\right) - \lambda_1 \sum_{j=1}^{N} \left(v(w_i)\pi_j^* - V(0)\right)$$

with constraints,

$$\sum_{j=1}^{N} v(w_i)\pi_j^* = V(0) \text{ and} \sum_{j=1}^{N} \pi_j^* = 1$$

Q, the canonical measure has to satisfy the following first-oder conditions:

$$ln\pi_j^* + 1 - \lambda_o - \lambda_1 v(w_j) = 0 \quad \text{for} j = 1, 2, ...N$$

equivalent to ,

$$\pi_j^* = exp(\lambda_0 + \lambda_1 v(w_j) - 1) \quad \text{for} j = 1, 2, ...N$$

Following from our second constraint that $\sum_{j=1}^{N} \pi_j^* = 1$, means that π_j^* is proportional to $exp(\lambda_1 v(w_j))$. Therefore, we can write ,

$$\pi_j^* = \frac{exp(\lambda_1 v(w_j)}{\sum_{j=1}^{N} exp(\lambda_1 v(w_j)}$$

What remains is the parameter λ_1 , which can be written as the following expression

$$\lambda_1 = \underset{\gamma}{argmin} \sum_{j=1}^{N} exp\left(\gamma(v(w_j)) - V(0)\right)$$

Once we have solved for λ_1, we can easily find the risk-adjusted probabilities that reflect the real-world distribution into a risk-neutral measure(Li, NG 2011)[22], so that

$$V(0) = \sum_{t=1}^{35} B(0,T)e^{St}E^{BE}[St] = \sum_{t=1}^{35} B(0,T)E^{RA}[St]$$

Figure 8 presents graphically the real world probabilities π_j^* computed using the canonical valuation. These will be the probabilities used from now on to obtain the risk-adjusted (RA) mortality distribution.

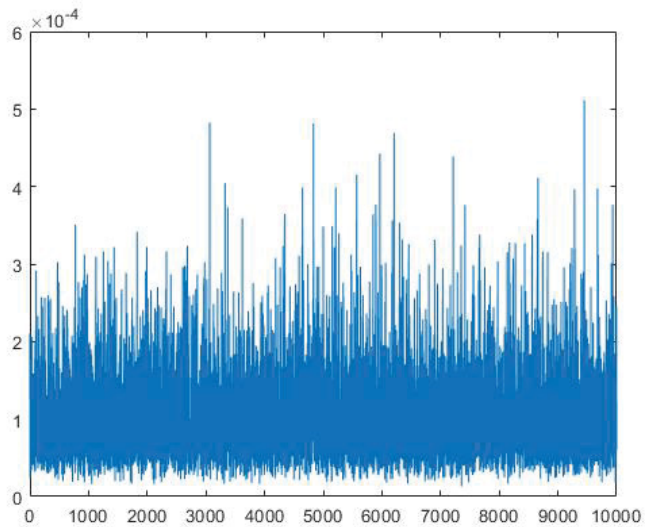

Figure 8: Canonical probabilities π_j^*

5.1.3 Assets and liabilities

To analyze the effectiveness of our hedge, we create a synthetic portfolio composed of 4 000 annuitants. Let us take, for example, a pension fund that offers a life annuity for which the annuitants pay an initial amount. Using Equation 7, we compute the risk-adjusted value of the longevity bond to be 13.2050$. We will use this measure to compute the assets of the pension plan's portfolio. That amount becomes an asset for the pension fund.

$$A = 4\ 000 \times \sum_{t=1}^{35} B(0,t) E_0^{RA}[e^{-\int_0^t m_x(v)dv}]$$

Contrary to the assets, the pension plan's liabilities are time dependant. The liability is the annual fixed amount, 1$ for example, that the fund has to pay the annuitant up until the year before his or her death. As the contract progresses in time, the liabilities tend to go down as fewer people are in the population. It is important to note that the portfolio's deaths are random and need to be estimated. Our basis condition is that the payments stop the year preceding the death of a subscriber. Therefore the discounted value of the $k-th$ person is given by

$$l_k = \sum_{t=1}^{\lfloor \tau_k \rfloor} B(0,t)$$

where τ_k is the stochastic time of death for annuitant k, with $k = [1, 2, ..., 4000]$. We obtain this variable using Monte-Carlo simulations and following the procedure suggested by Fung and al. (2015)[7]. To simulate death times, we use the Cox process that becomes an inhomogeneous Poisson, once the intensity of the mortality is obtained. Each process and the first jump time can be interpreted as death times (Fung and al., 2015)[7].

As presented in the Section 2.1, when applying the hedging scheme, we define the mortality rates as $m_x(t) = m_{x+t}(t)$.

1. We simulate 10 000 trajectories of mortality rates $m_x(t)$ for each of the

35 years of interest;

2. We calculate $I(t) = \sum_{i=1}^{t} m_x(i)$, the intensity at which deaths occur for each year, for $t = 1, 2, 3...T$ and for each simulation;

3. We generate random variables following an exponential distribution, such as $\epsilon \sim exp(1)$. We generate one per annuitant, for a total of 4 000.

4. For each of the previously generated random variable, ϵ, we determine τ_k, as the last t such that $I(t) < \epsilon$, obtaining the year of death of the k-th annuitant and

$$l_k = \sum_{n=1}^{\tau_k = t} B(0, n)$$

5. We can then compute the total liabilities for all the periods

$$L = \sum_{k=1}^{N} l_k$$

We can also compute the unhedged gains or losses for each of the periods

$$D_{unhedged} = A - L$$

and the average gain or loss

$$D_{\mu} = \frac{(A - L)}{N}$$

6. Step 1 through 5 are computed for the $j = 1 : 10000$ number of simulations

From Figure 9 below, we can see that the distribution of surpluses without the risk premium is centered at 0, whereas adding the risk premium puts the mean of the distribution to the right (around 0.25). The positive mean represents the discounted value of the risk premiums.

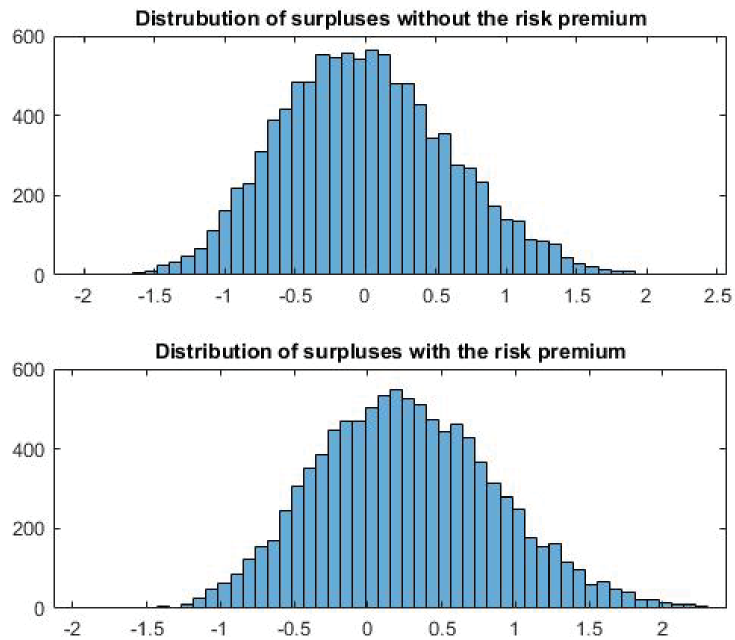

Figure 9: Compairaison of unhedged surpluses with and without the risk premium $delta = 0.002$

5.2 Longevity instruments

A pension plan has a short position on longevity; when the longevity of their portfolio increases, so do their liabilities. Therefore, it will want to hedge its position by being long on a longevity contract. This section will explore two longevity instruments, the S-Swap (a series of S-Forwards with different maturities) and the S-Cap (a call option). We will use the canonical probabilities calculated above in order to obtain proper pricing for our contracts.

Single cohort

5.2.1 S-Forward/S-Swap

A forward contract on the survival rate seems like the most intuitive hedging option for a pension plan. Survivor-forwards, or S-Forwards, developed by LLMA(2010), help the pension plan transfer part of its risk to an investor, and while it may reduce their potential losses, it can also reduce their potential gains. Typically, a forward is put in place at zero cost.

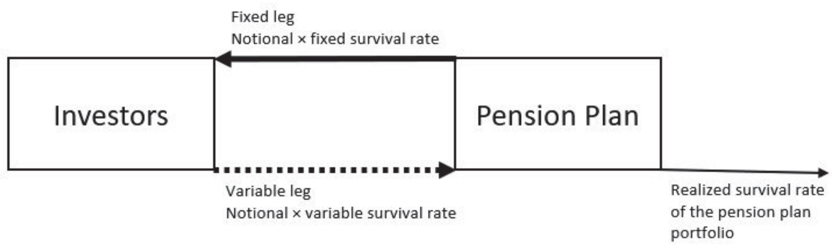

Figure 10: Settlement for a S-Forward contract between an investor and a pension plan

According to the above diagram, the pension plan enters the forward contract to receive an amount that is equal to the survival rate for a national cohort, and in exchange, it will pay a fixed rate. The swap rates are computed based on the national cohort, which leaves basis risk to the pension plan. This basis risk increases if we use a different national cohort than the pension plan (e.g., use the national American population to hedge a Canadian pension plan). If the survival rate for a given year T is higher than expected, the S-Forward will have a positive cash flow for the pension plan, reducing the loss on their portfolio. We can define S-Swaps as a series of S-Forwards with different maturities and fixed rates. The payoff of this longevity contract is

$$P_{S-swap} = (e^{-\int_0^T m_x(v)dv} - K_{Swap}(T))$$

where $K_{Swap}(T)$ is the fixed rate decided upon contract initiation (Fung and al., 2015)[7].

These contracts present many advantages, such as requiring no initial payment, and the settlement occurs at the end of each period. To respect the above mentioned constraint, we have the value of the portfolio such as,

$$B(0,T)E_0^{RA}(e^{-\int_0^T m_x(v)dv} - K_{Swap}(T)) = 0$$

Therefore we obtain the following equality,

$$K_{Swap}(T) = E_0^{RA}(e^{-\int_0^T m_x(v)dv})$$

Because we want the risk-adjusted mean of the mortality rates, we must apply the real-life probabilities obtained by the canonical valuation (Bélanger, 2020).

$$K_{Swap} = \sum_{j=1}^{10000} \pi_j^* S_x(T, j)$$

Where, $S_x(T) = e^{-\int_0^T m_x(v)dv}$ the survival rate for the population at a given period t. Therefore, the hedging of the portfolio becomes,

$$D = A - L + P_{S-Swap}$$

We will present the distribution of the final surpluses by dividing the total surplus by the number of participants in the portfolio,

$$\bar{D} = \frac{D}{4000}$$

5.2.2 S-Caplets

As mentioned, the S-Swap is a good way for a pension fund to transfer part of its risk, but it will wipe out its potential gains if longevity is lower than anticipated. An option type instrument, the S-Cap, gives the investors the right but not the obligation of exercising the option. Because no rational investor would choose to lose money, we assume that the Cap payoffs cannot be negative. Investors are ready to pay for this right. Therefore the Cap trades at a premium. (Fung and al., 2015)[7].

$$P_{S-Cap} = max\{(e^{-\int_0^T m_x(v)dv} - K_{Cap}), 0\}$$

In this instance, we take $K_{Cap} = E^{BE}[S_x(t)]$. The risk adjustments is taken into consideration, when calculating the premium paid for the derivative (Bélanger,2020)

$$V(0)_{S-Cap} = \sum_{t=1}^{T} B(0,t) \sum_{j=1}^{J} \pi_j^* \{S_0 x(j,t) - E_0^{BE}(S_x(t))\}^+$$

The derivative option's price depends on the distance between the realization of the mortality rates and its mean. As we go along in more advanced ages, where the variance is higher, we expect that gap to be positive on a higher number of realizations. Therefore, we expect the price of the derivative to positively correlate with its maturity.

Because the use of option derivatives is to diminish the potential losses when the realized survival rate is higher than the fixed survival rate, we expect to see a lower realization of losses than the S-Swaps. We also expect to see a difference between the distributions. The surpluses for this product will be,

$$D = A - L + (P_{S-Cap} - V(0)_{S-Cap})$$

Multi cohort

So far, we have looked at portfolios of annuitants turning all 65 in the same year. However, a plan is more likely to have simultaneously many other cohorts that begin at 66, 67, etc. To capture this more realistic presentation of a portfolio, we will look into hedging multiple cohorts using a synthetic multi-cohort portfolio. Our primary assumption that it is composed of 4 000 annuitants will remain. However, we will spread them through different age groups.

There are several methods of splitting the portfolio's participants into different groups. One way for example, would be to split evenly the participants throughout the cohorts at study. (Giannini 2020[8], Fontaine 2016[6]). To compute the surpluses using this specific distrubtion, we would take the weighted average of the surpluses for each of the cohorts. Let us take the portfolio of a pension plan and we hedge it is first 5 cohorts (65, 66, 67, 68 and 69). The hedged surpluses can be expressed such as,

$$D = \frac{N}{5}\sum_{x=65}^{69} a_x - \sum_{k=1}^{4000} l_k + \frac{N}{5}\left(\sum_{x=65}^{69}\sum_{t=1}^{T} B_0(t)(e^{-\int_0^t m_x(v)dv} - K)\right)$$

However, in this thesis, we decided to consider more cohorts in our synthetic multi-cohort portfolio composition. We follow the stair-like distribution as put forward Li-Luo [21]. We will compute the distribution of our surpluses using the weighted average from each cohort's surpluses (as in the equation above). However, the participants will no longer be equally distributed. Figure 11 is a visual representation of this new synthetic portfolio.

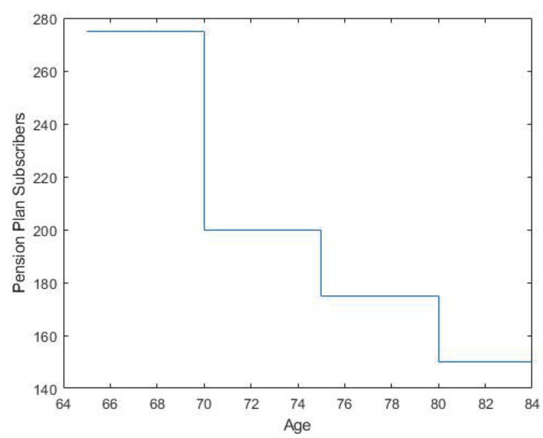

Figure 11: Pension Plan subscribers in a multi cohort hedging scheme

The stair-like composition means that we have a decreasing number of participants split into 20 cohorts. In Figure 11, in the age 65-69, the plan has 275 participants; from the ages 70-74, the plan has 200 participants. Furthermore, because these participants are not the same age at the time of contract initiation, they will pay a different amount which will change the value of the assets. The choice behind the method was to have a comparable basis for the results between the longevity and mortality instruments in a multi-cohort scheme. Therefore we will use this same method of portfolio composition in Section 5.3.3.

The remaining steps from Sections 5.2.1 and 5.2.2 on longevity instruments and their application do not change in the multi-cohort hedging. Each of the cohorts has its own specific K_{Swap}, K_{Cap}, and the 20 cohorts will have their individual option premium. The premiums paid for the caplet throughout different cohorts are available in Annex 5: Premiums for caps on the multi-cohort products (Section 8.5).

5.2.3 Results from hedging with longevity products

Results from hedging a single cohort

In the interest of computing the effectiveness of the hedging scheme, we compute the following:

$$D_{hedged} = A - L + H$$

Where,

1. D_{Hedged} is the distribution of the portfolio's surpluses after we apply the hedging scheme
2. A and L are the assets of the portfolio
3. H is the net payoff provided by each hedge.

The payoffs H for each of the hedging schemes depend on the maturity of their instrument, while the assets and liabilities are calculated for 35 years. We apply the hedge for 20 and 25 years, meaning that we cover the ages of 65 to 85 and 90, respectively. We see no need to extend the hedging age above 90 since the average life expectancy at birth for the Canadian population is 82 years old.

The hedging results using longevity instruments for a 20-year maturity are summarized in Table 3 and those for a 25-year in Table 4. These results are in concordance with those found by Fung and al (2005) [7].

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2688	0.4652	-0.0084	2.9219	-0.8016	-0.9640
Swap Hedged	0.1571	0.2829	-0.2241	2.9342	-0.5797	-0.6862
Cap Hedged	0.2106	0.3771	0.3527	3.1483	-0.5416	-0.6423

Table 3: Results from hedging a single cohort of age 65 with longevity instruments for a 20 year maturity

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2688	0.4652	-0.0084	2.9219	-0.8016	-0.9640
Swap hedged	0.1002	0.1700	-0.4509	3.0928	-0.4150	-0.4872
Cap Hedged	0.1698	0.3244	0.7815	3.6095	-0.3495	-0.4238

Table 4: Results from hedging a single cohort of age 65 with longevity instruments for a 25 year maturity

We can see in Tables 3 and 4 that adding hedging contracts improves the pension plan's risk profile. These results are in line with Fung and al. (2015)[7]. It is important to note that in order to be able to gain from these contracts, the institution has no other choice than to give up on the mean. For the 25 year hedge, for example, the unhedged mean is at 0.2688, whereas the swap is at 0.1002 and the cap at 0.1698. This mean reduction is compensated by reducing the standard deviation, reducing the VaR99 and the ES99.

We can see that the contracts will require more funds for a longer maturity; therefore, the distribution mean is higher for the 20-year maturity hedge than the 25. Nonetheless, the risk metrics for the instruments on the shorter maturity do not perform as well, since we stop the hedge at 85. The standard deviation for the swap contract is at 0.2829 for the 20-year maturity, whereas the 25-year maturity reduces it to 0.1700. We see similar results with the cap. Therefore, using 20-year maturity hedges leaves too much risk compared to the 25-year hedge. During those five years, our subscribers go from 80 to

85 years, meaning that most deaths occur within this period. Furthermore, there is no economic sense to hedge with a 30-year maturity, seeing that the number of participants remaining past 25 years is too low. From here on now, we will only compare and analyze the 25-year hedge. The remaining results for the 20-year maturity hedge are available in Annex : 3 Hedging results for a 20-year maturity hedge (Section 8.3).

The standard deviation reduction is more significant for the swap (0.1700) than the cap (0.3244), because the pension plan gives up profits and losses alike in this hedging scheme. On the other hand, the cap has a positive skewness (0.7815), meaning a longer or fatter tail on the right. This is a direct result of keeping only the positive payoffs. The premium paid for the caps is 0.2470$ for the 25-year hedge. In Table 4, we see that for the 25-year maturity hedge, both hedging instruments' risk metrics (VaR99 and ES99) perform better than the unhedged portfolio. Figures 12 and 13 below provide a visual representation of the various distributions.

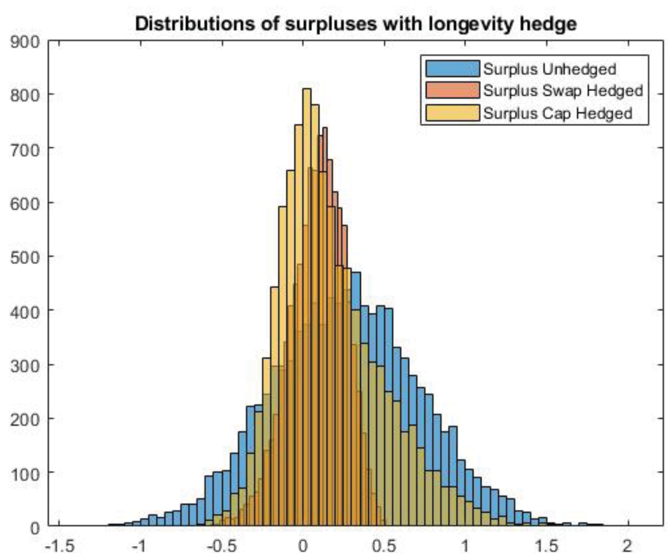

Figure 12: Distribution of hedging a single cohort of age 65 with longevity instruments for a 25 year maturity. This distribution refers to the results summarized in Table 4

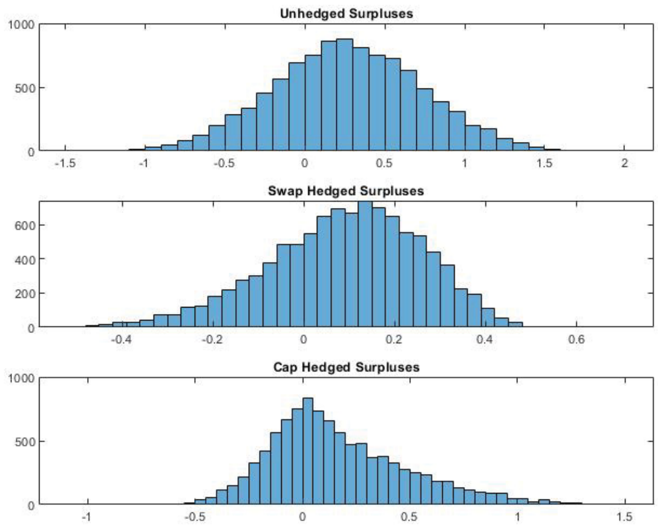

Figure 13: Individual distributions of hedging a single cohort of age 65 with longevity instruments for a 25-year maturity. This distribution refers to the results summarized in Table 4

Results from hedging multiple cohorts

We expect to see some differences in the multi-cohort hedge vs the single cohort. First, the unhedged multi-cohort portfolio should have a smaller mean because the assets are spread throughout time. The cohort beginning at age 65 will pay a higher price than the one beginning at 75. The asset pricing for multiple cohorts is available in Annex 4: Prices for longevity bonds on multi-cohort productions (Section 8.4).

We look into the hedge of every other cohort, beginning at age 65. Because our cohorts are spread out through time, we can apply a more parsimonious hedging scheme by hedging some key cohorts. We use S-Forwards with a 25-year maturity on specific cohorts. We read the following table: if we take the following contract $S^{25}_{65,67,69}$, we hedge ages 65, 67 and 69 for a maturity of 25 years.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1636	0.4021	-0.0554	3.0516	-0.7772	-0.9349
S^{25}_{65}	0.1515	0.3823	-0.0707	3.0451	-0.7474	-0.8981
$S^{25}_{65,67}$	0.1406	0.3624	-0.0882	3.0429	-0.7128	-0.8600
$S^{25}_{65,67,69}$	0.1313	0.3433	-0.1093	3.0407	-0.6854	-0.8205
$S^{25}_{65,67,69,71}$	0.1257	0.3298	-0.1226	3.0381	-0.6628	-0.7919
$S^{25}_{65,67,69,71,73}$	0.1210	0.3177	-0.1366	3.0364	-0.6447	-0.7668

Table 5: Various results from hedging a multiple cohorts beginning at age 65 with S-Swaps for a 25 year maturity.

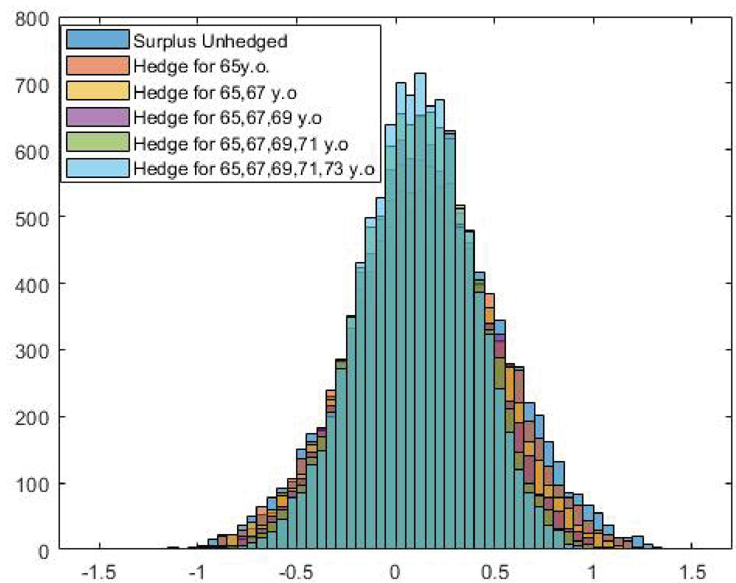

Figure 14: The distribution for various swap hedges, referring to the results presented in Table 5

Here are some interesting salient takeaways from Table 5 and Figure 14. We see that as we hedge more cohorts, the mean reduces, but so does the standard deviation of results. The mean reduction is not as apparent as for the previously discussed single cohort hedging scheme. This is because the assets are now spread throughout time, and the last participants pay a lower price since they are entering the pension's plan portfolio for a shorter period.

Like hedging a single cohort, we notice that the swap has a negative skew, and as expected, the risk metrics VaR99 and ES99 perform better as we add on more contracts to hedge the portfolio.

Table 6 below shows the results for the cap hedge.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1636	0.4021	-0.0554	3.0516	-0.7772	-0.9349
C_{65}^{25}	0.1602	0.3925	-0.0213	3.0496	-0.7502	-0.9016
$C_{65,67}^{25}$	0.1578	0.3830	0.0154	3.0542	-0.7158	-0.8670
$C_{65,67,69}^{25}$	0.1561	0.3737	0.0522	3.0630	-0.6950	-0.8313
$C_{65,67,69,71}^{25}$	0.1562	0.3672	0.0796	3.0694	-0.6725	-0.8054
$C_{65,67,69,71,73}^{25}$	0.1567	0.3614	0.1038	3.0768	-0.6575	-0.7826

Table 6: Various results from hedging a multiple cohorts beginning at age 65 with S-Caps for a 25 year maturity.

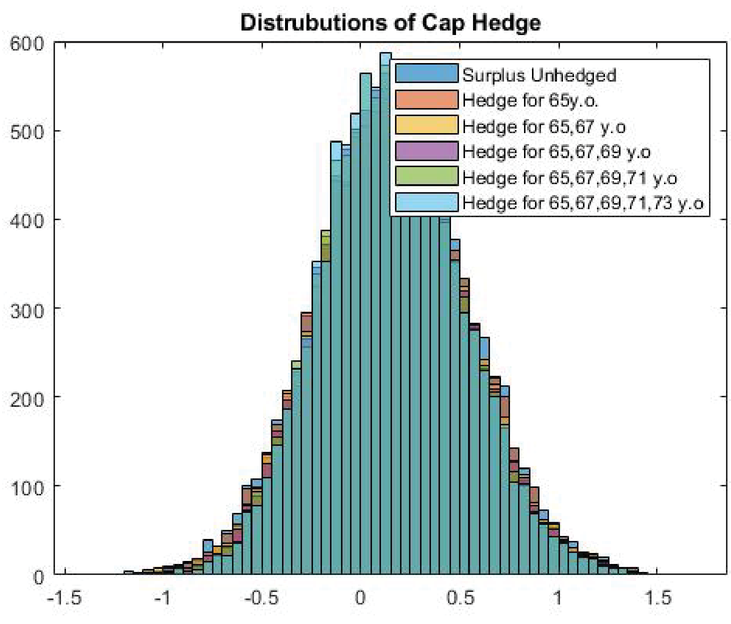

Figure 15: The distribution for various cap hedges, referring to the results presented in Table 6

Generally, the results resemble those of the single cohort. As we hedge using more contracts, the skewness becomes positive. When compared to the swaps, the mean is slightly higher and the standard deviation higher. On the other hand, we notice that the VaR99 and the ES99 do not perform as well as the single cohort paradigm. This is due to the premium paid for the cap for each cohort. Because we do not have the same level of positive skewness, the premium eats away from the potential upsides.

5.3 Mortality instruments

The longevity instruments presented in the previous section are known as cash-flow matching instruments, meaning we hedge the changes in cash inflow (or outflow) from liabilities. Cash-flow hedging instruments are easily implemented but can only fit the specific characteristic of the portfolio they hedge. Switching to a more general approach would mean looking at instruments that hedge fair value; we hedge the difference in value in liabilities. In our framework, assets are fixed at time 0; therefore, we hedge only the changes in liabilities.

In our framework, the instruments presented as fair value hedging are mortality instruments. Mortality instruments, put forward by Li-Luo [21], consider price sensitivity to the mortality curve. The methodology proposed by Li and Luo in their 2011 [21] paper is based on Ho's 1992 methodology for key rate durations [13], which is commonly used in hedging interest rate risk (Li, Luo 2011) [21].

It is fair to assume that mortality rates, much like interest rates, are correlated. If a mortality rate for a given year is subject to a spike, the years before and after will be affected too. Figure 16 illustrates this correlation.

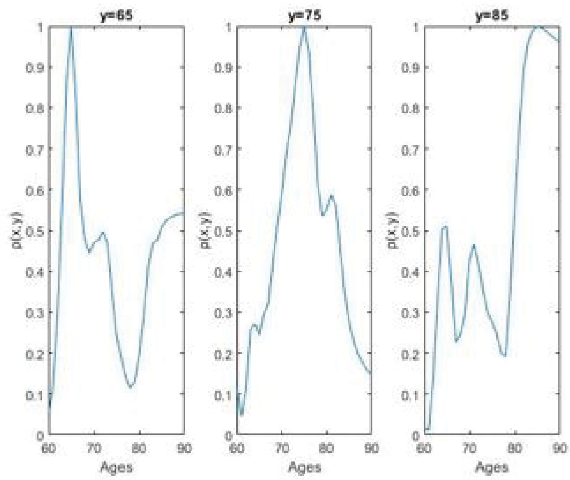

Figure 16: Values of $\rho(x, y)$ for ages 65,75,80

$\rho(x, y)$ from Figure 16 is computed by calculating the sample correlation between any given age x at year t, by taking the differential of the mortality rates between two periods.

Due to the correlation, evaluating the impact on liabilities from a change in the mortality curve has to be done through a vector of numbers rather than an isolated single spike. Because we are considering a vector rather than a single spike, we do not need to compute the sensitivity at every age; instead, we will choose key ages. This way, we can keep the hedging scheme parsimonious and use only a handful of contracts to hedge the changes in value in the liabilities.

Following the methodology put forward by Li-Luo, we will demonstrate how to construct an effective hedge using mortality derivatives. The contracts that we will use are called Q-Forwards. The weight we will put in each of the contracts depends on our portfolio's sensitivity towards that specific age.

5.3.1 Key q-durations

The key q-duration is defined as a portfolio's sensitivity to a change in the mortality rate. Suppose we have a spike of $\delta(j)$ on the j^{th} mortality rate. We consider a spike of $\delta(j) = 0.10\%$. For example, let us take a key age of 70, because of the relationship between neighbouring ages, we define the spike as,

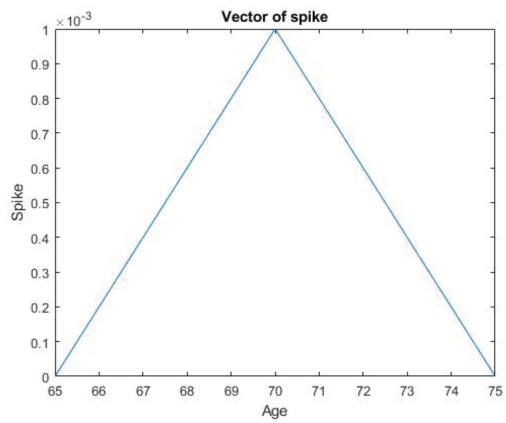

Figure 17: Exemple of shifts applied to the $j = 70$ key age and the impact on adjacent ages

The shift in the curve is meant to enhance the natural correlations found between neighbouring ages. As the age gap becomes widens, the correlations tend to diminish. We denote the shift $s(x, j, \delta(j))$ as the shift for a given age x, associated with the change of $\delta(j)$ for the jth key age. The shifts applied in Figure 17 can be expressed mathematically as,

$$s(x, j, \delta(j)) = \begin{cases} 0 & \text{if } x \leq x_{j-1} \\ \frac{\delta(j)(x - x_{j-1})}{x_j - x_{j-1}} & \text{if } x_{j-1} < x \leq x_j \\ \frac{\delta(j)(x_{j+1} - x)}{x_{j+1} - x_j} & \text{if } x_j < x < x_{j+1} \\ 0 & \text{if } x \geq x_{j-1} \end{cases}$$

For each age x between x_j and x_{j+1}, the impact on the mortality rate of age x is the weighted mean of $\delta(j)$ and $\delta(j+1)$, with the weights being $\frac{(x_{j+1} - x)}{x_{j+1} - x_j}$ and $\frac{(x - x_{j-1})}{x_j - x_{j-1}}$.

We apply the above shift to our original mortality curve, denoted q, to obtain $\tilde{q}(j)$. To properly evaluate the jth q-duration of the portfolio, we take the portfolio's value denoted $P(\tilde{q}(j))$ with the shift and $P(q)$ without the shift in the mortality curve.

$$KQD(P(q), j) = \frac{P(\tilde{q}(j)) - P(q)}{\delta j} \tag{8}$$

5.3.2 Q-Forward contracts

Let us take as an example a pension plan that would want to hedge its longevity risk by entering in a portfolio of mortality forwards. The pension plan will receive the fixed mortality rate, assumed at contract initiation, in exchange for the realized mortality rate of the underlying, which in this case will be a specific cohort. When mortality goes down, then the fixed leg payment is less than the variable one; therefore, it hedges the pension plan's position. In order to obtain a successful hedge, meaning that we reduce variance, we must first adjust the weight invested in each mortality forward (Li, Luo,2011)[21]. Figure 18 illustrates the position in the Q-Forwards.

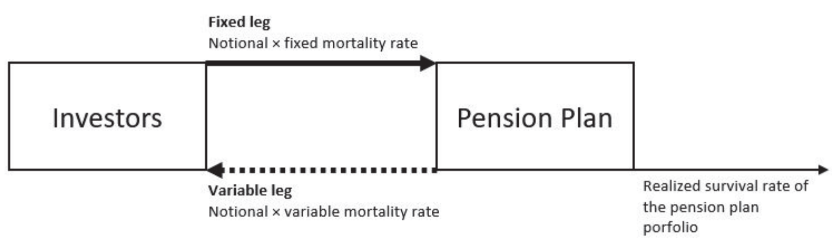

Figure 18: Settlement for a Q-Forward contract between an investor and a pension plan

To have an effective hedge, the forward contract and the underlying assets must have the same price sensitivity. We can measure this sensitivity using q-durations for key j ages. Let us take the value of the mortality forward, from a fixed receiver's perspective.

$$F_j(q) = 1(1+r)^{-T_j}(q^f(x_j,t_j) - q(x_j,t_j)) \tag{9}$$

Equation 9 reflects the differential between the fixed mortality rate and the realized mortality rate is discounted to the interest rate r. From Equation 9, we can compute the key q-duration of the mortality contract, such as,

$$\begin{aligned} KQD\left(F_j(q),j\right) &= \frac{F_j\left(\tilde{q}\left(\delta(j)\right)\right) - F_j(q)}{\delta(j)} \\ &= \frac{(1+r)^{-(T_j-t_0)}(q^f-\tilde{q}) - (1+r)^{-(T_j-t_0)}(q^f-q)}{\delta(j)} \\ &= \frac{(1+r)^{-(T_j-t_0)}(q-\tilde{q})}{\delta(j)} = \frac{(1+r)^{-(T_j-t_0)}(q-q-\delta(j))}{\delta(j)} \\ &= -(1+r)^{-(T_j-t_0)} \end{aligned}$$

Therefore, the key q-duration of the payoff of the mortality contract is

$$KQD(F_j(q),j) = -1(1+r)^{T_j}$$

To construct the hedge with the mortality instruments, we need to define a specific weight put in the n number of contracts we are using. To obtain the weight, we take the sensitivity of the key q-durations of the portfolio and the key q-durations for the key age j.

$$w(j) = \frac{KQD(P(q),j)}{KQD(F_j(q),j)} \tag{10}$$

To consider the importance of the correlation between adjacent ages, we will calculate the value at time zero by considering both the shifted and the non-shifted mortality curves. The following sections will subsequently explore how the single and multi-cohort shifts and the hedging schemes are applied. For this methodology, it is important to always keep the same random numbers used in both the shifted and non-shifted mortality curves. We want to take this additional step in order to reduce any statistical discrepancies.

Single cohort

We begin with a single cohort aged 65 in 2017. The key ages that will be considered in the single cohort hedging scheme are 70 in 2022, 75 in 2027, 80 in 2032, and 85 in 2037. Following the same hypothesis as for the longevity forwards, there is no economic need to hedge out further than 85, considering the small pool of population beyond that age.

Using Equation 8, we compute the key q-durations of the portfolio, whereas using Equation 10, we obtain the appropriate weights to invest in each of the forward contracts. Table 7 below shows the results of the computations.

	j=1	j=2	j=3	j=4
x_j	70	75	80	85
$(KQD(P(q), j)$	-47.7478	-30.37	-17.2215	-8.5927
$(KQD(F_j(q), j)$	-0.8219	-0.6756	-0.5553	-0.4564
w_j	58.0926	44.9551	31.0150	18.4027

Table 7: Key q-durations and weights for each of the Q-Forwards computed from the Canadian Total Population

These negative and descending key q-durations are in line with the results found in Li-Luo[21]. Recall that the key q-durations are computed by subtracting the $P(q)$ value from the $P(\tilde{q}(j))$. The value of the portfolio computed with the $\tilde{q}(j)$ curve will be higher than the one computed with the q curve because higher mortality rates lead to lower survival rates and therefore lower levels of liabilities.

The w_j are positive and descending, and they can be interpreted as the dollar amount to invest into each q-forward to replicate the sensitivity of the liabilities. The assets and liabilities are computed the same way as they were for the longevity instruments (from Section 5.1.3) , except the hedged payoffs are

$$H(q) = \sum_{j=1}^{N} w_j F_j(q)$$

where $F_j(q)$ as presented in Equation 9 is the individual payoff of the forward contract.

Multi cohort

Recall the synthetic portfolio introduced in Section 5.2 (Longevity instruments). The portfolio goes across 20 cohorts, the first one aged 65 in 2017 and the last one aged 85 in 2017. The key ages that will be at study for the multi-cohort setting are presented in Table 8.

k	n_k	Key ages
1	4	Ages 70, 75, 80 and 85
2	3	Ages 75, 80 and 85
3	2	Ages 80 and 85
4	1	Ages 85

Table 8: Key ages considered for the multi cohort mortality hedging

Not only do we want to enhance the natural correlation found between neighbouring ages, we also want to represent this relationship in between neighbouring cohorts. Therefore, the shift will no longer be a singular vector, bur rather a surface. We denote the shift function $s(x, c, (j, k)\delta(j, k))$ as the impact of a $\delta(j, k)$ change at the (j, k)th key rate for a given age x and a given birth year c. The birth years at study range from 1937 (portfolio participants of age 80 in 2017) to 1952 (portfolio participants of age 65 in 2017). The shift function is give by,

$$s(x, c, (j, k)\delta(j, k)) = \delta(j, k))\alpha(x, j, k)\beta(c, k)$$

The following relationship is derived from the year of birth, meaning that the shift declines with birth years that are further from the j^{th} key age of the kth cohort. In the above equation, α and β are calculated as such,

$$\alpha(x,j,k) = \begin{cases} 0 & \text{if } x \leq x_{j-1,k} \\ \frac{x - x_{j-1,k}}{x_j - x_{j-1}} & \text{if } x_{j-1,k} < x \leq x_{j,k} \\ \frac{x_{j+1,k} - x}{x_{j+1,k} - x_{j,k}} & \text{if } x_{j,k} < x < x_{j+1,k} \\ 0 & \text{if } x \geq x_{j-1,k} \end{cases}$$

and,

$$\beta(c,k) = \begin{cases} 0 & \text{if } c \leq c_{j-1,k} \\ \frac{c - c_{j-1,k}}{c_j - c_{j-1}} & \text{if } c_{j-1,k} < c \leq c_{j,k} \\ \frac{c_{j+1,k} - c}{c_{j+1,k} - c_{j,k}} & \text{if } c_{j,k} < c < c_{j+1,k} \\ 0 & \text{if } c \geq c_{j-1,k} \end{cases}$$

With these shifts we can apply the same methodology as for a single cohort, except now we will have a notional amount $w_{j,k}$. Visually, we obtain the following distribution of shifts and weights :

K cohort	2017 age	n$_k$	Age in time	$w_{j,k}$
1	65	1	(70,2022)	9.4233
		2	(75,2027)	7.5644
		3	(80,2032)	5.1849
		4	(85,2037)	3.0460
2	70	1	(75,2022)	9.9646
		2	(80,2027)	6.9993
		3	(85,2032)	4.7069
3	75	1	(75,2022)	5.2513
		2	(80,2027)	7.7613
4	80	1	(85,2022)	3.3964

Table 9: Weights applied to each contract throughout different cohorts

5.3.3 Results from hedging with mortality products

Results from hedging a single cohort

The key q-durations used to compute the weights to invest in forward contracts rely on the natural correlations found between mortality rates for neighbouring ages. Because our first step when applying the FDA model is to smooth the data, we expect to see the mortality hedging instruments perform better. This could lead to the desired results of giving up less of the mean for a better improvement of the risk metrics. Table 10 below summarizes our findings.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2688	0.4652	-0.0084	2.9219	-0.8016	-0.9640
Q-Forward Hedged	0.1015	0.2090	-0.1421	3.1000	-0.4127	-0.4953

Table 10: Results for hedging a single cohort at age 65 using 4 Q-Forwards.

The pension plan has to give up mean in order to obtain a better performance in the risks metrics: standard deviation (from 0.4652 to 0.2090), the VaR99 (from -0.8016 to -0.4127) and the ES99 (from -0.9640 to -0.4953). Basing our analysis on these metrics, we find mixed but comprable results. The Q-Forwards performs better than the S-Swap for a single cohort beginning at age 65 for a 25-year maturity. Recall that the S-Swap is a series of S-Forwards across different maturities. The S-Swap hedge mean is at 0.1002, its standard deviation is at 0.1700 and the VaR99 and ES99 are at -0.4150 and -0.4872, respectively. The standard deviation of the distribution is higher for the mortality instrument and the VaR99 and ES99 are slightly higher. This could be explained by the distribution's skewness which is closer to zero for the Q-Forwards (-0.1421) than the S-Swap (-0.2241).

Results from hedging multiple cohorts

Hedging multiple cohorts with Q-Forwards shows a bigger mean reduction compared to hedging with S-Swaps. However, the mean reduction is justified by a decrease in the standard deviation of results (from 0.4021 to 0.3098) and an improvement on the risk metrics VaR99 (from -0.7772 to -0.6312) and the ES99 (from -0.9349 to -0.7485).

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1636	0.4021	-0.0554	3.0516	-0.7772	-0.9349
Q-Forward Hedged	0.1193	0.3098	-0.1798	2.9832	-0.6312	-0.7485

Table 11: Results from hedging multiple cohorts using various Q-Forwards

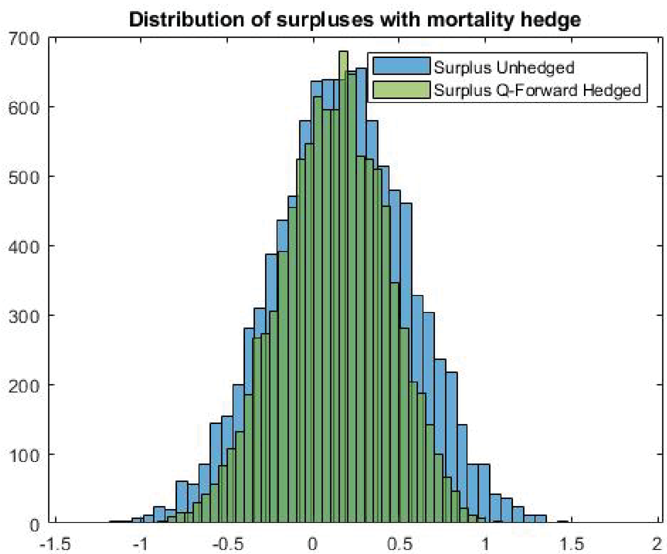

Figure 19: Distribution of hedging multiple cohorts using Q-Forwards. This distribution refers to the results summarized in Table 11.

We reach a similar conclusion when comparing results for the Q-Forwards and the S-Swap for hedging multiple cohorts. For example, our Q-Forwards reduce the mean to 0.1193 (compared to 0.1636 for the unhedged distribution), whereas the $S^{25}_{65,67,69,71,73}$ reduces it to 0.1210. The standard deviation for results is almost identical across both instruments (0.3177 for the S-Swap and 0.3098 for the Q-Forwards). The risk measure VaR99 and the ES99 perform better with the Q-Forwards. The VaR99 is at -0.6312 for the Q-Forwards and at -0.7485 for the S-Swap, whereas the ES99 is at -0.6312 for the Q-Forwards and at -0.7668 for the S-Swap.

Overview

We have now covered how the hedging instruments perform when we set up our pension plan's portfolio to mimic the historical composition of men and women in the Canadian population. We can see that mortality instruments are more efficient when analyzing the profits and loss after hedging.

In the following section, we will look at a portfolio with a specific composition of men and women. Every single pension plan portfolio of subscribers has its distribution of men and women, and shifting to a cross-gender study will give us a more realistic scenario. We will apply the same instruments but in different hedging, schemes to provide an exhaustive cross-gender study.

6 Cross gender hedging

So far, we have only considered the total population curve, which models the pension plan's portfolio according to the historical weights of men and women in the Canadian population. When we shift to a cross-gender study, we want to model the pension plan portfolio according to a specific composition. We choose a portfolio that is 65% women and 35% men; we will show that we need to apply the FDA paradigm to individual female and male mortality curves.

The models for each of the coefficients show something interesting. When we split the data into males and females, the first principal component, the β_1 model, is an MA model. Recall that under the total population curve, the first component was a random walk with drift. This finding reinforces our initial analysis that male and female individual mortality curves exhibit some degree of structure. Table 12 shows the models' detailed results from Figures 20 and 21.

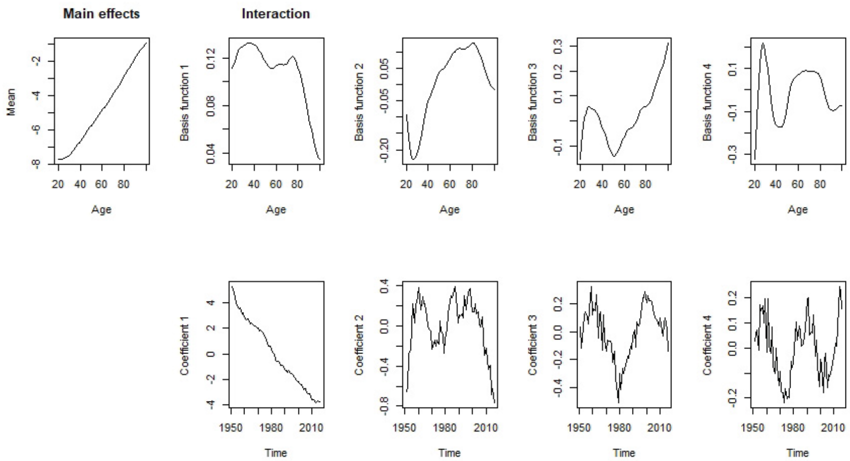

Figure 20: The first four interactions for the Canadian Female population

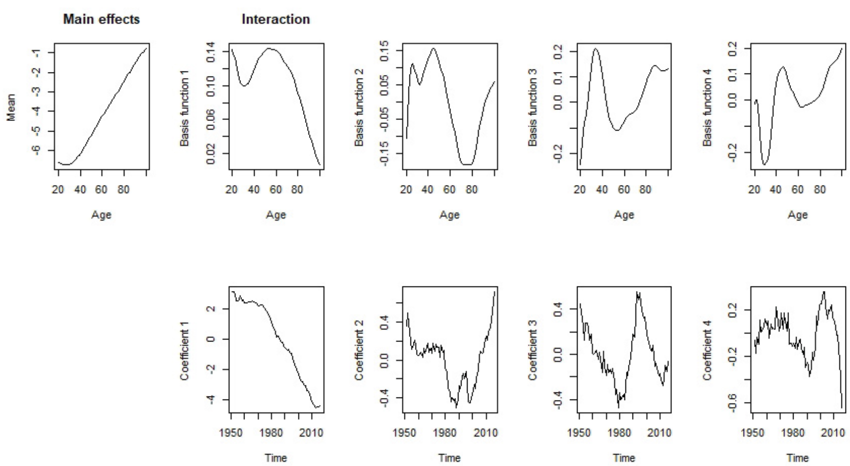

Figure 21: The first four interactions for the Canadian Male population

Model/Population	Female	Male
β_1	ARIMA(0,2,2)	ARIMA(0,2,1)
β_2	ARIMA(1,1,0)	ARIMA(0,2,1)
β_3	ARIMA(0,0,3)	ARIMA(2,0,2)
β_4	ARIMA(2,0,2)	ARIMA(1,0,0)

Table 12: Models of the beta coefficients for each segment of the population

Using these new coefficients and their basis, we can compute the same parameters for the hedging instruments as we did in previous sections.

To best evaluate our hedge's effectiveness, we apply three different types of hedging schemes. We begin with longevity and mortality hedging instruments based on the total population curve. We compare those results with instruments based on the male mortality curves and the female mortality curves. We finish our study by looking into a mixed hedging scheme, which is in the same proportion of men and women as the pension plan's underlying portfolio. We do this to provide benchmark-like results for comparison purposes. This means that we will compare hedging with instruments that are based on the female (or male mortality curve) to those in a mixed population instruments portfolio.

When working on this thesis, we realized that the total population method yielded unsatisfactory results. This method consists of taking the total population mortality curve to compute the parameters for the longevity and mortality instruments. This method's main disadvantage is that in the historical data, the proportion of men and women change throughout time, whereas in our paradigm, we keep the proportion constant. Furthermore, If the underlying portfolio diverges from the historical, therefore, the parameters of the hedging instruments are not specific enough and cannot hedge the population. These results confirm our initial analysis that the fundamental differences between male and female mortality curves are important. Because the results were unsatisfactory, we decided not to retain this method and, therefore, will not present the results in the essay. However, we provide these results for the single cohort longevity and mortality instruments in Annex 7 Cross gender hedging results using instruments based on the total Canadian mortality curve (Section 8.7). :

6.1 Unhedged portfolio and longevity hedging instruments

We build the pension plan portfolio using the methodology in Section 5.1. We use the same 4 000 annuitants, but we split them into 2 600 women (65%) and 1 400 men (35%). Each of the τ_k is generated based on the underlying mortality curve for each of the genders. We can compute the assets and liabilities individually for each gender. The longevity bond is 13.4028$ for a single woman at age 65 and 12.0492$ for a single man at age 65. The higher price for the longevity bond for women than men is consistent with the general findings that women tend to live longer and will therefore have higher survival rates in further ages.

Recall the hedging scheme we have applied so far,

$$H = A - L + P_{Method}$$

We define P_{Method} as the instrument-specific payoff.
Starting with the longevity swap, we have

$$P_{S-swap,\text{G}} = (e^{-\int_0^T m_{x,G}(v)dv} - K_G(T))$$

where

G is the gender of the instrument: men or women

K_G is the fixed rate of contract, computed upon the mortality curve of the given gender, computed as $E^{RA}[S_T]$ for $G =$ men or women.

We can compute the P_{Method} as,

$$P = \frac{\text{Women}}{\text{Total}} P_{S-swap,Women} + \frac{\text{Men}}{\text{Total}} P_{S-swap,Men}$$

We can set the Female/Total (Male/Total) ratio to 0 and the other ratio to 1 if we want to compute the hedging only from the male (female) S-Swap. To compute the mixed hedge, we input those ratios from the underlying portfolio of 65% women and 35% men.

When computing the Cap hedge, we proceed the same way:

$$P_{Cap,G} = max\{(e^{-\int_0^T m_{x,W}(v)dv} - k_G), 0\}$$

where the strike for the caplet is $k_G = E_0^{BE}[S_T]$, for G = men or women.

The total pay off is,

$$P = \frac{\text{Women}}{\text{Total}} P_{Cap,Women} + \frac{\text{Men}}{\text{Total}} P_{Cap,Men}$$

Using the same methodology as above, we can force one of the two arguments to equal zero, while the other one is equal to 1, to evaluate the hedge using a single mortality curve, let it be men or women.

$$P = \frac{\text{Women}}{\text{Total}} \sum_{j=1}^{N} w_{j,women} F_{j,women}(q) + \frac{\text{Men}}{\text{Total}} \sum_{j=1}^{N} w_{j,men} F_{j,men}(q)$$

From these formulas, we apply the index for males and females by separately setting one of the equation's terms to 0. When we look at the gender-specific mixed index, we keep the weights as those of the pension's plan portfolio.

6.2 Longevity hedging

6.2.1 Results from hedging a single cohort

We begin this section by presenting the results for the unhedged portfolio. Notice the difference between an unhedged mixed portfolio (Unhedged MP) that is 65% women and 35% men and the unhedged portfolio based on the historical weights in the data set (Unhedged HW) discussed in Section 5.2.3.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged MP	0.2513	0.3064	0.0318	3.0137	-0.4563	-0.5515
Unhedged HW	0.2688	0.4652	-0.0084	2.9219	-0.8016	-0.9640

Table 13: Comparaison of two unhedged portfolios, one that is computed with the male and female mortality curve independently and the other that is computed form the total population curve

Table 13 shows that both means are about the same, but the standard deviation of results, the VaR99, and the ES99 of the mixed portfolio are lower. The use of gender-specific mortality curves to compute the assets and liabilities generates this difference. This difference arises from the different inputs used, such as the bases ϕ_k and the coefficients $\beta_{t,k}$.

When applying the hedging scheme, we began by looking over the effectiveness of hedging only with instruments based on the male mortality curve. Our initial hypothesis is that these instruments will not provide a sufficient hedge for the underlying portfolio of 65% women. In Table 14, we see that the risk-reducing metrics (standard deviation, the VAR99, and the ES99) do not improve. This trend is maintained across all longevity and mortality instruments. So, we have decided not to present further results from hedging with male hedging instruments.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2513	0.3064	0.0318	3.0137	-0.4563	-0.5515
Swap Hedged	0.0612	0.3376	0.0305	3.0133	-0.5329	-0.6209
Cap Hedged	0.1433	0.2982	0.0837	3.0335	-0.7201	-0.8242

Table 14: Results from hedging a single cohort of age 65 for a 25-year maturity, with longevity instruments based on the male mortality curve

We obtain the following results when applying the female mortality instruments as a hedge to our mixed portfolio.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2513	0.3064	0.0318	3.0137	-0.4563	-0.5515
Swap Hedged	0.0789	0.1520	0.0156	2.9303	-0.2590	-0.3032
Cap Hedged	0.1566	0.2117	0.6422	3.7756	-0.2713	-0.3227

Table 15: Results from hedging a single cohort at age 65 for a 25-year maturity, with longevity instruments based on the female mortality curve

Because of our portfolio's composition, the results from hedging with female longevity instruments are remarkably better than if we hedged them with male longevity instruments. We see that the risk metrics are performing better, and we see that the cap surpluses have a positive skewness, meaning that the distribution will have a fat tail to the right.

However, the best results are those from a mixed hedging scheme that mimics the weights from the underlying portfolio. Table 16 below shows that for a mean reduction close to that of hedging with female longevity instruments, we cut the VaR from -0.4563 to -0.2562 for the S-Swap and -0.2832 for the Caplet. All other results presented in Table 16 are in line with those presented in previous sections of this thesis. The Swap's standard deviation is 0.1247 while the Cap's is 0.2194, whereas the Cap presents with positive skewness (0.6972 for the Cap and -0.1772 for the Swap).

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2513	0.3064	0.0318	3.0137	-0.4563	-0.5515
Swap Hedged	0.0727	0.1247	-0.1772	3.0526	-0.2562	-0.3061
Cap Hedged	0.1520	0.2194	0.6972	3.6221	-0.2313	-0.2832

Table 16: Results from hedging a single cohort at age 65, with longevity products, for a 25-year maturity, with longevity instruments that are based on a mix of 65% women and 35% men

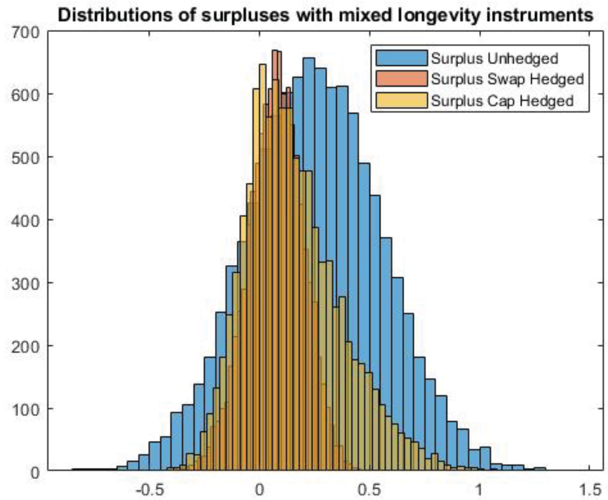

Figure 22: Distribution of single cohort, longevity hedge for a 25 year maturity , with Swaps and Caps. These distrubtions summarize the results found in Table 16.

6.2.2 Results from hedging multiple cohorts

As per the findings in Section 6.2.3, the unhedged multiple cohort portfolio's means are lower than the unhedged single cohort portfolio. A more detailed overview of the asset and instruments pricing for multiple cohorts is available in Annex 3. We present the results from the 25-year hedge.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1537	0.2061	-0.0525	2.9893	-0.3263	-0.4039
S_{65}^{25}	0.1479	0.1931	-0.0618	2.9968	-0.2994	-0.3768
$S_{65,67}^{25}$	0.1422	0.1816	-0.0732	3.0095	-0.2845	-0.3546
$S_{65,67,69}^{25}$	0.1373	0.1719	-0.0802	3.0160	-0.2687	-0.3346
$S_{65,67,69,71}^{25}$	0.1341	0.1659	-0.0861	3.0218	-0.2544	-0.3222
C_{65}^{25}	0.1508	0.1997	-0.0130	2.9924	-0.3058	-0.3829
$C_{65,67}^{25}$	0.1488	0.1940	0.0220	3.0037	-0.2904	-0.3642
$C_{65,67,69}^{25}$	0.1478	0.1891	0.0538	3.0101	-0.2780	-0.3474
$C_{65,67,69,71}^{25}$	0.1485	0.1861	0.0706	3.0169	-0.2683	-0.3358

Table 17: Various S-Forward and Caple hedges on different cohorts, with 25-year longevity instruments based on the female mortality curve.

Table 17 presents various results for multiple cohorts hedged with female instruments. We see that there is almost no change in the mean or the risk statistics when hedging only the first two cohorts. This is in line with our finding that women tend to live longer, and therefore the hedging needs are in further ages. When we look at the S-Swaps $S^{25}_{65,67,69,71}$, we see that we have an interesting reduction for the standard deviation, the VaR, and ES. As discussed in previous sections, because the skewness for the Caplet is near zero (while remaining positive for two or more Caplet contracts), the premium paid for the option eats away from the possible upside.

In Table 18 below, we apply mixed longevity instruments. We notice that the instruments perform marginally better than when using the female-only curve. However, we see similar characteristics between the Swap and Cap, and the risk metrics of the Swap perform better than those of the Cap.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1537	0.2061	-0.0525	2.9893	-0.3263	-0.4039
S^{25}_{65}	0.1482	0.1898	-0.0657	3.0017	-0.2972	-0.3693
$S^{25}_{65,67}$	0.1429	0.1760	-0.0789	3.0228	-0.2742	-0.3403
$S^{25}_{65,67,69}$	0.1382	0.1646	-0.0831	3.0289	-0.2491	-0.3163
$S^{25}_{65,67,69,71}$	0.1352	0.1578	-0.0905	3.0401	-0.2331	-0.3022
C^{25}_{65}	0.1510	0.1980	0.0025	2.9978	-0.3019	-0.3765
$C^{25}_{65,67}$	0.1491	0.1910	0.0501	3.0199	-0.2831	-0.3525
$C^{25}_{65,67,69}$	0.1483	0.1850	0.0920	3.0345	-0.2640	-0.3321
$C^{25}_{65,67,69,71}$	0.1491	0.1813	0.1115	3.0503	-0.2517	-0.3197

Table 18: Various S-Forward and Caple hedges on different cohorts, with a 25 year longevity instruments based on a mix of 65% women and 35% men

6.3 Mortality hedging

For comparison purposes, we will keep on using 4 Q-Forward contracts, spread throughout the ages of 70 and 85, just as we did in Section 5.3.

Following the methodology proposed in Section 5.3, we compute the weights for each of the forward contracts. We obtain two sets of results for both genders to apply to the curves. In Table 19, we can see that the hedging weights on the Q-Forwards based on the female mortality curve are higher than those based on the male mortality curve. As before, we calculate the sensitivity using a shift of $\delta = 0.10\%$. Recall how we computed the key q-durations sensitivities in Section 5.3.2,

$$KQD(P(q), j) = \frac{P(\tilde{q}(j)) - P(q)}{\delta j}$$

Because the female mortality rate is on average lower than men's, the women's KQDs tend to be higher than men's, resulting in higher weights for their Q-Forwards. In notional terms, the mortality instruments based on the mortality curve will be higher in the pension plan's hedged portfolio.

Weights	Women	Men
w_1	60.0625	51.0217
w_2	47.0699	37.1240
w_3	33.6594	23.8684
w_4	20.8291	12.6309

Table 19: Gender specific Q-Forward weights

6.3.1 Results from hedging a single cohort

When we compute the results of hedging the mixed portfolio, using only instruments based on the female mortality curve, we notice that they do not improve the pension plan's portfolio's risk profile. The VaR and the ES are higher than the unhedged portfolio. This can be explained by the highly negative skewness, meaning that we have extreme negative occurrences, resulting in more significant losses. This phenomenon can be explained by the contract weights computed in Table 19. Because of a potential mismatch between the composition of the hedging instrument and the portfolio, we expose ourselves to basis risk. When multiplying that basis risk by 60, we create greater discrepancies. The detailed results are in Table 20.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2513	0.3064	0.0318	3.0137	-0.4563	-0.5515
Q-Forward Hedge	0.0901	0.2058	-0.1958	3.4463	-0.4036	-0.5282

Table 20: Results from hedging a single cohort at age 65 with 4 Q-Forward contracts, based on the female mortality curve

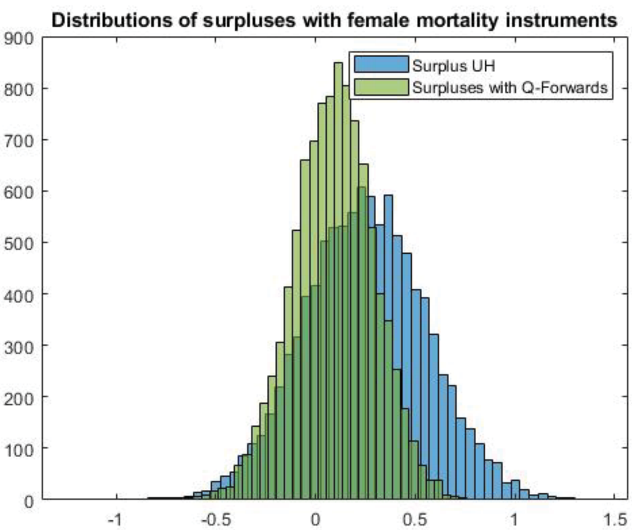

Figure 23: Distribution of single cohort at age 65, hedged with 4 female Q-Forwards. This distribution refers to the results summarized in Table 20

Although hedging with Q-Forwards with one gender only does not yield satisfactory results, using mixed instruments seems to provide a much better hedge. The mean reduction is compensated by a significant reduction in the standard deviation (from 0.3064 to 0.1343), the VaR99 (from -0.4563 to -

0.2314) and the ES99 (from -0.5515 to -0.2793). When comparing these results with the S-Swaps, we find that the portfolio hedged with Q-Forwards performs better. Recall that this was also the case when hedging with the total mortality curve. We believe that the Q-Forwards perform better due to the smoothing technique applied to the data. The results are summarized in Table 21.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2513	0.3064	0.0318	3.0137	-0.4563	-0.5515
Q-Forward Hedge	0.0944	0.1343	-0.1222	3.0092	-0.2314	-0.2793

Table 21: Results from single cohort mortality hedging with four forward contracts, using a portfolio that is a mix of 65% women and 35% men

6.3.2 Results from hedging multiple cohorts

For the final hedging results, we will look at the mortality multi-cohort hedging scheme, following the methodology in Section 5.3. We summarize the gender-specific weights we will apply to the forward contracts in Table 22.

K cohort	2017 age	n_k	Age in time	$w_{j,k,Women}$	$w_{j,k,Men}$
1	65	1	(70,2022)	9.8342	9.4233
		2	(75,2027)	8.0280	7.5644
		3	(80,2032)	5.5963	5.1849
		4	(85,2037)	3.3306	3.0460
2	70	1	(75,2022)	10.5600	9.9646
		2	(80,2027)	7.6144	6.9993
		3	(85,2032)	4.5140	4.7069
3	75	1	(75,2022)	5.7313	5.2513
		2	(80,2027)	7.7512	7.7613
4	80	1	(85,2022)	3.6552	3.1964

Table 22: Weights applied to each contract throughout different cohorts

Our findings for the multiple cohort hedging scheme using mortality products resemble the one for a single cohort. First, hedging using mortality instruments based on the female curve only does not yield satisfactory results.

For a mean reduction from 0.1537 to 0.1101, the VaR99 is higher for the Q-Forward hedge, whereas the ES99 remains practically the same. This can be explained by the weights applied to female contracts.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1537	0.2061	-0.0525	2.9893	-0.3263	-0.4039
Q-Forward Hedged	0.1101	0.1905	-0.0264	2.9484	-0.3345	-0.3999

Table 23: Results from hedging multiple cohorts beginning at age 65 with Q-Forwards based on the female mortality curve

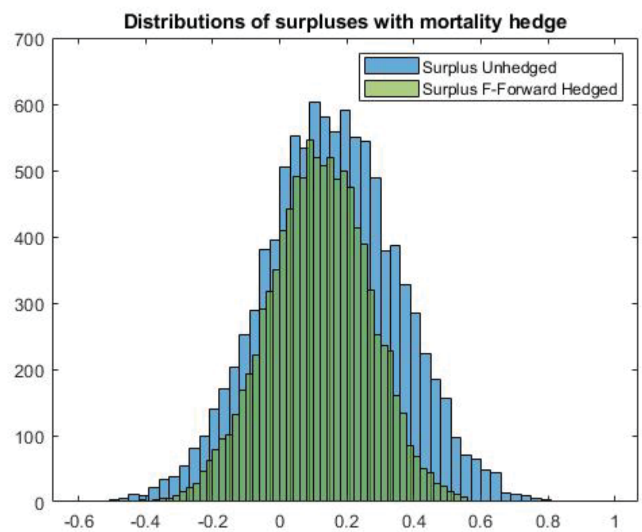

Figure 24: Distribution of surpluses from hedging multiple cohorts beginning at age 65 with Q-Forwards based on the female mortality curve. These distributions are linked to results in Table 23

When we use a mixed portfolio hedges, we get that the distribution of the portfolio profit and loss is better than when we use longevity instruments in multiple cohorts. The mean reduction (from 0.1537 to 0.1142) is compensated by an improvement of the risk metrics VaR99 (from -0.3263 to -0.2626) and the ES99 (from -0.4039 to -0.3159).

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1537	0.2061	-0.0525	2.9893	-0.3263	-0.4039
Q-Forward Hedged	0.1142	0.1584	-0.0894	2.9562	-0.2626	-0.3159

Table 24: Results from hedging multiple cohorts beginning at age 65 with Q-Forwards, using a portfolio that is a mix of 65% women and 35% men

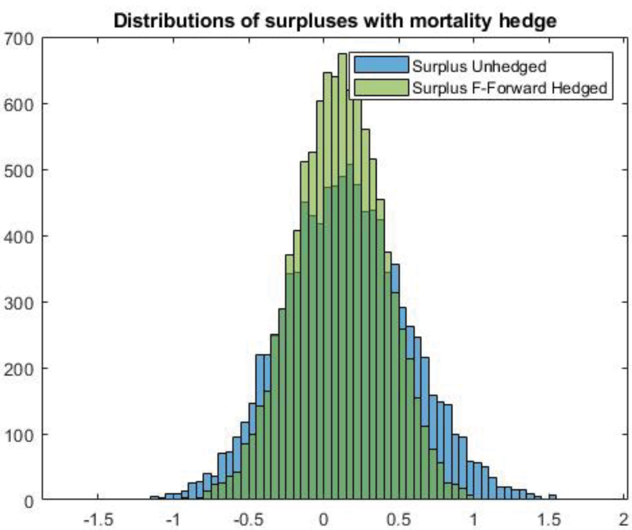

Figure 25: Distribution of surpluses from hedging multiple cohorts beginning at age 65 with Q-Forwards, using a portfolio that is a mix of 65% women and 35% men. These distributions are linked to results in Table 24

7 General conclusions, further discussions

In this thesis, we looked at a newer model to forecast mortality rates and apply hedging schemes. These models, although used in demographic studies, are new to hedging applications. Using the FDA model instead of the LC model allows us to capture most of the structure in the data set, which can augment the model's predictive power. Furthermore, this model has an extension : coherent FDA [14]. With this method, we can simultaneously forecast mortality rates of two or more populations (eg: Canadian and American) while letting each of the underlying mortality curves maintain their long-term patterns. This is interesting if we want to model and analyze the hedging scheme using a cross-population model.

However, one of the most significant disadvantages of the FDA paradigm is that it is sensitive to arbitrary decisions. For example, if we change the data set +/- 5 years, the ARIMA models can drastically change. Demographs will always prefer the ARIMA model that maximizes the AIC/BIC criteria because they want to ensure the best fit for the data. However, the model that fits best the criteria can be non-parsimonious (i.e. an ARIMA(3,1,4)). This is an important problem, especially if we want to have a financial product that is robust. Other models, such as CBD M5, have the property of being invariant to a change in the data frame. An extensive application of that model has been worked on by Giannini 2020 [8]. Furthermore, because the forecasting of the β coefficients is done through ARIMA models, we add to the uncertainty of the results. ARIMA processes are known to give better results for short-term predictions rather than long-term ones. These findings were discussed with Patrice Dion, from Statistique Canada, who convened that the FDA model does have some serious limitations.

We can conclude that mortality instruments perform better than longevity instruments in hedging a pension plan's portfolio regardless of the hedging scheme we applied. We believe that the quality of those results is due to the smoothing technique used in our paradigm. However, this point is not directly verified in our research, as we have not presented results without using a smoothing technique. The comparison could be an avenue for future research. Other hedging products using mortality curves exist, such a Q-puts; a Q-put gives the pension plan the right, but not the obligation, to exercise the option; this instrument (just like the S-Caps) trades at a premium.

From a cross-gender point of view, we can conclude that for longevity products (recall these are cash-flow matching instruments), the pension plan could rely on instruments based only on the female mortality curve if its composition of female annuitants is higher. On the other hand, when using mortality instruments, hedging using only the female mortality curve does not yield satisfactory results. Therefore, the best way for a pension plan to hedge its portfolio is to use a mix of mortality instruments based on the composition of men and women in its underlying portfolio.

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8 Appendixes

8.1 Results from fitting all available data

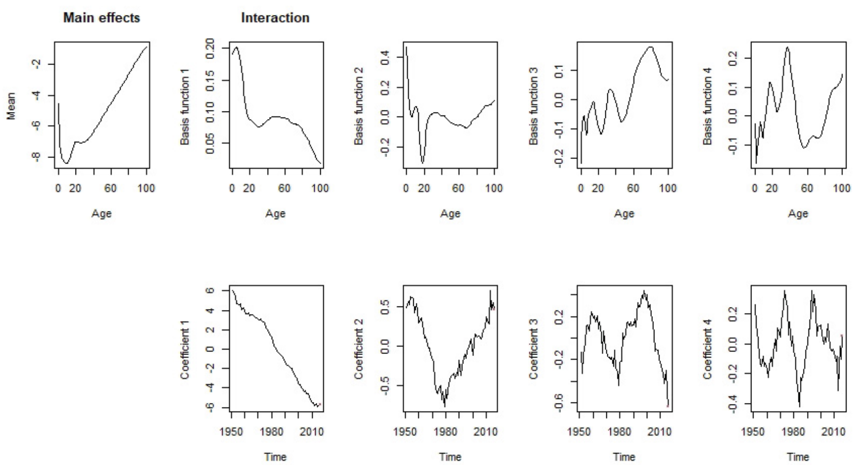

Figure 26: The first four interactions for the Canadian Population, when taking into account all available ages

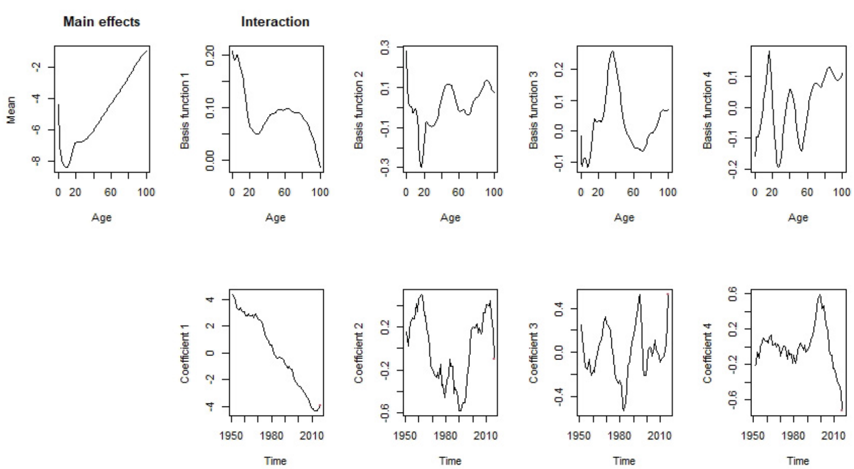

Figure 27: The first four interactions for the American Population, when taking into account all available ages

8.2 FDA Modelling for Slovenia and Estonia

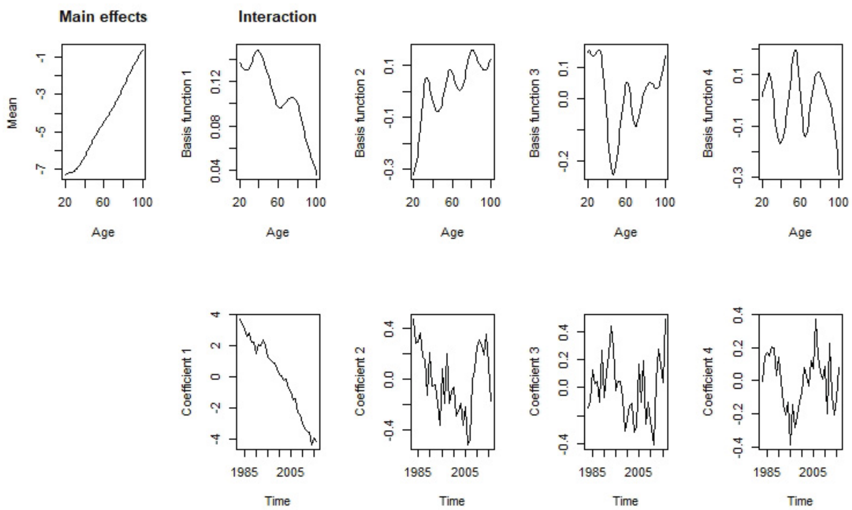

Figure 28: The first four interactions and the basis for Slovenia

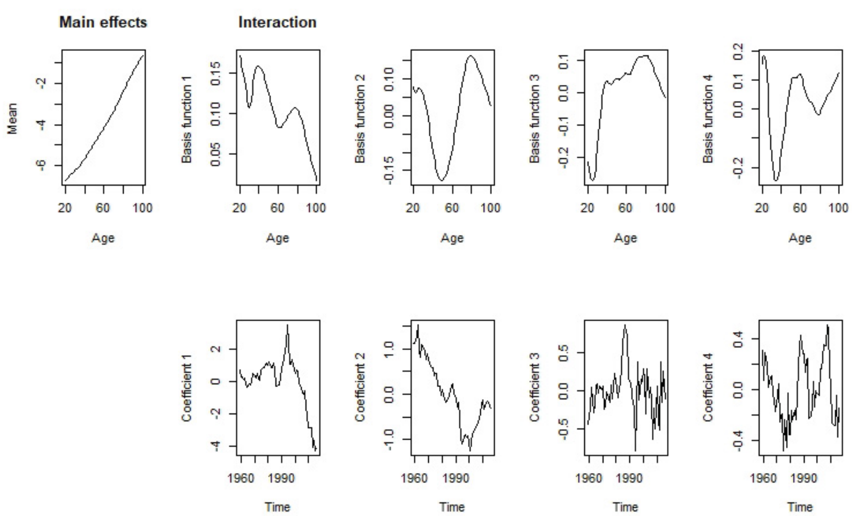

Figure 29: The first four interactions and the basis for Estonia

8.3 Hedging results for a 20 year maturity hedge

The following table shows the results for hedging multiple cohorts with longevity instruments for a 20 year hedge.

Cohort	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1643	0.3985	-0.0277	2.9214	-0.7446	-0.8883
S_{65}^{20}	0.1566	0.3859	-0.0409	2.9151	-0.7253	-0.8678
$S_{65,67}^{20}$	0.1502	0.3743	-0.0551	2.9143	-0.7060	-0.8510
$S_{65,67,69}^{20}$	0.1452	0.3635	-0.0668	2.9133	-0.6912	-0.8317
$S_{65,67,69,71}^{20}$	0.1424	0.3567	-0.0739	2.9087	-0.6778	-0.8165
$S_{65,67,69,71,73}^{20}$	0.1401	0.3508	-0.0805	2.9098	-0.6669	-0.8052
C_{65}^{20}	0.1633	0.3925	-0.0119	2.9240	-0.7248	-0.8696
$C_{65,67}^{20}$	0.1629	0.3870	0.0025	2.9328	-0.7105	-0.8544
$C_{65,67,69}^{20}$	0.1633	0.3818	0.0171	2.9393	-0.6981	-0.8369
$C_{65,67,69,71}^{20}$	0.1644	0.3785	0.0261	2.9414	-0.6861	-0.8231
$C_{65,67,69,71,73}^{20}$	0.1656	0.3756	0.0330	2.9451	-0.6728	-0.8128

8.4 Prices for longevity bonds on multi cohort products

8.4.1 Canadian Population

Cohort	Total Population	Female Population	Male Population
1	12.32	12.87	11.81
2	11.96	12.50	11.44
3	11.59	12.12	11.07
4	11.22	11.74	10.70
5	10.84	11.36	10.32
6	10.47	10.97	9.94
7	10.09	10.58	9.56
8	9.71	10.19	9.18
9	9.34	9.80	8.80
10	8.96	9.40	8.42
11	8.58	9.01	8.04
12	8.20	8.61	7.67
13	7.83	8.21	7.30
14	7.46	7.81	6.93
15	7.10	7.42	6.57
16	6.74	7.03	6.21
17	6.38	6.64	5.85
18	6.03	6.26	5.50
19	5.68	5.89	5.16
20	5.34	5.53	4.82

8.4.2 American Population

Cohort	Total Population	Female Population	Male Population
1	12.32	12.87	11.81
2	11.96	12.50	11.44
3	11.59	12.12	11.07
4	11.22	11.74	10.70
5	10.84	11.36	10.32
6	10.47	10.97	9.94
7	10.09	10.58	9.56
8	9.71	10.19	9.18
9	9.34	9.80	8.80
10	8.96	9.40	8.42
11	8.58	9.01	8.04
12	8.20	8.61	7.67
13	7.83	8.21	7.30
14	7.46	7.81	6.93
15	7.10	7.42	6.57
16	6.74	7.03	6.21
17	6.38	6.64	5.85
18	6.03	6.26	5.50
19	5.68	5.89	5.16
20	5.34	5.53	4.82

8.5 Premiums for caps on multi cohort products

8.5.1 Canadian Population

Cohort	Total Population	Female Population	Male Population
1	0.2478	0.2220	0.2333
2	0.2381	0.2081	0.2193
3	0.2282	0.1943	0.2053
4	0.2181	0.1807	0.1914
5	0.2078	0.1673	0.1778
6	0.1973	0.1542	0.1643
7	0.1867	0.1414	0.1511
8	0.1758	0.1289	0.1383
9	0.1646	0.1169	0.1258
10	0.1532	0.1052	0.1137
11	0.1416	0.0940	0.1019
12	0.1297	0.0832	0.0907
13	0.1177	0.0730	0.0799
14	0.1055	0.0634	0.0696
15	0.0933	0.0543	0.0599
16	0.0812	0.0458	0.0508
17	0.0695	0.0379	0.0424
18	0.0582	0.0306	0.0347
19	0.0475	0.0241	0.0278
20	0.0376	0.0183	0.0216

8.5.2 American Population

Cohort	Total Population	Female Population	Male Population
1	0.2590	0.3347	0.2481
2	0.2466	0.3228	0.2374
3	0.2341	0.3102	0.2268
4	0.2214	0.2968	0.2161
5	0.2086	0.2829	0.2055
6	0.1958	0.2683	0.1947
7	0.1831	0.2534	0.1839
8	0.1705	0.2383	0.1730
9	0.1580	0.2232	0.1620
10	0.1458	0.2082	0.1509
11	0.1338	0.1932	0.1396
12	0.1221	0.1782	0.1282
13	0.1107	0.1630	0.1169
14	0.0996	0.1476	0.1056
15	0.0887	0.1320	0.0944
16	0.0781	0.1161	0.0833
17	0.0678	0.1002	0.0725
18	0.0578	0.0846	0.0619
19	0.0481	0.0693	0.0516
20	0.0389	0.0546	0.0416

8.6 Hedging Results for the American population

8.6.1 American Total population

In the following annex, we summarize the results when we apply the longevity and mortality hedging scheme to the American Total Population, for a single and multiple cohorts. These results are in line with the found in this thesis, in Cairns and al [3] for the longevity instruments and in Lu-Luo for the mortality instruments [21].

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2352	0.4327	0.0811	2.9969	-0.7353	-0.8902
Swap Hedged 20 years	0.1019	0.1907	-0.1247	2.9861	-0.4019	-0.4738
Cap Hedged 20 years	0.1640	0.3215	0.8217	3.6704	-0.3555	-0.4270
Swap Hedge 25 years	0.0455	0.0869	-0.2813	3.0624	-0.2415	-0.2771
Cap Hedged 25 years	0.1312	0.2793	1.3337	4.6780	-0.1766	-0.2119

Table 25: Summary for the longevity hedge for the Amercain Total Population applying longevity instruments for different maturities

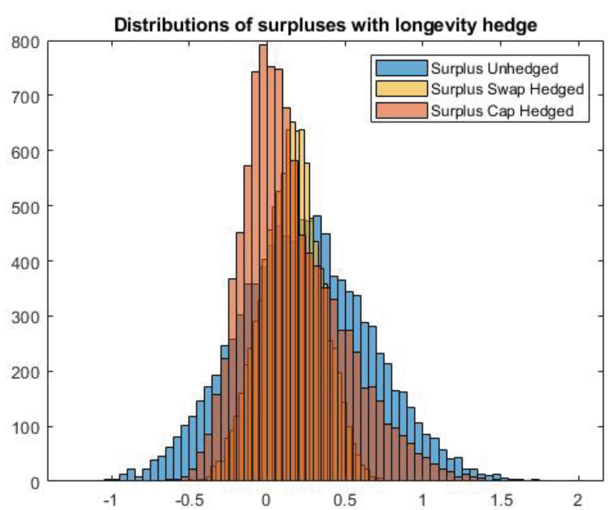

Figure 30: Distribution for the 25 year maturity hedge

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2359	0.42925	0.10317	2.9757	-0.71881	-0.85235
Q-Forward Hedged	0.082696	0.1844	-0.016134	3.054	-0.35307	-0.4100

Table 26: Results from hedging a single cohort at age 65 with 4 Q-Forwards

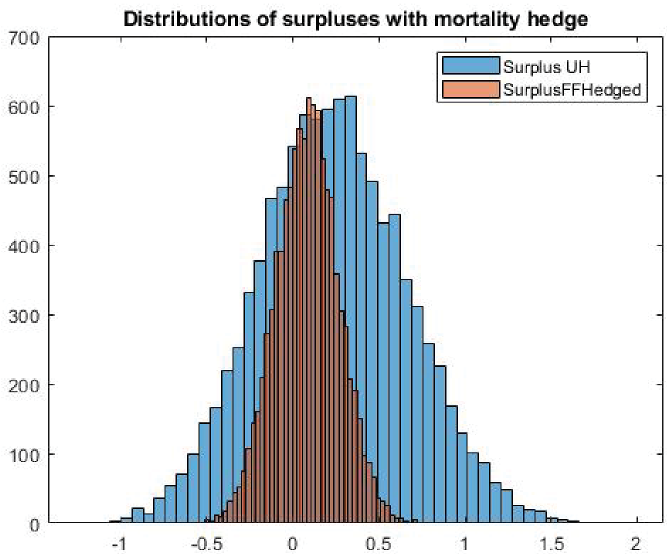

Figure 31: Distribution for mortality hedge on the American total population

Here are the results when applying the hedging scheme for the longevity and mortality instruments, on the total American population, for multiple cohorts.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1428	0.3165	0.0398	2.9811	-0.5783	-0.6837
S_{65}^{25}	0.1288	0.2932	0.0283	2.9849	-0.5432	-0.6367
$S_{65,67}^{25}$	0.1175	0.2714	0.0180	2.9862	-0.4771	-0.5965
$S_{65,67,69}^{25}$	0.1023	0.2507	0.0110	2.9717	-0.4284	-0.5511
C_{65}^{25}	0.1390	0.2997	0.0071	2.9717	-0.4283	-0.5616
$C_{65,67}^{25}$	0.1364	0.2892	0.1325	2.9669	-0.4205	-0.5374
$C_{65,67,69}^{25}$	0.1349	0.2674	0.1908	3.0159	-0.4101	-0.5130

Table 27: Various S-Forward hedges on different cohorts for 25 year maturity hedge

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1418	0.3165	0.0398	2.9811	-0.5783	-0.6837
Q-Forward Hedged	0.0914	0.2127	-0.1105	3.05	-0.4228	-0.4900

Table 28: Various S-Forward hedges on different cohorts for 20 year maturity hedge

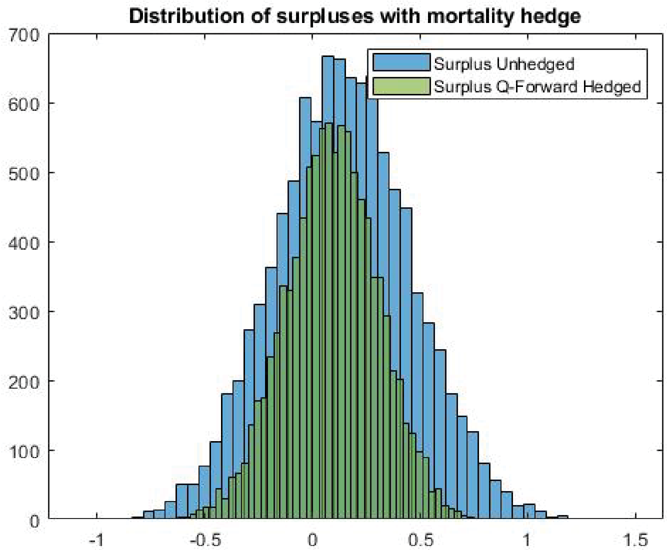

Figure 32: Distribution for mortality hedge on the American total population

8.6.2 Cross gender hedging of American population

This annex contains the results for a cross gender hedging scheme on the American population. We have decided to present only the results when using male and female hedging instruments that mimic a portfolio that is 65% women and 35% men.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2453	0.3681	0.0858	3.0310	-0.5937	-0.7041
Swap Hedged	0.1494	0.1258	-0.1398	2.9351	-0.3078	-0.3590
Cap Hedged	0.1442	0.2533	0.7315	3.6932	-0.1542	-0.1970

Table 29: Results from hedging a single cohort at age 65, with longevity products, for a 25 year maturity, with longevity instruments that are based on a mix of 65% women and 35% men

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2453	0.3681	0.0858	3.0310	-0.5937	-0.7041
Q-Forward Hedged	0.0851	0.1493	-0.0764	3.0704	-0.2740	-0.3247

Table 30: Results from single cohort mortality hedging with 4 forward contracts, using a portfolio that is 65% women and 35% men

When we apply the multiple cohort hedging scheme, we find the following results.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1537	0.2061	-0.0525	2.9893	-0.3263	-0.4039
S_{65}^{25}	0.1482	0.1898	-0.0657	3.0017	-0.2972	-0.3693
$S_{65,67}^{25}$	0.1429	0.1760	-0.0789	3.0228	-0.2742	-0.3403
$S_{65,67,69}^{25}$	0.1382	0.1646	-0.0831	3.0289	-0.2491	-0.3163
$S_{65,67,69,71}^{25}$	0.1352	0.1578	-0.0905	3.0401	-0.2331	-0.3022
C_{65}^{25}	0.1510	0.1980	0.0025	2.9978	-0.3019	-0.3765
$C_{65,67}^{25}$	0.1491	0.1910	0.0501	3.0199	-0.2831	-0.3525
$C_{65,67,69}^{25}$	0.1483	0.1850	0.0920	3.0345	-0.2640	-0.3321
$C_{65,67,69,71}^{25}$	0.1491	0.1813	0.1115	3.0503	-0.2517	-0.3197

Table 31: Various S-Forward and Caplet hedges on different cohorts for 20 year maturity hedge, using instruments based on the female mortality curve.

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.1481	0.2953	0.0406	2.9661	-0.5249	-0.6233
Q-Forward Hedged	0.1002	0.2133	-0.0642	2.9494	-0.4037	-0.4794

Table 32: Results from multiple cohorts mortality hedging with Q-forward contracts based on a mix of 65% women and 35% men

8.7 Cross gender hedging results using instruments based on the Canadian total population

In this annex, we provide the results for cross gender hedging using instruments that are based on the total population mortality curve.

Cohort	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2513	0.3064	0.0318	3.0137	-0.4563	-0.5515
Swap Hedged	-0.3318	0.2914	-0.1690	2.9752	-0.2404	-0.2404
Cap Hedged	-0.1599	0.1045	2.3356	9.2207	-1.0381	-1.1516

Table 33: Results from hedging a single cohort at age 65, with 25 year maturity longevity products that are based on the total population mortality curve

Portfolio	μ	std	Skewness	Kurtosis	VaR99	ES99
Unhedged	0.2513	0.3064	0.0318	3.0137	-0.4563	-0.5515
Q-Forward Hedged	0.0700	0.4961	-0.2527	3.2568	-1.1888	-1.4205

Table 34: Results from hedging a single cohort ate age 65, with 4 Q-Forwards based on the total population curve