TURUN YLIOPISTO
UNIVERSITY OF TURKU

ON USING DISTANCES TO LOCATE VERTICES

Resolving Sets and Metric Bases of Graphs, Two Generalisations and Their Forced Vertices

Anni Hakanen

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ISBN 978-951-29-8478-7 (PRINT)
ISBN 978-951-29-8479-4 (PDF)
ISSN 0082-7002 (PRINT)
ISSN 2343-3175 (ONLINE)
Painosalama Oy, Turku, Finland, 2021

UNIVERSITY OF TURKU
Faculty of Science
Department of Mathematics and Statistics
Mathematics
HAKANEN, ANNI: On Using Distances to Locate Vertices
Doctoral dissertation, 121 pp.
Doctoral Programme in Mathematics and Computer Sciences
May 2021

ABSTRACT

A graph consists of vertices that are connected by edges. A *resolving set* of a graph is a subset of its vertices that gives a unique combination of distances to every vertex of the graph. We can use the distances we are given to locate a vertex within the graph we are considering. Resolving sets were introduced by Slater in 1975 and independently by Harary and Melter in 1976. Robot navigation and network discovery and verification are examples of applications that have been suggested for resolving sets.

In this dissertation, we consider resolving sets and two of their generalisations that can be used to locate subsets of vertices instead of individual vertices. We consider how these generalisations are connected to other concepts such as locating-dominating sets and the boundary of a graph. We place special emphasis on studying the minimum cardinalities of resolving sets and the two generalisations. In addition to proving general bounds to these minimum cardinalities, we consider their exact values in some graph families. Natural decision problems arise from some of the concepts that we consider and we study their algorithmic complexities.

We also investigate which vertices of a graph must be included in an optimal resolving set or one of the two generalisations. For the resolving sets that can be used to locate subsets of vertices, there exist vertices that are in all such resolving sets. We call these vertices *forced vertices* of the graph. Such vertices do not exist for regular resolving sets. However, for minimum resolving sets they can exist, and we call them *basis forced vertices* of the graph. In this dissertation, we characterise the forced vertices of a graph, and consider some extremal properties of graphs that contain basis forced vertices.

Keywords: resolving set, metric dimension, forced vertex, algorithmic complexity.

TURUN YLIOPISTO
Matemaattis-luonnontieteellinen tiedekunta
Matematiikan ja tilastotieteen laitos
Matematiikka
HAKANEN, ANNI: On Using Distances to Locate Vertices
Väitöskirja, 121 s.
Matemaattis-tietotekninen tohtoriohjelma
Toukokuu 2021

TIIVISTELMÄ

Väitöskirjassa tarkastellaan objektien paikantamista tunnistinverkoissa etäisyyksien avulla. Verkko eli graafi koostuu viivoilla yhdistetyistä pisteistä ja paikannettavat objektit sijaitsevat pisteissä. Etäisyyksien avulla paikantavaksi koodiksi (*resolving set*) kutsutaan graafin pistejoukkoa, jonka alkiot antavat yksilöllisen yhdistelmän etäisyyksiä jokaiselle graafin pisteelle. Näitä etäisyyksiä käyttämällä tarkasteltavasta graafista voidaan paikantaa yksittäisiä pisteitä. Graafin pisteiden paikantamisen etäisyyksien avulla esittelivät toisistaan riippumatta sekä Slater vuonna 1975 että Harary ja Melter vuonna 1976. Etäisyyksien avulla paikantavia koodeja on ehdotettu käytettäväksi esimerkiksi robottien ohjaamisessa ja tietoverkkojen rakenteen selvittämisessä.

Tässä väitöskirjassa tarkastellaan tavanomaisten etäisyyksien avulla paikantavien koodien lisäksi niiden kahta yleistystä, joita käyttämällä graafista voidaan paikantaa yksittäisten pisteiden sijaan pistejoukkoja. Nämä yleistykset ovat yhteydessä muihin graafiteorian käsitteisiin, kuten paikantaviin peittokoodeihin (*locating-dominating sets*) ja graafin rajajoukkoon (*boundary of a graph*). Väitöskirjassa kiinnitetään erityistä huomiota optimaalisiin koodeihin eli koodeihin, joiden koko on mahdollisimman pieni. Optimaalisten koodien alkioiden lukumäärille todistetaan ylä- ja alarajoja, ja optimaalisia koodeja tarkastellaan tarkemmin monissa eri graafiperheissä. Väitöskirjassa tutkitaan myös näiden käsitteiden algoritmista kompleksisuutta.

Jotkin graafin pisteet on pakko sisällyttää käsiteltäviin koodeihin. Tavanomaiselle etäisyyksien avulla paikantavalle koodille tällaisia niin kutsuttuja pakotettuja pisteitä on olemassa vain kun koodi on optimaalinen. Tätä vastoin yleistetyille koodeille on olemassa pakotettuja pisteitä myös silloin kun koodi ei ole optimaalinen. Tässä väitöskirjassa karakterisoidaan pakotetut pisteet yleistetyille koodeille ja tutkitaan, millaisia ominaisuuksia on graafeilla, joiden etäisyyksien avulla paikantaville optimaalisille koodeille on olemassa pakotettuja pisteitä.

Asiasanat: etäisyyksien avulla paikantava koodi, metrinen dimensio, pakotettu piste, algoritminen kompleksisuus.

Acknowledgements

This dissertation would not have come to be without several supporters, and I would like to take this opportunity to thank those who have helped me in this endeavour.

Even though I would like to dedicate my life to mathematics the unfortunate fact remains that housing and food is not free. Therefore, I am grateful to University of Turku Graduate School UTUGS, Department of Mathematics and Statistics, and Magnus Ehrnrooth foundation for providing me with the necessary funding for my doctoral studies. In particular, I would like to thank our department head Iiro Honkala for giving me a place to do my research in even when he didn't have to.

The goal of my doctoral studies has been, above all, to obtain scientific competence. In this, my supervisors Tero Laihonen and Ville Junnila have been indispensable. I have been incredibly lucky to find supervisors in front of whom I don't need to censor much of what I say. I would like to thank my co-authors María Luz Puertas and Ismael G. Yero for working with me and thus letting me learn new ways to communicate my ideas. I am also grateful for the prudent scrutiny that the pre-examiners Cong X. Kang and Aline Parreau, and my opponent Cristina Dalfó have subjected this dissertation to.

I would also like to thank my colleagues at the department. Especially the coffee room regulars. I always looked forward to the conversations on all possible topics (sometimes even mathematics) during our coffee breaks. I would like to thank the members of our lunch gang, Jarkko Peltomäki, Outi Montonen and Kaisa Joki. Although our daily face-to-face lunches have reduced to weekly remote lunches during this last year, they have been even more crucial to my sanity. I would also like to thank all the office buddies I've had over the years. Joonatan Jalonen in particular has brightened my day many times with his good humour.

Hobbies and other recreational activities have been vital in maintaining a good balance between my work and other life. In this, Cantus Mercurialis has played a big role. Through CM, I have been lucky to meet good people that I would not have otherwise met, and I have been to places I would not have otherwise had access to. In particular, I am glad I have met Riikka, Saara and Kerttu in addition to Minna and all the other altos. I hope to see (and maybe even sing with) you all and many others again soon. *Totisesti.*

Lastly, I would like to thank my other friends and family who have supported me along the way. I am grateful to Veera for being my dearest friend and sister in all but blood. I would like to thank my family, Taina, Aarno, Tatu and Elina, for supporting and helping me in everything I do.

Pomarkku, May 2021
Anni Hakanen

Table of Contents

List of Original Publications

This dissertation is based on the following original publications, which are referred to in the text by their Roman numerals:

I Anni Hakanen and Tero Laihonen. *On $\{\ell\}$-Metric Dimensions in Graphs*. Fundamenta Informaticae, 2018; 162: 143-160.

II Anni Hakanen, Ville Junnila and Tero Laihonen. *The solid-metric dimension*. Theoretical Computer Science, 2020; 806: 156-170.

III Anni Hakanen, Ville Junnila, Tero Laihonen and María Luz Puertas. *On the Metric Dimensions for Sets of Vertices*. Discussiones Mathematicae Graph Theory. In press.

IV Anni Hakanen, Ville Junnila, Tero Laihonen and Ismael G. Yero. *On Vertices Contained in All or in No Metric Basis*. Submitted for publication.

1 Introduction

Graphs consist of vertices that are connected to each other by edges. A graph can be used as a model of many structures. The vertices of a graph may be thought to be computers or routers in a network, friends or followers in social media, or even rooms in a building, to name a few examples.

One widely studied problem in graph theory is to locate a vertex in a graph. One solution to this problem is the resolving set of a graph. Resolving sets locate vertices in a graph by giving the distances from its elements to the vertex that we are locating. A resolving set gives a unique combination of distances to each vertex of the graph. With these unique combinations, we can distinguish the vertices of a graph from one another. Resolving sets were first introduced in [40] and independently in [19].

There are, of course, other solutions to the locating problem. For example, locating-dominating sets (introduced in [36; 41; 42]) and identifying codes (introduced in [28]) can also be used to locate a vertex in a graph. The method of locating is different to resolving sets. Instead of distances, the information we have at our disposal is which elements of the locating-dominating set or identifying code are in the neighbourhood of the vertex we are locating. This set of elements is unique for all vertices of the graph if the locator set is an identifying code. If it is sufficient to locate only the vertices that are not in the locator set (sometimes called the non-codewords of the graph), then we are considering a locating-dominating set.

The vertex set of a graph is always a resolving set of the graph. Thus, instead of the existence of resolving sets, the objective of much of the research on resolving sets is focused on finding the smallest possible resolving set. The cardinality of the smallest resolving set of a graph is called the metric dimension of the graph. When given a random graph, finding the metric dimension is generally computationally difficult. Indeed, the decision problem whether the metric dimension of a graph is at most a given integer k is NP-complete. This result was mentioned without proof in [18]. Later in [31], the NP-completeness was shown by reducing the 3-satisfiability problem to this problem.

The research on resolving sets and the metric dimension of a graph has been very active in the last decades. Resolving sets have been connected to, for example, network discovery and verification [5], robot navigation [31], chemistry [9; 32], the coin weighing problem [8; 39] and the Mastermind game [8; 11].

Many new variants of resolving sets have been introduced and studied in the literature. Let us discuss the possible motivation behind some of these variants and their differences in an informal manner. Say, the elements of a resolving set are some sort of sensors that we place in a network. The cost of placing a sensor in a certain place can vary, perhaps due to a hard-to-reach location. In this situation, weighted resolving sets [13] could be the answer. If we place the sensors by visiting vertices, moving only between adjacent vertices, we can use a connected resolving set [14]. If we want to locate the edges of the graph instead of the vertices, then we can use edge resolving sets [30]. If we wish to locate both the edges and the vertices of the graph, then we can use mixed resolving sets [29]. Fault-tolerant resolving sets [21] and k-resolving sets [1; 16] are designed to locate vertices successfully even when one sensor fails. The functionality of the sensors may have some limitations. For example, the sensor may not sense objects that are further away than some threshold value. Then we may use resolving sets which use a bounded distance metric instead of the usual graphical distance [15; 25]. If the sensor cannot sense itself, then we may use strong total resolving sets [38]. There are many more variants of resolving sets besides the ones mentioned here.

In addition to the regular resolving set introduced in [40] and [19], we consider two other variants of resolving sets in this dissertation. These are generalisations of the regular resolving set built to distinguish between sets of vertices, that is, instead of locating one element at a time the goal is to locate multiple elements simultaneously. Similar generalisations of locating-dominating sets have been studied in [23; 27; 35] and of identifying codes in [3; 17; 22; 35], for example.

This dissertation consists of a summary and individual original publications. Chapters 2-4 summarise the results obtained in the four original publications or manuscripts that this dissertation is based on. In Chapter 2, we formally introduce the necessary concepts and discuss some of our results. More precisely, Section 2.1 is dedicated to introducing the regular resolving sets, and presenting some basic results and examples on the metric dimension of a graph. In Sections 2.2 and 2.3, similar introductions are given to $\{\ell\}$-resolving sets and ℓ-solid-resolving sets, respectively. We then discuss the connection of $\{\ell\}$-resolving sets and ℓ-solid-resolving sets in Section 2.4. An important tool we use in researching resolving sets and their generalisations is the concept of forced vertices. Everything pertaining to forced vertices is gathered to Chapter 3. In Section 3.1, we consider the forced vertices of $\{\ell\}$-resolving sets and ℓ-solid-resolving sets. A concept similar to forced vertices can be defined for resolving sets of minimum cardinality. This idea is explored in Section 3.2. Chapter 4 is a compact overview on the contents of the original publications, which are included at the end of this dissertation.

2 Resolving Sets

We begin by giving the basic definitions of resolving sets and the metric dimension of a graph. We go through some examples and present a couple well known results regarding resolving sets. In Sections 2.2 and 2.3 we introduce the two variants of resolving sets that can be used to locate multiple elements simultaneously. In Section 2.4, we discuss the connection of the concepts introduced in Sections 2.2 and 2.3, and also consider results that are similar between them.

2.1 Resolving Sets and the Metric Dimension of a Graph

Let G be a simple, undirected and finite graph with the vertex set $V(G)$ and the edge set $E(G)$. The *distance* between two vertices $v, u \in V(G)$ is the number of edges in a shortest path between v and u, and it is denoted by $d_G(v,u)$. If such a path does not exist (which is possible if the graph is not connected), then we consider the distance $d_G(v,u)$ to be infinite. If $d_G(v,u) = 1$ for distinct vertices $v, u \in V(G)$, then v and u are *adjacent* and we say that v and u are *neighbours* of each other. The *open neighbourhood* of $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) \mid d_G(v,u) = 1\}$, and the *closed neighbourhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If the graph we consider is clear from context, then we may omit the subscripts from these notations. We refer to [46] for further definitions and terminology regarding graph theory.

Let $R = \{r_1, \ldots, r_k\} \subseteq V(G)$. We consider the set R to be an ordered set. The *distance array* of the vertex $v \in V(G)$ with respect to the set R is

$$\mathcal{D}_R(v) = (d(r_1, v), \ldots, d(r_k, v)).$$

Definition 1. The set $R \subseteq V(G)$ is a *resolving set* of G if for all distinct vertices $v, u \in V(G)$ we have $\mathcal{D}_R(v) \neq \mathcal{D}_R(u)$. The smallest possible cardinality of a resolving set of G is the *metric dimension* of G, and it is denoted by $\beta(G)$. A resolving set of G of cardinality $\beta(G)$ is called a *metric basis* of G.

We say that a vertex $r \in V(G)$ *resolves* (or *distinguishes*) vertices $v, u \in V(G)$ if $d(r,v) \neq d(r,u)$. Using this terminology, we can also define resolving sets as follows: the set $R \subseteq V(G)$ is a resolving set of the graph G if for any distinct vertices $v, u \in V(G)$ there exists an element $r \in R$ that resolves v and u.

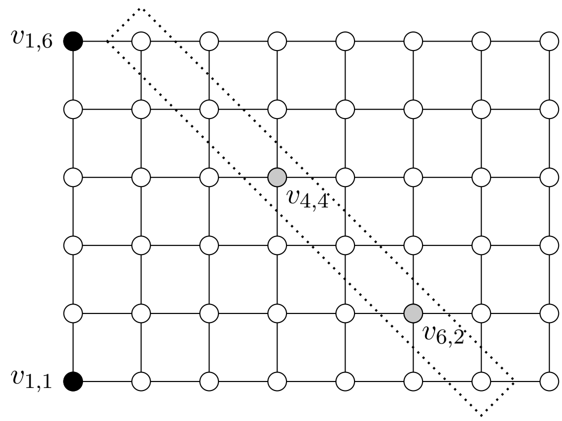


Figure 1. The 8×6 grid graph with a metric basis and an example case.

Over the years, varying terminology has been used when speaking of resolving sets and the metric dimension of a graph. The terms *resolving set* and *metric dimension* were introduced by Harary and Melter in [19]. In Slater's seminal work [40], resolving sets are *locating sets*, the distance array of a vertex is the *S-location* of a vertex, metric bases are *reference sets* and the metric dimension of a graph is the *location number* of a graph. Resolving sets are also called *metric generators* in [39; 47] for example, and the elements of a resolving set are called *landmarks* in [31]. The distance array of a vertex is called the *metric representation* or simply *representation* of a vertex in [9; 44].

The graph we consider in Example 2 is a 2-dimensional grid graph. The metric dimension of such graphs was shown to be 2 in [34].

Example 2. Consider the 8×6 grid graph G illustrated in Figure 1. We denote the vertices of this graph by $v_{i,j}$, where $i = 1, \ldots, 8$ and $j = 1, \ldots, 6$. If a vertex is in the corner of the graph, then it has only two neighbours. For example, the neighbours of the vertex $v_{1,1}$ are $v_{2,1}$ and $v_{1,2}$. The vertices at the frame of the graph have up to three neighbours. For example, the neighbours of the vertex $v_{1,2}$ are $v_{1,3}$, $v_{2,2}$ and $v_{1,1}$. If $i \neq 1, 8$ and $j \neq 1, 6$, then the vertex $v_{i,j}$ has four neighbours: $v_{i+1,j}$, $v_{i-1,j}$, $v_{i,j+1}$ and $v_{i,j-1}$. The distance between two vertices of the graph G can be easily calculated; we have $d(v_{i,j}, v_{k,l}) = |i-k|+|j-l|$. For example, the distance between the vertices $v_{1,6}$ and $v_{4,4}$ is $|1-4| + |6-4| = 3 + 2 = 5$.

No single vertex $v_{i,j}$ can be a resolving set of G, since there are multiple vertices at the same distance from $v_{i,j}$. For example, each $v_{i,j}$ has at least two neighbours, and both of these neighbours are at distance 1 from $v_{i,j}$. Thus, any resolving set of G must contain at least two vertices, and we have $\beta(G) \geq 2$.

The set $R_1 = \{v_{1,1}, v_{1,6}\}$ (illustrated as black vertices in Figure 1) is a metric basis of G. The fact that R_1 is a resolving set of G can be shown with a visual argument. The vertices that are at the same distance from $v_{1,1}$ form a diagonal in the illustration of the graph. For example, all vertices that are at distance 6 from $v_{1,1}$ are outlined with the dotted line in Figure 1. Similarly, the vertices that are at the same distance from $v_{1,6}$ form diagonal lines in the illustration of G. However, these diagonals are perpendicular compared to those of $v_{1,1}$. If the diagonals of $v_{1,1}$ and $v_{1,6}$ intersect, then there is only one element in the intersection. Thus, if two vertices are at the same distance from $v_{1,1}$, then they are at different distances from $v_{1,6}$, and vice versa. For example, the vertices $v_{4,4}$ and $v_{6,2}$ are both at distance 6 from $v_{1,1}$, however, we have $d(v_{1,6}, v_{4,4}) = 5$ and $d(v_{1,6}, v_{6,2}) = 9$. Thus, the set R_1 is a resolving set of G and $\beta(G) = 2$.

Every graph has at least one resolving set. For example, the set $V(G)$ is always a resolving set of G. Indeed, the position of the 0 in the distance array $\mathcal{D}_{V(G)}(v)$ implies which vertex v is. Similarly, the set $V(G) \setminus \{v\}$ is also a resolving set of G for any $v \in V(G)$. The vertex v is the only vertex that does not have a 0 in its distance array, and the distance array of any other vertex than v contains exactly one 0 given by the vertex itself. Thus, resolving sets exist for any graph G and we have

$$\beta(G) \leq |V(G)| - 1.$$

The graphs that attain this upper bound were characterised in [9] as is stated in the following theorem. We denote the *complete graph* with n vertices by K_n.

***Theorem* 3.** *[9] Let G be a connected graph with $n \geq 2$ vertices. We have $\beta(G) = n - 1$ if and only if $G = K_n$.*

The connected graphs with n vertices and metric dimension $n - 2$ were also characterised in [9]. Graphs with metric dimension $n - 3$ were characterised in [26; 45].

On the opposite end we have graphs with very small metric dimensions. The *path* with $n \geq 1$ vertices is denoted by P_n. It is easy to see that $\beta(P_n) = 1$. Indeed, either endpoint of the path gives a unique distance array to all vertices of P_n. The following theorem is a well known result in the area of resolving sets.

***Theorem* 4.** *[9; 31; 40] Let G be a connected graph with n vertices. We have $\beta(G) = 1$ if and only if $G = P_n$.*

Let us then consider trees. A *tree* is a connected graph that does not contain any cycles. The metric bases of trees have been studied extensively, see [9; 19; 31; 40], for example.

Example 5. Consider the tree T in Figure 2. A *leaf* (or a *pendant*) is a vertex that has only one neighbour. The leaves of T are labelled as the vertices l_i. Any tree has several metric bases that contain only leaves of the tree [9]. The tree T has, for example, the following metric bases: $\{l_1, l_2, l_5\}$, $\{l_1, l_3, l_6\}$ and $\{l_2, l_3, l_5\}$.

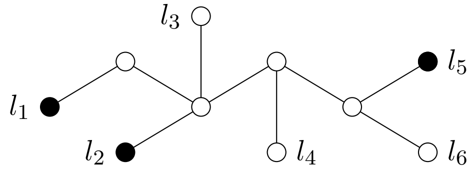


Figure 2. A tree with a metric basis illustrated as black vertices.

Due to the following remark, the graphs we consider in this dissertation are connected graphs most of the time.

Remark 6. Let G be a graph that is not connected. Let G_1 and G_2 be connected components of G. It is clear that no element of $V(G_1)$ resolves v and u when $v, u \in V(G_2)$. Let $R \subseteq V(G)$ be a resolving set of G. The set $R \cap V(G_i)$ is a resolving set of the connected component G_i when $G_i \neq K_1$. If we have $R \cap V(G_i) = \emptyset$ for a resolving set R and connected component G_i, then $G_i = K_1$ and $R \cap V(G_j) \neq \emptyset$ for all $j \neq i$. Due to this, we focus our research on resolving sets on connected graphs unless otherwise stated.

2.2 The $\{\ell\}$-Resolving Sets of a Graph

The resolving sets introduced in the previous section can be used to distinguish individual vertices from one another. In this section, we introduce a modification of resolving sets that can distinguish sets of vertices from one another. The variant we introduce in this section can locate multiple vertices simultaneously (i.e. it can locate vertex sets instead of individual vertices).

We define the distance between a vertex $v \in V(G)$ and a set of vertices $X \subseteq V(G)$ as $d(v, X) = \min_{x \in X} d(v, x)$. The distance array of the vertex set X with respect to the set $R = \{r_1, \ldots, r_k\} \subseteq V(G)$ is

$$\mathcal{D}_R(X) = (d(r_1, X), \ldots, d(r_k, X)).$$

Definition 7. Let ℓ be a positive integer. The set $R \subseteq V(G)$ is an $\{\ell\}$*-resolving set* (or ℓ*-set-resolving set*) of G if we have $\mathcal{D}_R(X) \neq \mathcal{D}_R(Y)$ for all nonempty and distinct sets $X, Y \subseteq V(G)$ such that $|X| \leq \ell$ and $|Y| \leq \ell$. The smallest possible cardinality of an $\{\ell\}$-resolving set of G is the $\{\ell\}$*-metric dimension* of G, and it is denoted by $\beta_\ell(G)$. An $\{\ell\}$-resolving set of cardinality $\beta_\ell(G)$ is called an $\{\ell\}$*-metric basis* of G.

The definition of a $\{1\}$-resolving set is exactly the same as Definition 1. Thus, instead of $\{1\}$-resolving sets and $\{1\}$-metric bases we simply speak of (regular) resolving sets and metric bases, as we do in Definition 1.

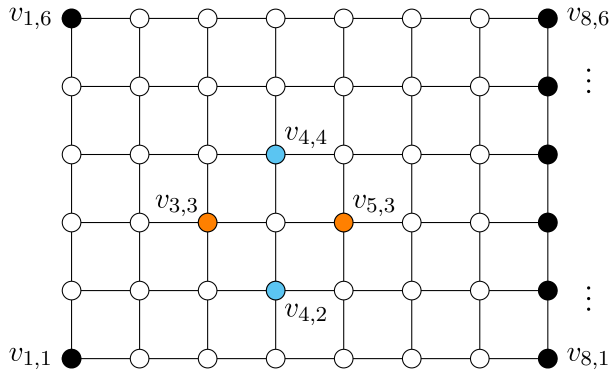


Figure 3. The 8×6 square grid with a $\{2\}$-metric basis illustrated as black vertices and an example case.

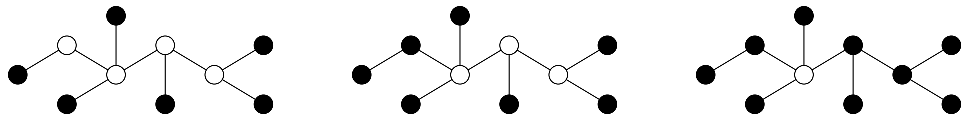
The concept of $\{\ell\}$-resolving sets was introduced in [33], where the $\{\ell\}$-metric dimension of a graph is called the ℓ-metric dimension of a graph. However, we have adopted the notation $\{\ell\}$-metric dimension in an effort to prevent any confusion with the k-metric dimension of a graph, which was introduced in [1] and independently in [16]. Moreover, our notation emphasises the fact that $\{\ell\}$-resolving sets can be used to resolve vertex sets instead of individual vertices.

The set $V(G)$ is an $\{\ell\}$-resolving set of G for any ℓ. Indeed, the positions of the 0's in the distance array $\mathcal{D}_{V(G)}(X)$ indicate the elements of X. For every graph G there exists an integer $\ell \leq |V(G)|$ such that $\beta_\ell(G) = |V(G)|$. If $\ell = |V(G)|$, then we certainly have $\beta_\ell(G) = |V(G)|$. Indeed, the only vertex that can distinguish the sets $V(G)$ and $V(G) \setminus \{v\}$ is v itself. Thus, we would need every vertex of G in order to distinguish between all subsets of vertices. The smallest value of ℓ for which $\beta_\ell(G) = |V(G)|$ depends on the structure of the graph G.

The following two examples consider the $\{\ell\}$-metric dimensions of the graphs already familiar from Examples 2 and 5.

Example 8. Let us consider again the 8×6 grid graph G as we did in Example 2. The $\{2\}$-metric dimension of an $n \times m$ grid graph was determined in [33]. According to [33], the set $R_2 = \{v_{1,1}, v_{1,6}\} \cup \{v_{8,i} \mid i = 1, \dots, 6\}$ (illustrated as black vertices in Figure 3) is a $\{2\}$-metric basis of G.

Let us consider an example case and compare the set R_2 and the metric basis $R_1 = \{v_{1,1}, v_{1,6}\}$ introduced in Example 2. Let $X = \{v_{4,2}, v_{4,4}\}$ and $Y = \{v_{3,3}, v_{5,3}\}$. The elements of X and Y are illustrated as cyan and orange vertices in Figure 3. As we saw in Example 2, the set R_1 is a metric basis of G. However, it cannot distinguish the sets X and Y. Indeed, we have $\mathcal{D}_{R_1}(X) = (4, 5) = \mathcal{D}_{R_1}(Y)$. Distinguishing the sets X and Y poses no difficulty to the set R_2; we have $d(v_{8,3}, X) = 5$ and $d(v_{8,3}, Y) = 3$. As it turns out in [33], the most difficult two element sets to distinguish from one another in the graph G are sets that form a cross-



(a) $\ell = 2$. **(b)** $\ell = 3$. **(c)** $\ell = 4$.

Figure 4. The unique $\{\ell\}$-metric bases of the tree T for $2 \leq \ell \leq 4$.

or a diamond-like shape – just like the sets X and Y do.

According to [33], the only $\{\ell\}$-metric basis of G is the set $V(G)$ when $\ell \geq 3$.

The *degree* of a vertex $v \in V(G)$ is defined as $\deg_G(v) = |N(v)|$.

Example 9. We consider the $\{\ell\}$-metric dimensions of trees in Publication I, which is one of the original publications included later on in this dissertation. We show that the unique $\{\ell\}$-metric basis of a tree T is the set $\{v \in V(T) \mid \deg(v) < \ell\}$. Let us consider the tree T familiar from Example 5. The $\{\ell\}$-metric bases of the tree T, when $2 \leq \ell \leq 4$, are illustrated in Figure 4. When $\ell \geq 5$, we have $\beta_\ell(T) = |V(T)|$.

In our original publications, we consider the $\{\ell\}$-metric dimensions of some specific graphs and graph families. For example, we consider finite king grids in addition to trees in Publication I. We determine the $\{\ell\}$-metric dimensions of king grids for all ℓ. The exact value of the metric dimension of a king grid was conjectured in [37] and later proved in [4]. In Publication I, we give a new proof for the metric dimension of a king grid. In Publication III, we consider the metric dimensions of Rook's graphs and flower snarks. As it turns out, the $\{2\}$-metric bases of Rook's graphs have an intriguing connection to combinatorial designs. In our original publications, we also show general results concerning $\{\ell\}$-resolving sets, which we will discuss later in Sections 2.4 and 3.

2.3 The ℓ-Solid-Resolving Sets of a Graph

The idea behind ℓ-solid-resolving sets is to locate up to ℓ elements – just like with an $\{\ell\}$-resolving set – but also detect if there are more than ℓ elements in a vertex set. In other words, ℓ-solid-resolving sets are designed to do the same as $\{\ell\}$-resolving sets but with the additional property that no vertex set with more than ℓ elements has the same distance array as a vertex set with up to ℓ elements.

We introduce 1-solid-resolving sets in Publication II, and the following, more general, definition of ℓ-solid-resolving sets and the ℓ-solid-metric dimension of a graph in Publication III.

Definition 10. Let ℓ be a positive integer. The set $R \subseteq V(G)$ is an *ℓ-solid-resolving set* of G if $\mathcal{D}_R(X) \neq \mathcal{D}_R(Y)$ for all nonempty and distinct sets $X, Y \subseteq V(G)$ such that $|X| \leq \ell$. The smallest possible cardinality of an ℓ-solid-resolving set of G is the

ℓ*-solid-metric dimension* of G, and it is denoted by $\beta_\ell^s(G)$. An ℓ-solid-resolving set of cardinality $\beta_\ell^s(G)$ is called an ℓ*-solid-metric basis* of G.

The difference between Definitions 7 and 10 is that the latter definition does not have the restriction $|Y| \leq \ell$ that the former definition has. However, this seemingly small difference can lead to a very large difference between the ℓ-solid-metric dimension and $\{\ell\}$-metric dimension of a graph. By Definitions 7 and 10, an ℓ-solid-resolving set of a graph G is an $\{\ell\}$-resolving set of G for a fixed ℓ. However, the converse is not true. We will consider the connection of ℓ-solid-resolving sets and $\{\ell\}$-resolving sets more closely in Section 2.4.

The following theorem provides an easy way to prove that a vertex set is an ℓ-solid-resolving set. Theorem 11 is in Publication III as Theorem 3.

***Theorem* 11.** *Let ℓ be a positive integer. The set $R \subseteq V(G)$ is an ℓ-solid-resolving set of G if and only if for all $x \in V(G)$ and nonempty $Y \subseteq V(G)$ such that $x \notin Y$ and $|Y| \leq \ell$ there exists an element $r \in R$ such that $d(r,x) < d(r,Y)$.*

When $\ell = 1$, Theorem 11 takes on a very simple form; the set $R \subseteq V(G)$ is a 1-solid-resolving set of G if and only if for all distinct $x, y \in V(G)$ there exists an element $r \in R$ such that $d(r,x) < d(r,y)$. We demonstrate the use of Theorem 11 for 1-resolving sets in the following example.

Example 12. Let us consider the 8×6 grid graph G familiar from Examples 2 and 8. As we saw in Example 2, the set $R_1 = \{v_{1,1}, v_{1,6}\}$ is a metric basis of G. It is not, however, a 1-solid-resolving set of G. This can be seen by considering the vertex $v_{2,4}$ and the set $Y = \{v_{4,6}, v_{5,1}\}$, for example. See Figure 5 for an illustration. We have $\mathcal{D}_{R_1}(v_{2,4}) = (4,3)$ and $\mathcal{D}_{R_1}(Y) = (4,3)$, and thus the set R_1 is not a 1-solid-resolving set of G according to Definition 10. The same conclusion can be reached using Theorem 11. Let $x = v_{5,1}$ and $Y = \{v_{2,4}\}$. We have $d(v_{1,1}, x) = d(v_{1,1}, Y)$ and $d(v_{1,6}, x) > d(v_{1,6}, Y)$. Thus, there does not exist an element $r \in R_1$ such that $d(r,x) < d(r,Y)$ and the set R_1 is not a 1-solid-resolving set of G according to Theorem 11.

Let us then consider the set $R_s = \{v_{1,1}, v_{1,6}, v_{8,1}, v_{8,6}\}$ (illustrated as black vertices in Figure 5). The set R_s is a 1-solid-resolving set of G. This can be seen by using Theorem 11 and the same visual argument we used in Example 2. Moreover, we show in Publication II that the set R_s is a 1-solid-metric basis of G. However, the set R_s is not a $\{2\}$-resolving set. For example, the sets $X = \{v_{2,4}, v_{7,3}\}$ and $Y = \{v_{4,6}, v_{5,1}\}$ have the same distance array.

Let then G be a finite 2-dimensional rectangular grid graph of any width and height. We show in Publication II that $\beta_1^s(G) = 4$ for any such grid graph G. However, the $\{2\}$-metric dimension of G is dependent on the width and height of G, as was shown in [33] and discussed in Example 8. Thus, in the case of grid graphs,

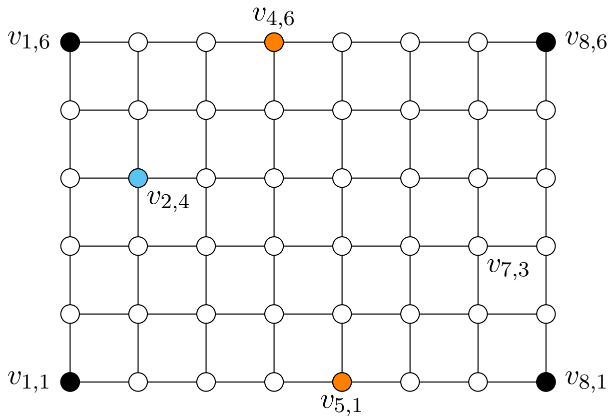


Figure 5. The 8×6 grid graph with a 1-solid-metric basis and an example case.

1-solid-resolving sets provide us a way to use some of the functionality of a $\{2\}$-resolving set without having to use the significantly larger $\{2\}$-resolving sets.

The grid graph of Example 12 is an example of a graph where the 1-solid- and $\{2\}$-metric dimensions do not coincide. However, it is possible for the ℓ-solid- and $\{\ell\}$-metric dimensions of a graph to be the same for some ℓ. For example, as we point out in Publication III, the ℓ-solid-metric dimension of a tree is the same as the $\{\ell+1\}$-metric dimension of the tree for all ℓ. However, we show in Publication II that a metric basis (that is, a $\{1\}$-metric basis) of a graph is never a 1-solid-resolving set of the graph. Therefore, we have $\beta(G) < \beta_1^s(G)$ for any graph G. The connection between ℓ-solid-resolving sets and $\{\ell\}$-resolving sets is discussed further in Section 2.4.

In Publication II, we consider the connection of 1-solid-resolving sets to other concepts. These concepts include the boundary of a graph (see [10; 20], for example), self-locating-dominating sets [27] and the Dilworth number of a graph [12]. We also prove that the decision problem whether $\beta_1^s(G) \leq k$ for a positive integer k is NP-complete.

2.4 The Connection of $\{\ell\}$-Resolving Sets and ℓ-Solid-Resolving Sets of a Graph

As we stated before in Section 2.3, an ℓ-solid-resolving set is an $\{\ell\}$-resolving set by definition. In Publication III, we show that an $\{\ell+1\}$-resolving set is an ℓ-solid-resolving set, and thus we have the following theorem.

***Theorem* 13.** *Let $R \subseteq V(G)$ and ℓ be a positive integer.*

(i) If R is an ℓ-solid-resolving set of G, then it is an $\{\ell\}$-resolving set of G.

(ii) If R is an $\{\ell+1\}$-resolving set of G, then it is an ℓ-solid-resolving set of G.

Due to Theorem 13 (i), we have $\beta_\ell^s(G) \geq \beta_\ell(G)$ for all ℓ and graphs G. Similarly, due to Theorem 13 (ii), we have $\beta_{\ell+1}(G) \geq \beta_\ell^s(G)$ for all ℓ and graphs G. Moreover, we have $\beta(G) < \beta_1^s(G)$ for all G as we show in Publication II.

Recall that $\beta(G) = 1$ if and only if $G = P_n$, as was stated in Theorem 4. This can be interpreted as the characterisation of the graphs that attain the obvious lower bound $\beta(G) \geq 1$. The following theorem states similar lower bounds for the ℓ-solid- and $\{\ell\}$-metric dimensions of a graph. We consider these lower bounds in Publications I-III.

***Theorem* 14.** *Let G be a connected graph, and let $\ell \geq 1$ be an integer. We have $\beta_\ell(G) \geq \ell$ and $\beta_\ell^s(G) \geq \ell + 1$.*

Let G be a graph with n vertices. Now, according to Theorem 14, we have $\beta_{n-1}^s(G) = \beta_n(G) = \beta_n^s(G) = n$. Due to this and what we discussed above, we have

$$\beta(G) < \beta_1^s(G) \leq \beta_2(G) \leq \beta_2^s(G) \leq \beta_3(G) \leq \ldots \leq \beta_{n-1}(G) \leq \beta_{n-1}^s(G) = n$$

for any graph G with n vertices.

The characterisations of the graphs that attain the lower bounds in Theorem 14 are collated in the following theorem. We prove these characterisations in Publications I-III.

We denote the *complete bipartite graph* by $K_{m,n}$. The graph $K_{1,n}$ is called the *star graph*.

***Theorem* 15.** *Let G be a connected graph, and let $\ell \geq 1$ be an integer. We have*

(i) $\beta_1^s(G) = 2$ if and only if $G = P_n$ for some $n \geq 2$,

(ii) $\beta_2(G) = 2$ if and only if $G = P_n$ for some $n \geq 2$,

(iii) for $\ell \geq 2$, we have $\beta_\ell^s(G) = \ell + 1$ if and only if either $|V(G)| = \ell + 1$ or $G = K_{1,\ell+1}$,

(iv) for $\ell \geq 3$, we have $\beta_\ell(G) = \ell$ if and only if either $|V(G)| = \ell$ or $G = K_{1,\ell}$.

Consider then a graph G with an infinite vertex set. The definitions of ℓ-solid- and $\{\ell\}$-resolving sets generalise to infinite graph in a natural manner. We show in Publication III that the 1-solid-metric dimension of an infinite graph cannot be finite. Thus, $\beta_\ell^s(G) = \infty$ for all ℓ and $\beta_\ell(G) = \infty$ for all $\ell \geq 2$. However, the metric dimension of an infinite graph may be finite or infinite [7]. The metric dimensions of infinite graphs have also been studied in [24; 43].

3 Forced Vertices

One way we can try to determine the ℓ-solid- or $\{\ell\}$-metric dimension of a graph is to find the vertices of the graph that must be in every such resolving set. In Section 3.1, we consider vertices that are in every ℓ-solid- or $\{\ell\}$-resolving set. In Section 3.2, we consider metric bases of graphs. We study vertices that are in every metric basis and their opposite; vertices that are in no metric basis.

3.1 Forced Vertices of ℓ-Solid- and $\{\ell\}$-Resolving sets

We begin by giving the definition for the forced vertices of ℓ-solid- and $\{\ell\}$-resolving sets of graphs.

Definition 16. Let ℓ be a positive integer. A vertex $v \in V(G)$ is a *forced vertex* of an $\{\ell\}$-resolving set (resp. ℓ-solid-resolving set) of G if every $\{\ell\}$-resolving set (resp. ℓ-solid-resolving set) of G contains v.

As we stated before in Section 2.1, the set $V(G) \setminus \{v\}$ is a resolving set of G for any $v \in V(G)$. Thus, no forced vertices of a resolving set exist. However, later in Section 3.2, we consider basis forced vertices – a concept similar to forced vertices, but with respect to metric bases instead of resolving sets. For the rest of this section we consider forced vertices of $\{\ell\}$-resolving sets for $\ell \geq 2$ and ℓ-solid-resolving sets for all ℓ.

In Publication I, we show that a vertex $v \in V(G)$ with $\deg(v) \leq \ell - 1$ is a forced vertex of an $\{\ell\}$-resolving set of G. Indeed, if the vertex v is not in the set $R \subseteq V(G)$, then the sets $N(v)$ and $N[v]$ have the same distance arrays with respect to R. Due to this, the vertex v is also a forced vertex of an $(\ell - 1)$-solid-resolving set.

Example 17. Consider the tree T and its $\{\ell\}$-metric bases introduced in Example 9. Notice that all elements of the $\{\ell\}$-metric bases for $2 \leq \ell \leq 4$ are forced vertices. When $\ell \geq 5$, all vertices of T are forced vertices of an $\{\ell\}$-resolving set. Let us then consider ℓ-solid-resolving sets. All vertices $v \in V(T)$ such that $\deg(v) \leq \ell$ are forced vertices of an ℓ-solid-resolving set of T. Thus, the leaves of the tree T are forced vertices of a 1-solid-resolving set of T. Notice that the set of all leaves of T is a $\{2\}$-resolving set of T. Thus, it is also a 1-solid-resolving set of T due to Theorem 13 (ii). The same holds for other ℓ. Namely, the forced vertices of an

ℓ-solid-resolving set form an $\{\ell+1\}$-resolving set of T, and thus also an ℓ-solid-resolving set.

In general, whether a vertex is a forced vertex is not solely dependent on the cardinality of its neighbourhood. In Publication III, we show the following two characterisations for forced vertices of ℓ-solid- and $\{\ell\}$-resolving sets of graphs. We denote $N[U] = \bigcup_{u \in U} N[u]$.

***Theorem* 18.** *Let $\ell \geq 1$. A vertex $v \in V(G)$ is a forced vertex of an ℓ-solid-resolving set of G if and only if there exists a set $U \subseteq V(G)$ such that $v \notin U$, $|U| \leq \ell$ and $N(v) \subseteq N[U]$.*

***Theorem* 19.** *Let $\ell \geq 2$. A vertex $v \in V(G)$ is a forced vertex of an $\{\ell\}$-resolving set of G if and only if there exists a set $U \subseteq V(G)$ such that $v \notin U$, $|U| \leq \ell - 1$ and $N(v) \subseteq N[U]$.*

If a vertex v is a forced vertex of an ℓ-solid-resolving set of G, then it is also a forced vertex of an $(\ell+1)$-solid-resolving set of G. The same holds also for $\{\ell\}$-resolving sets. Moreover, the forced vertices of ℓ-solid-resolving sets are exactly the same as those of $\{\ell+1\}$-resolving sets.

Example 20. Consider the 8×6 grid graph G from Example 2. The corner vertices $v_{1,1}, v_{1,6}, v_{8,1}, v_{8,6}$ are forced vertices of 1-solid-resolving sets and $\{2\}$-resolving sets of G; we have, for example, $N(v_{1,1}) = \{v_{2,1}, v_{1,2}\} \subseteq N[v_{2,2}]$ (see Figure 6).

Let us then consider the forced vertices of 2-solid-resolving sets. In addition to the vertices at the corners, the vertices at the frame of the graph (that is, vertices with at most three neighbours) are forced vertices. For example, we have $N(v_{6,1}) = \{v_{5,1}, v_{6,2}, v_{7,1}\} \subseteq N[\{v_{5,2}, v_{7,2}\}]$ (see Figure 6). The vertices in the middle of the graph are also forced vertices of 2-solid-resolving sets of G. We have $N(v_{i,j}) = \{v_{i+1,j}, v_{i-1,j}, v_{i,j+1}, v_{i,j-1}\} \subseteq N[\{v_{i+1,j+1}, v_{i-1,j-1}\}]$ for all $i \neq 1, 8$ and $j \neq 1, 6$. For example, we have $N(v_{4,5}) \subseteq N[\{v_{5,6}, v_{3,4}\}]$ (see Figure 6). In conclusion, all vertices of G are forced vertices of 2-solid-resolving sets of G. Consequently, we have $\beta_2^s(G) = |V(G)| = 48$ and $\beta_\ell^s(G) = \beta_\ell(G) = 48$ for all $\ell \geq 3$.

Let $\Delta(G)$ be the maximum degree of the graph G. If $\ell \geq \Delta(G)$, then every vertex of the graph G is a forced vertex of an ℓ-solid- or $\{\ell+1\}$-resolving set of G. Consequently, $\beta_\ell^s(G) = \beta_{\ell+1}(G) = |V(G)|$. All vertices of a graph can be forced vertices even when ℓ is much smaller than $\Delta(G)$. For example, all vertices of the complete graph K_n are forced vertices of a 1-solid-resolving set of K_n, and $\Delta(K_n) = n-1$. There exist graphs for which $\beta_\ell(G) \neq |V(G)|$ and $\ell = \Delta(G) - 1$. For example, consider the star graph $K_{1,n}$ with $n+1$ vertices. We have $\Delta(K_{1,n}) = n$ and $\beta_n(K_{1,n}) = n$ according to Theorem 15 (iv). The leaves of $K_{1,n}$ are forced vertices for all ℓ-solid- and $\{\ell\}$-resolving sets (except $\{1\}$-resolving sets). However,

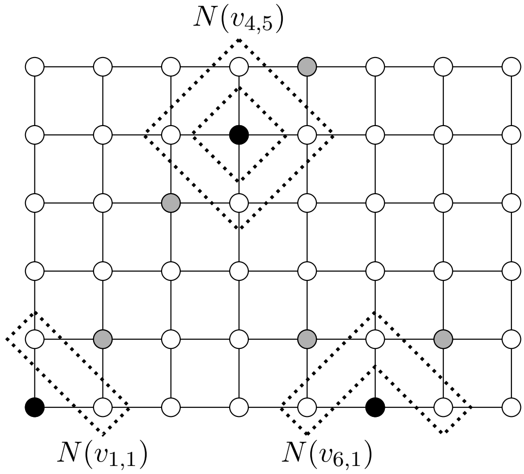


Figure 6. The open neighbourhoods of the black vertices are outlined with the dotted line. The closed neighbourhoods of the gray vertices cover the open neighbourhoods of the black vertices.

the center vertex of $K_{1,n}$ is not a forced vertex even for $\{n\}$-resolving sets. Notice that the star graph $K_{1,n}$ is also an example of a graph whose $\{\ell\}$-metric dimension is *not* equal to $|V(K_{1,n})| = n + 1$ for very large values of ℓ. Indeed, we have $\beta_\ell(K_{1,n}) \neq n + 1$ for all $\ell \leq n$

Although forced vertices are very helpful in providing lower bounds on the ℓ-solid- and $\{\ell\}$-metric dimensions of some graphs, different approaches are sometimes required. For example, we consider the ℓ-solid- and $\{\ell\}$-metric dimensions of flower snarks in Publication III. Flower snarks do not have any forced vertices of ℓ_1-solid- and $\{\ell_2\}$-resolving sets where $\ell_1 \leq 2$ and $\ell_2 \leq 3$. Thus, these ℓ_1-solid- and $\{\ell_2\}$-metric dimensions must be determined entirely by other means. For all other ℓ-solid- and $\{\ell\}$-resolving sets, all vertices are forced vertices, since flower snarks are 3-regular graphs.

3.2 Basis Forced and Void Vertices

In this section, we consider only regular metric bases of graphs. As we have stated before, forced vertices as they were defined in the previous section do not exist for resolving sets. However, although there are no vertices that are in every resolving set, there exist vertices that are in every metric basis of a graph.

Definition 21. A vertex $v \in V(G)$ is a *basis forced vertex* of the graph G if it is contained in every metric basis of G.

Basis forced vertices have been studied in [6], although basis forced vertices were not defined or drawn much attention to there. It was shown in [6] that for all

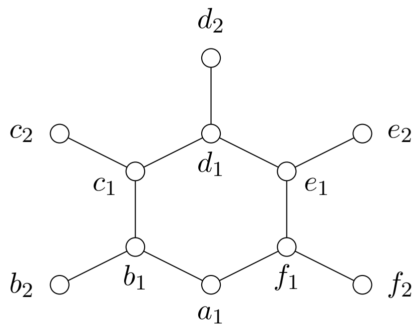


Figure 7. A graph that has a unique metric basis.

integers $k \geq 2$ and $r \geq 0$ such that $r \leq k$, there exists a graph with r basis forced vertices and metric dimension k.

The opposite of a basis forced vertex is a vertex that is in no metric basis of a graph. Such vertices are defined in the following definition.

Definition 22. A vertex $v \in V(G)$ is a *void vertex* of the graph G if it is in no metric basis of G.

In general, a graph may have three types of vertices; basis forced vertices, void vertices, and vertices that are in some metric bases but not all. The path P_n has $n-2$ void vertices and two vertices that are in some metric bases but not all. The complete graph K_n, however, has neither basis forced vertices nor void vertices. Each vertex $v \in V(K_n)$ is in multiple metric bases of K_n, but the set $V(G) \setminus \{v\}$ is a metric basis of K_n for any $v \in V(G)$. If each vertex of the graph G is either a basis forced vertex or a void vertex, then the graph G has exactly one metric basis; the set of basis forced vertices. Graphs with unique metric bases have been studied in [2; 6].

Example 23. Let us consider the graph G in Figure 7. We will show that the graph G has a unique metric basis, namely, $\{b_2, f_2\}$. Consequently, the vertices b_2 and f_2 are basis forced vertices of G, and the rest of the vertices are void vertices of G. For the purposes of this proof, let us divide the vertex set of G into pairs. We say that the vertices b_1 and b_2 are a pair, and similarly the vertices c_i, d_i, e_i and f_i (where $i = 1, 2$) form pairs. The only vertex with no pair is a_1. However, to simplify the language we use we think that a_1 forms a pair on its own.

Since the graph G is not a path, we have $\beta(G) > 1$ (due to Theorem 4, although it is quite clear that no one vertex can resolve the graph G). It is easy to check that the set $\{b_2, f_2\}$ is a resolving set of G. Thus, $\beta(G) = 2$.

Let R be a metric basis of G. Then $|R| = 2$. The two elements of R cannot be in antipodal (or opposite) pairs, because then we cannot resolve even the vertices of the cycle $a_1 b_1 \cdots f_1 a_1$. Indeed, if the set R consists of, say, the vertices b_1 and e_2, then

$\mathcal{D}_R(a_1) = \mathcal{D}_R(c_1)$, for example. Similarly, the elements of R cannot be in the same pair, since then we cannot resolve the vertices of the cycle.

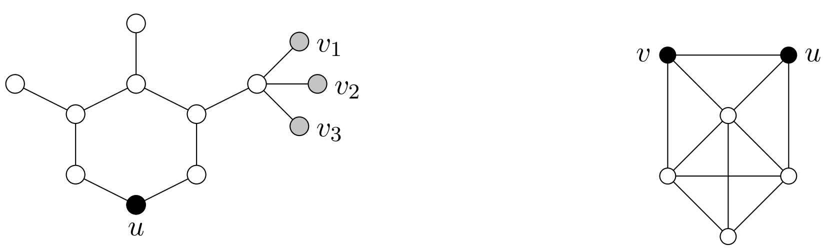
There exist three consecutive pairs that do not contain elements of R. However, there cannot be four consecutive pairs that do not contain elements of R. Suppose to the contrary that none of the elements a_1, b_i, c_i and d_i, where $i = 1, 2$, are in R (the cases for the other pairs go similarly). Now, we have $\mathcal{D}_R(c_1) = \mathcal{D}_R(d_2)$, since for any element of R there exist shortest paths to c_1 and d_2 that go through d_1. Regardless of what the four consecutive pairs are, there always exist vertices positioned similarly to c_1 and d_2 that share the same distance array. Thus, the set R is not a resolving set of G, a contradiction.

So far, we have shown that there exist three consecutive pairs with no elements of R, and that the two elements of R are not in neighbouring pairs. Suppose that $R = \{b_i, d_j\}$ for some $i, j \in \{1, 2\}$. Then the vertices c_2 and f_1 are not resolved, since $\mathcal{D}_R(c_2) = \mathcal{D}_R(f_1)$. Similarly, if $R = \{c_i, e_j\}$ for some $i, j \in \{1, 2\}$, then $\mathcal{D}_R(d_2) = \mathcal{D}_R(a_1)$. The same happens for all other such sets except $\{b_i, f_j\}$ (since there does not exist a vertex a_2 that would get mixed up with d_1). Therefore, we have $R = \{b_i, f_j\}$ for some $i, j \in \{1, 2\}$.

Finally, since the vertex b_1 does not resolve b_2 and c_1, and neither do the vertices f_1 and f_2, we have $b_2 \in R$. Similarly, the vertices f_1 and b_2 do not resolve the vertices f_2 and e_1, and thus $f_2 \in R$. In conclusion, we have $R = \{b_2, f_2\}$ and no other metric bases of G exist.

In Publication IV, we show that cut-vertices are not basis forced vertices, and most of them are in fact void vertices. We also consider pendants, and show that pendants attached to a tree-like structure (like the vertices v_i of the graph in Figure 8a) are not basis forced vertices. However, some pendants are basis forced vertices, for example, the vertices b_2 and f_2 of the graph in Figure 7 are basis forced vertices.

We also consider sparse and dense graphs. Trees do not have basis forced vertices, while some unicyclic graphs do have basis forced vertices. However, unicyclic



(a) **(b)**

Figure 8. **(a)** A graph whose metric bases are of the form $\{u, v_i, v_j\}$, where $i \neq j$. The vertex u is a basis forced vertex, whereas the white vertices are void vertices. **(b)** A graph with the unique metric basis $\{v, u\}$. The white vertices are void vertices, and the vertices v and u are, naturally, basis forced vertices.

graphs cannot have three or more basis forced vertices. In Figures 7 and 8a, we have two unicyclic graphs; the former has two basis forced vertices and the latter has one basis forced vertex. The densest graphs that have basis forced vertices have at most $\frac{n(n-1)}{2} - 2k$ edges, where $n = |V(G)|$ and k is the number of basis forced vertices the graph has. The graph in Figure 8b attains this bound for $k = 2$ and $n = 6$. In general, this bound is attained for all even k.

In Publication IV, we also show that deciding whether a vertex is a basis forced vertex is co-NP-hard, whereas deciding whether a vertex is a void vertex is NP-hard.

4 Overview of Original Publications

To conclude this summary, we give an overview of the results obtained in the original publications this work is based on.

In Publication I, we consider $\{\ell\}$-resolving sets of graphs. We take our first steps toward defining and characterising the forced vertices of $\{\ell\}$-resolving sets. We prove the lower bound $\beta_\ell(G) \geq \ell$, and characterise the graphs that attain this bound. In Theorem 2.6, we determine the $\{\ell\}$-metric dimensions of trees for all $\ell \geq 2$. We do this by giving a somewhat algorithmic proof of how we can uniquely determine the elements of X when we are given the distance array $\mathcal{D}_R(X)$. In addition to trees, we consider the $\{\ell\}$-metric dimension of the king grid $P_m \boxtimes P_n$, where $\boxtimes$ denotes the strong product of two graphs. The exact value of the regular metric dimension of a rectangular king grid was conjectured to be $\lceil \frac{m+n-2}{n-1} \rceil$ in [37]. The conjecture was later proved in [4]. We give another proof for this conjecture, and also show that $\beta_2(P_m \boxtimes P_n) = 2m + 2n - 4$ and $\beta_\ell(P_m \boxtimes P_n) = mn$ for all $\ell \geq 3$. The basic idea behind the proofs for the lower bounds $\beta_2(P_m \boxtimes P_n) \geq 2m + 2n - 4$ and $\beta_\ell(P_m \boxtimes P_n) \geq mn$ when $\ell \geq 3$ is to use the characterisation of forced vertices. However, the proofs are not written in such a way, since the characterisation of the forced vertices of $\{\ell\}$-resolving sets did not yet exist.

In Publication II, we introduce and study 1-solid-resolving sets. In addition to defining 1-solid-resolving sets, we provide an equivalent characterisation to 1-solid-resolving sets, which is easier to use than the definition. In Theorem 2.2, we characterise the forced vertices of a 1-solid-resolving set of a graph. We compare 1-solid-resolving sets to other close concepts such as self-locating-dominating sets, the boundary and the Dilworth number of a graph. From Theorems 3.1 and 3.3, it follows that $\beta(G) \leq |\partial(G)| - 1$, where $\partial(G)$ is the boundary of the graph G. This bound is sharp and an improvement of the bound $\beta(G) \leq |\partial(G)|$ obtained in [20]. In addition to showing general results, we consider the 1-solid-metric dimension of cycles, hypercubes, and Cartesian and strong product graphs. We consider the forced vertices of these product graphs and give bounds on the 1-solid-metric dimension of a product graph utilising the 1-solid-metric dimensions of the components of the graph product. In Theorems 7.2 and 7.4, we give upper bounds for the maximum degree and clique number of a graph when the 1-solid-metric dimension of the graph is known. Finally, we prove that deciding whether $\beta_1^s(G) \leq k$ for an integer k is an

NP-complete problem.

In Publication III, we introduce ℓ-solid-resolving sets, which are generalisations of 1-solid-resolving sets just like $\{\ell\}$-resolving sets are generalisations of the classical resolving sets. In Theorem 3, we give ℓ-solid-resolving sets an equivalent characterisation, which is a generalisation of the characterisation we gave to 1-solid-resolving sets in Publication II. In Section 2.2, we give full characterisations to the forced vertices of ℓ-solid- and $\{\ell\}$-resolving sets of graphs. We prove the lower bound $\beta_\ell^s(G) \geq \ell + 1$ and characterise the graphs that attain this bound for $\ell \geq 2$ (the case where $\ell = 1$ is considered in Publication II). In addition to these general results, we give bounds on the ℓ-solid- and $\{\ell\}$-metric dimension of Cartesian product graphs. In particular, we consider the $\{2\}$-metric dimension of Rook's graphs, and find an interesting connection to combinatorial designs. We also consider the ℓ-solid- and $\{\ell\}$-metric dimensions of flower snarks. The $\{2\}$-metric dimension and 1-solid-metric dimension of flower snarks are small constants, whereas the 2-solid-metric dimension and $\{3\}$-metric dimension are dependent on the size of the graph.

In Publication IV, we introduce the concepts of basis forced and void vertices for metric bases of graphs. We show that cut-vertices are void vertices in most cases, and even when a cut-vertex is not a void vertex, it is not a basis forced vertex. In addition to cut-vertices, we also consider pendants and show that in most cases pendants are not basis forced vertices. In Section 3.1, we provide a way to construct graphs with basis forced vertices from graphs that may not have basis forced vertices. Much of Publication IV is dedicated to studying basis forced vertices in sparse and dense graphs. We show that trees do not have basis forced vertices and give examples of unicyclic graphs that have basis forced vertices. Furthermore, we show that a unicyclic graph can have at most two basis forced vertices and that this bound is sharp. The densest graph that has basis forced vertices has $\frac{n(n-1)}{2} - 4$ edges, where n is the number of vertices of the graph. We show that if G is a graph with n vertices and $k > 0$ basis forced vertices, then $k \leq n - \beta(G) - 1$ and $|E(G)| \leq \frac{n(n-1)}{2} - 2k$. The former bound does not seem to be attainable, while the latter bound is attained for all even k. Lastly, we show that deciding whether a vertex is a basis forced vertex is co-NP-hard and deciding whether a vertex is a void vertex of a metric basis of a graph is NP-hard. Thus, both problems are algorithmically difficult.

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TURUN
YLIOPISTO
UNIVERSITY
OF TURKU

Painosalama Oy, Turku, Finland 2021

ISBN 978-951-29-8478-7 (PRINT)
ISBN 978-951-29-8479-4 (PDF)
ISSN 0082-7002 (PRINT)
ISSN 2343-3175 (ONLINE)