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Efficient Probabilistic Fines Under Negative Externalities

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Abstract

We introduce a probabilistic fine scheme into a simple model of a public bad with negative externalities. As the fine scheme is probabilistic, an agent's probability to be fined depends on its relative action level. This induces a counteracting positive externality into the model because the individual fine probability depends not only on own actions but also on the actions of other agents. In our analysis we derive conditions on the primitives of the model that guarantee the existence of an efficient equilibrium where the negative externality of the public bad is neutralised by the positive externality from the fine scheme. We also demonstrate that a fine scheme can always be designed in such a way that an efficient outcome is induced as a pure strategy equilibrium.

Key Words: Negative externalities, probabilistic fines, efficiency, equilibrium existence, lottery contest.

JEL classification: C72; D62; H23

1 Introduction

Several of the most pressing environmental and social problems are related to the existence of negative externalities (e.g., global warming caused by greenhouse gas emissions), where an agent obtains private profits through its actions but simultaneously induces negative externalities on all other agents. As agents typically do not take into account the fact that their actions are a 'public bad' for others, the resulting unregulated equilibrium outcome will be inefficient due to either over-production or over-consumption of the agents. If individual actions are observable, then a central authority is able to restore an efficient equilibrium outcome by resorting to simple policies and interventions like mandates or corrective Pigou-taxation that internalise these externalities. If individual actions are non-observable, however, efficiency-restoring policies (like Vickrey-Clarke-Groves mechanism and its variants) exist but are less intuitive and therefore harder to implement in practise.

In this paper we combine a public bad-framework with a simple and intuitive probabilistic fine scheme, where the probability that a specific agent is fined depends on its relative action in the sense that it is increasing in own action but decreasing in the aggregate action of all agents. This specific fine scheme can be interpreted as resulting from an imperfect monitoring system where agents are inspected randomly to evaluate their actions. If individual detection is more likely the 'higher' (in the sense of intensity and/or frequency) the respective individual action level but less likely the higher the aggregate action level, then the perceived probability to be detected (and therefore fined) has the mentioned properties. We discuss two applications to clarify this interpretation of a probabilistic fine scheme depending on the relative action level of the respective agent.

The first application considers policing in the context of the recent COVID-19 pandemic where several governments issued national restrictions on social interactions to control the spread of the virus. Naturally, violating these restrictions increases the utility of the violator but induces a negative externality on the community. In order to preserve rule-abiding behaviour, violations of these restrictions are monitored and sanctioned, if detected, by the police (in the UK, for instance, breaches of self-isolation rules can be fined up to GBP $10,000$ 'for the most egregious breaches'). In this context it makes sense to assume that the probability that a rule-breaking individual is detected and fined is, firstly, higher the more often this individual violates the rule, and secondly, lower the higher the total rate of rule-infringements in the community.[1] Hence, from the perspective of an individual the fine probability depends on her relative rate of rule-violation which is captured by the probabilistic fine schemes that we apply in our setup.

The second application is based on the so called non-point pollution control problem, compare Segerson (1988), as well as Shortle and Horan (2001) and Xepapadeas (2011) for surveys. Although perfect monitoring on the level of the individual polluter is assumed to be prohibitively costly in these types of non-point pollution problems, regulatory authorities are still able to resort to imperfect monitoring techniques like unexpected inspections of individual polluters. Also in this context it makes sense to assume that the fine probability is increasing in the individual emission level and decreasing in total emission by all firms. Hence, the ex-ante probability of a specific polluter to be detected and fined depends on the relative emission of this agent in the same ways as in our specification of a probabilistic fine scheme.

We now discuss our setup and approach in more detail and relegate the discussion of potential extensions and generalisations to the last section of this paper. In the following, we adopt an environmental economics terminology inspired by the non-point pollution control problem mentioned before to guide intuition; however, our approach is applicable to any public bad situation with negative externalities where probabilistic fine schemes might be of relevance. We consider a simple model of negative externalities induced through strategic emissions based on a quasi-linear framework with identical agents. Each agent derives positive profit from individual production which is linked one-to-one to individual emissions inducing a damage on the entire community. As each agent decides about its individual production/emission level strategically taking into account individual profit and individual damage instead of total damage, the unregulated equilibrium results in inefficient emission levels that are higher than the optimal amount.

[1]The last mentioned characteristic can be attributed to congestion in law enforcement, see Ferrer (2010), Freeman et al. (1996) and Ehrlich (1973), for contributions that make a similar assumption and discuss its implications.

We then introduce a probabilistic fine scheme into this setting where an agent's probability to be fined depends on a relative measure of its emission level. The fine scheme is modelled based on a modified lottery contest success function in the style of Tullock (1980), where we allow for affine transformations of individual emission levels as in Dasgupta and Nti (1998). Based on this specification, the fine probability is increasing and concave in own emission levels and decreasing in aggregated emission levels. This generates a positive externality (a higher individual fine probability also implies a lower fine probability for all other agents), which has the potential to counterbalance the negative externality stemming from the public bad characteristic of individual emissions. This mechanism has been applied in a 'dual' way in the literature that considers the financing of public goods by the means of lotteries or raffles, compare Morgan (2000) for the seminal contribution, where the positive externality of the public good is balanced to some extent by the negative externality of the lottery. However, there is no direct equivalence in the sense that one approach is the 'negative' reverse version of the other: Although we apply a similar (slightly more general) probability function as in Morgan (2000), the resulting payoff-function in our case is not globally concave (because the expected fine payment in convex in individual emission). This complicates the analysis because we cannot rely on standard techniques to characterise equilibria as the existence of pure-strategy equilibria is not guaranteed for this setup. In our analysis we therefore identify necessary and sufficient conditions on the primitives of the model that guarantee the existence of a symmetric pure-strategy equilibrium for a given level of total emissions. Using these conditions it is then straight-forward to verify whether the efficient emission levels can be induced as a pure-strategy equilibrium and to derive the corresponding efficient fine level.

We demonstrate that our characterization is tight by considering two special cases: If the fine probability is modelled as a simple Tullock lottery contest success function, then we can show that it is not possible to achieve the efficient emission levels as a pure-strategy equilibrium using this scheme. Nevertheless, the conditions are instrumental for a second-best approach because they facilitate the characterisation of the least inefficient total emission level that is still achievable as a pure-strategy equilibrium in this case. If the fine probability incorporates also affine transformations of individual emission levels, then an appropriately specified fine scheme can induce efficient emission levels in equilibrium. Comparing both cases suggests that an agent's fine probability has to be bounded away from zero even for small emission levels in order to allow for efficient emission in equilibrium. This insight is verified in an extended setup, where the central authority is assumed to have more discretionary power over the details of the fine scheme (beyond setting only the appropriate fine level). For this extended setup we can demonstrate that the efficient total emission level can be obtained as the unique pure strategy equilibrium for any well-behaved damage function using appropriate values for the parametrised fine scheme.

The rest of the paper is organised as follows. In section 2 we introduce our formal model, before we deal with equilibrium existence and characterisation for a given probabilistic fine scheme in section 3. In section 4 we analyse the question whether equilibrium existence is guaranteed if the fine scheme parameters can be designed appropriately. In section 5 we conclude by discussing the robustness of our results with respect to potential extensions and generalisations.

2 The Model

There is a group $N = \{1, \dots, n\}$ of identical agents that have access to a production process where individual output is denoted by $x_i \geq 0$ for each agent $i \in N$. Output can be sold on a competitive market at a fixed (normalized) prize of 1. The production of output is linked one-to-one to the emission of pollutants such that individual output and emission levels are strategically equivalent. Individual production induces (through individual emissions) negative externalities on all agents, which is captured by an individual damage function $D(X)$ with $X = \sum_{j \in N} x_j$ denoting total emission, that satisfies standard assumptions: $D'(X) > 0$, $D''(X) > 0$ and $D'''(X) \geq 0$. Individual payoff can therefore be expressed as a well-behaved quasi-linear function of individual emissions:

$$u(x_i, x_{-i}) = x_i - D(X) \text{ for all } i \in N.$$

As the payoff function is concave, the total equilibrium emission level X^* is characterized by first-order conditions:

$$D'(X^*) = 1. \tag{1}$$

Due to the quasi-linear form of the payoff function, the efficient total emission level $\hat{X}$ is obtained by maximizing the sum of individual payoff functions which yields the following Samuelson-condition:

$$D'(\hat{X}) = 1/n. \tag{2}$$

Comparing the two equations it is obvious that there is inefficient over-emission in equilibrium: $X^* > \hat{X}$.

We now consider a sanction mechanism that induces a probabilistic but fixed fine $F > 0$ on polluting agents. For each $x = (x_1, \ldots, x_n)$ we denote the set of polluting agents by $N_+^x = \{i : x_i > 0\}$ with corresponding cardinality $n_+^x = \left|N_+^x\right|$. The individual probability to be fined is proportional to an affine transformation of the respective emission levels:

$$Pr_i(x_i, x_{-i}) = \begin{cases} \frac{a+bx_i}{n_+^x a+bX} & \text{for } x_i > 0 \text{ where } a \geq 0, b > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note, that $Pr_i(x_i, x_{-i})$ is a well-behaved probability function, which is increasing and concave in x_i and decreasing in x_j for $j \neq i$.[2] This specification also contains (for $a = 0$ and $b = 1$) the simple proportional rule as special case, which coincides with the well-known lottery contest success function.

We assume that the proceeds of the sanction scheme are redistributed to polluting agents on a lump-sum basis, which yields the following expected payoff function:

$$u(x_i, x_{-i}; F) = \begin{cases} x_i - D(X) - \frac{a+bx_i}{n_+^x a+bX} F + \frac{F}{n_+^x} & \text{if } x_i > 0, \\ -D(X) & \text{otherwise.} \end{cases} \tag{3}$$

[2] For $a > 0$ the probability function has a discontinuity at $x_i = 0$. This property is one of the two technical assumptions needed to rule out asymmetric equilibria at the boundary.

A brief discussion of the properties of this modified payoff function should be in order at this point. Note first, that non-polluting agents are neither fined nor benefit from redistribution of the collected fine.[3] Note secondly, that due to the concavity of the probability function $Pr_i(x_i, x_{-i})$, the resulting function $u(x_i, x_{-i}; F)$ might be neither concave, nor quasi-concave in the interior of the strategy space. In fact, depending on the curvature of the damage function and the parameters (a, b, n, F), the payoff function can have several local maxima in the interior of the strategy space that might be payoff-dominated by strategies at the boundary. Hence, the existence of an equilibrium in pure strategies is not guaranteed without further restrictions on the primitives of the setup.

3 Equilibrium Analysis

Before addressing the question of efficiency, we first have to identify conditions that guarantee the existence of an equilibrium in pure strategies.[4] The following result provides these conditions for a given target level $\tilde{X}$ of total emissions. The idea behind this result is to bound the curvature of the damage function in such a way that the target level $\tilde{X}$ coincides with a local maximum of the payoff function while guaranteeing at the same time that unilateral deviations are not profitable. Based on this result it is then straight-forward to address the question whether the efficient level $\hat{X}$ can be induced as a pure-strategy equilibrium and to derive the corresponding fine level in the subsequent corollary.

Proposition 3.1 *Let $\tilde{X} > 0$ with $\tilde{x} = \frac{\tilde{X}}{n}$ be a given total emission level. There exists a finite fine level $\tilde{F} > 0$ such that $(\tilde{x}, \dots, \tilde{x})$ is the unique pure strategy equilibrium if and only if the following two conditions are satisfied:*

$$(i) \qquad \frac{2b}{na + b\tilde{X}} \leq \frac{D''(\tilde{X})}{1 - D'(\tilde{X})};$$

$$(ii) \qquad \frac{D(n\tilde{x}) - D((n-1)\tilde{x})}{n\tilde{x} - (n-1)\tilde{x}} \leq 1 - \frac{n(a + b\tilde{x})(1 - D'(n\tilde{x}))}{na + (n-1)b\tilde{x}}.$$

[3] This is the second technical assumption that is necessary to rule out asymmetric equilibria at the boundary. Without these two assumption most of our results except uniqueness would still be valid.

[4] The issue of mixed-strategy equilibria is discussed in the last section.

Proof. The proof consists of three parts. In the first part we show that if $\tilde{F}$ is a fine that induces $(\tilde{x}, \ldots, \tilde{x})$ as the unique pure strategy equilibrium, then the two conditions (i) and (ii) have to be satisfied. The second part proves that any unilateral deviation from $(\tilde{x}, \ldots, \tilde{x})$ satisfying these two conditions will induce a lower payoff. Hence, $(\tilde{x}, \ldots, \tilde{x})$ is a pure strategy symmetric equilibrium. The third part demonstrates that the equilibrium is unique.

Part 1: Suppose $\tilde{F}$ is a fine such that $(\tilde{x}, \ldots, \tilde{x})$ is the unique pure strategy equilibrium. Then it has to satisfy first-order conditions that can also be used to derive a closed form expression for $\tilde{F}$ using symmetry:

$$
\begin{aligned}
1 - D'(\tilde{X}) - \frac{(n-1)b\tilde{F}}{n^2(a+b\tilde{x})} &= 0 \qquad (4)\\
\Rightarrow \quad \tilde{F} &= \frac{n^2(a+b\tilde{x})(1-D'(n\tilde{x}))}{(n-1)b}.
\end{aligned}
$$

Second-order conditions have to be satisfied locally for $(\tilde{x}, \ldots, \tilde{x})$ to be an equilibrium. Using symmetry and substituting $\tilde{F}$ then yields condition (i).

$$
\begin{aligned}
-D''(\tilde{X}) + \frac{2b^2(n-1)\tilde{F}}{n^3(a+b\tilde{x})^2} &\leq 0 \qquad (5)\\
\Rightarrow \quad \frac{2b}{n(a+b\tilde{x})} &\leq \frac{D''(\tilde{X})}{1-D'(\tilde{X})}.
\end{aligned}
$$

Moreover, as $(\tilde{x}, \ldots, \tilde{x})$ is an equilibrium, a unilateral deviation of agent i sufficiently close to the lower boundary of the strategy space cannot induce a higher payoff.[5] Hence, the following inequality has to hold, which can be simplified further using symmetry and substituting $\tilde{F}$, which yields condition (ii):

$$
\begin{aligned}
& \lim_{x_i \to 0} u(x_i, \tilde{x}_{-i}; \tilde{F}) \leq u(\tilde{x}, \tilde{x}_{-i}; \tilde{F})\\
\Rightarrow \quad & 0 - D((n-1)\tilde{x}) - \left(\frac{a}{na + b(n-1)\tilde{x}}\right)\tilde{F} + \frac{\tilde{F}}{n} \leq \tilde{x} - D(n\tilde{x})\\
\Rightarrow \quad & D(n\tilde{x}) - D((n-1)\tilde{x}) \leq \tilde{x} - \left(\frac{1}{n} - \frac{a}{na + b(n-1)\tilde{x}}\right)\tilde{F}
\end{aligned}
$$

[5] In the second part of the proof we also show that any other unilateral deviation to the interior of the strategy space induces lower payoff.

$$\Rightarrow \quad D(n\tilde{x}) - D((n-1)\tilde{x}) \leq \left(1 - \frac{n(a+b\tilde{x})(1-D'(n\tilde{x}))}{na+(n-1)b\tilde{x}}\right)\tilde{x}. \tag{6}$$

Note, that neither a unilateral deviation to zero can be profitable because it can be verified that:

$$\begin{aligned} u(0, \tilde{x}_{-i}; \tilde{F}) &= 0 - D((n-1)\tilde{x}) \\ &< 0 - D((n-1)\tilde{x}) - \left(\frac{a}{na+b(n-1)\tilde{x}}\right)\tilde{F} + \frac{\tilde{F}}{n} \\ &= \lim_{x_i \to 0} u(x_i, \tilde{x}_{-i}; \tilde{F}) \leq u(\tilde{x}, \tilde{x}_{-i}; \tilde{F}) \end{aligned}$$

Part 2: Let $\tilde{F} = \frac{n^2(a+b\tilde{x})(1-D'(n\tilde{x}))}{(n-1)b}$ and assume that conditions (i) and (ii) are satisfied. Then it can be checked that $(\tilde{x}, \ldots, \tilde{x})$ corresponds to a local maximum of the payoff function because eq. (4) and (5) are satisfied. Moreover, eq. (6) is satisfied as well implying that unilateral deviations to the lower boundary of the strategy space are not profitable. It remains to be shown that other unilateral deviations to the interior of the strategy space are not profitable as well. If there is a profitable deviation in the interval $(0, \tilde{x})$, then there must also exist another local maximum in $(0, \tilde{x})$ because we already ruled out profitable deviations at the boundary. We will therefore first demonstrate (by contradiction) that a local maximum at $\bar{x} \in (0, \tilde{x})$ does not exist. We will then demonstrate that deviations $\bar{x} > \tilde{x}$ can neither be profitable.

Suppose there exists a local maximum for agent i at $\bar{x} \in (0, \tilde{x})$ given that all other agents $j \neq i$ choose $\tilde{x}$. Then it must be the case that

$$-D''((n-1)\tilde{x} + \bar{x}) + \frac{2b^2(n-1)(a+b\tilde{x})}{(na+b((n-1)\tilde{x}+\bar{x}))^3}\tilde{F} \leq 0,$$

which implies together with $D''' \geq 0$ that for all $x > \bar{x}$ the following inequality holds:

$$-D''((n-1)\tilde{x} + x) + \frac{2b^2(n-1)(a+b\tilde{x})}{(na+b((n-1)\tilde{x}+x))^3}\tilde{F} < 0.$$

Note that both $\tilde{x}$ and $\bar{x}$ correspond to local maxima; hence, there must exist at least one $x' \in (\bar{x}, \tilde{x})$ where the payoff function is (at least weakly) convex in a neighbourhood of this point:

$$-D''((n-1)\tilde{x}+x') + \frac{2b^2(n-1)(a+b\tilde{x})}{(na+b((n-1)\tilde{x}+x'))^3}\tilde{F} \geq 0.$$

As $x' > \bar{x}$ this is a contradiction to the previously established strict inequality

We now prove that there cannot exist a profitable deviations $x > \tilde{x}$. As $\tilde{x}$ corresponds to a local maximum, it must be the case that the payoff function is strictly concave on $(\tilde{x}, \infty)$ because a similar argument to the one above gives us

$$-D''((n-1)\tilde{x}+x) + \frac{2b^2(n-1)(a+b\tilde{x})}{(na+b((n-1)\tilde{x}+x))^3}\tilde{F} < 0 \text{ for all } x > \tilde{x}.$$

Hence, payoff must be lower for all $x > \tilde{x}$ which implies that a deviation from $\tilde{x}$ to $x > \tilde{x}$ cannot be profitable.

Part 3: Let $\tilde{F}$ be as defined at the beginning of Part 2 such that $(\tilde{x}, \ldots, \tilde{x})$ is a pure strategy equilibrium.

Step 1: Suppose $\overline{\overline{x}} = (\overline{\overline{x}}_1, \ldots, \overline{\overline{x}}_n) \neq (\tilde{x}, \ldots, \tilde{x})$ is also a pure strategy equilibrium. Let $\overline{\overline{X}} = \sum_{j \in N} \overline{\overline{x}}_j$. Clearly, $\overline{\overline{X}} > 0$. We first show by contradiction that $\overline{\overline{x}}_j > 0$ for all $j \in N$. Suppose to the contrary that there exists j such that $\overline{\overline{x}}_j = 0$. Then it can be verified that

$$\begin{aligned} u(0, \overline{\overline{x}}_{-j}; \tilde{F}) &= 0 - D(\overline{\overline{X}}) \\ &< 0 - D(\overline{\overline{X}}) - \left(\frac{a}{(n_+^{\overline{\overline{x}}}+1)a + b\overline{\overline{X}}}\right)\tilde{F} + \frac{\tilde{F}}{n_+^{\overline{\overline{x}}}+1} \\ &= \lim_{x_j \to 0} u(x_j, \overline{\overline{x}}_{-j}; \tilde{F}), \end{aligned}$$

which implies that it is profitable for agent j to unilaterally deviate from $\overline{\overline{x}} = 0$, a contradiction. Hence, $\overline{\overline{x}}_i > 0$ for all $i \in N$. Thus, it follows from the first-order conditions that $\overline{\overline{x}}_1 = \ldots = \overline{\overline{x}}_n$, i.e., $\overline{\overline{x}}$ is a symmetric pure strategy equilibrium. Therefore, every pure strategy equilibrium corresponding to the fine $\tilde{F}$ is symmetric.

Step 2: Given Step 1, we only need to show that there does not exist a symmetric pure strategy equilibrium that is distinct from $(\tilde{x}, \ldots, \tilde{x})$.

Suppose $0 < \tilde{Z} \neq \tilde{X}$ with $\tilde{z} = \frac{\tilde{Z}}{n}$ is such that $(\tilde{z}, \ldots, \tilde{z})$ is also a pure strategy equilibrium for the fine level $\tilde{F}$. Then we have the following FOCs and SOCs for a local maximum:

$$1 - D'(\tilde{X}) - \frac{b(n-1)\tilde{F}}{n^2(a + b\tilde{x})} = 0 \tag{7}$$

$$1 - D'(\tilde{Z}) - \frac{b(n-1)\tilde{F}}{n^2(a + b\tilde{z})} = 0 \tag{8}$$

$$-D''(\tilde{X}) + \frac{2b^2(n-1)\tilde{F}}{n^3(a + b\tilde{x})^2} \leq 0 \tag{9}$$

$$-D''(\tilde{Z}) + \frac{2b^2(n-1)\tilde{F}}{n^3(a + b\tilde{z})^2} \leq 0. \tag{10}$$

Define $G(X) = 1 - D'(X) - \frac{b(n-1)\tilde{F}}{n(na+bX)}$ for all $X > 0$. Then we have:

$$G'(X) = -D''(X) + \frac{b^2(n-1)\tilde{F}}{n(na + bX)^2} \qquad \text{for all } X > 0 \tag{11}$$

$$G''(X) = -D'''(X) - \frac{2b^3(n-1)\tilde{F}}{n(na + bX)^3} < 0 \qquad \text{for all } X > 0. \tag{12}$$

Now, (7) and (8) imply:

$$G(\tilde{X}) = 1 - D'(\tilde{X}) - \frac{b(n-1)\tilde{F}}{n(na + b\tilde{X})} = 1 - D'(\tilde{X}) - \frac{b(n-1)\tilde{F}}{n^2(a + b\tilde{x})} = 0$$

$$G(\tilde{Z}) = 1 - D'(\tilde{Z}) - \frac{b(n-1)\tilde{F}}{n(na + b\tilde{Z})} = 1 - D'(\tilde{Z}) - \frac{b(n-1)\tilde{F}}{n^2(a + b\tilde{z})} = 0.$$

Also, (9), (10) and (11) imply:

$$\begin{aligned} G'(\tilde{X}) = -D''(\tilde{X}) + \frac{b^2(n-1)\tilde{F}}{n(na + b\tilde{X})^2} &= -D''(\tilde{X}) + \frac{b^2(n-1)\tilde{F}}{n^3(a + b\tilde{x})^2} \\ &< -D''(\tilde{X}) + \frac{2b^2(n-1)\tilde{F}}{n^3(a + b\tilde{x})^2} \leq 0 \\ G'(\tilde{Z}) = -D''(\tilde{Z}) + \frac{b^2(n-1)\tilde{F}}{n(na + b\tilde{Z})^2} &= -D''(\tilde{Z}) + \frac{b^2(n-1)\tilde{F}}{n^3(a + b\tilde{z})^2} \\ &< -D''(\tilde{Z}) + \frac{2b^2(n-1)\tilde{F}}{n^3(a + b\tilde{z})^2} \leq 0 \qquad . \end{aligned}$$

However, $G'(\tilde{X}) < 0$, $G'(\tilde{Z}) < 0$ and $G''(X) < 0$ for all $X > 0$ imply that it is not possible to have both $G(\tilde{X}) = 0$ and $G(\tilde{Z}) = 0$, a contradiction. Therefore, there does not exist a symmetric pure strategy equilibrium that is distinct from $(\tilde{x}, \ldots, \tilde{x})$. □

Proposition 3.1 does not address the issue of efficiency. However, using the Samuelson condition from eq. (2) to simplify conditions (i) and (ii) leads to the following corollary which provides the respective conditions for the existence of an equilibrium that yields the efficient emission level.

Corollary 3.2 *Given fine $\hat{F} = \frac{n(a+b\hat{x})}{b}$ with $\hat{x} = \frac{\hat{X}}{n}$, the efficient total emission level $\hat{X}$ can be achieved as the unique pure strategy equilibrium $(\hat{x}, \ldots, \hat{x})$ if and only if the following two conditions are satisfied:*

$$(iii) \qquad \frac{2b(n-1)}{n^2(a+b\hat{x})} \leq D''(\hat{X});$$

$$(iv) \qquad \frac{D(n\hat{x}) - D((n-1)\hat{x})}{n\hat{x} - (n-1)\hat{x}} \leq \frac{a}{na + (n-1)b\hat{x}} = \lim_{x_i \to 0} Pr_i(x_i, \hat{x}_{-i}).$$

Condition (iv) can be used to derive the following negative result with respect to a specific class of probabilistic fine schemes that include linear (instead of affine) transformations of emissions.

Corollary 3.3 *A probabilistic fine scheme with $a = 0$ (which includes the simple lottery contest success function as special case) cannot induce the efficient total emission level $\hat{X}$ as equilibrium outcome.*

Proof. For $a = 0$ condition (iv) reduces to $\frac{D(n\hat{x}) - D((n-1)\hat{x})}{n\hat{x} - (n-1)\hat{x}} \leq 0$. This is a contradiction because the left-hand side of this inequality is positive as $D(\cdot)$ is increasing. □

In order to gain further intuition with respect to these conditions, we now consider two special cases. The first case is related to Corollary 3.3 and considers the standard lottery contest success function (or, alternatively, a simple proportional fine probability). While Corollary 3.3 implies that the efficient emission level cannot be obtained as a pure strategy equilibrium, the two conditions (i) and

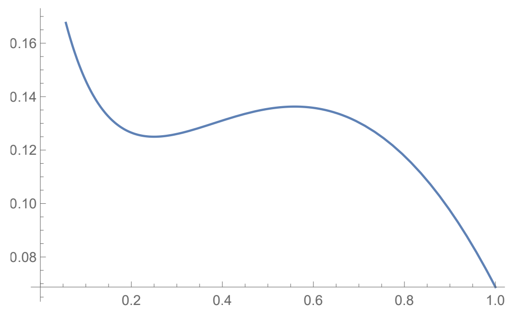

Figure 1: Case 1 with $u(x_1, \hat{x}_2; \hat{F})$

(ii) can be used to identify a fine that leads to the most efficient emission level that is achievable as a pure strategy equilibrium. The second case demonstrates that a simple modification of the lottery contest success function is sufficient to restore efficiency in equilibrium. Hence, we are able to demonstrate that there exist well-behaved damage functions and fine schemes such that conditions (i) and (ii), and also (iii) and (iv) respectively, are both satisfied.

Case 1. Consider the following specification that corresponds to a proportional fine probability function: $(a, b, n) = (0, 1, 2)$ with $D(X) = \frac{1}{2}X^2$. For this specification the efficient emission level is $\hat{X} = \frac{1}{2}$ which is lower than the equilibrium emission level without fine: $X^* = 1$. Condition (iii) of Corollary 3.2 is violated $(2 \leq 1)$ as well as condition (iv) $(3/8 \leq 0)$. Figure 1 shows the payoff function of agent 1, given that agent 2 extracts the efficient amount $\hat{x}_2 = 1/4$. Note, that $\hat{x}_1 = 1/4$ corresponds to a critical point, which is neither a local maximum nor a global maximum (instead, $\hat{x}_1 = 1/4$ corresponds to a local minimum).

Nevertheless, Proposition 1 can be used to derive the lowest total emission level that is achievable as a pure strategy equilibrium. Setting $\tilde{X} = 4/5$ implies that condition (i) is satisfied with strict inequality $(5/2 < 5)$, while condition (ii) is satisfied by equality. Hence, $\tilde{X} = 4/5$ is the lowest total emission level that is achievable in this case. The corresponding fine level is $\tilde{F} = 8/25$ and the respective payoff function for agent 1 is presented in Figure 2. Note that the second-best emission level $\tilde{x}_1 = 0.4$ now corresponds to a global maximum.

Case 2. Consider the following slight modification of the previous specification: $(a, b, n) = (1, 1, 2)$ with $D(X) = \frac{1}{2}X^2$. Now the fine probability is bounded

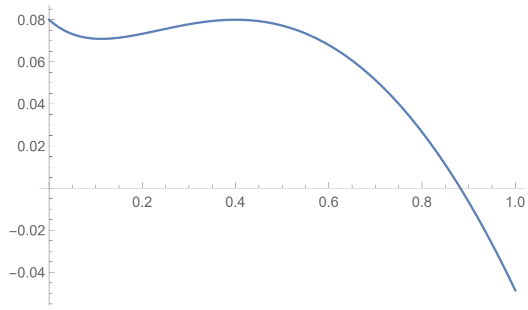

Figure 2: Case 1 with $u(x_1, \tilde{x}_2; \tilde{F})$

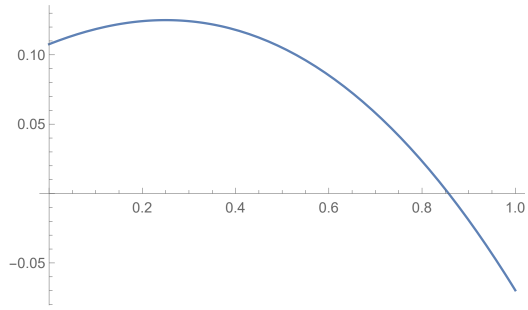

Figure 3: Case 2. $u(x_1, \hat{x}_2; \hat{F})$

away from zero for any individual emission level: $Pr_i(x_i, x_{-i}) = \frac{1+x_i}{2+X} > 0$ for all $x_i > 0$. Note that the efficient emission level from Case 1 remains unaltered because the damage function is the same. For this specification it can be verified that conditions (iii) and (iv) from Corollary 3.2 are satisfied. Hence, based on the corresponding fine $\hat{F} = 5/2$ the efficient emission level can be achieved as a symmetric equilibrium in pure strategies. Figure 3 shows the respective payoff function for this case and demonstrates that the individual emission level $\hat{x}_1 = 1/4$ corresponds to a global maximum.

4 Designing An Efficient Fine Scheme

Until now all parameters except the fine level F have been assumed to be given. Alternatively, the parameters (a, b) that determine the affine transformation in the fine scheme could be interpreted as additional choice variables in a design problem of the central authority. In this case the design problem is to choose $(a, b, F) \in \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$ in order to implement a specific emission level $Z \in (0, X^*)$ as a unique pure strategy equilibrium.[6] The next result demonstrates that this is possible for any emission level $Z \in (0, X^*)$ using appropriate values for the choice variables.

Proposition 4.1 *Let Z be any total emission level in the interval* $(0, X^*)$ *and let* $z = \frac{Z}{n}$. *Then there exists* $(a^Z, b^Z, F^Z) \in \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$ *such that the symmetric emission vector* $(z, \ldots, z)$ *is the unique pure strategy equilibrium.*

Proof. Let $Z \in (0, X^*)$ and $z = \frac{Z}{n}$. Because we have derived a closed form expression for the fine level in the proof of Proposition 1, it is sufficient to show that there exists $(a^Z, b^Z) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ such that conditions (i) and (ii) in Proposition 1 are satisfied.

Let us pick any $b^Z \in \mathbb{R}_{++}$ and consider each of the two conditions in turn. Since $\lim_{a\to\infty} \frac{2b^Z}{na+b^Z Z} = 0$, condition (i) is satisfied for sufficiently large a. Next, it can be verified that $\lim_{a\to\infty} \left[1 - \frac{n(a+b^Z z)(1-D'(nz))}{na+(n-1)b^Z z}\right] = D'(nz)$. Also, it follows from strict convexity of $D(\cdot)$ that $\frac{D(nz)-D((n-1)z)}{nz-(n-1)z} < D'(nz)$. Hence, condition (ii) is also satisfied for sufficiently large a. Therefore, given any $b^Z \in \mathbb{R}_{++}$, there exists sufficiently large a^Z such that conditions (i) and (ii) are satisfied. □

The insights from the proof of this proposition confirm the intuition from the two cases provided in the previous section of the paper: Setting parameter a sufficiently high (bounding the fine probability away from zero even for small individual emission levels) is crucial to induce an efficient equilibrium outcome.[7]

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[6] The efficient emission level $\hat{X}$ derived in eq. (2) is contained in this set $(0, X^*)$, where X^* is the equilibrium emission level without any fine characterised in eq. (1). Hence, the following result holds also for the efficient emission level.

[7] This result has a dual expression in the contest literature: In Dasgupta and Nti (1998) it is demonstrated that setting a positive value for a will lead to less total effort in the respective contest game.

5 Concluding Discussion

Our approach demonstrates that probabilistic fine schemes can be designed in such a way that efficient outcomes result in equilibrium under negative externalities. Naturally, the simple setup applied here facilitates the analysis but also raises issues of robustness with respect to potential generalisations and extensions. In the following we discuss some potential extensions regarding other types of negative externalities, heterogeneity of agents, and the issue of mixed-strategy equilibria, which open up further research possibilities.

In our setup negative externalities are captured by an additive-separable damage function which allows for sufficient tractability. However, our approach is potentially also applicable in the context of alternative specifications. Consider, for instance, the classical common pool resource extraction game as specified in Ostrom et al. (1992), where the following functional form (adapted to our notation and ignoring wealth constraints) is applied: $u_i(x) = \frac{x_i}{X} f(X) - x_i$, where $f(X)$ is a concave function. In this specification individual profit (i.e. the first term) is increasing in own resource extraction but negatively affected by aggregate extraction levels. In contrast to our specification individual profit and aggregate damage are not additively separable. However, the resulting tragedy of the commons implies that there is still inefficient over-extraction in the symmetric Nash-equilibrium.[8] As the setup is well-behaved with a concave payoff-function, it can be modified accordingly by adding the same probabilistic fine scheme as in our setup. We conjecture that the exact conditions to guarantee the existence of an efficient symmetric pure strategy equilibrium will be more complex but could be derived using the same approach: Firstly, guaranteeing local concavity in equilibrium and, secondly, excluding unilateral deviations to the boundary by restricting the curvature of the profit function.

Our analysis relies on equilibrium symmetry which will not hold if agents are heterogeneous. Hence, it is unlikely that a symmetric probabilistic fine scheme can restore efficiency in a setup with heterogeneous agents[9] because this would

[8]Ostrom (1990) analyses real instances of successfully governed commons and demonstrates that these mechanisms typically entail monitoring of extraction levels and sanctioning of over-extracting agents through fines.

[9]Applying a symmetric probabilistic fine scheme in a model with heterogeneous agent might

require either idiosyncratic fines specifically designed for each agent (using a Lindahl-pricing approach) or biased probability functions (as used in Franke and Leininger (2014) in a public good context). Designing those idiosyncratic fines or biases would require complete information with respect to individual preference parameters of the agents. This leads to the typical problem of truthful preference revaluation which goes beyond the scope of this paper.

In our analysis we focused on the identification of conditions that guarantee the existence of an equilibrium in pure strategies and did not address the issue of mixed-strategy equilibria. Although Glicksberg (1952) is not directly applicable due to the discontinuity of the payoff function at $x_i = 0$ (comp., footnote 2), there might still exist equilibria in mixed strategies if the conditions for a pure-strategy equilibrium are not met. Unfortunately, the characterisation of such an equilibrium is not trivial (comp. Ewerhart (2015) for some recent work on mixed-strategy equilibrium analysis in the case of standard Tullock contest games). Analysing whether probabilistic fine schemes might restore efficiency in specifications where mixed-strategy equilibria exist might therefore constitute an interesting research possibility that we plan to address in the future.

still lead to pareto-improvements, see Morgan (2000) for the case of public good provision through lotteries.

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