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Capital Market Risk and the Dynamics of the Income Distribution

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Abstract:

This paper introduces a dynamic model of the wealth distribution with aggregate risk in the capital market; the model combines credit rationing and portfolio selection decisions. In a closed economy the long-run behaviour of wealth is independent of the initial income distribution when there is aggregate uncertainty, although further restrictions are necessary when there is no aggregate uncertainty. There can be credit rationing at the long-run equilibrium. In poor economies aggregate risk in the capital market slows growth, whereas in richer economies a risky capital market is good for income growth.

Keywords: Wealth Distribution, Dynamics, Uncertainty.

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Non-Technical Summary:

Inequalities in wealth, or in income, mean that there is a distribution of incomes; some people are rich and others are poor. The presence of inequality is intimately related to economic growth, as countries grow the incomes of individuals may become more, or less, equal. One economic puzzle is why inequality persists when there is growth, why are the great-grandchildren of the poor child still poor? One reason for this is that the poor are unable to borrow to the same extent as richer individuals and so are unable to make the investments that generate higher life-time incomes. (We say that such people are credit rationed.) This paper studies a dynamic model of the income distribution where there is aggregate uncertainty and credit rationing. It first considers a country which cannot borrow or lend to others. (A closed capital market.) It shows that in such a country aggregate uncertainty makes inequality less persistent. Then it considers what happens when a country is able to borrow and lend to the rest of the world (an open capital market). This has very important implications for borrowers and savers in the country, it reduces the rate of interest and it decreases the riskiness of some types of savings. All savers must save at the world rate of interest. We show that allowing an open capital market can harm the growth of the economy.

1. Introduction

In dynamic models of the income distribution the functioning of capital markets is the key issue. One class of models (for example Banerjee and Newman (1991), Galor and Zeira (1993)) assumes the existence of a perfect world capital market and an open economy, so domestic banks can borrow and lend at the world interest rate as much or as little as desired. A second and more recent class of models (for example Aghion and Bolton (1997) and Piketty (1997)) assumes closed capital markets, so the market interest rate adjusts to equate the supply and demand for funds within a given economy. The common theme in these models is that growth is determined by the ability of agents to borrow to finance investment in new technology. There is credit rationing so the demand side of the capital market is modelled in considerable detail. However, the supply of funds to the capital market is considered in less detail: either there is an infinite supply at the prevailing world interest rate, or all savings in the economy get placed on the domestic capital market.

This paper aims to redress this balance by considering savings decisions in greater detail. Agents have alternative ways of saving and make portfolio choices. In developing countries the supply of funds to the domestic capital market is just as important as the demand for loans. An undersupply of capital may arise from various forms of capital flight, or from individuals choosing to store wealth in "safe" storable commodities such as land and gold. (The term capital flight is used very loosely here to describe any means of storing wealth outside the domestic economy: it may be that domestic capital is held in overseas accounts or it may be that domestic capital is held in foreign-denominated notes (e.g. dollars) within the country which cannot be used as a basis for making loans.) Broadly speaking, the domestic capital market is just one asset the individual can invest in and portfolio decisions determine the allocation of savings among available means of storing wealth. The model below contains individuals who allocate their wealth amongst a portfolio of assets (a safe commodity, the risky domestic capital market, and investing in capital intensive technology) and we investigate how these portfolio choices affect the capital markets and thereby growth.

The key features of the model below are risk aversion and aggregate risk in the capital market. The source of the capital market risk is the individual investment projects undertaken by borrowers. In previous models the returns to each individual's technology have independent and idiosyncratic risk. There are a large number of borrowers, so the law of large numbers applies and the idiosyncratic risk vanishes at the aggregate level. In these models the lenders feel no aggregate effect from the idiosyncratic risks and are able to pay a fixed rate of return to individuals who supply funds to lending institutions. In the model below there is correlation between the payoffs to individuals' technology. This correlation does not vanish at the aggregate level so lenders also bear aggregate risk. It follows that these lenders cannot pay a certain rate of return to savers. When there is aggregate uncertainty in the returns to capital, the capital supply from risk-averse agents will generally be less than that observed in the absence of aggregate uncertainty. Risk averse agents will also hedge risk present in the domestic capital market by storing their wealth in a portfolio of assets, thereby holding some of their wealth in a safe, storable commodity and some of their wealth in the country's capital market. Moreover, there is a significant distributional component in individuals' portfolio choices. Agents' propensity to avoid the capital market will depend upon their location in the wealth distribution. In general the very poor (on fixed subsistence incomes) will choose to hold all of their endowment in the risky capital market, because a large part of their life-time wealth is certain. Those with minimal inherited assets and a fixed lifetime's income can only invest this minimal amount in the capital market. They will choose to do this, because their lifetime's portfolio is dominated by the fixed element of their incomes.[1] [2] As individuals become richer the fixed component of their life-time wealth becomes smaller, so they allocate a smaller proportion of their total wealth to the capital market. Thus, (for a given rate of interest) the undersupply of funds to the capital market in the presence of aggregate uncertainty becomes more pronounced when the economy gets richer.

We will find that in the early stages of development growth is slower in presence of aggregate uncertainty. This is because savers, as well as investors, bear some risk and

[1]This is a similar argument to economic justifications for gambling among the very poor.
[2]If labour income is also risky, this argument will not apply.

this risk tends to increase the rate of interest and slows development. Thus aggregate uncertainty will generally raise the barriers to development that an economy faces. We will also show that income growth in richer economies may actually be improved by the presence of aggregate uncertainty. Aggregate uncertainty generally raises interest rates and thus increases the values of agents' expected future wealth. When there are many rich lenders and few relatively poor borrowers, the net effect of higher interest rates on average growth is positive.

Our main result is that the stochastic aggregate shocks generate long-run behaviour of the income distribution which is independent of the initial income distribution. This will not generally be the case when there is no aggregate uncertainty. Thus the presence of noise in the system decreases the dependence of the income distribution on the initial state. When there is no aggregate randomness, we can show that the initial income distribution does not affect the long-run and growth of the system only if more restrictive assumptions are made. This suggests that the case for non-ergodic growth is only convincing in models with aggregate stochastic shocks, because non-ergodic growth in deterministic models is not generally robust to the inclusion of aggregate shocks.

In the last section of this work we will compare the open and closed versions of this economy and we find that there are not clear benefits from opening capital markets. In particular we will show that in economies with no credit rationing there is always faster growth with closed capital markets than with open capital markets. So if capital is in excess supply in the domestic economy it is harmful to growth to open the capital market. The explanation for this is a form of crowding out. Opening capital markets drives down the domestic rate of interest which makes the domestic capital market a less attractive asset to domestic savers. Consequently, domestic savers tend to switch from the domestic market to the world capital market and make lower returns on their portfolio. As savers are making lower returns their wealth grows less quickly. This is consistent with the experience of the Japanese economy, which experienced rapid growth fuelled by a high rate of domestic savings and closed capital markets and slower growth with open capital markets.

2. The Model

This section begins with a bald description of the preferences of agents and the technology available to them. Then the capital market is described. The first subsection describes the nature of the credit rationing in this model and how there will usually be three types of agents present in our model. There are the poor, who are credit rationed and obliged to use the subsistence technology. There is the middle class, who are borrowers and use the capital intensive technology, and there are the rich, who can afford to finance the use of capital intensive technology out of their own endowments. In each case we will describe the portfolio choices of the three types of individuals and how they make their bequests. The final subsection is devoted to writing down the (random) dynamic system for the wealth distribution that results from behaviour of the agents in this model. To do this it is necessary to describe the equilibrium in the asset markets.

In this model there is one good (a consumption good), a continuum of identical individuals with mass 1 (a generation) and each generation lives for one period. The consumption good is storable and does not depreciate between periods. An individual begins its existence at the start of a period and is endowed with one unit of labour and a wealth bequest from its ancestor. The individual then chooses how to invest its assets and which technology to use. At the end of the period an individual realises the returns from her investments and her productive activity, consumes, makes a bequest of the consumption good to its child and finally dies. Time is denoted $t = 0, 1, 2, ...$ One individual's wealth will be denoted by $x \in \Re$. To describe a wealth distribution at time t we will use a probability measure λ_t defined on the real line $\Re$.[3] (In what follows it will usually be sufficient to think of λ_t as having a distribution function F_t, however.) We will define Λ to be the space of all probability measures λ_t on the compact interval $[0, X]$, where X is chosen to be larger than any feasible income level generated by this system. An agent's preferences, u, are defined on its consumption

[3] To be precise we will define λ_t as a probability measure on the measurable space $(\Re, \mathcal{B})$, where $\mathcal{B}$ is the Borel sigma-algebra.

c_t and its bequest of the consumption good to its successor x_{t+1};

$$u(c_t, x_{t+1}) = \left(\frac{c_t^{\alpha} x_{t+1}^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \right)^{\gamma},$$

where $0 < \alpha < 1$ and $0 < \gamma < 1$. If an agent realises z units of the consumption good at the end of its life, then these preferences imply that the individual leaves a bequest of size $(1-\alpha)z$ and consumes αz. Substitution shows that the individual's indirect utililty for the quantity of consumption good at the end of its life is $v(z) = z^{\gamma}$, and the individual has constant relative risk aversion.

There are two sources of income in the model. There is a subsistence technology. This technology does not require any labour it just provides individuals with y units of income at the end of their life. Every individual receives y simply by being alive, this subsistence income is specific to the individual and cannot be removed or seized by creditors. We will treat y as being small and it represents the lowest level of income people can be certain of. There are also two types of capital-intensive technology one is more efficient than the other. Each capital-intensive technology requires k units of the consumption good and one unit of labour. The output level of both technologies is risky and, unlike previous studies, correlated across projects. The returns to the capital intensive technologies are as follows. In the bad state of the world both capital-intensive technologies produce nothing; the bad state occurs with probability $1-\beta$. In the good state of the world technology 1 produces G units of the consumption good with probability ϕ/β and 0 units with probability $1-\phi/\beta$, where $0 \leq \phi \leq \beta$. In the good state of the world technology 2 produces B units of the consumption good with probability π/β and 0 units with probability $1-\pi/\beta$. Technology 1 has a lower maximum level of output than technology 2. But, technology 1 gives a higher expected utility than storage which in turn gives a higher expected utility than technology 2. These assumptions are summarised in the following conditions

$$(1) \qquad B > G, \qquad \phi(G+y)^{\gamma} + (1-\phi)y^{\gamma} > k + y > \pi(B+y)^{\gamma} + (1-\pi)y^{\gamma}.$$

The output of the capital-intensive technologies are correlated across agents. The amount of correlation is determined by the parameter β; as $\beta \to 1$ the amount of correlation shrinks to zero and as $\beta \to \phi$ the correlation approaches unity. Notice that as β varies the probability distribution over returns to the technology does not vary, so all technology choices are independent of β. Thus the role of β is to represent the aggregate risk in making loans. As $\beta \to 1$ so all projects become independent of each other and the law of large numbers ensures that there is no aggregate risk in providing loans. However, when $\beta \to \phi$ the aggregate risk in providing loans is the same as the risk undertaken by the borrower.

The capital market in this economy is closed and is described by a mutual fund. Capital is supplied to the mutual fund by individuals allocating some of their wealth endowment to it. The demand for capital from the mutual fund comes from individuals endowed with less than k units of the consumption good who, nevertheless, want to use a capital-intensive technology and must borrow sufficient units of the good to embark on the technology. Investing in this mutual fund is risky, because in bad states all of the assets supplied to the fund will be lost. Let r denote the rate of interest paid by the fund in good states, so every unit of the good supplied to the fund is repaid with $1+r$ units at the end of the period and every unit borrowed from the fund is repaid by $1+r$ units. The expected rate of return from the mutual fund is $\beta(1+r)$ and as $\beta \to 1$ the mutual fund becomes a risk-free investment.

2.1 Credit Rationing and the Types of Agent

In this economy there will be three types of individuals: those who only use the subsistence technology, those who borrow but use the capital-intensive technology and those who can use the capital-intensive technology without borrowing. Below we will describe each of these types' portfolio decisions, that is, how they allocate their inherited wealth between investing in the capital intensive technology, saving in the safe storage technology and investing in the risky mutual fund. And as agents' decisions to join each of these three groups is endogenous it is also necessary to describe what determines agents' decisions to become borrowers and lenders. We will treat each of these types in turn below.

First, we will study those individuals who borrow and use the capital-intensive technology. We will assume that all borrowers from the fund are obliged to use their entire bequest x to fund their investment, k, in the technology; the bequest made to an individual is observable by the lenders. However, the fund is unable to observe which project the borrowers use or what the returns to the project actually are. They do have at their disposal a liquidation technology which does not retrieve any of the borrower's assets, but simply ensures that borrowers receive only their subsistence income if the loan is not repaid. Thus the fund liquidates all loans that are not paid back and it is an optimal strategy for all borrowers to repay loans when ever they are able to. The borrowers can choose which technology to use. They prefer technology 1 to technology 2 when

$$(2)\quad \phi[G+y+(1+r)(x-k)]^{\gamma}+(1-\phi)y^{\gamma} > \pi[B+y+(1+r)(x-k)]^{\gamma}+(1-\pi)y^{\gamma}.$$

When $x=k$ this inequality is satisfied (by (1)), however as the left increases faster than the right there exists $\tilde{x}(r)<k$ such that

$$(3)\quad \phi[G+y+(1+r)(\tilde{x}(r)-k)]^{\gamma}-\phi y^{\gamma}=\pi[B+y+(1+r)(\tilde{x}(r)-k)]^{\gamma}-\pi y^{\gamma}.$$

Thus, as in Stiglitz and Weiss (1981), there is credit rationing for all individuals with wealth less than $\tilde{x}(r)$. The fund will not lend to borrowers with a bequest less than $\tilde{x}(r)$, because such borrowers will choose to use the inefficient technology. Only individuals with bequests satisfying $\tilde{x}(r)\le x\le k$ will be borrowers; when their project is successful their bequest is $(1-\alpha)(G+y+(x-k))$ and otherwise it is $(1-\alpha)y$.[4]

The individuals who use only the subsistence technology must decide how much of their bequest to allocate to the mutual fund and how much to keep in the consumption good, given their indirect utility is of the constant relative risk aversion form described above. We will describe the solution to this portfolio problem. The very poorest allocate their entire bequest to the mutual fund and above a threshold level of bequest the investment in the mutual fund is an affine function of the bequest. Let θ denote

[4] The debt contract described here is not optimal, because the borrowers are risk averse. Generally, an optimal contract would offer the borrowers some insurance against the project being unsuccessful, but not complete insurance to prevent the borrower from choosing the bad technology. Such a contract would tend to increase the risk borne by the mutual fund and therefore increase the aggregate risk borne by the savers in this economy. Thus a truly optimal contract would tend to increase the effects ascribed to aggregate risk described in this model.

the proportion of the endowment in the mutual fund. The individual's expected payoff is $\beta\gamma^{-1}[y+x+\theta xr]^{\gamma}+(1-\beta)\gamma^{-1}[y+x-\theta x]^{\gamma}$. Maximising this with respect to θ we find that, provided $0 \leq \theta \leq 1$, the optimal investment decision by individuals using the subsistence technology gives

$$x\theta = \frac{(1-\psi)(y+x)}{1+r\psi}, \qquad \psi = \left(\frac{1-\beta}{\beta r}\right)^{\frac{1}{1-\gamma}}. \tag{4}$$

Thus individuals using the subsistence technology invest an affine function of their endowment in the mutual fund. Provided the expected payoff from the mutual fund is greater than unity, $\beta(1+r) > 1$, it follows that $\psi < 1$ and all individuals using the subsistence technology allocate some of their wealth endowment to the mutual fund: $x\theta > 0$. Moreover, the very poorest individuals will allocate all of their wealth endowment to the mutual fund, that is $\theta = 1$ when

$$x \leq \underline{x} := \frac{1-\psi}{\psi(1+r)}y. \tag{5}$$

The final type in our model are those individuals who are sufficiently wealthy to invest in the capital-intensive technology without borrowing $x > k$. It is always optimal for these individuals to choose the technology with the highest expected return, but they again must decide what proportion of their remaining assets to allocate to the mutual fund. The poorest of these individuals do not invest in the mutual fund, because they are already exposed to a lot of risk in their technology choice. However, as these individuals become richer they allocate more of their inheritance to this fund. Let θ denote the proportion of their endowment (net of their investment costs) that they allocate to the mutual fund. Their expected payoff is

$$\phi[G+y+(x-k)(1+r\theta)]^{\gamma}+(\beta-\phi)[y+(x-k)(1+r\theta)]^{\gamma}+(1-\beta)[y+(x-k)(1-\theta)]^{\gamma}.$$

So when $0 < \theta < 1$ the optimal value of θ, denoted $\theta(x)$, satisfies.

$$\begin{aligned} 0 \;=\;& \phi\left[\frac{G+y}{x-k}+1+r\theta\right]^{\gamma-1} \\ &+(\beta-\phi)\left[\frac{y}{x-k}+1+r\theta\right]^{\gamma-1}-\frac{1-\beta}{r}\left[\frac{y}{x-k}+1-\theta\right]^{\gamma-1}. \end{aligned} \tag{6}$$

The right is decreasing in θ and is negative as $\theta \to 1$ for y small, so a sufficient condition for the individual to allocate a positive proportion of its wealth to the mutual fund is that the right hand side is positive when $\theta = 0$. There exists a threshold level of wealth $\bar{x}$ such that all individuals with bequests $k \leq x \leq \bar{x}$ invest in capital-intensive technology and store the rest of their bequest, that is $\theta(x) = 0$ and they do not invest in the mutual fund. We will have $\bar{x} > k$ when the returns to the mutual fund are positive but not large. To be precise, when $0 < \beta(1+r) - 1 < r\phi$ the threshold, $\bar{x}$, satisfies

$$\left(\frac{\phi r + 1 - \beta(1+r)}{\phi r}\right)^{\frac{1}{1-\gamma}} = \frac{y + \bar{x} - k}{G + y + \bar{x} - k}.$$

When the returns to the mutual fund are positive and sufficiently large $\beta(1+r)-1 > r\phi$, then all of the rich individuals want to invest in the mutual fund $\bar{x} = k$ and $\theta(x) > 0$. In summary we will define the function $\theta(x)$ for $x \in [k, \infty)$ to represent these individuals' asset holdings. This function will satisfy (6) for $x > \bar{x}$ and for $x \in [k, \bar{x}]$ it will have $\theta(x) = 0$. It will be useful to have some bounds on the amount the rich keep in the safe asset. These bounds are described in the following Lemma.

Lemma 1

$$\frac{1-\psi}{1+r\psi}(y+x-k) \geq \theta(x)(x-k) \geq \frac{1-\psi}{1+r\psi}(y+x-k) - \frac{G\psi}{1+r\psi}$$

<u>Proof:</u> See the Appendix.

We will now give sufficient conditions for the existence of credit rationing in this model. That is, we show that all individuals with $0 \leq x \leq k$ prefer to borrow rather than use the subsistence technology and lend some of their assets to the mutual fund. Although individuals' expected income must rise as result of undertaking the risky project, it is not immediately obvious that they strictly prefer to use the capital intensive technology, because this requires them to allocate all of their wealth to the risky technology and none to the safe storage technology. Thus by borrowing to use the risky technology they are forced to make sub-optimal portfolio choices. The conditions for the existence of credit rationing are an upper bound on the interest rate r. When r is sufficiently high it is more attractive to invest in the mutual fund rather than to borrow and use technology 1.

Lemma 2 *When $\phi^{1/\gamma}(G+y) > \beta(1+r)(y+k)$ each individual with $x < k$ prefers technology 1 and borrowing to using the subsistence technology and lending.*

Proof: See the Appendix.

It is clearly preferable to fund an investment in technology from an inheritance rather than borrowing from the mutual fund, this is a sufficient condition for all individuals with inheritance. $x > k$ to use the capital-intensive technology.

2.2 A Dynamic Process for Wealth and Capital Market Equilibrium

The dynamics of the income distribution are determined by the map from the current income distribution to future income distributions. The optimal behaviour of the classes described in the previous section determines their optimal bequest. So this section begins with a formal description of the state-dependent map from current wealth to future wealth in (7) and (8) below, then by writing down the expected evolution of the average level of wealth in (9). This is an incomplete description of the dynamic process for the income distribution, however, because the equilibrium interest rate is a function of the wealth distribution. So, this Section ends by formally describing how the equilibrium interest rate is determined in (11).

In bad states the behaviour in the previous section induces the map (7) from current inheritance x_t to next period's bequest x_{t+1}.

$$x_{t+1} = (1-\alpha)\begin{cases} y, & x_t \leq \underline{x} \\ \psi(1+r_t)(y+x_t)(1+r\psi)^{-1}, & \underline{x} \leq x_t < \tilde{x} \\ y, & \tilde{x} \leq x_t < k \\ y + (1-\theta(x_t))(x_t - k), & k \leq x_t. \end{cases} \tag{7}$$

The map (7) takes each wealth level in period t to a wealth level in period $t+1$. This map thus takes an income distribution $\lambda_t \in \Lambda$ and maps it to an income distribution tomorrow $\lambda_{t+1} \in \Lambda$ conditional on a bad state having occurred. We will define $f : \Lambda \rightarrow \Lambda$ to be the map from today's income distribution to tomorrow's income distribution in the bad states. In good states the map from today's income level to tomorrow's depends upon whether the individual technology used was successful and produced output G or failed and produced nothing.

$$(8) \qquad x_{t+1} = (1-\alpha)\begin{cases} y + (1+r_t)x_t, & x_t \leq \underline{x} \\ (1+r_t)(y+x_t)(1+r\psi)^{-1}, & \underline{x} \leq x_t < \tilde{x} \\ G + y + (1+r_t)(x_t - k), & \text{successful, } \tilde{x} \leq x_t < k \\ y, & \text{fails, } \tilde{x} \leq x_t < k \\ G + y + (x_t - k)(1 + r\theta(x_t)), & \text{successful, } k \leq x_t \\ y + (1 + r\theta(x_t))(x_t - k), & \text{fails } k \leq x_t. \end{cases}$$

This map again induces a map from today's income distribution $\lambda_t \in \Lambda$ to tomorrow's income distribution $\lambda_{t+1} \in \Lambda$ conditional on a bad state having occurred. We will define $F : \Lambda \to \Lambda$ to be this map.

A state of our system at time t is a probability measure $\lambda_t \in \Lambda$. One way of summarising this measure is its mean or average $E_{\lambda_t} x := \int x d\lambda_t$. The expectations, E_{λ_t}, are taken relative to the information available at the start of period t. Tomorrow's distribution of wealth is random from today's point of view, because it depends upon whether a good or a bad state occurred. Similarly, tomorrow's average wealth is random because it depends upon the state. We will let $E_{\lambda_t} x_{t+1}$ denote the expected value of the average wealth tomorrow, where expectations are taken relative to period t's state, from above we can write down the following relation for tomorrow's expected average wealth.

$$\begin{aligned}(9)\, E_{\lambda_t} x_{t+1} \;=\;& (1-\alpha)y + \beta(1+r)(1-\alpha)E_{\lambda_t} x_t \\ & -(1-\alpha)\frac{\psi(1+r)}{1+r\psi}(\beta(1+r)-1)\int_{\underline{x}}^{\tilde{x}} (x - \underline{x}) d\lambda_t \\ & +(1-\alpha)\int_{\tilde{x}}^{k} \phi G - \beta(1+r)k + (\beta - \phi)(1+r)(k - x) d\lambda_t \\ & +(1-\alpha)\int_{k}^{\infty} \phi G - \beta(1+r)k + (1-\theta)(1-\beta(1+r))(x-k) d\lambda_t.\end{aligned}$$

When $\beta = 1$ no bad states occur and (9) describes a deterministic relationship between the average of the current income distribution and its average next period.

At the moment the model is incomplete, because we haven't specified how the interest rate is determined. (Note: both of the maps (7) and (8) are dependent on r.) To start we assume that the capital market is closed and so the equilibrium of the capital market in period t determines the rate of interest r_t as a function of the current wealth distribution λ_t. The supply of capital to the mutual fund comes from the subsistence individuals who invest and the very rich who invest.

$$(10) \qquad S_t(r) := \int_0^{\underline{x}} x_t d\lambda_t + \int_{\underline{x}}^{\tilde{x}} \frac{(1-\psi)(y+x_t)}{1+r\psi} d\lambda_t + \int_k^{\infty} \theta(x_t)(x_t-k)d\lambda_t$$

The function $S_t(r)$ increases in r.[5] As no individual will enter the capital market when it yields a return less than the storage technology we must have $S_t(r) = 0$ when $r+1 < 1/\beta$. However, as $r+1$ approaches $1/\beta$ from above capital may still be in positive supply when individuals are risk neutral. The demand for capital comes from the borrowers

$$D_t(r) := \int_{\tilde{x}}^{k} (k - x_t) d\lambda_t.$$

If there is a positive mass of individuals that want to borrow, $\int_{\tilde{x}}^{k} d\lambda_t > 0$, then demand is a decreasing function of r (as $\tilde{x}$ increases in r). When the rates of interest are low the demand will be positive and Lemma 1 shows that when is r sufficiently high using the risky technology is less attractive than investing in the mutual fund. At this point the demand for capital jumps to zero and the supply of capital jumps up.

The capital market can be in two states: There is a unique interest rate that equates the demand and supply of capital. There is autarky where there is zero demand for capital at any interest rate that savers are willing to supply it. So either there exists a unique interest rate $r_t > (1/\beta) - 1$ such that $D_t(r) = S_t(r)$, or there is zero demand for capital at any interest rate such that $(1+r)\beta > 1$. It will prove useful to re-write the equation for $S_t(r) = D_t(r)$ in the following way

$$(11) \quad E_{\lambda_t} x_t = \frac{\psi(1+r)}{1+r\psi} \int_{\underline{x}}^{\tilde{x}} (x - \underline{x}) d\lambda_t + k \int_{\tilde{x}}^{\infty} d\lambda_t + \int_k^{\infty} (1-\theta(x))(x-k) d\lambda_t.$$

The equations (7), (8) and (11) together with an initial position λ_0 describe a stochastic process for the wealth distribution λ_t. With probability $1-\beta$ a current state $\lambda_t \in \Lambda$ is mapped to a new wealth distribution $f(\lambda_t)$ described by the map (7) and capital market equilibrium (9). With probability β current state $\lambda_t \in \Lambda$ is mapped to a new wealth distribution $F(\lambda_t)$ described by the map (8) and capital market equilibrium (9). The quadruple (λ_0, β, f, F) defines a stochastic process on the state space Λ starting at the initial distribution λ_0.

[5] As $\underline{x}$ decreases in r, $\tilde{x}$ increases in r, ψ decreases in r, $\bar{x}$ decreases in r and $\theta(x)$ increases in r.

We end this section by presenting an equation that combines the capital market clearing condition with the equations for expected growth. The expected change in income is described by (11) subject to the capital market clearing (9). If (9) is used to substitute for $E_t x_t$ in the dynamic equation (11) we get

$$
\begin{aligned}
(12) E_{\lambda_t} x_{t+1} - E_{\lambda_t} x_t \quad = \quad & (1-\alpha)y - \frac{\alpha\psi(1+r)}{1+r\psi} \int_{\underline{x}}^{\tilde{x}} (x - \underline{x}) d\lambda_t \\
& + \int_{\tilde{x}}^{k} (1-\alpha)\phi G - k + (\beta - \phi)(1+r)(1-\alpha)(k-x) d\lambda_t \\
& + \int_{k}^{\infty} (1-\alpha)\phi G - k + \alpha(1-\theta)(k-x) d\lambda_t
\end{aligned}
$$

This will, occasionally, be useful in the calculations below.

3. The Evolution of the Income Distribution

This section starts by providing some discussion on the short-run effects of aggregate uncertainty. We will show that small amounts of aggregate uncertainty have no effect on equilibrium interest rates in poor economies, although it increases the equilibrium interest rate in general. Then we study how the short-run rate of growth is affected by the presence of aggregate risk. In particular we show that "trickle up" growth (when savers are mainly poor and borrowers are mainly rich) is quite weak when there is small amounts of aggregate uncertainty. Whereas, "trickle down" growth (when the reverse is true and savers are generally rich while borrowers are poor) can be strengthened by aggregate risk. It is even possible to find situations where aggregate risk is beneficial for income growth. This partial analysis is followed by two propositions, which show that the stochastic process for wealth distributions described above converges through time to unique limiting behaviour. Thus the Propositions give conditions for there to be no indeterminacy of long run behaviour, this contrasts with the results of Piketty (1997). The second of these Propositions deals with the model when there is no aggregate risk and shows that the unique limiting behaviour is consistent with credit rationing in the limit. This contrasts with the result of Aghion and Bolton (1997), which requires no credit rationing when $r = 0$ to get unique limiting behaviour.

First we show that the equilibrium interest rate does not increase as the level of aggregate uncertainty reduces (β increases). However, for small levels of aggregate risk in poor economies the interest rate is independent of the level of aggregate uncertainty. The demand for capital at a given state λ_t is always independent of aggregate uncertainty, because the amount of credit rationing, $\tilde{x}$, depends on the idiosyncratic risk of projects, but not on the level of aggregate risk. It is the idiosyncratic risk that affects agent's choice between projects and the agent's adverse selection problem not the aggregate risk. The capital demand schedule is a decreasing function of the interest rate, because the amount of credit rationing increases as r increases. The supply of capital, at a given interest rate, does depend on the amount of aggregate uncertainty. Investors will tend to reduce the supply of savings to the mutual fund as the level of uncertainty increases, thus the capital supply curve shifts inwards as uncertainty rises while the upward sloping demand curve stays fixed. Consequently, the equilibrium rate of interest cannot fall. The exception to this arises when the economy is poor and the aggregate risk is small. In this case the subsistence class are the only suppliers of capital and are unable to adjust their portfolio of assets optimally because of credit rationing. A large part of their income is certain, so they are willing to allocate their entire inheritance to the mutual fund provided the level of aggregate risk is small. The supply of capital to the mutual fund will, therefore, also be independent of the level of aggregate risk when there are no rich individuals and aggregate risk is small. As both the demand and supply of funds are independent of β in this case so is r. This discussion is now summarised in the following Lemma.

> LEMMA 3 *The equilibrium interest rate, r, does not increase when aggregate uncertainty decreases(β increases). There exists a $\bar{\beta} < 1$, such that if $\int_k^\infty d\lambda_t = 0$ the equilibrium interest rate is independent of β, for $\beta \in [\bar{\beta}, 1]$.*

Proof: See the Appendix.

The conditions in this Lemma are also sufficient for the rate of interest to be independent of the amount of aggregate risk. If there are any individuals with wealth greater than k, or any members of the subsistence class who do not allocate all of their wealth to the capital market, then an increase in aggregate risk will push the interest rate down.

The next Lemma says that in poor economies the expected rate of growth is lower in the presence of small amounts of aggregate uncertainty, or when the investment projects are highly correlated, $\beta \to \phi$. For the reasons described above, the interest rate is independent of small amounts of risk in poor economies, so when aggregate risk increases the interest rate does not alter and its only effect is to increase the probability that the subsistence investors lose their savings. Thus aggregate risk slows growth through its effects on the growth of savings of the subsistence class. When investment projects are highly correlated the positive effect on growth of higher asset returns (r) is not present, because the only benefit from higher interest rate on growth occurs in in which fail although the state is good, see (9). In other cases, however, the effects on growth are ambiguous; it is possible that aggregate uncertainty is beneficial to income growth in the short run.

LEMMA 4 *Assume that* $\int_k^\infty d\lambda_t = 0$ *and* $\phi(1-\alpha)G \geq k$*:* (1) *When* $\beta \in [\bar{\beta}, 1]$ *then expected short-run growth is strictly bounded away from zero,* $\Delta_t := E_{\lambda_t} x_{t+1} - E_{\lambda_t} x_t > (1-\alpha)y$*, and is increasing in* β*.* (2) *When* $\lambda(t)$ *has a continuous density and projects are highly correlated (*$\beta \to \phi$*), then* Δ_t *is increasing in* β*.*

PROOF: See the Appendix.

A small amount of risk can be beneficial for income growth when there is no credit rationing. Lemma 5 shows that in sufficiently rich economies, where there is enough capital for every low income individual, aggregate uncertainty raises income growth. When there is no credit rationing the rich supply the funds for the poor borrowers, an increase in aggregate risk leads the rich to demand a higher rate of interest on these loans. Consequently, aggregate uncertainty (when there is no credit rationing) leads to a re-distribution of wealth from borrowers to savers. This redistribution is favourable, when aggregate risk is small.

LEMMA 5 *If there is no credit rationing in the state* λ_t *and average wealth is high and* λ_t *has a continuous density, then a decrease in aggregate* *risk (increase in β) decreases the expected average income next period ($E_{\lambda_t} x_{t+1}$).*

Proof: See the Appendix.

To quantify the effect of aggregate risk on the long-run behaviour of this model is difficult, because the long-run behaviour of the model is described by a distribution over Λ. We will show that the limiting behaviour of the income distribution λ_t is *independent* of the initial income distribution and that the rate of convergence to this limiting behaviour is exponential. Thus the long-run behaviour of the income distribution is unaffected by the initial state of the system. That is not to say, however, that the income distribution or the rate of interest becomes constant as time passes. This could never be the case in a model where there is aggregate uncertainty that continues to shock the wealth distribution. Instead there is a stationary distribution of the states $\lambda_t \in \Lambda$ and each realisation of the stochastic process converges exponentially fast to this stationary distribution. The convergence of the states λ_t to a stationary distribution is not sufficient, however, to show that individuals within the economy have equal chances of being rich and being poor. In general, it is possible for the states λ_t converge to behaviour that is independent of the initial distribution, but for individual's incomes within the distribution to depend on the initial condition. (For example, if the individual with the lowest income always had the lowest income.) Our proof will show that not only will the long-run behaviour of λ be independent of the initial state, but so will individuals incomes within the distribution be independent of their initial position. We will treat the case with aggregate shocks $\beta < 1$ and without aggregate shocks $\beta = 1$ separately as somewhat different methods are needed for each of these cases. Of course, when $\beta = 1$ the law of large numbers implies that in the aggregate the system is deterministic and in this case the system does converge to a unique income distribution. The rate of convergence is again exponentially fast.

We will be able to show that the nature of the convergence is stronger than that described in Aghion and Bolton (1997) and Piketty (1997). The convergence of the income distributions will be in the strong topology rather than the weak topology.[6]

[6]For descriptions of the strong and weak topology on distributions see Stokey and Lucas (1989)

If λ, λ' are two income distributions in Λ we will measure the distance between these distributions as

$$\|\lambda - \lambda'\| := 2 \sup_{B \in \mathcal{B}} |\lambda(B) - \lambda(B')|.$$

Thus the distance between two distributions is an upper bound on the difference between the proportion of the population in any (Borel) set of incomes.[7] The distance measure above can be used to define open sets in the state space Λ, we will let $\mathcal{L}$ denote the Borel sigma-algebra generated by these open sets.

Proposition 1 establishes the convergence for the case where bad states occur with positive probability $\beta < 1$. The proof relies on the *coupling* of stochastic processes to achieve its conclusions. This is a simple and intuitive approach to proving limit theorems for Markov processes and is explained in Grimmett and Stirzaker (1982), for example. The basic idea behind the proof is to show that irrespective of the initial distribution the future behaviour of the system must eventually be identical. The first step in our proof is to show that there is a finite number N, such that if there are N consecutive bad states the entire income distribution is concentrated at the point $(1-\alpha)y$. To establish this we show that all individuals must have wealth less than k after a finite number of periods, because successive failures of the capital intensive technology will eventually destroy the richest generations asset stock. The richest individuals borrow and the poorest lend, once all individuals have insufficient wealth to finance the capital intensive project without borrowing. We show that individuals with inherited wealth $(1-\alpha)y$ use the subsistence technology and invest all of their wealth in the mutual fund, so their savings are constantly being destroyed and can never leave a bequest of more than $(1-\alpha)y$. The borrowers also end up at the lowest wealth after one bad state, thus the stock of people at the lowest inherited wealth level grows and includes the entire population in a finite number of periods. Once this first step is established it follows that for any two initial income distributions there is a probability β^{2N} (the probability that they both have N successive bad states) that after N periods they are both concentrated at $(1-\alpha)y$. Once they are both concentrated at $(1-\alpha)y$ the future evolution of these distributions must be identical

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[7] This metric induces a complete topology on the state space Λ.

because they have both started from the same point. This implies there is a probability $0 < 1 - \beta^{2N} < 1$, that after N periods the future evolution of the income distribution is not identical. After MN periods, therefore, there is a probability $(1 - \beta^{2N})^M$ that the distributions of the income distributions are not identical. As M tends to infinity there is a zero probability that the income distributions are not identical! This argument shows that, independent of the initial distribution, ultimately all income distributions must be evolving in an identical fashion and that the rate of convergence is exponential. Three assumptions are necessary for this argument to work. The first is that bad states do not occur with probability greater than one half, which ensures the capital market continues to open. The second is that the bequest of an individual with one period's subsistence income is insufficient to finance a capital purchase. The final condition ensures that all individuals with inherited wealth $(1-\alpha)y$ put all their inheritance in the mutual fund.

Conditional on an initial state, λ_0 say, the state of the system at time t can take a finite number of values, which are determined by the sequence of good and bad states that actually occur. In the Proposition below we will use the expression E_0 to denote expectations taken over the stochastic process governing good and bad states for a given initial state. Thus $E_0\|\lambda_t - \lambda'_t\|$ is the average distance (using the strong topology) between the states at time t for two different initial conditions.

PROPOSITION 1 *Assume* $1 > \beta > 1/2$, $(1-\alpha)y < k$ *and*

$$\frac{k - y(1-\alpha)}{\beta - (1-\alpha)(1-\beta)(1-\gamma)} < \frac{\phi G - \pi B}{\phi - \pi}.$$

Let λ_t *be the state of the stochastic process* (λ_0, β, f, F) *at time* t *and let* λ'_t *be the state of the stochastic process* $(\lambda'_0, \beta, f, F)$ *at time* t*, for* $\lambda_0, \lambda'_0 \in \Lambda$*. Then* $E_0\|\lambda_t - \lambda'_t\| \to 0$*.*

Proof: See the Appendix.

The proof of the previous proposition relies heavily on there being bad states, so it would be reasonable to wonder whether a similar result also holds when there are no bad states. This is now proved for the case where $\beta = 1$. In this proof more assumptions are necessary to establish that the long-run behaviour of the system is independent of the initial state. The proof proceeds in two stages (as in Aghion and Bolton 1997). In the first stage the rate of interest is shown to fall to zero in finite time and capital is in permanent excess supply. It is this part of the proof that requires extra assumptions, because at some initial conditions it is possible that that the economy does not grow sufficiently quickly to drive interest rates to zero (this issue is studied in Picketty 1997). Thus it is necessary to ensure that the returns to the investment project are sufficiently large for growth to be self-sustaining. Once this has happened the wealth distribution evolves according to a linear Markov process and for any two dynasties with initial levels of wealth, x and x' say, there is a positive probability, $\omega > 0$, that after n periods both of these dynasties have been mapped to $(1-\alpha)y$. This requires all their investment projects to be successful initially, so both of the dynasties' wealth levels converge to close to the maximum, then a sequence of unsuccessful projects until they are both borrowers with an unsuccessful project. An argument based on coupling can then be used, because once the dynasties have been simultaneously mapped to the wealth level $(1-\alpha)y$ the future distributions of wealth for these two dynasties are identical (by the Markov structure). For any two individuals there is a fixed probability $\omega > 0$ that their their successors' wealth levels have identical distributions after n periods. It follows that any two individuals eventually have identical wealth distributions, and that convergence exponentially fast to this limit. We define $\tilde{x}_0$ to be the level of $\tilde{x}$ when $r = 0$. The proposition does not assume that there is no credit rationing in when interest rates fall to zero, $\tilde{x}_0 < 0$. Instead it assumes that $\tilde{x}_0 < (2-\alpha)(1-\alpha)y$ which is sufficient for the subsistence class to acquire enough savings in one life time even when the interest rate is zero. If this assumption fails eventually the economy may get stuck at a position where everyone is in the subsistence class.

PROPOSITION 2 *Suppose* $\beta = 1$, $(1-\alpha)(2-\alpha)y > \tilde{x}_0$, $(1-\phi)^2 k < \phi(1-\alpha)y$ *and* $(1-\alpha)\phi G - (2-\phi)k > 0$, *then:* $r_t \to 0$ *in finite time with probability one. For any* $\lambda_0, \lambda'_0 \in \Lambda$ *then* $\|\lambda_t - \lambda'_t\| \to 0$ *when* λ_t (λ'_t) *is the state of the process at time* t *when it starts in state* λ_0 (λ'_0).

Proof: See the Appendix.

This result establishes a minor extension of the convergence result of Aghion and Bolton (1997). In their model the interest rate falls to zero and the resultant linear Markov process converges to a unique ergodic distribution. In their paper it is essential for there to be no credit-rationing at zero interest rates, because otherwise there is a non-monotone map from current wealth to future bequests and the results of Hopenhayn and Prescott (1992) do not apply. The above Proposition shows that even if credit rationing continues to hold at zero interest rates, there is still convergence to a unique limiting distribution. Thus the presence of credit rationing can be consistent with unique long-run behaviour. This contrasts with Piketty (1997) who derives multiple ergodic distributions for income when for a model with long-run credit rationing.

4. World Capital Markets

In the previous section we studied the evolution on the income distribution under the assumption that the capital market in this economy was closed. Now we consider a different extreme case — where the economy has an open capital market and is small relative to the world capital market.

We will model the rest of the world as being made up of a very large number of risk neutral individuals who are willing to supply infinite capital at any positive rate of return. We assume, therefore, that the rest of the world has developed to a situation where there are no credit constraints, there is an excess supply of capital and equilibrium in the world capital market occurs at a zero rate of interest. Given this assumption, the opening of the domestic capital market will lead to foreign investors driving down the equilibrium domestic rate of interest, so that the mutual fund gives a zero expected rate of return, that is, $\beta(1+r) = 1$. The assumption of risk neutrality for the participants in the world capital market is not necessary for there to be an infinite supply of capital at any positive rate of return. There would still be an arbitrarily large supply of capital at any positive rate of return provided we assume our economy is small relative to the rest of the world and that the aggregate risk in our economy *is not* perfectly correlated with any risk experienced by the rest of the world. In this case diversification of portfolios by sufficiently many risk averse individuals will generate an arbitrarily large supply of capital to the small economy at any positive expected rate of return.

The individuals in our small economy are also able to participate in the capital markets. These individuals (potentially) have three different assets they can invest in: the storable commodity, the domestic capital market, the world capital market. In fact, however, the world capital market and the storable commodity both guarantee the same rate of return, because equilibrium in the world capital market drives the world rate of interest down to zero. All individuals in the small economy will avoid investing directly in their small economy, because they are risk averse and the mutual fund is a risky investment with the same rate of return as the storable commodity. Thus opening the capital market leads to domestic savers transferring all of their savings from the domestic capital market to the world capital market (or the storable commodity) — a rudimentary form of capital flight. This change in savings is driven by large world capital inflows that crowd out domestic savings.

The above assumptions on the world capital market have the following implications for the parameters of our model: the poor invest no assets in the mutual fund ($\underline{x} = 0$ and $\psi = 0$), the rich never supply any of their assets to the mutual fund ($\theta(x) = 0$), the rate of interest at which borrowers can obtain funds satisfies $\beta(1+r) = 1$, the threshold level of income at which credit rationing applies, $\tilde{x}$, is determined by this rate of interest and we will denote this $\hat{x} \leq \tilde{x}$ for all $1+r \geq \beta^{-1}$. One element of the aggregate uncertainty has disappeared, because no small individual is willing to allocate any of its portfolio to the mutual fund. There is no aggregate uncertainty in the returns to saving, however, there is still some aggregate uncertainty because the the output of the economy is correlated in the bad states. The maps analogous to (7) and (8) in the open economy case are given below. In bad states the map from current bequests x_t to next period's bequest, x_{t+1}, is

$$x_{t+1} = (1-\alpha) \begin{cases} y + x_t, & x_t < \hat{x} \\ y, & \hat{x} \leq x_t < k \\ y + x_t - k, & k \leq x_t \end{cases} . \tag{13}$$

In good states the map from today's income is

$$(14) \qquad x_{t+1} = (1-\alpha)\begin{cases} y + x_t, & \underline{x} \le x_t < \hat{x} \\ G + y + \beta^{-1}(x_t - k), & \text{successful, } \hat{x} \le x_t < k \\ y, & \text{fails, } \hat{x} \le x_t < k \\ G + y + x_t - k, & \text{successful, } k \le x_t \\ y + x_t - k, & \text{fails } k \le x_t \end{cases}.$$

When world capital markets are open $\hat{x}$ is independent of the income distribution and so these maps are linear in the income distribution.

The effects of opening the capital markets to the world are not clear cut — there are different effect for different parts of the income distribution. The net gainers are the middle classes, the net losers are the poor and the rich may also lose. The people who unambiguously gain from the easier conditions for borrowing are the middle class who do not save at all. These pay a lower rate of interest for their loan when capital markets are opened. A second group who benefit from the opening of capital markets are individuals who can borrow at the lower world interest rate, although in the closed capital market they used the subsistence technology and were credit rationed at the higher equilibrium rate of interest. The individuals who lose when capital markets are opened are those who must make savings decisions. A consequence of opening of capital markets is the driving down of the domestic interest rate to the world level, so no risk averse saver is willing to allocate any of her savings to the mutual fund. Thus savers in the domestic economy have lost an asset that they could previously have included in their portfolio. All savers must, therefore, be unambiguously worse off as a consequence of the opening of the capital markets. Those on subsistence incomes are worse off, so are the rich individuals who would have placed their savings on the domestic capital market. Moreover, individuals are worse off in terms of utility and in terms of expected income (because savers expected savings income is certain when capital markets are open so they will only use a risky portfolio in preference to this if it ensures them a higher expected return as well as higher expected utility). Thus the incomes of the poor and of the rich grow less quickly when capital markets are opened.

We will now present a result which compares the rate of growth when there are open capital markets with the rate of growth when there are closed capital markets.

In particular we show that growth can be less when the capital markets are opened up. The Proposition below gives a sufficient and necessary condition for opening capital markets to slow the rate of growth. The Proposition then describes states where opening capital markets does harm growth. Opening asset markets generally has two opposing effects on growth which are described above. The effect on growth of opening capital markets is unambiguous in states where there is no credit rationing in the closed economy. In this case, the beneficial effect on growth of opening asset markets is considerably weakened, because there are no new borrowers that arise when the lower world interest rates prevail. In fact we are able to show that the effect on expected growth is unambiguously bad. By continuity it follows that when there is little credit rationing, or when relatively few new borrowers are created by opening credit markets, then opening credit markets is not good for growth. This first set of states roughly corresponds to models where there are large number of rich individuals, capital is plentiful, and it is the poor who are borrowing from the rich. So the Proposition shows that trickle-down growth is weaker in open capital markets than in closed capital markets. We also show that when projects are highly correlated and there is a lot of aggregate risk, opening credit markets is beneficial to average growth. The reason is that this provides the economy with a lot of insurance against

PROPOSITION 3 *The opening of capital markets slows expected income growth at the income distribution λ_t if and only if*

$$(15)\quad (\beta(1+r)-1)\int_{\tilde{x}}^{k}(k-x)d\lambda_t - \int_{\hat{x}}^{\tilde{x}}\left(\frac{\beta(\phi G-k)}{\beta-\phi}+k-x\right)d\lambda_t > 0.$$

This implies that: (1) growth is always faster with closed capital markets when there is no credit rationing, (2) growth is slower in closed capital markets when projects are highly correlated $\phi \to \beta$.

Proof : See the Appendix.

5. Conclusion

Aggregate risk in dynamic models of the income distribution impacts on agents capital supply decisions, but makes ergodic growth more likely. To convincingly make a case for non-ergodic growth we must consider models with aggregate shocks.

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Appendix

PROOF OF LEMMA 1

From (6) we have

$$\psi^{1-\gamma}[y+(x-k)(1-\theta)]^{\gamma-1} = \frac{\phi}{\beta}[G+y+(1+r\theta)(x-k)]^{\gamma-1}+\frac{\beta-\phi}{\beta}[y+(1+r\theta)(x-k)]^{\gamma-1}.$$

The two terms in square brackets are upper and lower bounds for the left as the right is a convex combination. By rearranging these upper and lower bounds we get the following bounds on θ.

$$\frac{y(\psi-1)}{x-k}+\psi-1 \leq -\theta(1+\psi r) \leq \psi-1+\frac{G\psi}{x-k}+\frac{(\psi-1)y}{x-k}.$$

A further rearrangement is then needed to establish the result. *Q.E.D.*

PROOF OF LEMMA 2

The expected utility from borrowing is $\phi[G+y+(1+r)(x-k)]^\gamma+(1-\phi)y^\gamma$. The expected utility from using the subsistence technology is $\beta(y+x)^\gamma(1+r)^\gamma(1+r\psi)^{1-\gamma}$, when $x>\underline{x}$. When $x\in[0,\underline{x}]$ this over estimates the utility obtained from using the subsistence technology, because these individuals are constrained in their portfolio decisions. Thus, for all $x\in[0,k]$ a sufficient condition for borrowing to be better than subsistence is

$$\text{(A.1)}\qquad \phi[G+y+(1+r)(x-k)]^\gamma \geq \beta(1+r\psi)\left(\frac{(y+x)(1+r)}{1+r\psi}\right)^\gamma, \qquad x\in[0,k].$$

Raise both sides to the power $1/\gamma$, then the left and the right of (A.1) are linear functions of x, and the slope of the function on the left is less than the slope of the function on the right (as $\phi<\beta$ and $\psi>0$). A necessary and sufficient condition for (A.1) is, therefore, found by setting $x=k$ in (A.1). Some rearranging of (A.1) with $x=k$ gives the condition

$$\phi^{1/\gamma}(G+y)\geq\beta(y+x)(1+r)(\beta+(1-\beta)\psi^\gamma)^{\frac{1-\gamma}{\gamma}},$$

and as $\psi,\gamma,\beta\in[0,1]$ the condition in the Lemma is sufficient for this. *Q.E.D.*

PROOF OF LEMMA 3

The proof that r does not increase in β follows from the discussion, so it remains to prove the rest of the Lemma. Since $\lim_{\beta\to 1}\underline{x}=\infty$ and $\tilde{x}\leq k$ is independent of β, for all states λ_t there is an interval $\bar{\beta}\leq\beta\leq 1$ with strictly positive Lebesgue measure such that $\underline{x}\geq k\geq\tilde{x}$. When $\underline{x}>\tilde{x}$ and $\int_k^\infty d\lambda_t=0$ the condition for credit market equilibrium is

$$E_{\lambda_t}x_t=k\int_{\tilde{x}}^\infty d\lambda_t.$$

Both sides of this are independent of β, so the equilibrium value of r is too. *Q.E.D.*

PROOF OF LEMMA 4

Part (1): When $\lambda_t([k,\infty)) = 0$ (12) reduces to

$$\begin{aligned} E_{\lambda_t}x_{t+1} - E_{\lambda_t}x_t &= (1-\alpha)y - \frac{\alpha\psi(1+r)}{1+r\psi}\int_{\underline{x}}^{\tilde{x}}(x-\underline{x})d\lambda_t \\ &\quad + \int_{\tilde{x}}^{k}(1-\alpha)\phi G - k + (\beta-\phi)(1+r)(1-\alpha)(k-x)d\lambda_t. \end{aligned}$$

When $\beta \in [\underline{\beta}, 1]$ (and $\tilde{x} < \underline{x}$) the second term on the right vanishes. As $(1-\alpha)G\phi > k$, the right is strictly larger than $(1-\alpha)y$ and increases in β, as r and $\tilde{x}$ are constant.

Part 2: When $\lambda_t([k,\infty)) = 0$ capital market equilibrium implies.

$$\frac{\alpha\psi(1+r)}{1+r\psi}\int_{\underline{x}}^{\tilde{x}}(x-\underline{x})d\lambda_t = \alpha\left(E_{\lambda_t}x_t - k\int_{\tilde{x}}^{k}d\lambda_t\right).$$

If this is substituted into (9) when $\lambda_t([k,\infty)) = 0$ we get

$$\frac{E_{\lambda_t}x_{t+1}}{1-\alpha} = y + E_{\lambda_t}x_t + \int_{\tilde{x}}^{k}\phi G - k + (\beta-\phi)(1+r)(k-x)d\lambda_t.$$

λ_t has a continuous density function (say $f(.)$), so we can differentiate this with respect to β.

$$\begin{aligned} \frac{\partial E_t x_{t+1}}{\partial\beta} &= (1-\alpha)\int_{\tilde{x}}^{k}(k-x)\left(1+r+(\beta-\phi)\frac{\partial r}{\partial\beta}\right)d\lambda_t \\ &\quad -\frac{\partial r}{\partial\beta}\frac{\partial\tilde{x}}{\partial r}(1-\alpha)(\phi G - k + (1+r)(\beta-\phi)(k-\tilde{x}))f(\tilde{x}) \end{aligned}$$

As $\frac{\partial r}{\partial\beta} \leq 0$ and $\frac{\partial\tilde{x}}{\partial r} > 0$ the second term on the right is positive. The whole of the right is positive provided the term in the integral is positive, which will be true for β sufficiently close to ϕ. *Q.E.D.*

Proof of Lemma 5

In state λ_t there is no credit rationing so $\tilde{x} \leq (1-\alpha)y$ and (11) becomes

$$E_{\lambda_t}x_t = k + \int_k^{\infty}(1-\theta(x))(x-k)d\lambda_t.$$

By differentiating this with respect to β we get

(A.2)
$$\frac{\partial r}{\partial\beta} = -\frac{\int_k^{\infty}\frac{\partial\theta(x)}{\partial\beta}(x-k)d\lambda_t}{\int_k^{\infty}\frac{\partial\theta(x)}{\partial r}(x-k)d\lambda_t}.$$

There is no credit rationing, so (9) can be re-written as

$$\begin{aligned}\frac{E_{\lambda_t} x_{t+1}}{1-\alpha} &= y + \phi G - \phi \int_0^k (1+r)(k-x)d\lambda_t + \int_k^\infty (x-k)(1-\theta+\theta\beta(1+r))d\lambda_t \\ &= y + \phi G + \int_k^\infty (x-k)d\lambda_t + [(\beta-\phi)(1+r)-1]\int_0^k (k-x)d\lambda_t.\end{aligned}$$

(Capital market equilibrium implies $\int_0^k (k-x)d\lambda_t = \int_k^\infty \theta(x-k)d\lambda_t$ and this gives the second line.) When λ_t has a continuous density the above can be differentiated with respect to β.

$$\frac{\partial E_t x_{t+1}}{\partial \beta} = (1-\alpha)\int_0^k (k-x)d\lambda_t \left[1 + r + (\beta-\phi)\frac{\partial r}{\partial \beta}\right]$$

When there is no credit rationing the effect of β on growth depends on the sign of $1 + r + (\beta - \phi)\frac{\partial r}{\partial \beta}$. The derivative $\frac{\partial r}{\partial \beta}$ is negative, as an increase in β shifts capital supply outwards, so the effect is ambiguous.

By Lemma 1 as $x - k \to \infty$ so $\theta(x) \to (1-\psi)/(1+r\psi)$. For x large we can approximate $\theta(x)$ by $(1-\psi)/(1+r\psi)$ which is independent of x. We can, therefore, approximate $\frac{\partial r}{\partial \beta}$, using (A.2), by

$$\frac{\partial r}{\partial \beta} \approx -\frac{\frac{\partial \theta}{\partial \beta}}{\frac{\partial \theta}{\partial r}}, \qquad \text{where} \qquad \theta = \frac{1-\psi}{1+r\psi}.$$

Some elementary calculus gives

$$\begin{aligned}\frac{\partial r}{\partial \beta}(\beta-\phi) + 1 + r &\approx \frac{-r(1+r)}{\beta(1-\beta)[1+r(1-(1-\gamma)\psi)]}(\beta-\phi) + 1 + r \\ &= \frac{(1+r)\{\beta(1-\beta) + r\phi - r\beta[\beta + (1-\beta)(1-\gamma)\psi]\}}{\beta(1-\beta)\{1+r[1-\psi(1-\gamma)]\}}.\end{aligned}$$

This is negative when β is close to unity. *Q.E.D.*

<u>PROOF OF PROPOSITION 1</u>

Let λ be an initial state. We will first show that there exists a finite integer N, independent of λ such that N consecutive bad states concentrates the income distribution $f^N(\lambda)$ at a point mass at $(1-\alpha)y$.

Given $(1-\alpha)y < k$ there exists a finite N' such that no individual has wealth greater than k after N' consecutive bad states. By (7) no individual can move from

$x_t \leq k$ to $x_{t+1} > k$ in a bad state when: $(1-\alpha)y < k$ and

$$\frac{\psi(1+r)}{1+r\psi}(y+k) < k.$$

(A sufficient condition for the equation above is $\underline{x} < k$ which is true provided y is small.) Therefore, it is sufficient to show that after N' consecutive bad states all individuals with income greater than k are mapped to incomes less than k. For $x_t > k$ the bequest in bad states is $x_{t+1} = (1-\alpha)(y+(1-\theta(x_t))(x_t-k)) \leq (1-\alpha)(y-k+x_t)$. By iterating we get $x_{N'} \leq (y-k)(1-(1-\alpha)^{N'})(1-\alpha)\alpha^{-1} + (1-\alpha)^{N'}x_0$. A sufficient condition for no individual to have wealth greater than k after N' bad states is

$$k > (y-k)(1-(1-\alpha)^{N'})(1-\alpha)\alpha^{-1} + (1-\alpha)^{N'}X,$$

for N' finite and a sufficient condition for this is $k > (1-\alpha)y$.

When no individual has wealth greater than k there are a finite number, N'', of consecutive bad states before the income distribution is concentrated at $(1-\alpha)y$. The first step is to show that all individuals with the lowest wealth put all their savings into the mutual fund, $\underline{x} > (1-\alpha)y$. From (5) an equivalent condition for this is $1 > \psi(1+(1+r)(1-\alpha))$, and as the right is decreasing in r this describes a lower bound on the interest rate. The value $\tilde{x}$ is adjusted to equate demand and supply of capital to the mutual fund. $\tilde{x} \geq (1-\alpha)y$ when there are no individuals with $x > k$, because otherwise every individual with $x \in [0,k)$ wishes to borrow and there is no supply of capital. But $\tilde{x} \geq (1-\alpha)y$ implies that individuals with $x = (1-\alpha)y$ do not satisfy the strict credit-rationing condition (2).

$$\phi[G+y+(1+r)(y(1-\alpha)-k)]^\gamma + (1-\phi)y^\gamma \leq \pi[B+y+(1+r)(y(1-\alpha)-k)]^\gamma + (1-\pi)y^\gamma$$

This defines another lower bound on the interest rate. We will show that if r satisfies this second lower bound then $\underline{x} > (1-\alpha)y$. By monotonicity, it is sufficient to show that $1 = \psi(1+(1+r)(1-\alpha))$ implies

$$G+y+(1+r)(y(1-\alpha)-k) > \left(\frac{\pi}{\phi}[B+y+(1+r)(y(1-\alpha)-k)]^\gamma + (1-\frac{\pi}{\phi})y^\gamma\right)^{1/\gamma}.$$

It is, therefore, sufficient to show that

$$1 = \psi(1+(1+r)(1-\alpha)) \implies G+(1+r)(y(1-\alpha)-k) > \frac{\pi}{\phi}[B+(1+r)(y(1-\alpha)-k)],$$

or

$$\text{(A.3)} \qquad 1 = \psi(1 + (1+r)(1-\alpha)) \implies (1+r)(k - y(1-\alpha)) < \frac{\phi G - \pi B}{\phi - \pi}.$$

Since the interest rate that solves $1 + (1-\alpha)(1+r) = \psi^{-1} = [r\beta/(1-\beta)]^{1/(1-\gamma)}$ must be smaller than $(\beta - (1-\alpha)(1-\beta)(1-\gamma))^{-1} - 1$, a sufficient condition for (A.3) is

$$\frac{k - y(1-\alpha)}{\beta - (1-\alpha)(1-\beta)(1-\gamma)} < \frac{\phi G - \pi B}{\phi - \pi}.$$

This is true by assertion.

The final step is to show that when everyone has wealth less than k and $\underline{x} > (1-\alpha)y$ then there are a finite number of bad states before the income distribution is concentrated at $(1-\alpha)y$. As $\underline{x} > (1-\alpha)y$ all individuals with wealth in $[0, \underline{x}]$ and $[\tilde{x}, k]$ are mapped to $(1-\alpha)y$ in bad states, by (7). Once at wealth level $(1-\alpha)y$ they stay there. To prove the final step it is sufficient to show that there are only a finite number of periods when $\tilde{x} > (1-\alpha)y$, because when $(1-\alpha)y = \tilde{x}$ a bad state implies the bequest distribution is concentrated at $(1-\alpha)y$. Suppose that $\tilde{x} > (1-\alpha)y$, and all the population have inheritances in the interval $[(1-\alpha)y, k]$. For a given supply, S, of capital the number of borrowers ω is minimised by assuming that all borrowers have wealth $(1-\alpha)y$ so $\omega \geq S[k - (1-\alpha)y]^{-1}$. The supply of capital S is minimised by assuming that all savers have wealth $(1-\alpha)y$, so $S \geq (1-\omega)(1-\alpha)y$. Eliminating S from these inequalities gives a lower bound on the number of borrowers $\omega \geq (1-\alpha)y/k$. Thus after one bad state there is a strictly positive measure $(1-\alpha)y/k$ of the population with the inheritance $(1-\alpha)y$. The number of people with the inheritance $(1-\alpha)y$ cannot fall in bad states, so $(1-\alpha)y/k$ is a lower bound on the proportion of the population with inheritance $(1-\alpha)y$. These individuals supply capital each period (as $\tilde{x} > (1-\alpha)y$); a lower bound on the capital supply is $(1-\alpha)^2y^2/k$. There must, therefore, be a fraction of $\delta > (1-\alpha)^2y^2k^{-1}[k-(1-\alpha)y]^{-1}$ in the population who are borrowing the funds. δ is strictly bounded from zero and a proportion of at least δ are mapped to $(1-\alpha)y$ after each bad state. This can happen for only a finite number of times, so $(1-\alpha)y = \tilde{x}$ in a bounded and finite number of periods.

From above, there exists a finite number, N say, such that after N bad states any income distribution is mapped to Dirac distribution at $(1-\alpha)y$. The probability of there not being N consecutive bad states is $1-\beta^N$, so in a sequence of $t > nN$ periods the probability of there not being N consecutive bad states is at most $(1-\beta^N)^n$. The probability that $\{\lambda_t\}$ and $\{\lambda'_t\}$ do not have N periods where they both have bad states is at least $(1-\beta^{2N})^n$. Thus

$$E_0\|\lambda_t - \lambda_t\| \leq (1-\beta^{2N})^n 1, \qquad t > Nn;$$

because $\|.\| \leq 1$ and once both process have had N consecutive bad states in the same periods $\|\lambda_t - \lambda'_t\| = 0$. Letting t (and therefore n) tend to infinity proves the proposition. *Q.E.D.*

<u>Proof of Proposition 2</u>

If $\beta = 1$ then $\underline{x} = \tilde{x}$ and $\theta(x) = 1$ for all x. The condition for equilibrium in the capital market (11) is

$$E_{\lambda_t} x_t = k \int_{\tilde{x}}^{\infty} d\lambda_t. \tag{A.4}$$

The equality (A.4) can only be satisfied if $E_{\lambda_t} x_t \leq k$. (If (A.4) does not hold, then the supply of capital exceeds the demand at any positive interest rate and $r = 0$.) When the capital market clears equation (12) holds, so

$$E_{\lambda_t} x_{t+1} - E_{\lambda_t} x_t = (1-\alpha)[y + (1+r)(1-\phi)\int_{\tilde{x}}^{k}(k-x)d\lambda_t] + ((1-\alpha)\phi G - k)\int_{\tilde{x}}^{\infty} d\lambda_t. \tag{A.5}$$

No aggregate uncertainty implies that tomorrow's average income is known with certainty today, $E_{\lambda_{t+1}} x_{t+1} = E_{\lambda_t} x_{t+1}$. (A.5), therefore, implies that average income grows by at least $(1-\alpha)y$ each period that capital markets clear. There can be at most $T = ky^{-1}(1-\alpha)^{-1}$ successive periods when the capital market clears, average income grows by $(1-\alpha)y$ and $E_{\lambda_t} x_t \leq k$, since after T periods average income must be greater than k which contradicts the condition for capital markets clearing. Let t denote a period where there is excess supply of capital, so $r_t = 0$ and $E_{\lambda_t} x_t > k \int_{\tilde{x}_0}^{\infty} d\lambda_t$ where $\tilde{x}_0$ denotes the level of credit rationing at zero interest rates. The next period's average wealth is described by (9) with $r = 0$

$$E_{\lambda_t} x_{t+1} = (1-\alpha)\left(y + E_t x_t + (1-\phi)\int_{\tilde{x}_0}^{k}(k-x)d\lambda_t + \int_{\tilde{x}_0}^{\infty} \phi G - k d\lambda_t\right).$$

Capital will be in excess supply in period $t+1$ if $E_{\lambda_t} x_{t+1} > k \int_{\tilde{x}_0}^{\infty} d\lambda_{t+1}$. That is if

$$k \int_{\tilde{x}_0}^{\infty} d\lambda_{t+1} < (1-\alpha) \left(y + E_t x_t + (1-\phi) \int_{\tilde{x}_0}^{k} (k-x) d\lambda_t + \int_{\tilde{x}_0}^{\infty} \phi G - k d\lambda_t \right).$$

As $E_{\lambda_t} x_t > k \int_{\tilde{x}_0}^{\infty} d\lambda_t$ in period t a sufficient condition for this is

$$k \int_{\tilde{x}_0}^{\infty} d\lambda_{t+1} < (1-\alpha) \left(y + (1-\phi) \int_{\tilde{x}_0}^{k} (k-x) d\lambda_t + \int_{\tilde{x}_0}^{\infty} \phi G d\lambda_t \right).$$

The people with bequests in the interval $[\tilde{x}_0, \infty)$ in period $t+1$ are children of: individuals with a successful project in period t, individuals who were on subsistence incomes in period t, or individuals with unsuccessful projects in period t but were sufficiently wealthy to leave a bequest greater than $\tilde{x}_0$. The individuals in this last class have at least wealth $x^{\dagger}$, where $\tilde{x}_0 = (1-\alpha)(y + x^{\dagger} - k)$. Thus we can get an upper bound for $\int_{\tilde{x}_0}^{\infty} d\lambda_{t+1}$ in terms of the state λ_t, and a sufficient condition for the equation above is

$$\begin{aligned} k \left(\phi \int_{\tilde{x}_0}^{\infty} d\lambda_t + \int_{0}^{\tilde{x}_0} d\lambda_t + (1-\phi) \int_{x^{\dagger}}^{\infty} d\lambda_t \right) &< (1-\alpha) \left(y + (1-\phi) \int_{\tilde{x}_0}^{k} (k-x) d\lambda_t \right) \\ &\quad + (1-\alpha) \int_{\tilde{x}_0}^{\infty} \phi G d\lambda_t. \end{aligned}$$

Finally, subtracting $k \int_{\tilde{x}_0}^{\infty} d\lambda_t$ and $k(1-\phi) \int_{x^{\dagger}}^{\infty} d\lambda_t$ from each side, we get

$$\begin{aligned} k \left[-(1-\phi) \int_{\tilde{x}_0}^{\infty} d\lambda_t + \int_{0}^{\tilde{x}_0} d\lambda_t \right] &< (1-\alpha) y + (1-\alpha)(1-\phi) \int_{\tilde{x}_0}^{k} (k-x) d\lambda_t \\ &\quad + [(1-\alpha)\phi G - k] \int_{\tilde{x}_0}^{x^{\dagger}} d\lambda_t \\ &\quad + [(1-\alpha)\phi G - (2-\phi)k] \int_{x^{\dagger}}^{\infty} d\lambda_t. \end{aligned}$$

The terms on the right of this expression are all positive, by the assumptions we make on the parameters, when there is no credit rationing in the limit $\tilde{x}_0 < (1-\alpha)y$ the term on the right is negative. We must show that this will hold in finite time.

We will now show that there exists a T such that for all $t > T$ the term on the left is smaller than $(1-\alpha)y$. We will do this by finding an upper bound for $\int_{0}^{\tilde{x}_0} d\lambda_t$ for all t large. When $\beta = 1$ the state evolves deterministically and there is a deteministic sequence $\{r_t\}$ of equilibrium interest rates. For this sequence of interest rates define a second deterministic process, starting at λ_0, on Λ where all individuals with failing projects leave a bequest of $(1-\alpha)y$, no matter how much wealth they actually have (the rest of their intended bequest vanishes). This is the system described by the time-dependent map.

$$x_{t+1} = (1-\alpha)\begin{cases} y + (1+r_t)x_t, & x_t \leq \tilde{x} \\ G + y + (1+r_t)(x_t - k), & \text{successful, } \tilde{x} \leq x_t < k \\ y, & \text{fails, } \tilde{x} \leq x_t < k \\ G + y + (x_t - k)(1+r), & \text{successful, } k \leq x_t \\ y, & \text{fails } k \leq x_t. \end{cases}$$

The image of any state λ under the above map is first-order stochastic dominated by the image of λ under the original map. (Because the only difference between the two maps is the increased numbers of individuals mapped to the wealth level $(1-\alpha)y$.)

Let p_t (q_t)denote the number of individuals with wealth less than $\tilde{x}_0$ (greater than $\tilde{x}_0$ who use the subsistence technology) at time t under the above map. Individuals with wealth less than $\tilde{x}_0$ have children with wealth greater than $\tilde{x}_0$, so $p_{t+1} = (1-\phi)(1-p_t-q_t)$.

$$\begin{aligned} p_{t+1} &= (1-\phi)(1-p_t-q_t) \\ &= (1-\phi)(\phi - q_t + (1-\phi)q_{t-1}) + (1-\phi)^2 p_{t-1} \\ &\leq (1-\phi) + (1-\phi)^2 p_{t-1} \\ &= (1-\phi)\sum_{s=0}^{t/2}(1-\phi)^{2s} + (1-\phi)^{t+2} p_1 \end{aligned}$$

Letting $t \to \infty$ the upper bound for p_{t+1} converges to $(1-\phi)/(2\phi - \phi^2)$. For any $\epsilon > 0$ there exists a T such that $p_t \leq (1-\phi)/(2\phi-\phi^2) + \epsilon$ for all $t > T$. This process is stochastically dominated by the original so

$$\begin{aligned} \int_0^{\tilde{x}_0} d\lambda_t \quad \leq p_t \leq & \quad \frac{1-\phi}{\phi(2-\phi)} + \epsilon \\ \int_0^{\tilde{x}_0} d\lambda_t - (1-\phi)\int_{\tilde{x}_0}^{\infty} d\lambda_t \quad \leq & \quad \frac{1-\phi}{\phi} - (1-\phi) + \epsilon \\ = & \quad \frac{(1-\phi)^2}{\phi} + \epsilon \end{aligned}$$

By the assumption on the parameters this upper bound implies that for t sufficiently large the inequality holds and $r = 0$ in finite time.

Once $r = 0$ the income distribution evolves according to the linear map

$$x_{t+1} = (1-\alpha)\begin{cases} y + x, & x_t < \tilde{x}_0 \\ G + y + x_t - k, & \text{successful, } \tilde{x}_0 \le x_t < k \\ y, & \text{fails, } \tilde{x}_0 \le x_t < k \\ G + y + x_t - k, & \text{successful, } k \le x_t \\ y + x_t - k, & \text{fails } k \le x_t \end{cases}.$$

There exists a finite N such that N successive failures of the technology drives a dynasty with wealth $(1-\alpha)(G+y-k)/\alpha$ to an endowment $x \in (\tilde{x}_0, k)$. This is because successive failures eventually drives a dynasty's wealth below $\tilde{x}_0$ and the assumption on $\tilde{x}_0$ and k imply that individuals with wealth in the interval $[0, \tilde{x}_0)$ must be borrowers next period. For $\epsilon > 0$ sufficiently small, N successive failures of the technology drives a dynasty with initial wealth within ϵ of $(1-\alpha)(G+y-k)/\alpha$ to an endowment in $(\tilde{x}_0, k)$.

Consider a dynasty with initial income $x \in [0, X]$. It takes any dynasty only one lifetime to become a borrower if it is on a subsistence income, so after $M+1$ successes the dynasty has current generation has an endowment x_M which satisfies.

$$\text{(A.6)} \qquad |x_M - \frac{1-\alpha}{\alpha}(G+y-k)| \le (1-\alpha)^M |(1-\alpha)(x+y) - \frac{1-\alpha}{\alpha}(G+y-k)|$$

(Recall that $(1-\alpha)(x+y)$ is the bequest after one period of subsistence.) There exists a finite value of M so that the right of (A.6) is less than ϵ for all $x \in [0, X]$. Thus for a finite M and finite N (from the last paragraph) sequence of $M+1$ successes and then N failures of technology drives a dynasty with initial wealth x to an endowment in the interval $(\tilde{x}_0, k)$. At this point the dynasty is a borrower and a further failure of the technology drives it to an endowment $(1-\alpha)y$. There is a probability $\zeta := \phi^{M+1}(1-\phi)^{N+1}$ that in any $M+N+2$ periods the dynasty is driven to endowment $(1-\alpha)y$.

Let $r = 0$ and consider two dynasties with initial wealth x and x' respectively. Let $\mu_x^t \in \Lambda$ (respectively $\mu_{x'}^t \in \Lambda$) denote the distribution of the dynasty's endowment at time t when it had initial wealth x (respectively x'). Then for $t = M+N+2$ there is a probability of at least ζ^2 that both dynasties have been mapped to endowment level $(1-\alpha)y$ in period t. Once this has happened the Markov property implies that the future distributions of these dynasty's wealth are identical.

$$\|\mu_x^t - \mu_{x'}^t\| \le (1-\zeta^2)1 \qquad \forall t \ge M+N+2$$

Of course, if $t \ge d(M+N+2)$ there are at least d opportunities for the two processes to be driven to the endowments $(1-\alpha)y$. By iterating this idea we find that, whatever the initial state, the Markov process converges exponentially to a unique stationary distribution.

$$\|\mu_x^t - \mu_{x'}^t\| \le (1-\zeta^2)^d 1 \qquad \forall t \ge d(M+N+2)$$

Q.E.D.

<u>PROOF OF PROPOSITION 3</u>

The expected income in period $t+1$ when the capital markets are closed is given by (9). When capital markets are open $\theta = 0$, $1+r = \beta^{-1}$, and all individuals with wealth greater than $\hat{x}$ borrow, so the expected income in period $t+1$ satisfies

$$\frac{E_{\lambda_t} x_{t+1}}{1-\alpha} = y + E_{\lambda_t} x_t + \int_{\hat{x}}^{k} \phi G - k + (\frac{\beta-\phi}{\beta})(k-x) d\lambda_t + \int_k^\infty \phi G - k d\lambda_t.$$

Define Δ_{t+1} to be the difference between expected income in period $t+1$ when capital markets are closed and expected income in period $t+1$ when capital markets are open. (Thus $\Delta_{t+1} > 0$ implies that expected income tomorrow is higher when capital markets are closed.) Subsititution from (9) and above gives

$$\begin{aligned}\frac{\Delta_t}{1-\alpha} \quad := \quad & (\beta(1+r)-1)\left(E_{\lambda_t} x_t - \frac{\psi(1+r)}{1+r\psi}\int_{\underline{x}}^{\tilde{x}} (x-\underline{x}) d\lambda_t\right) \\ & - \int_{\hat{x}}^{\tilde{x}} \phi G - k + \frac{\beta-\phi}{\beta}(k-x) d\lambda_t + \\ & + (\beta(1+r)-1)\left(\int_{\tilde{x}}^{k} (k-x)\frac{\beta-\phi}{\beta} - k d\lambda_t - \int_k^\infty k + (1-\theta)(x-k) d\lambda_t\right).\end{aligned}$$

Capital market equilibrium in the closed economy, (11), allows us to substitute out for $E_{\lambda_t} x_t$ in the above expression.

$$\Delta_t = (1-\alpha)\frac{\beta-\phi}{\beta}\left((\beta(1+r)-1)\int_{\tilde{x}}^{k}(k-x)d\lambda_t - \int_{\hat{x}}^{\tilde{x}} \frac{\beta(\phi G - k)}{\beta - \phi} + k - x d\lambda_t\right)$$

It is this that gives (15). Statement (1) in the Proposition is true, because the equilibrium interest rate in the closed economy with aggregate risk always satisfies

$\beta(1+r) > 1$ and when there is no credit rationing there is no-one in the economy with wealth less than $\tilde{x}$, that is $\int_0^{\tilde{x}} d\lambda_t = 0$ which makes the second term in (15) zero. Statement (2) is true because as $\phi \to \beta$ so $\Delta_t \to -\beta(\beta G - k) \int_{\hat{x}}^{\tilde{x}} d\lambda_t$. *Q.E.D.*