GEORGIA INSTITUTE OF TECHNOLOGY — OFFICE OF CONTRACT ADMINISTRATION

PROJECT ADMINISTRATION DATA SHEET

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Title: Propagation of Finite Amplitude Sound Beams, With Applications to Parametric Acoustic Arrays

ADMINISTRATIVE DATA

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RESTRICTIONS

See Attached NSF Supplemental Information Sheet for Additional Requirements.

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SPONSORED PROJECT TERMINATION/CLOSEOUT SHEET

Date 3/14/85

Project No. E-25-649 School/~~Lab~~ ME

Includes Subproject No.(s) N/A

Project Director(s) Dr. J. H. Ginsberg GTRC / ~~GIT~~

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Title Propagation of Finite Amplitude Sound Beams, With Applications to Parametric Acoustic Arrays

Effective Completion Date: 3/31/84 (Performance) 6/30/84 (Reports)

Grant/Contract Closeout Actions Remaining:

- [ ] None
- [X] Final Invoice or Final Fiscal Report
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E-25-649

STATUS OF THE RESEARCH PROJECT

The overall goal of the project is to obtain an analytical model for the sound beam that is created by a transducer, such as a piston, when the amplitude of the signal is relatively high. Such a model will provide important improvements in current predictive capabilities, which cannot describe a variety of phenomena. The framework for this research is a singular perturbation method for nonlinear acoustics that has evolved from previous investigations by the Principal Investigator, as well as from the current project. A major innovation in the current effort is the utilization of integral transforms in the context of the overall method.

1. Axisymmetric Sound Beams

The feasibility of the research method was demonstrated in an earlier study of finite amplitude sound beams that originate from an infinitely long strip which is situated in a plane. This is the two dimensional analog of the classical "infinite baffle problem", in which a circular piston vibrates within a planar boundary. The latter produces an axisymmetric sound beam. The two dimensional analog was studied first because it features a Cartesian geometry, for which a Fourier cosine transform is suitable.

In contrast, a Hankel transform, which features Bessel functions, would have been too difficult to employ simultaneously with investigating how the general procedure should be developed.

One of the basic tasks for the current project was to obtain results for the circular geometry. This matter has been assigned the highest priority because success in the effort would have immediate usefulness to practitioners, particularly in the area of sonar systems. The results of this work have been presented at several meetings [1-2], as well as in an extensive paper [3], all of which are appended to this report.

Any study using the "direct method" follows a series of readily identifiable steps, which the Principal Investigator will soon be reporting [4, 5]. First, one must formulate the linear solution for the system. This involves solving the partial differential equations and boundary conditions conventionally encountered in acoustics. The next step is to use the linear solution to form the source terms, which are the excitations in the partial differential equation for the second order velocity potential. This step sometimes requires obtaining an alternative representation of the linear solution, particularly when the linear solution contains higher transcendental functions. (Such is the case for the piston problem, where Bessel functions are encountered.)

After the excitation of the second order potential has been characterized, the next step is to solve for the terms in this potential which grow as the signal propagates. It is in this way that the distortion effects of nonlinearity are manifested. The final step in the analysis is to deduce from the total potential function expressions for the particle velocity and pressure. This part of the analysis involves evaluating a coordinate straining transformation which cancels the growth tendencies of the second order terms in the response variables.

In regard to the piston problem (the axisymmetric sound beam), each of these tasks has been achieved. The formulation of the linear solution which was appropriate for the following steps was found to be the King integral. This is essentially the inversion of a Hankel transform solution. Such a representation is seldom mentioned in texts on acoustics, and papers which have used it usually derive it by mathematical manipulations of the Rayleigh integral which is obtained from source theory.

The task of describing the source terms for the second order potential was much more difficult than it was for the earlier two dimensional analog. Success was achieved by focusing on the region off the beam axis, where the asymptotic (approximate) representation of the Bessel functions could be employed. The corresponding solution for the potential was achieved by using an integration according to the method of stationary phase. This procedure was suggested by the two dimensional analysis.

The last step in the solution procedure is the evaluation of the coordinate straining. The initial result was achieved by following the same steps as in the two dimensional analysis; this revealed an unexpected difficulty. The two dimensional analog considered the situation where the pressure on the boundary is known, whereas the velocity at the boundary is the known quantity in the circular piston problem. The latter leads to a mathematical singularity. Treatment of this singularity was successfully performed by recasting the transformation using functions which are like the original transformation in only a portion of the spatial domain of interest.

The difficulty with evaluating the coordinate straining also pertains to the two dimensional strip problem. For this reason, it was decided to forego one of the goals in the initial research proposal. Specifically, the effect of different spatial patterns on the two dimensional strip will not be examined. Instead, the same analysis will be performed for the axisymmetric sound beam, after the solution has been validated by comparison with the results of experiments to which the Principal Investigator has contributed [6]. The foregoing decision was made in order to conserve computational resources for evaluations which have practical application.

2. General Sound Beams

Although a circular piston leads to an axisymmetric sound beam, it is not the only one encountered in practice. For example, piston transducers are often arranged in an array. One method of describing the signal is to employ a full set of cylindrical coordinates, allowing for variation in the circumferential direction. Formulation of this problem commenced recently. It involves a combination of a Fourier series for the circumferential direction and a Hankel transform for the transverse direction.

3. Multi-Harmonic Excitations

The excitation on the boundary need not be harmonic. It could consist of several different frequencies, or even an infinite set forming a Fourier series for a periodic excitation. An initial treatment of this question in the two dimensional strip problem considered the case of two frequencies. The analysis was successful up to the point of evaluating the velocity potential. Completion of the analysis, which involves determination of the coordinate straining, was not achieved because of inconsistencies which arose in each attempt. However, the alternative coordinate transformation which was developed in the analysis of the axisymmetric sound beam seems to offer a good opportunity for completing the analysis of the two-frequency excitation.

4. Propagation in Circular Waveguides

The formulation of the circular piston problem leads to a representation of the signal in terms of a continuous spectrum of infinitesimal propagation modes. Propagation in a circular waveguide (duct) involves a finite sum of such modes. The similarities, as well as differences, between this system and that of the circular piston signal are potentially very instructive. Accordingly, it was decided to explore how the research techniques could be applied to this problem.

One investigation had studied this question earlier, under the assumption that the nonlinear source terms do not resonate any of the modes of the waveguide. This assumption has been shown by the current project to be erroneous. The analysis of a simple type of excitation, which produces a single axisymmetric mode in a linearized analysis, is nearing completion [7]. This effort will be continued by studying more general types of excitations in which numerous duct modes are excited simultaneously. Success in this effort is likely to lead to important applications in the realm of noise propagation in jet engines.

NATIONAL SCIENCE FOUNDATION
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FINAL PROJECT REPORT
NSF FORM 98A

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PART I—PROJECT IDENTIFICATION INFORMATION

1. Institution and Address	2. NSF Program	3. NSF Award Number
Georgia Institute of Technology School of Mechanical Engineering Atlanta, GA 30332	Mechanical Systems	MEA-8101106
	4. Award Period From 9-81 To 6-84	5. Cumulative Award Amount 94,993

6. Project Title
Propagation of Finite Amplitude Sound Beams, with Application to Parametric Acoustic Arrays

PART II—SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

The goal of this project was the development of an analytical model for the acoustic signal generated by a transducer in a passive planar boundary. At sufficiently high frequencies the signal becomes a beam of sound that is confined to the region around the axis of symmetry. The particular concern was situations where the amplitude of the piston velocity is a large fraction of the speed of sound, in which case nonlinearities associated with finite amplitude effects create substantial distortion of the waveform. These effects accompany the diffraction and spherical spreading phenomena that arise in the linearized description of the signal, which is valid only for very low signal strength.

The method by which the program objectives were met involved several extensions of the Principal Investigator's direct method for nonlinear acoustics. The nonlinear partial differential equation for the velocity potential was solved with the aid of a Hankel transform to represent the spatial dependence on the position transverse to the axis. The inversion of the solution for the potential at the second perturbation order was obtained by an asymptotic integration technique, and the description of the corresponding pressure signal was obtained by generalizing the method of strained coordinates. The mathematical expression for the signal, and the quantitative evaluation of that result, revealed that the unusual distortion phenomena are produced by diffraction effects that shift the phase of the higher harmonics differently from the way in which the fundamental frequency signal is affected. Additional evaluations, which were carried out in order to validate the predictions, showed that the result was very accurate in predicting the waveform. Extensions of the method to treat asymmetrical and mixed-frequency excitations were nearing completion at the time that the project terminated.

PART III—TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

1. ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM: Check (✓)	Approx. Date
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c. Data on Scientific Collaborators		X			
d. Information on Inventions	X				
e. Technical Description of Project and Results		X			
f. Other *(specify)*					

2. Principal Investigator/Project Director Name *(Typed)*	3. Principal Investigator/Project Director Signature	4. Date
Jerry H. Ginsberg		2/18/85

TECHNICAL DESCRIPTION OF RESEARCH ON FINITE AMPLITUDE EFFECTS FOR CONFINED ACOUSTIC BEAMS

SPONSORED BY THE
NATIONAL SCIENCE FOUNDATION
MECHANICAL SYSTEMS PROGRAM
GRANT NO. MEA-8101106

PRINCIPAL INVESTIGATOR:
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I. BACKGROUND

Numerous evaluations of the acoustic field radiating from a baffled transducer have appeared in the published literature. The papers by Harris [1] and Greenspan [2] provide excellent surveys of the state-of-the-art for linear theories. An important feature is that these theories are applicable for a wide range of parameters. Employing approximations, such as those describing an axisymmetric sound beam in the far field (Fraunhofer zone) can substantially reduce computational cost, but it is not necessary to use approximations. Linear theory is valid when the source level is sufficiently low. Even then, diffraction effects in the near field, which lead to localized cancellations and reinforcements, complicate the task of correlating near field measurements to far field propagation properties. The situation becomes even more complicated when one tries to increase the propagation range by raising the source level. It is logical to try to overcome effects such as dissipation and scattering by generating higher level signals. Such attempts inevitably lead to a greater role for nonlinear effects. One of the effects of nonlinearity is to divert energy from the fundamental signal to higher harmonics, which is equivalent to lowering the efficiency of the transducer. In the face of these conflicts it is apparent that developing a unified theory for nonlinear effects in sound beams is a challenging matter. However, such a theory is necessary if an understanding of the distortion phenomena is to be gained.

Concern for the effects of nonlinearity in the sound beam radiating from a baffled transducer certainly predates the experiments performed by Browning and Mellen [3] and the theoretical investigation by Ingenito and Williams [4]. However, both works led to questions that are only now beginning to be answered. The measurements revealed that nonlinearity distorts a CW signal in a manner that is not symmetrical between the compression and rarefaction phases. This observation is radically different from nonlinear phenomena known to exist in regular geometries, such as planar, spherical and cylindrical waves [5].

The investigation performed by Ingenito and Williams was a first attempt. It employed a perturbation procedure whose leading term was the familiar Rayleigh integral governing a sound beam in linear theory. The difficulties encontered in the determination of the second order terms required the introduction of simplifications. Restricting the ka parameter to be very large ($ka > 100$) led to an assumption that the sound beam was a planar wave. Also, backscattering effects were neglected, and only the nonlinear generation of a second harmonic was considered. Comparison of the prediction of this harmonic on-axis with prior experiments showed order of magnitude agreement for amplitudes. However, the disparities increased with increasing range. The analysis predicted that the second harmonic is in-phase with the fundamental, but this disagreed with the observed asymmetries between compression and rarefaction. It was logical to wonder whether the failure of the analysis to predict this primary phenomenon meant that other aspects of the nonlinear distortion process were also not being modeled.

Somewhat before the publication of the work by Ingenito and Williams, papers by Khokhlov and Zabolotskaya [6,7] deriving a general parabolic wave equation for sound beams appeared in the Soviet literature. These works were followed by numerous studies, primarily Soviet as typified by References [8-12], which solved the equation numerically. The model equation itself is a modified version of Burgers' equation, in which the transverse terms in the Laplacian operator are added in order to account for diffraction and spherical spreading. The numerical solutions of this equation apparently employed the same computer program, but the specifics of this program have barely been explained in the available publications.

A significant aspect of these analyses is that they do predict asymmetrical distorition between compression and rarefaction. Unfortunately, their results were not compared to experiments, and their parameters do not coincide with those for experiments done elsewhere. Hence, their accuracy was uncertain. From a scientific viewpoint, two fundamental assumptions leading to the modified Burgers' equation are questionable. The rate of variation of the signal transverse to the axis was considered to depend on a length scale that is intermediate to a short scale associated with the axial wavelength and a long scale corresponding to the distance at which significant distortion occurs. This type of behavior is likely to occur in only selected regions. For example, diffraction effects result in very rapid transverse variation in the spatial distribution close to the transducer. Furthermore, spherical spreading in the far field means that the rate of variation in the transverse direction (perpendicular to the axis) is slow only in the region around the axis. The other questionable aspect of the model equation is that it is based on the planar wave assumption that $p = c_0 v_z$. A relation such as this cannot be obtained in the near field, and it is valid in the far field only in the region near the axis. Thus, the modified Burgers' equation was an improvement, but it still left many significant questions.

A different method for solving the modified Burgers' equation was recently developed by J. N. Tjotta, S. Tjotta et al [13]. Their approach uses the temporal periodicity of the nonlinear signal to expand the pressure in a Fourier series. Making this expression satisfy the partial differential equation leads to a coupled set of differential equations for the spatial dependence of each harmonic. Solving these equations requires truncation of the series, which means that repeated efforts might be necessary before convergence is achieved. Also the complicated nature of the coupled equations for the amplitudes, accompanied by the fact that these equations must be solved numerically, eliminates the possibility of using the analysis and its results to understand the physical reasons for the unique nonlinear phenomena.

The results obtained by the Fourier series display the qualitative features of the observed phenomena. However, the parameters for the examples do not match measured data, so the study does not permit an assessment of the influence of the approximation embedded in the modified Burgers' equation. A quasilinear approximation introduced by Tjotta et al. was explored in a séparate paper [14]. The disagreement between measured [15] and predicted second harmonic levels was so large (more than 10 dB and increasing with range) that one must wonder whether it is the approximations in the solution or in the basic equations that are at fault.

Another approach, tried by Lockwood, Muir and Blackstock [16], employed Lockwood's solution for finite amplitude nonuniform spherical waves [17]. Their idea was that if the signal level is sufficiently low, then the sound beam will be well approximated by its linear version until the far field is attained. Evaluation of the linear solution at some nominal distance in the far field then provides the boundary excitation for the spherical wave.

The predictions of this analysis for the spatial dependence of the amplitudes compared favorably with measured data. However, the limitation to low input levels avoided the question of asymmetrical distortion, which seems to arise from an interaction of near field diffraction and nonlinear distortion. Also, the distance for the transition to a nonlinear spherical wave is a crucial parameter for the analysis. This gives rise to an anomoly because there is no specific distance at which the wave attains sphericity -- the transition is relatively gradual. As a consequence, the spherical propagation model relies on an arbitrary empirical parameter, rather than being self-sufficient.

II. RESEARCH TECHNIQUE

The individual limitations of each of the earlier approaches has been overcome by the techniques developed in this project. The primary result was an analytical description of the signal radiating from a baffled transducer. The result is valid for all regions of interest (i.e., near field or far field), and it gives equal attention to the relative significance of the various phenomena (spreading, distortion, and diffraction). The basis for the analysis is the King integral [19] for a linear sound beam.

In this treatment, the sound beam is a modal superposition of waves having differing transverse wavelength. The Rayleigh integral formulation used by Ingenito and Williams [4] describes the linear signal as the superposition of signals emanating from point sources. In essence, by recombining these sources to form modes, the King integral provides a viewpoint for the resonant interaction of linear sources that is much more conducive to finding the second order terms.

Another inherent advantage of the analytical technique is that it employs a nonlinear wave equation that is valid for any isentropic process in an inviscid fluid [20]. It is not necessary to introduce assumptions regarding the physics of a sound beam. Indeed, the few assumptions that are employed are typical of those employed to evaluate nonlinear sound radiation in simpler systems, such as cylinders and spheres [5,17].

The starting point for the analysis of an axisymmetric sound beam is a nonlinear wave equation governing the velocity potential. The derivation of this equation [20] is quite ingenious, for it uses first integrals of the equations for conservation of energy and of mass to eliminate the density and the pressure from the field equations. For a harmonic process at frequency ω in a medium whose linearized speed of sound is c_0, convenient length and time scales for nondimensionalizing are c_0/ω and $1/\omega$, respectively. The corresponding differential equations is

$$\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = \frac{\partial}{\partial t}\left[(\beta_0 - 1)\left(\frac{\partial\phi}{\partial t}\right)^2 + \nabla\phi \cdot \nabla\phi\right] + O(\phi^3) \qquad (1)$$

where β_0 is the coefficient of nonlinearity [21]. In order to study the influence of alternative designs of transducers, the motion on the boundary is represented as

$$\left.\frac{\partial\phi_1}{\partial t}\right|_{z=0} = \frac{1}{2i} f(R)\exp(it) + CC \qquad (2)$$

where f(R) is an arbitrary function of the transverse distance from the center of the transducer and CC, which denotes complex conjugate, is a reminder that ϕ is real. If desired, f(R) may be complex, so that even the case of boundary excitation in the form of transversely propagating waves, may be studied.

The wave equation (1) is solved by a perturbation procedure utilizing the acoustic Mach number ε in eq. (2) as the perturbation parameter. Note that even for a very intense signal (e.g., 220 dB max. SPL on-axis in water relative to 1 uPa), the value of is very small compared to unity. The perturbation expansion for the velocity potential is

$$\phi = \varepsilon\phi_1 + \varepsilon\phi_2 + \cdots \qquad (3)$$

Several general remarks, arising from the solution of other wave propagation problems, are pertinent to eq. (3). First, there are two kinds of nonlinear effects. Some produce terms in ϕ_2 that never exceed a set maximum as the signal propagates. (One such effect is associated with the fact that the input from the transducer originates from a moving boundary, rather than the much simpler description, $z = 0$.) The smallness of the parameter leads to the conclusion that these fixed magnitude effects cannot account for measured levels of distortion. The other group of nonlinear effects arise from resonance-like phenomena. These terms lead to distortion that grows with increasing distance. Shocks ultimately form from this effect, unless dissipation is adequate to overcome the nonlinear distortion process. It is this cumulative growth effect that needs to be evaluated.

The second aspect of eq. (3) that needs to be noted is that the velocity potential does not directly represent the state of the fluid. In contrast, the expressions for pressure and particle velocity derived from the potential must have proper behavior at all locations. This means that a perturbation expansion for the state variables must never violate the requirement that the second order terms remain small compared to the first order ones. There seems to be a conflict between this requirement and the fact that the major effect of nonlinearity is to cause cumulative growth of the second order terms. The resolution of this conflict will involve coordinate straining transformations, from which a unified and consistent description of the signal is derived. The fact that cumulative growth is the primary manifestation of nonlinearity also applies to the sound beam. Consequently, it is necessary to determine only the growing terms in the second order potential.

The first order term ϕ_1 is the King integral. That representation arises from using a Hankel (Fourier-Bessel) transform to represent the transverse dependence. Specifically, it is found that

$$\phi_1 \quad - \int_0^\infty \left(\frac{n F_n}{\mu_n}\right) \exp(it - \mu_n z)\, J_0(nr)\, dn \qquad (4)$$

where n is a transverse wave number and

$$\mu_n = i(1 - n^2)^{1/2}; \; n < 1$$
$$\mu_n = (n^2 - 1)^{1/2}; \; n > 1 \qquad (5)$$

The features of the transducer vibration pattern, such as a piston or a plate mode or a wavelike input, are contained in the complex coefficient F_n, which is the transform of the distribution function f(R),

$$F_n = \frac{1}{2\, i} \int_0^\infty R\, f(r)\, J_0(nr)\, dr \qquad (6)$$

The next step in the analysis is to use the first order potential ϕ_1 to describe the excitation driving the second order term ϕ_2. The nonlinear terms in eq. (1) lead to quadratic products of ϕ_1. The inhomogeneous wave equation governing ϕ_2 is

$$\nabla^2\phi_2 - \frac{\partial^2\phi_2}{\partial t^2} = -4i\int_0^\infty\int_0^\infty \left(\frac{mnF_mF_n}{\mu_m\mu_n}\right)\Big[(\beta_0 - 1 - \mu_m\mu_n)\, J_0(mr)\, J_0(nr) - mn\, J_1(mr)\, J_1(nr)\Big] \times \exp[2it - (\mu_m + \mu_n)z]\, dm\, dn \tag{7}$$

Recall that only the part of ϕ_2 that grows with increasing distance needs to be evaluated. The evaluation of this effect commences by writing the solution for ϕ_2 as

$$\phi_2 = \int_0^\infty \Phi\, dn \tag{8a}$$

$$\Phi = n\int_0^\infty m\, \Phi_{21}(z,t,m,n)\big[J_0(mR)\, J_0(nR) - J_1(mR)\, J_1(nR)\big]\, dm + n\int_0^\infty m\, \Phi_{22}(z,t,m,n)\big[J_0(mR)\, J_0(nR) + J_1(mR)\, J_1(nR)\big]\, dm \tag{8b}$$

The essence of these expressions is that ϕ_2 is defined in eq. (8a) to be the second order contribution to the mode associated with the wave number n in the linear King integral, eq. (4). Then eq. (8b) defines how all modes couple nonlinearly with this mode. The conditions governing the unspecified functions a and b are found by making eqs. (8) satisfy eq. (7). The analysis of the resulting partial differential equations for a and b shows that a process of constructive interaction occurs in the region around m = n. Away from that region, there is destructive interference between the linear m and n modes that form the nonlinear source terms. As a result of such interference, only the region around m = n generates the resonance effect that causes cumulatively growing distortion. In order to evaluate the growing terms in ϕ_2, it is necessary to account for all terms in the region around m = n in eq. (8b). This is achieved by using an asymptotic integration according to Laplace's method [22], based on

$$m = n - q\Delta \tag{9}$$

where Δ is a small fixed parameter and q measures the detuning.

The asymptotic integration based on this procedure produces the following description for modes away from n = 1 (the transition from propagating to evanescent modes).

$$\Phi = i\beta_0 \frac{nF_n^{\,2}}{\mu_n \mu_n} (\pi\mu_n z)^{1/2} \exp(2it - 2\mu_n z)\left[J_0(nR)^2 - J_1(nR)^2\right] \tag{10}$$

The combination of eqs. (3), (4), (8a), and (10) yield an expression for the potential function that may be differentiated to obtain expressions for the pressure and particle velocity. Ingenito and Williams [4] stopped their analysis at this stage, so they failed to satify the second general requirement for a perturbation solution; the growth of the second order terms with increasing z (see eq. (10)) requires correction. This is achieved by introducing a coordinate straining transformation based on the idea that the derived expressions are premature truncations of properly behaved analytical series. The effect of nonlinearity in this viewpoint is to shift the space-time location at which a specified signal arises. The coordinate transformation of the mode associated with transverse number n is dependent on that number. The transformed spatial coordinates, denoted as ζ_n and α_n, are related to the (physical) cylindrical coordinates by the following implicit functional dependence.

$$\begin{aligned} z &= \zeta_n - 2\pi\varepsilon\beta_0\zeta_n\{i(nF_n/\bar{\mu}_n)\exp(it)\,\mathrm{erfc}[(\mu_n\zeta_n)^{1/2}] \\ &\quad + CC\}\, J_0(n\alpha_n) \\ R &= \alpha_n + 2\pi\varepsilon\beta_0\zeta_n\{i(F_n\mu_n/\bar{\mu}_n)\exp(it)\,\mathrm{erfc}[(\mu_n\zeta_n)^{1/2}] \\ &\quad + CC\}\, J_1(n\alpha_n) \end{aligned} \tag{11}$$

It is necessary to solve these equations for specified values of the spatial coordinates (z, R) at a specified instant t. Then the contribution to the pressure of the corresponding mode n is found by evaluating the integrand in

$$p = \rho_0 c_0^{\,2}\varepsilon \int_0^\infty \left(\frac{inF_n}{\mu_n}\right) \exp(it - \mu_n\zeta_n)\, J_0(n\alpha_n)\, dn + CC \tag{12}$$

Because the integrand cannot be evaluated in closed form due to the transcendental nature of eqs. (11), the pressure is obtained by integrating numerically.

Hence, the analytical results lead to a numerical algorithm for evaluating the pressure at a specified instant at a specified location. Spatial wave patterns or temporal waveforms are readily obtained by repeated application of the integral. Predictions for the amplitudes and phase angles of the harmonics are obtained by Fast Fourier Transform analysis of the temporal waveforms.

III. RESEARCH ACHIEVEMENTS

A. Analysis, Evaluation, and Interpretation

The analysis described in the previous section involved a large effort to place some of the steps on a firm mathematical foundation. This effort has also clarified the physical understanding of the manner in which a continuous spectrum of modes interact to create distortion. The initial presentation [24, 25] was improved substantially to the version that recently appeared in the Journal of the Acoustical Society of America [26, 27]. Those works compared the analytical results to data previously reported by Gallego-Jaurez and Gaete-Garreton [28]. The predictions for amplitude levels of the harmonics was well within the identifiable experimental error. Furthermore, a comparison of waveforms showed that the analysis does describe the asymmetrical distortion discussed earlier. Indeed, one comparison confirmed a measured waveform that displayed a change in this asymmetry associated with the transition from the near field to the far field. No prior analysis had anticipated this phenomenon.

The original version of the computer program for this model was quite inefficient. It required a relatively large amount of computer memory and long execution times. Consequently, performing extensive parameter studies required a prohibitive allocation of resources on a mainframe computer. Extensive reprogramming has led to a version that can run efficiently on the VAX minicomputer that is owned by the School of Mechanical Engineering at Georgia Tech. Further improvements now nearing completion will produce a version that will be suitable for execution on an IBM PC microcomputer.

B. Validation

An initial series of experiments was performed by Dr. Moffett in July 1982 at the NUSC facility at Newport, Rhode Island in order to obtain data for a comparison with the theoretical model [29]. Agreement between theory and experiment for amplitudes of the harmonics was reasonably good. The hydrophones employed for measurement showed substantial irregularity in their frequency response and no phase calibration was performed. The discrepancies between theory and experiment were shown to be less than the uncertainty in the response of the transducer.

In September 1983, Moffett endeavored to improve his measurements by resurrecting the special purpose transducers that were utilized by Browning and Mellen. The data were sent to Georgia Tech in its original digital form in order to provide a complete data base for comparison. There was a great deal of difficulty in reading the tapes due to limitations of the available equipment. Eventually, the data were displayed as waveforms and analyzed for frequency content.

It was soon apparent that the receiving hydrophone exhibited extremely anamolous behavior at high sound pressure levels, for reasons that have not yet been identified. However, an extremely important verification was obtained at a far field location. The signal there had decayed to a level that was within the tolerance of the receiver, even though the source was being driven to a high amplitude. As shown in Figure 3 of Appendix D, the measured and predicted signals coincide! (The full set of data have not yet been compiled into a report, so even Dr. Moffett does not yet know how good the agreement is.)

C. Other Viewpoints

The description of the far field as a nonuniform spherical wave performed by Lockwood, et al. [16] was discussed earlier. One of the difficulties with that result was that it relied on knowledge of the signal at some reference location. That location had to also be in the far field because the propagation theory excluded near field effects. The original analysis using this concept employed linear theory to evaluate the reference signal.

It was reasoned that an alternative was to use the project's nonlinear King integral to generate the reference signal. That would account for near field distortion. The reference signal driving the spherical wave is computed from the King integral as a multiharmonic periodic wave. The spherical propagation model was modified by the Principal Investigator to permit the reference signal to be an arbitrary periodic input. The signal generated in this manner was then compared to those predicted directly by the King integral. The results of that study [18] disclosed some significant aspects of the distortion process. A faithful reproduction of the amplitudes of the harmonics could be obtained by using the Rayleigh distance ($1/2\ ka^2$) as the reference position. However the phase angles for the harmonics could not be reproduced well unless the reference postiion was at several multiples of the Rayleigh distance.

The questions of how the modes interact nonlinearly was addressed in another manner. It was recognized that the continuous spectrum of modes in the King integral arises because the baffled system is essentially a circular wave guide of infinite diameter. A finite diameter waveguide has a discrete spectrum of modes. By exciting only one such mode in a linear sense, it would be possible to follow closely the manner in which resonant interactions take place nonlinearly. The analysis of this problem [30] followed steps that were suggested by the analysis of sound beams. It disclosed the existence of an internal resonance that had not been identified in earlier investigations of circular ducts, such as those done by Keller and Millman [31] and Nayfeh [32]. Despite this result, it only shed light on the formation of harmonics in a sound beam by what the signal did not display. Specifically, all of the harmonics were in phase. The logical conclusion is that the phase shifting phenomenon is produced in open geometries, such as the baffled transducer, by the interaction of closely spaced wavelets.

D. Parameter Studies

The main computer program for sound beams was modified to provide predictive capabilities for an assortment of transducer vibration patterns f(R). Mappings of the spatial dependence of the amplitudes and phase angles in the entire region from the transducer to the far field have been carried out for a piston transducer. In this conventional geometry piston vibration pattern f(R) is a step function. The spatial dependencies of the linearized response and of the nonlinear signal were compared in order to learn more about the influence of nonlinearity.

A noteworthy aspect of this study is the fact that there had been no prior extensive evaluations of phase shifts for the higher harmonics in the near field and the transition region. The evaluations revealed an interesting trend for the phase angles. The higher harmonics are close in phase to the fundamental near the source. As the wave propagates, each harmonic tends to lag further behind its predecessor, until there is a 90_0 phase difference in the far field. This is significant because a true spherical wave also undergoes a 90_0 phase shift in the transition from the near field to the far field. The implication of this observation is that asymmetrical distortion results from spherical transitions for higher harmi=onics which are delayed by the higher frequencies.

E. Extensions of the Theory

Efforts are now underway to extend the King integral to systems that have not previously been explored for nonlinear effects. One study is exploring the effects of asymmetry in the transducer vibration pattern. The transducer in this model is being driven by a pair of terms, one axisymmetric and the other associated with a sectorial variation around the azimuth of the transducer. This study will also provide useful knowledge about the linear signal, which has not been widely explored for asymmetry.

A primary motivation for this study lies in measurements taken by Gallego-Juarez and Gaeta-Garreton [28], as well as by Moffett in his recent effort. Certain anamolies in the amplitude levels on-axis have no obvious explanation in terms of asixymmetric phenomena. It is conveivable that the response on-axis is highly sensitive to asymmetries.

The project recently completed the analytical phase, which involved deriving the second order potential, and the associated coordinate strainings. The development of computer algorithms and quantitative evaluations still remain to be performed.

Another effort, not yet completed, is attempting to describe the nonlinear interaction of collinear sound beams at differing frequencies. The parametric array, which is the limiting case in which the primary frequencies are close, has already been studied extensively. The present study had not made such a restriction, so it will permit studying the resonant interaction in a variety of situations. This effort has thus far succeeded in carrying out the asymptotic integration. The results were recently reported [33]. A key observation suggesting additional study was that it seemed to be possible to enhance the conversion efficiency of the second harmonic. This would be achieved by giving each primary transducer a different diameter.

REFERENCES

1. G. R. Harris, "Review of transient field theory for a baffled planar piston," J. Acoust. Soc. Am. 70, 10-20 (1980).

2. M. Greenspan, "Piston radiator: Some extensions of the theory," J. Acoust. Soc. Am. 65, 608-621 (1979).

3. D. G. Browning and R. H. Mellen, "Finite-Amplitude distortion of 150 kHz waves in water," J. Acoust. Soc. Am. 44, 644-646 (1968).

4. F. Ingenito and A. O. Williams, Jr., "Calculation of second harmonic generation in a piston beam," J. Acoust. Soc. Am. 49, 319-328 (1971).

5. D. T. Blackstock, "On plane, spherical, and cylindrical sound waves of finite amplitude in lossless fluids," J. Acoust. Soc. Am. 36, 217-219 (L) (1964).

6. E. A. Zabolotskaya and R.V. Khokhlov, "Quasi-plane waves in the nonlinear acoustics of confined beams," Sov. Phys. Acoust. 15, 35-40 (1969).

7. E. A. Zabolotskaya and R.V. Khokhlov, "Convergent and divergent sound beams in nonlinear media," Sov. Phys. Acoust. 16, 39-42 (1970).

8. N. S. Bakhvalov, Ya. M Zhileiken, E. A. Zabolotskaya and R.V. Khokhlov, "Nonlinear propagation of a sound beam in a nondissipative medium," Sov. Phys. Acoust. 22, 272-274 (1978).

9. N. S. Bakhvalov, Ya. M Zhileiken, E. A. Zabolotskaya and R.V. Khokhlov, "Propagation of finite amplitude sound beams in a dissipative medium," Sov. Phys. Acoust. 24, 271-275 (1978).

10. V. E. Kunitsyn and O. V. Rudenko, "Second-harmonic generation in the field of a piston radiator," Sov. Phys. Acoust. 24, 310-313 (1978).

11. N. S. Bakhvalov, Ya. M Zhileiken, E. A. Zabolotskaya and R.V. Khokhlov, "Harmonic generation in sound beams," Sov. Phys. Acoust. 24, 101-106 (1979).

12. N. S. Bakhvalov, Ya. M Zhileiken, and E. A. Zabolotskaya, "Nonlinear propagation of sound beams with a uniform amplitude distribution," Sov. Phys. Acoust. 26, 95-100 (1980).

13. S. I. Aanonson, T.Barkve, J. N. Tjotta, and S. Tjotta, "Distortion and harmonic generation in the nearfield of a finite amplitude sound beam," J. Acoust. Soc. Am. 75, 749-768 (1984).

14. J. Berntsen, J. N. Tjotta, and S. Tjotta, "Nearfield of a large acoustic transducer. Part IV: Second harmonic and sum frequency radiation," J. Acoust. Soc. Am. 75, 1383-1391 (1984).

15. M. B. Moffett, "Measurement of fundamental and second harmonic pressures in the field of a circular piston source," J. Acoust. Soc. Am. 65, 318-323 (1979).

16. J. C.Lockwood, T.G. Muir, and D. T. Lockwood, "Directive harmonic generation in the radiation field of a circular piston," J. Acoust. Soc. Am. 53, 1148-1153 (1973).

17. J. C. Lockwood, "Two problems in high-intensity sound," U. Texas Austin, Appl. Res. Lab. ARL-TR-26 (1971).

18. J. H. Ginsberg, "Transition from the nonlinear King integral to spherical propagation for a finite amplitude sound beam," J. Acoust. Soc. Am. 74,, S24 (1983); presented at the 106th Meeting of the Acoustical Society of America, San Diego, California (7-11 November 1983).

19. E. Skudrzyk, The Foundations of Acoustics (Springer-Verlag, New York, 1971), 429-430.

20. S. Goldstein, Lectures in Fluid Mechanics (Wiley-Interscience, New York, 1960), Ch. 4.

21. A. D. Pierce, Acoustics (McGraw-Hill,New York, 1981), Ch. 5.

22. C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers (McGraw-Hill, New York, 1978), Ch. 7.

23. G. B. Whitham, Linear and Nonlinear Waves (Wiley-Interscience, New York, 1974), Ch. 8.

24. J. H. Ginsberg, "A singular perturbation analysis of axisymmetric finite amplitude sound beams," 9th International Symposium on Nonlinear Acoustics, University of Leeds, England (20-24 July 1981), and 9th U. S. National Congress of Applied Mechanics, Cornell University, Itahca, NY (21-25 June 1982).

25. J. H. Ginsberg, "An Improved King integral for finite amplitude sound beams", J. Acoust. Soc. Am. 73, S82 (1983); presented at the 105th Meeting of the Acoustical Society of America, Cincinnati, Ohio (9-13 May 1983).

26. J. H. Ginsberg, "Nonlinear King integral for arbitrary sound beams at finite amplitude - I. Asymptotic evaluation of the velocity potential," J. Acoust. Soc. Am. 76, 1201-1207 (1984).

27. J. H. Ginsberg, "Nonlinear King integral for arbitrary sound beams at finite amplitude - II. Derivation of uniformly accurate expressions," J. Acoust. Soc. Am. 76, 1208-1214 (1984).

28. J. A. Gallego-Juarez and L. Gaete-Garreton, "Propagation of finite-amplitude ultrasonic waves in air - I. Spherically diverging waves in the free field," J. Acoust. Soc. Am. 73, 761-765 (1983).

29. M. B. Moffett and J. H. Ginsberg, "Finite amplitude waveforms produced by a piston projector," J. Acoust. Soc. Am. 72, S40 (1982); presented at the 104th Meeting of the Acoustical Society of America, Orlando, Florida (9-12 November 1982).

30. J. H. Ginsberg and H. C. Miao, "Finite amplitude wave propagation in a cylindrical waveguide," 11th International Congress on Acoustics, Paris, France (19-27 July 1983).

31. J. B. Keller and M. H. Millman, "Finite amplitude sound-wave propagation in a waveguide," J. Acoust. Soc. Am. 49, 329-333 (1971).

32. A. H. Nayfeh,"Nonlinear propagation of a wave packet hard-walled circular duct," J. Acoust. Soc. Am. 57, 803-810 (1975).

33. M. A. Foda and J. H. Ginsberg, "Analysis of nonlinear harmonic generation for arbitrary dual frequency transducer excitation," J. Acoust. Soc. Am. 75, S92 (1984); presented at the 107th Meeting of the Acoustical Society of America, Norfolk, Virginia (7-10 May 1984).

34. N. S. Bakhvalov, Ya. M Zhileiken, E. A. Zabolotskaya and R.V. Khokhlov, "Focused high-amplitude sound beams" Sov. Phys. Acoust. 24, 10-15 (1978).

35. D. T. Blackstock, "Thermoviscous attenuation of plane, periodic, finite-amplitude sound waves," J. Acoust. Soc. Am. 36, 534-542 (1964).

J. H. Ginsberg, "A singular perturbation analysis of axisymmetric finite amplitude sound beams," 9th International Symposium on Nonlinear Acoustics, University of Leeds, England (20-24 July 1981), and 9th U. S. National Congress of Applied Mechanics, Cornell University, Itahca, NY (21-25 June 1982).

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Nonlinear King integral for arbitrary axisymmetric sound beams at finite amplitudes. I. Asymptotic evaluation of the velocity potential

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This paper initiates the derivation of a general analytical model for nonlinear effects in sound beams driven at high source pressure levels. The excitation is generated by a planar transducer that is in harmonic motion in an arbitrary axisymmetric pattern. The analysis develops a perturbation solution of a nonlinear equation for the velocity potential. The first-order term, which is derived with the aid of a Hankel transform to represent the transverse dependence, is the King integral for a linear sound beam. Using this integral to form the source terms exciting the second-order potential leads to a dual Hankel transform. Reduction to a single integral is achieved with the aid of an asymptotic integration following Laplace's method. The second-order term that is derived in this manner describes the tendency for the second harmonic to grow with increasing distance from the source. This result is an intermediate step in the overall development, because the integrand loses validity in the spectrum of transverse wavenumbers near the transition between evanescent and propagating wavelets, as well as for increasing distance from the transducer.

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INTRODUCTION

Recent surveys[1,2] have noted that the "infinite baffle" problem has been described by several alternative formulations. The specific concern in this subject is the signal generated within a fluid by small amplitude oscillations of a transducer which is contained within an infinite planar boundary. Such results are valid for very weak signals, in which case material and convective nonlinearities are negligible effects. Two basic formulations of the linear problem are the Rayleigh and King integrals.

The Rayleigh integral[3] treats the signal as a superposition of spherical wavelets which are generated by infinitesimal sources on the transducer face. In contrast, the King integral[4] results from a Hankel (Fourier–Bessel) integral transform transverse to the axis of symmetry. The acoustical medium in such an analysis becomes a waveguide of infinite diameter. The transducer then seems to generate a spectrum of guided planar mode wavelets whose strength varies with the transverse position. The significant aspect of both quadrature solutions is that they provide a convenient framework for quantitative evaluations of the signal at any location. They also lead to analytical approximations that are valid in certain ranges, such as the Fraunhofer (farfield) region.

The same is not true for treatments of nonlinear effects which arise when the transducer is driven to large amplitudes. One type of analysis of this question was performed by Lockwood *et al.*[5] They considered the case where the excitation is reasonably small, so that nonlinearity is not significant in the nearfield. Such a restriction leads to a farfield description based on Lockwood's analysis of nonuniform spherical waves.[6] Obviously, such an analysis provides no information regarding nonlinear effects in the nearfield.

The nearfield was the specific concern of the analysis performed by Ingenito and Williams.[7] They employed a perturbation series for the potential function, in which the leading term was described by the Rayleigh integral. That result was then used to evaluate the source terms exciting the second-order potential. Neglecting backscattering at the second order and introducing some additional approximations then led to a description of second harmonic formation that had a quadrature form.

One limitation of that analysis is that it is valid only for very high frequencies: $ka > 100$ according to Ref. 7. Another shortcoming is one that is often encountered in perturbation analyses. Specifically, if a dependent variable is expanded in a perturbation series, then the results are only valid when the second-order terms are very small compared to the first order ones. The analysis performed by Ingenito and Williams indicated that the second harmonic grows with increasing axial distance, whereas the first harmonic (i.e., linear result) shows no such growth. It follows that these results shed light on how harmonics begin to form, but further extrapolation might lead to errors. (This seems to be the case for their Fig. 1.)

Another formulation of finite amplitude sound beams, which has been prominent in the Soviet literature, employs a modified Burgers' equation. The basic assumption made in the derivation of this equation[8,9] is that there are three spatial scales for the signal. The shortest scale is the axial wavelength and the longest scale describes the development of nonlinear effects axially. The intermediate scale describes the variation transverse to the axis of symmetry.

These approximations seem reasonable for the high-frequency limit. Unfortunately, solutions for monochromatic transducer motion have only been obtained by finite-difference techniques. Typical of such investigations are Refs. 10–

12, which all seem to have employed the same (incompletely described) computer code. It is significant that this group of investigations have only considered situations where the boundary excitation is a prescribed pressure. Extending those analyses to cases where the particle velocity on the boundary is known, as is the case for most transducers, requires a relation between pressure and velocity that is not contained in the basic theory. Also, it should be noted that the small scale of some diffraction effects[13] introduces some doubt regarding the length scales assumed to derive the modified Burgers' equation.

The present investigation is a perturbation analysis, as was the work by Ingenito and Williams, although the King integral is used here to describe the first-order term. The analysis is founded on the recognition that nonlinearities arise in two forms when the signal level is moderately high. Some nonlinear effects maintain their level or die out as the signal propagates away from the transducer. Typical of such an effect is the fact that the transducer face represents a moving, rather than a fixed boundary, for satisfying continuity of particle velocity.

Order of magnitude considerations indicate that such effects are too small to describe the levels of higher harmonics that have been measured.[14,15] As is the case for planar and other one-dimensional waves,[16] significant distortion phenomena stem from cumulative growth of higher harmonics. Such action is a result of the fact that the acoustic medium is nondispersive, so that higher harmonics propagate with, and resonantly interact with, the primary signal.

The present analysis consistently accounts for cumulative growth effects. The sole assumption introduced in the course of the analysis is that the nonlinear mechanism causing harmonic generation has the same behavior at all locations in the acoustic field. The first part of the investigation, described in this paper, obtains an expression for the first two orders of approximation of the velocity potential. The second part of the investigation[17] will employ coordinate straining transformations to correct irregularities in the response associated with the derived potential function. The acoustic signal will be described in a quadrature form that is reminiscent of the linear King integral. A quantitative example will compare the harmonic content of the waveform to measurements recently reported by Gallego-Juarez and Gaete-Garreton.[15]

The overall analytical procedure may be traced back to the author's previous investigation of two-dimensional radiation from a boundary.[18,19] However, the use of complex functional forms and the introduction of Hankel transforms to treat the axisymmetrical geometry require substantial alterations from previous work.

I. FUNDAMENTAL EQUATIONS

The propagation speed of infinitesimal planar waves is denoted as c_0, and $(\beta_0 - 1)$ is the nonlinearity parameter in the pressure-density relation at fixed entropy[3]

$$p = \rho_0 c_0^2 \left[\left(\frac{\rho - \rho_0}{\rho_0} \right) + (\beta_0 - 1) \left(\frac{\rho - \rho_0}{\rho_0} \right)^2 + \cdots \right]. \tag{1}$$

Thus $\beta_0 = (\gamma + 1)/2$ for an ideal gas, where γ is the ratio of specific heats. Let (R, z) denote nondimensional cylindrical coordinates (axisymmetric), with $R = 0$ corresponding to the center of the transducer which is on the boundary $z = 0$. Also, denote the nondimensional time variable as t. The corresponding dimensional position coordinates are $(R/k, z/k)$ and the dimensional time is t/ω, where ω is the frequency of the monochromatic excitation and $k = \omega/c_0$ is the wavenumber for a nominal planar wave.

The dimensionless velocity potential ϕ is related to the particle velocity components by

$$v_z = c_0 \frac{\partial \phi}{\partial z}, \quad v_R = c_0 \frac{\partial \phi}{\partial R}. \tag{2}$$

The boundary condition corresponding to the axisymmetric motion of an arbitrary transducer may then be written in complex form as

$$\left. \frac{\partial \phi}{\partial z} \right|_{z=0} = \frac{\epsilon}{2i} f(R) \exp(it) + \text{c.c.}, \tag{3}$$

where $f(R)$ is any complex function whose magnitude approaches zero with increasing R. In general, c.c. will be used to denote the complex conjugate of the preceding term. (Forming products of complex functions necessitates retaining conjugate parts, rather than identifying only real parts.) For weakly nonlinear waves, the acoustic Mach number ϵ is a finite parameter with $|\epsilon| \ll 1$. The nonlinear wave equation governing ϕ is[20]

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = 2(\beta_0 - 1) \frac{\partial \phi}{\partial t} \nabla^2 \phi + \frac{\partial}{\partial t} (\nabla \phi \cdot \nabla \phi) + O(\phi^3), \tag{4a}$$

where

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R} \frac{\partial \phi}{\partial R} + \frac{\partial^2 \phi}{\partial z^2}, \qquad \nabla \phi \cdot \nabla \phi = \left(\frac{\partial \phi}{\partial R} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2. \tag{4b}$$

In addition to Eq. (3), the other boundary condition for ϕ is that the signal should be either an outgoing wave or an evanescent wave at large z, and that it show suitable decay with increasing R.

The velocity potential is expanded in a perturbation series

$$\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots. \tag{5}$$

The equations governing ϕ_1 and ϕ_2 are found by collecting like powers of ϵ in Eqs. (3) and (4a). The first-order equations are

$$\nabla^2 \phi_1 - \frac{\partial^2 \phi_1}{\partial t^2} = 0, \qquad \left. \frac{\partial \phi_1}{\partial z} \right|_{z=0} = \frac{1}{2i} f(R) \exp(it) + \text{c.c.} \tag{6}$$

Equations (6) are the statement of the linearized problem. The nonlinear effects are contained in ϕ_2 and succeeding terms. A complete solution for ϕ_2 requires satisfaction of the boundary conditions, which involves evaluation of the complementary solution, as well as of the particular solution associated with the source terms arising from ϕ_1. However, the complementary solution is bounded and therefore represents a noncumulative $O(\epsilon^2)$ contribution to the signal at all locations. As noted earlier such effects are usually insignificant compared to the observed levels of nonlinear distortion. Thus it is only necessary to find a particular solution of the second-order equation. The first of Eqs. (6) provides a simple identity for $\nabla^2\phi_1$. The resulting second-order equation arising from Eq. (4a) is

$$\nabla^2\phi_2 - \frac{\partial^2\phi_2}{\partial t^2} = \frac{\partial}{\partial t}\left[(\beta_0 - 1)\left(\frac{\partial\phi_1}{\partial t}\right)^2 + \nabla\phi\cdot\nabla\phi\right]. \quad (7)$$

II. LINEARIZED SOLUTION

Two approaches that have been employed to solve the linearized problem, Eqs. (6), are the Rayleigh integral and the King integral. The latter, which is essentially an inversion of a Hankel (Fourier–Bessel) transform, is more suitable for the task of evaluating ϕ_2. Hence let

$$\phi_1 = \int_0^\infty n\Phi_1(n,z,t)J_0(nR)dn + \text{c.c.} \quad (8)$$

Substituting this expression into Eqs. (6) and using the fact that $J_0(nR)$ is a solution of Bessel's equation leads to

$$\frac{\partial^2\Phi_1}{\partial z^2} - \frac{\partial^2\Phi_1}{\partial t^2} - n^2\Phi_1 = 0,$$

$$\left.\frac{\partial\Phi_1}{\partial z}\right|_{z=0} = F_n \exp(it), \quad (9)$$

where

$$F_n = \frac{1}{2i}\int_0^\infty R f(R)J_0(nR)dR \quad (10)$$

is the transform of the (complex) spatial excitation function $f(R)$.

The solution of Eqs. (9) is a propagating wave when $0<n<1$, or an evanescent wave when $n>1$. This solution is

$$\Phi_1 = -(F_n/\mu_n)\exp(it - \mu_n z), \quad (11)$$

where

$$\mu_n^2 = n^2 - 1. \quad (12a)$$

Satisfying the radiation condition as $z\to\infty$ leads to the following choice for the branch cuts:

$$\mu_n = \begin{cases} i(1-n^2)^{1/2}, & 0<n<1, \\ (n^2-1)^{1/2}, & n>1. \end{cases} \quad (12b)$$

The result of substituting Eq. (11) into Eq. (8) is

$$\phi_1 = -\int_0^\infty \left(\frac{nF_n}{\mu_n}\right)\exp(it - \mu_n z)J_0(nR)dn + \text{c.c.} \quad (13)$$

This is the King integral representation of the potential function for the linearized sound beam.

III. SECOND-ORDER POTENTIAL

The first-order solution ϕ_1 in Eq. (13) is used to form the source terms driving ϕ_2 in Eq. (7). Forming quadratic products of ϕ_1 requires that different symbols be used to represent the transverse wavenumber forming each term in the product. Also, care must be taken to include the complex conjugate parts in the product. A quadratic product of sinusoidal terms generally leads to a term having a nonzero mean value, but the time derivative appearing on the right side of Eq. (7) removes such an effect. The result is

$$\begin{aligned}\nabla^2\phi_2 &- \frac{\partial^2\phi_2}{\partial t^2} \\ &= -4i\int_0^\infty\int_0^n\left(\frac{mnF_mF_n}{\mu_m\mu_n}\right)[(\beta_0 - 1 - \mu_m\mu_n) \\ &\quad\times J_0(mR)J_0(nR) - mnJ_1(mR)J_1(nR)] \\ &\quad\times\exp[2it - (\mu_m+\mu_n)z]dm\,dn + \text{c.c.}, \end{aligned} \quad (14)$$

where the symmetry of the integrand has been exploited to reduce the integration over the first wavenumber to a finite range. (This introduces an additional factor of two.)

It is consistent with the form of Eq. (14) to try to construct the particular solution for ϕ_2 as the sum of two dual Hankel transforms. The kernel of one transform would be $mn\, J_0(mR)J_0(nR)$ and the kernel of the second would be $mn\, J_1(mR)J_1(nR)$. The following equivalent form, which utilizes linear combinations of the aforementioned kernel functions, leads to significant analytical simplifications.

$$\phi_2 = \int_0^\infty \Phi_2\, dn, \quad (15a)$$

where

$$\begin{aligned}\Phi_2 &= n\int_0^n m\Phi_{21}(z,t,m,n)[J_0(mR)J_0(nR) \\ &\quad - J_1(mR)J_1(nR)]dm + n\int_0^n m\Phi_{22}(z,t,m,n) \\ &\quad\times[J_0(mR)J_0(nR) + J_1(mR)J_1(nR)]dm. \end{aligned} \quad (15b)$$

The following identities, which are derived from the recursion relations for Bessel functions,[21] are useful for evaluating the transverse derivatives of ϕ_2.

$$\begin{aligned}&\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}\right)J_0(mR)J_0(nR) \\ &\quad = -(m^2+n^2)J_0(mR)J_0(nR) + 2mnJ_1(mR)J_1(nR), \\ &\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}\right)J_1(mR)J_1(nR) \\ &\quad = 2mnJ_0(mR)J_0(nR) - (m^2+n^2-4/R^2)J_1(mR) \\ &\quad\quad\times J_1(nR) - 2(m/R)J_0(mR)J_1(nR) \\ &\quad\quad - 2(n/R)J_1(mR)J_0(nR). \end{aligned} \quad (16)$$

In view of these relations, the result of substituting Eqs. (15) into Eq. (14) and matching the integrands on either side of the equality sign is

$$\left[\frac{\partial^2\Phi_{21}}{\partial z^2} - \frac{\partial^2\Phi_{21}}{\partial t^2} - (m+n)^2\Phi_{21}\right][J_0(mR)J_0(nR) - J_1(mR)J_1(nR)]$$
$$+\left[\frac{\partial^2\Phi_{22}}{\partial z^2} - \frac{\partial^2\Phi_{22}}{\partial t^2} - (m-n)^2\Phi_{22}\right][J_0(mR)J_0(nR) + J_1(mR)J_1(nR)]$$
$$-[(4/R^2)J_1(mR)J_1(nR) - 2(m/R)J_0(mR)J_1(nR) - 2(n/R)J_1(mR)J_0(nR)](\Phi_{21}-\Phi_{22})$$
$$= -4i(F_mF_n/\mu_m\mu_n)[(\beta_0 - 1 - \mu_m\mu_n)J_0(mR)J_0(nR) - mnJ_1(mR)J_1(nR)]\exp[2it - (\mu_m+\mu_n)z] + \text{c.c.} \tag{17}$$

If the last bracketed term on the left side of this equation was not present, it would be a simple matter to match like functions of R on either side. In order to address this matter, note that the functions Φ_{21} and Φ_{22} are independent of R. Conditions governing them in a specific region of R should be applicable for all R. This is significant because the bracketed term causing difficulty in Eq. (17) decreases at a rate $1/R$ faster than the other terms as R increases. The foregoing argument suggests that because the term is negligible at large R, it should be negligible in the evaluation of Φ_{21} and Φ_{22} at any R.

The validity of this hypothesis might be questioned for situations where n and m are small. In that case, the asymptotic decay of Bessel functions having arguments mR or nR might be approached at unacceptably large values of R. This question may be examined by using the series expansions of the Bessel functions for small arguments. Specifically, when $mR \ll 1$ and $nR \ll 1$, it may be shown that

$$\frac{4}{R^2}J_1(mR)J_1(nR) - 2\frac{m}{R}J_0(mR)J_1(nR) - 2\frac{n}{R}J_1(mR)\times J_0(nR) = -mn[1 - \tfrac{1}{2}(m^2+n^2)R^2 + \tfrac{1}{64}(5m^2n^2 - m^4 - n^4)R^4 - \cdots]\,. \tag{18}$$

Thus the third bracketed term in Eq. (17) is $O(mn)$ when mR and nR are small, whereas other terms in that equation are order unity. Thus the troublesome bracketed term should have negligible influence in this region also.

Neglecting the aforementioned term has a physical justification. Recall that in the King integral formulation, the acoustic signal is viewed as a spectrum of modes in an infinite waveguide. The wavenumbers m and n are merely parameters characterizing the transverse rate of variation of these modes. It is reasonable to expect that the nonlinear mechanism generating the second-order contributions to these modes are described by the same differential equations at all values of R, and for all values of m and n.

When the third bracketed term in Eq. (17) is ignored, matching like functions of R in that equation leads to

$$\begin{aligned}\frac{\partial^2\Phi_{21}}{\partial z^2} - \frac{\partial^2\Phi_{21}}{\partial t^2} - (m+n)^2\Phi_{21} &= -2i\frac{F_nF_m}{\mu_n\mu_m}(\beta_0 - 1 - \mu_n\mu_m + nm)\\ &\quad\times\exp[2it - (\mu_n+\mu_m)z] + \text{c.c.},\\ \frac{\partial^2\Phi_{22}}{\partial z^2} - \frac{\partial^2\Phi_{22}}{\partial t^2} - (m-n)^2\Phi_{22} &= -2i\frac{F_nF_m}{\mu_n\mu_m}(\beta_0 - 1 - \mu_n\mu_m - nm)\\ &\quad\times\exp[2it - (\mu_n+\mu_m)z] + \text{c.c.}\end{aligned} \tag{19}$$

The virtue of constructing the solution for ϕ_2 in the form of Eqs. (15) is now evident; any other form would not have resulted in uncoupling of the equations for the transform variables Φ_{21} and Φ_{22}.

The form of the solution of Eqs. (19) is suggested by the physical implication of the linear King integral, which consists of a continuous spectrum of modes in a circular waveguide of infinite extent. The axis of the sound beam (i.e., the z axis) is the direction in which these modes propagate. In general, nonlinear generation of harmonics increases with increasing propagation distance. Hence the particular solutions may be written as

$$\begin{aligned}\Phi_{21} &= a(z,m,n)\exp[2it - (\mu_n+\mu_m)z] + \text{c.c.},\\ \Phi_{22} &= b(z,m,n)\exp[2it - (\mu_n+\mu_m)z] + \text{c.c.},\end{aligned} \tag{20}$$

where cumulative growth will be manifested by increasing values of the amplitudes a and b. Substitution of Eqs. (20) into Eqs. (19) leads to a set of uncoupled ordinary differential equations for these amplitudes.

$$\frac{d^2a}{dz^2} - 2(\mu_n+\mu_m)\frac{da}{dz} + [(\mu_n+\mu_m)^2 - (m+n)^2 + 4]a = -2i\frac{F_nF_m}{\mu_n\mu_m}(\beta_0 - 1 - \mu_n\mu_m + nm), \tag{21a}$$

$$\frac{d^2b}{dz^2} - 2(\mu_n+\mu_m)\frac{db}{dz} + [(\mu_n+\mu_m)^2 - (m-n)^2 + 4]b = -2i\frac{F_nF_m}{\mu_n\mu_m}(\beta_0 - 1 - \mu_n\mu_m - nm). \tag{21b}$$

At this stage, it is appropriate to recall that the analysis of ϕ_2 requires evaluation of only the portion that exhibits growth. If the values of m and n are such that the coefficient of a or b in Eqs. (21) does not vanish, then the particular solution is independent of z. In contrast, if this coefficient should vanish, then the corresponding particular solution for a or b is proportional to z. (Vanishing of the coefficient is equivalent to secularity in perturbation analyses of nonlinear vibrations.) It is found with the aid of Eq. (12b) that the condition of a vanishing coefficient only occurs when $m = n$ in Eq. (19a). Therefore only the contribution of a needs to be evaluated. The magnitude of b is bounded at all z, which means that b does not represent a cumulative distortion effect.

IV. INTEGRATION BY LAPLACE'S METHOD

The condition where the solution to Eq. (21a) grows with increasing z has been shown to arise as $m \to n$. In contrast, regions far from $m = n$ represent contributions that do not change in overall magnitude with increasing distance.

The contribution of the region around $m = n$ may be determined by following Laplace's asymptotic integration method[22] based on an expansion using

$$m = n - q\Delta, \quad \Delta \ll 1, \quad q = O(1), \tag{22a}$$

where Δ and q are positive because $m < n$ for the integrals in Eqs. (15). Note that Δ is a fixed parameter indicating the scale of the difference between m and n. The Taylor series for the coefficient μ_m defined in Eq. (12b) is found to be

$$\mu_m = \mu_n(1 - nq\Delta/\mu_n^2 - q^2\Delta^2/2\mu_n^4 + \cdots). \tag{22b}$$

These expressions for m and μ_m lead to the following representation of Eq. (21a) in the region where $m \approx n$:

$$\frac{d^2a}{dz^2} - 2\mu_n\left(2 - \frac{nq\Delta}{\mu_n^2} - \frac{q^2\Delta^2}{2\mu_n^4} + O(\Delta^3)\right)\frac{da}{dz} - [q^2\Delta^2/\mu_n^2 + O(\Delta^3)]a_1 = -2i\beta_0 F_n^2/\mu_n^2. \tag{22c}$$

Now observe that when $q = 0$ $(m = n)$, the particular solution of this equation is

$$a|_{q=0} = (i\beta_0 F_n^2/2\mu_n^3)z. \tag{23}$$

In contrast, the general solution for a when $q \neq 0$ has the form

$$a = A_1\exp(\sigma_1 z) + A_2\exp(\sigma_2 z) + 2i\beta_0 F_n^2/q^2\Delta^2, \tag{24}$$

where the coefficients σ_1 and σ_2 are the roots of the characteristic equation governing the complementary solution:

$$\sigma^2 - 2\mu_n(2 - nq\Delta/\mu_n^2 - q^2\Delta^2/2\mu_n^4)\sigma - q^2\Delta^2/\mu_n^2 = 0. \tag{25}$$

Solving this quadratic equation yields

$$\sigma_1 = -q^2\Delta^2/4\mu_n^3 + O(\Delta^3), \quad \sigma_2 = 4\mu_n + O(\Delta). \tag{26}$$

As $q \to 0$, Eq. (24) must approach Eq. (23). Because σ_2 is $O(1)$ and Eq. (23) has no term that varies exponentially in z, set $A_2 = 0$. Also, because σ_1 is $O(\Delta^2)$, $\exp(\sigma_1 z)$ may be replaced by the leading terms in a series. The condition that Eq. (24) approach Eq. (23) then leads to

$$\lim_{q\to 0}[A_1(1 + \sigma_1 z) + 2i\beta_0 F_n^2/q^2\Delta^2] = (i\beta_0 F_n^2/2\mu_n^3)z. \tag{27}$$

The first of Eqs. (26) shows that this condition is satisfied at all values of z by

$$A_1 = -2i\beta_0 F_n^2/q^2\Delta^2. \tag{28}$$

Substituting this expression into Eq. (24) yields the general solution

$$a = (2i\beta_0 F_n^2/q^2\Delta^2)[1 - \exp(-q^2\Delta^2 z/4\mu_n^3)]. \tag{29}$$

The next step is to evaluate the total contribution to Φ_2 of the value of a associated with all wavenumbers m. For this determination the first wavenumber n is held constant at an arbitrary value. The combined effect is defined by Eqs. (15) and (20) to be

$$\Phi_2 = n\int_0^n m\{a(z,m,n)\exp[2it - (\mu_n + \mu_m)z] + \text{c.c.}\} \times [J_0(mR)J_0(nR) - J_1(mR)J_1(nR)]dm + O(1). \tag{30}$$

Equation (29) gives the behavior of a in the region where $m \approx n$. According to Laplace's integration technique, the behavior at large z may be found by using that relation only. In order to demonstrate this feature, the region of integration is broken into two intervals. The boundary between these intervals is defined to be $m = n - \delta$, where δ is a small finite quantity. Then

$$\Phi_2 = \int_0^{n-\delta} d\Phi_2 + \int_{n-\delta}^n d\Phi_2. \tag{31}$$

Because the first domain of integration does not contain the secularity condition, the oscillatory nature of the integrand results in boundedness of the first integral for all z.[23] This term, like comparable effects that occurred earlier, does not contribute to the cumulative distortion process, so it is discarded. In essence, this region features destructive interference between the m and n waves.

In order to evaluate the second integral, Eqs. (22) and (29) are substituted into Eq. (30). The difference between m and n is less than δ in the region of integration, so m may be replaced by n in the Bessel function. Similarly, μ_m may be replaced by μ_n in the exponential term in Eq. (30). The integral arising from Eq. (31) then simplifies to

$$\Phi_2 = \{2i\beta_0 n^2 F_n^2 E(\delta,\Delta,z/\mu_n^3)\exp(2it - 2\mu_n z) + \text{c.c.}\} \times [J_0(nR)^2 - J_1(nR)^2], \tag{32}$$

where

$$E(\delta,\Delta,z/\mu_n^3) = \int_0^{\delta/\Delta}\left(\frac{1 - \exp(-q^2\Delta^2 z/4\mu_n^3)}{q^2\Delta^2}\right)(\Delta dq). \tag{33}$$

Evaluation of the function E introduces a square root of z/μ_n^3, for which it is important to account for the fact that μ_n is imaginary when $n < 1$. Specifically,

$$\left(\frac{z}{\mu_n^3}\right)^{1/2} = \left(\frac{\mu_n z}{|n^2 - 1|^2}\right)^{1/2} = \frac{(\mu_n z)^{1/2}}{\mu_n\bar{\mu}_n} \tag{34}$$

where an overbar denotes the complex conjugate of the marked quantity. Integration by parts of Eq. (33) then gives

$$E(\delta,\Delta,z/\mu_n^3) = \left(\frac{(\pi\mu_n z)^{1/2}}{2\mu_n\bar{\mu}_n}\right)\text{erf}\left(\frac{\delta(\mu_n z)^{1/2}}{2\mu_n\bar{\mu}_n}\right) - (1/\delta)[1 - \exp(-\delta^2 z/4\mu_n^3)], \tag{35}$$

where erf denotes the error function.

One noteworthy feature of Eqs. (32) and (35) is that the only remaining parameter associated with the asymptotic integration is the integration limit δ, which is finite value. Consider Eq. (35) as z increases while δ is held fixed. An expansion of the error function for large arguments leads to

$$\text{erf}\left(\frac{\delta(\mu_n z)^{1/2}}{2\mu_n\bar{\mu}_n}\right) \sim 1 - \frac{2\mu_n\bar{\mu}_n}{\delta(\pi\mu_n z)^{1/2}}\exp\left(\frac{-\delta^2 z}{4\mu_n^3}\right), \tag{36}$$

$$E(\delta,\Delta,z/\mu_n^3) \sim [(\pi\mu_n z)^{1/2} 2\mu_n\bar{\mu}_n] - 1/\delta.$$

The growth effect comes from the first term in E above. In general, the behavior as $z \to \infty$ is said to be the "dominant term."[22] The dominant term in Φ_2 originates from the portion of the particular solution associated with the region of secularity, $m \simeq n$. Subdominant terms, such as the particular solution associated with $m \neq n$, have already been neglected because they do not represent a growth effect. Thus the function E in Eq. (32) may be replaced by its dominant term, as given by Eq. (36) when $1/\delta$ is neglected. When the resulting expression for Φ_2 is used to form the second-order potential according to Eq. (15a), and then combined with the first-order potential in Eq. (13), the result is

$$\phi = \int_0^\infty \Phi \, dn, \tag{37a}$$

where

$$\Phi = -\epsilon \frac{nFn}{\mu_n} \exp(it - \mu_n z) J_0(nR) + \epsilon^2 i\beta_0 \frac{n^2 F_n^2}{\mu_n \bar{\mu}_n} (\pi \mu_n z)^{1/2} \times \exp(2it - 2\mu_n z)[J_0(nR)^2 - J_1(nR)^2] + \text{c.c.} + \text{subdominant terms.} \tag{37b}$$

V. CLOSURE

An expression for the pressure may be obtained by differentiating Eqs. (37) with respect to time, but the result has some problematical aspects. First, it is clear that the $O(\epsilon^2)$ term grows as $z^{1/2}$, while the $O(\epsilon)$ term remains bounded. Thus the second-order term satisfies the smallness assumptions inherent to a perturbation series only when z is small compared to $1/\epsilon^2$.[24] From a practical viewpoint this limits the validity of the result to distances that are very small compared to the shock formation distance.

Another aspect relating to the validity of the result is less obvious. Consider the situation where $n \to 1$, in which case $\mu_n \to 0$. The expression for pressure derived from Eq. (37) has μ_n in the denominator, so it is singular at $n = 1$. The key aspect of the singularity is that the $O(\epsilon)$ term will contain a factor $1/\mu_n$, while the $O(\epsilon^2)$ term will contain a factor $1/\bar{\mu}_n \mu_n^{1/2}$. Thus, the $O(\epsilon^2)$ term grows more rapidly than the $O(\epsilon)$ term as $n \to 1$. This is another instance where the magnitude of the second-order term grows relative to the first-order term. As was true for the $z^{1/2}$ dependence, nonuniform accuracy limits the usefulness of the pressure derived from Eqs. (37).

The lack of uniform accuracy in z is not surprising, because it is the equivalent of secular terms in nonlinear vibration analyses. The nonuniform accuracy in the wavenumber n is a result of the analytical procedure that was followed. The integration by Laplace's method assumed that $|\mu_n|$ is not very small. This is most clearly indicated in Eq. (22b), where the truncation of the series expansion is appropriate only if $|q\Delta / \mu_n^2| \ll 1$. Only very small values of q satisfy this criterion when $n \to 1$ ($\mu_n \to 0$). Therefore, the contribution to the second-order potential from the region around $n = 1$ is not well described asymptotically.

There are other shortcomings in the form of Eq. (37b). First, the $O(\epsilon)$ term is the same as that obtained from linear theory, i.e., it is the conventional King integral. Thus the relation does not indicate that there is depletion of the fundamental harmonic as energy is transferred to higher harmonics.[25] Another important limitation is that Eq. (37b) describes only the second harmonic, but higher harmonics are known to be significant to the distortion process.

The aforementioned items lead to concern regarding the validity of any prediction of pressure. This would certainly be the case if Eqs. (37) were to be used directly. The analysis in the next part of this investigation[17] overcomes these difficulties. It treats the response obtained from Eqs. (37) as the asymptotic representation for small $\epsilon z^{1/2}$ and $n \neq 1$ of functional forms that are uniformly accurate. Such an analysis is not applied directly to the velocity potential because there are situations where a portion of the potential may exhibit nonuniform growth while the pressure and other state variable do not.[26]

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[1]G. R. Harris, "Review of transient field theory for a baffled planar piston," J. Acoust. Soc. Am. **70**, 10–20 (1981).

[2]R. New, R. I. Becker, and P. Wilhelmig, "A limiting form for the nearfield of the baffled piston," J. Acoust. Soc. **70**, 1518–1526 (1981).

[3]A. D. Pierce, *Acoustics* (McGraw- Hill, New York, 1981), Chap. 5.

[4]E. Skudrzyk, *The Foundations of Acoustics* (Springer-Verlag, New York, 1971), pp. 429–430.

[5]J. C. Lockwood, T. G. Muir, and D. T. Blackstock,"Directive harmonic generation in the radiation field of a circular piston," J. Acoust. Soc. Am. **53**, 1148–1153 (1973).

[6]J. C. Lockwood, "Two problems in high-intensity sound," Univ. Texas at Austin, Appl. Res. Lab., ARL-TR-71-26 (1971).

[7]F. Ingenito and A. O. Williams, "Calculation of second-harmonic generation in a piston beam," J. Acoust. Soc. Am. **49**, 319–328 (1971).

[8]E. A. Zabolotskaya and R. V. Khokhlov, "Quasi-plane waves in the nonlinear acoustics of confined beams," Sov. Phys. Acoust. **15**, 35–40 (1969).

[9]E. A. Zabolotskaya and R. V. Khokhlov, "Convergent and divergent sound beams in nonlinear media," Sov. Phys. Acoust. **19**, 39–42 (1970).

[10]N. S. Bakhvalov, Ya. M. Zhileikin, E. A. Zabolotskaya, and R. V. Khokhlov, "Nonlinear propagation of a sound beam in a nondissipative medium," Sov. Phys. Acoust. **22**, 272–274 (1976).

[11]N. S. Bakhvalov, Ya. M. Zhileikin, E. A. Zabolotskaya, and R. V. Khokhlov, "Propagation of finite amplitude sound beams in a dissipative medium," Sov. Phys. Acoust. **24**, 271–275 (1978).

[12]N. S. Bakhvalov, Ya. M. Zhileikin, E. A. Zabolotskaya, and R. V. Khokhlov, "Harmonic generation in sound beams," Sov. Phys. Acoust. **25**, 101–106 (1979).

[13]J. Zemanek, "Beam behavior within the nearfield of a vibrating piston," J. Acoust. Soc. Am. **49**, 181–191 (1971).

[14]M. B. Moffett, "Measurement of fundamental and second harmonic pressures in the field of a circular piston source," J. Acoust. Soc. Am. **65**, 318–323 (1979).

[15]J. A. Gallego-Juarez and L. Gaete-Garreton, "Experimental study of nonlinearity in free progressive acoustic waves in air at 20 kHz," J. Physique **40** (C8), 336–340 (1978); "Propagation of finite-amplitude ultrasonic waves in air —I. Spherically diverging waves in the free field," J. Acoust. Soc. Am. **73**, 761–765 (1983).

[16]D. T. Blackstock, "On plane, spherical, and cylindrical sound waves of finite amplitude in lossless fluids," J. Acoust. Soc. Am. **36**, 217–219 (1964).

[17]J. H. Ginsberg, "Nonlinear King integral for arbitrary axisymmetric sound beams at finite amplitudes—II. Derivation of uniformly accurate expressions," J. Acoust. Soc. Am. **76**, 1208–1214 (1984).

[18]J. H. Ginsberg, "On the nonlinear generation of harmonics in sound radiation from a vibrating planar boundary," J. Acoust. Soc. Am. **69**, 60–65 (1981).

[19]J. H. Ginsberg, "Uniformly accurate description of finite amplitude sound radiation from a harmonically vibrating planar boundary," J. Acoust. Soc. Am. **69**, 929–936 (1981).

[20]S. Goldstein, *Lectures in Fluid Mechanics* (Wiley-Interscience, New York, 1960), Chap. 4.

[21] *Handbook of Mathematical Foundations*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), Chap. 7.

[22] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineering* (McGraw-Hill, New York, 1978), pp. 261–267.

[23] I. N. Sneddon, *Fourier Transforms* (McGraw-Hill, New York, 1951), Chaps. 1–2.

[24] A. H. Nayfeh, *Perturbation Methods* (Wiley-Interscience, New York, 1973), pp.16–18.

[25] W. Keck and R. T. Beyer, "Frequency spectrum of finite amplitude ultrasonic waves in liquids," Phys. Fluids **3**, 346–352 (1960).

[26] A. H. Nayfeh and A. Kluwick, "A comparison of three perturbation methods for non-linear waves," J. Sound Vib. **48**, 293–299 (1976).