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Hanif D. Sherali
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Dedicated to

Our Parents and Teachers

PREFACE

The disjunctive cut principle of Balas and Jeroslow, and the related polyhedral annexation principle of Glover, provide new insights into cutting plane theory. This has resulted in its ability to not only subsume many known valid cuts but also improve upon them. Originally a set of notes were written for the purpose of putting together in a common terminology and framework significant results of Glover and others using a geometric approach, referred to in the literature as convexity cuts, and the algebraic approach of Balas and Jeroslow known as Disjunctive cuts. As it turned out subsequently the polyhedral annexation approach of Glover is also closely connected with the basic disjunctive principle of Balas and Jeroslow. In this monograph we have included these results and have also added several published results which seem to be of strong interest to researchers in the area of developing strong cuts for disjunctive programs. In particular, several results due to Balas [4,5,6,7], Glover [18,19] and Jeroslow [23,25,26] have been used in this monograph. The appropriate theorems are given without proof. The notes also include several results yet to be published [32,34,35] obtained under a research contract with the National Science Foundation to investigate solution methods for disjunctive programs.

The monograph is self-contained and complete in the sense that it attempts to pool together existing results which the authors viewed as important to future research on optimization using the disjunctive cut approach. However, we have not attempted to record and discuss all important known valid inequalities, and methods to develop them. We have also listed only a minimum of references. An interested researcher will find readily a larger and more meaningful list of references in [4,5,6,7,18,19,20,23,25,26].

In writing this monograph and in reporting the research results, the authors found the works of Egon Balas, Fred Glover and Bob Jeroslow fundamental and extremely thought provoking. These publications initiated this study, and

we are deeply indebted to them. We are also indebted to the National Science Foundation for supporting the research endeavor on Disjunctive Programming under their grant No. ENG 77-23683 and to Mike Thomas, Director of the School of Industrial Engineering at the Georgia Institute of Technology, for the support we have received in successfully completing this project. Finally, we are thankful to Mrs. Joene Owen for her cooperation and an excellent typing of this manuscript.

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Chapter I

INTRODUCTION

1.1 Basic Concepts

A disjunctive program is an optimization problem where the constraints represent logical conditions. In this monograph we are concerned with such conditions expressed as linear constraints. The methods associated with disjunctive programming are by no means novel. Some of the methods proposed over two decades ago to solve integer programming problems used cutting planes derived from logical statements implying integrality. It can be shown that these problems can be viewed as disjunctive programs and the cutting planes used in integer programming are special applications of the principal theorem in disjunctive programming. As amply demonstrated by the recent works of Balas, Glover and Jeroslow, the disjunctive programming approach has provided a powerful unifying theory of all cutting plane solution strategies. Furthermore, it has provided a completely different perspective to examine this theory and has enabled one to derive deeper insights into existing knowledge. In the exposition that follows, we will be presenting the existing and new thoughts on disjunctive programming so that a reader can readily understand the developments thus far, and appreciate the potentials for research in this area.

Let us first introduce some fundamental concepts involved in our investigation. By the term <u>disjunctive program</u>, we mean a linear or nonlinear program which contains logical conditions stated as linear constraints. In our context, logical conditions include the following operations, stated in terms of say, conditions A and B.

(i) <u>Conjunction</u> - denoted by $A \wedge B$, this asserts that both conditions A and B must hold. As an example, a polyhedral set may be viewed as a conjunction of several linear inequalities or half spaces.

(ii) <u>Disjunction</u> - denoted by $A \vee B$, this asserts that either condition A or B (or both) must hold. A common example of this, as mentioned above, arises in linear zero-one programs. There, in the presence of the restriction $0 \leq x \leq 1$ on each variable, one has the disjunction that either $x \leq 0$ or $x \geq 1$ must hold.

(iii) Negation or Complement - denoted by $\bar{A}$ this asserts that condition A must not hold. For example, one might assert in some context that the total cost $2x_1 + 3x_2$, say, must not exceed 7 units. Thus, condition A is $2x_1 + 3x_2 > 7$ and the relevant negation is $2x_1 + 3x_2 \leq 7$.

(iv) Implication - denoted by A => B, this asserts that if A holds, then B must hold. As an example one might say in some context that if a plant i is located at a certain potential site, then the total output from it must be at least p_i units. Letting $y_i = 1$ or 0 according as the plant i is located or not, and letting $\sum_j x_{ij}$ denote the total output from it, the implication condition is

$$y_i = 1 \Rightarrow \sum_j x_{ij} \geq p_i$$

Note that this implication is equivalent to the disjunction

$$\{y_i = 0\} \vee \{\sum_j x_{ij} \geq p_i\}$$

In general, A => B is equivalent to the disjunction $\bar{A} \vee B$. Hence, examining the above logical conditions, one may note that conjunctions and negations stated in terms of linear inequalities lead to polyhedral sets which are, as it is well known, convex. Moreover, implications are essentially disjunctions as shown above. Now, it is the operation of disjunction which leads to nonconvexities and renders the problem of interest to us.

Let us now proceed to formulate a disjunctive program in a general setting and then cite and briefly examine several important problems which are special cases of this problem. The notations we use throughout this study are, as far as possible, consistent with those in existing literature.

Consider the following constraint sets S_h, where $h \in H$, an index set which may or may not be of finite cardinality.

$$S_h = \{x: A^h x \geq b^h, x \geq 0\}, h \in H \qquad (1.1)$$

In terms of the sets S_h, one may state a disjunction

$$x \in \bigcup_{h \in H} S_h \text{ or simply } \bigvee_{h \in H} \{A^h x \geq b^h, x \geq 0\} \qquad (1.2)$$

This disjunction may be imbedded into a general problem called a disjunctive program as follows:

$$\text{DP:} \quad \text{minimize} \quad f(x)$$
$$\text{subject to} \quad x \in X$$
$$x \in \bigcup_{h \in H} \{A^h x \geq b^h, x \geq 0\}$$

where f: $R^n \to R$ is lower semicontinuous and where X, is a closed subset of the nonnegative orthant of R^n.

The application of disjunctive methods to solve problems of type DP above, involve the derivation of suitable cutting planes or valid linear inequalities defined as follows:

Definition - An inequality $\pi x \geq \pi_0$ is said to be a valid inequality for the disjunction $x \in \bigcup_{h \in H} S_h$ if

$$x \in S = \bigcup_{h \in H} S_h \text{ implies } \pi x \geq \pi_0 \qquad (1.3)$$

Before proceeding any further, let us pause and examine some special cases of Problem DP which have been of interest to researchers. These problems, discussed along with their applications in the next section, include the generalized lattice point problem, the cardinality constrained linear program, the binary mixed-integer linear program, the extreme point optimization problem, the linear complementarity problem, and numerous others. Later in Chapter VIII we will discuss in some detail certain specific problems viewed as disjunctive programs.

1.2 Special Cases of Disjunctive Programs and Their Applications

1.2.1 The Generalized Lattice Point Problem

This problem may be stated mathematically as follows

GLPP: minimize $c^t x$

$$\text{subject to} \quad v = d - Dx \geq 0 \qquad (1.4)$$
$$u = b - Ax \geq 0$$

$$\left\{ \begin{array}{l} \text{and at least } q \text{ components of } u, \text{ corresponding to linearly} \\ \text{independent rows of } A, \text{ must be zero.} \end{array} \right\} \qquad (1.5)$$

Note that the superscript t will, unless otherwise stated, be used to denote the matrix transpose operation. Here, we assume that D is of dimension mxn and A is of dimension pxn. Now let us take different combinations of q out of p components of u which correspond to linearly independent rows of A. Thus, suppose that there are $\hat{h} \leq \binom{p}{q}$ such combinations and let H denote the index set $\{1,\ldots,\hat{h}\}$. For any such combination, say $h \in H$, define the set

$S_h = \{u\colon u_i = 0$ if i is the index of one of the q components of u corresponding to $h\}$, for $h \in H$ (1.6)

Then, Problem GLPP may be restated in a form, usually referred to as the <u>disjunctive normal form</u>, as follows:

$$\text{minimize} \quad c^t x$$
$$\text{subject to} \quad v = d - Dx \geq 0$$
$$u = b - Ax \geq 0$$
$$u \in \bigcup_{h \in H} S_h \qquad (1.7)$$

Note that constraints (1.5) or equivalently, constraint (1.7) with S_h defined in (1.6), essentially states that u must be an interior point with respect to at most a p-q dimensional face of the set $U = \{u\colon u_i \geq 0$ for each $i\}$. We remark that one may relax constraint (1.5) to simply assert that at least q of the p components of u must be zero, whence, $\hat{h} = \binom{p}{q}$.

Problem GLPP has been used as a special subroutine for minimizing a concave function over a convex region and for determining the most degenerate solution to a linear programming problem. In the latter context, such a solution is desirable, for example, in a fixed charge problem which has large fixed costs and linear variable costs. In this case, the most degenerate linear programming solution yields a good lower bound and/or starting point for any other scheme. Among other applications, the multiple choice problem is of significant importance.

1.2.2 The Cardinality Constrained Linear Program

This problem is a special case of the generalized lattice point problem, and may be stated as follows,

<u>CCLP</u>:

$$\begin{aligned} \text{minimize} \quad & c^t x \\ \text{subject to} \quad & Dx \leq d \\ & x \geq 0 \\ & |x|^+ \leq n-q \end{aligned} \tag{1.8}$$

where D is of dimension mxn and $|x|^+$ denotes the number of positive components of the vector x. Again, as before, we may transform constraint (1.8) to restate the problem in the normal disjunctive form. For this purpose, define the index set $H = \{1,\ldots,\binom{n}{q}\}$ and let each $h \in H$ correspond to q particular components of x such that set H exhausts all such combinations. Hence, define

$$S_h = \{x\text{: the q components of x corresponding to h are equal to zero}\} \tag{1.9}$$

Thus, Equation (1.8) may be replaced by

$$x \in \bigcup_{h \in H} S_h \tag{1.10}$$

As an application, one may consider the manufacture of several (n) items at a production facility and let x_i denote the volume of production for item or product i, i=1,...,n. The constraints $Dx \leq d$ may represent resource limitations and the disjunctive constraints (1.8), or equivalently (1.10) with (1.9), may restrict the production to at most (n-q) items.

In a like manner, one may be concerned with the location of (n-q) facilities at a subset of a number of potential sites. These facilities are required to satisfy a certain demand. The problem then may be to optimally locate these facilities and determine their capacity so as to minimize costs while satisfying demands.

There is a generalization of Problem CCLP known as the Element Constrained Linear Program (ECLP). Here, decision variables y_i, $i=1,\dots,n$ are defined according to

$$y_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } i=1,\dots,n$$

The decision vector $y=(y_1,\dots,y_n)$ then is restricted according to a constraint set $Fy \leq f$. Thus, Problem CCLP is a special case of Problem ECLP with $Fy \leq f$ denoting the single constraint

$$\sum_{i=1}^{n} y_i \leq n-q$$

Hence, in the first example cited above for Problem CCLP, one may have certain contingency constraints between products or certain products may be mutually exclusive. Such interactions between products would convert the problem into an element constrained linear program.

1.2.3 The Binary Mixed Integer Linear Program

This problem is also a special case of Problem GLPP. It may be formulated mathematically as follows:

$$\text{BMILP:} \quad \begin{aligned} \text{minimize} \quad & c_1^t x_1 + c_2^t x_2 \\ \text{subject to} \quad & D_1 x_1 + D_2 x_2 \leq d \\ & x_1 \geq 0 \\ & x_{2i} = 0 \text{ or } 1 \text{ for each } i=1,\dots,n \end{aligned} \tag{1.11}$$

where D_2 is of dimension mxn. To write Problem BMILP in the disjunctive normal form, note that (1.11) is equivalent to the following constraints:

$$x_{2i} + u_i = 1 \qquad i=1,\ldots,n$$
$$x_{2i},\ u_i \geq 0 \qquad i=1,\ldots,n$$
$$\{\text{At least n components of } (x_2,u) \text{ are zero}\} \tag{1.12}$$

Now one may transform Equation (1.12) in a manner identical to that used for transforming (1.8) to (1.10) through the definition (1.9). Problem BMILP has several well known applications such as the multiple choice programming problem, the knapsack problem, the fixed-charge location-allocation problem, and others.

1.2.4 The Extreme Point Optimization Problem

This problem is closely related to Problem GLPP, and may be stated as follows:

$$\underline{\text{EPP}}: \qquad \text{minimize} \quad c^t x$$
$$\text{subject to} \quad Dx = d$$
$$x \text{ is an extreme point of } P = \{x\colon Ax = b,\ x \geq 0\} \tag{1.13}$$

Hence, D is of dimension mxn and A is of dimension (pxn). Let us attempt to re-write Equation (1.13). Consider any point $x \in X$ and identify those components x_j of this point x which satisfy $x_j = 0$. Let $J = \{j\colon x_j = 0\} \subseteq \{1,\ldots,n\}$. Now construct a matrix $\tilde{I}$ whose rows are comprised of unit vectors e_j, each row corresponding to a $j \in J$, where e_j has all components zero except for a unity in position j. Then consider the matrix $\begin{pmatrix} A \\ \tilde{I} \end{pmatrix}$. Then one may easily see that a point $x \in X$ is an extreme point of X if and only if $\begin{pmatrix} A \\ \tilde{I} \end{pmatrix}$ has rank n.

Thus, to write Problem EPP in a disjunctive normal form, consider an enumeration of all subsets J_h of the set $\{1,\ldots,n\}$ such that if one constructs a matrix I_h for each such J_h, where I_h has rows comprised of vectors e_j for $j \in J_h$, then the matrix $\begin{pmatrix} A \\ I_h \end{pmatrix}$ has rank n. Further, let H contain the indices h corresponding to such sets J_h. Then let us define

$$S_h = \{x: \; x_j \leq 0 \text{ for each } j \in J_h, \; x \geq 0\} \text{ for each } h \in H \tag{1.14}$$

Using (1.14), we may now re-write (1.13) to formulate Problem EPP in a disjunctive normal form as follows:

$$\begin{aligned} \text{minimize} \quad & c^t x \\ \text{subject to} \quad & Dx = d \\ & Ax = b \\ & x \in \bigcup_{h \in H} S_h \end{aligned}$$

Applications of Problem EPP include several bilinear programming problems such as the location-allocation problem using rectilinear distance measure. The problem of minimizing inventory and changeover costs for a single machine scheduling situation has also been formulated as Problem EPP. Another application is its use as a subroutine in a cutting plane procedure to find an extreme point of a set which is also feasible to a system of cuts generated at any stage.

1.2.5 The Linear Complementarity Problem

This problem may be stated mathematically as

LCP:

$$\begin{aligned} \text{minimize} \quad & c^t x \\ \text{subject to} \quad & Dx = d \\ & x \geq 0 \\ & x_p x_q = 0 \quad \text{for each } (p,q) \in Z \end{aligned}$$

where Z is an appropriate set of two-tuple indices. Now, consider the construction of $2^{|Z|}$ distinct sets J_h, $h \in H = \{1,\ldots,2^{|Z|}\}$, where each J_h has exactly one of the indices p,q for each $(p,q) \in Z$. Define

$$S_h = \{x: \; x_j \leq 0 \text{ for } j \in J_h, \; x \geq 0\} \quad \text{for each } h \in H.$$

Then Problem LCP may be restated in the disjunctive normal form as

$$\begin{aligned} \text{minimize} \quad & c^t x \\ \text{subject to} \quad & Dx = d \\ & x \in \bigcup_{h \in H} S_h \end{aligned}$$

When the cardinality of the set H is small, Problem DP can easily be solved using the solution of $|H|$ problem as shown by Theorem 1.1 below. When this direct approach is not available, we need more sophisticated tools. This is the subject of discussion over the next few chapters.

Theorem 1.1.

Consider Problem DP stated above and assume $|H| < \infty$. Define problems

$$DP_h: \text{ minimize } \{f(x): \ x \in X \cap S_h\} \quad \text{for each } h \in H \tag{1.15}$$

Let x^h solve DP_h. Then x^{h*} solves DP, where

$$f(x^{h*}) = \underset{h \in H}{\text{minimum}} \ \{f(x^h)\} \tag{1.16}$$

Proof. By contradiction, suppose x^* solves DP with $f(x^*) < f(x^{h*})$, and assume that $x^* \in S_{\hat{h}}$ for some $\hat{h} \in H$. Since x^* is feasible to $DP_{\hat{h}}$ and $x^{\hat{h}}$ solves $DP_{\hat{h}}$, we must have $f(x^*) \geq f(x^{\hat{h}}) \geq f(x^{h*})$, a contradiction. This completes the proof.

Essentially, Theorem 1.1 involves the solution of $|H|$ problems in order to recover an optimal solution to Problem DP. This may be a viable approach for some special problems for which the cardinality of H is not too large. For example, one may be considering a production planning problem in which each set S_h may represent the restrictions on the process accruing from the implementation of production method $h \in H$. On the other hand, for zero-one linear integer programs for example, the application of Theorem 1.1 is tantamount to total enumeration and for a complementarity problem which requires, say, $u_j v_j = 0$ for $j=1,\ldots,m$, one would need to solve 2^m problems to obtain an optimal solution. It is for the solution of such problems, that we devote this study.

We now discuss some basic concepts and principles involved in disjunctive programming methods. An attempt is made in this chapter to present thoughts and ideas and to derive results so that the development is intuitively appealing. Thereafter, we discuss in a general context, the derivation of deep disjunctive cuts and also look at certain specializations. We then digress momentarily to demonstrate how the depth of cut that can be derived depends upon the formulation of the disjunctive statement. Based on this exposition, we discuss procedures for strengthening given valid cuts. This is then related to the supports and facets of the convex hull of feasible points. Following this, we show that disjunctive cutting planes subsume all other types of cutting planes by recovering several known cutting planes from a general form of a disjunctive cutting plane. Finally, we treat special cases of disjunctive programs. First, we demonstrate how the notion of the convex hull of feasible points admits two finitely convergent procedures for a special class of disjunctive programs known as facial disjunctive programs. Thereafter, we discuss, some specific applications.

1.3 Notes and References

Owen [29] considered a class of problems where at least one variable from each of several sets is required to be equal to zero. Applications of this formulation include integer programming, the linear complementary problem and the concave minimization problem. Since valid inequalities were derived from certain logical disjunctions, Owen called these valid inequalities disjunctive constraints. He noted that the cuts derived are indeed special cases of valid inequalities derived by Glover and Klingman [15] in the context of generalized Lattice Point problems. In his paper, Owen has really given a primitive algorithm, but the spirit of the approach is that of the cutting plane algorithms proposed by Gomory, Balas, Young and others for Integer Programming, by Tui, Balas, and Ritter in the context of nonconvex problems with linear constraints. It is covered by the general theory of convexity cuts of Glover [16,18], and overlaps with the work of Balas [2,3] and Burdet [10,12] using polarity.

An excellent survey of disjunctive programming principles and applications in the spirit of this chapter may be found in the several works of Balas [6,7], and Jeroslow [23,25].

Chapter II

BASIC CONCEPTS AND PRINCIPLES

2.1 Introduction

In this chapter, we will lead the reader to the most important and fundamental results in disjunctive programming. In order to enable the reader to appreciate the subject matter and to gain better insight into it, we will develop these results from first principles through well known facts. Toward this end, let us commence our discussion with the following well known concept.

2.2 Surrogate Constraints

Let us consider the following constraint set

$$S_1 = \{x: \sum_{j \in N} a_{ij}x_j \geq b_i \text{ for each } i \in Q_1,\ x \geq 0\} \tag{2.1}$$

where $N = \{1,\dots,n\}$ is index set for the x-variables and Q_1 is an index set for the linear constraints in S_1 aside from the nonnegativity restrictions. Now let us multiply each of these linear constraints by corresponding nonnegative parameters λ_i, $i \in Q_1$. Then clearly, $x \in S_1$ implies that

$$\sum_{j \in N} \lambda_i a_{ij} x_j \geq \lambda_i b_i \text{ for each } i \in Q_1 \tag{2.2}$$

By simply summing up the constraints (2.2), the following well known result is easily established.

Lemma 2.1

Let S_1 be the constraint set of Equation (2.1). Then $x \in S_1$ implies that

$$\sum_{j \in N} \left\{ \sum_{i \in Q_1} \lambda_i a_{ij} \right\} x_j \geq \sum_{i \in Q_1} \lambda_i b_i \tag{2.3}$$

for any set of nonnegative parameters λ_i, $i \in Q_1$.

Let us consider the converse of Lemma 2.1. In doing so, we are addressing the following question. Suppose that we are given an inequality $\pi x \geq \pi_0$ which is implied by the constraint set S_1. That is, $x \in S$ implies $\pi x \geq \pi_0$. Then, does there exist a surrogate constraint of the form (2.3) obtained through suitable parameters $\lambda_i \geq 0$, $i \in Q_1$ such that this surrogate constraint uniformly dominates the given inequality? The answer is yes. We are able to specify parameters $\lambda_i \geq 0$, $i \in Q_1$ such that if x satisfies (2.3) using these parameters $\lambda_i \geq 0$, $i \in Q_1$, then x must satisfy $\pi x \geq \pi_0$. We establish this result below and then illustrate it through an example.

Lemma 2.2

Let $\pi x \geq \pi_0$ be any inequality implied by S_1 of Equation (2.1) and suppose that S_1 is consistent. Then, there exists a set of nonnegative multipliers λ_i, $i \in Q_1$ such that

$$\sum_{i \in Q_1} \lambda_i a_{ij} \leq \pi_j \text{ for each } j \in N \text{ and } \pi_0 \leq \sum_{i \in Q_1} \lambda_i b_i \qquad (2.4)$$

Proof. Consider the following linear program P and its dual D

$$\underline{P}: \qquad \text{minimize} \left\{ \sum_{j \in N} \pi_j x_j : \sum_{j \in N} a_{ij} x_j \geq b_i,\ i \in Q_i,\ x \geq 0 \right\}$$

$$\equiv \underset{x \in S_1}{\text{minimize}} \left\{ \sum_{j \in N} \pi_j x_j \right\}$$

$$\underline{D}: \qquad \text{maximize} \left\{ \sum_{i \in Q_1} \lambda_i b_i : \sum_{i \in Q_1} \lambda_i a_{ij} \leq \pi_j,\ j \in N,\ \lambda_i \geq 0 \right\}$$

$$\text{for each } i \in Q_1$$

Now, since $x \in S_1$ implies $\sum_{j \in N} \pi_j x_j \geq \pi_0$, the primal problem is bounded below by π_0 and hence the feasible region of D is non-empty. Further, since S_1 is consistent, there exists an optimal solution to Problem D. It is easy to see now that the required result holds for any set of dual optimal variables λ_i, $i \in Q_1$. This completes the proof.

Let us illustrate the above result with an example. Consider

$$S_1 = \{x:\ x_1 + 2x_2 \geq 2,\ 3x_1 + x_2 \geq 3,\ x_1, x_2 \geq 0\}$$

Now $x_1 + 2x_2 \geq 2$, x_1, $x_2 \geq 0$ imply that $2x_1 + 2x_2 \geq 2$ or that $x_1 + x_2 \geq 1$. Alternatively, $3x_1 + x_2 \geq 3$, x_1, $x_2 \geq 0$ imply that $3x_1 + 3x_2 \geq 3$ or that $x_1 + x_2 \geq 1$. Hence, the inequality $x_1 + x_2 \geq 1$ is implied by S_1. Can we find a surrogate constraint which uniformly dominates this constraint? For this purpose, we consider Problem D in the proof of Lemma 2.2, namely,

$$\text{D:} \qquad \begin{aligned} \text{maximize} \quad & 2\lambda_1 + 3\lambda_2 \\ \text{subject to} \quad & \lambda_1 + 3\lambda_2 \leq 1 \\ & 2\lambda_1 + \lambda_2 \leq 1 \\ & \lambda_1, \lambda_2 \geq 0 \end{aligned}$$

One may readily verify that $\lambda_1 = \frac{2}{5}$, $\lambda_2 = \frac{1}{5}$ solves this problem. The surrogate constraint resulting from this is $x_1 + x_2 \geq \frac{7}{5}$. It is also interesting to note that this was a unique optimal solution to Problem D above. Thus, in this case what Problem D essentially did was to translate the cutting plane $x_1 + x_2 \geq 1$ parallel to itself until it supported the feasible region S_1. This is illustrated in Figure 2.1 below.

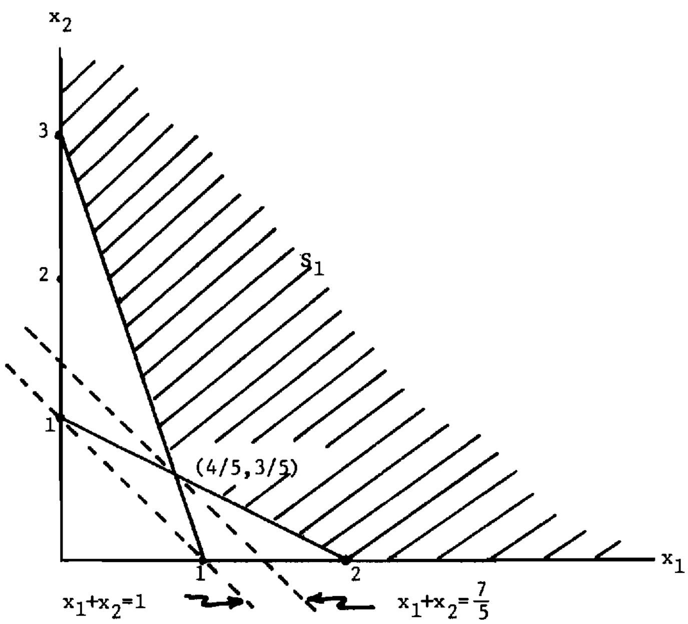

Figure 2.1. Dominance of a Surrogate Constraint

In fact, if $\pi x = \pi_0$ does not support S_1 and $\pi x \geq \pi_0$ is a valid inequality for S_1, then clearly, $x \in S_1$ implies $\pi x > \pi_0$. Thus, if one solves the problem P defined in the proof of Lemma 2.2 and obtains therefrom $\hat{\pi}_0 = \underset{x \in S_1}{\text{minimum}} (\pi x)$ then $\hat{\pi}_0 > \pi_0$ and hence, $\pi x \geq \hat{\pi}_0$ strictly dominates $\pi x \geq \pi_0$. Moreover, $\pi x \geq \hat{\pi}_0$ is a valid inequality for S_1. Thus, Lemma 2.2 would then yield a surrogate constraint which uniformly dominates $\pi x \geq \pi_0$. In other words, non-dominated surrogate constraints support S_1.

2.3 Pointwise-Supremal Cuts

We now proceed to recall another well known, pertinent concept. Suppose that the sets S_h of Equation (1.1) are comprised of a single linear constraint and are given by

$$S_h = \left\{ x: \sum_{j \in N} a^h_{1j} x_j \geq b^h_1,\ x \geq 0 \right\} \text{ for each } h \in H \qquad (2.5)$$

We use the above notation so as to be consistent with the case wherein each S_h, $h \in H$ may contain several constraints as introduced later. We are considering at this point the disjunction $x \in \bigcup_{h \in H} S_h$.

For example, let $S_1 = \{x: 2x_1 - 3x_2 \geq 5,\ x \geq 0\}$ and $S_2 = \{x: x_1 + 3x_2 \geq 4,\ x \geq 0\}$ and consider the statement that x satisfies S_1 or S_2. Then, it is a well known fact that x must satisfy $[\max\{2,1\}]x_1 + [\max\{-3,3\}]x_2 \geq \min\{5,4\}$ or $2x_1 + 3x_2 \geq 4$. Thus $2x_1 + 3x_2 \geq 4$ is a valid inequality for the disjunction $x \in \bigcup_{h \in H} S_h$ since it is implied by each of S_1 and S_2. This fact is generalized and formalized below. First consider the following definition. Then, Lemma 2.3 establishes the required result.

Definition

Consider a set of vectors $\{v^h: h \in H\}$ where for each $h \in H$, $v^h = v^h_1, \ldots, v^h_n$. Then, the pointwise supremum of this set of vectors, denoted by $\sup_{h \in H} v^h$, is a vector $v = (v_1, \ldots, v_n)$ with components

$$v_j = \underset{h \in H}{\text{supremum}} \{v^h_j\} \text{ for } j = 1, \ldots, n$$

In a like manner, we define the pointwise infimum of a set of vectors.

Lemma 2.3

Consider the constraint sets S_h, $h \in H$ as defined in Equation (2.5). Then, for the disjunction $x \in \bigcup_{h \in H} S_h$, the following inequality is valid.

$$\sum_{j \in N} \gamma_j x_j \geq \gamma_0, \text{ where } \gamma = \sup_{h \in H} \{a_1^h\},\ \gamma_0 = \inf_{h \in H} \{b_1^h\} \tag{2.6}$$

Proof. Consider any $\bar{x} \in \bigcup_{h \in H} S_h$. Hence, there exists an $\hat{h} \in H$ for which

$$\sum_{j \in N} a_{1j}^{\hat{h}} \bar{x}_j \geq b_1^{\hat{h}},\ \bar{x} \geq 0$$

This implies that

$$\sum_{j \in N} \gamma_j \bar{x}_j = \sum_{j \in N} \left\{ \sup_{h \in H} a_{1j}^h \right\} \bar{x}_j \geq \sum_{j \in N} a_{1j}^{\hat{h}} \bar{x}_j \geq b_1^{\hat{h}} \geq \inf_{h \in H} \left(b_1^h \right) = \gamma_0$$

and the proof is complete.

We will now put Lemmas (2.1), (2.2) and (2.3) together to show that this leads to the fundamental result of disjunctive programming.

2.4 Basic Disjunctive Cut Principle

Suppose that we have constraint sets of the form (1.1), that is.

$$S_h = \left\{ x: \sum_{j \in N} a_{ij}^h x_j \geq b_i^h \text{ for each } i \in Q_h,\ x \geq 0 \right\},\ h \in H \tag{2.7}$$

where Q_h is an appropriate index set for the constraints in S_h, $h \in H$.

Consider the disjunction $x \in \bigcup_{h \in H} S_h$. Let us now use our discussion in the two proceeding sections to derive valid inequalities for this disjunction.

First of all, note that when each S_h, $h \in H$ has only a single linear constraint, then from Section 2.3 we are able to derive valid inequalities for the disjunction $x \in \bigcup_{h \in H} S_h$. Hence, let us use surrogate constraints to trans-from the given sets S_h into singleton constraint sets and then use the concepts of Section 2.3.

More specifically, let λ_i^h, $i \in Q_h$, $h \in H$ be any set of nonnegative parameters. For each $h \in H$, let us use the corresponding multipliers λ_i^h, $i \in Q_h$, and from the surrogate constraints

$$\sum_{j \in N} \left\{ \sum_{i \in Q_h} \lambda_i^h a_{ij}^h \right\} x_j \geq \sum_{i \in Q_h} \lambda_i^h b_i^h \quad \text{for each } h \in H \tag{2.8}$$

Next, let us define sets $\hat{S}_h$, $h \in H$ as follows

$$\hat{S}_h = \{x: \text{ Equation (2.8) is satisfied, } x \geq 0\} \tag{2.9}$$

Now clearly, $x \in S_h$ implies that $x \in \hat{S}_h$. Hence, the disjunction $x \in \bigcup_{h \in H} S_h$ may be replaced by the (weaker) disjunction $x \in \bigcup_{h \in H} \hat{S}_h$. But then, from Lemma (2.3), valid inequalities for the latter disjunction are of the form

$$\sum_{j \in N} \left[\sup_{h \in H} \left\{ \sum_{i \in Q_h} \lambda_i a_{ij}^h \right\} \right] x_j \geq \inf_{h \in H} \left\{ \sum_{i \in Q_h} \lambda_i^h b_i^h \right\} \tag{2.10}$$

This result is known as the forward part of the Basic Disjunctive Cut Principle. To arrive at the converse statement, consider any valid inequality $\sum_{j \in N} \pi_j x_j \geq \pi_0$ implied by $x \in \bigcup_{h \in H} S_h$, and assume that each S_h is consistent. Thus, since $x \in S_h$ implies $x \in \bigcup_{h \in H} S_h$, then $\pi x \geq \pi_0$ is a valid inequality for each S_h, $h \in H$. Now, applying Lemma 2.2 for each $h \in H$, we may hence assert that there exist nonnegative parameters λ_i^h, $i \in Q_h$ such that

$$\left\{ \sum_{i \in Q_h} \lambda_i^h a_{ij}^h \leq \pi_j \quad \text{for each } j \in N \text{ and } \pi_0 \leq \sum_{i \in Q_h} \lambda_i^h b_i^h \right\} \text{for each } h \in H$$

This in turn implies that

$$\sup_{h \in H} \left[\sum_{i \in Q_h} \lambda_i^h a_{ij}^h \right] \leq \pi_j \text{ for each } j \in N \text{ and } \pi_0 \leq \inf_{h \in H} \left[\sum_{i \in Q_h} \lambda_i^h b_i^h \right] \tag{2.11}$$

This result is known as the reverse part of the Basic Disjunctive Cut Principle. Hence, the forward part gives us a set of valid inequalities for the disjunction $x \in \bigcup_{h \in H} S_h$, one for each choice of λ_i^h, $i \in Q_h$, $h \in H$. The reverse part then asserts that if each S_h is consistent, then any valid inequality may be uniformly dominated by a disjunctive cut of the type (2.10). These results are stated formally below.

Theorem 2.1 (Basic Disjunctive Cut Principle)

Let S_h, $h \in H$ be constraint sets given by Equation (1.1). Here, $|H|$ may or may not be finite. Suppose that at least one of the linear inequality systems S_h, $h \in H$ must hold. Then, for any choice of nonnegative vectors $\lambda^h = (\lambda_i^h,\ i \in Q_h)$, the inequality

$$\left[\sup_{h\in H} (\lambda^h)^t A^h\right] x \geq \inf_{h\in H} (\lambda^h)^t b^h$$

is a valid disjunctive cut, where the superscript t denotes the transpose operation.

Furthermore, if every system S_h is consistent, then for any valid inequality $\sum_{j\in N} \pi_j x_j \geq \pi_0$, there exist nonnegative vectors λ^h, $h \in H$, such that $\pi_0 \leq \inf_{h\in H} (\lambda^h)^t b^h$ and for each $j \in N$, the jth component of $\sup_{h\in H}(\lambda^h)^t A^h$ does not exceed π_j.

Thus far, we have demonstrated that (2.10) yields valid inequalities with no mention being made regarding the selection of values for the parameters λ_i^h, $i \in Q_h$, $h \in H$. This is the subject matter of the next chapter.

2.5 Notes and References

The basic disjunctive cut discussed in Section 2.4 is due to Balas, Glover, and Jeroslow. The forward part appears in Balas [4,6] and the converse in Jeroslow [25]. The same result in a different setting was given by Glover [18,19].

Chapter III

GENERATION OF DEEP CUTS USING THE FUNDAMENTAL DISJUNCTIVE INEQUALITY

3.1 Introduction

Recall from Chapter I that our motivation in using disjunctive programming methods is to aid us in solving nonconvex problems of the type

$$\begin{array}{llll} \text{DP:} & \text{minimize} & f(x) & \\ & \text{subject to} & x \in X & (3.1) \\ & & x \in \bigcup_{h \in H} S_h & (3.2) \end{array}$$

where f: $R^n \to R$ is lower semicontinuous, X is a closed subset of the nonnegative orthant of R^n and each S_h, $h \in H$ is given by Equation (1.1).

Adopting a relaxation strategy to solve Problem DP suppose we relax constraint (3.2). If a solution to the resulting problem is feasible to (3.2), then it solves Problem DP. Otherwise, we have a point infeasible to the disjunction (3.2). We thus derive a cut which is valid in the sense that it deletes the current point, but deletes no point satisfying (3.2). We add this inequality to the relaxed problem and update the current solution. Thus, at any stage, we solve the problem to minimize f(x) subject to $x \in X$ and x satisfies the linear disjunctive inequalities or cuts generated thus far. The procedure terminates when a solution to such a problem satisfies (3.2).

Now in Chapter II we demonstrated that (2.10) defines valid cuts for the disjunctive statement (3.2). Given the current point infeasible to (3.2), we now address the question of selecting nonnegative values for the parameters λ_i^h, $i \in Q_h$, $h \in H$ in the inequality (2.10) so as to derive a deep disjunctive cut. We will be devoting our attention to the following two disjunctions titled DC1 and DC2. We remark that numerous disjunctive statements can be cast in the format of DC1 or DC2.

DC1:

Suppose that each system S_h is comprised of a single linear inequality, that is, let

$$S_h = x: \left\{ \sum_{j=1}^{n} a_{1j}^{h} x_j \geq b_1^h, \quad x \geq 0 \right\} \text{ for } h \in H = \{1, \ldots, \hat{h}\} \tag{3.3}$$

where we assume that $\hat{h} = |H| < \infty$ and that each inequality in S_h, $h \in H$ is stated with the origin as the current point at which the disjunctive cut is being generated. Then, the disjunctive statement DC1 is that at least one of the sets S_h, $h \in H$ must be satisfied. Since the current point (origin) does not satisfy this disjunction, we must have $b_i^h > 0$ for each $h \in H$. Further, we will assume, without loss of generality, that for each $h \in H$, $a_{1j}^h > 0$ for some $j \in \{1, \ldots, n\}$ or else, S_h is inconsistent and we may disregard it.

DC2:

Suppose each system S_h is comprised of a set of linear inequalitites, that is, let

$$S_h = \left\{ x: \sum_{j=1}^{n} a_{ij}^{h} x_j \geq b_i^h \text{ for each } i \in Q_h, \quad x \geq 0 \right\} \text{ for } h \in H = \{1, \ldots, \hat{h}\} \tag{3.4}$$

where Q_h, $h \in H$ are appropriate constraint index sets. Again, we assume that $\hat{h} = |H| < \infty$ and that the representation in (3.4) is with respect to the current point as the origin. Then, the disjunctive statement DC2 is that at least one of the sets S_h, $h \in H$ must be satisfied. Although it is not necessary here for $b_i^h > 0$ for all $i \in Q_h$ one may still state a valid disjunction by deleting all constraints with $b_i^h \leq 0$, $i \in Q_h$ from each set S_h, $h \in H$. Clearly a valid cut for the relaxed constraint set is valid for the original constraint set. We will thus obtain a cut which possibly is not as strong as may be derived from the original constraints. To aid in our development, we will therefore assume henceforth that $b_i^h > 0$, $i \in Q_h$, $h \in H$. Figure 3.1 below illustrates the possible weakening of the cuts derived by such a deletion of constraints. Observe that since a valid cut defines a closed half-space which contains $\bigcup_{h \in H} S_h$, this half space must also contain the

the closure of the convex hull of $\bigcup_{h \in H} S_h$. Since the closure of the convex hull of the union of the sets S_h, $h \in H$ resulting after the deletion of the constraints as above contains the closure of the convex hull of the union of the original sets S_h, $h \in H$, the family of valid cuts derived by the new disjunction are a subset of those that are valid for the original disjunction. Incidentally, one may also note that facets of the closure of the convex hull of feasible points are desirable deep cuts.

Before proceeding with our analysis, let us briefly comment on the need for deep cuts. Although intuitively desirable, it is not always necessary to seek a deepest cut. For example, if one is using cutting planes to implicitly search a feasible region of discrete points, then all cuts which delete the same subset of this discrete region may be equally attractive irrespective of their depth relative to the convex hull of this discrete region. On the other hand, if one is confronted with the problem of iteratively exhausting a feasible region which is not finite, then indeed deep cuts are meaningful and desirable.

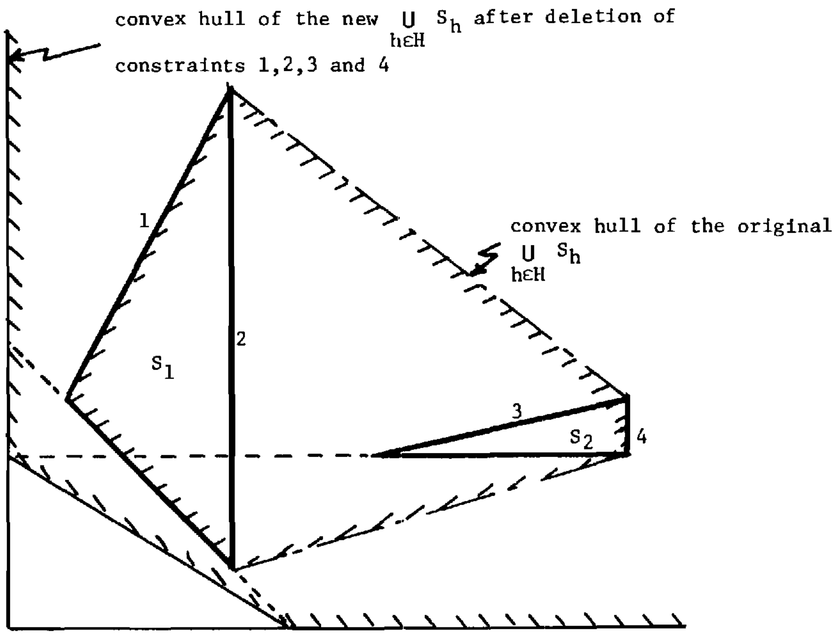

Figure 3.1. Formulation of the Disjunction DC2

3.2 Defining Suitable Criteria for Evaluating the Depth of a Cut

In this section, we will lay the foundation for the concepts we propose to use in deriving deep cuts. Specifically, we will explore the following two criteria for deriving a deep cut:

(i) Maximize the euclidean distance between the origin and the nonnegative region feasible to the cutting plane

(ii) Maximize the rectilinear distance between the origin and the nonnegative region feasible to the cutting plane.

Let us briefly discuss the choice of these criteria. Referring to Figure 3.2(a) below, one may observe that simply attempting to maximize the euclidean distance from the origin to the cut can favor a weaker cut over stronger cuts. However, since one is only interested in the subset of the nonnegative orthant feasible to the cuts, the choice of criterion (i) above avoids such anamolies. Of course, as Figure 3.2(b) indicates, it is possible for this criterion to be unable to recognize dominance and treat one cut and another one which dominates it as alternative optimal cuts.

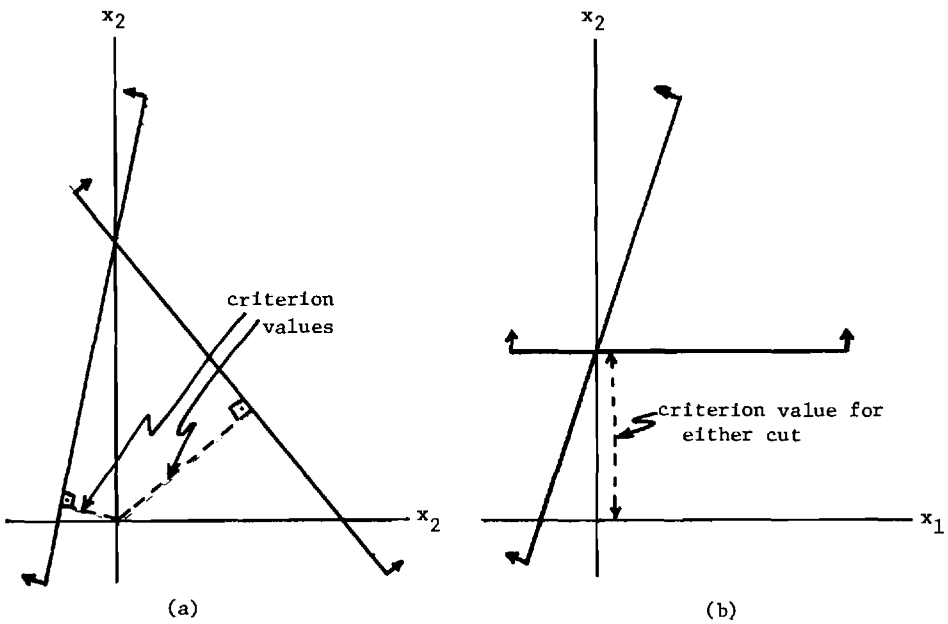

Figure 3.2. Recognition of Dominance

Let us now proceed to characterize the euclidean distance from the origin to the nonnegative region feasible to a cut

$$\sum_{j=1}^{n} z_j x_j \geq z_0, \text{ where } z_0 > 0,\ z_j > 0 \text{ for some } j \varepsilon \{1,\ldots,n\} \tag{3.5}$$

The required distance is clearly given by

$$\theta_e = \text{minimum } \{\|x\| : \sum_{j=1}^{n} z_j x_j \geq z_0,\ x \geq 0\} \tag{3.6}$$

where $\|x\| = \sqrt{\sum_{j=1}^{n} x_j^2}$. Consider the following result

Lemma 3.1

Let θ_e be defined by Equation (3.5) and (3.6). Then

$$\theta_e = \frac{z_0}{\|y\|} \tag{3.7}$$

where,

$$y = (y_1,\ldots,y_n),\ y_j = \text{maximum } \{0, z_j\},\ j=1,\ldots,n \tag{3.8}$$

Proof. Note that the solution $x^* = \left(\frac{z_0}{\|y\|^2}\right) y$ is feasible to the problem in (3.6) with $\|x^*\| = \frac{z_0}{\|y\|}$. Moreover, for any x feasible to (3.6), we have, $z_0 \leq \sum_{j=1}^{n} z_j x_j \leq \sum_{j=1}^{n} y_j x_j \leq \|y\| \ \|x\|$, or that, $\|x\| \geq \frac{z_0}{\|y\|}$. This completes the proof.

Now, let us consider the second criterion. The motivation for this criterion is similar to that for the first criterion and moreover, as we shall see below, the use of this criterion has intuitive appeal. First of all, given a cut (3.5), let us characterize the rectilinear distance from the origin to the nonnegative region feasible to this cut. This distance is given by

$$\theta_r = \text{minimum } \{|x|: \sum_{j=1}^{n} z_j x_j \geq z_0,\ x \geq 0\} \tag{3.9}$$

where $|x| = \sum_{j=1}^{n} |x_j|$. Consider the following result.

Lemma 3.2

Let θ_r be defined by Equations (3.5) and (3.9). Then,

$$\theta_r = \frac{z_0}{z_m} \text{ where } z_m = \underset{j=1,\ldots,n}{\text{maximum}}\ z_j \tag{3.10}$$

Proof. Note that the solution $x^* = (0,\ldots,\frac{z_0}{z_m},\ldots,0)$, with the m^{th} component being non-zero, is feasible to the problem in (3.9) with $|x^*| = \frac{z_0}{z_m}$. Moreover, for any x feasible to (3.9), we have,

$$\frac{z_0}{z_m} \leq \sum_{j=1}^{n} \frac{z_j}{z_m} x_j \leq \sum_{j=1}^{n} x_j = |x|$$

This completes the proof.

Note from Equation (3.10) that the objective of maximizing θ_r is equivalent to finding a cut which maximizes the smallest positive intercept made on any axis. Hence, the intuitive appeal of this criterion.

3.3 Deriving Deep Cuts for DC1

It is very encouraging to note that for the disjunction DC1 we are able to derive a cut which not only simultaneously satisfies both the criterion of Section 3.2, but which is also a facet of the set S defined by (3.11) below.

$$S = \text{closure convex hull of } \bigcup_{h\in H} S_h \tag{3.11}$$

This is a powerful statement since all valid inequalities are given through (2.10) and none of these can strictly dominate a facet of S.

We will find it more convenient to state our results if we normalize the linear inequalities (3.3) by dividing through by their respective, positive, right-hand-sides. Hence, let us assume without loss of generality that

$$S_h = \left\{ x: \sum_{j=1}^{n} a_{1j}^h x_j \geq 1, \ x \geq 0 \right\} \quad \text{for } h\varepsilon H=\{1,\ldots,\hat{h}\} \qquad (3.12)$$

Then the application of Theorem 2.1 to the disjunction DC1 yields valid cuts of the form:

$$\sum_{j=1}^{n} \left\{ \max_{h\varepsilon H} \lambda_1^h a_{1j}^h \right\} x_j \geq \min_{h\varepsilon H}\{\lambda_1^h\} \qquad (3.13)$$

where λ_1^h, $h \varepsilon H$ are nonnegative scalars. Again, there is no loss of generality in assuming that

$$\sum_{h\varepsilon H} \lambda_1^h = 1, \ \lambda_1^h \geq 0, \ h\varepsilon H=\{1,\ldots,\hat{h}\} \qquad (3.14)$$

since we will not allow all λ_1^h, $h \varepsilon H$ to be zero. This is equivalent to normalizing (3.13) be dividing through by $\sum_{h\varepsilon H} \lambda_1^h$.

Theorem 3.1 below derives two cuts of the type (3.13), both of which simultaneously achieve the two criteria of the foregoing section. However, the second cut uniformly dominates the first cut. In fact, no cut can strictly dominate the second cut since it is shown to be a facet of S defined by (3.11).

Theorem 3.1

Consider the disjunctive statement DC1 where S_h is defined by (3.12) and is assumed to be consistent for each $h \varepsilon H$. Then the following results hold

(a) Both the criteria of Section 3.2 are satisfied by letting

$$\lambda_1^h = 1/\hat{h} = \lambda_1^{h*}, \text{ say, for } h \varepsilon H \qquad (3.15)$$

in inequality (3.13) to obtain the cut

$$\sum_{j=1}^{n} a_{1j}^* x_j \geq 1, \text{ where } a_{1j}^* = \max_{h\varepsilon H} a_{1j}^h, \ j=1,\ldots,n \qquad (3.16)$$

(b) Further, defining

$$\gamma_1^h = \underset{j:a_{1j}^h>0}{\text{minimum}} \{a_{1j}^*/a_{1j}^h\} > 0, \ h \in H \tag{3.17}$$

and letting

$$\lambda_1^h = \gamma_1^h \Big/ \sum_{p\in H} \gamma_1^p = \lambda_1^{h**}, \text{ say, for } h \in H \tag{3.18}$$

in inequality (3.13), we obtain a cut of the form

$$\sum_{j=1}^{n} a_{1j}^{**} x_j \geq 1 \tag{3.19}$$

which again satisfies both the criteria of Section 3.2.

(c) The cut (3.19) uniformly dominates the cut (3.16); in fact,

$$a_{1j}^{**} \begin{cases} = a_{1j}^* \text{ if } a_{1j}^* > 0 \\ \leq a_{1j}^* \text{ if } a_{1j}^* \leq 0 \end{cases}, \quad j=1,\ldots,n \tag{3.20}$$

(d) The cut (3.19) is a facet of the set S of Equation (3.11).

Proof.

(a) Clearly, $\lambda_1^h = 1/\hat{h}$, $h \in H$ leads to the cut (3.16) from (3.13). Now consider the euclidean distance criterion of maximizing θ_e (or θ_e^2) of Equation (3.7). For cut (3.16), the value of θ_e^2 is given by

$$(\theta_e^*)^2 = 1 \Big/ \sum_{j=1}^{n} (y_j^*)^2 > 0 \text{ where } y_j^* = \max\{0, a_{1j}^*\}, \ j=1,\ldots,n \tag{3.21}$$

Now, for any choice λ_1^h, $h \in H$,

$$\theta_e^2 = \left[\min_{h\in H}(\lambda_1^h)\right]^2 \Big/ \sum_{j=1}^{n} y_j^2 = (\lambda_1^p)^2 \Big/ \sum_{j=1}^{n} y_j^2, \text{ say}, \tag{3.22}$$

where $y_j = \max\{0, \max_{h\in H} \lambda_1^h a_{1j}^h\}$. If $\lambda_1^p = 0$, then $\theta_e = 0$ and noting (3.21), such a choice of parameters λ_1^h, $h \in H$ is suboptimal. Hence, $\lambda_1^p > 0$, whence (3.22) becomes

$\theta_e^2 = 1 \Big/ \sum_{j=1}^{n} \left(\frac{y_j}{\lambda_1^p}\right)^2$. But since $(\lambda_1^h/\lambda_1^p) \geq 1$ for each $h \in H$, we get

$$y_j/\lambda_1^p = \max\{0, \max_{h\varepsilon H} \frac{\lambda_1^h}{\lambda_1^p} \; a_{1j}^h\} \geq \max\{0, \max_{h\varepsilon H} a_{1j}^h\} = y_j^*$$

Thus $\theta_e^2 \leq (\theta_e^*)^2$ or that the first criterion is satisfied.

Now consider the maximization of θ_r of Equations (3.9), (3.10). For the choice (3.15), the value of θ_r is given by

$$\theta_r^* = \frac{1}{\max_j a_{1j}^*} > 0 \tag{3.23}$$

Now, for any choice λ_1^h, $h \in H$, from Equations (3.10), (3.6) we get

$$\theta_r = \left(\min_{h\varepsilon H} \lambda_1^h\right) \Big/ \left(\max_j \max_{h\varepsilon H} \lambda_1^h a_{1j}^h\right) = \lambda_1^p \Big/ \max_j \max_{h\varepsilon H} \lambda_1^h a_{1j}^h, \text{ say.}$$

As before, $\lambda_1^p = 0$ implies a value of θ_r inferior to θ_r^*. Thus, assume $\lambda_1^p > 0$. Then, $\theta_r = 1 \Big/ \max_j \max_{h\varepsilon H} \left(\frac{\lambda_1^h}{\lambda_1^p}\right) a_{1j}^h$. But $(\lambda_1^h/\lambda_1^p) \geq 1$ for each $h \in H$ and in evaluating θ_r, we are interested only in those $j\varepsilon\{1,\ldots,n\}$ for which $a_{1j}^h > 0$ for some $h \in H$. Thus $\theta_r \leq 1/\max_j \max_{h\varepsilon H} a_{1j}^h = \theta_r^*$, or that the second criterion is also satisfied. This proves part (a).

(b) and (c). First of all, let us consider the values taken by γ_1^h, $h \in H$. Note from the assumption of consistency that γ_1^h, $h \in H$ are well defined. From (3.16), (3.17), we must have $\gamma_1^h \geq 1$ for each $h \in H$. Moreover, if we define from (3.16)

$$H^* = \{h\varepsilon H\colon a_{1k}^h = a_{1k}^* > 0 \text{ for some } k\varepsilon\{1,\ldots,n\}\} \tag{3.24}$$

then clearly $H^* \neq \{\phi\}$ and for $h\varepsilon H^*$, Equation (3.17) implies $\gamma_1^h \leq 1$. Thus,

$$\gamma_1^h \begin{cases} = 1 & \text{for } h\varepsilon H^* \\ > 1 & \text{for } h\notin H^* \end{cases} \tag{3.25}$$

Hence,

$$\min_{h \in H} \gamma_1^h = 1 \tag{3.26}$$

or that, using (3.18) in (3.13) yields a cut of the type (3.19), where,

$$a_{1j}^{**} = \max_{h \in H} a_{1j}^h \gamma_1^h \ , \ j=1,\ldots,n \tag{3.27}$$

Now, let us establish relationship (3.20). Note from (3.16) that if $a_{1j}^* \leq 0$, then $a_{1j}^h \leq 0$ for each $h \in H$ and hence, using (3.25), (3.27), we get that (3.20) holds. Next, consider $a_{1j}^* > 0$ for some $j \in \{1,\ldots,n\}$. From (3.24), (3.25), (3.27), we get

$$a_{1j}^{**} = \max\{\max_{h \in H} a_{1j}^h, \max_{\substack{h \in H^* \\ a_{1j}^h > 0}} a_{1j}^h \gamma_1^h\} \tag{3.28}$$

where we have not considered $h \notin H^*$ with $a_{1j}^h \leq 0$ since $a_{1j}^{**} > 0$. But for $h \notin H^*$ with $a_{1j}^h > 0$, we get from (3.16), (3.17)

$$a_{1j}^h \gamma_1^h = a_{1j}^h \left[\min_{k: a_{1k}^h > 0} \left\{ \frac{\max_{r \in H} a_{1k}^r}{a_{1k}^h} \right\} \right] \leq a_{1j}^h \left\{ \frac{\max_{r \in H} a_{1j}^r}{a_{1j}^h} \right\} = \max_{r \in H} a_{1j}^r \tag{3.29}$$

Using (3.29) in (3.28) yields $a_{1j}^{**} = a_{1j}^*$, which establishes (3.20).

Finally, we show that (3.19) satisfies both the criteria of Section 3.2. This part follows immediately from (3.20) by noting that the cut (3.16) yields $\theta_e = \theta_e^*$ of (3.21) and $\theta_r = \theta_r^*$ of (3.23). This completes the proofs of parts (b) and (c).

(d) Note that since (3.19) is valid, any $x \in S$ satisfies (3.19). Hence, in order to show that (3.19) defines a facet of S, it is sufficient to identify n affinely independent points of S which satisfy (3.19) as an equality, since clearly, S is of dimension n. Define

$$J_1 = \{j \in \{1,\ldots,n\}: a_{1j}^{**} > 0\} \text{ and let } J_2 = \{1,\ldots,n\} - J_1 \tag{3.30}$$

Consider any $p \epsilon J_1$, and let

$$e_p = (0,\ldots,\frac{1}{a_{1p}^{**}},\ldots,0), \quad p \epsilon J_1 \tag{3.31}$$

have the non-zero term in the p^{th} position. Now, since $p \epsilon J_1$, (3.20) yields

$$a_{1p}^{**} = a_{1p}^{*} = \max_{h \epsilon H} a_{1p}^{h} = a_{1p}^{h_p}, \text{ say.}$$

Hence, $e_p \epsilon S_{h_p}$ and so, $e_p \epsilon S$ and moreover, e_p satisfies (3.19) as an equality. Thus, $e_p, p \epsilon J_1$ qualify as $|J_1|$ of the n affinely independent points we are seeking.

Now consider a $q \epsilon J_2$. Let us show that there exists an S_{h_q} satisfying

$$\gamma_1^{h_q} a_{1p}^{h_q} = a_{1p}^{**} \text{ for some } p \epsilon J_1$$

and (3.32)

$$\gamma_1^{h_q} a_{1q}^{h_q} = a_{1q}^{**}$$

From Equation (3.27), we get $a_{1q}^{**} = \max_{h \epsilon H} a_{1q}^{h} \gamma_1^{h} = a_{1q}^{h_q} \gamma_1^{h_q}$, say. Then for this $h_q \epsilon H$, Equation (3.17) yields $\gamma_1^{h_q} = \underset{j: a_{1j}^{h_q} > 0}{\text{minimum}} \{a_{1j}^{*}/a_{1j}^{h_q}\} = a_{1p}^{*}/a_{1p}^{h_q}$, say. Or, using (3.20) $\gamma_1^{h_q} a_{1p}^{h_q} = a_{1p}^{*} = a_{1p}^{**} > 0$. Thus (3.32) holds. For convenience, let us rewrite the set S_{h_q} below as

$$S_{h_q} = \{x: a_{1p}^{h_q} x_p + a_{1q}^{h_q} x_q + \sum_{j \neq p,q} a_{1j}^{h_q} x_j \geq 1, \ x \geq 0\} \tag{3.33}$$

Now, consider the direction

$$d_q = \begin{cases} (0,\ldots,\frac{1}{a_{1p}^{**}},\ldots, -\frac{1}{a_{1q}^{**}},\ldots,0) & \text{if } a_{1q}^{**} < 0 \\ (0,\ldots, 0 ,\ldots, \Delta ,\ldots,0) & \text{if } a_{1q}^{**} = 0 \end{cases} \tag{3.34}$$

where $\Delta > 0$. Let us shown that d_q is a direction for S_{h_q}. Clearly, if $a_{1q}^{**} = 0$, then from (3.32) $a_{1q}^{h_q} = 0$ and thus (3.33) establishes (3.34). Further, if $a_{1q}^{**} < 0$ then one may easily verify from (3.32), (3.33), (3.34) that

$$\hat{e}_p = (0,\ldots,\gamma_1^{h_q}/a_{1p}^{**},\ldots,0) \in S_{h_q} \text{ and } \hat{e}_p + \delta[\gamma_1^{h_q}\, d_q] \in S_{h_q} \text{ for each } \delta \geq 0$$

where $\hat{e}_p$ has the non-zero term at position p. Thus, d_q is a direction for S_{h_q}. It can be easily shown that this implies d_q is a direction for S. Since $e_p = (0,\ldots,\frac{1}{a_{1p}^{**}},\ldots,0)$ of Equation (3.31) belongs to S, then so does $(e_p + d_q)$. But $(e_p + d_q)$ clearly satisfies (3.19) as an equality. Hence, we have identified n points of S, which satisfy the cut (3.19) as an equality, of the type

$$\left.\begin{aligned} e_p &= (0,\ldots,\frac{1}{a_{1p}^{**}},\ldots,0) \text{ for } p\varepsilon J_1 \\ e_q &= d_q + e_p \text{ for some } p\varepsilon J_1, \text{ for each } q\varepsilon J_2 \end{aligned}\right\} \quad (3.35)$$

where d_q is given by (3.34). Since these n points are clearly affinely independent, this completes the proof.

Thus, in view of Theorem 3.1, it is "optimal" to derive a cut (3.19) for the disjunction DC1. In generalizing this to disjunction DC2, we find that such an ideal situation no longer exists. Nevertheless, we are able to obtain some meaningful results. But before proceeding to DC2, let us illustrate the above concepts through an example.

<u>Example</u>

Let H = {1,2}, n=3 and let DC1 be formulated through the sets

$$S_1 = \{x: x_1 + 2x_2 - 4x_3 \geq 1,\ x \geq 0\},\ S_2 = \{x: \frac{x_1}{2} + \frac{x_2}{3} - 2x_3 \geq 1,\ x \geq 0\}.$$

The cut (3.16), i.e., $\sum a_{1j}^{*} x_j \geq 1$, is $x_1 + 2x_2 - 2x_3 \geq 1$. From (3.17),

$$\gamma_1^1 = \min \{\frac{1}{1}, \frac{2}{2}\} = 1 \text{ and } \gamma_1^2 = \min\{\frac{1}{1/2}, \frac{2}{1/3}\} = 2$$

Thus, through (3.18), or more directly, from (3.27), the cut (3.19), i.e., $\sum a_{1j}^{**} x_j \geq 1$ is $x_1 + 2x_2 - 4x_3 \geq 1$. This cut strictly dominates the cut (3.16) in this example, though both have the same values $1/\sqrt{5}$ and 1/2 respectively for θ_e and θ_r of Equations (3.6) and (3.9) respectively.

3.4 Deriving Deep Cuts for DC2

To begin with, let us make the following interesting observation. Suppose that for convenience, we assume without loss of generality as before, that $b_i^h = 1$, $i \varepsilon Q_h$, $h \varepsilon H$ in Equation (3.3). Thus, for each $h \varepsilon H$, we have the constraint set

$$S_h = \left\{ x: \sum_{j=1}^{n} a_{ij}^h x_j \geq 1, \; i \varepsilon Q_h, \; x \geq 0 \right\} \tag{3.36}$$

Now for each $h \varepsilon H$, let us multiply the constraints of S_h by corresponding scalars $\delta_i^h \geq 0$, $i \varepsilon Q_h$ and add them up to obtain the surrogate constraint

$$\sum_{j=1}^{n} \left\{ \sum_{i \varepsilon Q_h} \delta_i^h a_{ij}^h \right\} x_j \geq \sum_{i \varepsilon Q_h} \delta_i^h, \; h \varepsilon H \tag{3.37}$$

Or, assuming that not all δ_i^h are zero for $i \varepsilon Q_h$, i.e., letting each set S_h, $h \varepsilon H$ govern the cut, (3.37) may be re-written as

$$\sum_{j=1}^{n} \left\{ \sum_{i \varepsilon Q_h} \left[\frac{\delta_i^h}{\left(\sum_{p \varepsilon Q_h} \delta_p^h \right)} \right] a_{ij}^h \right\} x_j \geq 1, \; h \varepsilon H \tag{3.38}$$

Finally, denoting $\delta_i^h \Big/ \sum_{p \varepsilon Q_h} \delta_p^h$ by λ_i^h for $i \varepsilon Q_h$, $h \varepsilon H$, we may write (3.38) as

$$\sum_{j=1}^{n} \left(\sum_{i \varepsilon Q_h} \lambda_i^h a_{ij}^h \right) x_j \geq 1 \text{ for each } h \varepsilon H \tag{3.39}$$

where,

$$\sum_{i \varepsilon Q_h} \lambda_i^h = 1 \text{ for each } h \varepsilon H, \; \lambda_i^h \geq 0 \text{ for } i \varepsilon Q_h, \; h \varepsilon H \tag{3.40}$$

Observe that by surrogating the constraints of (3.36) using parameters λ_i^h, $i\varepsilon Q_h$, $h \varepsilon H$ satisfying (3.40), we have essentially represented DC2 as DC1 through (3.39). In other words, since $x \varepsilon S_h$ implies x satisfies (3.39) for each $h \varepsilon H$, then given λ_i^h, $i\varepsilon Q_h$, $h \varepsilon H$, DC2 implies that at least one of (3.39) must be satisfied. Now, whereas Theorem 2.1 would directly employ (3.37) to derive a cut, since we have normalized (3.37) to obtain (3.39), we know from the previous section that the optimal strategy is to derive a cut (3.19) using inequalities (3.39).

Now let us consider in turn the two criteria of Section 3.2.

3.4.1 Euclidean Distance-Based Criterion

Consider any selection of values for the parameters λ_i^h, $i\varepsilon Q_h$, $h \varepsilon H$ satisfying (3.40) and let the corresponding disjunction DC1 derived from DC2 be that at least one of (3.39) must hold. Then, Theorem 3.1 tells us through Equations (3.16), (3.21) that the euclidean distance criterion value for the resulting cut (3.19) is

$$\theta_e(\lambda) = 1 \Bigg/ \sqrt{\sum_{j=1}^{n} y_j^2} \tag{3.41}$$

where,

$$y_j = \max\{0, z_j\}, \quad j=1,\ldots,n \tag{3.42}$$

and

$$z_j = \max_{h\varepsilon H} \{ \sum_{i\varepsilon Q_h} \lambda_i^h a_{ij}^h \}, \quad j=1,\ldots,n \tag{3.43}$$

Thus, the criterion of Section 3.2 seeks to

$$\text{maximize}\{\theta_e(\lambda): \lambda = (\lambda_i^h) \text{ satisfies } (3.40)\} \tag{3.44}$$

or equivalently, to

$$\text{minimize } \{ \sum_{j=1}^{n} y_j^2 : (3.40), (3.42), (3.43) \text{ are satisfied}\}. \tag{3.45}$$

It may be easily verified that the problem of (3.45) may be written as

$$\underline{PD_2}: \qquad \text{minimize} \quad \sum_{j=1}^{n} y_j^2 \tag{3.46}$$

$$\text{subject to} \quad y_j \geq \sum_{i \in Q_h} \lambda_i^h a_{ij}^h \quad \text{for each } h \in H \quad \text{for each } j=1,\ldots,n \tag{3.47}$$

$$\sum_{i \in Q_h} \lambda_i^h = 1 \qquad \text{for each } h \in H \tag{3.48}$$

$$\lambda_i^h \geq 0 \quad i \in Q_h,\ h \in H \tag{3.49}$$

The equivalence follows by noting that any optimal solution to PD_2 must satisfy (3.42) as an equality. In particular, we have deleted the constraints $y_j \geq 0$, $j=1,\ldots,n$ since for any feasible λ_i^h, $i \in Q_h$, $h \in H$, there exists a dominant solution with nonnegative y_j, $j=1,\ldots,n$. This relaxation is simply a matter of convenience in our solution strategy.

Before proposing a solution procedure for Problem PD_2, let us make some pertinent remarks. Note that Problem PD_2 has the purpose of generating parameters λ_i^h, $i \in Q_h$, $h \in H$ which are to be used to obtain the surrogate constraints (3.39). Thereafter, the cut that we derive for the disjunction DC_2 is the cut (3.19) obtained from the statement that at least one of (3.39) must hold. Hence, Problem PD_2 attempts to find values for λ_i^h, $i \in Q_h$, $h \in H$, such that this resulting cut achieves the euclidean distance criterion.

Problem PD_2 is a convex quadratic program for which the Karush-Kuhn-Tucker conditions are both necessary and sufficient. Several efficient simplex-based quadratic programming procedures are available to solve such a problem. However, these procedures require explicit handling of the potentially large number of constraint in Problem PD_2. On the other hand, the subgradient optimization procedure discussed below takes full advantage of the problem structure. We are first able to write out an almost complete solution to the Karush-Kuhn-Tucker system. We will refer to this as a partial solution. In case we are unable to either actually construct a complete solution or to assert that a feasible completion exists, then through the construction procedure itself, we will show that a subgradient direction is available. Moreover, this latter direction is very likely to be a direction of ascent. We therefore propose to move in the negative of this direction and if necessary, project back onto the feasible region. These iterative steps are now repeated at this new point.

3.4.1 Karush-Kuhn-Tucker Systems for PD_2 and Its Implications

Letting u_j^h, $h \in H$, $j=1,\ldots,n$ denote the lagrangian multipliers for constraints (3.47), t_h, $h \in H$ those for constraints (3.48), and w_i^h, $i \in Q_h$, $h \in H$ those for constraints (3.49), we may write the Karush-Kuhn-Tucker optimality conditions as

$$\sum_{h \in H} u_j^h = 2y_j \qquad j=1,\ldots,n \tag{3.50}$$

$$\sum_{j=1}^{n} u_j^h a_{ij}^h + t_h - w_i^h = 0 \text{ for each } i \in Q_h, \text{ and for each } h \in H \tag{3.51}$$

$$u_j^h \left\{ \sum_{i \in Q_h} \lambda_i^h a_{ij}^h - y_j \right\} = 0 \quad \text{for each } j=1,\ldots,n, \text{ and each } h \in H \tag{3.52}$$

$$\lambda_i^h w_i^h = 0 \text{ for } i \in Q_h,\ h \in H \tag{3.53}$$

$$w_i^h \geq 0 \quad i \in Q_h,\ h \in H \tag{3.54}$$

$$u_j^h \geq 0 \quad j=1,\ldots,n,\ h \in H \tag{3.55}$$

Finally, Equations (3.47), (3.48), (3.49) must also hold.

We will now consider the implications of the above conditions. This will enable us to construct at least a partial solution to these conditions, given particular values of λ_i^h, $i \in Q_h$, $h \in H$. First of all, note that Equations (3.42), (3.45) and (3.55) imply that

$$y_j \geq 0 \text{ for each } j=1,\ldots,n \tag{3.56}$$

$$y_j = \max\left\{0,\ \sum_{i\in Q_h} \lambda_i^h a_{ij}^h,\ h\in H\right\} \text{ for } j=1,\ldots,n \tag{3.57}$$

Now, having determined values for $y_j = j=1,\ldots,n$, let us define the sets

$$H_j = \begin{cases} \{\Phi\} \text{ if } y_j = 0 \\ \{h\in H\colon y_j = \sum_{i\in Q_h} \lambda_i^h a_{ij}^h > 0\} \end{cases} \text{ for } j=1,\ldots,n \tag{3.58}$$

Now, consider the determination of u_j^h, $h \in H$, $j=1,\ldots,n$. Clearly, Equations (3.50), (3.52) and (3.55) along with the definition (3.58) imply that for each $j=1,\ldots,n$

$$u_j^h = 0 \text{ for } h\in H/H_j \text{ and that } \sum_{h\in H_j} u_j^h = 2y_j,\ u_j^h \geq 0 \text{ for each } h\in H_j \tag{3.59}$$

Thus, for any $j\in\{1,\ldots,n\}$, if H_j is either empty or a singleton, the corresponding values for u_j^h, $h \in H$ are uniquely determined. Hence, we have a choice in selecting values for u_j^h, $h \in H_j$ only when $|H_j| \geq 2$ for any $j\in\{1,\ldots,n\}$. Next, multiplying (3.51) by λ_i^h and using (3.53), we obtain

$$\sum_{j=1}^{n} \left[u_j^h \sum_{i\in Q_h} \{\lambda_i^h a_{ij}^h\}\right] + t_h \sum_{i\in Q_h} \lambda_i^h = 0 \text{ for each } h\in H \tag{3.60}$$

Using Equations (3.48), (3.52), this gives us

$$t_h = -\sum_{j=1}^{n} u_j^h y_j \text{ for each } h\in H \tag{3.61}$$

Finally, Equations (3.51), (3.61) yield

$$w_i^h = \sum_{j=1}^{n} u_j^h\,[a_{ij}^h - y_j] \text{ for each } i\in Q_h,\ h\in H \tag{3.62}$$

Notice that once the variables u_j^h, $h \in H$, $j=1,\ldots,n$ are fixed to satisfy (3.59), all the variables are uniquely determined. We now show that if the variables w_j^h, $i \in Q_h$, $h \in H$ so determined are nonnegative, we then have a Karush-Kuhn-Tucker solution. Since the objective function of PD_2 is convex and the constraints are linear, this solution is also optimal.

Lemma 3.3

Let a primal feasible set of values for λ_i^h, $i \in Q_h$, $h \in H$ be given. Determine values for all variables y_j, u_j^h, t_h, w_i^h using Equations (3.57) through (3.62), selecting an arbitrary solution in the case described in Equation (3.59) if $|H_j| \geq 2$. If $w_i^h \geq 0$, $i \in Q_h$, $h \in H$, then λ_i^h, $i \in Q_h$, $h \in H$ solves Problem PD_2.

Proof. By construction Equations (3.47) through (3.52), and (3.55) clearly hold. Thus, noting that in our problem the Kuhn-Tucker conditions are sufficient for optimality, all we need to show is that if $w = (w_i^h) \geq 0$ then (3.53) holds. But from (3.52) and (3.62) for any $h \in H$, we have,

$$\sum_{i \in Q_h} \lambda_i w_i^h = \sum_{i \in Q_h} \lambda_i^h \left\{ \sum_{j=1}^{n} u_j^h \, [a_{ij}^h - y_j] \right\} = \sum_{j=1}^{n} \left\{ u_j^h \, [\sum_{i \in Q_h} \lambda_i^h a_{ij}^h - y_j] \right\} = 0$$

for each $h \in H$.

Thus, $\lambda_i^h \geq 0$, $w_i^h \geq 0$ $i \in Q_h$, $h \in H$ imply that (3.53) holds and the proof is complete.

The reader may note that in Section 3.4.1(d) we will propose another stronger sufficient condition for a set of variables λ_i^h, $i \in Q_h$, $h \in H$ to be optimal. The development of this condition is based on a subgradient optimization procedure discussed below.

3.4.1(b) Subgradient Optimization Scheme for Problem PD_2

For the purpose of this development, let us use (3.57) to rewrite Problem PD_2 as follows. First of all define

$$\Lambda = \{\lambda = (\lambda_i^h): \text{constraints (3.48) and (3.49) are satisfied}\} \qquad (3.63)$$

and let F: $\Lambda \to R$ be defined by

$$F(\lambda) = \sum_{j=1}^{n} [\text{maximum } \{0, \sum_{i \in Q_h} \lambda_i^h a_{ij}^h, h \in H\}]^2 \qquad (3.64)$$

Then, Problem PD_2 may be written as

$$\text{minimize } \{F(\lambda): \lambda \in \Lambda\}$$

Note that for each j=1,...,n, $g_j(\lambda) = \max \{0, \sum_{i \in Q_h} \lambda_i^h a_{ij}^h, h \in H\}$ is convex and nonnegative. Thus, $[g_j(\lambda)]^2$ is convex and so $F(\lambda) = \sum_{j=1}^{n} [g_j(\lambda)]^2$ is also convex.

The main thrust of the proposed algorithm is as follows. Having a solution $\bar{\lambda}$ at any stage, we will attempt to construct a solution to the Karush-Kuhn-Tucker system using Equations (3.50) through (3.55). If we obtain nonnegative values $\bar{w}_i^h$ for the corresponding variables w_i^h, $i \in Q_h$, $h \in H$, then by Lemma 3.3 above, we terminate. Later in Section 3.4.1(g), we will also use another sufficient condition to check for termination. If we obtain no indication of optimality, we continue. Theorem 3.2 below established that in any case, the vector $w = \bar{w}$ constitutes a subgradient of f(•) at the current point $\bar{\lambda}$. We hence take a suitable step in the negative subgradient direction and project back onto the feasible region Λ of Equation (3.63). This completes one iteration. Before presenting Theorem 3.2, consider the following definition.

Definition 3.1

Let F: $\Lambda \to R$ be a convex function and let $\lambda \in \Lambda \subset R^m$. Then $\xi \in R^m$ is a subgradient of F(•) at $\bar{\lambda}$ if

$$F(\lambda) \geq F(\bar{\lambda}) + \xi^t(\lambda - \bar{\lambda}) \text{ for each } \lambda \in \Lambda.$$

Theorem 3.2

Let $\bar{\lambda}$ be a given point in Λ defined by (3.63) and let $\bar{w}$ be obtained from Equations (3.57) through (3.62), with an arbitrary selection of a solution to (3.59).

Then, $\bar{w}$ is a subgradient of $F(\cdot)$ at $\bar{\lambda}$, where $F: \Lambda \to R$ is defined in Equation (3.64).

Proof. Let y and $\bar{y}$ be obtained through Equation (3.57) from $\lambda \in \Lambda$ and $\bar{\lambda} \in \Lambda$ respectively. Hence,

$$F(\lambda) = \sum_{j=1}^{n} y_j^2 \text{ and } F(\bar{\lambda}) = \sum_{j=1}^{n} \bar{y}_j^2$$

Thus, from Definition 3.1, we need to show that

$$\sum_{h\in H} \sum_{i\in Q_h} \bar{w}_i^h(\lambda_i^h - \bar{\lambda}_i^h) \leq \sum_{j=1}^{n} y_j^2 - \sum_{j=1}^{n} \bar{y}_j^2 \tag{3.65}$$

Noting from Equations (3.52), (3.62) that $\sum_{h\in H} \sum_{i\in Q_h} \bar{w}_i^h \bar{\lambda}_i^h = 0$, we have,

$$\sum_{h\in H} \sum_{i\in Q_h} \bar{w}_i^h(\lambda_i^h - \bar{\lambda}_i^h) = \sum_{h\in H} \sum_{i\in Q_h} \bar{w}_i^h\lambda_i^h = \sum_{h\in H} \sum_{i\in Q_h} \sum_{j=1}^{n} \bar{u}_j^h\lambda_i^h\,[a_{ij}^h - \bar{y}_j]$$

$$= \sum_{h\in H} \sum_{j=1}^{n} \bar{u}_j^h \left(\sum_{i\in Q_h} \lambda_i^h a_{ij}^h\right) - \sum_{h\in H} \sum_{j=1}^{n} [\bar{u}_j^h\bar{y}_j \sum_{i\in Q_h} \lambda_i^h]$$

Using (3.48) and (3.50), this yields

$$\sum_{h\in H} \sum_{i\in Q_h} \bar{w}_i^h\,(\lambda_i^h - \bar{\lambda}_i^h) = \sum_{h\in H} \sum_{j=1}^{n} \bar{u}_j^h\left(\sum_{i\in Q_h} \lambda_i^h a_{ij}^h\right) - 2\sum_{j=1}^{n} \bar{y}_j^2$$

Combining this with (3.65), we need to show that

$$\sum_{h\in H} \sum_{j=1}^{n} \bar{u}_j^h \left(\sum_{i\in Q_h} \lambda_i^h a_{ij}^h\right) \leq \sum_{j=1}^{n} y_j^2 + \sum_{j=1}^{n} \bar{y}_j^2 \tag{3.66}$$

But Equations (3.50), (3.55), (3.57) imply that

$$\sum_{h\in H} \sum_{j=1}^{n} \bar{u}_j^h \left(\sum_{i\in Q_h} \lambda_j^h a_{ij}^h\right) \leq \sum_{h\in H} \sum_{j=1}^{n} \bar{u}_j^h y_j = 2\sum_{j=1}^{n} y_j\bar{y}_j$$

$$\leq 2\|y\|\,\|\bar{y}\| \leq \|y\|^2 + \|\bar{y}\|^2$$

so that Equation (3.66) holds. This completes the proof.

Although, given $\bar{\lambda} \in \Lambda$, any solution to Equations (3.57) through (3.62) will yield a subgradient of F(•) at the current point $\bar{\lambda}$, we would like to generate, without expending much effort, a subgradient which is hopefully a direction of ascent. This would hence accelerate the cut generation process. Later in Section 3.4.1(b) we describe one such scheme to determine a suitable subgradient direction. For the present moment, let us assume that we have generated a subgradient $\bar{w}$ and have taken a suitable step size $\bar{\theta}$ in the direction $-\bar{w}$ as prescribed by the subgradient optimization scheme. Let

$$\bar{\bar{\lambda}} = \bar{\lambda} - \bar{\theta}\,\bar{w} \tag{3.67}$$

be the new point thus obtained. To complete the iteration, we must now project $\bar{\bar{\lambda}}$ onto Λ, that is, we must determine a new $\bar{\lambda}$ according to

$$\bar{\lambda}_{new} \equiv P_{\Lambda}(\bar{\bar{\lambda}}) = \text{minimum } \{||\lambda - \bar{\bar{\lambda}}|| : \lambda \in \Lambda\} \tag{3.68}$$

The method of accomplishing this efficiently is presented in the next subsection.

3.4.1(c) Projection Scheme

For convenience, let us define the following linear manifold

$$M_h = \left\{ \lambda_i^h,\ i\varepsilon Q_h : \sum_{i\varepsilon Q_h} \lambda_i^h = 1 \right\},\ h\varepsilon H \tag{3.69}$$

and let $\bar{M}_h$ be the intersection of M_h with the nonnegative orthant, that is,

$$\bar{M}_h = \{\lambda_i^h,\ i\varepsilon Q_h : \sum_{i\varepsilon Q_h} \lambda_i^h = 1,\ \lambda_i^h \geq 0,\ i\varepsilon Q_h\} \tag{3.70}$$

Note from Equation (3.63) that

$$\Lambda = \bar{M}_1 \times \ldots \times \bar{M}_{|H|} \tag{3.71}$$

Now, given $\bar{\bar{\lambda}}$, we want to project it onto Λ, that is, determine $\bar{\lambda}_{new}$ from Equation (3.68). Towards this end, for any vector $\alpha = (\alpha_i, i\varepsilon I)$, where I is a suitable index set for the $|I|$ components of α, let $P(\alpha, I)$ denote the following problem

$$\underline{P(\alpha,I):}\ \text{minimize } \{\frac{1}{2} \sum_{i\varepsilon I} (\lambda_i - \alpha_i)^2 : \sum_{i\varepsilon I} \lambda_i = 1,\ \lambda_i \geq 0,\ i\varepsilon I\} \tag{3.72}$$

Then to determine $\bar{\lambda}_{new}$, we need to find the solutions $(\bar{\lambda}^h_{new})_i$, $i\varepsilon Q_h$ as projections onto $\bar{M}_h$ of $\bar{\bar{\lambda}}^h = (\bar{\bar{\lambda}}^h, i\varepsilon Q_h)$ through each of the $|H|$ separable problems $P(\bar{\bar{\lambda}}^h, Q_h)$. Thus, henceforth in this section, we will consider only one such $h\varepsilon H$. Theorem 3.3 below is the basis of a finitely convergent iterative scheme to solve Problem $P(\bar{\bar{\lambda}}^h, Q_h)$.

<u>Theorem 3.3</u>

Consider the solution of Problem $P(\beta^k, I_k)$, where $\beta^k = (\beta^k, i\varepsilon I_k)$, with $|I_k| \geq 1$. Define

$$\rho_k = (1 - \sum_{i\varepsilon I_k} \beta^k_i) \Big/ |I_k| \tag{3.73}$$

and let

$$\bar{\beta}^k = \beta^k + (\rho_k) 1_k \tag{3.74}$$

where 1_k denotes a vector of $|I_k|$ elements, each equal to unity. Further, define

$$I_{k+1} = \{i\varepsilon I_k : \bar{\beta}^k_i > 0\} \tag{3.75}$$

Finally, let β^{k+1} defined below be a subvector of $\bar{\beta}^k$,

$$\beta^{k+1} = (\beta^{k+1}_i,\ i\varepsilon I_{k+1}) \tag{3.76}$$

where, $\beta^{k+1}_i = \bar{\beta}^k_i$, $i\varepsilon I_{k+1}$. Now suppose that $\hat{\beta}^{k+1}$ solves $P(\beta^{k+1}, I_{k+1})$.

(a) If $\bar{\beta}^k \geq 0$, then $\bar{\beta}^k$ solves $P(\beta^k, I_k)$

(b) If $\bar{\beta}^k \not\geq 0$, then β solves $P(\beta^k, I_k)$, where β has components given by

$$\beta_i = \begin{cases} \hat{\beta}_i^{k+1}, & \text{if } i\varepsilon I_{k+1} \quad \text{for each } i\varepsilon I_k \\ 0 & \text{otherwise} \end{cases} \tag{3.77}$$

Proof. For the sake of convenience, let RP(α,I) denote the problem obtained by relaxing the nonnegativity restrictions in P(α,I). That is, let

$$\underline{\text{RP}(\alpha,\text{I})}: \quad \text{minimize } \{\frac{1}{2} \sum_{i\varepsilon I} (\lambda_1 - \alpha_i)^2 : \sum_{i\varepsilon I} \lambda_i = 1\}$$

First of all, note from Equation (3.73), (3.74) that $\bar{\beta}^k$ solves RP(β^k,I_k) since $\bar{\beta}^k$ is the projection of β^k onto the linear manifold

$$\{\lambda = (\lambda_i, i\varepsilon I_k): \sum_{i\varepsilon I_k} \lambda_i = 1\} \tag{3.78}$$

which is the feasible region of RP(β^k,I_k). Thus, $\bar{\beta}^k \geq 0$ implies that $\bar{\beta}^k$ also solves P(β^k,I_k). This proves part (a).

Next, suppose that $\bar{\beta}^k \ngeq 0$. Observe that β is feasible to $\underline{P(\beta^k,I_k)}$ since from (3.77), we get $\beta \geq 0$ and $\sum_{i\varepsilon I_k} \beta_i = \sum_{i\varepsilon I_{k+1}} \hat{\beta}^{k+1} = 1$ as $\hat{\beta}^{k+1}$ solves P(β^{k+1}, I_{k+1}).

Now, consider any $\lambda = (\lambda_i, i\varepsilon I_k)$ feasible to P(β^k,I_k). Then, by the Pythagorem Theorem, since $\bar{\beta}^k$ is the projection of β^k onto (3.78), we get

$$|| \lambda - \beta^k ||^2 = || \lambda - \bar{\beta}^k ||^2 + || \bar{\beta}^k - \beta^k ||^2$$

Hence, the optimal solution to P($\bar{\beta}^k$,I_k) is also optimal to P(β^k,I_k). Now, suppose that we can show that the optimal solution to Problem P($\bar{\beta}^k$,I_k) must satisfy

$$\lambda_i = 0 \text{ for } i \notin I_{k+1} \tag{3.79}$$

Then, noting (3.76), (3.77), and using the hypothesis that $\hat{\beta}^{k+1}$ solves P(β^{k+1},I_{k+1}), we will have established part (b). Hence, let us prove that (3.79) must hold. Towards this end, consider the Karush-Kuhn-Tucker equations for Problem $P(\bar{\beta}^k, I_k)$ with t and w_i, $i \varepsilon I_k$ as the appropriate langrangian multipliers.

$$\sum_{i \varepsilon I_k} \lambda_i = 1, \ \lambda_i \geq 0 \text{ for each } i \varepsilon I_k \tag{3.80}$$

$$(\lambda_i - \bar{\beta}_i^k) + t - w_i = 0 \text{ and } w_i \geq 0 \text{ for each } i \varepsilon I_k \tag{3.81}$$

$$\lambda_i w_i = 0 \text{ for each } i \varepsilon I_k \tag{3.82}$$

Now, since $\sum_{i \varepsilon I_k} \bar{\beta}_i^k = 1$, we get from (3.80), (3.81) that

$$t = \sum_{i \varepsilon I_k} w_i \Big/ |I_k| \geq 0 \tag{3.83}$$

But from (3.81), (3.82) we get for each $i \varepsilon I_k$,

$$0 = w_i \lambda_i = \lambda_i(\lambda_i + t - \bar{\beta}_i^k)$$

which implies that for each $i \varepsilon I_k$, we must have,

either $\lambda_i = 0$, whence from (3.81), $w_i = t - \bar{\beta}_i^k$ must be nonnegative

or $\lambda_i = \bar{\beta}_i^k - t$, whence from (3.81), $w_i = 0$.

In either case above, noting (3.80), if $\bar{\beta}_i^k \leq 0$, that is, if $i \notin I_{k+1}$, we must have $\lambda_i = 0$. This completes the proof.

Using Theorem 3.3, one may easily validate the following procedure for finding $\bar{\bar{\lambda}}_{new}^h$ of Equation (3.68), given $\bar{\bar{\lambda}}^h$. This procedure has to be repeated separately for each $h \varepsilon H$.

<u>Initialization</u>

Set k=0, $\beta^0 = \bar{\bar{\lambda}}^h$, $I_0 = Q_h$. Go to Step 1.

<u>Step 1</u>

Given β^k, I_k, determine ρ_k and $\bar{\beta}^k$ from (3.73), (3.74). If $\beta^k \geq 0$, then terminate with $\bar{\lambda}_{new}^h$ having components given by

$$(\bar{\lambda}^h_{new})_i = \begin{cases} \bar{\beta}^k_i & \text{if } i \varepsilon I_k \\ 0 & \text{otherwise} \end{cases}$$

Otherwise, proceed to Step 2.

Step 2

Define I_{k+1}, β^{k+1} as in Equations (3.75), (3.76), increment k by one and return to Step 1.

Note that this procedure is finitely convergent as it results in a strictly decreasing, finite sequence $|I_k|$ satisfying $|I_k| \geq 1$ for each k, since $\sum_{i\varepsilon I_k} \bar{\beta}^k = 1$ for each k.

Example

Suppose we want to project $\bar{\bar{\lambda}}^h = (-2,3,1,2)$ on to $\Lambda \subset R^4$. Then the above procedure yields the following results.

Initialization

$k=0$, $\beta^0 = (-2,3,1,2)$, $I_0 = \{1,2,3,4\}$.

Step 1

$\rho_0 = -3/4$, $\bar{\beta}^0 = (-\frac{11}{4}, \frac{9}{4}, \frac{1}{4}, \frac{5}{4})$

Step 2

$k=1$, $I_1 = \{2,3,4\}$, $\beta^1 = (\frac{9}{4}, \frac{1}{4}, \frac{5}{4})$

Step 1

$\rho_1 = -\frac{11}{12}$, $\bar{\beta}^1 = (\frac{4}{3}, -\frac{2}{3}, \frac{1}{3})$

Step 2

$k=2$, $I_2 = \{2,4\}$, $\beta^2 = (\frac{4}{3}, \frac{1}{3})$

Step 1

$\rho_2 = -\frac{1}{3}$, $\bar{\beta}^2 = (1,0) \geq 0$

Thus, $\bar{\lambda}^h_{new} = (0,1,0,0)$.

3.4.1(d) A Second Sufficient Condition for Termination

As indicated earlier in Section 3.4.1(b), we will now derive a second sufficient condition on $\bar{w}$ for $\bar{\lambda}$ to solve PD_2. For this purpose, consider the following lemma.

Lemma 3.4

Let $\bar{\lambda} \in \Lambda$ be given and suppose we obtain $\bar{w}$ using Equations (3.57) through (3.62). Let $\hat{w}$ solve the problem.

$$\underline{PR_h}: \text{ minimize } \left\{ \frac{1}{2} \sum_{i \in Q_h} (\bar{w}_i^h - w_i^h)^2 : \sum_{i \in Q_h} w_i^h = 0,\ w_i^h \leq 0 \text{ for } i \in J_h \right\} \quad \text{for each } h \in H$$

where,

$$J_h = \{i \in Q_h : \bar{\lambda}_i^h = 0\},\ h \in H \tag{3.84}$$

Then, if $\hat{w} = 0$, $\bar{\lambda}$ solves Problems PD_2.

Proof. Since $\hat{w} = 0$ solves PR_h, $h \in H$, we have for each $h \in H$,

$$\sum_{i \in Q_h} (\bar{w}_i^h)^2 \leq \sum_{i \in Q_h} (\bar{w}_i^h - w_i^h)^2 \tag{3.85}$$

for all w_i^h, $i \in Q_h$ satisfying $\sum_{i \in Q_h} w_i^h = 0$, $w_i^h \leq 0$ for $i \in J_h$. Given any $\lambda \in \Lambda$ and given any $\mu > 0$ define

$$w_i^h = (\bar{\lambda}_i^h - \lambda_i^h) / \mu,\ i \in Q_h,\ h \in H \tag{3.86}$$

Then, $\sum_{i \in Q_h} w_i^h = 0$ for each $h \in H$ and since $\bar{\lambda}_i^h = 0$ for $i \in J_h$, $h \in H$, we get $w_i^h \leq 0$ for $i \in J_h$, $h \in H$. Thus, for any $\lambda \in \Lambda$, but substituting (3.86) into (3.85), we have,

$$\mu^2 \sum_{i \in Q_h} (\bar{w}_i^h)^2 \leq \sum_{i \in Q_h} (\lambda_i^h - \bar{\lambda}_i^h + \mu \bar{w}_i^h)^2 \text{ for each } h \in H \tag{3.87}$$

But equation (3.87) implies that for each $h \in H$, $\lambda^h = \bar{\lambda}^h$ solves the problem

$$\text{minimize} \left\{ \sum_{i \in Q_h} \{[\lambda_i^h - (\bar{\lambda}_i^h - \bar{w}_i^h)]^2 : \sum_{i \in Q_h} \lambda_i^h = 1, \lambda_i^h \geq 0, i \in Q_h \right\} \text{for each } h \in H$$

In other words, the projection $P_\Lambda(\bar{\lambda} - \bar{w}\mu)$ of $(\bar{\lambda} - \bar{w}\mu)$ onto Λ is equal to $\bar{\lambda}$ for any $\mu > 0$. From the theory of subgradient optimization, since $\bar{w}$ is a subgradient of $F(\cdot)$ at $\bar{\lambda}$, then $\bar{\lambda}$ solves PD_2. This completes the proof.

Note that Lemma 3.4 above states that if the "closest" feasible direction $-w$ to $-\bar{w}$ is a zero vector, then $\bar{\lambda}$ solves PD_2. Based on this result, we derive through Lemma 3.5 below a second sufficient condition for $\bar{\lambda}$ to solve PD_2.

Lemma 3.5

Suppose w=0 solves Problems PR_h, $h \in H$ as in Lemma 3.4. Then for each $h \in H$, we must have

$$\text{(a) } \bar{w}_i^h = t_h, \text{ a constant, for each } i \notin J_h$$
$$\text{(b) } \bar{w}_i^h \leq t_h \text{ for each } i \in J_h \tag{3.88}$$

where J_h is given by Equation (3.84).

Proof. Let us write out the Karush-Kuhn-Tucker conditions for Problem PR_h, for any $h \in H$. We obtain

$$(w_i^h - \bar{w}_i^h) + t_h = 0 \text{ for } i \notin J_h$$

$$(w_i^h - \bar{w}_i^h) + t_h - u_i^h = 0 \text{ for } i \in J_h$$

$$u_i^h \geq 0, \ i \in J_h, \ u_i^h w_i^h = 0 \quad i \in J_h$$

$$\sum_{i \in Q_h} w_i^h = 0, \ w_i^h \geq 0 \text{ for } i \in J_h, \ t_h \text{ unrestricted}$$

If w=0, solves PR_h, $h \in H$, then since PR_h has a convex objective function and linear constraints, then there must exist a solution to

$$\bar{w}_i^h = t_h \text{ for each } i \notin J_h$$

and

$$u_i^h = (t_h - \bar{w}_i^h) \geq 0 \text{ for each } i \varepsilon J_h.$$

This completes the proof.

Thus Equation (3.88) gives us another sufficient conditon for $\bar{\lambda}$ to solve PD_2. We illustrate the use of this condition through an example in Section 3.4.1(g).

3.4.1(e) Schema of an Algorithm to Solve Problem PD_2

The procedure developed above is depicted schematically in Figure 3.3. In block 1 an arbitrary, or preferably a good heuristic solution, $\bar{\lambda} \varepsilon \Lambda$ is sought. For example, one may use $\bar{\lambda}_i^h = 1 / |Q_h|$ for each $i \varepsilon Q_h$, for $h \varepsilon H$. For blocks 4 and 6, we recommend the standard procedural steps adopted for the sub-gradient optimization scheme.

3.4.1(f) Derivation of a Good Subgradient Direction

In our discussion in Section 3.4.1(a), we saw that given a $\lambda \varepsilon \Lambda$ of Equation (3.63), we were able to uniquely determine $\bar{y}_j$, j=1,...,n through Equation (3.57). Thereafter, once we fixed values $\bar{u}_j^h$ for u_j^h, j=1,...,n, $h \varepsilon H$ satisfying Equation (3.59), we were able to uniquely determine values for the other variables in the Karush-Kuhn-Tucker System using Equations (3.61), (3.62). Moreover, the only choice in determining $\bar{u}_j^h$, j=1,...,n, $h \varepsilon H$ arose in case $|H_j| \geq 2$ for some $j \varepsilon \{1,...,n\}$ in Equation (3.60). We also established that no matter what feasible values we selected for u_j^h, $j \varepsilon \{1,...,n\}$, $h \varepsilon H$, the corresponding vector w obtained was a subgradient direction. In order to select the best such subgradient direction, we are interested in finding a vector $\bar{w}$ which has the smallest euclidean norm among all possible vectors corresponding to the given solution $\bar{\lambda} \varepsilon \Lambda$. However, this problem is not easy to solve. Moreover, since this step will merely be a subroutine at each iteration of the proposed scheme to solve PD_2, we will present a heuristic approach to this problem.

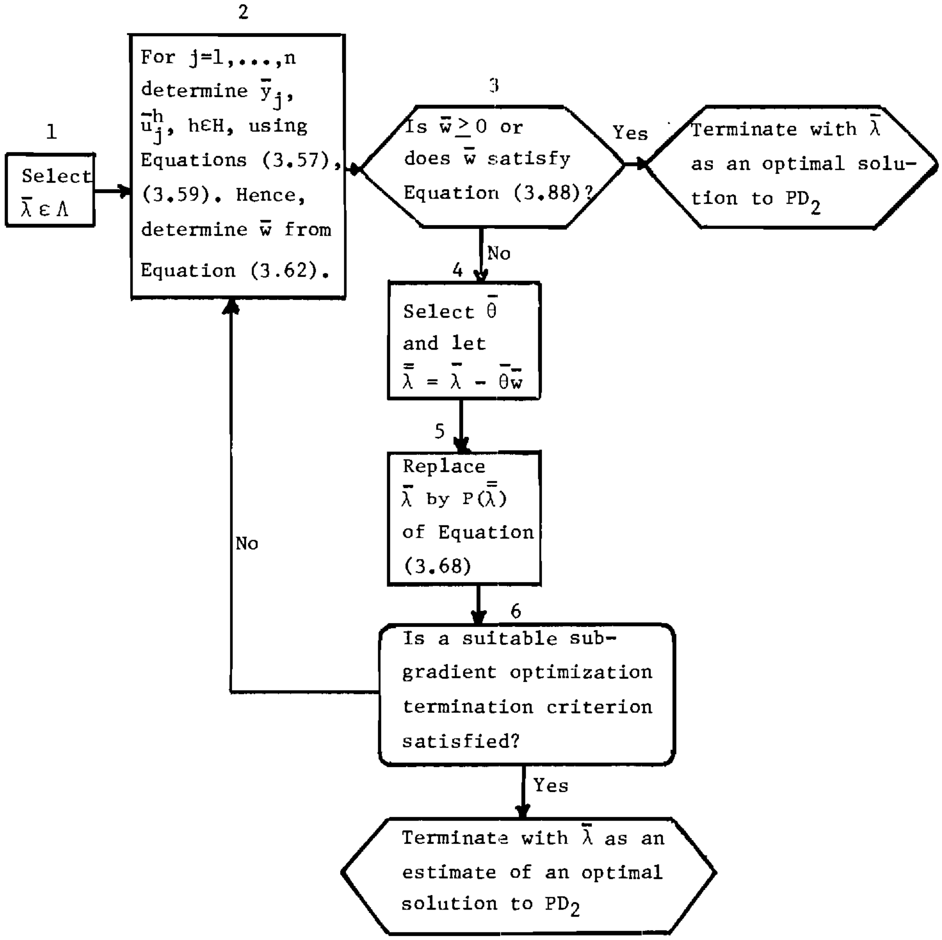

Figure 3.3. Schema of an Algorithm for Problem PD_2

Towards this end, let us define for convenience, mutually exclusive but not uniquely determined sets N_h, $h \in H$ as follows

$$N_h \subset \{j \in \{1,\ldots,n\}: h \in H_j \text{ of Equation (3.58)}\} \qquad (3.89)$$

$$N_i \cap N_j = \{\phi\} \text{ for any } i, j \in H \text{ and } \bigcup_{h \in H} N_h = \{j \in \{1,\ldots,n\}: \bar{y}_j > 0\} \qquad (3.90)$$

In other words, we take each $j\varepsilon\{1,\ldots,n\}$ which has $\bar{y}_j > 0$, and assign it to some $h\varepsilon H_j$, that is, assign it to a set N_h, where $h\varepsilon H_j$. Having done this, we let

$$\bar{u}_j^h = \begin{cases} 2\bar{y} & \text{if } j\varepsilon N_h \\ 0 & \text{otherwise} \end{cases} \quad \text{for each } j\varepsilon\{1,\ldots,n\},\ h\varepsilon H. \tag{3.91}$$

Note that Equation (3.91) yields values $\bar{u}_j^h$ for u_j^h, $j\varepsilon\{1,\ldots,n\}$, $h\,\varepsilon\,H$ which are feasible to (3.59). Hence, having defined sets N_h, $h\,\varepsilon\,H$ as in Equations (3.89), (3.90), we determine $\bar{u}_j^h$, $j\varepsilon\{1,\ldots,n\}$, $h\,\varepsilon\,H$ through (3.91) and hence $\bar{w}$ through (3.62).

Thus, the proposed heuristic scheme commences with a vector w obtained through an arbitrary selection of sets N_h, $h\,\varepsilon\,H$ satisfying Equations (3.89), (3.90). Thereafter, we attempt to improve (decrease) the value of w^tw in the following manner. We consider in turn each $j\varepsilon\{1,\ldots,n\}$ which satisfies $|H_j| \geq 2$ and move it from its current set N_{h_j}, say, to another set N_h with $h\varepsilon H_j$, $h\neq h_j$, if this results in a decrease w^tw. If no such single movements result in a decrease in w^tw, we terminate with the incumbent solution w as the sought subgradient direction. This procedure is illustrated in the example given below.

3.4.1(g) Illustrative Example

The purpose of this subsection is to illustrate the technique of the foregoing section for determining a good subgradient direction as well as the termination criterion of Section 3.4.1(d).

Thus, let $H = \{1,2\}$, $n=3$, $|Q_1| = |Q_2| = 3$ and consider the constraint sets

$$S_1 = \left\{ x: \begin{array}{r} 2x_1 - 3x_2 + x_3 \geq 1 \\ -x_1 + 2x_2 + 3x_3 \geq 1 \\ 3x_1 - x_2 - x_3 \geq 1 \\ x_1, x_2, x_3 \geq 0 \end{array} \right\} \text{ and } S_2 = \left\{ x: \begin{array}{r} 3x_1 - x_2 - x_3 \geq 1 \\ 2x_1 + x_2 - 2x_3 \geq 1 \\ -x_1 + 3x_2 + 3x_3 \geq 1 \\ x_1, x_2, x_3 \geq 0 \end{array} \right\}$$

Further, suppose we are currently located at a point $\bar{\lambda}$ with

$$\bar{\lambda}_1^1 = 0,\ \bar{\lambda}_2^1 = 5/12,\ \bar{\lambda}_3^1 = 7/12;\ \bar{\lambda}_1^2 = 7/12,\ \bar{\lambda}_2^2 = 0,\ \bar{\lambda}_3^2 = 5/12.$$

Then the associated surrogate constraints are

$$\frac{4}{3}x_1 + \frac{1}{4}x_2 + \frac{2}{3}x_3 \geq 1 \quad \text{for } h=1$$

and (3.92)

$$\frac{4}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \geq 1 \quad \text{for } h=2$$

Using Equations (3.57), (3.60), we find

$$\bar{y}_1 = \frac{4}{3} \text{ with } H_1 = \{1,2\},\ \bar{y}_2 = \frac{2}{3} \text{ with } H_2 = \{2\} \text{ and } \bar{y}_3 = \frac{2}{3} \text{ with } H_3 = \{1, 2\}.$$

Note that the possible combinations of N_1 and N_2 are as follows:

(i) $N_1 = \{1\}$, $N_2 = \{2,3\}$,

(ii) $N_1 = \{\phi\}$, $N_2 = \{1,2,3\}$,

(iii) $N_1 = \{1,3\}$, $N_2 = \{2\}$, and

(iv) $N_1 = \{3\}$, $N_2 = \{1,2\}$.

A total enumeration of the values of u obtained for these sets through (3.91) and the corresponding values for w are shown below.

		u_j^h, $j\varepsilon\{1,\ldots,n\}$						w_i^h, $i\varepsilon Q_h$, $h\varepsilon H$						
N_1	N_2	u_1^1	u_2^1	u_3^1	u_1^2	u_2^2	u_3^2	w_1^1	w_2^1	w_3^1	w_1^2	w_2^2	w_3^2	w^tw
{1}	{2,3}	8/3	0	0	0	4/3	4/3	16/9	-56/9	40/9	-40/9	-28/9	56/9	129.78
{ϕ}	{1,2,3}	0	0	0	8/3	4/3	4/3	0	0	0	0	- 4/3	0	1.78
{1,3}	{2}	8/3	0	4/3	0	4/3	0	20/9	-28/9	20/9	-20/9	4/9	28/9	34.37
{3}	{1,2}	0	0	4/3	8/3	4/3	0	-4/9	28/9	-20/9	20/9	20/9	-28/9	34.37

Thus, according to the proposed scheme, if we commence with $N_1 = \{1\}$, $N_2 = \{2,3\}$, then picking j=1 which has $|H_j| = 2$, we can move j=1 into N_2 since $2 \varepsilon H_1$. This leads to an improvement. As one can see from above, no further improvement is possible. In fact, the best solution shown above is accessible by the proposed scheme by all except the third case which is a "local optimal".

We now illustrate the sufficient termination condition of Section 3.4.1(d). The vector $\bar{w}$ obtained above is $(\overset{h=1}{0,0,0} \,|\, 0,\frac{-4}{3},0)$. Further the vector $\bar{\lambda}$ is $(\overset{h=1}{0, \frac{5}{12}, \frac{7}{12}} \,|\, \overset{h=2}{\frac{7}{12}, 0, \frac{-5}{12}})$. Thus, even though $\bar{w} \not\geq 0$, we see that the conditions (3.88) of Lemma 3.5 are satisfied for each $h \varepsilon H = \{1,2\}$ and thus the given $\bar{\lambda}$ solves PD_2.

The disjunctive cut (3.19) derived with this optimal solution $\bar{\lambda}$ is obtained through (3.92) as

$$\frac{4}{3} x_1 + \frac{2}{3} x_2 + \frac{2}{3} x_3 \geq 1 \tag{3.93}$$

3.4.2 Maximizing the Rectilinear Distance Between the Origin and the Disjunctive Cut

In this section, we will briefly consider the case where one desires to use rectilinear instead of euclidean distances. Extending the developments of Sections 3.2, 3.3 and 3.4.1, one may easily see that the relevant problem is

$$\text{minimize}\{\max_{j\varepsilon\{1,\ldots,n\}} y_j: \text{constraints (3.47), (3.48), (3.49) are satisfied}\}.$$

The reason why we consider this formulation is its intuitive appeal. To see this, note that the above problem is separable in $h \varepsilon H$ and may be rewritten as

$$\text{PD}_1: \text{minimize}\{\xi^h: \xi^h \geq \sum_{i\varepsilon Q_h} \lambda_i^h a_{ij}^h \text{ for each } j=1,\ldots,n, \sum_{i\varepsilon Q_h} \lambda_i^h = 1, \lambda_i^h \geq 0$$

$$\text{for each } i\varepsilon Q_h, \xi^h \geq 0\} \quad \text{for each } h \varepsilon H.$$

Thus, for each $h \varepsilon H$, PD_1 seeks λ_i^h, $i\varepsilon Q_h$ such that the largest of the surrogate constraint coefficients is minimized. Once such surrogate constraints are obtained, the disjunctive cut (3.19) may be derived using the principles of Section 3.3.

As far as the solution of Problem PD_1 is concerned, we merely remark that one may either solve it as a linear program or rewrite it as the minimization of a piecewise linear convex function subject to linear constraints and use a subgradient optimization technique. We further note that the structure of Problem PD_1 may render it more amenable to the latter solution technique.

3.5 Other Criteria for Obtaining Deep Cuts

In this section, we will briefly deal with some other plausible criteria which one may adopt. Since DC1 is a specia case of DC2, we will treat only the latter case. Further, we will consider the original disjunction DC2, that is, we will not require $b_i^h > 0$ for each $i \varepsilon Q_h$, h H. Note that the basic disjunctive cut for DC2 is given by Equation (2.10) rewritten below for convenience

$$\sum_{j=1}^{n} [\max_{h\varepsilon H} \{\sum_{i\varepsilon Q_h} \lambda_i^h a_{ij}^h\}] x_j \geq \min_{h\varepsilon H} \{\sum_{i\varepsilon Q_h} \lambda_i^h b_i^h\} \qquad (3.94)$$

Now, a criterion which may be suggested would be to maximize the surplus with respect to the origin. This would mean that the right-hand-side of (3.94) should be made as large as possible, with, of course, some restriction on the overall magnitude of the minimand such as

$$\sum_{i\varepsilon Q_h} \sum_{h\varepsilon H} \lambda_i^h b_i^h = \hat{h}, \text{ say} \tag{3.95}$$

where,

$$\hat{h} = |H|.$$

One may easily verify that this implies it is optimal to select

$$\sum_{i\varepsilon Q_h} \lambda_i^h b_i^h = 1 \quad \text{for each } h\varepsilon H. \tag{3.96}$$

Other than the restriction (3.96), we are still free to select nonnegative values for λ_i^h, $i\varepsilon Q_h$, $h \varepsilon H$. Since the resultant cut (3.94) should at least support S (Equation 3.11)), we may simply select a set of positive coefficients λ_j, $j\varepsilon N$ and solve the linear program.

$$\begin{aligned}
\text{LP:} \quad & \text{minimize} \quad \sum_{j\varepsilon N} \delta_j y_j \\
& \text{subject to} \quad \sum_{i\varepsilon Q_h} \lambda_i^h b_i^h = 1 \qquad \text{for each } h\varepsilon H \\
& \qquad \sum_{i\varepsilon Q_h} \lambda_i^h a_{ij}^h \leq y_j \qquad \text{for each } h\varepsilon H, \text{ for } j=1,\ldots,n \\
& \qquad \lambda_i^h \geq 0, \ i\varepsilon Q_h, \ h\varepsilon H \\
& \qquad y_j, \ j\varepsilon N \text{ are unrestricted in sign.}
\end{aligned}$$

Essentially, the constraints of Problem LP conform with those of Problem PD_2 (Equations (3.47), (3.48), (3.49)). Choosing different values for $\delta_j > 0$, $j\varepsilon N$ in the objective function of LP would yield different cuts (through the parameters λ_i^h, $i\varepsilon Q_h$, $h\varepsilon H$), all of which would be nondominated supports of S, including facets of S.

3.6 Some Standard Choices of Surrogate Constraint Multipliers

We now present two standard procedures for selecting values for the parameters λ_i^h, $i\varepsilon Q_h$, $h\varepsilon H$ for the disjunction DC2. Although not strongly motivated, these solution procedures have intuitive appeal.

The first of these methods prescribes that the constraint sets S_h, $h \varepsilon H$ be first represented in the form given in Equation (3.36), and then one may select $\lambda_i^h = 1 / |Q_h|$ for each $i\varepsilon Q_h$, for $h \varepsilon H$.

As a second method, one may identify for each $h \varepsilon H$, a constraint which has the largest number of minimal column elements. That is, for each $h \varepsilon H$, we compute minimum $\{a_{ij}^h: i\varepsilon Q_h\}$ for each $j\varepsilon N$ and identify the constraint $\hat{i}\varepsilon Q_h$ which contains the maximum number of these $|N|$ minimal coefficients. On the other hand, we may let $\hat{i}$ be the constraint with the most number of negative coefficients for each $h \varepsilon H$. We then set $\lambda_{\hat{i}}^h = 1$ and $\lambda_i^h = 0$ for $i\varepsilon Q_h$, $i \neq \hat{i}$, for each $h \varepsilon H$.

Before illustrating the above two methods, we draw the readers' attention to an obvious, though pertinent, fact. Suppose we are given constraint sets S_h, $h \varepsilon H$ of Equation (3.12) as in DC1 and the disjunction states that at least k of these constraints must be satisfied, where $k < |H|$. Then, by grouping the $|H|$ sets k at a time, we may equivalently represent this disjunction as DC2, stating that at least one of the resulting $\binom{|H|}{k}$ sets must be satisfied. A disjunctive cut may now be derived based on the statement DC2. There is, however, an alternative approach. Note that we may choose to delete any (k-1) of the $|H|$ constraint sets and then assert that at least one of the remaining sets must be satisfied. This would then represent the given disjunction as DC1. Of course, for k=1, both the alternatives are identical. For smaller values of k, the latter technique is likely to be superior to the former technique since by deleting the rows which contain the largest number of column-maxima, one can usually do better than say, by averaging coefficients. On the other hand, for k close to $|H|$, the former is likely to be better since by taking the average, say, of several k numbers of arbitrary sign would tend to produce smaller cut coefficients. These are simply general rules of thumb and clearly one may produce examples which indicate the contrary. We now illustrate the two methods proposed above for selecting values for λ_i^h, $i \in Q_h$, $h \in H$.

Example

Consider the example of Section 3.4.1(g). The first method discussed above yields $\lambda_i^h = 1/3$ for each $i \in Q_h$, $h \in H$. This gives surrogate constraints

$$\frac{4}{3} x_1 - \frac{2}{3} x_2 + x_3 \geq 1 \text{ for } h = 1 \text{ and } \frac{4}{3} x_1 + x_2 \geq 1 \text{ for } h = 2.$$

Hence, the disjunctive cut is

$$\frac{4}{3} x_1 + x_2 + x_3 \geq 1 \tag{3.97}$$

This cut is uniformly dominated by the cut (3.93) derived through Problem PD2.

The second method discussed above suggests that we should use $\lambda_3^1 = \lambda_1^2 = 1$ and $\lambda_i^h = 0$ otherwise for h=1,2, and i=1,2,3. This yields the cut

$$3x_1 - x_2 - x_3 \geq 1 \tag{3.98}$$

Neither (3.98) nor (3.93) uniformly dominate the other. However, the values of the euclidean and the rectilinear distance criterion for each of these cuts (3.93) and (3.98) are respectively 0.6123, 0.75 and 0.111, 0.333.

3.7 Note and References

The question of how to specify the cut parameters λ_i^h has been addressed before in the general context of cutting plane theory. However, Balas [7] and Glover [19] have addressed the question of finding these parameters in the context of disjunctive programs. The parameters defined in equation (3.15) was used by Balas [6]. The cut defined by (3.19) and (3.20) was motivated by a similar result due to Glover [18] in the context of convexity cuts. This chapter contains several results in the context of disjunctive programming due to appear in [ ].

Chapter IV

EFFECT OF DISJUNCTIVE STATEMENT FORMULATION ON DEPTH OF CUT AND POLYHEDRAL ANNEXATION TECHNIQUES

4.1 Introduction

In this chapter, we wish to emphasize two important salient features of disjunctive programming methods. Both these features basically relate to the issue of depth of cut. More specifically, we will first illustrate that one can derive cuts differing in depth through different formulations of a given disjunctive statement. Secondly, we will exhibit some connections between disjunctive programming technqiues and known polyhedral annexation methods. Based on the latter exposition, as well as on some further development, we will exhibit how one may strike a reasonable tradeoff between the effort involved to generate a cut and its depth.

The organization of this chapter is as follows. First, we illustrate the above tradeoff involved through a numerical example. Thereafter, we make some general remarks and in particular we relate these ideas to two specific cases, namely, the generalized lattice point problem and the linear complementarity problem. Next, as in Chapter III, we consider two situations - one in which each set S_h, $h \in H$ (Equation (1.1)) contains exactly one constraint and a second case in which each set S_h, $h \in H$ may contain more than one constraint. Using the first case, we establish connections between disjunctive programming methods and polyhedral annexation techniques. Following this, we demonstrate two schemes by which improved disjunctive cuts may be derived through suitable disjunction formulations. Finally, we present extensions of these developments to the second case.

4.2 Illustration of the Tradeoff Between Effort for Cut Generation and the Depth of Cut

Consider the problem

$$\text{maximize} \quad 2x_1 + 3x_2$$

$$\text{subject to} \quad \begin{aligned} x_1 + x_2 &\le 10 \quad \text{or} \quad x_1 + x_2 + s_1 = 10 \\ x_1 &\le 8 \quad \text{or} \quad x_1 + s_2 = 8 \\ x_2 &\le 5 \quad \text{or} \quad x_2 + s_3 = 5 \\ x_1, x_2 &\ge 0, \qquad s_1, s_2, s_3 \ge 0 \end{aligned}$$

Further, suppose that the following disjunctive statement must hold:

{Either x_1 or x_2 must equal zero, i.e., $x_1 x_2 = 0$}

Relaxing the disjunctive statement and solving the corresponding linear program, we obtain the solution depicted in Figure 4.1.

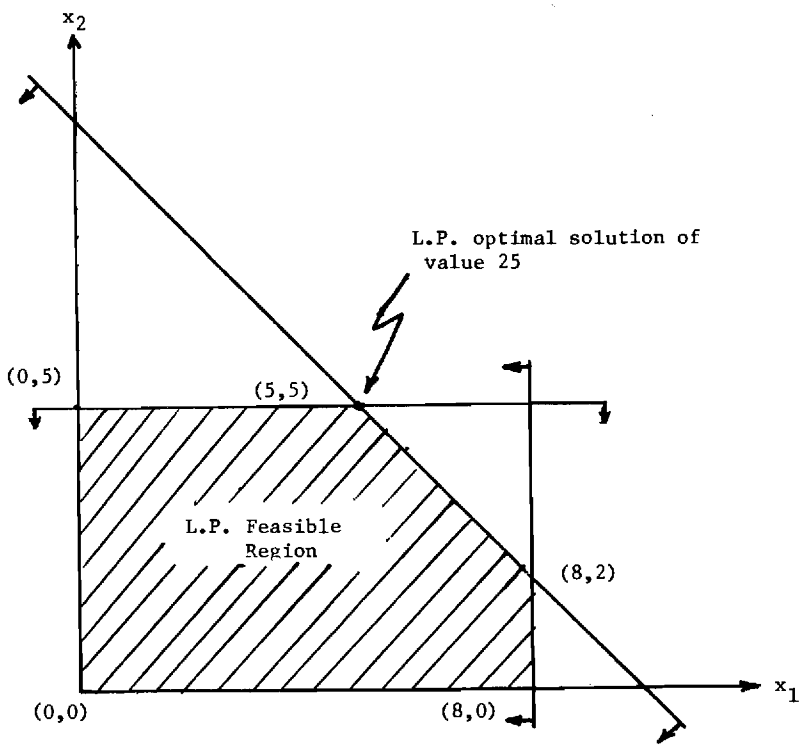

Figure 4.1. LP Solution with the Disjunction Relaxed

The simplex tableau corresponding to this optimal linear programming solution, which incidentally is feasible to the disjunction, is given below.

		Non-basic variables		RHS
		s_1	s_3	
objective function		2	1	25
Basic Variables	x_1	1	-1	5
	s_2	-1	1	3
	x_2	0	1	5

From this tableau, we can represent x_1 and x_2 in terms of the non-basic variables as

$$x_1 = 5 - s_1 + s_3$$
$$x_2 = 5 - s_3 \tag{4.1}$$

Hence, the disjunctive statement $x_1x_2 = 0$ may be restated as follows. At least one of the constraints $x_1 \leq 0$ or $x_2 \leq 0$ must be satisfied along with nonnegativity restrictions. In terms of the current nonbasic variables, this may be restated as implying that at least one of the following constraint sets must be satisfied

$$\{(s_1,s_3)\colon s_1 - s_3 \geq 5,\ s_1,s_3 \geq 0\}$$
$$\{(s_1,s_3)\colon \quad s_3 \geq 5,\ s_1,s_3 \geq 0\}. \tag{4.2}$$

Now, from our analysis in Chapter III, we know that the best cut which one may derive from this disjunction is

$$[\max\{\frac{1}{5}\ ,\ 0\}]s_1 + [\max\{-\ \frac{1}{5}\ ,\ \frac{1}{5}\}]s_3 \geq 1$$

$$\text{i.e.} \quad s_1 + s_3 \geq 5 \tag{4.3}$$

This cut may now be appended to the above tableau and the optimization procedure continued. The reader may note that, we can use Equation (4.1) to re-write the cut (4.3) as

$$x_1 + 2x_2 \leq 10 \tag{4.4}$$

Now, let us get a geometric interpretation as to why (4.3) is indeed the deepest cut. Note that in specifying the disjunction (4.2), we have neglected nonnegativity on s_2 in the tableau representing the current point. Effectively, we neglected the constraint $x_1 \leq 8$ and used only "local" information. As a result, we implied that the feasible region of the problem is $\{x_1 = 0,\ 0 \leq x_2 \leq 5\} \cup \{x_2 = 0,\ 0 \leq x_1 \leq 10\}$. The convex hull of this region is S' and is depicted in Figure 4.2 below. One may observe from S' that the best corresponding cut is precisely cut (4.4)

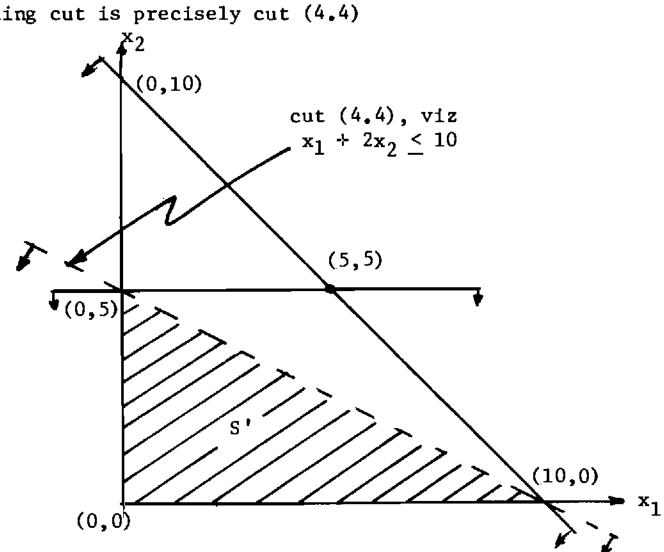

Figure 4.2. Deepest Cut for the Formulation (4.2) of the Given Disjunction

Now, let us specify the disjunction using additional information. We know for the above example problem that we must have either $x_1 = 0$ whence, $0 \leq x_2 \leq 5$ or we must have $x_2 = 0$ whence, $0 \leq x_1 \leq 8$. This feasible region to the example problem is shown darkened in Figure 4.3. Now, any valid cut should not delete any points in this feasible region. Since the half-space feasible to a cut is a closed convex set, the cut must not delete any point in the convex hull S of this feasible region shown thatched in Figure 4.3. Hence, a deep cut can at best support S, and the best cut in the present context is clearly $5x_1 + 8x_2 \leq 40$. This cut as well as the cut (4.4) is shown in Figure 4.3.

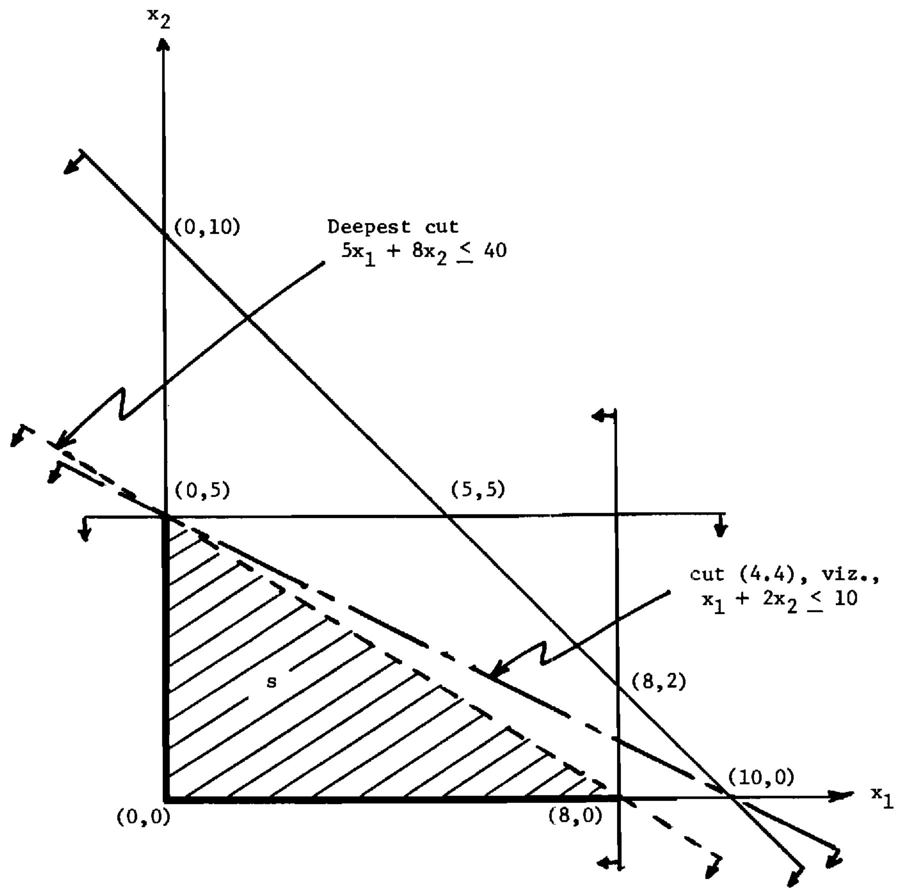

Figure 4.3. Deepest Cut

We will now see how the deepest cut $5x_1 + 8x_2 \leq 40$ can be derived algebraically. Obviously, to obtain this cut we must consider all the constraints of the original problem. In other words, the nonnegativity restrictions on s_2 must be included as well. Since

$$s_2 = 3 + s_1 - s_3 \tag{4.5}$$

in the tableau representing the current point, the given disjunctive statement implies that at least one of the following constraint sets must be satisfied.

$$\begin{aligned} S_1: \{(s_1,s_3):\ & s_1 - s_3 \geq 5 \\ & s_1 - s_3 \geq -3 \\ & s_1, s_3 \geq 0\} \\ S_2: \{(s_1,s_3):\ & s_3 \geq 5 \\ & s_1 - s_3 \geq -3 \\ & s_1, s_3 \geq 0\} \end{aligned} \tag{4.6}$$

Using multipliers 5 and 0 for the constraints of S_1 and multipliers 8 and 5 for the constraints of S_2, we obtain the surrogate constraints

$$5s_1 - 5s_3 \geq 25 \quad \text{and} \quad 5s_1 + 3s_3 \geq 25$$

This yields the disjunctive cut

$$5s_1 + 3s_3 \geq 25 \tag{4.7}$$

or, using Equations (4.1), this may be rewritten as

$$5x_1 + 8x_2 \leq 40 \tag{4.8}$$

As depicted in Figure 4.3, the cut (4.8) is the deepest possible for the given disjunction. Later in Sections 4.4 and 4.5, we will show how (4.7) or (4.8) may be derived conveniently through (4.6).

So far we have illustrated the cuts in the (x_1,x_2) space. In the space defined by the nonbasic variables (s_1,s_3), the cuts are as given by (4.3) and (4.7). These cuts are illustrated in Figures 4.4(a) and 4.4(b) respectively.

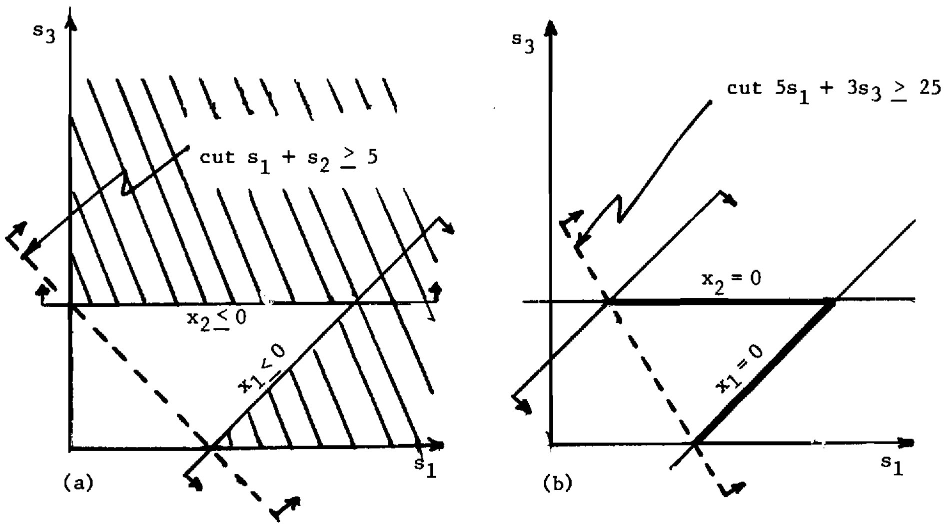

Figure 4.4. Illustration in the Nonbasic Variable Space

Through this example, we have demonstrated that one needs to consider all the constraints of the problem if a deepest cut is to be recovered through the disjunctive principles. However, this would lead to greater effort in the cut generation process. We now indicate the implication of this on the Generalized Lattice Point Problem, and the Linear Complementarity Problem.

4.3 Some General Comments with Applications to the Generalized Lattice Point and the Linear Complementarity Problems

Very briefly, we will illustrate through the Generalized Lattice Point and the Linear Complementarity problems the tradeoff which may be involved in the formulation of disjunctions. Recall our formulation of the Generalized Lattice Point Problem (GLPP) in Section 1.2.1 and let us rewrite it in a slightly different manner.

Note that if one selects q of the p components of u corresponding to linearly independent rows of A and restricts these q components to be zero, then one is confining the point u to some p-q dimensional facet of the set

$$U = \{u: u_i \geq 0, \ i=1,\ldots,p\} \tag{4.9}$$

Letting $F_1,\ldots,F_{\hat{h}}$ be the set of such (p-q) dimensional facets of U and denoting $H=\{1,\ldots,\hat{h}\}$, the set S_h of Equation (1.6) may be alternatively written as

$$S_h = \{u: u \in F_h\}, \ h \in H \tag{4.10}$$

Hence, Problem GLPP may be written as

$$\begin{aligned} \text{minimize} \quad & c^t x \\ \text{subject to} \quad & v = d - Dx \geq 0 \\ & u = b - Ax \geq 0 \\ & u \in \bigcup_{h \in H} S_h \end{aligned}$$

where S_h, $h \in H$ is given by Equation (4.10). If the rank of A is less than the number of rows A, that is, if A is not of full rank, then Problem GLPP poses an extra difficulty. One may relax the disjunctive statement in such a case to read

$$\{\text{at least } q \text{ of the } p \text{ components of } u \text{ are zero}\} \tag{4.11}$$

Now, one may unambiguously let $H=\{1,\ldots,\binom{p}{q}\}$ and correspondingly define sets S_h, $h \in H$, each corresponding to a unique combination of q out of p components of u restricted to zero. This modification makes the cut generation process much simpler and hence faster. However, this is at the expense of the depth of cut that can be derived therefrom.

Similarly, in the linear complementarity problem LCP considered in Section 1.2.5, one may be considering the violation of a particular disjunction $x_p x_q = 0$, say, in a solution to a relaxation of this problem. Hence, as exhibited in the foregoing section, one may simply use the constraints corresponding to $x_p \leq 0$ and $x_q \leq 0$ in order to derive a cut. Alternatively, one may choose to incorporate in the disjunction formulation the nonnegativity restrictions on the other basic variables as well.

Let us now generalize this concept to the situation of interest to us, namely the case where the objective function f is quasiconcave, and the set X of Problem DP (Section 1.1) is polyhedral. In order to establish connections between disjunctive programming techniques and polyhedral annexation methods, and to simplify the presentation, we will initially assume that each of the sets S_h, $h \in H$ is comprised of only a single constraint. Again, we will assume that a relaxation strategy is being adopted to solve Problem DP, so that currently, we have an extreme point optimal solution to the problem minimize $\{f(x): x \in X, x \geq 0\}$, which violates the disjunction $x \in \bigcup_{h \in H} S_h$. Here, we are assuming that the set X is comprised of the original linear constraints along with any valid inequalities which may have been generated over previous iterations. Accordingly, in terms of the current nonbasic variables, let the sets X and S_h, $h \in H$ be given by

$$X = \{x: Gx \leq g\} \equiv \{x: \sum_{j \in J} g_{ij} x_j \leq g_i \text{ for } i=1,\ldots,m\} \tag{4.12}$$

$$S_h = \{x: \sum_{j \in J} a_j^h x_j \geq 1, \ x \geq 0\}, \ h \in H \tag{4.13}$$

where J is the index set of the nonbasic variables. For each set S_h, $h \in H$, we have normalized the single constraint by its respective right-hand-side which must be positive since the origin violates each such constraint. Now, in order to derive a valid inequality which deletes the origin, one may invoke the disjunction

$$x \in \bigcup_{h \in H} S_h \tag{4.14}$$

However, we propose to derive stronger cuts by invoking the alternate disjunction

$$x \in \bigcup_{h \in H} XS_h \tag{4.15}$$

where,

$$XS_h \equiv X \cap S_h = \{x: \ Gx \leq g, \ \sum_{j \in J} a_j^h x_j \geq 1, \ x \geq 0\} \tag{4.16}$$

Note that one may invoke other valid disjunctions between the extremes (4.14) and (4.15) by adding on a subset of the constraints of X to each of the sets S_h, $h \in H$. As we have see, in the formulation of the disjunction, there is a tradeoff involved between the strength of the inequalities derived and the effort expended in generating these inequalities. Now, one viable approach is to commence with the disjunction (4.14) to obtain an initial cut, and then to sequentially add on constraints of X, attempting at each step to improve the current cut. This is basically the central point of the discussion of the following section.

4.4 Sequential Polyhedral Annexation

In this section, we will first briefly discuss the polyhedral annexation technique as is relevant to the present exposition. We will then demonstrate how an algorithmic scheme called sequential polyhedral annexation may be implemented to use the set X defined by (4.12) in order to improve the fundamental cut (3.19) available from the disjunction (4.14). We will also indicate some drawbacks of this method which lead us to proposing a variation of the scheme. This variation, however, involves additional effort in generating a cut.

Let us begin our discussion by making the observation that a disjunction which stipulates that at least one of the sets S_h of Equation (4.13) must be satisfied is equivalent to the statement that the interior of the polyhedron

$$S_H = \{x: \sum_{j\in J} a_j^h x_j \leq 1, \text{ for each } h\in H\} \tag{4.17}$$

contains no feasible points in the nonnegative orthant. Henceforth, for the sake of convenience, we will call a polyhedron NFIP if its interior contains no feasible points in the nonnegative orthant. Thus, the polyhedral annexation procedure essentially does the following. Given several NFIP polyhedra, the technique suitably annexes them to each other in order to derive a new NFIP polyhedron of the type (4.17). Then, based on the constraints of this polyhedron, a cut of the type (3.19) is generated. The annexation scheme is based on the following main result.

Theorem 4.1

Let the polyhedra

$$S_P = \{x: \sum_j a_j^p x_j \leq b^p \quad \text{for each } p\in P\} \tag{4.18}$$

and

$$S_Q = \{x: \sum_j a_j^q x_j \leq b^q \quad \text{for each } q\in Q\} \tag{4.19}$$

be NFIP. Then, for any $k\in P$, and for any nonnegative parameters μ_{kq}, μ_q, $q\in Q$, the following polyhedron is NFIP:

$$S_R = \{x: \sum_j a_j^r x_j \leq b^r \text{ for each } r\varepsilon R\}$$

$$\equiv \{x: \sum_j a_j^p x_j \leq b^p \text{ for each } p\varepsilon P - \{k\} \tag{4.20}$$

$$\sum_j (\mu_{kq} a_j^k + \mu_q a_j^q) x_j \leq (\mu_{kq} b^k + \mu_q b^q) \text{ for each } q\varepsilon Q\}$$

Proof. By contradiction, suppose S_R is not NFIP. Then, there exists a feasible, nonnegative x satisfying

$$\sum_j a_j^p x_j < b^p \quad \text{for each } p \varepsilon P - \{k\}$$

$$\sum_j (\mu_{kq} a_j^k + \mu_q a_j^q) x_j < \mu_{kq} b^k + \mu_q b^q \text{ for each } q\varepsilon Q$$

The first of these inequalities implies that $\sum_j a_j^k x_j \geq b^k$ or else, S_P would not be NFIP. This along with the second inequality implies that $\mu_q\{ \sum_j a_j^q x_j - b^q\} < \mu_{kq}\{b^k - \sum_j a_j^k x_j\} \leq 0$, or that S_Q is not NFIP, a contradiction. This completes the proof.

In terms of the traditional disjunctive programming methods, Thoerem 4.1 has the following interpretation. The condition that at least one of the constraint sets

$$S_P = \{x: \sum_j a_j^p x_j \geq b^p,\ x \geq 0\},\ p\varepsilon P \tag{4.21}$$

and at least one of the constraint sets

$$S_q = \{x: \sum_j a_j^q x_j \geq b^q,\ x \geq 0\},\ q\varepsilon Q \tag{4.22}$$

must be satisfied, implies the weaker condition that at least one of the following constraint sets must be satisfied for some $k\varepsilon P$

$$S_p \text{ for } p \in P - \{k\},$$

$$S_{k,q} = \{x: \sum_j a_j^k x_k \geq b^k, \sum_j a_j^q x_j \geq b^q, x \geq 0\} \text{ for } q \in Q \tag{4.23}$$

Given any set of nonnegative surrogate multipliers μ_{kq}, μ_q for the two constraints in each of the sets $S_{k,q}$, $q \in Q$, this in turn implies that at least one of the constraint sets

$$S_p \text{ for } p \in P - \{k\},$$

$$S_{kq} = \{x: \sum_j (\mu_{kq} a_j^k + \mu_q a_j^q) x_j \geq (\mu_{kq} b^k + \mu_q b^q), x \geq 0\} \text{ for } q \in Q \tag{4.24}$$

must be satisfied, or that S_R of Equation (4.20) must be NFIP.

Clearly, the choice of $k \in P$ for the purpose of annexation is crucial with regard to the strength of the inequality which may be derived from the disjunction (4.24). We will now discuss this choice in the context of a method known as sequential polyhedral annexation, as applied to the concepts introduced in Section 4.2.

Thus, suppose one has derived the following cut (3.19) from the disjunction that at least one of the sets S_h, $h \in H$ of Equation (4.13) must be satisfied

$$\sum_{j \in J} \bar{\pi}_j x_j \geq 1 \tag{4.25}$$

The question addressed at this point is whether or not a given cut coefficient $\bar{\pi}_k$, $k \in J$ can be improved (decreased) without worsening (increasing) the other coefficients. (In the discussion below, the reader may note that the sets X, S_H, S_P and S_Q are defined by (4.12), (4.17), (4.18) and (4.19) respectively). The manner in which the sequential method proposes to accomplish this is to commence with the NFIP polyhedron S_H and annex constraints of X one at a time. During this annexation process, that constraint which is a "blocking hyperplane", i.e., forms a "block", for the k^{th} edge extension, is chosen to be surrogated with the newly added constraints. That is the cut coefficient $\bar{\pi}_k$ is determined by that particular constraint through (3.19). In other words, the surrogation serves the purpose of attempting to rotate this blocking hyperplane so as to permit an improved edge intercept. Of course, if more than one constraint form a block for the k^{th} edge extension, then this process will have to be repeated for each of the blocking hyperplanes. Thus, starting with S_P equal to S_H, a set S_Q with $|Q| = 1$ is chosen to contain a single constraint of X. Let us assume that a constraint $k\varepsilon P$ of S_p forms a block for the k^{th} edge extension. Then, S_P and S_Q are annexed through nonnegative parameters μ_{kq} and μ_q as follows.

Note that since the origin is infeasible to each S_P, $p\varepsilon P$ of Equation (4.21), we may assume as before without loss of generality that $b^p = 1$, $p\varepsilon P$. To maintain consistency, we may also stipulate without loss of generality that the surrogation makes the right hand side of the constraint in S_{kq} of Equation (4.24) equal to unity, i.e., $\mu_{kq} + \mu_{kq}b^q = 1$. Thus, under the restriction that the cut derived from the disjunction (4.24) improves the k^{th} edge intercept without worsening the other edge intercepts, we are searching for parameters μ_{kq}, μ_q satisfying

$$\mu_q \geq 0, \ \mu_{kq} = 1 - \mu_q b^q \geq 0 \tag{4.26}$$

$$\bar{\pi}_j \geq \mu_{kq} a_j^k + \mu_q a_j^q \quad \text{for each } j\varepsilon J \tag{4.27}$$

One may easily deduce from this that the appropriate choice reduces to finding the largest $\mu_q \geq 0$ satisfying

$$\mu_q \leq \underset{j\varepsilon J}{\text{minimum}} \left\{ \frac{\bar{\pi}_j - a_j^k}{(a_j^q - a_j^k b^q)} : (a_j^q - a_j^k b^q) > 0 \right\}$$

and (4.28)

$$\mu_q b^q \leq 1$$

Then μ_{kq} is given through (4.26) and thus, the resulting NFIP polyhedron S_R of Equation (4.20) becomes the new polyhedron of the type (4.18). The cut (3.19) is updated, if necessary, with this new NFIP polyhedron (or disjunction) and the process is similarly repeated until the improvement of all edge intercepts have been attempted using all the constraints of X one at a time. Note that at each annexation, if the corresponding parameter μ_q obtained through (4.28) turns out to be zero, then this implies that $S_R \equiv S_P$ so that no improvement is possible with the current annexation.

Now, there is one principal drawback of this technique and that is, the final cut derived is dependent on the order in which one considers the constraints of X of Equation (4.12) to be used as sets S_Q of Equation (4.19). We illustrate this fact below through an example and then proceed to propose an alternative method.

Illustrative Example

Let us modify the example of Section 4.2 by adding an additional constraint to the set X of Equation (4.12). Hence, let the sets of Equation (4.13) or (4.21) be

$$S_1 = \{(s_1,s_3): \frac{s_1}{5} - \frac{s_3}{5} \geq 1,\ s_1,s_3 \geq 0\},$$

$$S_2 = \{(s_1,s_3): \frac{s_3}{5} \geq 1,\ s_1,s_3 \geq 0\} \qquad (4.29)$$

and suppose X is given by

$$X = \{(s_1,s_3): - s_1 + s_3 \leq 3,\ - s_1 + 3s_3 \leq 12\} \qquad (4.30)$$

The sets XS_1 and XS_2 of Equation (4.16) as well as the best cut available from the disjunction (4.15) are depicted in Figure 4.5.

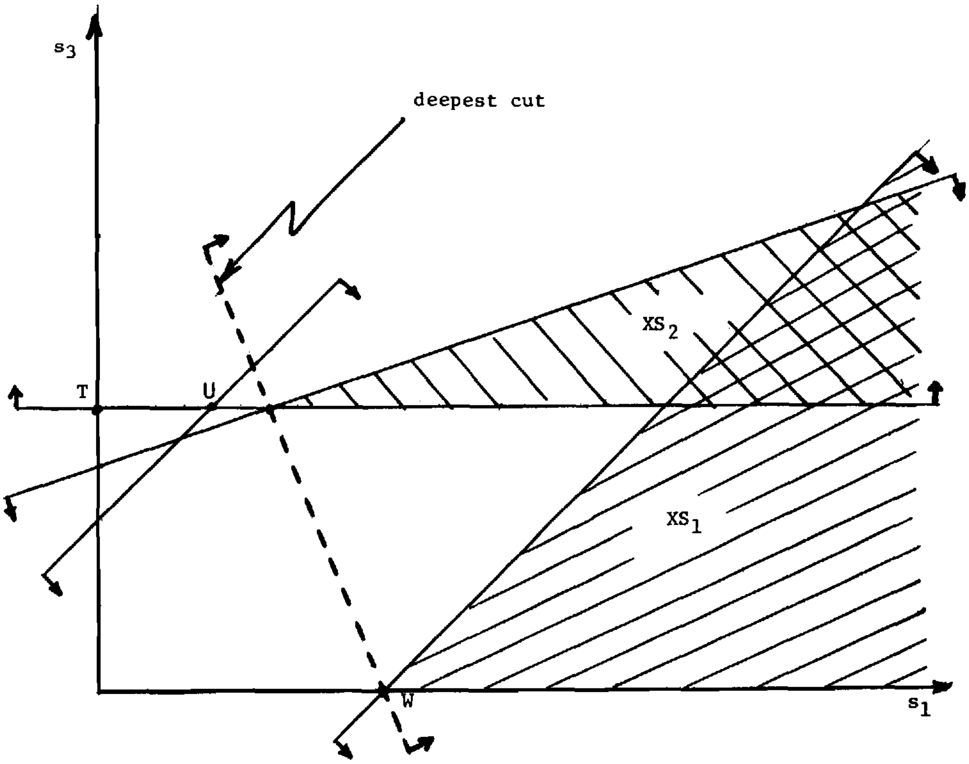

Figure 4.5. Deepest Cut From the Improved Formulation

Now, the cut (3.19) available from the disjunction $(s_1,s_3) \in S_1 \cup S_2$ is $\frac{s_1}{5} + \frac{s_3}{5} \geq 1$. This cut passes through the points T and W of Figure 4.5. One can see that the extension corresponding to edge s_1 cannot be improved. Hence, let us attempt to improve the edge intercept corresponding to s_3 using the sequential polyhedral annexation scheme. Towards this end, note that the constraint of S_2 represents the blocking hyperplane. Using the first constraint of X in the initial set S_Q of Equation (4.19), (with the inequality reversed) the relationships (4.28) yield

$$\mu_q \leq \text{minimum}\left(\frac{\frac{1}{5} - 0}{1 - (0)(-3)}, \cdot\right), \; -3\mu_q \leq 1, \; \mu_q \geq 0$$

The largest μ_q satisfying this is $\mu_q = \frac{1}{5}$, whence (4.26) gives $\mu_{kq} = 1 - (\frac{1}{5})(-3) = \frac{8}{5}$. Thus, the disjunction (4.24) is $(s_1, s_3) \in S_1 \cup S_{2q}$ where,

$$S_{2q} = \{(s_1,s_3)\colon \frac{1}{5} s_1 + \frac{3}{25} s_3 \geq 1, \; s_1,s_3 \geq 0\} \equiv \text{New } S_2\text{, say} \qquad (4.31)$$

The cut (3.19) from this disjunction is

$$\frac{1}{5} s_1 + \frac{3}{25} s_3 \geq 1 \qquad (4.32)$$

which passes through points U and W in Figure 4.5, and is also shown in Figure 4.4(b). Now let us repeat this by taking S_1 as in (4.29), S_2 as given by (4.31), the second constraint of X forming the set S_Q, and the constraint of S_2 representing the blocking hyperplane for the edge s_3 in the cut (4.32). The relationships (4.28) yield

$$\mu_q \leq \text{minimum}\left(\frac{\frac{1}{5} - \frac{1}{5}}{1 - (\frac{1}{5})(-12)}, \; \cdot\right), \; -12\mu_q \leq 1, \; \mu_q \geq 0$$

which implies, that $\mu_q = 0$ or that no further improvement is possible.

In this example, if one had considered the constraints of X in the reverse order then one would have obtained the deepest cut as shown in Figure 4.5. However, the appropriate ordering of the constraints of X is a combinatorial problem. Furthermore, conceivably it may be possible in some instances that the best cut is not recoverable no matter in which order the constraints of X are considered.

The method we propose to employ in the next section considers all the constraints of X simultaneously, that is, examines the disjunction (4.15) itself in an attempt to improve edge intercepts one at a time, holding other edge intercepts fixed at each stage. This technique is easy to implement and directly yields the best cut coefficients, the corresponding appropriate surrogate multipliers being available, if required, as a set of optimal dual variables.

4.5 A Supporting Hyperplane Scheme for Improving Edge Extensions

Suppose as before that we are given sets S_h, $h \in H$ defined by Equation (4.13) with the stipulation that at least one of these sets must be satisfied. We re-emphasize here that we continue to assume that each set S_h has only one constraint merely for convenience. In addition, we are given a constraint set X (Equation (4.12)) which must also be satisfied by an feasible point. The disjunction under consideration is that $x \in \bigcup_{h \in H} XS_h$ (Equation (4.15)) where, as in Equation (4.16), $XS_h = X \cap S_h$, $h \in H$.

Thus, assume that currently, we have a cut of the form

$$\sum_{j \in J} \bar{\pi}_j x_j \geq 1 \qquad (4.33)$$

which is valid for the disjunction (4.15). Note that initially, (4.33) may be taken as the cut (3.19) derived from the disjunction $x \in \bigcup_{h \in H} S_h$.

Now, consider a $k \in J$ and suppose that we are presently trying to improve the k^{th} edge intercept, that is, decrease $\bar{\pi}_k$. Towards this end, let us assume that we are able to solve for each $h \in H$

$$\begin{aligned} P_{kh}: \quad & \text{minimize} \quad \pi_{kh} \\ & \text{subject to} \quad \pi_{kh} x_k + \sum_{\substack{j \in J \\ j \neq k}} \bar{\pi}_j x_j \geq 1 \text{ for each } x \in XS_h \\ & \text{and} \\ & \pi_{kh} x_k + \sum_{\substack{j \in J \\ j \neq k}} \bar{\pi}_j x_j = 1 \text{ supports } XS_h \end{aligned} \qquad (4.34)$$

Let

$$\bar{\pi}_k^* = \underset{h \in H}{\text{maximum}} \{\bar{\pi}_{kh}\} \tag{4.35}$$

where $\bar{\pi}_{kh}$ is the solution to problem P_{kh}. Now consider the cut

$$\bar{\pi}_k^* x_k + \sum_{\substack{j \in J \\ j \neq k}} \bar{\pi}_j x_j \geq 1 \tag{4.36}$$

Clearly, (4.36) is satisfied by each $x \in \bigcup_{h \in H} XS_h$, that is, (4.36) is a valid cut for the disjunction (4.15). Moreover, any inequality $\sum_{j \in J} \pi_j x_j \geq 1$ with $\pi_j = \bar{\pi}_j$ for $j \in J - \{k\}$ and $\pi_k < \bar{\pi}_k^*$ is not valid because it deletes a point $\hat{x}$ of $XS_{\hat{h}}$ at which the corresponding hyperplane $\bar{\pi}_{k\hat{h}} x_k + \sum_{\substack{j \in J \\ j \neq k}} \bar{\pi}_j x_j = 1$ supports $XS_{\hat{h}}$, where $\hat{h} \in H$ is an index for which equality holds in (4.35). To see this, it is sufficient to show that if $\bar{\pi}_{kh} > -\infty$ in (4.34), then a point of support referred to in (4.34) occurs at an $\bar{x}$ satisfying $\bar{x}_k > 0$. This is clearly so, for if not, then $\bar{\pi}_{kh}$ can be reduced still further. Thus, (4.36) gives the best intercept possible for the k^{th} edge when all other intercepts are held fixed. Hence, replacing $\bar{\pi}_k$ of (4.33) by $\bar{\pi}_k^*$, we would obtain a (possibly) new valid cut (4.33). This process may now be repeated for each edge in turn till no further improvement is possible. Of course, different cuts may be obtained by considering the edges in different orders, but each of these cuts cannot be uniformly dominated by any other cut.

We will now proceed to discuss the determination of $\bar{\pi}_{kh}$, the coefficient of x_k in the cutting plane under consideration, given through (4.34). The problem we formulate below to accomplish this, has the following motivation. Observe that the cut hyperplane is constrained to pass through (n-1) linearly independent points of the form $(0, \ldots, \frac{1}{\bar{\pi}_j}, \ldots, 0)$ for $j \in J - \{k\}$. In order to uniquely define the cutting plane, we need to identify a suitable point $\bar{x}$ which has $\bar{x}_k > 0$. Now, according to Equation (4.34), this cutting plane will need to support the set XS_h with each point of XS_h being feasible to it. Hence, in order to determine $\bar{\pi}_{kh}$, we may hold the intercepts on the axes $j \in J - \{k\}$ fixed and decrease the intercept on the k^{th} axis (increase π_{kh}) until the hyperplane merely supports XS_h at some point $\bar{x}$ with $\bar{x}_k > 0$. This problem is mathematically stated below. Theorem 4.2 later establishes that an optimal solution to this problem yields $\pi_k = \bar{\bar{\pi}}_{kh}$

$\bar{P}_{kh}$: maximize π_k

$$\text{subject to} \quad \pi_k x_k + \sum_{\substack{j \varepsilon J \\ j \neq k}} \bar{\pi}_j x_j = 1 \tag{4.37}$$

$$x \; \varepsilon \; XS_h \tag{4.38}$$

$$x_k > 0 \tag{4.39}$$

Note that π_k is unrestricted in sign. Now using Equations (4.12), (4.13), (4.16) and solving for π_k through Equation (4.37), we may rewrite the above problem as

$$\text{maximize} \quad \frac{1}{x_k} - \sum_{\substack{j \varepsilon J \\ j \neq k}} \bar{\bar{\pi}}_j \left(\frac{x_j}{x_k}\right)$$

$$\text{subject to} \quad \sum_{j \varepsilon J} g_{ij}\left(\frac{x_j}{x_k}\right) \leq \frac{g_i}{x_k} \qquad \text{for } i=1,\ldots,m$$

$$\sum_{j \varepsilon J} a_j^h \left(\frac{x_j}{x_k}\right) \geq \frac{1}{x_k}$$

$$\left(\frac{x_j}{x_k}\right) \geq 0, \; x_k > 0$$

Finally, letting

$$\xi = \frac{1}{x_k} \text{ and } y_j = \frac{x_j}{x_k} \quad \text{for each } j \varepsilon J \tag{4.40}$$

we obtain the following linear programming problem in $|J|$ variables

$$\text{LP}_{kh}: \quad \text{maximize} \quad z(\xi,y) = \xi - \sum_{\substack{j\varepsilon J \\ j\neq k}} \bar{\pi}_j y_j$$

$$\text{subject to} \quad \sum_{\substack{j\varepsilon J \\ j\neq k}} g_{ij}y_j - g_i\xi \leq -g_{ik} \quad \text{for } i=1,\ldots,m$$

$$\xi - \sum_{\substack{j\varepsilon J \\ j\neq k}} a_j^h y_j \leq a_k^h$$

$$\xi \geq 0, \; y_j \geq 0 \text{ for } j\varepsilon J - \{k\}$$

Consider the following result.

Theorem 4.2

If Problem LP_{kh} is feasible, then it has an optimal solution $\bar{\xi}$, $\bar{y}_j$, $j\varepsilon j - \{k\}$ with $\bar{\xi} < \infty$. Moreover, the optimal solution values of Problems LP_{kh} and Problem P_{kh} (defined by 4.34)) are equal.

Proof. Note that the constraints of Problem LP_{kh} may be rewritten as $\sum_{j\varepsilon J} g_{ij}y_j - g_i\xi < 0$ for $i=1,\ldots,m$; $-\sum_{j\varepsilon J} a_j^h y_j + \xi \leq 0$ and $y_k = 1$, with ξ, $y \geq 0$. Letting μ_i, $i=1,\ldots,m$, γ and β_k be the respective dual variables associated with these constraints, the dual to Problem LP_{kh} may be written as

$$\text{DLP}_{kh}: \quad \text{minimize} \quad \beta_k$$

$$\text{subject to} \quad a_j^h\gamma - \sum_{i=1}^{m} g_{ij}\mu_i \leq \bar{\pi}_j \quad \text{for } j\varepsilon J - \{k\} \tag{4.41}$$

$$a_k^h\gamma - \sum_{i=1}^{m} g_{ik}\mu_i \leq \beta_k \tag{4.42}$$

$$\gamma - \sum_{i=1}^{m} g_i\mu_i \geq 1 \tag{4.43}$$

$$\gamma, \; \mu \geq 0$$

Letting $\bar{\beta}_k$ denote the minimum value of β_k, we will show that $\bar{\beta}_k = \bar{\pi}_{kh}$. We have from (4.34)

$$\bar{\pi}_{kh} x_k + \sum_{\substack{j \varepsilon J \\ j \neq k}} \bar{\pi}_j x_j \geq 1 \qquad \text{for each } x \varepsilon XS_h \tag{4.44}$$

$$\bar{\pi}_{kh} \hat{x}_k + \sum_{\substack{j \varepsilon J \\ j \neq k}} \bar{\pi}_j \hat{x}_j = 1 \qquad \text{for some } \hat{x} \varepsilon XS_h \tag{4.45}$$

Hence (4.44) is implied by XS_h and Lemma 2.2 asserts the existence of $\hat{\gamma} \geq 0$, $\hat{\mu}_i \geq 0$ satisfying (4.41) through (4.43) with $\beta_k = \bar{\pi}_{kh}$. That is $\hat{\gamma}$, $\hat{\mu}_i$ and $\bar{\pi}_{kh}$ is feasible to DLP_{kh}. Thus LP_{kh} is bounded. Hence, $\bar{\beta}_k \leq \bar{\pi}_{kh} < \infty$. Now let $\bar{\gamma}$, $\bar{\mu}_i$ and $\bar{\beta}_k$ solve DLP_{kh}. Then, $x \varepsilon XS_h$ implies

$$\sum_{j \varepsilon J} [a_j^h \bar{\gamma} - \sum_{i=1}^{m} g_{ij} \bar{\mu}_i] x_j \geq [\bar{\gamma} - \sum_{i=1}^{m} g_i \bar{\mu}_i]$$

Then noting (4.41) through (4.43), we get

$$\bar{\beta}_k \hat{x}_k + \sum_{\substack{j \varepsilon J \\ j \neq k}} \bar{\pi}_j \hat{x}_j \geq 1 \qquad \text{for each } x \varepsilon XS_h \tag{4.46}$$

We have shown that $\bar{\beta}_k \leq \bar{\pi}_{kh}$. Now if $\bar{\beta}_k < \bar{\pi}_{kh}$, then from (4.45)

$$\bar{\beta}_k x_k + \sum_{\substack{j \varepsilon J \\ j \neq k}} \bar{\pi}_j x_j < 1$$

contradicting (4.46). Hence $\bar{\beta}_k = \bar{\pi}_{kh}$.

Finally, since LP_{kh} is bounded, there exists an optimal extreme point solution $(\bar{\xi}, \bar{y})$ with $\bar{\xi}$ finite. This completes the proof.

<u>Corollary</u>

Let $\bar{\xi}$, $\bar{y}_j$, $j \varepsilon J - \{k\}$ solve LP_{kh} with $\bar{\xi} < \infty$, and with $\bar{\beta}_k$ as the corresponding objective function value. Then, $\pi_k = \bar{\beta}_k$, $x_k = \frac{1}{\bar{\xi}}$ and $x_j = \frac{\bar{y}_j}{\bar{\xi}}$ for $j \varepsilon J - \{k\}$ solves

P_{kh}.

It is easy to show that the following expedient for determining $\bar{\pi}_k^*$ of Equation (4.35) through the solutions of Problems LP_{kh}, $h \varepsilon H$ for a given $k \varepsilon J$, is a valid scheme.

Step 1

Consider the cut (3.19) derived for the disjunction $x \varepsilon \bigcup_{h \varepsilon H} S_h$. Let $h \varepsilon H$ be a "blocking hyperplane" for the k^{th} edge as defined in Section 4.4.

Step 2

Solve LP_{kh}. If LP_{kh} is infeasible, then select any $h \varepsilon H$ not considered thus far and repeat Step 2. (If LP_{kh} is infeasible for each $h \varepsilon H$, then $x_k = 0$ for each $x \varepsilon \bigcup_{h \varepsilon H} XS_h$ and the variable x_k may be disregarded from the problem). Otherwise, obtain an optimal solution value $\bar{\pi}_{kh}$. If $\bar{\pi}_{kh} = \bar{\pi}_k$, terminate with $\bar{\pi}_k^* = \bar{\pi}_k$. If each $h \varepsilon H$ has been considered, terminate with $\bar{\pi}_k^*$ given through Equation (4.35). Otherwise, select an $h \varepsilon H$ not considered thus far and repeat Step 2 by solving Problem LP_{kh}' with the added restriction that its objective value be at least equal to the largest of the objective values of problems LP_{kh} that have been solved before this for other $h \varepsilon H$.

We remark at this point that the development of the present section may be easily extended to a broader class of problems in which each of the sets S_h, $h \varepsilon H$ may contain more than one constraint. This is accomplished by simply writing the second constraint of LP_{kh} for each of the constraints in S_h. Further, Step 1 of the scheme discussed above may be started from an arbitrary h or from one determined heuristically.

We also draw the readers' attention to the linear programming approach (Problem LP) of Section 3.5 for deriving deep cuts. This formulation is similar in thrust to Problem DLP_{kh} of the present section and may be used to obtain supports of the closure of the convex hull of $\{x: x \varepsilon \bigcup_{h \varepsilon H} S_h\}$ which are valid cuts for the corresponding disjunction $x \varepsilon \bigcup_{h \varepsilon H} S_h$.

4.6 Illustrative Examples

Suppose that we are given

$$S_1 = \{x: -x_1 + x_2 \geq 1, x \geq 0\} \text{ and } S_2 = \{x: x_1 \geq 1, x \geq 0\}$$

Consider the following three examples of the set X:

(i) $X = \{x: 2x_1 - 2x_2 \leq 1\}$

(ii) $X = \{x: 2x_1 - x_2 \leq 1\}$, and

(iii) $X = \{x: 6x_1 - 2x_2 \leq 3\}$.

These three cases are depicted in Figure 4.6. Also shown in this figure are the respective best cuts available.

Now, the cut (3.19) from the disjunction $x \in \bigcup_{h \in H} S_h$ is $x_1 + x_2 \geq 1$.

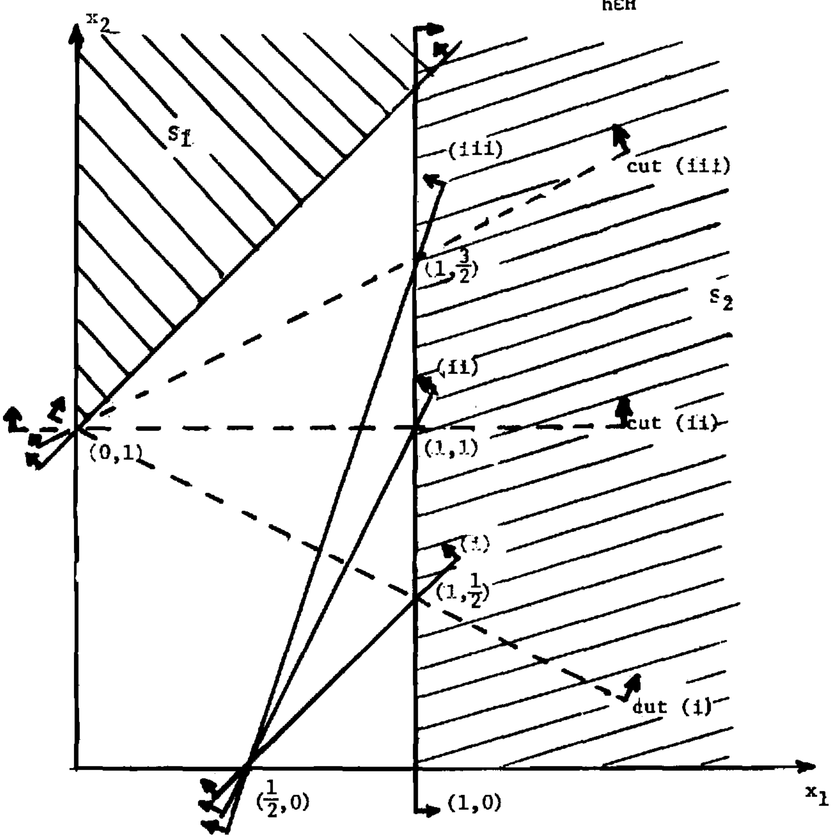

Figure 4.6. Cuts for the Three Illustrative Examples

Using the development of the previous section to improve the extension corresponding to the variable x_1 holding the other extensions fixed yields the following problem. Note that h=2 is selected at Step 1 in each case.

	Example (i)	Example (ii)	Example (iii)
Problems LP_{kh} with k=1, h=2	maximize $\xi - y_2$ subject to $2y_2 + \xi \geq 2$ $\xi \leq 1$ $\xi, y_2 \geq 0$	maximize $\xi - y_2$ subject to $y_2 + \xi \geq 2$ $\xi \leq 1$ $\xi, y_2 \geq 0$	maximize $\xi - y_2$ subject to $2y_2 + 3\xi \geq 6$ $\xi \leq 1$ $\xi, y_2 \geq 0$
Solution Value:	1/2	0	-1/2
ξ	1	1	1
y_2	1/2	1	3/2

One may easily verify that the Problem LP_{kh} with h=1 yields smaller objective values in each case. Further, the edge extension corresponding to x_2 cannot be improved. Hence, the best cuts obtained in each case are

Example (i) $\quad 1/2\, x_1 + x_2 \geq 1$

Example (ii) $\quad x_2 \geq 1$

Example (iii) $\quad -1/2\, x_1 + x_2 \geq 1$

These cuts are illustrated in Figure 4.6.

We will now proceed to discuss another important concept closely related to the depth of disjunctive cuts. Specifically, as briefly mentioned on several occasions thus far, facets of the closure of the convex hull of the set $\bigcup_{h \in H} S_h$ constitute highly desirable disjunctive cuts. In fact, if one replaces the nonconvex set $\bigcup_{h \in H} S_h$ by the closure of its convex hull, which is polyhedral whenever S_h are defined by a set of linear constraints with $|H| < \infty$, then the nonconvex disjunctive program may be effectively replaced by its convex equivalent.

A method of identifying and generating such facets is the topic of our discussion in the following chapter.

4.7 Notes and References

Recall that a valid cut is nondominated if it supports the closure of the convex hull of the feasible region. This chapter first shows how the strength of a disjunctive cut depends on the formulation of the disjunctive statement. Thus, one can conceivably derive a number of different nondominated cuts, each corresponding to a different formulation. A related thought that is pursued in this chapter is as follows. Given a valid cut, possibly a nondominated cut corresponding to one formulation, can we improve upon this cut. In particular, the improved cut is a nondominated cut corresponding to a new formulation obtained from the previous one by adding one constraint. The answer is affirmative, and it is precisely the polyhedral annexation scheme of Glover [19]. A further thought that arises is whether the strength of the final cut thus derived is dependent upon the order in which the constraints are added. The answer again is affirmative. In Section 4.5 therefore a scheme is developed whereby a given valid cut is improved by considering all the constraints simultaneously.

Chapter V

GENERATION OF FACETS OF THE CLOSURE OF THE CONVEX HULL OF FEASIBLE POINTS

5.1 Introduction

In this Chapter, we examine a procedure for replacing the disjunctive statement in a problem by linear inequalities which represent the facets of the closure of the convex hull of points feasible to the disjunction. In particular, we present necessary and sufficient conditions for an inequality to define a facet of the closure of the convex hull of feasible points. The actual generation of such facets is a hard problem. However, for a special class of problems (called "facial problems" in this chapter) it is possible to obtain the closure of the convex hull of points satisfying disjunctions, in a sequence of q steps, where each step generates the closure of the convex hull of points satisfying one disjunction only.

To simplify the presentation, we will avoid proving results and will simply state them and illustrate them through a numerical example. To begin with, let us state the form of the disjunctive program DP of Chapter I which we will be working with in the present context

DP:

$$\begin{aligned} &\text{minimize} && f(x) = c^t x \\ &\text{subject to} && x \in X = \{x: Dx \geq d,\ x \geq 0\} \\ & && \bigvee_{h \in H} \{A^h x \geq b^h,\ x \geq 0\} \end{aligned}$$

The linear program obtained by relaxing the disjunctive statement of Problem DP is

LP:

$$\begin{aligned} &\text{minimize} && c^t x \\ &\text{subject to} && Dx \geq d \\ & && x \geq 0 \end{aligned}$$

We will assume henceforth that both DP and LP are stated above in terms of the nonbasic variables at the current optimal solution to LP. Thus, the current solution is $x = 0$ with feasibility implying that $d \leq 0$ and optimality implying that $c \geq 0$.

To illustrate, let us work with the numerical problem of Section 4.2 throughout this chapter. From the optimal tableau for Problem LP given in Section 4.2, we deduce

$$
\text{DP:} \qquad \begin{array}{rl} \text{minimize} & 2s_1 + s_3 \\ \text{subject to} & -s_1 + s_3 \geq -5 \\ & s_1 - s_3 \geq -3 \\ & - s_3 \geq -5 \\ & s_1, s_3 \geq 0 \end{array}
$$

$$
\left\{ \begin{array}{c} s_1 - s_3 \geq 5 \\ s_1,\ s_3 \geq 0 \end{array} \right\} \vee \left\{ \begin{array}{c} s_3 \geq 5 \\ s_1,\ s_3 \geq 5 \end{array} \right\}
$$

5.2 A Linear Programming Equivalent of the Disjunctive Program

In this section, we will write a linear program ELP which is in a defined sense equivalent to the nonconvex problem DP given above. For this purpose let us define the following sets.

For each $h \in H$, let

$$
F_h = \{x \epsilon R^n\colon D^h x \geq d^h,\ x \geq 0\} \equiv \{x \epsilon R^n\colon Dx \geq d,\ A^h x \geq b^h,\ x \geq 0\} \tag{5.1}
$$

represent the points feasible to LP which are also feasible to the h^{th}, $h \in H$, disjunctive constraint. Also let

$$
F = \bigcup_{h \epsilon H} F_h = \left\{ x \epsilon R^n\colon \bigvee_{h \epsilon H} \{D^h x \geq d^h,\ x \geq 0\} \right\} \tag{5.2}
$$

Finally, let us denote the feasible region of LP as

$$F_0 = \{x \in R^n: Dx \geq d, \ x \geq 0\} \tag{5.3}$$

Let us assume that $|H| < \infty$ and let us define

$$H^* = \{h \in H: F_h \neq \{\phi\}\} \tag{5.4}$$

Now, let us characterize the closure of the convex hull of F, denoted by <u>$c\ell$ conv F</u>. Note that any $x \in F$ may be written as

$$x = \sum_{h \in H^*} \xi_0^h u^h, \text{ where } u^h \in F_h, \ h \in H^*, \text{ and where,}$$

$$\sum_{h \in H^*} \xi_0^h = 1, \ \xi_0^h \geq 0$$

Hence, substituting $\xi^h = \xi_0^h u^h$ and noting that $D^h u^h \geq d^h$, $u^h \geq 0$, we get

$$c\ell \text{ conv } F = \left\{ \begin{array}{l} x \in R^n: x = \sum_{h \in H^*} \xi^h \\ D^h \xi^h \geq d^h \xi_0^h, \ h \in H^* \\ \sum_{h \in H^*} \xi_0^h = 1 \\ (\xi_1^h, \xi_0^h) \geq 0, \ h \in H^* \end{array} \right\} \tag{5.5}$$

It may be shown that if the feasible region of LP is bounded, then (5.5) is true with H* replaced by H. Now, if LP has a finite optimal solution, then directly using the characterization of Equation (5.5), we may write a linear program ELP equivalent to Problem DP in the sense of Theorem 5.1 stated below.

<u>ELP</u>: minimize $\sum_{h\varepsilon H} c^t \xi^h$

subject to $D^h \xi^h \geq d^h \xi_0^h, \ h\varepsilon H$

$$\sum_{h\varepsilon H} \xi_0^h = 1$$

$$(\xi^h, \xi^h) \geq 0, \ h\varepsilon H$$

Let us denote the feasible region of Problem ELP as P. Then consider the following result stated without proof.

<u>Theorem 5.1</u>

Problems DP and ELP are equivalent in the following sense

(i) For every extreme point x of cℓ conv F, there corresponds an extreme point of P with components

$$\begin{cases} (\xi^k, \xi_0^k) = (x, 1) \text{ for some } k\varepsilon H \\ 0 \quad \text{otherwise} \end{cases}$$

(ii) All extreme points of P have components of the following form

$$\begin{cases} (\xi^k, 1) \quad \text{for some } k\varepsilon H \\ 0 \quad \text{otherwise} \end{cases}$$

where, $x = \xi^k$ is an extreme point of F_k

(iii) x is optimal to DP if and only if the corresponding extreme point of P defined in (i) above is optimal to ELP.

For the moment, we will not involve ourselves with the description of specialized solution procedures for Problem ELP. We merely remark that there do exist simplex-based decomposition solution schemes which exploit the structure of ELP. We will now proceed to give an alternate characterization of the set cℓ conv F which permits the explicit generation of the facets of the convex hull of feasible points. Before that, let us illustrate some of the concepts introduced in this section through our numerical example. Figure 5.1 is provided below for this purpose, and may be referred to along with the statement of Problem DP.

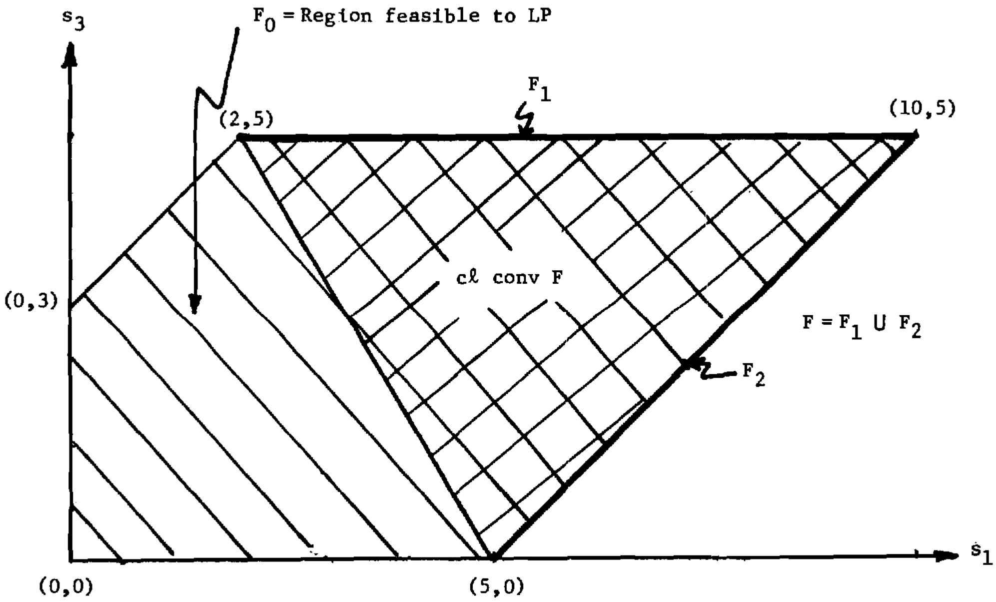

Figure 5.1. Illustration of the sets F_0, F_1, F_2, F, cℓ conv F

5.3 Alternative Characterization of the Closure of the Convex Hull of Feasible Points

In this section, we will lay the foundations for the procedure which will generate facets of the set cℓ conv F. In particular, we will present an alternative characterization for the latter set. To begin with, let us informally or conceptually introduce certain definitions which we will find necessary to use in the following exposition.

By the polar set of F, we mean the set

$$F^0 = \{\pi \in R^n: \pi x \leq 1 \text{ for each } x \in F\} \qquad (5.6)$$

By the scaled polar of F with scale parameter π_0, we mean the set

$$F^0(\pi_0) = \{\pi \in R^n: \pi x \leq \pi_0 \quad \text{for each } x \in F\} \tag{5.7}$$

The scaled reverse polar of F with scale parameter π_0 is the set

$$F^{\#}(\pi_0) = \{\pi \in R^n: \pi x \geq \pi_0 \quad \text{for each } x \in F\} \equiv - F^0(-\pi_0) \tag{5.8}$$

Given two sets S and T, their Minkowski Sum is the set

$$S + T = \{x: x = s + t,\ s \in S,\ t \in T\} \tag{5.9}$$

The conical hull of S is the set

$$\text{cone } S = \{x: x = \sum_i \lambda_i x^i,\ x^i \in S \text{ for each } i,\ \lambda_i \geq 0\} \tag{5.10}$$

The linear hull of S is the smallest subspace of R^n containing S, that is,

$$\ell h\ S = \{x \in R^n: x = \sum_i \lambda_i x^i,\ x^i \in S \text{ for each } i\} \tag{5.11}$$

The affine hull of S is the set

$$\text{aff } S = \{x \in R^n: x = \sum_i \lambda_i x^i,\ x^i \in S \text{ for each } i,\ \sum_i \lambda_i = 1\} \tag{5.12}$$

The orthogonal complement of S is the set

$$S^{\perp} = \{\pi \in R^n: \pi x = 0 \text{ for each } x \in S\} \tag{5.13}$$

The recession cone of S is the closed cone which is a closure of the set of directions of S, that is,

$$C(S) = c\ell\{v \in R^n: x \in S \text{ implies that } x + \lambda v \in S \text{ for all } \lambda \geq 0\} \tag{5.14}$$

The linearity of S is the dimensionality of the largest subspace contained in C(S), that is,

$$\ell\text{in } S = \dim \{R: R \text{ is a subspace of } C(S) \text{ and}$$
$$Q \subset R \text{ for each subspace } Q \text{ of } C(S)\} \tag{5.15}$$

Note that for a set T, the dimension of T, denoted dim T, is the dimension of the linear hull of T.

Let us turn our attention to the scaled reverse polar of F, $F^{\#}(\pi_0)$, (Equation (5.8)) which we will find very important in the present context. Note that $F^{\#}(\pi_0)$ is the set of all normals to the hyperplanes which define valid cuts for the stated disjunction. Hence, two alternative ways of writing $F^{\#}(\pi_0)$ are given below

$$\text{(a) } F^{\#}(\pi_0) = \{\pi \in R^n: \pi x^i \geq \pi_0 \text{ for each } x^i \in \text{vert } c\ell \text{ conv } F$$
$$\pi d^i \geq 0 \text{ for each } d^i \in \text{dir } c\ell \text{ conv } F\} \tag{5.16}$$

$$\text{(b) } F^{\#}(\pi_0) = \{\pi \in R^n: \pi \geq \theta^h D + \sigma^h A^h,\ h \in H^* \text{ for some } \theta^h, \sigma^h \geq 0,$$
$$h \in H^* \text{ such that } \theta^h d + \sigma^h b^h \geq \pi_0\} \tag{5.17}$$

Above, for a polyhedral set S, vert S denotes the set of extreme points of the set S and dir S denotes the set of directions of S.

Note that the sign of π_0 is important in this context since we can always scale $\pi x \geq \pi_0$ so that π_0 is either +1 or -1 or 0. These latter three cases will henceforth be of primary interest to us. Further, whenever the sign of π_0 is inconsequential, we will simply write $F^{\#}$ instead of $F^{\#}(\pi_0)$. Finally, note that the characterization (5.16) of $F^{\#}(\pi_0)$ is conceptual whereas that of Equation (5.17) is accessible. To aid our understanding of $F^{\#}$, let us actually construct it using (5.16) for our problem, using values of 1, -1, and 0 for π_0.

As depicted in Figure 5.1, $c\ell$ conv F is a polytope with extreme points (2,5), (5,0), (10,5). The set dir $c\ell$ conv F is vacuous. Using Equation (5.16), we have,

$$F^{\#}(\pi_0) = \left\{ \begin{array}{r} \pi = (\pi_1, \pi_2):\ 2\pi_1 + 5\pi_2 \geq \pi_0 \\ 5\pi_1 \geq \pi_0 \\ 10\pi_1 + 5\pi_2 \geq \pi_0 \\ \pi_1,\ \pi_2 \text{ unrestricted} \end{array} \right\}$$

Figure 5.2 illustrates the sets $F^{\#}(1)$, $F^{\#}(-1)$ and $F^{\#}(0)$.

Some useful properties of the reverse polar are stated below without proof.

Lemma 5.1

Let S and T be arbitrary sets. Then,

$$(\lambda S)^{\#} = \frac{1}{\lambda}(S^{\#}),\ -\infty < \lambda < \infty$$

$$S \subseteq T \text{ implies } S^{\#} \supseteq T^{\#}$$

$$(S \cup T)^{\#} = S^{\#} \cap T^{\#}$$

Based on these properties, one may establish the following important relationships between $F^{\#}$ and $c\ell$ conv F.

Theorem 5.2

(i) If $\pi_0 > 0$, then

$$0 \in c\ell \text{ conv } F \iff F^{\#} = \{\phi\} \iff F^{\#} \text{ is bounded}$$

(ii) If $\pi_0 \leq 0$, then $F^{\#} \neq \{\phi\}$ and

$$0 \in \text{int conv } F \iff F^{\#} \text{ is bounded.}$$

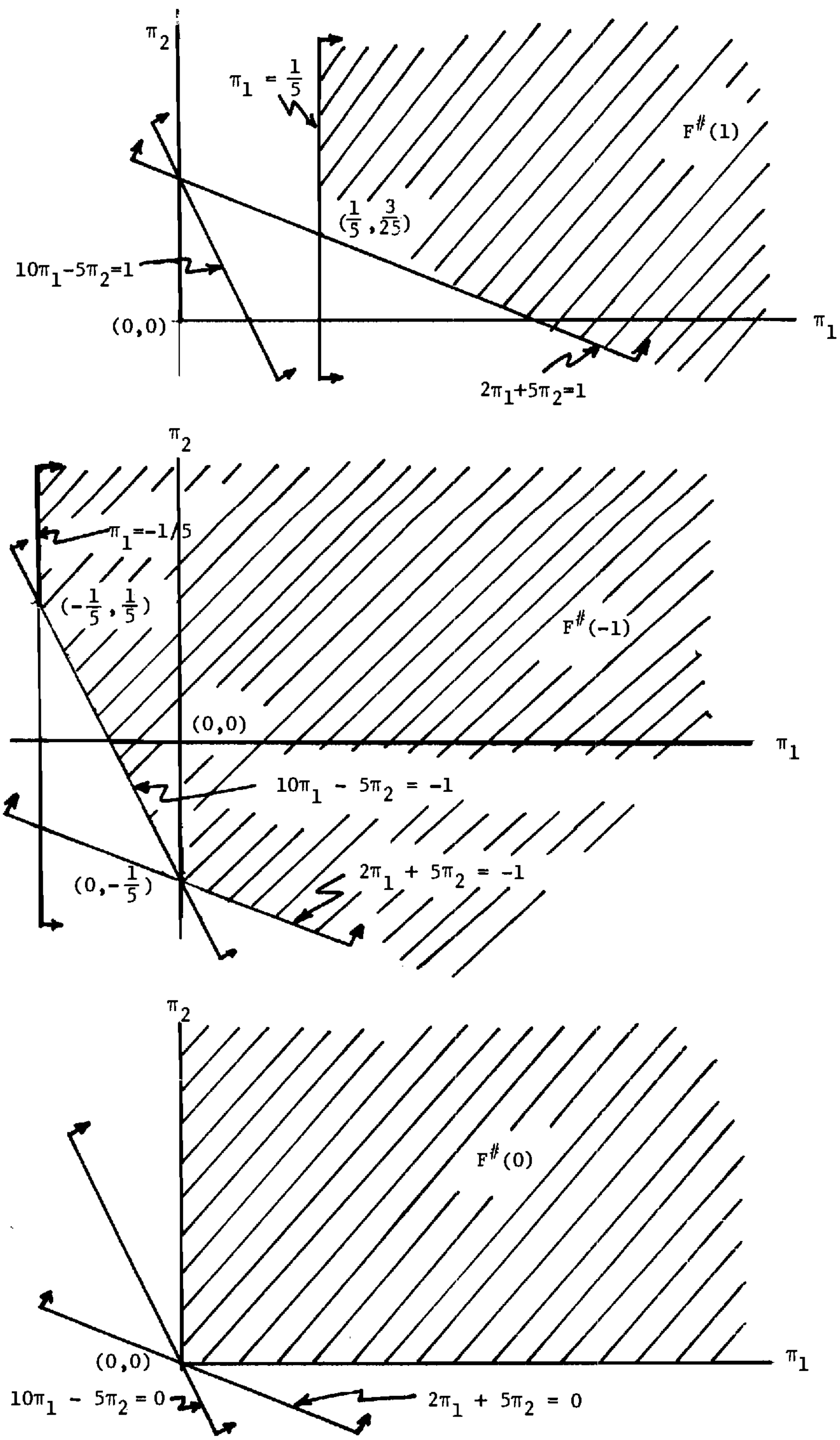

Figure 5.2. Construction of $F^{\#}(\pi_0)$ for $\pi_0 = 1, -1, 0$

However, the set more closely related with cℓ conv F is the set $(F^{\#})^{\#}$ denoted by $F^{\#\#}$. Suppose that $F^{\#} \neq \{\phi\}$. That is, from Theorem 5.2 above, assume that $0 \notin$ cℓ conv F. In other words, the point we are currently located at (assumed to be the origin) is infeasible to the disjunction. Then, note from Equation (5.8) and the definition of valid inequalities that

$$F^{\#\#}(\pi_0) = \{x \in R^n = x \text{ is feasible to all valid inequalities of the form } \pi x \geq \pi_0\} \tag{5.18}$$

Thus, clearly, we have,

$$c\ell \text{ conv } F = \bigcap_{\pi_0 = 1,-1,0} F^{\#\#}(\pi_0) \tag{5.19}$$

Since the set of all valid inequalities are jointly determined by $F^{\#}(1)$, $F^{\#}(-1)$, $F^{\#}(0)$, then from (5.18), the intersection (5.19) defines the closure of the convex hull of feasible points. Two alternative ways of rewriting $F^{\#\#}$ are given below

$$\text{(a)}\quad F^{\#\#}(\pi_0) = \left\{ \begin{array}{l} x \in R^n\text{: } \pi^i x \geq \pi_0 \text{ for all } \pi^i \in \text{vert } F^{\#}(\pi_0) \\ \qquad d^i x \geq 0 \text{ for all } d^i \in \text{dir } F^{\#}(\pi_0) \end{array} \right\} \tag{5.20}$$

(b) From definitions (5.9), (5.10) and (5.19) one may alternatively show that

$$F^{\#\#}(\pi_0) = \begin{cases} c\ell \text{ (conv F + cone F)} & \text{if } \pi_0 > 0 \\ c\ell \text{ cone F} & \text{if } \pi_0 = 0 \\ c\ell \text{ conv (F} \cup \{0\}) & \text{if } \pi_0 < 0 \end{cases} \tag{5.21}$$

As before, we may use the characterization (5.16) to construct $F^{\#\#}(\pi_0)$ from the sets $F^{\#}(\pi_0)$ of Figure 5.2. These sets are depicted in Figure 5.3 below. The reader may find it interesting to verify diagramatically the definition of Equation (5.21).

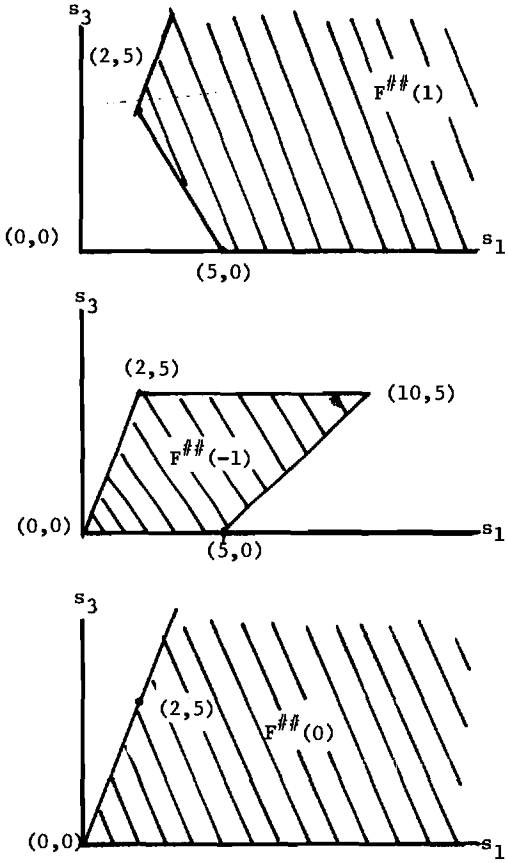

Figure 5.3. Construction of $F^{\#\#}(\pi_0)$ for π_0 = 1,-1,0

Next, we state some useful properties of the set $F^{\#}$ and some important results concerning $F^{\#}$ and $F^{\#\#}$. These are based on the definitions (5.8) through (5.15) introduced earlier.

Lemma 5.2

(a) If $F^{\#} = \{\phi\}$ (which necessarily means that $\pi_0 > 0$), then $F^{\#\#} = R^n$

(b) $F^{\#\#\#} = F^{\#}$

(c) aff $F^{\#\#} = \ell h\ F^{\#\#} = \ell h\ F$

(d) $\ell h\ F = L^{\perp}$ where

$$L = \text{largest subspace contained in the recession cone } C(F^{\#}) \text{ of } F^{\#} \equiv \text{linearity space of } F^{\#} \tag{5.22}$$

(e) $\dim F^{\#\#} + \ell\text{in}\ F^{\#\#} = n$

(f) $\dim F^{\#\#} = \begin{cases} \dim F \text{ if } 0 \in \text{aff } F \\ \dim F + 1 \text{ if } 0 \notin \text{aff } F \end{cases}$

(g) $F^{\#} = (F^{\#} \cap \ell h\ F) + L$. (This follows from (d) above and fact that L defined in (5.22) is a subset of $F^{\#}$).

(h) Lowest dimensional faces of $F^{\#}$ are of dimension $(n - \dim F^{\#\#})$.

For our example problem,

$\text{aff } F^{\#\#} \equiv \ell h\ F^{\#\#} \equiv \ell h\ F \equiv L^{\perp} \equiv R^2$

$\ell in\ F^{\#} \equiv L = \{\phi\}$ for each π_0

$\dim F^{\#\#} = 2$ (=n)

Also, $0 \in \text{aff } F$ and $\dim F = 2$

Since $L = \{\phi\}$, $\ell h\ F = R^2$, $F^{\#} \subset R^2$, $F^{\#} = (F^{\#} \cap \ell h\ F) + L = F^{\#} \cap R^2$.

Lowest dimensional faces of $F^{\#}$ are of dimension $(n - \dim F) = (2-2) = 0$

i.e., lowest dimensional faces of $F^{\#}$ are extreme points.

5.4 Generation of Facets of the Closure of the Convex Hull of Feasible Points

In this section, we will characterize the facets of the set $c\ell$ conv F in terms of the sets $F^{\#}$ and $F^{\#\#}$ discussed in the foregoing section. For the sake of completeness, let us define a facet.

Definition

$\pi x \geq \pi_0$ is a <u>facet</u> of a d-dimensional set S if

$\pi x \geq \pi_0$ for each $x \in S$ and

$\pi x = \pi_0$ for exactly d affinely independent points of S.

Theorem 5.3 stated next characterizes the facets of the set $F^{\#\#}$ which play an instrumental role in determining the facets of $c\ell$ conv F as will be seen shortly.

Theorem 5.3

$\pi x \geq \pi_0$ is a facet of $F^{\#\#}$ <u>and</u> $\pi \in \ell h\ F$

if and only if

$$\left\{ \begin{array}{l} \pi_0 \neq 0,\ \pi \neq 0,\ \pi \in \text{vert } (F^{\#} \cap L^{\perp}) \\ \text{or } \ \pi_0 = 0,\ \pi \neq 0,\ \pi \in \text{dir } (F^{\#} \cap L^{\perp}) \end{array} \right\}$$

<u>Note</u>: Recall from Lemma 5.2(d) that $L^{\perp} = \ell h\ F$. In this theorem, the statement $\pi \in \ell h\ F$ becomes necessary for the following technicality. Suppose $F^{\#\#}$ is less

than full dimensional. Then a facet of $F^{\#\#}$ defined by a hyperplane H can also be defined by a family of hyperplanes H' such that

$$(\ell h\ F) \cap H' = (\ell h\ F) \cap H$$

Thus, we specify a particular hyperplane from this family which has its normal π lying in ℓh F.

Since for our problem, the largest subspace contained in the recession cone of $F^{\#}$ is of zero dimension, i.e., $\ell in\ F^{\#} = 0$. Thus, (5.22) yields $L = \{\phi\}$ or that $\ell h\ F \equiv L^{\perp} = R^2$. Hence, $F^{\#} \cap L^{\perp} = F^{\#}$ in this case. Thus, in the present case, Theorem 5.3 above follows from the fact that $F^{\#\#}$ may be defined as in Equation (5.20).

Let us now consider an important result for computing facets of $F^{\#\#}$.

<u>Theorem 5.4</u>

Let $g \in R^n$, $\pi x \geq \pi_0$ be a facet of $F^{\#\#}$ and let $\pi \in \ell h$ F. Then, $g \in \{x \in F^{\#\#}: \pi x = \pi_0\} \iff \pi g = \pi_0$ supports $F^{\#}$ and contains π, where,

$$\pi \in \begin{cases} \text{vert } F^{\#} \cap L^{\perp} \text{ if } \pi_0 \neq 0 \\ \\ \text{dir } F^{\#} \cap L^{\perp} \text{ if } \pi_0 = 0 \end{cases}$$

For example, in our illustrative problem, consider $\pi_0 = 0$. Let us examine $F^{\#\#}(0)$. The facet $5s_1 - 2s_3 \geq 0$ has $\pi = (5,-2)$ lying in $\ell h\ F \equiv R^2$. Further, $g = (2,5)^t$ say, belongs to $F^{\#\#}(0)$ with $\pi g = 0$. (Note that $\pi_0 = 0$ here). Then the theorem implies that $2x_1 + 5x_2 = 0$ supports $F^{\#}(0)$ and contains the point $\pi = (5,-2)$ where, as seen in Figure 5.2, $\pi = (5,-2) \in$ dir $F^{\#}(0)$ (since $L^{\perp} = R^2$ here). The converse result is also true.

Now, let us consider the main result of this section. This result characterizes the facets of $c\ell$ conv F in terms of the sets $F^{\#}$ and $F^{\#\#}$. Two cases are considered, e.g., $\pi_0 \neq 0$ and $\pi_0 = 0$. A discussion of this theorem and its implications follow after its statement.

Theorem 5.5

(i) Suppose $\pi_0 \neq 0$. Further, assume that $0 \in \text{aff } F$. Then,

$$\{\pi x \geq \pi_0 \text{ is a facet of } c\ell \text{ conv } F\} \iff \{\pi x \geq \pi_0 \text{ is a facet of } F^{\#\#}\}$$

Corollary

Assume $0 \in \text{aff } F$, $\pi \in \ell h\ F$. Then,

$$\{\pi x \geq \pi_0 \text{ is a facet of } c\ell \text{ conv } F\} \iff \{\pi \neq 0,\ \pi \in \text{vert } F^{\#} \cap L^{\perp}$$

(ii) Suppose $\pi_0 = 0$. If $\pi x \geq 0$ is a facet of $c\ell$ conv F, $\pi \in \ell h\ F$, then this implies that

$$\pi \neq 0, \text{ and that } \pi \text{ is an extreme direction of } F^{\#} \cap L^{\perp}$$

Conversely, if $\pi \neq 0$, $\pi \in \text{dir } (F^{\#} \cap L^{\perp})$, and $d = \dim F^{\#\#}$, then either

(a) $\pi x \geq 0$ is a facet of $c\ell$ conv F, or

(b) $\pi x \geq 0$ is a (d-2) dimensional face of $c\ell$ conv F such that this (d-2) dimensional face is an intersection of two adjacent facets of the type

$$\pi^1 x \geq \pi_0^1,\ \pi_0^1 > 0$$

and

$$\pi^2 x \geq \pi_0^2,\ \pi_0^2 < 0$$

with

$$\pi = \left(\frac{\pi^1}{\pi_0^1} - \frac{\pi^2}{\pi_0^2}\right)$$

Note: Let us consider case (i) first, e.g., $\pi_0 \neq 0$. Note that if $0 \notin$ aff F, then this implies that dim F = d-1 and that each facet $\pi x = \pi_0$ of $F^{\#\#}$ contains all of cℓ conv F, instead of being a facet of cℓ conv F as when $0 \in$ aff F. Thus, in case $0 \notin$ aff F, a facet of cℓ conv F is given by the intersection of $\pi x = \pi_0$ and a (d-2) dimensional face of $F^{\#\#}$ corresponding to an edge or a 1 dimensional face of $F^{\#} \cap L^{\perp}$.

Secondly, note from Lemma 5.2(c), (d) that if $F^{\#\#}$ is full dimensional, then $L = \{\phi\}$, or, $L^{\perp} = R^n$. This implies that one may examine the non-zero vertices of $F^{\#}$ itself to obtain facets of cℓ conv F.

Thirdly, if d = dim $F^{\#\#} < n$, then ℓin $F^{\#} = (n-d) > 0$ (see Lemma 5.2 (e)). Hence, $F^{\#}$ has no extreme points. However, there is a one-to-one correspondence between vertices π of $F^{\#} \cap L^{\perp}$ and (n-d) dimensional faces of $F^{\#}$ which are of the form $\alpha + L$, where L is given by (5.22). These are the lowest dimensional faces of $F^{\#}$ (Lemma 5.2(h)).

Again, the corollary in case (i) of the theorem designates $\alpha + L$ to be an (n-d) dimensional face of $F^{\#}$. In particular, if $F^{\#\#}$ is of full dimension, i.e., d=n, then this Corollary states that

$$\{\pi x \geq \pi_0 \text{ is a facet of c}\ell \text{ conv F}\} \iff \{\pi \neq 0 \text{ is an extreme point of } F^{\#}\} \quad (5.23)$$

Case (ii) of the theorem states that if $\pi \neq 0$ belongs to the set dir $F^{\#} \cap L^{\perp}$ or, if $F^{\#\#}$ is of full dimension, and $\pi \neq 0$ belongs to dir $(F^{\#})$ itself, then $\pi x \geq 0$ may be used in a system of linear inequalities which characterize cℓ conv F in either case (a) or (b) of the converse.

Let us illustrate some aspects of this theorem through our numerical example first, and then discuss the utility of this theorem.

Let us begin with Case (i). Note that $0 \in$ aff F and that $F^{\#} \cap L^{\perp} = F^{\#}$ in our problem. The facets of cℓ conv F of the type $\pi x \geq \pi_0$, $\pi_0 \neq 0$, are

(1) $5s_1 + 3s_3 \geq 25$, i.e., $\frac{s_1}{5} + \frac{3s_3}{25} \geq 1$, which is a facet of $F^{\#\#}(1)$ with

$\left(\frac{1}{5}, \frac{3}{25}\right)$ being the only extreme point of $F^{\#}(1)$

(2) $-s_1 + s_3 \geq -5$, i.e., $\frac{-s_1}{5} + \frac{s_3}{5} \geq -1$, which is a facet of $F^{\#\#}(-1)$ with

$\left(\frac{-1}{5}, \frac{1}{5}\right)$ being an extreme point of $F^{\#}(-1)$

and

(3) $-s_3 \geq -5$, i.e., $\frac{-s_3}{5} \geq -1$, which is also a facet of $F^{\#\#}(-1)$ with

$\left(0, \frac{-1}{5}\right)$ being the only other extreme point of $F^{\#}(-1)$.

To verify Case (ii), observe that extreme directions π of $F^{\#}(0) \cap L^{\perp} \equiv F^{\#}(0)$ here, are (0,1) and (5,-2). Hence $\pi x \geq 0$ has the form $x_2 \geq 0$ and $5x_1 - 2x_2 \geq 0$. Since d=2 in our problem, both these define d-2=0 dimensional faces of $c\ell$ conv F, i.e., they define extreme points (5,0) and (2,5) respectively of $c\ell$ conv F. Hence case (ii) (b) of the converse applies. Further, (5,0) is the intersection of adjacent facets $5s_1 + 3s_3 \geq 25$ and $-s_1 + s_3 \geq -5$ with $\frac{1}{25}(5,3) + \frac{1}{5}(-1,+1) = (0,\frac{8}{25})$ which defines $\frac{8}{25}x_2 \geq 0$ or $x_2 \geq 0$. Similarly, (2,5) is the intersection of adjacent facets $5s_1 + 3s_3 \geq 25$ and $-s_3 \geq -5$ with $\frac{1}{25}(5,3) + \frac{1}{5}(0,-1) = (\frac{1}{5},-\frac{2}{25})$ which defines $\frac{1}{5}x_1 - \frac{2}{25}x_2 \geq 0$ or $5x_1 - 2x_2 \geq 0$.

Implementation

Now, we know that if we obtain an extreme point $\pi(\neq 0)$ of $F^{\#} \cap L^{\perp}$, for $\pi_0 \neq 0$, then $\pi x \geq \pi_0$ defines a facet of $c\ell$ conv F if $0 \in$ aff F (or contains $c\ell$ conv F if $0 \notin$ aff F). Further, if $\pi(\neq 0)$ is an extreme direction of $F^{\#} \cap L^{\perp}$ with $\pi_0 = 0$, then $\pi x \geq 0$ either defines a facet of $c\ell$ conv F or again, contains $c\ell$ conv F. In any case, $\pi x \geq \pi_0$ is a valid inequality in the system defining $c\ell$ conv F. By virtue of Theorem 5.5, it is also sufficient to represent $c\ell$ conv F by the system of inequalities of the type $\pi x \geq \pi_0$ where π is either an extreme point of $F^{\#} \cap L^{\perp}$ if $\pi_0 \neq 0$ and is an extreme direction of $F^{\#} \cap L^{\perp}$ if $\pi_0 = 0$.

The basic problem at hand then, is to identify extreme points or extreme directions of $F^{\#} \cap L^{\perp}$, as appropriate. The concept utilized in accomplishing this is that if one minimizes a linear function g, say, over $F^{\#}$, and if the minimum value is finite, then one will have detected a lowest dimensional face of $F^{\#}$.

Hence, assume as before that $0 \notin c\ell$ conv F, that is, the current point (origin) is infeasible to the disjunction. Then the problem we wish to examine is

$$P1(g,\pi_0): \qquad \text{minimize} \quad \{g\pi: \pi \in F^{\#}(\pi_0)\}$$

Alternatively, using the characterization of $F^{\#}(\pi_0)$ given through Equation (5.17), we have

$$\begin{aligned} P1(g,\pi_0): \qquad & \text{minimize} && g\pi \\ & \text{subject to} && u^h D^h \leq \pi \\ &&& u^h d^h \geq \pi_0 \\ &&& u^h \geq 0, \ \pi \text{ unrestricted} \end{aligned}$$

where $u^h = (\theta^h, \sigma^h)$ and $D^h = \binom{D_h}{A}$.

The dual of this problem may be written as

$$\begin{aligned} P2(g,\pi_0): \qquad & \text{maximize } z = && \sum_{h \in H^*} \pi_0 \xi_0^h \\ & \text{subject to} && D^h \xi^h \geq d^h \xi_0^h, \ h \in H^* \\ &&& \sum_{h \in H^*} \xi^h = g \\ &&& \xi_0^h, \ \xi^h \geq 0, \ h \in H^* \end{aligned}$$

For our example problem, $P1(g,\pi_0)$, $P2(g,\pi_0)$ may be written as follows

$$P1(g,\pi_0): \quad \text{minimize} \quad g_1\pi_1 + g_2\pi_2$$

$$\text{subject to} \quad -u_1^1 + u_2^1 + u_4^1 \leq \pi_1$$

$$u_1^1 - u_2^1 - u_3^1 - u_4^1 \leq \pi_2$$

$$-u_1^2 + u_2^2 \leq \pi_1$$

$$u_1^2 - u_2^2 - u_3^2 + u_4^2 \leq \pi_2$$

$$-5u_1^1 - 3u_2^1 - 5u_3^1 + 5u_4^1 \geq \pi_0$$

$$-5u_1^2 - 3u_2^2 - 5u_3^2 + 5u_4^2 \geq \pi_0$$

$$u_j^h \geq 0, \; h=1,2, \; j=1,2,3,4$$

$$\pi_1, \pi_2 \text{ unrestricted}$$

and

$$P2(g,\pi_0): \quad \text{maximize} \quad \xi_0^1 + \xi_0^2 \;\; \pi_0$$

$$\text{subject to} \quad -\xi_1^1 + \xi_2^1 \geq -5\xi_0^1$$

$$\xi_1^1 - \xi_2^1 \geq -3\xi_0^1$$

$$-\xi_2^1 \geq -5\xi_0^1$$

$$-\xi_2^1 \geq -5\xi_0^1$$

$$-\xi_1^2 + \xi_2^2 \geq -5\xi_0^2$$

$$\xi_1^2 - \xi_2^2 \geq -3\xi_0^2$$

$$-\xi_2^2 \geq -5\xi_0^2$$

$$\xi_2^2 \geq 5\xi_0^2$$

$$\xi_1^1 + \xi_1^2 = g_1$$

$$\xi_2^1 + \xi_2^2 = g_2$$

$$\xi_0^h, \; \xi_j^h \geq 0, \; h=1,2; \; j=1,2$$

The task at hand then, is to find a characterization of g such that $P1(g,\pi_0)$ has a finite minimum, that is, $P2(g,\pi_0)$ is feasible with a finite maximum. Since $F^{\#} \neq \{\phi\}$, P1 is finite if and only if P2 is feasible. Such a characterization of g is given through the following theorem.

Theorem 5.6 (Characterization of g)

(i) If $g \in c\ell \text{ conv } F$, $g \neq 0$, then for every $\lambda > 0$ such that $\lambda g \in c\ell \text{ conv } F$ (and such a λ exists), we get

$$P2(g,\pi_0) \text{ has a feasible solution } \bar{\xi} = (\bar{\xi}^h, \bar{\xi}_0^h), \text{ with } \sum_{h \in H^*} \bar{\xi}_0^h = 1/\lambda$$

Conversely, if $\bar{\xi}$ is feasible to $P2(g,\pi_0)$ with $1/\lambda = \sum_{h \in H^*} \bar{\xi}_0^h$, then $g \in c\ell \text{ conv } F$, $g \neq 0$, $\lambda g \in c\ell \text{ conv } F$.

(ii) If $\pi_0 \neq 0$, then a solution $\bar{\xi}$ feasible to $P2(g,\pi_0)$ is also optimal if and only if the objective function value $\bar{z} = \pi_0(1/\bar{\lambda})$, where

$$\bar{\lambda} = \begin{cases} \text{minimum } \{\lambda: \lambda g \in c\ell \text{ conv } F\} \text{ if } \pi_0 > 0 \\ \text{maximum } \{\lambda: \lambda g \in c\ell \text{ conv } F\} \text{ if } \pi_0 < 0 \end{cases}$$

(iii) If $\pi_0 = 0$, any feasible solution $\bar{\xi}$ to $P2(g,0)$ is optimal with value $\bar{z} = 0$. However, $P1(g,0)$ has an optimal solution $(\bar{\pi},\bar{u})$ with $\bar{\pi}g = 0$ and $\bar{\pi} \neq 0$ if and only if $g \in$ boundary of $c\ell \text{ conv } F$.

Illustration

For the sake of illustration, let us consider part (ii) of the above theorem. Consider $\pi_0 = 1$ and $g = (1,1)$, say. Then,

$$\bar{\lambda} = \text{minimum } \{\lambda: \lambda(1,1) \in c\ell \text{ conv } F\} = 25/8$$

Further, $\underset{\pi \in F^{\#}(1)}{\text{minimum}} \{\pi_1 + \pi_2\}$ occurs at $(\frac{1}{5},\frac{3}{25})$ with objective value

$8/25 = \pi_0/\tilde{\lambda} = 1/(25/8)$.

For part (iii) above, note that $g \varepsilon$ boundary $c\ell$ conv F and $\pi_0 = 0$ implies that P1(g,0): min $\{\lambda g: \pi \varepsilon F^{\#}(0)$ has an infinite number of alternative optimal solutions along some extreme direction of $F^{\#}(0) \cap L^{\perp}$. Note that $F^{\#}(0)$ is a polyhedral cone with the vertex at the origin. Hence (0,0) is also optimal and the optimal value is therefore always $\bar{\xi} = 0$.

Next, we give a characterization of solutions of P1(g,π_0) for a given g,π_0 such that these solutions contribute towards determining facets of $F^{\#\#}$ (and hence of $c\ell$ conv F) by detecting extreme points and extreme directions of $F^{\#}$. Such solutions are called <u>regular solutions</u>. Again, we consider cases $\pi_0 \neq 0$ and π_0 separately.

<u>Theorem 5.7 (Regular Optimal Solutions)</u>

(i) Suppose $\pi_0 \neq 0$. Then,

$\{\bar{\pi} \varepsilon F^{\#}$ is an extreme point of $F^{\#} \cap L^{\perp}\}$ <=> {(a) $\bar{\pi} \varepsilon L^{\perp}$ and (b) there exists a $p \varepsilon L^{\perp}$ such that $\bar{\pi} \varepsilon L^{\perp}$ is the unique point which minimizes πp on $F^{\#}\}$

Accordingly, if $(\bar{\pi},\bar{u})$ is optimal to P1(g,π_0) for some $g \varepsilon c\ell$ cone F, then

$\{\bar{\pi} \varepsilon$ vert $F^{\#} \cap L^{\perp}\}$ <=> {(1) $\bar{\pi} \varepsilon L^{\perp}$ and (2) there exists a $\gamma \varepsilon L^{\perp}$ such that if (π,u) solves P1$(g + \gamma,\pi_0)$ and if $\pi \varepsilon L^{\perp}$ then this implies that $\pi = \bar{\pi}\}$

If $(\bar{\pi},\bar{u})$ satisfies these latter conditions (1) and (2), we call it a <u>regular solution</u>.

(ii) Suppose $\pi_0 = 0$. (Note: In this case, $F^{\#}$ is a cone and $F^{\#} \cap L^{\perp}$ is a pointed cone, that is, has an extreme point at the origin). Then,

$\{\bar{\pi} \in F^{\#}$ is an extreme direction of $F^{\#} \cap L^{\perp}\}$ <=> {(a) $\bar{\pi} \neq 0$, $\bar{\pi} \in L^{\perp}$ and (b) there exists a $p \in L^{\perp}$ such that up to a positive multiplier, $\bar{\pi} \in L^{\perp}$ is the unique point which minimizes πp on $F^{\#}$}

Accordingly, if some $(\bar{\pi},\bar{u})$ is optimal to $P1(g,0)$ for some $g \in$ boundary $c\ell$ cone F (recall that this gives rise to extreme directions of $F^{\#}(0) \cap L^{\perp}$, then,

$\{\bar{\pi} \in$ dir $F^{\#} \cap L^{\perp}\}$<=> (1) $\bar{\pi} \neq 0$, $\bar{\pi} \in L^{\perp}$ and (2) there exists a $\gamma \in L^{\perp}$ such that if (π,u) solves $P1(g + \gamma,0)$, and $\pi \neq 0$, $\pi \in L^{\perp}$, then this implies that $\pi = \lambda\bar{\pi}$, $\lambda > 0$.

If $(\bar{\pi},\bar{u})$ satisfies these latter conditions (1) and (2), we call it a regular solution.

Interpretation

Consider Case (i). This simply says that for every extreme point $\bar{\pi}$ of $F^{\#} \cap L^{\perp}$, there exists a vector p which one may use in min $\{\pi p: \pi \in F^{\#}\}$ such that this extreme point is a unique minimizing point. Moreover, given an extreme point optimal solution $(\bar{\pi},\bar{u})$ to $P1(g,\pi_0)$, we can always perturb the objective function so that this extreme point is the unique optimal solution.

For example, in our illustrative problem, consider $\pi_0 = 1$. Then $(\frac{1}{5},\frac{3}{25})$ is an extreme point of $F^{\#} \cap L^{\perp}$. Further, clearly,

(a) $(\frac{1}{5},\frac{3}{25}) \in L^{\perp} \equiv R^2$, and

(b) taking $p = (5,3)$, say, the problem min $5\pi_1 + 3\pi_2$: $\pi \in F^{\#}(1)$ has $(\frac{1}{5},\frac{3}{25})$ as its unique optimal solution.

Similar remarks hold for Case (ii). Here, the uniqueness of the optimal solution is upto a positive multiplier since we always encounter alternative optimal minimizing points along extreme directions in this case.

An important and relevant result which ties in the above statements is given next.

Lemma 5.3

If $Pl(g,\pi_0)$ has an optimal solution, then it has a regular optimal solution.

We will now characterize facets of $F^{\#\#}$ (and hence of $c\ell$ conv F) in terms of regular optimal solutions to $Pl(g,\pi_0)$.

Theorem 5.8 (Characterization of facets of $F^{\#\#}$, and hence facets of $c\ell$ conv F, in terms of regular optimal solutions)

(i) Suppose $\pi_0 \neq 0$. Let $g \in c\ell$ cone F, $g \neq 0$ and let

$$\bar{\lambda} = \begin{cases} \text{minimum } \{\lambda: \lambda g \in c\ell \text{ conv } F\} \text{ if } \pi_0 > 0 \\ \text{maximum } \{\lambda: \lambda g \in c\ell \text{ conv } F\} \text{ if } \pi_0 < 0 \end{cases}$$

Then, $\pi x > \pi_0$, $(\pi_0 \neq 0)$, $\pi \in \ell h$ F is a facet of $F^{\#\#}$ containing the point $\bar{\lambda} g$, if and only if $\pi = \bar{\pi}$ for some regular optimal solution $(\bar{\pi},\bar{u})$ to $Pl(g,\pi_0)$.

(ii) Suppose $\pi_0 = 0$. Let $g \in$ boundary $c\ell$ conv F, $g \neq 0$. Then, $\pi x \geq 0$, $\pi \in \ell h$ F is a facet of $F^{\#\#}$ containing the point g if and only if $\pi = \lambda\bar{\pi}$ for some $\lambda > 0$ and some regular optimal solution $(\bar{\pi},\bar{u})$ to $Pl(g,0)$.

Illustration

Consider Case (i) above, and let $\pi_0 = 1$. Consider the facet $\frac{5s_1}{25} + \frac{3s_3}{25} \geq 1$ of $F^{\#\#}$ and recall that for our problem $\ell h\ F = R^2$. Now, since this is also a facet of $c\ell$ conv F, hence for any $g \in c\ell$ conv F, we can find the appropriate $\bar{\lambda}$ such that $\bar{\lambda}$ g lies on this facet. Moreover, $\pi = (\frac{1}{5},\frac{3}{25})$ here is a regular optimal solution to $Pl(g,1)$ for any $g \in c\ell$ conv F.

Next, consider Case (ii). Here, the vector $g = (2,5)$, say, belongs to the boundary of $c\ell$ conv F, $g \neq 0$. Then, $5s_1 - 2s_3 \geq 0$ is a facet of $F^{\#\#}(0)$ containing (2,5). Moreover, taking $\lambda = 1$, we see that $\pi = (5,-2) = \bar{\pi}$ solves the problem to minimize $2\pi_1 + 5\pi_2$ subject to $\pi \in P^{\#}(0)$. In fact, any point along the ray satisfying $2\pi_1 + 5\pi_2 = 0$ solves this latter problem. Hence, $\pi = \lambda\bar{\pi}$, $\lambda = 1$ for some regular optimal solution $(\bar{\pi},\bar{u})$ to $P1(g,0)$.

Summary and Notes: Using Theorem 5.7 and 5.8 above, all the facets of $F^{\#\#}$ may be obtained by solving $P1(g,\pi_0)$ (or its dual) for various vectors $g \in c\ell$ conv F. Further, from Theorem 5.5, each such facet is, or yields in conjunction with some other facets of $F^{\#\#}$, a facet of $c\ell$ conv F. The following points are worth noting:

(i) If $\pi_0 \neq 0$, $\bar{\lambda}$ is as defined in Theorem 5.8(i), and if $\bar{\lambda}$ g is the convex combination of k extreme points and extreme directions of $c\ell$ conv F ($1 \leq k \leq n$) then this implies that each of these vertices and extreme direction vectors are contained in each facet of $c\ell$ conv F that contains $\bar{\lambda}$ g. Moreover, each such facet can be obtained by solving $P1(g,\pi_0)$ (or its dual).

For example, refer to Figure 5.4 below. $\bar{\lambda}$ g is the convex combination of extreme points (2,5) and (5,0) of $c\ell$ conv F. Only one such facet contains these extreme points and also $\bar{\lambda}$ g, and this facet is generated by solving $P1(g,\pi_0)$ as demonstrated in the illustrative example solved below in Section 5.5.

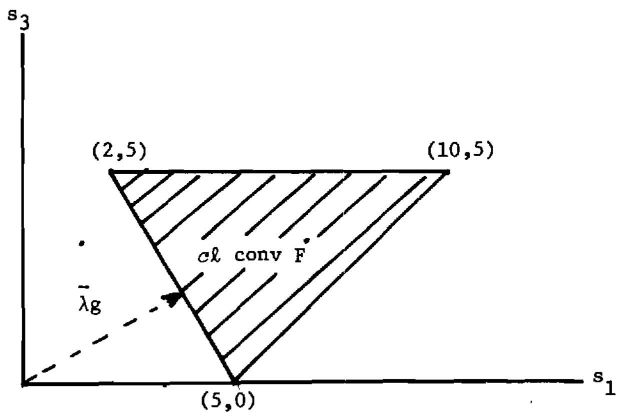

Figure 5.4. Representation of $\bar{\lambda}$g

(ii) Analogous remarks hold for the case $\pi_0 = 0$ as in (i) above.

(iii) If g corresponds to an extreme point of cℓ conv F, then solving $P1(g,\pi_0)$ with $\pi_0 = 1, -1$ and 0 gives all the facets of cℓ conv F containing the vertex g. This is a special case of remark (i) above. Furthermore, for a given π_0, the associated facets all correspond to alternative regular optimal solutions of $P1(g,\pi_0)$. Thus, if one facet containing g is found, the others are easily obtained therefrom.

Specialized schemes for solving $P1(g,\pi_0)$ and $P2(g,\pi_0)$ may be devised and are in fact available. However, since this leads away from the motivation of our present disucssion, we avoid these schemes here. We next illustrate in detail the method of determining facets of cℓ conv F for our example problem.

5.5 Illustrative Example

Consider the example problem we introduced in Section 5.1. We will first obtain facets of the "$\geq$ 1 type", then of the "$\leq$ -1 type" and finally of the "$\geq$ 0 type for cℓ conv F.

1. "$\geq$ 1 type" facets (π_0=1). Here, we are interested in solving P1(g,1) for various vectors $g \in$ cℓ cone F. If $(\bar{\pi},\bar{u})$ is obtained as a regular optimal solution, then we put $\pi = \bar{\pi}$ and derive $\pi x \geq 1$ as a facet of $F^{\#\#}$ and hence as a facet of cℓ conv F.

   We note here, that for any $g \in$ cℓ cone F, we get $\bar{\pi} = (\frac{2}{5},\frac{3}{25})$ in the regular optimal solution. Also, note that as illustrated in Figure 5.4, defining $\bar{\lambda}$ as in Theorem 5.8(i), for $g \in$ cℓ cone F, $\bar{\lambda}g$ lies on the facet $5s_1 + 3s_3 \geq 25$ of cℓ conv F. Indeed, this is the facet we obtain here using $\pi = \bar{\pi}$ in $\pi x \geq 1$, that is $\frac{s_1}{5} + \frac{3s_3}{25} \geq 1$ or $5s_1 + 3s_3 \geq 25$. Moreover, this is the only "$\geq$ 1 type" facet cℓ conv F.

2. "$\geq$ -1 type" facets (π_0=-1). Now again, we need to solve P1(g,-1) for various vectors $g \in$ cℓ cone F. In our problem, depending on the values of g selected, we obtain either $(-\frac{1}{5},\frac{1}{5})$ or $(0, -\frac{1}{5})$ as $\bar{\pi}$ in regular optimal solutions. Thus,

$\frac{-s_1}{5} + \frac{s_3}{5} \geq -1$ and $\frac{-s_3}{5} \geq -1$ are facets of $F^{\#\#}(-1)$ and hence of cℓ conv F.

Thus, the only "≥ -1 type" facets of cℓ conv F are $s_1 - s_3 \leq 5$ and $s_3 \leq 5$.

3. "≥ 0 type" facets ($\pi_0 = 0$). Now, we need to solve P1(g,0) for vectors $g \in$ boundary cℓ cone F, $g \neq 0$. Thus, we may either select

$$g = \lambda(2,5),\ \lambda > 0$$

or

$$g = \lambda(5,0),\ \lambda > 0$$

The first choice of g yields

$\bar{\pi} = \beta(5,-2),\ \beta > 0$ in the regular optimal solution to P1(g,0)

and the second choice yields

$$\bar{\pi} = \beta(0,1),\ \beta > 0$$

This gives us $5s_1 - 2s_3 \geq 0$ and $s_3 \geq 0$ as facets of $F^{\#\#}(0)$. (Note that (5,-2) and (0,1) are extreme directions of $F^{\#}(0) \cap L^{\perp} = F^{\#}(0)$ here, since $L^{\perp} = R^2$). Now, according to case (ii) of Theorem 5.5, consider the facet $5s_1 - 2s_3 \geq 0$. (Similar remarks hold for the facet $s_3 \geq 0$). Either this is a facet of cℓ conv F or it defines a 0-dimensional (d-2 dimensional in general) intersection of facets of cℓ conv F. As observed earlier, in either case, we can use these inequalities as defining half-spaces for cℓ conv F without any inhibitions. At most, as seen in Figure 5.5 below, we will have introduced degeneracy in the problem. Thus, the set cℓ conv F obtained above may be defined by the inequalities.

$$5s_1 + 3s_3 \geq 25$$

$$s_1 - s_3 \leq 5$$

$$s_3 \leq 5$$

$$5s_1 - 2s_3 \geq 0$$

$$s_3 \geq 0$$

These inequalities may now replace the disjunctive statement in Problem DP. The set $c\ell$ conv F is depicted in Figure 5.5 below.

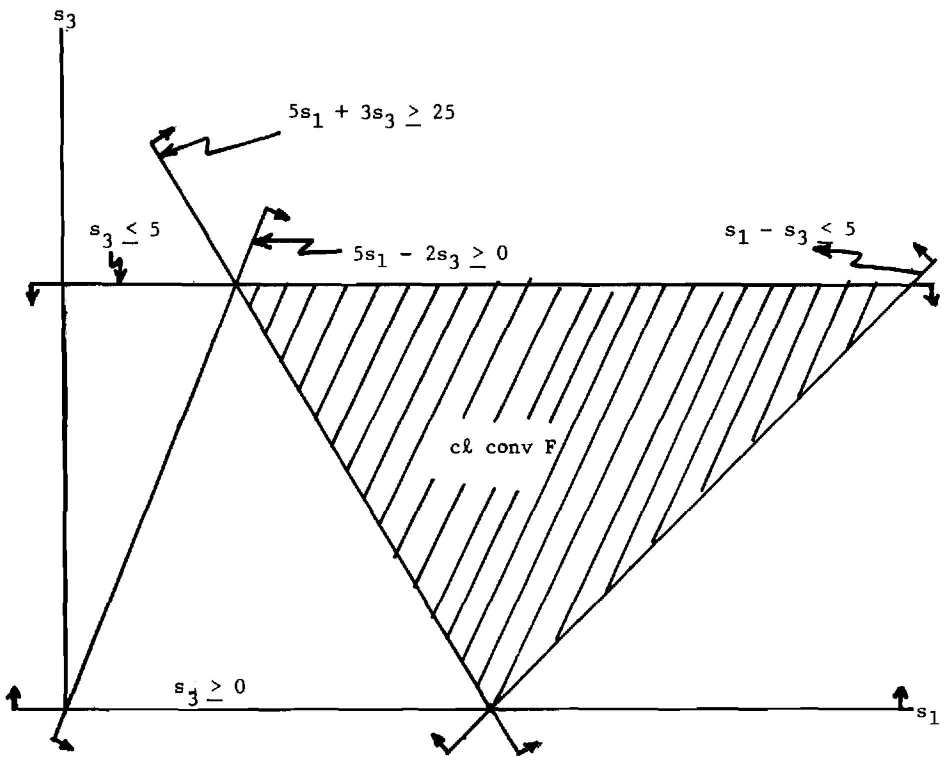

Figure 5.5. Inequalities Defining $c\ell$ conv F

5.6 Facial Disjunctive Programs

As may be apparent from our foregoing discussion, if $|H|$ is large, then the solution of Problem DP through the generation of facets of $c\ell$ conv F is prohibitive. For example most mixed integer linear programs would be intractable by this technique due to the size and complexity of Problems $P1(g,\pi_0)$ and $P2(g,\pi_0)$. However, this technique is attractive for small $|H|$.

Hence, for a large $|H|$ there is a need to relax some disjunction and in some manner, use facets obtained for some enforced disjunctions to generate facets for other disjunctions when they are also simultaneously enforced. It turns out that such a procedure is possible for special disjunctive programs called facial disjunctive problems.

Before we discuss this, let us consider Problem DP. This problem has been stated in the so called disjunctive normal form. There is another way of writing Problem DP, which we will find more convenient in the present context. Suppose that for each $h \in H$, $A^h x \geq b^h$ has $|Q_h = Q|$ inequalities, where Q is the index set of these inequalities. Then, we may construct $|Q|$ sets H_j, $j \in Q$ such that for each $h \in H$, exactly one of the inequalities in $A^h x \geq b^h$ is placed distinctly in each of the sets H_j, $j \in Q$. Thus, H is the cartesian product of the H_j, $j \in Q$, that is,

$$H = \prod_{j \in Q} H_j \qquad (5.24)$$

Problem DP may now be stated in the so-called conjunctive normal form

$$\begin{aligned} \text{minimize} \quad & f(x) = c^t x \\ \text{subject to} \quad & x \in F_0 = \{x: Dx \geq d,\ x \geq 0\} \\ & \bigwedge_{j \in Q} \left[\bigvee_{i \in H_j} a_i^h x \geq b_i^h \right] \end{aligned}$$

where,

$$\{x: A^h x \geq b^h, x \geq 0\} \equiv \{x: a_i^h x \geq b_i^h, i \in Q, x \geq 0\}, h \in H \qquad (5.25)$$

and F_0 is as defined in Equation (5.3). Now consider the set F defined in Equation (5.2) and let us denote it by F^Q. Accordingly, for a set $T \subset Q$, let us denote the corresponding set of feasible points to the (relaxed) disjunction as F^T. Thus,

$$T \subset Q \Rightarrow F^T \supseteq F^Q \qquad (5.26)$$

The set $c\ell \text{ conv } F^T$ will be called a partial convex hull of F for $T \subset Q$. Now, suppose we use the disjunctions in some set $T \subset Q$ alone and as before generate all facets of F^T. Further, suppose that we now replace the disjunctions in T with these facets in the original problem. Does the solution of the resulting problem satisfy the disjunctions in T? The answer is yes for special problems called <u>facial disjunctive programs</u> described below.

Consider the following definitions.

Definition 5.1

Let F_0 be a convex set. A subset F of F_0 (possibly empty) is called a <u>face</u> of F_0 if there exists a supporting hyperplane of F_0 whose intersection with F_0 defines F.

Definition 5.2

A disjunction $\bigvee_{i \in H_j} \{a_i^h x \geq b_i^h\}$ is called <u>facial</u> with respect to F_0, if

$$F_i^h = F_0 \cap \{x \in R^n: a_i^h x \geq b_i^h\} \qquad (5.27)$$

is a face of F_0 for each $i \in H_j$. (Note that a face may be an extreme point, an edge, ..., a facet or the entire set). A disjunctive program is said to be a <u>facial disjunctive program</u> if F_1 is a face of F_0 for each $i \in H_j$ and for each $j \in Q$.

<u>EXAMPLE</u>. In our problem, we have,

$$F_1^1 = F_0 \cap \{(s_1,s_3): s_3 \geq 5\}.$$

and

$$F_1^2 = F_0 \cap \{(s_1,s_3): s_1 - s_3 \geq 5\}.$$

These are depicted as F_1 and F_2 respectively in Figure 5.1. As one may see in that figure, F_1^1 and F_1^2 are faces (facets in this case) of F_0. Hence, our problem is <u>facial</u>.

<u>Necessary and Sufficient Condition for $a_i^h x \geq b_i^h$ to be a Face of F_0: Theorem 5.9.</u> Let F_i^h be as defined in Equation (5.27). If there exist $(\mu,\nu) \in R^m \times R^n$ satisfying

$$\left\{\begin{array}{ll} \mu(-D) + \nu(-I) & = a_i^h \\ \mu(-d) & = b_i^h \\ (\mu,\nu) \geq 0 & \end{array}\right\} \qquad (5.28)$$

then F_i^h is a face of F_0, namely,

$$F_i^h = \{x\epsilon F_0: a_i^h x = b_i^h\} = \{x\epsilon F_0: D_i x = d_i \text{ for each } i\epsilon M^+,$$

$$x_j = 0 \text{ for each } j\epsilon N^+\} \qquad (5.29)$$

where D_i is the i^{th} row of D and

$$M^+ = \{i\epsilon M: \mu_i > 0\},\ N^+ = \{j\epsilon N: \nu_j > 0\} \qquad (5.30)$$

Conversely, if F_j^h is a face of F_0, and $F_0 \neq F_i^h \neq \{\phi\}$, then there exist $(\mu,\nu)\epsilon R^m \times R^n$

satisfying the property (5.28).

Henceforth, we will assume that F_0 is bounded (we may regularize it if necessary) and that DP is facial. This implies that F_h of Equation (5.1) is a polytope for each $h \in H$, that is conv F^S ($\equiv$conv F) is a polytope.

<u>Consequence of the Facial Property:</u>

<u>Theorem 5.10</u>

If DP is facial and F_0 is bounded, then

$$\{\text{Extreme points of conv } F^Q\} = \bigcup_{h \in H} \{\text{extreme points of } F_h\}$$

$$\subseteq \{\text{extreme points of } F_0\}$$

<u>For example</u>, in our problem, referring to Figure 5.1, we have,

$$\{\text{extreme points of conv } F\} = \{(2,5),(10,5),(5,0)\}$$

$$\{\text{extreme points of } F_1\} = \{(2,5),(10,5)\}$$

$$\{\text{extreme points of } F_2\} = \{(10,5),(5,0)\}$$

$$\{\text{extreme points of } F_0\} = \{(0,0),(0,3),(2,5),(10,5),(5,0)\}$$

Practically speaking, the most important consequence of the facial property is that conv F^Q may be obtained in as many steps ($|Q|$), as there are disjunctions in the conjunctive normal form, by applying the disjunctions one at a time alone. Now, when DP is facial and F_0 is bounded, it turns out that if $T \subset Q$, $i \in H_j$ and $j \in Q-T$, then,

$$F_i^h \cap \text{conv } F^T = \text{conv}[F_i^h \cap F^T] \qquad (5.31)$$

In other words, having conv F^T, for some $T \subset Q$, we select an F_i^h not yet considered and compute conv$[F_i^h \cap F^T]$ simply as $F_i^h \cap$ conv F^T. This leads to another important result.

Theorem 5.11

Assume that DP is facial and that F_0 is bounded. Then, for any $T \subset Q$,

$$\text{conv}[F^{Q-T} \cap \text{conv } F^T] = \text{conv } [F^Q]$$

This main result is used to compute conv$[F^Q]$ in the following manner. Let $Q=\{j_1,j_2,\ldots,j_q\}$ where $q = |Q|$. Then as a corollary to the above theorem, we have,

$$\text{conv } F^Q = \text{conv}[F^{\{j_q\}} \cap \text{conv}\{\ldots \cap \text{conv}(F^{\{j_2\}} \cap \text{conv } F^{\{j_1\}})\ldots)\}] \qquad (5.32)$$

We may now apply (5.31) to the decomposition (5.32) in order to compute conv F^Q in $q = |Q|$ steps.

We terminate a brief discussion of facial disjunctive programs at this point. Later, in Chapter VII, we will return to facial disjunctive programs as a special case of Problem DP and will present two finitely covergent algorithms to solve such problems. One of these procedures is based on Theorem 5.10 whereas the other is based on Theorem 5.11. Both of these procedures solve Problem DP by generating facets of cℓ conv F^Q as and when needed till either an optimal solution is obtained or all the facets of cℓ conv F^Q have been generated, whence the problem is necessarily solved.

Thus far, we have addressed the question of generating deep disjunctive cuts. In the next chapter, we will examine some of the cutting planes available in the literature and identify them as basically disjunctive cutting planes by putting them in the general format of the latter type of cuts.

5.7 Notes and References

This chapter is heavily based on the results of Balas in [5]. If the facets of the closure of the convex hull of feasible points are known, clearly the problem of solving disjunctive programs is trivial. Balas' study takes an important step in characterizing them. Furthermore, for a special important case of disjunctive programs, the study opens up the possibility of generating the facets sequentially.

Chapter VI

DERIVATION AND IMPROVEMENT OF SOME EXISTING CUTS THROUGH DISJUNCTIVE PRINCIPLES

6.1 Introduction

In discussing the basic disjunctive cut principle, we indicated that it subsumes all other cut generation principles. In this chapter, we will demonstrate this to a certain extent by actually deriving some existing cutting planes as disjunctive cuts. In the process, it will be seen that the disjunctive principles may be used to actually improve upon three cuts. In fact, for the first type of cut we discuss below, we will utilize the concepts of Chapter IV to obtain an improved version of the existing cut.

6.2 Gomory's Mixed Integer Cuts

Consider a mixed integer program where only certain variables are constrained to be integral. Suppose we have a simplex tableau representation of a basic feasible solution to the corresponding problem with integrality relaxed. Further, assume that this solution does not satisfy the integrality constraints. In particular, let us identify a basic, integer-constrained variable x_i whose current value, a_{i0}, is non-integral. Let us write the representation of x_i in terms of the non-basic variables t_j, $j \in J$ in the tableau representing the solution at hand as follows

$$x_i = a_{i0} + \sum_{j \in J} a_{ij}(-t_j) \qquad (6.1)$$

Now, partition J as

$$J = J_1 \cup J_2 \qquad (6.2)$$

where

$$J_1 = \{j\varepsilon J: t_j \text{ is integer-constrained}\} \tag{6.3}$$

and

$$J_2 = J - J_1 \tag{6.4}$$

Further, denoting the largest integer less than or equal to a given real number μ as $[\mu]$ and denoting the smallest integer greater than or equal to a given real number μ as $<\mu>$, we may write

$$a_{i0} = [a_{i0}] + f_{i0},\ f_{i0} > 0 \tag{6.5}$$

$$a_{ij} = [a_{ij}] + f_{ij}, \text{ for } j\varepsilon J \tag{6.6}$$

Substituting (6.5), (6.6) into (6.1), we obtain

$$x_i = [a_{i0}] + f_{i0} + \sum_{j\varepsilon J_1} ([a_{ij}] + f_{ij})(-t_j) + \sum_{j\varepsilon J_2} a_{ij}(-t_j)$$

or

$$x_i - [a_{i0}] - \sum_{j\varepsilon J_1} [a_{ij}](-t_j) = f_{i0} + \sum_{j\varepsilon J_1} f_{ij}(-t_j) + \sum_{j\varepsilon J_2} a_{ij}(-t_j) \tag{6.7}$$

Now, let us introduce a new set of parameters ϕ_{ij}, $j\varepsilon J \cup \{0\}$ defined as follows

$$\phi_{i0} = f_{i0}$$

$$\phi_{ij} = a_{ij} \quad \text{for } j\varepsilon J_2$$

$$\phi_{ij} = \begin{cases} f_{ij} & \text{if } f_{ij} \leq f_{i0} \\ f_{ij}-1 & \text{if } f_{ij} > f_{i0} \end{cases} \qquad \text{for } j\varepsilon J_1 \tag{6.8}$$

These parameters are merely notational expedients. Substituting (6.8) into (6.7), we obtain

$$x_i - [a_{i0}] - \sum_{j\varepsilon J_1} [a_{ij}](-t_j) - \sum_{\substack{j\varepsilon J_1 \\ f_{ij}>f_{i0}}} (-t_j) = \phi_{i0} + \sum_{j\varepsilon J} \phi_{ij}(-t_j) \quad (6.9)$$

Finally, denote

$$y_i = \phi_{i0} + \sum_{j\varepsilon J} \phi_{ij}(-t_j) \quad (6.10)$$

Observe in Equation (6.9) that the left hand side is necessarily integral and hence, so is the right hand side. That is, y_i of Equation (6.10) should necessarily be integral. In particular, the following disjunction must hold

$$\{y_i \le 0\} \vee \{y_i \ge 1\} \quad (6.11)$$

that is,

$$\{\sum_{j\varepsilon J} \phi_{ij}t_j \ge \phi_{i0}\} \vee \{\sum_{j\varepsilon J} -\phi_{ij}t_j \ge 1 - \phi_{i0}\}$$

Noting from (6.5), (6.8) that $0 < \phi_{i0} = f_{i0} < 1$ whenever x_i is fractional, we may use the basic disjunctive cut principle along with the deep cut notion of Section 3.2 to first write the above disjunction as

$$\{\sum_{j\varepsilon J} \frac{\phi_{ij}}{\phi_{i0}} t_j \ge 1\} \vee \{\sum_{j\varepsilon J} \frac{-\phi_{ij}}{1-\phi_{i0}} t_j \ge 1\}$$

and then derive the cut

$$\sum_{j\varepsilon J} \max\{\frac{\phi_{ij}}{\phi_{i0}}, \frac{-\phi_{ij}}{1-\phi_{i0}}\} t_j \ge 1 \quad (6.12)$$

Note in Equation (6.12), for each $j \in J$, the cut coefficient is determined by the nonnegative element of the pair. Thus, if we define

$$J_k^+ = \{j \in J_k:\ \phi_{ij} \geq 0\},\ J_k^- = \{j \in J_k:\ \phi_{ij} < 0\} \text{ for } k=1,2 \tag{6.13}$$

we may write (6.12) as

$$\sum_{j \in J_1^+} \frac{\phi_{ij}}{\phi_{i0}} t_j + \sum_{j \in J_1^-} \frac{-\phi_{ij}}{1-\phi_{i0}} t_j + \sum_{j \in J_2^+} \frac{\phi_{ij}}{\phi_{i0}} t_j + \sum_{j \in J_2^-} \frac{-\phi_{ij}}{1-\phi_{i0}} t_j \geq 1$$

Finally, substituting (6.8) into this cut, we obtain Gomory's mixed integer cut as

$$\sum_{j \in J_1^+} \frac{f_{ij}}{f_{i0}} t_j + \sum_{j \in J_1^-} \frac{1-f_{ij}}{1-f_{i0}} t_j + \sum_{j \in J_2^+} \frac{a_{ij}}{f_{i0}} t_j + \sum_{j \in J_2^-} \frac{-a_{ij}}{1-f_{i0}} t_j \geq 1 \tag{6.14}$$

Now, let us consider improving this cut. The concept of the strategy we employ is basically that of Chapter IV. More specifically, we reformulate the disjunction (6.11) to incorporate additional constraints as follows

$$\begin{Bmatrix} y_i \leq 0 \\ x_h \geq 0,\ h \in I \end{Bmatrix} \bigvee \begin{Bmatrix} y_i \geq 1 \\ x_h \geq 0,\ h \in I \end{Bmatrix} \tag{6.15}$$

where I denotes the set of basic variables. Hence, letting

$$x_h = a_{h0} + \sum_{j \in J} a_{hj}(-t_j) \quad \text{for } h \in I \tag{6.16}$$

we may rewrite (6.15) by using (6.10), (6.16) as

$$\left\{\begin{array}{l}\sum_{j\epsilon J} \phi_{ij}t_j \geq \phi_{i0} \\ \sum_{j\epsilon J} (-a_{hj})t_j \geq -a_{h0},\ h\epsilon I\end{array}\right\} \vee \left\{\begin{array}{l}\sum_{j\epsilon J} (-\phi_{ij})t_j \geq 1 - \phi_{i0} \\ \sum_{j\epsilon J} (-a_{hj})t_j \geq -a_{h0},\ h\epsilon I\end{array}\right\} \tag{6.17}$$

Letting λ_0^1, λ_h^1, $h\epsilon I$ be the nonnegative multipliers for the first set of constraints in (6.17) and λ_0^2, λ_h^2, $h\epsilon I$ those for the second set, we may write the appropriate surrogate constraints as

$$\sum_{j\epsilon J} (\lambda_0^1\phi_{ij} - \sum_{h\epsilon I} \lambda_h^1 a_{hj})t_j \geq (\lambda_0^1\phi_{i0} - \sum_{h\epsilon I} \lambda_h^1 a_{h0})$$

and

$$\sum_{j\epsilon J} (-\lambda_0^2\phi_{ij} - \sum_{h\epsilon I} \lambda_h^2 a_{hj})t_j \geq \lambda_0^2(1-\phi_{i0}) - \sum_{h\epsilon I} \lambda_h^2 a_{h0}$$

Using the concepts of Chapter III, the disjunctive cut we derive is

$$\sum_{j\epsilon J} \max\{\lambda_0^1\phi_{ij} - \sum_{h\epsilon I} \lambda_h^1 a_{hj},\ -\lambda_0^2\phi_{ij} - \sum_{h\epsilon I} \lambda_h^2 a_{hj}\}t_j \geq 1 \tag{6.18}$$

where

$$\lambda_0^1\phi_{i0} - \sum_{h\epsilon I} \lambda_h^1 a_{h0} = 1$$

$$\lambda_0^2(1-\phi_{i0}) - \sum_{h\epsilon I} \lambda_h^2 a_{h0} = 1$$

$$\lambda_0^1,\ \lambda_h^1,\ \lambda_0^2,\ \lambda_h^2 \geq 0 \qquad h\epsilon I \tag{6.19}$$

learly, (6.18) can be made to uniformly dominate Gomory's mixed integer cut 6.12) (or(6.14)) since the latter is obtained from the former by selecting $^1_0 = 1/\phi_{i0}$, $\lambda_0^2 = 1/(1-\phi_{i0})$ and $\lambda_h^k = 0$, k=1,2, $h\epsilon I$. Again, for the appropriate selection of parameters in (6.18), (6.19), one may resort to the concepts of Chapter III. Alternatively, one may handle the constraints $x_h \geq 0$, $h \in I$ of Equation (6.15) in a manner similar to that recommend in Chapter IV.

6.3 Convexity or Intersection Cuts with Positive Edge Extensions

In this section, we will discuss the general setting through which convexity or intersection cuts are derived with the purpose of demonstrating how the disjunctive cut principle is capable of generating such cuts. Hence, consider a convex set

$$C = \{x: a^h x \leq b^h,\ h \in H\} \tag{6.20}$$

defined by certain hyperplanes $a^h x \leq b^h$, $h \in H$, where $a^h = (a_1^h, \ldots, a_n^h)$. Further, let $b^h > 0$, $h \in H$ and suppose that we are currently located at the origin. In this setting, let us assume that there exists a subset S of the nonnegative orthant of R^n which contains points of interest to us. Suppose we have identified a set C which contains the origin, but no point of S, i.e., $S \cap C = \{\phi\}$. The point we are currently located at, viz., the origin, is not of (further) interest to us. Our intention now is to use the set C to generate a cut which deletes the origin but no point of S.

Accordingly, let us identify the n half lines

$$\xi^j = \{x: x = \lambda e_j,\ \lambda \geq 0\} \quad j=1,\ldots,n \tag{6.21}$$

where e_j is the j^{th} unit vector. These half lines are defined by the coordinate axes incident at the origin. Let us now proceed along each of the half lines (6.21) in turn and compute the maximum distance $\bar{\lambda}_j$ we can traverse along this direction and still remain within the set C. In other words,

$$\bar{\lambda}_j = \sup\{\lambda \geq 0: (a^h)(\lambda e_j) \leq b^h,\ h \in H\} \quad \text{for } j=1,\ldots,n \tag{6.22}$$

Then, it can be shown that a valid convexity cut is given by

$$\sum_{j=1}^{n} (1/\bar{\lambda}_j)x_j \geq 1 \tag{6.23}$$

Typically, one works with the simplex tableau representation of the current point, whence, x_j, j=1,...,n are the nonbasic variables. Accordingly, C is defined in terms of nonbasic variables and at the current point, $x_j=0$, j=1,...,n.

Now, let us apply the disjunctive principle to this situation. Observe that since $S \cap C = \{\phi\}$, we are only interested in those points $x \geq 0$ which violate at least one of the inequalities defining C. Thus, we may stipulate that at least one of the systems

$$a^h x \geq b^h \qquad h \in H$$

$$x \geq 0$$

holds. Using Theorem 3.1, a suitable cut which one may derive through Equation (3.16) is given by

$$\sum_{j=1}^{n} \{\sup_{h \in H}(a_j^h/b^h)\}x_j \geq 1 \tag{6.24}$$

Note that one may have preferably chosen the cut given by Equation (3.19) instead. However, we will work with the above cut to preserve simplicity as well as to derive certain known cuts in the literature. Returning to Equation (6.22), we observe that

$$\bar{\lambda}_j = \begin{cases} \inf_{h \in H}\{b^h/a_j^h : a_j^h > 0\} \\ \infty \text{ if } a_j^h \leq 0 \text{ for each } h \in H \end{cases} \qquad j=1,\ldots,n$$

Hence, the cut (6.23) has coefficients

$$1/\bar{\lambda}_j = \begin{cases} \sup_{h\varepsilon H}\{a_j^h/b^h:\ a_j^h > 0\} \text{ if at least one } a_j^h > 0 \\ \\ 0 \text{ if all } a_j^h \leq 0 \end{cases} \qquad j=1,\ldots,n \qquad (6.25)$$

Comparing (6.24) and (6.25), we note that if $\xi^j \not\subset C$ for any $j\varepsilon\{1,\ldots,n\}$, then at least one $a_j^h > 0$, $h\varepsilon H$ for each $j\varepsilon\{1,\ldots,n\}$. In this case, (6.24) and (6.25) are identical. However, if $\xi^j \subset C$ for at least one $j\varepsilon\{1,\ldots,n\}$, then the corresponding coefficient $1/\bar{\lambda}_j$ is zero for the cut (6.23) but may be negative for the cut (6.24). Hence, (6.24) uniformly dominates (6.23), and may strictly dominate it. This latter case is depicted in Figure 6.1 below.

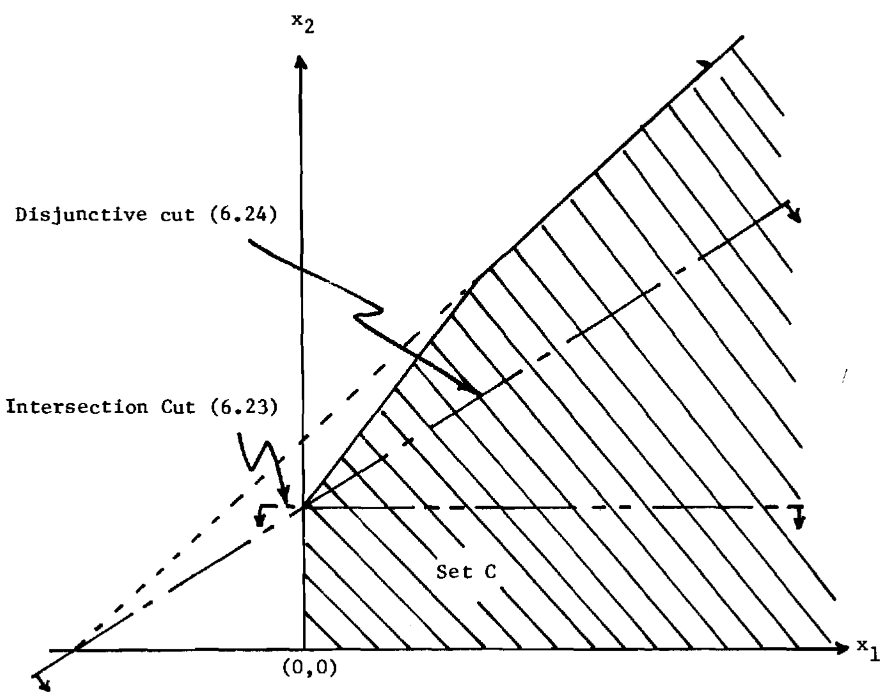

Figure 6.1. Illustration of Disjunctive and Intersection Cuts

Several existing cuts are subsumed under the category just described. Notice that we have not required the set H to be of finite cardinality. Thus, C need not be polyhedral. Therefore, hypercylindrical or spherical intersection cuts are also recovered under the discussed framework. In this situation, a hypercylinder or sphere containing the origin but no integer valued points (integral in terms of original variables) may be defined. Accordingly, the constraints in C represent tangential hyperplanes to the hypercylinder or sphere. If one uses a stronger condition and requires each constraint of C to simply correspond to a tangent to the sphere at some integer point, then one recovers octahedral cuts. Similarly, to obtain diamond cuts, one may use the condition that at least $|I|$ of the $2|I|$ constraints $x_i \leq 0$, $x_i \geq 1$, $i \in I$ must hold. This latter cut may be further strengthened by replacing x_i by y_i, $i \in I$ as defined in Equation (6.10).

Again, various polar cutting planes may be recovered from the above discussion by letting C be an appropriate reverse polar set. In this connection, the reader may note that the negative edge extension cut and the reverse polar cut would be identical to the disjunctive cut derived above.

We will now proceed to discuss one such special case in the next section. We will show how the convexity cuts generated in this case are subsumed under the disjunctive cut principle and how these cuts may be further strengthened.

6.4 Reverse Outer Polar Cuts for Zero-One Programming

Consider a program in (x,y) of the form

$$\begin{aligned} \text{minimize} \quad & cx + dy \\ \text{subject to} \quad & A(x,y) = b \\ & x \in S \\ & x,y \geq 0 \end{aligned}$$

ere, S is some set of points of interest to us. Also, currently, suppose we ave a basic solution $(\bar{x},\bar{y})$ to the linear program with the constraint $x \in S$ relaxed.

Further, assume that $\bar{x} \notin S$. Now, let us define the set C. For this purpose, let us say that we can identify some bilinear function

$$f(x,z): R^n \times R^n \to R \tag{6.26}$$

and a scalar k which are such that if we let C be the level set

$$C = L(k) = \{x: f(x,x) \leq k\} \tag{6.27}$$

then C contains $\bar{x}$, but int C contains no point in $S \cap X$, where,

$$X = \{x \geq 0: A(x,y) = b \text{ for some } y \geq 0\} \tag{6.28}$$

One may now polarize the function f(.,.) by replacing its argument (x,x) by (x,z) in (6.27) and thereby define a <u>reverse polar set</u>, alternatively called the <u>scaled generalized reverse polar</u> of X, with scalar k, as

$$X^o(k) = \{z \in R^n: f(x,z) \leq k, \text{ for each } x \in X\}$$

$$= \{z \in R^n: f(x,z) \leq k, \text{ for each } x \in \text{vert } X\} \tag{6.29}$$

where we have assume that X is bounded and that vert X represents the set of extreme points X. Note that although f(.,.) or L(k) may be nonconvex, $X^o(k)$ is a convex polyhedral set. Further, by suitably defining f(.,.) and k, we can have $\bar{x}$ lying in the interior of $X^o(k)$ with $X^o(k)$ containing no point in $X \cap S$. That is, any $z \in X \cap S$ must satisfy $f(x,z) \geq k$ for at least one vector x of X. Since f(x,z) is linear in z for a fixed X, we may use the basic disjunctive cut principle on this statement. Let us now illustrate the application of this to 0-1 integer programming. In this context, S is the set of integer valued points and currently, $\bar{x} \notin S$. Also, among other constraints, X contains the constraints $x_j \leq 1$ for each variable x_j. Given the simplex tableau representing the current point $\bar{x}$, we may identify the half lines

$$\xi^j = \{x: x = \bar{x} - a^j\lambda_j,\ \lambda_j \geq 0\},\ j\varepsilon J \qquad (6.30)$$

where J is the index set associate with n nonbasic variables. The function f(.,.) which we select is given by

$$f(x,z):\ (x - \frac{1}{2}\,e)^t G(z - \frac{1}{2}\,e),\ (x,z)\ \varepsilon\ R^n \times R^n \qquad (6.31)$$

where e=(1,...,1) and where

$$G = \begin{bmatrix} g_1 & & & 0 \\ & g_2 & & \\ & & \ddots & \\ 0 & & & g_n \end{bmatrix},\ \sum_{i=1}^{n} g_i = n,\ g_i \geq 0,\ i=1,\dots,n \qquad (6.32)$$

and $g_i > 0$ for at least one $i\varepsilon\{1,\dots,n\}$ for which $0 < \bar{x}_i < 1$. Further, we take the value of k to be $\frac{n}{4}$. Then, the set C is the level set

$$C = L(\tfrac{n}{4}) = \{x:\ (x - \frac{1}{2}\,e)^t G(x - \frac{1}{2}\,e) \leq \tfrac{n}{4}\} \qquad (6.33)$$

This may be simplified to

$$C = L(\tfrac{n}{4}) = \{x:\ \sum_{i=1}^{n} g_i x_i (x_i - 1) < 0\} \qquad (6.34)$$

Hence, all 0-1 points are contained in the boundary of the set $L(\frac{n}{4})$ while from (6.32), we observe that $\bar{x}\ \varepsilon$ int $L(\frac{n}{4})$. Thus, int $L(\frac{n}{4})$ contains $\bar{x}$ but no point in $X \cap S$. Continuing, we have from Equation (6.29),

$$X^o(\tfrac{n}{4}) = \{z:\ (x - \frac{1}{2}\,e)^t G(z - \frac{1}{2}\,e) \leq \frac{n}{4} \text{ for each } x\varepsilon X\}$$

or

$$X^{o}(\tfrac{n}{4}) = \{z: (x - \tfrac{1}{2} e)^{t} Gz \leq \tfrac{1}{2} x^{t}Ge \quad \text{for each } x\varepsilon X\} \tag{6.35}$$

One may show that int $X^{o}(\frac{n}{4})$ contains $\bar{x}$ but no point in $X \cap S$. Hence, any point $z \in X \cap S$ must satisfy at least one of the inequalities

$$(x - \tfrac{1}{2} e)^{t} G z \geq \tfrac{1}{2} x^{t} G e, \text{ for } x\varepsilon X. \tag{6.36}$$

Now, since $z \varepsilon X$, from (6.30), we may write it as

$$z = \bar{x} - \sum_{j\varepsilon J} a^{j} t_{j} \tag{6.37}$$

where t_j, $j \varepsilon J$ are the current nonbasic variables. Substituting (6.37) into (6.36), we must have at least one of the following inequalities holding

$$\sum_{j\varepsilon J} (\tfrac{1}{2} e - x)^{t} G\, a^{j} t_{j} \geq \tfrac{1}{2}(x + \bar{x})^{t} G e - x^{t} G \bar{x}, \text{ for } x\varepsilon X \tag{6.38}$$

But note that

$$\tfrac{1}{2}(x + \bar{x})^{t} G e - x^{t} G \bar{x} = \tfrac{1}{2} \Sigma(x_i + \bar{x}_i) g_i - \Sigma x_i \bar{x}_i g_i$$

$$> \tfrac{1}{2} \Sigma(x_1^2 + \bar{x}_1^2) g_i - \Sigma x_i \bar{x}_i g_i = \tfrac{1}{2} \Sigma(x_i - \bar{x}_i)^2 g_i \geq 0 \tag{6.39}$$

since $x_i^2 \leq x_i$, $\bar{x}_i^2 \leq \bar{x}_i$, $g_i \geq 0$ for each $i=1,\ldots,n$ and $g_i > 0$ for some i for which $0 < \bar{x}_i < 1$ (see (6.32)).

Thus, the right hand side of (6.38) is positive and we may normalize (6.38) by its right hand side for each $x\varepsilon X$. Thus, applying Theorem 3.1, Equation (3.16), we may derive the disjunctive cut

$$\sum_{j\varepsilon J} (1/\bar{\lambda}_j)t_j \geq 1$$

where, $\bar{\lambda}_j$, $j\varepsilon J$ is given by

$$(1/\bar{\lambda}_j) = \left[\max_{x\varepsilon X} \frac{(\frac{1}{2}\ e - x)^t G\ a^j}{\frac{1}{2}\ (x+\bar{x})^t Ge - x^t G\bar{x}}\right],\ j\varepsilon J \tag{6.40}$$

Now, let us determine the intersection cut based on $X^o(\frac{n}{4})$ of (6.35). This cut is given as

$$\sum_{j\varepsilon J} (1/\hat{\lambda}_j)t_j \geq 1 \tag{6.41}$$

where

$$\hat{\lambda}_j = \max\{\lambda_j:\ z = (\bar{x} - a^j\lambda_j)\ \varepsilon\ X^o(\tfrac{n}{4})\}$$

$$= \max\{\lambda_j:\ (x - \tfrac{1}{2}\ e)^t G(\bar{x} - a^j\lambda_j) \leq \tfrac{1}{2}\ x^t Ge \quad \text{for each } x\varepsilon X\}$$

$$= \max\{\lambda_j:\ \lambda_j(\tfrac{1}{2}\ e - x)^t Ga^j \leq \tfrac{1}{2}(x + \bar{x})^t Ge - x^t G\bar{x} \text{ for each } x\varepsilon X\}$$

But noting (6.39), we obtain

$$\hat{\lambda}_j = \max\{\lambda_j:\ \lambda_j\left[\frac{(\frac{1}{2}\ e - x)^t Ga^j}{\frac{1}{2}(x + \bar{x})^t Ge - x^t G\bar{x}}\right] \leq 1 \text{ for each } x\varepsilon X\}$$

or

$$\hat{\lambda}_j = \max\{\lambda_j:\ \lambda_j \max_{x\varepsilon X}\left[\frac{(\frac{1}{2}\ e - x)^t Ga^j}{\frac{1}{2}\ (x + \bar{x})^t Ge - x^t G\bar{x}}\right] \leq 1\} \tag{6.42}$$

Thus, if $(1/\bar{\lambda}_j) > 0$ in (6.40), then from (6.42), we observe that $1/\hat{\lambda}_j = 1/\bar{\lambda}_j$. On the other hand, if $1/\bar{\lambda}_j \leq 0$ in (6.40), then $\hat{\lambda}_j \to \infty$ or $1/\hat{\lambda}_j = 0$. Hence, the cut $\sum_{j \varepsilon J} \frac{1}{\bar{\lambda}_j} t_j \geq 1$ uniformly dominates $\sum_{j \varepsilon J} \left(\frac{1}{\hat{\lambda}_j}\right) t_j \geq 1$ and in fact, the former (disjunctive) cut implies the latter (intersection) cut.

Before concluding, we note that polar cut results analogous to those given for 0-1 programming above, may be obtained for (nonconvex) quadratic programming problems as well.

In the final two chapters of these notes, we will consider some special cases of disjunctive programs. To begin with, in the next chapter, we will treat facial disjunctive programs as introduced in Chapter V. Specifically, we will discuss two finitely convergent schemes for solving such problems. Thereafter, in Chapter VIII, we will dwell briefly on specific applications of some of the classes of disjunctive programs introduced in Chapter I.

6.5 Notes and References

In view of Theorem 2.1, any valid cut for a disjunctive program should be recoverable or can be dominated by a disjunctive cut of the theorem. Balas [4,6] has discussed this relationship in some detail for integer and nonlinear programs. As noted by Balas [4,6] and Glover [18,19], the new cuts clearly have the capability of improving some of the well-known cuts, and this is demonstrated by the discussion in the chapter.

Chapter VII

FINITELY CONVERGENT ALGORITHMS FOR FACIAL DISJUNCTIVE PROGRAMS WITH APPLICATIONS TO THE LINEAR COMPLEMENTARITY PROBLEM

7.1 Introduction

In our discussion of Chapter V, we had introduced a special class of disjunctive programs called facial disjunctive programs, examples of which included the zero-one linear integer programming problem and the linear complementarity problem. We had seen that for this special class of problems, it was relatively easy to generate the convex hull of feasible points. In this chapter, we will discuss two finitely convergent schemes which solve facial disjunctive programs by generating facets of the convex hull of feasible points as and when needed, until such time as either a suitable termination criterion is met or the problem is solved through the generation of the entire convex hull.

The organization of this chapter is as follows. We first briefly discuss how Theorem 5.11 may be exploited to develop a finite scheme for facial disjunctive programs. Thereafter, we present in greater detail, a second alternative method based on Theorem 5.10. This technique is the principal thrust of this chapter. Finally, we demonstrate how this latter method may be specialized for the linear complementarity problem.

7.2 Principal Aspects of Facial Disjunctive Programs

For the sake of completeness and convenience, let us re-introduce certain notations and concepts to be used in this chapter. The facial disjunctive program under consideration is

$\underline{\text{FDP}}$: minimize $c^t x$

subject to $x \in X = \{x: Dx = d,\ x \geq 0\}$ (7.1)

$x \in Y = \bigcap_{h \in H} [\bigcup_{i \in Q_h} \{x: a_i^h x \geq b_i^h\}]$ (7.2)

Here, c is a 1xn real vector, $x=(x_1,\ldots,x_n)$ is a (nx1) vector of variables, X is assumed to be a non-empty and bounded polyhedral set (regularized as assumed in Chapter V, if necessary). Further, D is an mxn real matrix and d is an nx1 real vector. The set Y is a conjunction of $|H| < \infty$ disjunctions, with $H=\{1,\ldots,\hat{h}\}$, say. Observe that we have deviated in consistency as regards to notation in this chapter so as to make the presentation more readable. Since this chapter is written to be basically self contained, we hope that this will not lead to any confusion. Continuing, the set Y defines for each $h \in H$, a disjunction which states that at least one of the constraints $a_i^h x \geq b_i^h$ must be satisfied for some $i \in Q_h$. Here, a_i^h is a 1xn real vector and b_i^h is a real scalar for each $i \in Q_h$, $h \in H$. The constraint index sets Q_h, $h \in H$ may contain common elements correpsonding to common constraints, and are otherwise disjoint.

Recall from Chapter V that the disjunction $x \in \bigcup_{i \in Q_h} \{a_i^h x \geq b_i^h\}$ is called _facial_ with respect to X if $X \cap \{x: a_i^h x \geq b_i^h\}$ is a face of X for each $i \in Q_h$. In addition, the disjunctive program FDP is said to be _facial_ if each of the disjunctions $h \in H$ is facial with respect to X. As before, by a _face_ of X we imply a subset of X defined by the intersection of X with a hyperplane which supports it.

Now, with our assumption of X being a bounded polyhedral set and with Y as specified in (7.2), we have,

$$F = c\ell \text{ conv } X \cap Y = \text{conv } X \cap Y \quad (7.3)$$

Further, let us inductively define

$$K_0 = X$$

$$K_h = \text{conv}[\bigcup_{i \in Q_h} (K_{h-1} \cap \{x: a_i^h x \geq b_i^h\})] \text{ for } h=1,\ldots,\hat{h} \quad (7.4)$$

Then, Theorem 5.11 and 5.10 are respectively re-stated below as properties P1 and P2.

P1: $K_{\hat{h}}$ of Equation (7.4) is equal to F of Equation (7.3)

P2: {Extreme points of F of Equation (7.3)} $\subseteq$ {Extreme points of X of Equation (7.1)}

We will now proceed to discuss the skeleton of a procedure for solving Problem FDP based on property P1 above. This discussion will also serve to lay the foundations for the second procedure which is treated at length in this chapter.

7.3 Stepwise Approximation of the Convex Hull of Feasible Points

Essentially, this scheme for solving Problem FDP is a relaxation strategy. To begin with, the constraints (7.2) are relaxed and the resulting linear program is solved. If the optimal solution $\bar{x}$, say, satisfies $\bar{x} \varepsilon Y$, then this solution is also optimal to FDP. Otherwise, a disjunction for some $h \varepsilon H$ is violated. Based on a violated disjunction, a cutting plane which deletes $\bar{x}$ but no point of X satisfying this disjunction and hence, no point of F, is now generated. This cutting plane is imposed as an additional constraint and the optional solution $\bar{x}$ is hence updated. This process is repeated till can optimal solution to some relaxed problem is feasible to (7.2).

Finiteness of the scheme is based on a result which is basically a strengthened version of the reverse part of Theorem 2.1, namely, the fundamental disjunctive cut principle. The result is stated below (without proof).

Theorem 7.1

Let $S_r = \{x: A^r x \geq b^r,\ x \geq 0\}$ for each $r \varepsilon R$ be non-empty sets and consider the disjunctive $x \varepsilon \bigcup_{r \varepsilon R} S_r$. Further, let $|R| = \tau$, say, and define the set

$$E = \{(\lambda^1,\ldots,\lambda^\tau,\alpha,\alpha_0):\ \begin{array}{ll} \lambda^r A^r - \alpha = 0 & \text{for } r=1,\ldots,\tau \\ \lambda^r b^r - \alpha_0 \geq 0 & \text{for } r=1,\ldots,\tau \\ \sum_r \sum_i \lambda_i^r = 1 & \\ \lambda^r \geq 0 \quad \text{for } r=1,\ldots,\tau\} & \end{array} \tag{7.5}$$

where for each $r=1,\ldots,\tau$, the vector λ^r has as many columns as A^r has rows, and where α is of the same dimension as x, with α_0 being a scalar. Then,

$$c\ell \text{ conv} \bigcup_{r\in R} S_r \subseteq \{x: \alpha^i x \geq \alpha_{i0} \text{ for each } i \text{ such that } (\lambda^{1i},\ldots,\lambda^{\tau i},\alpha^i,\alpha_{i0}) \text{ is an extreme point of } E\} \qquad (7.6)$$

In other words, if we had an enumeration of the extreme points of E of the form $(\lambda^{1i},\ldots,\lambda^{\tau i},\alpha^i,\alpha_{i0})$ indexed by i, then we could construct the closure of the convex hull of points feasible to the disjunction $x \in \bigcup_{r\in R} S_r$ as the intersection of the half-spaces $\alpha^i x \geq \alpha_{i0}$. This fact along with Property P1 may be used roughly as follows.

Initially, let us solve the problem of minimizing cx over the set $K_0 \equiv X$. Assume for the sake of simplicity that the optional solution $\bar{x}$ found violates the disjunction corresponding to h=1. Then, $\bar{x} \notin K_1$ and one may derive a cut corresponding to an extreme point of E of Equation (7.5) which deletes $\bar{x}$. Here, the constraint sets $A^r x \geq b^r$, for $r \in R$ correspond to $K_0 \cap \{x: a_i^h x \geq b_i^h\}$ for $i \in Q_1$. The cut may be simply derived by maximizing $\alpha_0 - \alpha x$ over this set E. Now, during the course of the procedure, whenever the disjunction for h=1 is violated, this step may be repeated. Clearly, from Theorem 7.1, this can happen only finitely often, the entire set K_1 being constructed in the worst case. In a similar manner, one may inductively argue that subsequent disjunction violations considered can be repeated only a finite number of times. Again, assuming for the sake of simplicity that these disjunction violations occur and are considered in the order h=1,2,... one may note that when deriving cuts for the j^{th} disjunction, the constraints $A^r x \geq b^r$ for $r \in R$ used in the set E of Equation (7.5) correspond to the intersection of the set K_0, the cuts generated for the disjunction violations 1,2,...,j-1 and the disjunctive constraints indexed by Q_j. For algorithmic purposes, whenever an updated solution violates more than one disjunction which has been previously considered, the cut derived is based on the most recent one of these disjunctions. In this manner, at worse, one would construct the sets $K_0,\dots,K_{\hat{h}}$ in their entirety. Typically, the actual sequence of sets constructed may be only an approximation of these sets in the vicinity of an optional solution.

7.4 Approximation of the Convex Hull of Feasible Points through an Extreme Point Characterization

The second procedure (which we shall call the Extreme Point Method) for solving Problem FDP is basically the same type of relaxation scheme as discussed in the foregoing section. Hence, a series of cutting planes and updated solutions to the relaxed problems are generated till such time as an updated solution is found which satisfies the disjunction (7.2). Whereas we had to specify restrictions on the type of and manner in which the cutting planes were generated in the previous section in order to ensure finiteness, we have some flexibility in this respect in the present approach. Instead, in order to invoke Property P2, we place specific restrictions on the type of points at which the cuts are generated. Specifically, these points are required to be so called extreme faces of the set X with respect to cuts generated at any stage of the procedure. This concept of extreme faces is discussed in the following subsection.

7.4.1 Extreme Faces and Their Detection

Let us assume that at a particular stage s cuts, $Gx \leq g$, have been generated in the space of the x-variables. Let

$$\Lambda = \{x \in R^n: Gx + Ix_s = g,\ x_s \geq 0\} \qquad (7.7)$$

be the subset of R^n feasible to these cuts. Here, $x_s = (x_{n+1},\dots,x_{n+s})$ denotes the vector of slack variables (with the superscript t being used to designate the matrix transpose operation), and I is an identity matrix of size s. Further, let $N = \{1,\dots,n\}$ denote the index set of the original x-variables, which we will call key variables. Also, let $S = \{n+1,\dots,n+s\}$ denote the index set of the slack variables for the s cuts, which we will call as nonkey variables. For a set $Z \subseteq N$, let

$$F_Z = \{x \varepsilon X: x_j = 0 \text{ for } j \varepsilon Z\} \tag{7.8}$$

Note that all faces of X can be represented as F_Z for some suitable set Z. Finally, for any point $x \varepsilon F_Z$, let the zero components of x be denoted by

$$Z(x) = \{j \varepsilon N: x_j = 0\} \tag{7.9}$$

Definition 4.1

Let F_Z be a face of X defined by some $Z \subseteq N$ such that $F_Z \cap \Lambda \neq \phi$. Then F_Z is an extreme face of X relative to Λ if for any two points $x^1, x^2 \varepsilon F_Z \cap \Lambda$, we have $Z(x^1) = Z(x^2)$.

In other words, an extreme face F_Z satisfies the property that $F_Z \cap \Lambda$ does not contain any point in a lower dimensional face of X. Examples of extreme faces of X relative to Λ are extreme points of X feasible to Λ, or an edge of X not disjoint with Λ but with neither of the two extreme points of X defining this edge being feasible to Λ.

Now, observe that Property P2 essentially directs that the search for an optimal solution to Problem FDP may be restricted to a search among the extreme points of X. However, we will find it simpler to restrict our search to a larger set, namely, the extreme faces of X. Since extreme faces of X relative to some Λ are also faces of X, the number of such extreme faces of X (relative to all Λ's) is finite. Hence, a procedure which detects and deletes in a finite number of steps at least one extreme face per iteration is finitely covergent. This is indeed the principal thrust of the present scheme.

Given a simplex tableau representation of an extreme point of $X \cap \Lambda$ at any stage, a simple procedure to find an extreme face of X relative to Λ utilizes the following restricted basis entry rule:

"Only a nonkey variable x_j, $j \varepsilon S$, is eligible to enter the basis" (7.10)

Based on this, the method outlined below either finds an extreme face or indicates that no such face exists.

Step 1

Let x_r denote the largest valued basic key variable in the current solution which has not yet been considered at a previous iteration. If no such variable exists, go to Step 3. Otherwise, proceed to Step 2.

Step 2

Solve the Problem P_r: minimize $\{x_r: x \in X \cap \Lambda\}$ as a linear program subject to the restricted basis entry rule (7.10). If the solution yields $x_r = 0$ and x_r is basic, pivot it out of the basis, if possible, by exchanging it with a nonkey, nonbasic variable. Return to Step 1.

Step 3

If all key variables are basic, there is no extreme face of X relative to Λ. Otherwise, the current set Z of indices of nonbasic key variables defines, through Equation (7.8), an extreme face F_Z of X relative to Λ. In particular, if all nonbasic variables are key variables, then F_Z represents an extreme point of X.

7.4.2 Schema of the Proposed Approach

The procedure we adopt operates as follows. At any stage, given the set Λ of Equation (7.7), we solve the relaxes problem

$$P(\Lambda): \quad \text{minimize} \quad \{cx: x \in X \cap \Lambda\} \qquad (7.11)$$

If an optimal solution $\bar{x}$ to this problem satisfies $\bar{x} \in Y$ of Equation (7.2), we stop wtih $\bar{x}$ as an optimal solution to Problem FDP. Otherwise, we generate a cut based on a violated disjunction, and then after updating the tableau, we use the routine of Section 7.4.1 to find an extreme face F_Z of X relative to Λ. If no extreme faces exist, then we terminate with the current best known solution as optimal to FDP. Otherwise, depending on the dimension of F_Z, two possible routes are open to us. If F_Z is of dimension greater than zero, then a disjunctive face cut is developed which deletes F_Z but no extreme point of X feasible to Λ. Details of such a cut are presented in Section 7.5. On the other hand, if F_Z represents an extreme point of X then we check if this extreme point is feasible to Y. If it is, then we update the current best known solution, if necessary, and again generate a disjunctive face cut which deletes only this particular extreme point of X. If the extreme point is infeasible to Y, however, a stronger disjunctive cut may be developed as discussed in Chapter IV and re-iterated in Section 7.5. In any case, after the appropriate cut has been generated and Λ has been updated, we say that an iteration has been completed. A new iteration is now commenced by solving Problem P(Λ) of Equation (7.11).

As an additional expedient, we will also impose the cost cut

$$c^t x \leq \hat{\nu}$$

where $\hat{\nu}$ is the current best known objective value of Problem FDP. Hence, the right hand side of this cut is simply updated each time an improved solution is detected. Although this cut will not affect the solution of Problems P(Λ), it will assist in confining the search to improving solutions during the extreme face finding routine. This is essential because otherwise, the extreme face finding routine would simply concentrate on feasibility, regardless of objective function values.

Figure 7.1 gives a flow chart of the proposed scheme. The collection of extreme faces of X relative to all possible sets Λ being finite, this method is clearly finitely convergent. Instead of reviewing in detail the general concepts involved in generating disjunctive face cuts, we discuss its generation for the linear complementarity problem in particular and merely allude subsequently to the extension of this to the general case.

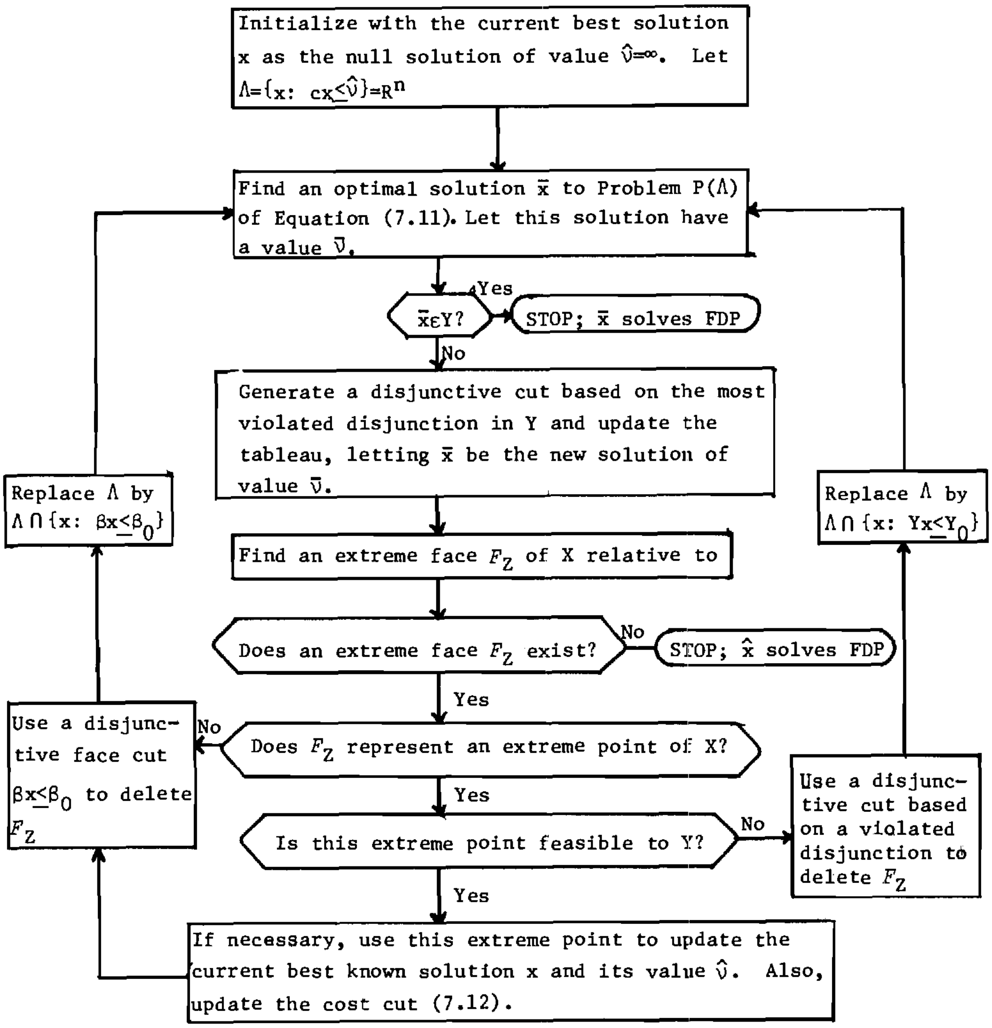

Figure 7.1. Flow-Chart for the Proposed Scheme

7.5 Specializations of the Extreme Point Method for the Linear Complementarity Problem

In this section, we will demonstrate how the cutting planes to be used in the procedure depicted in Figure 7.1 may be generated for the linear complementarity problem. Alongside this discussion, we will also make remarks for the handling of the general case.

7.5.1 Disjunctive Face Cut at an Extreme Face F_Z Which is not an Extreme Point of X

Suppose that the current tableau represents an extreme point $x^0=(x_1^0,\ldots,x_n^0)$ of $X \cap \Lambda$ with $x^0 \in F_Z$, where

$$Z = \{j \in N: x_j \text{ is currently nonbasic}\} \tag{7.13}$$

Let us assume that the disjunction $x_p x_q=0$ is violated by x^0.

Now, consider the solution of Problem P_p defined in Step 2 of the extreme face finding routine. Recall that this problem is solved subject to the restricted basis entry rule (7.10). At optimality, let

$$N_p = \{j \in N: x_j \text{ is nonbasic}\} \tag{7.14}$$

$$S_p = \{j \in S: x_j \text{ is nonbasic}\} \tag{7.15}$$

and let the canonical representation of x_p in terms of the nonbasic variables x_j, $j \in N_p \cup S_p$ be

$$x_p + \sum_{j \in N_p} a_{pj}x_j + \sum_{j \in S_p} a_{pj}x_j = b_p$$

Since $N_p \subseteq Z$, by adding suitable zero coefficients, the above equation may be written as

$$x_p + \sum_{j \in Z} a_{pj}x_j + \sum_{j \in S_p} a_{pj}x_j = b_p \tag{7.16}$$

In a similar manner, after solving P_q, we would have an equation

$$x_q + \sum_{j\varepsilon Z} a_{qj}x_j + \sum_{j\varepsilon S_q} a_{qj}x_j = b_q \tag{7.17}$$

It is easy to show that

$$a_{pj} \leq 0 \quad j\varepsilon S_p, \; a_{qj} \leq 0 \quad j\varepsilon S_q, \; b_p > 0, \; b_q > 0 \tag{7.18}$$

Now, the requirement that at least one of $x_p \leq 0$, $x_q \leq 0$ must hold may be written as requiring that at least one of the following constraint sets must be satisfied

$$\sum_{j\varepsilon Z \cup S_{pq}} \left(\frac{a_{rj}}{b_r}\right) x_j \geq 1; \; x_j \geq 0 \text{ for } j\varepsilon Z \cup S_{pq} \quad \text{for } r=p,q \tag{7.19}$$

where $S_{pq} = S_p \cup S_q$ and again, we have suitably defined zero coefficients wherever necessary. From Theorem 2.1, a valid cut is

$$\sum_{j\varepsilon Z \cup S_{pq}} \left(\max\left\{\frac{a_{pj}}{b_p}, \frac{a_{qj}}{b_q}\right\}\right)x_j \geq 1 \tag{7.20}$$

Observe from (7.18) that (7.20) implies

$$\sum_{j\varepsilon Z} \left(\max\left\{\frac{a_{pj}}{b_p}, \frac{a_{qj}}{b_q}\right\}\right)x_j \geq 1$$

and hence, (7.20) deletes F_Z since any $x\varepsilon F_Z$ satisfies $x_j=0$ for $j\varepsilon Z$. Finally, note that either in the general case of facial disjunctive programs or in the present application when $x^0 \varepsilon Y$, one may obtain an equation of the type (7.16) for each $r\varepsilon N$ such that $x_r^0 > 0$ and then derive a cut based on the disjunction that at least one of these variables x_r must be zero at any extreme point of x feasible to Λ, if such a point exists.

7.5.2 Disjunctive Cut at an Extreme Face F_Z which is an Extreme Point of X

Let x^0 be the extreme point of X represented by the current tableau as the extreme face F_Z. Again, if $x^0 \in Y$, then we develop a disjunctive face cut as in the general case of the foregoing discussion. On the other hand, if some disjunction $x_p x_q = 0$ is violated, then a deeper cut may be generated in the following manner.

Let the canoical representation of the (positive) basic variables x_p and x_q in the current tableau be given by

$$\begin{aligned} x_p + \sum_{j \in Z} a_{pj} x_j &= b_p > 0 \\ x_q + \sum_{j \in Z} a_{qj} x_j &= b_q > 0 \end{aligned} \tag{7.21}$$

where Z is currently also the index set for nonbasic variables. Now, the disjunction that at least one of the variables x_p, x_q equals zero may be written as the requirement that at least one of the constraint sets

$$\left\{ \sum_{j \in Z} \left(\frac{a_{pj}}{b_p}\right) x_j \geq 1,\ x_j \geq 0 \text{ for } j \in Z \right\}, \left\{ \sum_{j \in Z} \left(\frac{a_{qj}}{b_q}\right) x_j \geq 1,\ x_j \geq 0 \text{ for } j \in Z \right\} \tag{7.22}$$

must be satisfied. Through Theorem 2.1, a valid cut based on this statement is

$$\sum_{j \in Z} \bar{\pi}_j x_j \geq 1 \tag{7.23}$$

where,

$$\bar{\pi}_j = \max\left\{\frac{a_{pj}}{b_p}, \frac{a_{qj}}{b_q}\right\} \text{ for each } j \in Z \tag{7.24}$$

which clearly deletes $F_Z = x^0$. Now, in Chapter IV, we discussed how this cut may further strengthened by considering nonnegativity conditions on the other basic variables also. Hence, if we let B_{pq} be the index set of basic variables x_r, $r \neq p$, $r \neq q$ then we may replace the disjunction (7.22) by the following disjunction, where we have used a canonical representation for the basic variables x_r, $r \in B_{pq}$ similar to that in (7.21). This disjunction states that at least one of the following constraint sets must be satisfied

$$\{ \sum_{j \in Z} a_{rj}x_j \leq b_r \text{ for } r \in B_{pq}, \sum_{j \in Z} a_{pj}x_j \geq b_p, \; x_j \geq 0 \text{ for } j \in Z\}$$
$$\{ \sum_{j \in Z} a_{rj}x_j \leq b_r \text{ for } r \in B_{pq}, \sum_{j \in Z} a_{pj}x_j \geq b_q, \; x_j \geq 0 \text{ for } j \in Z\} \tag{7.25}$$

The improvement technique proposed in Chapter IV essentially attempts to derive a cut in terms of the nonbasic variables x_j, $j \in Z$ such that this cut is a support for the closure of the convex hull of the union of the two sets in (7.25). The method accomplishes this by commencing with the cut (7.23), say, and attempting to improve (decrease) as much as possible each cut coefficient one at a time, holding the other cut coefficients fixed. Hence, if at any stage, if (7.23) represents the current cut and if one is trying to reduce the coefficient of x_k, then as in Chapter IV this coefficient is given by the larger of the optimal values of the two linear programs LP_{kp} and LP_{kq}, where

$$LP_{kh}: \quad \text{maximize} \quad \xi - \sum_{\substack{j \in Z \\ j \neq k}} \bar{\pi}_j y_j$$

$$\text{subject to} \quad \sum_{\substack{j \in Z \\ j \neq k}} a_{rj}y_j - b_r\xi \leq -a_{rk} \quad \text{for } r \in B_{pq}$$

$$b_h\xi - \sum_{\substack{j \in Z \\ j \neq k}} a_{hj}y_j \leq a_{hk} \tag{7.26}$$

$$\xi \geq 0, \; y_j \geq 0 \text{ for } j \in Z - \{k\}$$

Again, as indicated in Chapter IV, both LP_{kp} and LP_{kq} need not necessarily be solved independently. Thus, for instance, if Equation (7.24) yields $\bar{\pi}_k = \frac{a_{pk}}{b_p}$, say, then one may solve LP_{kp} first. If the optimal value $\bar{\pi}_{kp}$ of LP_{kp} equals $\bar{\pi}_k$, then LP_{kq} need not be solved. Otherwise, LP_{kq} may be solved with the added constraint that its objective value exceeds $\bar{\pi}_{kp}$.

Of course, to reduce the effort in the generation of such cuts, one need not include all the constraints for $r \epsilon B_{pq}$ in the sets of Equation (7.25). Instead, some heuristic rule made be used to select a subset of these contraints. For example, one may select those constraints from B_{pq} which delete at least one of the finite intercepts which the cut (7.23) makes on some axis. Hence, one may select

$$\{r \epsilon B_{pq} : \frac{a_{rk}}{\bar{\pi}_k} > b_r \text{ for some } k \epsilon \bar{\pi}_k \neq 0\}$$

7.6 Notes and References

The term "facial disjunctive programs" was first used by Balas [5] where the principal result, namely Properties P1 and P2, discussed in this chapter, are proved. Those results have led to two finitely convergent algorithms for facial programs presented in this chapter. The first is due to Jeroslow [26] and makes use of one of the properties. The second algorithm is based on the other property and uses the concept of "extreme faces" first presented by Majthay and Whinston [28].

CHAPTER VIII

SOME SPECIFIC APPLICATIONS OF DISJUNCTIVE PROGRAMMING PROBLEMS

8.1 Introduction

In Chapter I we discussed in general the major applications of disjunctive programming problems. These included the generalized lattice point and related problems, the cardinality constrained problem, the extreme point programming problem and the binary mixed integer linear programming problem. In this chapter, we will present some specific applications which are subsumed under these general classes of problems.

8.2 Examples of Bi-Quasiconcave Problems

The Bi-Quasiconcave problem may be written as

$$\text{minimize } \{f(x,y): x \varepsilon S_x,\ y \varepsilon S_y\}$$

where S_x and S_y are polyhedral sets in variables x and y respectively and $(.,.)$ is a real valued function such that $f(.,y)$ and $f(x,.)$ are quasiconcave for any fixed x and y. It is easy to see that this latter property guarantees that an optimal solution is obtained at an extreme point of $S_x \text{ X } S_y$. Thus these problems are essentially extreme point optimization problems. We will now discuss some practical Bi-Quasiconcave problems.

8.2.1 Orthogonal Production Scheduling - A Multiperiod Activity Analysis Model

Let us first of all consider the class of problems known as the Multi-period Activity Analysis problems or the multistage production problems. These problems have the mathematical form

$$\text{minimize} \quad \sum_{k=1}^{K} (c^k)^t x^k$$

$$\text{subject to} \quad Ax^k \geq b^k \text{ for } k=1,\ldots,K$$

$$x^k \geq 0$$

Here, we have, say n activities producing m commodities over K periods. Thus, x^k is a vector representing the activity levels at period k, k=1,...,K, for activities 1,...,n, say. Further, A_{mxn} is the matrix of input-output or technological coefficients, b^k is a vector which denotes the requirements for various commodities 1,...,m in period k and c^k is a vector which represents the unit cost associated with each activity 1,...,n in period k.

However, certain physical considerations may require that certain orthogonal constraints of the form $x_j^{k-1} \cdot x_j^k = 0$, k=2,...,K hold for some activities j. For example, this may arise in the context of machine scheduling wherein due to maintenance considerations, certain activities cannot be scheduled in two consecutive periods. As another example, they may arise in an agricultural production situation wherein certain crops cannot be raised in two consecutive periods to preserve specific nutrients in the soil.

This problem may be transformed into a Bi-Quasiconcave Programming problem as follows. Let us assume, merely for convenience, that each activity is restricted by the orthogonal scheduling constraint mentioned above. Then we may use the penalty function method to ascribe a high cost to any schedule which is infeasible to these orthogonal constraints. That is, letting M be a large constant, we may formulate the orthogonal production problem as

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{K} (c^k)^t x^k + M \sum_{k=2}^{K} (x^{k-1})^t x^k \\ \text{subject to} \quad & Ax^k \geq b^k, \; k=1,\ldots,K \\ & x^k \geq 0 \end{aligned}$$

For the case K=2, this problem is clearly a Bi-Quasiconcave Program; in fact, for K=2, it is a Bilinear Programming Problem with an optimal solution being an extreme point of $X_1 \times X_2$ where $X_1 = \{x^1 \geq 0\colon Ax^1 \geq b^1\}$, $X_2 = \{x^2 \geq 0\colon Ax^2 \geq b^2\}$.

8.2.2 Application to Game Theory

Consider a two player game where player P_1 selects his strategy first as a vector x from the set $X = \{x: A_1x \leq b, x \geq 0\}$. Depending on the strategy x selected by P_1, let us say that player P_2 selects a strategy y from the set $Y(x) = \{y: A_2y \leq d + Cx, y \geq 0\}$. Here, Y(x) is assumed bounded and nonempty for each $x \in X$. Further, let us say that when P_1 selects strategy x and P_2 selects strategy y, there is an associated payoff $f(x,y) = p^tx + q^ty$ from P_1 to P_2, where p and q are given cost vectors. Thus, given x, P_2 will solve the problem

$$\begin{array}{ll} \text{maximize} & q^ty \\ \text{subject to} & y \in Y(x) \end{array}$$

Let y(x) denote an optimal solution to the above problem. Hence, knowing the technique to be adopted by P_2, P_1 will try to select a strategy $x \in X$ which minimizes $f(x,y(x))$, that is, he will solve the problem

$$\begin{array}{ll} \text{minimize} & \{p^tx + \text{minimum}\{(d + Cx)^tz: A_2z \geq q, \ z \geq 0\}\} \\ \text{subject to} & A_1x \leq b, \ x \geq 0 \end{array}$$

where $q^ty(x) = \max\{q^ty: A_2y \leq d + Cx, y \geq 0\}$ has been rewritten as $q^ty(x) = \min\{(d + Cx^tz: A_2z \geq q, z \geq 0\}$. Hence, the above problem may equivalently be written as

$$\begin{array}{ll} \text{minimize} & d^tz + p^tx + z^tCx \\ \text{subject to} & A_2z \geq q, \ z \geq 0 \\ & A_1z \leq b, \ x \geq 0 \end{array}$$

This problem is again a Bi-Quasiconcave Programming Problem; in fact, it is a bilinear problem.

8.2.3 Multi-Stage Assignment Problem

For the sake of simplicity, consider a two-stage assignment problem. The development given below may easily be generalized to the multi-stage problem. Hence, suppose we have N jobs and N machines with the stipulation that at each of the two stages, one and only one machine should be assigned to each job. The profit of assigning machine i to job k at the first stage is simply p_{ik}. However, the profit of assigning job i to machine j at the second stage depends on the job k to which machine i was assigned at the first stage. This profit is accordingly given by $\bar{p}_{ij} + q_{ijk}$. Thus, the total two-stage profit is given by

$$\sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} x_{ij}^1 + \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{p}_{ij} x_{ij}^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} q_{ijk} x_{ij}^2 x_{ik}^1$$

where

$$x^r \varepsilon X_r = \{x^r : \sum_{i=1}^{N} x_{ij}^r = 1,\ j=1,\ldots,N,\ \sum_{j=1}^{N} x_{ij}^r = 1,\ i=1,\ldots,N,\ x_{ij}^r = 0, 1$$

$$\text{for } i,\ j \varepsilon \{1,\ldots,N\} \text{ and for } r=1,2$$

That is X_1 and X_2 represent the assignment constraints at stages one and two respectively. Hence defining N^2 vectors $p = (p_{ij})$, $\bar{p} = (\bar{p}_{ij})$ and letting Q be an appropriate matrix made up of zeroes and coefficients q_{ijk}, we may formulate this problem as

$$\begin{aligned} \text{minimize} \quad & p^t x^1 + \bar{p}^t x^2 + (x^1)^t Q x^2 \\ \text{subject to} \quad & x^1 \varepsilon X_1 \\ & x^2 \varepsilon X_2 \end{aligned}$$

This is again a bilinear programming problem.

8.2.4 Rectilinear Distance Location-Allocation Problems

As a final example of Bi-Quasiconcave Programming Problem, we consider this problem which is again a Bilinear Programming Problem. Specifically, consider a multifacility location-allocation problem which involves the distribution of several products among some new facilities to be located and between these new and other already existing facilities. Thus, suppose n new facilities are to be located. Let the variables denoting their location in a two-dimensional layout be (x_i, y_i), i=1,...,n. Further, let there be m existing facilities currently located at (x_i, y_i), i=n+1,...,n+m. Let a_{ik} denote the availability of product k, k=1,...,p, say, at a new facility i, for i=1,...,n and let b_{ik} denote the requirement of product k at a new or existing facility i, k=1,...,p, i=1,...,n+m. Let us also assume that each unit of product k supplied from new facility i to new or existing facility j costs c_{ijk} with a corresponding transportation cost per unit distance of t_{ijk}. Here, the distances are taken to be measured using the rectilinear norm. This distance measure is appropriate in the context of movement along city streets or in a grid of aisles in a factory or a warehouse.

The problem is to determine the locations (x_i, y_i) for the new facilities i=1,...,n and to find feasible allocations u_{ijk} of product k from new facility i to new or existing facility j so as to minimize the total purchase (or manufacture) and transportation costs. Mathematically, this problem may be written as

$$\text{minimize} \quad \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{j=1}^{n+m} \{c_{ijk} + t_{ijk} (|x_i - x_j| + |y_i - y_j|)\} u_{ijk}$$

$$\text{subject to} \quad u \in U = \{u = (u_{111}, \ldots, u_{n,n+m,p}):$$

$$\sum_{j=1}^{n+m} u_{ijk} \leq a_{ik} \quad \text{for } i=1,\ldots,n,\ k=1,\ldots,p$$

$$\sum_{i=1}^{n} u_{ijk} = b_{jk} \quad \text{for } j=1,\ldots,n+m;\ k=1,\ldots,p$$

$$u_{ijk} \geq 0 \quad \text{for } i=1,\ldots,n;\ j=1,\ldots,n+m,\ k=1,\ldots,p\}.$$

One may now use the usual transformation on the absolute value terms in the objective function above to write $|x_i - x_j| + |y_i - y_j|$ as $(x_{ij}^+ + x_{ij}^- + y_{ij}^+ + y_{ij}^-)$, where the restrictions on these new variables may be denoted as $z \in Z$, say, where,

$$z = (x_{11}^+, \ldots, x_{n,n+m}^+, x_{11}^-, \ldots, x_{n,n+m}^-, y_{11}^+, \ldots, y_{n,n+m}^+, y_{11}^-, \ldots, y_{n,n+m}^-, x_1, \ldots, x_n, y_1, \ldots, y_n)^t$$

and where,

$$Z = \{z:\ x_i - x_j - x_{ij}^+ + x_{ij}^- = 0 \quad \text{for } i=1,\ldots,n,\ j=1,\ldots,n+m$$
$$y_i - y_j - y_{ij}^+ + y_{ij}^- = 0 \quad \text{for } i=1,\ldots,n,\ j=1,\ldots,n+m$$
$$x_{ij}^+,\ x_{ij}^-,\ y_{ij}^+,\ y_{ij}^- \geq 0 \quad \text{for } i=1,\ldots,n,\ j=1,\ldots,n+m\}$$

Then, the rectilinear distance location-allocation problem may be written as the bilinear programming problem

$$\begin{aligned} \text{minimize} \quad & c^t u + z^t T u \\ \text{subject to} \quad & u \in U \\ & z \in Z \end{aligned}$$

where c and T are appropriate cost vectors and matrices respectively. Note that the orthogonality constraints of the type $x_j^+ x_{ij}^- = y_{ij}^+ y_{ij}^- = 0$ are not explicitly needed since the columns of x_{ij}^+ and x_{ij}^- (as also of y_{ij}^+ and y_{ij}^-) are linearly dependent in Z.

8.3 Load Balancing Problem

The load balancing problem is one which involves the allocation of m jobs of given "weights" w_i to n departments such that the total resulting work loads, L_j, j=1,...,n are as equally balanced as possible. Hence, if we define 0-1 variables x_{ij} as

$$x_{ij} = \begin{cases} 1 \text{ if job 1 is assigned to department } j \\ 0 \text{ otherwise} \end{cases} \qquad i=1,\ldots,m,\ j=1,\ldots,n$$

then the work load at station j is given by

$$L_j = \sum_{i=1}^{m} w_i x_{ij}$$

The concept of an equitable balance of load between stations is subjective. One may choose to minimize the difference between the minimum and the maximum work load at any station. Or, one may examine the average work load $L = \frac{1}{n} \sum_{i=1}^{m} w_i$ and choose to

$$\text{minimize} \quad \{ \sum_{j=1}^{n} |L_j - L| \}$$

Using this latter alternative one may adopt two types of formulations. Firstly, one may introduce an (m+1)th dummy job and stipulate the following constraints

$$x \varepsilon X = \{x=(x_{ij}): \sum_{j=1}^{n} x_{ij} = 1,\ i=1,\ldots,m$$

$$\sum_{j=1}^{n} x_{m+1,j} = m(n-1)$$

$$\sum_{i=1}^{m+1} x_{ij} = m,\ j=1,\ldots,n$$

$$x_{ij} \geq 0\} \qquad (8.1)$$

These above constraints constitute a transportation constraint set in which there are (m+1) supply points, m of which have a unit supply and the $(m+1)^{st}$ has a supply of m(n-1). Further, there are n demand points, each of them having a demand of m units. Moreover, every basic feasible solution to this problem is integer, and specifically, zero-one. In fact, there is a one-to-one correspondence between the assignment of jobs to departments and basic feasible solutions of this constraint set. Hence, this is now an extreme point problem wherein one searches for the best extreme point of X.

In another equivalent form, we may let

$$L_j - L = y_j^+ - y_j^-, \; y_j^+, \; y_j^- \quad 0, \; y_j^+ y_j^- = 0$$

and recalling that $L_j = \sum_{i=1}^{m} w_i x_{ij}$, we may formulate the load balancing problem as

$$\text{minimize} \quad \sum_{j=1}^{n} (y_j^+ + y_j^-)$$

$$\text{subject to} \quad \sum_{i=1}^{m} w_i x_{ij} - y_j^+ + y_j^- = L \qquad j=1,\ldots,n$$

$$y_j^+, y_j^- \geq 0 \qquad j=1,\ldots,n$$

and x is an extreme point of X of Equation (8.1). Note that the orthogonal constraints $y_{ij}^+ y_j^- = 0$, $j=1,\ldots,n$ may be omitted in solution procedures which set the above problem up as linear programs since then the columns of y_j^+ and y_j^- are linearly dependent for each $j=1,\ldots,n$.

8.4 The Segregated Storage Problem

This problem considers a certain resource which is available in quantities $S_1,\ldots,S_m$ at m sources and is to be allocated to meet the demands $D_1,\ldots,D_n$ of n users with the added restriction that the requirement of each of (n-1) users, say, 1,2,...,n-1 is to be met from one and only one source. The last, or the n^{th}, user can be supplied from any of the sources. In a storage context, the first (n-1) users correspond to private (special) storage facilities and the n^{th} user corresponds to public (common or general) storage facility. Mathematically, we may let x_{ij} denote the quantity shipped from supply point i to demand point j at a cost of say, c_{ij} per unit and formulate the problem under the assumption that $\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$ as

$$\text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} \quad x \varepsilon X = \{x = (x_{ij}): \sum_{i=1}^{m} x_{ij} \leq D_j,\ j=1,\dots,n$$

$$\sum_{j=1}^{n} x_{ij} = S_i,\ i=1,\dots,m$$

$$x_{ij} \geq 0\}$$

$$\sum_{i=1}^{m} \delta_{ij} \leq 1,\ j=1,\dots,n-1$$

where,

$$\delta_{ij} = \begin{cases} 0 & \text{if } x_{ij} = 0 \\ 1 & \text{if } x_{ij} > 0 \end{cases} \quad \text{for } j=1,\dots,n-1$$

This problem is also an extreme point optimization problem since it can be shown that there exists an optimal solution to it which is an extreme point of the set X of Equation (8.2).

8.5 Production Scheduling on N-Identical Machines

Consider a firm which manufactures K products, each of which must be processed on the same machine. The machine has N rings of dies and is capable of processing N products simultaneously. Also, it is assumed that $N < K$. However, the entire machine must be shut down to change from one set of N products to another set. Thus, this problem may be viewed as one involving N identical machines, each with a single ring of dies, which are coupled by demand constraints. This demand is available as a forecast of each of the K products over the next T periods, with the current (initial) inventory level being known. Changeovers are permitted only at the end of each period with the cost being proportional to the number of rings whose dies must be changed. The problem is to determine an optimal production schedule for the K products over T weeks so as to minimize the total changeover and inventory costs while complying with the demand requirements.

Thus, let

x_{kt} = the number of rings producing product k in time period t, k=1,...,K, t=1,...,T.

d_{kt} = integral demand for product k at the end of time period t, k=1,...,K, t=1,...,T.

y_{kt} = inventory of product k at the end of period t, k=1,...,K, t=0,...,T, with t=0 yielding initial inventory.

c = cost of changing a single ring of dies.

c_k = inventory carrying cost for a single period for product k, k=1,...,K.

Now, it is clear that

$$\sum_{k=1}^{K} |x_{k,t+1} - x_{kt}|$$

represents twice the number of rings changed from manufacturing one product to another at the end of period t, so that the total changeover cost over T periods is

$$\frac{c}{2} \sum_{t=1}^{T-1} \sum_{k=1}^{K} |x_{k,t+1} - x_{kt}| \qquad (8.3)$$

Further, the inventory of product k during the t^{th} period is obtaine through the cumulative occurance over t-1 periods as

$$y_{k,t-1} = y_{k0} + \sum_{i=1}^{t-1} x_{ki} - \sum_{i=1}^{t-1} d_{ki} \qquad (8.4)$$

The production constraints require all the N rings to be busy in each time period, or,

$$\sum_{k=1}^{K} x_{kt} = N, \text{ for } t=1,\ldots,T \qquad (8.5)$$

Further, to stipulate that the demands are all met with, we need the constraints

$$y_{k,t-1} + x_{kt} - y_{kt} = d_{kt} \text{ for } t=1,\ldots,T,\ k=1,\ldots,K \qquad (8.6)$$

Then, the problem at hand is to

$$\begin{aligned} \text{minimize} \quad & \frac{c}{2} \sum_{t=1}^{T-1} \sum_{k=1}^{K} x_{k,t+1} - x_{kt} + \sum_{t=1}^{T} \sum_{k=1}^{K} c_k y_{k,t-1} \\ \text{subject to} \quad & \sum_{k=1}^{K} x_{kt} = N \qquad t=1,\ldots,T \\ & y_{k,t-1} + x_k - y_{kt} = d_{kt} \qquad t=1,\ldots,T,\ k=1,\ldots,K \\ & x_{kt},\ y_{kt} \geq 0, \text{ and integer, } k=1,\ldots,K,\ t=1,\ldots,T \end{aligned}$$

To convert this problem into one with network constraints, a redundant constraint of the following form may be added

$$\begin{aligned} \sum_{k=1}^{K} y_{kT} &= \text{total inventory at end of planning horizon} \\ &= \sum_{k=1}^{K} y_{k0} + NT - \sum_{t=1}^{T} \sum_{k=1}^{K} d_{kt} \end{aligned}$$

Utilizing the usual transformation of representing the absolute value of a variable as the difference between two nonnegative variables, the above problem may be converted into an integer linear program. The integrality restrictions may then be replaced with the equivalent requirements that the solution should be an extreme point of the network constraints given above. Hence, this problem may be represented as an extreme point optimization problem.

8.6 Fixed Charge Problem

This type of problem is a mathematical programming problem which involves a fixed cost to be added if a variable is non-zero. More specifically, the problem may be stated mathematically as

$$\text{minimize} \qquad c^t x + \sum_{j=1}^{n} \delta_j f_j : \; x \in X$$

where X is a polyhedral set and

$$\delta_j = \begin{cases} 0 & \text{if } x_j = 0 \\ 1 & \text{if } x_j > 0 \end{cases} \qquad \text{for } j=1,\ldots,n$$

and where c denotes the vector of variable cost coefficients. Here f_j is the fixed charge incurred if $x_j > 0$. It can be shown that the above objective function is concave and hence there exists an optimal solution which is an extreme point of X. Thus, this too is of the class of extreme point optimization problems.

As an example of a fixed charge problem, one may think of a transportation-locatio- situation wherein the fixed cost is associated with the construction of a supply facility or a warehouse at a potential site. As another example, the fixed charge may arise as a fixed set-up cost in a scheduling problem if the decision to manufacture a certain product is adopted. Fixed charges also arise in passenger transportation models wherein the introduction of each additional transport facility involves an extra fixed cost.

8.7 Project Selection/Portfolio Allocation/Goal Programming

Consider the problem

$$\text{minimize} \quad c^t x$$

$$\text{subject to} \quad x \in S = \{x: Ax \leq b_0\}$$

$$x \in \text{extreme point of } X = \{x: Bx \leq b,\ x \geq 0\}$$

where B is block-angular with blocks $B_1, \ldots, B_p$, say.

These types of problems arise, for example, in the context of project selection problems wherein the extreme points of X correspond to projects being proposed by the p "subordinate" units. These units must be coordinated by the "superordinate" whose stipulations/restrictions are expressed by the set S. If convex combinations of projects proposed by the subordinates is not meaningful, one is restricted to selecting an extreme point of X, that is, one needs to investigate the extreme point optimization problem given above. Similar structures arise in Portfolio selection and in Goal Programming. In the latter case, the objective is to obtain a solution as "close" as possible to the preset goals.

8.8 Other Applications

In several production planning problems, one is confronted with a profit function which is convex due to economies of scale. That is, as the level of production is increased, the profits increase more rapidly than in direct proportion at first and then level off due to diminishing marginal returns. Hence the problem of maximizing a convex (often quadratic) function over linear constraints is essentially an extreme point optimization problem.

In decision theory problems, a decision tree is constructed wherein each path through the tree represents a strategy with a utility value associated with it. The objective is to maximize the expected utility over a finite set of vectors, each vector denoting the values associated with a strategy. The problem may be reduced to that of maximizing a linear function over a polytope, where the polytope is defined as the convex hull of a finite set of points. Thus this is a special case of an extreme point optimization problem where the extreme points are a subset of a known finite set of discrete points.

Finally, we note that 0-1 linear integer programming problems can be converted into problems of minimizing a concave function over a polyhedral set. This may be accomplished by simply incorporating a penalty term of the form $\sum_{j=1}^{n} M x_j(1-x_j)$ into the objective function where M is a suitably large constant, and x_j, j=1,...,n are variables restricted to be zero or one in value. The problem is hence an extreme point optimization problem. However, due to the ill-conditioning effects of M, usually implicit enumeration schemes have been known to permit more efficient solution procedures than the implementation of the above transformation.

8.9 Notes and References

This section elaborates on some of the problems that can be represented as disjunctive programs. The reader may note that theoretically integer-programming problems in general can be cast in a disjunctive programming format. Only some of the special practical cases, particularly with 0-1 variables, that seem more amenable to solution procedures using disjunctive programming principles are discussed in this chapter. The thought proposed and discussed by Balas [4] and Glover [19] of incorporating disjunctive programming/polyhedral annexation principles within a branch-and-bound approach is particularly significant in the context of developing viable solution procedures.

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A FINITELY CONVERGENT ALGORITHM FOR BILINEAR PROGRAMMING PROBLEMS USING POLAR CUTS AND DISJUNCTIVE FACE CUTS†

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A finite algorithm is presented in this study for solving Bilinear programs. This is accomplished by developing a suitable cutting plane which deletes at least a face of a polyhedral set. At an extreme point, a polar cut using negative edge extensions is used. At other points, disjunctive cuts are adopted. Computational experience on test problems in the literature is provided.

Key words: Bilinear Programming, Polar Cuts, Disjunctive Cuts, Cutting Plane.

1. Introduction

Mathematically, the Bilinear Programming Problem may be stated as

$$\text{BLP:}\quad \text{minimize}\quad \phi(x, y) = c^t x + d^t y + x^t C y,$$

$$\text{subject to}\quad x \in X_0 = \{x \in R^n : Ex = e, x \geq 0\}, \tag{1.1}$$

$$y \in Y_0 = \{y \in R^m : Fy = f, y \geq 0\} \tag{1.2}$$

where X_0 and Y_0 are bounded polyhedral sets.

Cutting plane procedures have been previously proposed to solve such problems [13, 22]. The one proposed by Konno [13] leads to an ϵ optimal solution, that is, a solution differing in value from the global minimum value by no more than a prespecified positive quantity ϵ. On the other hand, the one proposed by Vaish and Shetty [22] yields a global optimal solution. Both these methods are convergent, though not necessarily finitely convergent. In this paper, we propose a cutting plane algorithm which deletes at least one face of X_0 at each iteration and hence converges finitely. In order to accomplish this, we employ two types of cuts.

(1) *Polar cut.* This is introduced at an extreme point of X_0 feasible to the cuts generated thus far, and is based on the theory of generalized polars [2, 5]. Here we employ the concept of negative-edge extensions in the spirit of the work of Glover [7, 8]. The cuts thus derived are subsumed under the general theory of

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cutting planes proposed by Burdet [6] and Jeroslow [10]. Incidentally, in this study we also improve the technique proposed in [21] to solve the parametric problems involved in the generation of polar cuts.

(2) *Disjunctive cut.* This is introduced at a suitable point which is not an extreme point of X_0. Under certain specific conditions, a disjunctive cut will be used at a degenerate extreme point of X_0. The cut is based upon the principles of valid inequalities as discussed by Owen [16], Balas [3], Glover [7, 8], and Jeroslow [11]. The cuts permit coefficients of either sign, and hence, as noted by Balas [3], they tend to circumvent problems associated with dual degeneracy.

A host of practical problems may be modeled as Problem BLP [12, 20, 21]. These include the two-stage or multi-stage assignment problems [21], the maximization of a convex function over a polytope [2, 14, 20], some game theory, production scheduling and decision theory problems [12, 21] and the multi-facility, multi-product, rectilinear distance location–allocation problem [19].

Problem BLP is essentially a nonconvex problem and hence a local minimum need not be a global minimum. Several previous studies [1, 13, 21, 22, 23] have investigated the structural properties of Problem BLP and we refer the interested reader to these for such a discussion. We will, however, indicate that for either a fixed $x \in X_0$ or $y \in Y_0$, Problem BLP is linear in the other variable set. Hence, there exists an optimal solution to Problem BLP which is an extreme point of $X_0 \times Y_0$.

In developing a cutting plane algorothm for Problem BLP, it is clear that in order to preserve the separability of the variables associated with X_0 and Y_0, the cut should be introduced in either the set X_0 or the set Y_0. Further, as we will see later, introducing the cuts in one set will involve solving several linear programs over the other set. Consequently, if one of the sets, say Y_0, has a special structure, we can preserve this set and introduce the cuts in the set X_0. As an example, in the rectilinear distance location–allocation problem [19], it is worth preserving the transportation problem constraints, as efficient solution procedures for the transportation problem are available.

2. Extreme faces of X_0 relative to the cuts

At a particular stage, suppose that s cuts, $Dx \le d$, have been generated in the space of the x-variables. Let the set of points feasible to these cuts be

$$Q = \{x \in R^n : Dx + Ix_S = d, x_S \ge 0\} \tag{2.1}$$

where x_S denotes the vector of slack variables $(x_{n+1}, \dots, x_{n+s})^t$, and I is an identity matrix of size s. Following Majthay and Whinston [15], we will now define an extreme face of X_0 relative to Q. If a point on this face is not an extreme point of X_0, we will develop a suitable cut in Section 3. On the other hand, if the extreme face is an extreme point of X_0, we will attempt to develop a deeper cut in Section 4, failing which, we will revert to the cut of Section 3.

Definition 2.1 [9]. Let X_0 be a convex subset in R^n. A nonempty subset F of X_0 is called a (proper) *face* of X_0 if there exists a supporting hyperplane H of X_0 such that $F = X_0 \cap H$.

Now let $N = \{1, \dots, n\}$ denote the index set of the original set of variables, which we will call *key variables*. Also, let $S = \{n+1, \dots, n+s\}$ denote the index set of the slack variables of the s cuts, which we will call *nonkey variables*. For a set $Z \subset N$, let

$$F_Z = \{x \in X_0 \colon x_j = 0 \text{ for } j \in Z\}. \tag{2.2}$$

Note that all the faces of X_0 can be represented as above for a suitable specification of the set Z.

Definition 2.2. Let F_Z be a face of X_0 such that $F_Z \cap Q \neq \phi$. Then F_Z is an *extreme face* of X_0 relative to Q if for each $k \in N$, $x \in F_{Z \cup \{k\}} \neq F_Z$ implies $x \notin Q$.

In other words, an extreme face F_Z satisfies the property that $F_Z \cap Q$ does not contain any point in a lower dimensional face of X_0. Note that an extreme point of X_0 feasible to Q qualifies for an extreme face. Likewise, an edge of X_0 whose defining extreme points are not in Q is also an extreme face of X_0 relative to Q, whereas an edge with one defining extreme point in Q will not qualify.

Given a set $Z_0 \subset N$, an extreme face of X_0 can be identified by sequentially adding indices to the set Z_0. The simple procedure by Majthay and Whinston [15] which accomplishes this adopts the following *restricted basis entry rule*:

"Only a nonkey variable x_j, $j \in S$, is eligible to enter the basis." (2.3)

In [15] it is proved that the procedure described below either finds an extreme face or indicates that no such face exists.

Initialization

Consider a standard simplex tableau representation of a basic feasible solution to $X_0 \cap Q$.

Step 1: Let x_r denote the most positive basic key variable in the current solution not yet considered at a previous iteration. If no such variable exists, go to Step 3. Otherwise, proceed to Step 2.

Step 2: Solve the Problem P_r: minimize $\{x_r \colon x \in X_0 \cap Q\}$ subject to the restricted basis entry rule in (2.3). If the solution yields $x_r = 0$ and x_r is basic, pivot it out of the basis, if possible, by exchanging it with a nonkey, nonbasic variable. Return to Step 1.

Step 3: If all key variables are basic, there is no extreme face of X_0 relative to Q. Otherwise, the current set Z of indices of nonbasic key variables defines, through Eq. (2.2), an extreme face F_Z of X_0 relative to Q. In particular, if all nonbasic variables are key variables, then F_Z is an extreme point of X_0.

The above procedure yields a simplex tableau corresponding to an extreme point x^0 of $X_0 \cap Q$ which lies on an extreme face F_Z of X_0 relation to Q, if such a face exists. If F_Z is of dimension greater than zero, then we will develop a disjunctive cut to delete the entire extreme face. On the other hand, if the extreme face is an extreme point of X_0, a cut better than the disjunctive can be developed. In order to do this, starting from the current point, we locate another extreme point with suitable properties and generate a polar cut with negative edge extensions as discussed in Section 4. If such a suitable point is not available, a disjunctive cut is developed. In either case, the cut generated is appended to the system of cuts to update the set Q. The procedure thus continues until the extreme faces (and hence the extreme points) of X_0 relative to some Q are all deleted.

3. Development of the disjunctive face cut

Suppose that the procedure of the foregoing section terminates with an extreme face F_Z of X_0 relative to Q and a point $x^0 = (x_1^0, \dots, x_n^0)$ on this face which is not an extreme point of X_0. Since we are primarily interested in extreme points of X_0 (recall the extreme point optimality property of Problem BLP), we may introduce the disjunction that at least one of the basic key variables which is currently positive must be zero in order to yield an extreme point of X_0 feasible to Q, if such a point exists. Accordingly, define

$$B_N = \{r \in N : x_r^0 > 0\}. \tag{3.1}$$

Further, let

$$Z = \{j \in N : x_j \text{ is currently nonbasic}\}. \tag{3.2}$$

Now consider an index $r \in B_N$ and suppose Problem P_r is solved. At optimality, let

$$\begin{aligned} &N_r = \{j \in N : x_j \text{ is nonbasic}\} \\ \text{and}\quad &S_r = \{j \in S : x_j \text{ is nonbasic}\} \end{aligned} \tag{3.3}$$

and let the canonical representation of x_r in terms of the nonbasic variables x_j, $j \in N_r \cup S_r$ be

$$x_r + \sum_{j \in N_r} a_{rj} x_j + \sum_{j \in S_r} a_{rj} x_j = b_r.$$

Since $N_r \subset Z$, by adding suitable zero coefficients, the above equation can be written as

$$x_r + \sum_{j\in Z} a_{rj}x_j + \sum_{j\in S_r} a_{rj}x_j = b_r. \tag{3.4}$$

Note that Problem P_r minimizes x_r, and hence a_{rj} is the coefficient of x_j in the objective function at optimality. Hence, even with the restricted basis entry rule, we may stipulate that

$$a_{rj} \leq 0 \quad \text{for } j \in S_r. \tag{3.5}$$

Also, note that if $b_r = 0$ and $a_{rj} \neq 0$ for some $j \in S_r$ implies that x_r could be made nonbasic. Hence $b_r = 0$ implies $a_{rj} = 0$ for $j \in S_r$. Thus, under the restricted basis entry rule, no pivot will ever occur in the row of x_r and moreover, at termination, we will have $x_r^0 = 0$, contradicting $r \in B_N$. Hence,

$$b_r > 0 \quad \text{for } r \in B_N. \tag{3.6}$$

Rewriting Eq. (3.4) for each $r \in B_N$, and suitably defining zero coefficients, we may state the disjunction $x_r \leq 0$ for at least one $r \in B_N$ as:

The following set of inequalities must hold for at least one $r \in B_N$

$$\left\{\sum_{j\in Z}\left(\frac{a_{rj}}{b_r}\right)x_j + \sum_{j\in S_N}\left(\frac{a_{rj}}{b_r}\right)x_j \geq 1,\ x_j \geq 0 \text{ for } j \in Z \cup S_N\right\} \tag{3.7}$$

where $S_N = \bigcup_{r\in B_N} S_r$.

Now, for any $r \in B_N$, if $a_{rj} \leq 0$ for all $j \in Z$, then the corresponding system in Eq. (3.7) is inconsistent and we may delte it. Let B_N^* denote the set of consistent systems. If $B_N^* = \{\emptyset\}$, then clearly no extreme point of X_0 feasible to Q exists and we may terminate. Otherwise, according to the development of Owen [16] or more generally (and more rigorously), that of Balas [3] or Glover [7] or Jeroslow [11], the following is a valid cut

$$\sum_{j\in Z}\left\{\max_{r\in B_N^*}\left(\frac{a_{rj}}{b_r}\right)\right\}x_j + \sum_{j\in S_N^*}\left\{\max_{r\in B_N^*}\left(\frac{a_{rj}}{b_r}\right)\right\}x_j \geq 1 \tag{3.8}$$

where $S_N^* = \bigcup_{r\in B_N^*} S_r$. We note that the cut (3.8) may be strengthened in the following manner through the use of a result due to Glover (see [7, Theorem 2]). Rewriting with obvious notation the consistent inequalities in (3.7) as $\sum_{j\in Z\cup S_N^*} \alpha_{rj}x_j \geq 1$ for each $r \in B_N^*$ and inequality (3.8) as $\Sigma\ \alpha_j^* x_j \geq 1$, let us define

$$\gamma_r = \underset{j:\,\alpha_{rj}>0}{\text{minimum}}\{\alpha_j^*/\alpha_{rj}\} \quad \text{for each } r \in B_N^*.$$

Then, using Glover's [7] results, one may easily validate the following inequality or cut

$$\sum_{j\in Z\cup S_N^*}\left[\underset{r\in B_N^*}{\text{maximum}}\{\gamma_r\alpha_{rj}\}\right]x_j \geq 1. \tag{3.9}$$

Sherali and Shetty [17] have shown that (3.9) dominates any valid inequality based on the disjunctive statement (3.7) since (3.9) defines a facet of the closure of the convex hull of

$$\bigcup_{r\in B_N^*}\left\{x\colon \sum_{j\in Z\cup S_N^*} \alpha_{rj}x_j \geq 0,\ x\geq 0\right\}.$$

Now, consider the following assertion.

Lemma 3.1. *The disjunctive cut of Eq.* (3.9) *deletes the extreme face F_Z of X_0 relative to the set Q.*

Proof. Since the cut (3.9) uniformly dominates the cut (3.8) (see [7]), it is sufficient to show that (3.8) deletes the extreme face F_Z of X_0 relative to Q. From Eq. (3.5), the cut of Eq. (3.8) implies

$$\sum_{j\in Z}\left\{\max_{r\in B_N^*}\left(\frac{a_{rj}}{b_r}\right)\right\}x_j \geq 1.$$

Further, by the definition of B_N^*, for each $r\in B_N^*$, $a_{rj}>0$ for some $j\in Z$, and hence, the coefficients of the above cut are not all nonpositive. The proof follows from the fact that for any $x\in F_Z$, $x_j=0$ for each $j\in Z$. It is worthwhile noting again that since the cut (3.9) has coefficients of both signs, problems associated with dual degeneracy tend to be reduced [3]. Also, the reader may note that Majthay and Whinston [15] have also proposed some facet cuts of the type $\sum_{j\in Z} p_jx_j \geq p$ where the coefficients $p_j>0$, $j\in Z$ are prechosen and the cut generation routines attempt to make p as large as possible. However, since we are only interested in deleting F_Z without regard to the depth of the cut, we have elected to use above an efficient scheme which will simply accomplish this for us at a low computational cost. Finally, the reader may observe that a similar valid disjunctive cut may be generated in an even more straightforward manner when one obtains an extreme face of X_0 which is also an extreme point of X_0. Despite this fact, we prefer to expend more effort and generate significantly deeper cuts, if possible, by finding intersection points of suitable positive and negative edge extensions with the boundaries of the polar set, rather than with the boundaries of the smaller set which is the closure of the complement of the union of the disjunctive sets. This is the subject of the next section.

4. Development of the negative-edge extension polar cut

In this section, we assume that we have found an extreme point x^0 of X_0 feasible to the set Q. In order to present the cutting plane techniques, we will first introduce two concepts. The first is merely an artifice in the implementation of the procedure. It exhibits the relationship of the procedure with convexity cuts [7] and is based on the generalized reverse polar of a set [5, 6]. The second concept is more crucial to our development, and concerns what we call a weak pseudo-global minimum as opposed to the pseudo-global minimum discussed in [22].

Definition 4.1. Given a set Y_0 and a scalar α, the *generalized reverse polar* of Y_0 relative to α is the set

$$Y_0(\alpha) = \{x \in R^n: \min_{y \in Y_0} \phi(x, y) \geq \alpha\} \tag{4.1}$$

where as before,

$$\phi(x, y) = c^t x + d^t y + x^t C y.$$

Note that $Y_0(\alpha)$ is a polyhedral set; in fact, we may rewrite $Y_0(\alpha)$ as

$$Y_0(\alpha) = \bigcap_{i=1}^{u} H_i^+(x) = \bigcap_{i=1}^{u} \{x \in R^n: c^t x + d^t y^i + x^t C y^i \geq \alpha\} \tag{4.2}$$

where $y^1, \dots, y^u$ are the u extreme points of the polytope Y_0 and $H_i^+(x)$, $i = 1, \dots, u$ are the corresponding u closed halfspaces.

The basic idea behind the cutting plane method is to let α be the current best objective function value (CBOFV) and to define a cutting plane which will delete as large a subset of $Y_0(\alpha) \cap X_0$ as possible. In order to be able to develop a cutting plane which deletes the extreme point from which it is generated but no point in X_0 which does not belong to $Y_0(\alpha)$, we will find it expeditious to generate it from an extreme point of X_0 which is a weak pseudo-global minimum. Theorem 4.1 below establishes the validity of the cut. First consider the following definition where $A(\bar{x})$ represents the set of extreme points of X_0 adjacent to the extreme point of $\bar{x}$ of X_0.

Definition 4.2. Let Q be the region feasible to the s cuts generated thus far. Further, let $(\bar{x}, \bar{y})$ be an extreme point of $X_0 \times Y_0$ such that $\bar{x} \in Q$ and $\min_{y \in Y_0} \phi(\bar{x}, y) = \phi(\bar{x}, \bar{y})$. Consider a basis B of (1.1) representing $\bar{x}$. Then $(\bar{x}, \bar{y})$ is said to be a *weak pseudo-global minimum (WPGM)* relative to the basis B if for each $\hat{x} \in A(\bar{x})$ such that $\hat{x} \in Q$ and $\hat{x}$ is obtainable from $\bar{x}$ through a single pivot on B, we have $\min_{y \in Y_0} \phi(\hat{x}, y) \geq \phi(\bar{x}, \bar{y})$. Note that whenever a particular basis B is not of relevance to us, we will simply call $(\bar{x}, \bar{y})$ as a WPGM.

Now, let $(\bar{x}, \bar{y})$ be a WPGM. Consider the extended simplex tableau (in Tucker form) corresponding to the extreme point solution $\bar{x}$. Identify the p (i.e., n minus the number of rows in E of Eq. (1.1)) edges incident at $\bar{x}$ corresponding to the p nonbasic variables $x_j, j \in J$, where $J \subset N$ denotes the set of indices of the nonbasic variables, all of which are currently key variables. Note that X_0 is contained in the cone defined by these edges with vertex at $\bar{x}$. Let e^j be the extended column of the nonbasic variable $j \in J$ with components representing the negative rate of change of the n key x-variables with x_j. Denote the half-lines emanating from $\bar{x}$ along the above p edges by

$$\zeta^j = \{x\colon\ x = \bar{x} - e^j\lambda_j,\ \lambda_j \geq 0\} \quad \text{for } j \in J. \tag{4.3}$$

Let α be the CBOFV and let

$$J_1 = \{j \in J\colon\ \min_{y \in Y_0} \phi(x, y) < \alpha\} \quad \text{for some } x \in \zeta^j \tag{4.4}$$

and let

$$J_2 = J - J_1. \tag{4.5}$$

Finally, let

$$\hat{\lambda}_j = \text{supremum}\{\lambda_j\colon\ \phi(\bar{x} - e^j\lambda_j, y) \geq \alpha \text{ for all } y \in Y_0\} \quad \text{for } j \in J, \tag{4.6}$$

$$\hat{\hat{\lambda}}_j = \text{supremum}\{\lambda_j\colon\ \phi(\bar{x} + e^j\lambda_j, y) \geq \alpha \text{ for some } y \in Y_0\} \quad \text{for } j \in J_2 \tag{4.7}$$

and set

$$\bar{\lambda}_j = \begin{cases} \hat{\lambda}_j & \text{if } j \in J_1, \\ -\hat{\hat{\lambda}}_j & \text{if } j \in J_2. \end{cases} \tag{4.8}$$

Before proceeding, based on the above notation, let us explain the conceptual idea behind the cutting plane generation. Given a WPGM $(\bar{x}, \bar{y})$ and the CBOFV α, we consider the polyhedral cone with vertex at $\bar{x}$ and edges ζ^j, $j \in J$. We then move a distance $\hat{\lambda}_j$ along each of these edges until we intersect a facet of $Y_0(\alpha)$. This is so because from Eq. (4.6) and the definition of $Y_0(\alpha)$,

$$\hat{\lambda}_j = \text{supremum}\{\lambda_j\colon (\bar{x} - e^j\lambda_j) \in Y_0(\alpha)\} \quad \text{for } j \in J. \tag{4.9}$$

If $\hat{\lambda}_j$ is finite and non-zero, then we continue with the cut generation procedure. However, if $\hat{\lambda}_j = 0$ for some j, then we abort the polar cut and develop a disjunctive cut at $\bar{x}$ by using the rows of the basic, positive-valued key variables in the current tableau as Eqs. (3.7), with $S_N = \{\emptyset\}$. Observe that, by virtue of $(\bar{x}, \bar{y})$ being a WPGM, this situation can arise only if the edge under consideration leads to a degenerate pivot, the objective value of $(\bar{x}, \bar{y})$ is equal to α and $\zeta^j \cap Y_0(\alpha) = \bar{x}$. Empirical results indicate that this situation rarely arises.

Now, if $\hat{\lambda}_j = \infty$ for any $j \in J$, i.e., if $\zeta^j \subset Y_0(\alpha)$, then we have $j \in J_2$ and we consider the negative extension of ζ^j from $\bar{x}$. We move as far as we can $(\hat{\hat{\lambda}}_j)$ along this negative extension, so long as we still lie in at least one half space $H_i^+(x)$ defining $Y_0(\alpha)$ (Eq. (4.2)). Theorem 4.1 below establishes that if any $\hat{\hat{\lambda}}_j = 0$ for $j \in J_2$, then we may terminate with the current best solution as optimal. In this manner, provided $\hat{\lambda}_j > 0, j \in J$, we identify p distinct points of intersection. Since in terms of the current nonbasic variables, the polytope X_0 is imbedded in $\boldsymbol{R}^p$, these p intersection points define a unique hyperplane in $\boldsymbol{R}^p$ as

$$\sum_{j \in J} (x_j/\bar{\lambda}_j) = 1.$$

Theorem 4.1 below either verifies optimality or specifies a *valid cut* which deletes the point $\bar{x}$ but no point $\hat{x}$ such that $\min_{y \in Y_0} \phi(\hat{x}, y) < \alpha$.

Theorem 4.1. *Let Q be the region feasible to the s cuts generated thus far, let α be the CBOFV, and let $(\bar{x}, \bar{y})$ be a WPGM. Further, assume that in Eq.* (4.6), $\hat{\lambda}_j > 0$ *for each $j \in J$.*

(a) *If $J_1 = \{\emptyset\}$, then a feasible solution yielding the objective function value α is optimal.*

(b) *If $J_1 \neq \{\emptyset\}$, then $\bar{\lambda}_j \neq 0$ for each $j \in J$.*

(c) *If $J_1 \neq \{\emptyset\}$, then a valid cut is given by*:

$$\sum_{j \in J} (x_j / \bar{\lambda}_j) \geq 1 \tag{4.10}$$

where x_j, $j \in J$ are the nonbasic variables.

Proof. Under the hypothesis $\hat{\lambda}_j > 0$ for each $j \in J$, $J_1 = \{\emptyset\}$ clearly implies that $X_0 \subset Y_0(\alpha)$. This proves part (a).

Hence, suppose that $J_1 \neq \{\emptyset\}$ and consider any $j \in J_2$, which implies that $\hat{\lambda}_j = \infty$. Since $J_1 \neq \{\emptyset\}$, there exists a $q \in J_1$ and a point (x^q, y^q), $x^q \in \zeta^q$ and $y^q \in Y_0$ such that

$$\phi(x^q, y^q) = \alpha \quad \text{and} \quad \phi(\hat{x}, y^q) < \alpha \quad \text{for some } \hat{x} \in \zeta^q. \tag{4.11}$$

Observe that $\phi(\bar{x}, y^q) \geq \min_{y \in Y_0} \phi(\bar{x}, y) \geq \alpha$. Then for points of the form $\gamma x^q + (1-\gamma)\bar{x}$, we get from Eq. (4.11),

$$\phi(\gamma x^q + (1-\gamma)\bar{x}, y^q) = \gamma \phi(x^q, y^q) + (1-\gamma)\phi(\bar{x}, y^q) = \alpha.$$

In other words, $\phi(x, y^q) = \alpha$ for all $x \in \zeta^q$, contradicting (4.11). Hence $\phi(\bar{x}, y^q) > \alpha$ and by continuity of ϕ for a fixed $y = y^q$, we conclude that $\hat{\hat{\lambda}}_j > 0$ for $j \in J_2$. From (4.8) then, we have $\bar{\lambda}_j \neq 0$.

Finally, part (c) follows from Glover [7, Theorem 2] by noting that the only use Glover makes of his assumption, in our terminology, that $\bar{x} \in$ interior $Y_0(\alpha)$, is to ensure $\bar{\lambda}_j \neq 0$ for any $j \in J$. This completes the proof.

In order to complete our presentation, we need to show:

(i) How to find a WPGM from an extreme point x^0 of X_0 feasible to Q.

(ii) How to solve for the parameters $\bar{\lambda}_j$, $j \in J$ of Eq. (4.8). These aspects are considered in the next two subsections.

4.1. Determination of a WPGM

Suppose $Q \subseteq R^n$ is the region feasible to the s cuts generated thus far and further, suppose that x^0 is the extreme point of X_0 detected extreme face of X_0 relate to Q. Recall that our need for finding a WPGM was to ensure that $\hat{\lambda}_j > 0$ for at least those $j \in J$ which correspond to non-degenerate pivots in the current simplex tableau. Besides, this forms an improvement routine for detecting better quality solutions. Accordingly, given a simplex tableau representing an extreme point x^e of X_0, let $\bar{A}(x^e)$ denote the set of extreme points of X_0 which are accessible from x^e through single, non-degenerate pivots on this tableau. Observe that if $x^e \in Q$, the set $A(x^e) \cap Q$ is easily obtained from the current tableau as points resulting from single non-degenerate pivots which involve the exchange of a key-variable for another key-variable. Hence, consider the following procedure.

Initialization

Let $k = 0$, and go to Step 1.

Step 1: Let $\hat{x} \in \bar{A}(x^k) \cap Q$ be such that

$$\min_{y \in Y_0} \phi(\hat{x}, y) < \min_{y \in Y_0} \phi(x^k, y) = \phi(x^k, y^k).$$

If no such point exists, (x^k, y^k) is a WPGM. Otherwise, go to Step 2.

Step 2: Let $\hat{x} = x^{k+1}$, increment k by 1 and return to Step 1.

The algorithm clearly yields a WPGM. Moreover, since the cardinality of $\bar{A}(x^e)$ is finite for any basic representation of an extreme point x^e of X_0, and since every pass through Step 2 results in a strict decrease in the objective function value, the procedure is finitely convergent. Note that for the first iteration when $s = 0$, it is obviously advantageous, but not necessary, to let x^0 be a local star minimum. A local star minimum can be located by solving the problems $\min_{y \in Y_0} \phi(x, y)$ and $\min_{y \in Y_0} \phi(x, y)$ iteratively as, for example, in [22].

4.2. Determination of the parameters $\bar{\lambda}_j$, $j \in J$ of Eq. (4.8)

To specify a valid cut through (4.6), (4.7) and (4.8) we need to compute $\hat{\lambda}_j$ or $\hat{\hat{\lambda}}_j$ for each $j \in J$. These quantities may be determined by solving the following parametric problems.

Problem PAR 1.

$$\hat{\lambda}_j = \text{supremum}\{\lambda : \psi_j(\lambda) \geq \alpha\} \tag{4.12}$$

where

$$\psi_j(\lambda) = \min_{y \in Y_0} \phi(\bar{x} - e^j\lambda, y). \tag{4.13}$$

Problem PAR 2.

$$\hat{\hat{\lambda}}_j = \text{supremum}\{\lambda : \bar{\psi}_j(\lambda) \geq \alpha\} \tag{4.14}$$

where

$$\bar{\psi}_j(\lambda) = \max_{y \in Y_0} \phi(\bar{x} + e^j\lambda, y). \tag{4.15}$$

It is easy to show that $\psi_j(\cdot)$ and $-\bar{\psi}_j(\cdot)$ of Eqs. (4.13), (4.15) are piecewise linear and concave with breakpoints occurring whenever the extreme point solution optimizing $\psi_j(\cdot)$ (or $\bar{\psi}_j(\cdot)$) changes. We will now describe an efficient modification of Newton's procedure to solve for $\hat{\lambda}_j$ and $\hat{\hat{\lambda}}_j$. The effectiveness of this scheme over the Bolzano search procedure was demonstrated for a special case of the bilinear program in [19], and is again evident from the computational experience of Section 7.

Consider the system of Eqs. (4.12), (4.13) for obtaining $\hat{\lambda}_j$. Given an extreme point y^i of Y_0, the slope of the corresponding linear portion of $\psi_j(\cdot)$ is easily obtained through Eq. (4.13) as

$$m_i = -(c^t e^j + (e^j)^t C y^i). \tag{4.16}$$

Hence, we may proceed according to the following algorithm. Fig. 4.1 below illustrates a typical sequence of steps taken by this procedure.

Initialization

Set $k = 1$ and $\lambda^1 = L$, a large number. Determine $\psi_j(L)$ and let y^1 be a corresponding minimizing extreme point solution. If $\psi_j(L) \geq \alpha$, then $j \in J_2$ and

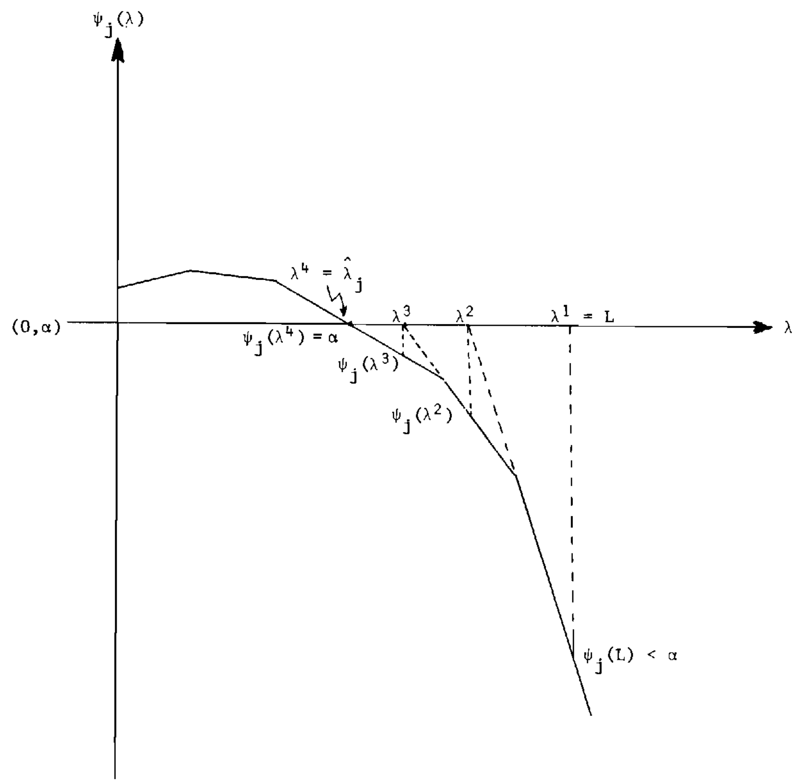

Fig. 4.1

we need to solve the Eqs. (4.14), (4.15) for $\hat{\lambda}_j$. Else, go to Step 1.

Step 1: Determine the slope m_k through Eq. (4.16) corresponding to y^k, and compute

$$\lambda^{k+1} = (\alpha - \psi_j(\lambda^k) + m_k\lambda^k)/m_k. \tag{4.17}$$

If $\lambda^{k+1} = 0$, abort the present cut generation scheme and develop a disjunctive cut at $\bar{x}$ as in Section 3. Otherwise, proceed to Step 2.

Step 2: Determine $\psi_j(\lambda^{k+1})$ and a corresponding minimizing extreme point solution y^{k+1}. If $\psi_j(\lambda^{k+1}) = \alpha$, terminate with $\hat{\lambda}_j = \lambda^{k+1}$. Else, $\psi_j(\lambda^{k+1}) < \alpha$ and incrementing k to $k+1$, return to Step 1.

Finite convergence of the above procedure is guaranteed by the fact that each pass through Steps 1 and 2 leads to a different linear part of $\psi_j(\cdot)$ with a strictly smaller slope and the number of such linear portions are finite since Y_0 is a polytope.

The method for solving Eqs. (4.14), (4.15) for $\hat{\hat{\lambda}}_j$, $j \in J_2$ is identical to the above procedure except that if $\bar{\psi}_j(L) \geq \alpha$ in the Initialization step, then we set $\hat{\hat{\lambda}}_j = \infty$ and also, Eq. (4.17) in Step 1 is replaced by

$$\lambda^{k+1} = (\alpha - \bar{\psi}_j(\lambda^k) - m_k\lambda^k)/(-m_k). \tag{4.18}$$

Finally in Step 2, $\psi_j(\lambda^{k+1}) < \alpha$ should be replaced by $\bar{\psi}_j(\lambda^{k+1}) > \alpha$.

Before proceeding, we address briefly the special case $X_0 = Y_0$ which arises, for example, when minimizing a concave quadratic function subject to linear constraints [14]. In this situation, at Step 2 of the above scheme, one may check the objective value of the solution (y^{k+1}, y^{k+1}). If this solution is better than the current best, then one may update α, and the cut generation may be re-initialized at the new solution. As shown by Balas and Burdet [4], when a cut is finally obtained in this manner, the corresponding intersection points are defined with respect to, what they call, a reverse outer polar, and leads to a deepening of the cuts. Konno [13, 14] has also demonstrated how his cuts may be strengthened in a similar context.

We will now state a schema for solving Problem BLP.

5. A finitely convergent algorithm for Problem BLP

Initialization

Set $Q = R^n$, $\alpha = \infty$.

Step 1: Determine an extreme face of X_0 relative to Q. If none exists (i.e., all key variables are basic), then terminate with the current best solution as optimal. On the other hand, if this extreme face is not an extreme point of X_0, go to Step 2. Otherwise, go to Step 3.

Step 2: Generate a face cut of the form in Eq. (3.7). If $B_N^* = \{\emptyset\}$, terminate with the current best solution as optimal. Otherwise, augment the set Q with this cut and return to Step 1.

Step 3: Starting from this extreme point of X_0, determine a WPGM and update the current best solution and its value, if necessary, using this WPGM. Solve the parametric Problems PAR1 and PAR2 at the WPGM. If $\hat{\lambda}_j = 0$ for any $j \in J$, then go to Step 2. If either $J_1 = \{\emptyset\}$ or if $J_1 \neq \{\emptyset\}$ and $\hat{\hat{\lambda}}_j = 0$ for some $j \in J_2$, terminate with the current best solution as optimal. Otherwise, generate a negative edge extension polar cut (4.10). Augment the set Q with this cut and return to Step 1.

Observe that Steps 2 and 3 are finite procedures and they delete an extreme face of X_0 determined in Step 1. Since an extreme face of X_0 relative to any Q is also a face of X_0 and since the faces of X_0 are finite, the proposed scheme is finitely convergent.

Before concluding this section, we point out that in some cases (for example, see [19]) it is possible to directly locate an extreme point of X_0 feasible to Q due to some special structure in the problem. Such a possibility must be first sought out in any application as it obviously leads to significant simplifications.

6. Illustrative example

Below, we illustrate our procedure by solving an example both, graphically by explicitly using the polar sets, and also by the scheme prescribed by the proposed algorithm. The example also illustrates how the cutting planes are deeper than those of Vaish and Shetty [22].

Hence, consider the problem

$$\text{minimize} \quad \phi(x, y) = (2x_1 - x_2)y_1 + (2x_2 - 3x_1)y_2 + (8y_1 - 6y_2),$$

subject to

$$x \in X_0 = \left\{ \begin{array}{rl} x: & -2x_1 + 5x_2 \le 18 \\ & -3x_1 - 2x_2 \le -11 \\ & -x_2 \le -1 \\ & 3x_1 + 2x_2 \le 62 \\ & 2x_1 + 12x_2 \le 84 \\ & x_1, x_2 \ge 0 \end{array} \right\}, \quad y \in Y_0 = \left\{ \begin{array}{rl} y: & -y_1 + y_2 \le -1 \\ & -3y_1 + 4y_2 \le -1 \\ & 4y_1 - 5y_2 \le 3 \\ & y_1, y_2 \ge 0 \end{array} \right\}.$$

Note that here, although the key-variables are x_1, x_2 and the slack variables in the constraints of X_0, for brevity we will simply write x as $(x_1, x_2)^t$. Accordingly, $c = (0, 0)^t$, $d = (8, -6)^t$ and

$$C = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Let us start with the point $x^1 = (1,4)^t$ Solving

$$\min_{y \in Y_0} \phi(x^1, y) \equiv \min_{y \in Y_0} [6y_1 - y_2]$$

we obtain a value of 11 at $y^1 = (2, 1)^t$. Testing the adjacent extreme points, it is easily verified that this represents a WPGM. Hence, currently, $\alpha = 11$.

We will now generate a cutting plane from the point $(1,4)^t$. Eq. (4.2) defines $Y_0(11)$ through the constraints

$$x_1 \geq 1, \qquad x_2 \geq -1 \quad \text{and} \quad -x_1 + 3x_2 \geq -15.$$

Fig. 6.1 shows the polar set superimposed on X_0 and exhibits the intersection points $P = (\frac{13}{3}, -1)^t$ and $W = (-129, -48)^t$ for the polar cut. (Note that W is obtained through a negative edge extension.) The cut defined by these intersection points is

$$141x_1 - 400x_2 \geq 1011. \tag{6.1}$$

Observe that Vaish and Shetty [22] would have obtained intersection points at $P = (\frac{13}{3}, -1)$ and at infinity along the ray through $(6, 6)^t$. Hence, their cut would have been

$$6x_1 - 15x_2 \geq 41. \tag{6.2}$$

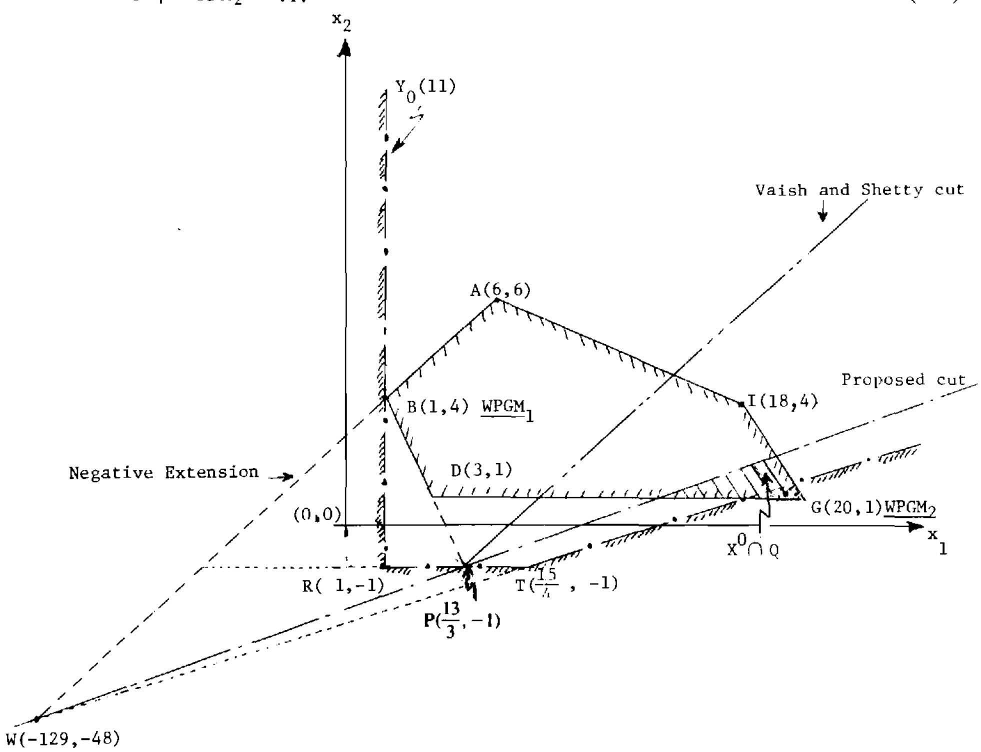

Fig. 6.1

Note that (6.1), unlike (6.2), deletes the extreme point $(18,4)^t$ of X_0.

We will now demonstrate how to generate the cutting plane without the explicit use of the polar set, i.e., without having to enumerate the extreme points of Y_0.

At the extreme point $(1,4)^t$ the current tableau has the basis and its inverse as given below, where $x_3, \dots, x_7$ are the slack variables in the constraints of X_0.

$$B = \left[\begin{array}{cc|ccc} x_1 & x_2 & x_5 & x_6 & x_7 \\ -2 & 5 & & & \\ -3 & -2 & & 0 & \\ \hline 0 & -1 & & & \\ 3 & 2 & & I & \\ 2 & 12 & & & \end{array}\right], \qquad B^{-1} = \left[\begin{array}{cc|c} -\frac{2}{19} & -\frac{5}{19} & \\ \frac{3}{19} & -\frac{2}{19} & 0 \\ \hline \frac{3}{19} & -\frac{2}{19} & \\ 0 & 1 & I \\ -\frac{32}{19} & \frac{34}{19} & \end{array}\right].$$

The nonbasic variables for which we have to compute $\bar{\lambda}$, are x_3 and x_4. For the sake of illustration, we will determine $\bar{\lambda}_4$ for x_4 since this involves a negative edge extension and let the reader verify that $\bar{\lambda}_3$ for x_3 is $\frac{95}{3}$ (i.e., $\{3\} \in J_1$).

$\bar{\lambda}_4$ *for* x_4: From B^{-1},

$$\phi(\bar{x} - e^4\lambda, y) = (6 + \tfrac{8}{19}\lambda)y_1 - (1 + \tfrac{11}{19}\lambda)y_2.$$

For $\lambda^1 = L$,

$$\psi_4(L) = \min_{y \in Y_0}[(6 + \tfrac{8}{19}L)y_1 - (1 + \tfrac{11}{19}L)y_2] = (37 + \tfrac{1}{19}L) > \alpha$$

and hence, $\{4\} \in J_2$.

Now,

$$\phi(\bar{x} + e^4\lambda, y) = (6 - \tfrac{8}{19}\lambda)y_1 - (1 - \tfrac{11}{19}\lambda)y_2$$

along the negative-edge extension. For $\lambda^1 = L$,

$$\bar{\psi}_4(L) = \max_{y \in Y_0}[(6 - \tfrac{8}{19}L)y_1 - (1 - \tfrac{11}{19}L)y_2] = (37 - \tfrac{1}{19}L) < \alpha$$

at $y^1 = (7,5)^t$.

Step 1: $m_1 = -e_4^t C\bar{y} = \frac{1}{19}$ and $\lambda^2 = (11 - 37 + \frac{1}{19}L - \frac{1}{15}L)/(-\frac{1}{19}) = 494$.

Step 2: $\bar{\psi}_4(494) = \max_{y \in Y_0}[-202y_1 + 285y_2] = 11 = \alpha$ at $y^2 = (7,5)^t$.

Thus $\hat{\lambda}_4 = 494$ or $\bar{\lambda}_4 = -494$.

The cut is hence $x_3/(\frac{95}{3}) - x_4/(494) \ge 1$ or in terms of x_1, x_2,

$$141x_1 - 400x_2 \ge 1011 \quad \text{as in Eq. (6.1).}$$

Note that the polar cut of Vaish and Shetty [22] would have found $\bar{\lambda}_3 = \frac{95}{3}$, $\bar{\lambda}_4 = \infty$ and hence would have been

$$x_3/(95/3) \ge 1 \quad \text{or} \quad 6x_1 - 15x_2 \ge 41$$

as in Eq. (6.2).

It is interesting to note that the face cut available at the extreme point $(1,4)$ of X_0 is $\frac{1}{19}x_3 + \frac{1}{19}x_4 \geq 1$ or $5x_1 - 3x_2 \geq 12$ which is uniformly dominated by the cuts of both Eqs. (6.1) and (6.2).

Now, $X_0 \cap Q$ is as shown in Fig. 6.1. The extreme face finding routine easily leads to the extreme point $x^0 = (20, 1)^t$ of X_0 which also represents a WPGM of value 9 along with $\bar{y} = (7, 5)^t$. One may verify that the polar cut generated from this point with $\alpha = 9$ exhausts the feasible region. The current best solution $x_1 = 20$, $x_2 = 1$, $y_1 = 7$, $y_2 = 5$ is hence optimal.

7. Computational experience

The proposed algorithmic scheme was coded in FORTRAN IV. Table 7.1 gives our computational experience on the CDC CYBER 70 Model 74-28/CDC 6400 computer using test problems in the literature. The first four problems are the illustrative examples taken from the references indicated in the first column. Problems 5 through 8 are the special structured test problems of Konno [14]. These problems have as many local minima equal to global minima as the number of constraints. Also, the sets X_0 and Y_0 are identical. Konno's method [13, 14] takes advantage of this structure and as discussed earlier, yields deeper cuts and leads to computational savings. Since our procedure was not specialized to handle such situations, we did not attempt to solve larger sized problems of this class. Finally, problems 9, 10 and 11 are the test problems of Zwart [25]. As observed in [26], these problems are of medium level of difficulty. These problems also have $X_0 = Y_0$, and again, no advantage was taken of this structure. The reader may note that, in addition to the proposed scheme of generating both polar and disjunctive cuts, we have attempted to use only polar cuts whenever possible, irrespective of whether the extreme face obtained at any iteration is an extreme point of X_0 are not. Computational results presented support the contention that polar cuts are generally deeper than our disjunctive cuts if the former can be used. Thus the disjunctive cuts serve the purpose of ensuring that a cut can always be generated and that the procedure is finite.

Finally, we have also recorded the average number of iterations required for the solution of the parametric Problems PAR1 and PAR2. Our experience clearly indicates the advantage of using the proposed search technique over, say, the Balzano bisection search procedure.

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Table 7.1

Problem						Use of polar and disjunctive cuts*						Use of polar cuts alone				
Index no.	Source	Size n	Number of X constraints	m	Number of Y constraints	a	b	c	d	e	f	a	c	e	f	g
1	Section 6	7	5	5	3	2	0	0.0055	—	0.023	1.25	2	0.0055	0.023	1.25	0
2	Konno [13]	5	3	5	3	1	0	0.0060	—	0.012	1.00	1	0.0060	0.012	1.00	0
3	Konno [14]	6	4	6	4	3	0	0.0076	—	0.053	1.33	3	0.0076	0.053	1.33	0
4	Zwart [24]	4	2	4	2	1	0	0.0020	—	0.010	1.00	1	0.0020	0.010	1.00	0
5	Konno [14]	4	2	4	2	2	0	0.0040	—	0.018	1.00	2	0.0040	0.018	1.00	0
6	Konno [14]	6	3	6	3	3	0	0.0080	—	0.050	1.22	3	0.0080	0.050	1.22	0
7	Konno [14]	8	4	8	4	5	8	0.0156	0.0135	0.484	1.40	10	0.0168	0.352	1.45	0
8	Konno [14]	10	5	10	5	8	42	0.0459	0.1014	10.988	1.825	21	0.039	2.383	1.67	0
9	Zwart [25]	10	5	10	5	1	0	0.0220	—	0.058	1.00	1	0.0220	0.058	1.00	0
10	Zwart [25]	20	10	20	10	4	0	0.4760	—	4.780	1.025	19	0.3950	11.243	1.00	0
11	Zwart [25]	40	20	40	20	8	92	5.4625	1.3923	372.375	1.025	35	5.525	235.82	1.34	1

[a] Number of polar cuts.
[b] Number of disjunctive cuts.
[c] Average time (cpu seconds) for polar cuts.
[d] Average time (cpu seconds) for disjunctive cuts.
[e] Total execution time (cpu seconds).
[f] Average number of iterations for solving the parametric problems PAR1 or PAR2.
[g] Number of times a polar cut was aborted and a disjunctive cut was generated at an extreme point of X_0.
* Problems 10 and 11 were run with this technique using "ϵ/ϕ_{max}" = 0.05 in Konno's [13, 14] terminology, with Problem 11 being prematurely terminated after total of 100 cuts.

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ON THE GENERATION OF DEEP DISJUNCTIVE CUTTING PLANES*

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ABSTRACT

In this paper we address the question of deriving deep cuts for nonconvex disjunctive programs. These problems include logical constraints which restrict the variables to at least one of a finite number of constraint sets. Based on the works of Balas, Glover, and Jeroslow, we examine the set of valid inequalities or cuts which one may derive in this context, and defining reasonable criteria to measure depth of a cut we demonstrate how one may obtain the "deepest" cut. The analysis covers the case where each constraint set in the logical statement has only one constraint and is also extended for the case where each of these constraint sets may have more than one constraint.

1. INTRODUCTION

A Disjunctive Program is an optimization problem where the constraints represent logical conditions. In this study we are concerned with such conditions expressed as linear constraints. Several well-known problems can be posed as disjunctive programs, including the zero-one integer programs. The logical conditions may include conjunctive statements, disjunctive statements, negation and implication as discussed in detail by Balas [1,2]. However, an implication can be restated as a disjunction, and conjunctions and negations lead to a polyhedral constraint set. Thus, this study deals with the harder problem involving disjunctive restrictions which are essentially nonconvex problems.

It is interesting to note that disjunctive programming provides a powerful unifying theory for cutting plane methodologies. The approach taken by Balas [2] and Jeroslow [14] is to characterize all valid cutting planes for disjunctive programs. As such, it naturally leads to a statement which subsumes prior efforts at presenting an unified theory using convex sets, polar sets and level sets of gauge functions [1,2,5,6,8,13,14]. On the other hand, the approach taken by Glover [10] is to characterize all valid cutting planes through relaxations of the original disjunctive program. Constraints are added sequentially, and when all the constraints are considered Glover's, result is equivalent to that of Balas and Jeroslow. Glover's approach is a constructive procedure for generating valid cuts, and may prove useful algorithmically.

The principal thrust of the methodologies of disjunctive programming is the generation of cutting planes based on the linear logical disjunctive conditions in order to solve the corresponding nonconvex problem. Such methods have been discussed severally by Balas [1,2,3], Glover [8], Glover, Klingman and Stutz [11], Jeroslow [14] and briefly by Owen [17]. But the most fundamental and important result of disjunctive programming has been stated by

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Balas [1,2] and Jeroslow [14], and in a different context by Glover [10]. It unifies and subsumes several earlier statements made by other authors and is restated below. This result not only provides a basis for unifying cutting plane theory, but also provides a different perspective for examining this theory. In order to state this result, we will need to use the following notation and terminology.

Consider the linear inequality systems S_h, $h \in H$ given by

$$S_h = \{x\colon A^h x \geqslant b^h,\ x \geqslant 0\},\ h \in H \tag{1.1}$$

where H is an appropriate index set. We may state a disjunction in terms of the sets S_h, $h \in H$ as a condition which asserts that a feasible point must satisfy at least one of the constraint S_h, $h \in H$. Notationally, we imply by such a disjunction, the restriction $x \in \bigcup_{h \in H} S_h$. Based on this disjunction, an inequality $\pi' x \geqslant \pi_0$ will be considered a *valid inequality* or a *valid disjunctive cut* if it is satisfied for each $x \in \bigcup_{h \in H} S_h$. (The superscript t will throughout be taken to denote the transpose operation). Finally, for a set of vectors $\{v^h\colon h \in H\}$, where $v^h = (v_1^h, \ldots, v_n^h)$ for each $h \in H$, we will denote by $\sup_{h \in H} (v^h)$, the pointwise supremum $v = (v_1, \ldots, v_n)$ of the vectors v^h, $h \in H$, such that $v_j = \sup_{h \in H} \{v_j^h\}$ for $j = 1, \ldots, n$.

Before proceeding, we note that a condition which asserts that a feasible point must satisfy at least p of some q sets, $p \leqslant q$, may be easily transformed into the above disjunctive statement by letting each S_h denote the conjunction of the q original sets taken p at a time. Thus, $H = \left\{1, \ldots, \binom{q}{p}\right\}$ in this case. Now consider that following result.

THEOREM 1: (Basic Disjunctive Cut Principle) — Balas [1,2], Glover [10], Jeroslow [14]

Suppose that we are given the linear inequality systems S_h, $h \in H$ of Equation (1.1), where $|H|$ may or may not be finite. Further, suppose that a feasible point must satisfy at least one of these systems. Then, for any choice of nonnegative vectors λ^h, $h \in H$, the inequality

$$\left[\sup_{h \in H} (\lambda^h)' A^h\right] x \geqslant \inf_{h \in H} (\lambda^h)' b^h \tag{1.2}$$

is a valid disjunctive cut. Furthermore, if every system S_h, $h \in H$ is consistent, and if $|H| < \infty$, then for any valid inequality $\sum_{j=1}^{n} \pi_j x_j \geqslant \pi_0$, there exist nonnegative vectors λ^h, $h \in H$ such that $\pi_0 \leqslant \inf_{h \in H} (\lambda^h)' b^h$ and for $j = 1, \ldots, n$, the jth component of $\sup\ (\lambda^h)' A^h$ does not exceed π_j.

The forward part of the above theorem was originally proved by Balas [2] and the converse part by Jeroslow [14]. This theorem has also been independently proved by Glover [10] in a somewhat different setting. The theorem merely states that given a disjunction $x \in \bigcup_{h \in H} S_h$, one may generate a valid cut (1.2) by specifying any nonnegative values for the vectors λ^h, $h \in H$. The versatility of the latter choice is apparent from the converse which asserts that so long as we can identify and delete any inconsistent systems, S_h, $h \in H$, then given any valid cut $\pi' x \geqslant \pi_0$, we may generate a cut of the type (1.2) by suitably selecting values for the parameters λ^h, $h \in H$ such that for any x belonging to the nonnegative orthant of R^n, if (1.2) holds then we must have $\pi' x \geqslant \pi_0$. In other words, we can make a cut of the type (1.2) uniformly dominate any given valid inequality or cut. Thus, any valid inequality is either a special case of

(1.2) or may be strictly dominated by a cut of type (1.2). In this connection, we draw the reader's attention to the work of Balas [1] in which several convexity/intersection cuts discussed in the literature are recovered from the fundamental disjunctive cut. Note that since the inequality (1.2) defines a closed convex set, then for it to be valid, it must necessarily contain the polyhedral set

$$S = \text{convex hull of } \bigcup_{h \in H} S_h. \tag{1.3}$$

Hence, one may deduce that a desirable deep cut would be a facet of S, or at least would support it. Indeed, Balas [3] has shown how one may generate with some difficulty cuts which contain as a subset, the facets of S when $|H| < \infty$. Our approach to developing deep disjunctive cuts will bear directly on Theorem 1. Specifically, we will be indicating how one may specify values for parameters λ^h to provide supports of S, and will discuss some specific criteria for choosing among supports. We will be devoting our attention to the following two disjunctions titled DC1 and DC2. We remark that most disjunctive statements can be cast in the format of DC2. Disjunction DC1 is a special case of disjunction DC2, and is discussed first because it facilitates our presentation.

DC1:

Suppose that each systems S_h is comprised of a single linear inequality, that is, let

$$S_h = \left\{x: \sum_{j=1}^{n} a_{1j}^h x_j \geqslant b_1^h,\ x \geqslant 0\right\} \text{ for } h \in H = \{1, \ldots, \hat{h}\} \tag{1.4}$$

where we assume that $\hat{h} = |H| < \infty$ and that each inequality in S_h, $h \in H$ is stated with the origin as the current point at which the disjunctive cut is being generated. Then, the disjunctive statement DC1 is that at least one of the sets S_h, $h \in H$ must be satisfied. Since the current point (origin) does not satisfy this disjunction, we must have $b_1^h > 0$ for each $h \in H$. Further, we will assume, without loss of generality, that for each $h \in H$, $a_{1j}^h > 0$ for some $j \in \{1, \ldots, n\}$ or else, S_h is inconsistent and we may disregard it.

DC2:

Suppose each system S_h is comprised of a set of linear inequalities, that is, let

$$S_h = \left\{x: \sum_{j=1}^{n} a_{ij}^h x_j \geqslant b_i^h \text{ for each } i \in Q_h,\ x \geqslant 0\right\} \text{ for } h \in H = \{1, \ldots, \hat{h}\} \tag{1.5}$$

where Q_h, $h \in H$ are appropriate constraint index sets. Again, we assume that $\hat{h} = |H| < \infty$ and that the representation in (1.5) is with respect to the current point as the origin. Then, the disjunctive statement DC2 is that at least one of the sets S_h, $h \in H$ must be satisfied. Although it is not necessary here for $b_i^h > 0$ for all $i \in Q_h$ one may still state a valid disjunction by deleting all constraints with $b_i^h \leqslant 0$, $i \in Q_h$ from each set S_h, $h \in H$. Clearly a valid cut for the relaxed constraint set is valid for the original constraint set. We will thus obtain a cut which possibly is not as strong as may be derived from the original constraints. To aid in our development, we will therefore assume henceforth that $b_i^h > 0$, $i \in Q_h$, $h \in H$.

Before proceeding with our analysis, let us briefly comment on the need for deep cuts. Although intuitively desirable, it is not always necessary to seek a deepest cut. For example, if one is using cutting planes to implicitly search a feasible region of discrete points, then all cuts which delete the same subset of this discrete region may be equally attractive irrespective of their depth relative to the convex hull of this discrete region. Such a situation arises, for example, in the work of Majthay and Whinston [16]. On the other hand, if one is confronted with the problem of iteratively exhausting a feasible region which is not finite, as in [20] for example, then indeed deep cuts are meaningful and desirable.

2. DEFINING SUITABLE CRITERIA FOR EVALUATING THE DEPTH OF A CUT

In this section, we will lay the foundation for the concepts we propose to use in deriving deep cuts. Specifically, we will explore the following two criteria for deriving a deep cut:

(i) Maximize the euclidean distance between the origin and the nonnegative region feasible to the cutting plane

(ii) Maximize the rectilinear distance between the origin and the nonnegative region feasible to the cutting plane.

Let us briefly discuss the choice of these criteria. Referring to Figure 1(a) and (b), one may observe that simply attempting to maximize the euclidean distance from the origin to the cut can favor weaker over strictly stronger cuts. However, since one is only interested in the subset of the nonnegative orthant feasible to the cuts, the choice of criterion (i) above avoids such anamolies. Of course, as Figure 1(b) indicates, it is possible for this criterion to be unable to recognize dominance, and treat two cuts as alternative optimal cuts even through one cut dominates the other.

Let us now proceed to characterize the euclidean distance from the origin to the nonnegative region feasible to a cut

$$\sum_{j=1}^{n} z_j x_j \geqslant z_0, \text{ where } z_0 > 0,\ z_j > 0 \text{ for some } j \in \{1, \ldots, n\}. \tag{2.1}$$

The required distance is clearly given by

$$\theta_e = \text{minimum } \{||x||: \sum_{j=1}^{n} z_j x_j \geqslant z_0,\ x \geqslant 0\}. \tag{2.2}$$

Consider the following result.

LEMMA 1: Let θ_e be defined by Equations (2.1) and (2.2). Then

$$\theta_e = \frac{z_0}{||y||} \tag{2.3}$$

where,

$$y = (y_1, \ldots, y_n),\ y_j = \text{maximum } \{0, z_j\},\ j = 1, \ldots, n. \tag{2.4}$$

PROOF: Note that the solution $x^* = \left(\frac{z_0}{||y||^2}\right) y$ is feasible to the problem in (2.2) with $||x^*|| : \frac{z_0}{||y||}$. Moreover, for any x feasible to (2.2), we have, $z_0 \leqslant \sum_{j=1}^{n} z_j x_j \leqslant \sum_{j=1}^{n} y_j x_j \leqslant ||y||\ ||x||$, or that, $||x|| \geqslant \frac{z_0}{||y||}$. This completes the proof.

Now, let us consider the second criterion. The motivation for this criterion is similar to that for the first criterion and moreover, as we shall see below, the use of this criterion has intuitive appeal. First of all, given a cut (2.1), let us characterize the rectilinear distance from the origin to the nonnegative region feasible to this cut. This distance is given by

$$(2.5) \qquad \theta_r = \text{minimum } \{|x|: \sum_{j=1}^{n} z_j x_j \geqslant z_0,\ x \geqslant 0\}, \text{ when } |x| \sum_{j=1}^{n} x_j.$$

Consider the following result.

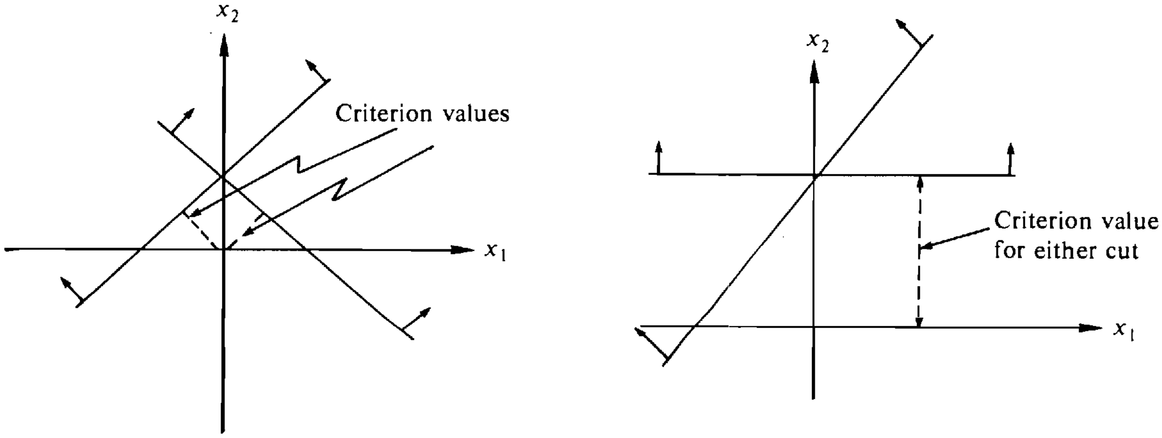


FIGURE 1. Recognition of dominance

LEMMA 2: Let θ_r be defined by Equations (2.1) and (2.5). Then,

$$(2.6) \qquad \theta_r = \frac{z_0}{z_m} \text{ where } z_m = \underset{j=1,\dots,n}{\text{maximum}}\ z_j.$$

PROOF: Note that the solution $x^* = (0, \dots, \frac{z_0}{z_m}, \dots 0)$, with the mth component being non-zero, is feasible to the problem in (2.5) with $|x^*| = \frac{z_0}{z_m}$. Moreover, for any x feasible to (2.5), we have,

$$\frac{z_0}{z_m} \leqslant \sum_{j=1}^{n} \frac{z_j}{z_m} x_j \leqslant \sum_{j=1}^{n} x_j = |x|.$$

This completes the proof.

Note from Equation (2.6) that the objective of maximizing θ_r is equivalent to finding a cut which maximizes the smallest positive intercept made on any axis. Hence, the intuitive appeal of this criterion.

3. DERIVING DEEP CUTS FOR DC1

It is very encouraging to note that for the disjunction DC1 we are able to derive a cut which not only simultaneously satisfies both the criterion of Section 2, but which is also a facet of the set S of Equation (1.3). This is a powerful statement since all valid inequalities are given through (1.2) and none of these can strictly dominate a facet of S.

We will find it more convenient to state our results if we normalize the linear inequalities (1.4) by dividing through by their respective, positive, right-hand-sides. Hence, let us assume without loss of generality that

$$S_h = \left\{x : \sum_{j=1}^{n} a_{1j}^h x_j \geq 1,\ x \geq 0\right\} \text{ for } h \in H = \{1, \ldots, \hat{h}\}. \tag{3.1}$$

Then the application of Theorem 1 to the disjunction DC1 yields valid cuts of the form:

$$\sum_{j=1}^{n} \left\{\max_{h \in H} \lambda_1^h a_{1j}^h\right\} x_j \geq \min_{h \in H} \{\lambda_1^h\} \tag{3.2}$$

where λ_1^h, $h \in H$ are nonnegative scalars. Again, there is no loss of generality in assuming that

$$\sum_{h \in H} \lambda_1^h = 1,\ \lambda_1^h \geq 0,\ h \in H = \{1, \ldots, \hat{h}\} \tag{3.3}$$

since we will not allow all λ_1^h, $h \in H$ to be zero. This is equivalent to normalizing (3.2) by dividing through by $\sum_{h \in H} \lambda_1^h$.

Theorem 2 below derives two cuts of the type (3.2), both of which simultaneously achieve the two criteria of the foregoing section. However, the second cut uniformly dominates the first cut. In fact, no cut can strictly dominate the second cut since it is shown to be a facet of S defined by (1.3).

THEOREM 2: Consider the disjunctive statement DC1 where S_h is defined by (3.1) and is assumed to be consistent for each $h \in H$. Then the following results hold:

(a) Both the criteria of Section 2 are satisfied by letting $\lambda_1^h = \lambda_1^{h*}$ where

$$\lambda_1^{h*} = 1/\hat{h} \qquad \text{for } h \in H \tag{3.4}$$

in inequality (3.2) to obtain the cut

$$\sum_{j=1}^{n} a_{1j}^{*} x_j \geq 1, \text{ where } a_{1j}^{*} = \max_{h \in H} a_{1j}^h, \text{ for } j = 1, \ldots, n. \tag{3.5}$$

(b) Further, defining

$$\gamma_1^h = \underset{j: a_{1j}^h > 0}{\text{minimum}} \{a_{1j}^{*}/a_{1j}^h\} > 0,\ h \in H \tag{3.6}$$

and letting $\lambda_1^h = \lambda_1^{h**}$, where

$$\lambda_1^{h**} = \gamma_1^h / \sum_{p \in H} \gamma_1^p \qquad \text{for } h \in H \tag{3.7}$$

in inequality (3.2), we obtain a cut of the form

$$\sum_{j=1}^{n} a_{1j}^{**} x_j \geq 1, \text{ where } a_{1j}^{**} = \max_{h \in H} a_{1j}^h \gamma_1^h \text{ for } j = 1, \ldots, n \tag{3.8}$$

which again satisfies both the criteria of Section 2.

(c) The cut (3.8) uniformly dominates the cut (3.5); in fact,

$$a_{1j}^{**} \begin{cases} = a_{1j}^{*} \text{ if } a_{1j}^{*} > 0 \\ \leq a_{1j}^{*} \text{ if } a_{1j}^{*} \leq 0 \end{cases},\ j = 1, \ldots, n. \tag{3.9}$$

(d) The cut (3.8) is a facet of the set S of Equation (1.3).

PROOF:

(a) Clearly, $\lambda_1^h = 1/\hat{h}$, $h \in H$ leads to the cut (3.5) from (3.2). Now consider the euclidean distance criterion of maximizing θ_e (or θ_e^2) of Equation (2.3). For cut (3.5), the value of θ_e^2 is given by

(3.10) $$(\theta_e^*)^2 = 1\Big/\sum_{j=1}^{n} (y_j^*)^2 > 0 \text{ where } y_j^* = \max\{0, a_{1j}^*\},\ j = 1, \ldots, n.$$

Now, for any choice λ_1^h, $h \in H$,

(3.11) $$\theta_e^2 = \left[\min_{h \in H}(\lambda_1^h)\right]^2 \Big/ \sum_{j=1}^{n} y_j^2 = (\lambda_1^p)^2 \Big/ \sum_{j=1}^{n} y_j^2, \text{ say,}$$

where $y_j = \max\{0, \max_{h \in H} \lambda_1^h a_{1j}^h\}$. If $\lambda_1^p = 0$, then $\theta_e = 0$ and noting (3.10), such a choice of parameters λ_1^h, $h \in H$ is suboptimal. Hence, $\lambda_1^p > 0$, whence (3.11) becomes $\theta_e^2 = 1\Big/\sum_{j=1}^{n} \left(\frac{y_j}{\lambda_1^p}\right)^2$. But since $(\lambda_1^h/\lambda_1^p) \geqslant 1$ for each $h \in H$, we get

$$y_j/\lambda_1^p = \max\left\{0, \max_{h \in H}\left(\frac{\lambda_1^h}{\lambda_1^p}\right) a_{1j}^h\right\} \geqslant \max\left\{0, \max_{h \in H} a_{1j}^h\right\} = y_j^*.$$

Thus $\theta_e^2 \leqslant (\theta_e^*)^2$ so that the first criterion is satisfied.

Now consider the maximization of θ_r of Equation (2.5), or equivalently Equation (2.6). For the choice (3.4), the value of θ_r is given by

(3.12) $$\theta_r^* = \frac{1}{\max_j a_{1j}^*} > 0.$$

Now, for any choice λ_1^h, $h \in H$, from Equations (2.6), (3.2) we get

$$\theta_r = \left\{\min_{h \in H} \lambda_1^h\right\} \Big/ \left\{\max_j \max_{h \in H} \lambda_1^h a_{1j}^h\right\} = \lambda_1^p \Big/ \max_j \max_{h \in H} \lambda_1^h a_{1j}^h, \text{ say.}$$

As before, $\lambda_1^p = 0$ implies a value of θ_r inferior to θ_r^*. Thus, assume $\lambda_1^p > 0$. Then, $\theta_r = 1\Big/\max_j \max_{h \in H} \left\{\frac{\lambda_1^h}{\lambda_1^p}\right\} a_{1j}^h$. But $(\lambda_1^h/\lambda_1^p) \geqslant 1$ for each $h \in H$ and in evaluating θ_r, we are interested only in those $j \in \{1, \ldots, n\}$ for which $a_{1j}^h > 0$ for some $h \in H$. Thus, $\theta_r \leqslant 1/\max_j \max_{h \in H} a_{1j}^h = \theta_r^*$, so that the second criterion is also satisfied. This proves part (a).

(b) and (c). First of all, let us consider the values taken by γ_1^h, $h \in H$. Note from the assumption of consistency that γ_1^h, $h \in H$ are well defined. From (3.5), (3.6), we must have $\gamma_1^h \geqslant 1$ for each $h \in H$. Moreover, if we define from (3.5)

(3.13) $$H^* = \{h \in H : a_{1k}^h = a_{1k}^* > 0 \text{ for some } k \in \{1, \ldots, n\}\}$$

then clearly $H^* \neq \{\phi\}$ and for $h \in H^*$, Equation (3.6) implies $\gamma_1^h \leqslant 1$. Thus,

$$(3.14)\qquad \gamma_1^h \begin{cases} = 1 & \text{for } h \in H^* \\ > 1 & \text{for } h \notin H^*. \end{cases}$$

Hence,

$$(3.15)\qquad \min_{h \in H} \gamma_1^h = 1$$

or that, using (3.7) in (3.2) yields a cut of the type (3.8), where,

$$(3.16)\qquad a_{1j}^{**} = \max_{h \in H} a_{1j}^h \gamma_1^h, \; j = 1, \ldots, n.$$

Now, let us establish relationship (3.9). Note from (3.5) that if $a_{1j}^* \leqslant 0$, then $a_{1j}^h \leqslant 0$ for each $h \in H$ and hence, using (3.14), (3.16), we get that (3.9) holds. Next, consider $a_{1j}^* > 0$ for some $j \in \{1, \ldots, n\}$. From (3.13), (3.14), (3.16), we get

$$(3.17)\qquad a_{1j}^{**} = \max\{\max_{h \in H} a_{1j}^h, \; \max_{\substack{h \in H^* \\ a_{1j}^h > 0}} a_{1j}^h \gamma_1^h\}$$

where we have not considered $h \notin H^*$ with $a_{1j}^h \leqslant 0$ since $a_{1j}^{**} > 0$. But for $h \notin H^*$ with $a_{1j}^h > 0$, we get from (3.5), (3.6)

$$(3.18)\qquad a_{1j}^h \gamma_1^h = a_{1j}^h \left[\min_{k: a_{1k}^h > 0} \left\{ \frac{\max_{r \in H} a_{1k}^r}{a_{1k}^h} \right\} \right] \leqslant a_{1j}^h \left\{ \frac{\max_{r \in H} a_{1j}^r}{a_{1j}^h} \right\} = \max_{r \in H} a_{1j}^r.$$

Using (3.18) in (3.17) yields $a_{1j}^{**} = a_{1j}^*$, which establishes (3.9).

Finally, we show that (3.8) satisfies both the criteria of Section 2. This part follows immediately from (3.9) by noting that the cut (3.5) yields $\theta_e = \theta_e^*$ of (3.10) and $\theta_r = \theta_r^*$ of (3.12). This completes the proofs of parts (b) and (c).

(d) Note that since (3.8) is valid, any $x \in S$ satisfies (3.8). Hence, in order to show that (3.8) defines a facet of S, it is sufficient to identify n affinely independent points of S which satisfy (3.8) as an equality, since clearly, dim $S = n$. Define

$$(3.19)\qquad J_1 = \{j \in \{1, \ldots, n\}: a_{1j}^{**} > 0\} \text{ and let } J_2 = \{1, \ldots, n\} - J_1.$$

Consider any $p \in J_1$, and let

$$(3.20)\qquad e_p = \left(0, \ldots, \frac{1}{a_{1p}^{**}}, \ldots, 0\right), \; p \in J_1$$

have the non-zero term in the p^{th} position. Now, since $p \in J_1$, (3.9) yields

$$a_{1p}^{**} = a_{1p}^* = \max_{h \in H} a_{1p}^h = a_{1p}^{h_p}, \text{ say,}$$

Hence, $e_p \in S_{h_p}$ and so, $e_p \in S$ and moreover, e_p satisfies (3.8) as an equality. Thus, e_p, $p \in J_1$ qualify as $|J_1|$ of the n affinely independent points we are seeking.

Now consider a $q \in J_2$. Let us show that there exists an S_{h_q} satisfying

$$\gamma_1^{h_q} a_{1p}^{h_q} = a_{1p}^{**} \text{ for some } p \in J_1$$

and

(3.21) $$\gamma_1^{h_q} a_{1q}^{h_q} = a_{1q}^{**}.$$

From Equation (3.16), we get $a_{1q}^{**} = \max_{h \in H} a_{1q}^h \gamma_1^h = a_{1q}^{h_q} \gamma_1^{h_q}$, say. Then for this $h_q \in H$, Equation (3.6) yields $\gamma_1^{h_q} = \underset{j: a_{1j}^{h_q} > 0}{\text{minimum}} \{a_{1j}^{*}/a_{1j}^{h_q}\} = a_{1p}^{*}/a_{1p}^{h_q}$, say. Or, using (3.9), $\gamma_1^{h_q} a_{1p}^{h_q} = a_{1p}^{*} = a_{1p}^{**} > 0$. Thus (3.21) holds. For convenience, let us rewrite the set S_{h_q} below as

(3.22) $$S_{h_q} = \{x: a_{1p}^{h_q} x_p + a_{1q}^{h_q} x_q + \sum_{j \neq p,q} a_{1j}^{h_q} x_j \geq 1,\ x \geq 0\}.$$

Now, consider the direction

(3.23) $$d_q = \begin{cases} \left(0, \ldots, \dfrac{1}{a_{1p}^{**}}, \quad -\dfrac{1}{a_{1q}^{**}}, \ldots, 0\right) & \text{if } a_{1q}^{**} < 0 \\ (0, \ldots, 0, \ldots, \quad \Delta, \ldots, 0) & \text{if } a_{1q}^{**} = 0 \end{cases}$$

where $\Delta > 0$. Let us show that d_q is a direction for S_{h_q}. Clearly, if $a_{1q}^{**} = 0$, then from (3.21) $a_{1q}^{h_q} = 0$ and thus (3.22) establishes (3.23). Further, if $a_{1q}^{**} < 0$ then one may easily verify from (3.21), (3.22), (3.23) that

$$\hat{e}_p = (0, \ldots, \gamma_1^{h_q}/a_{1p}^{**}, \ldots, 0) \in S_{h_q} \text{ and } \hat{e}_p + \delta[\gamma_1^{h_q} d_q] \in S_{h_q} \text{ for each } \delta \geq 0$$

where $\hat{e}_p$ has the non-zero term at position p. Thus, d_q is a direction for S_{h_q}. It can be easily shown that this implies d_q is a direction for S. Since $e_p = \left(0, \ldots, \dfrac{1}{a_{1p}^{**}}, \ldots, 0\right)$ of Equation (3.20) belongs to S, then so does $(e_p + d_q)$. But $(e_p + d_q)$ clearly satisfies (3.8) as an equality. Hence, we have identified n points of S, which satisfy the cut (3.8) as an equality, of the type

(3.24) $$\left.\begin{aligned} & e_p = \left(0, \ldots, \frac{1}{a_{1p}^{**}}, \ldots, 0\right) \quad \text{for } p \in J_1 \\ & e_q = d_q + e_p \text{ for some } p \in J_1, \text{ for each } q \in J_2 \end{aligned}\right\}$$

where d_q is given by (3.23). Since these n points are clearly affinely independent, this completes the proof.

It is interesting to note that the cut (3.5) has been derived by Balas [2] and by Glover [9, Theorem 1]. Further, the cut (3.8) is precisely the strengthened negative edge extension cut of Glover [9, Theorem 2]. The effect of replacing λ_1^{h*} defined in (3.4) by λ_1^{h**} defined in (3.7) is equivalent to the translation of certain hyperplanes in Glover's theorem. We have hence shown through Theorem 2 how the latter cut may be derived in the context of disjunctive programming, and be shown to be a facet of the convex hull of feasible points. Further, both (3.5) and (3.8) have been shown to be alternative optima to the two criteria of Section 2.

In generalizing this to disjunction DC2, we find that such an ideal situation no longer exists. Nevertheless, we are able to obtain some useful results. But before proceeding to DC2, let us illustrate the above concepts through an example.

EXAMPLE: Let $H = \{1,2\}$, $n = 3$ and let DC1 be formulated through the sets

$$S_1 = \{x: x_1 + 2x_2 - 4x_3 \geqslant 1,\ x \geqslant 0\}, S_2 = \{x: \frac{x_1}{2} + \frac{x_2}{3} - 2x_3 \geqslant 1,\ x \geqslant 0\}.$$

The cut (3.5), i.e., $\Sigma a^{\bullet}_{1j}x_j \geqslant 1$, is $x_1 + 2x_2 - 2x_3 \geqslant 1$. From (3.6),

$$\gamma_1^1 = \min\left\{\frac{1}{1}, \frac{2}{2}\right\} = 1 \text{ and } \gamma_1^2 = \min\left\{\frac{1}{1/2}, \frac{2}{1/3}\right\} = 2.$$

Thus, through (3.7), or more directly, from (3.16), the cut (3.8), i.e., $\Sigma\, a^{\bullet\bullet}_{1j}x_j \geqslant 1$ is $x_1 + 2x_2 - 4x_3 \geqslant 1$. This cut strictly dominates the cut (3.5) in this example, though both have the same values $1/\sqrt{5}$ and $1/2$ respectively for θ_e and θ_r of Equations (2.2) and (2.5).

4. DERIVING DEEP CUTS FOR DC2

To begin with, let us make the following interesting observation. Suppose that for convenience, we assume without loss of generality as before, that $b_i^h = 1$, $i \in Q_h$, $h \in H$ in Equation (1.4). Thus, for each $h \in H$, we have the constraint set

$$S_h = \left\{x: \sum_{j=1}^{n} a_{ij}^h x_j \geqslant 1,\ i \in Q_h,\ x \geqslant 0\right\}. \tag{4.1}$$

Now for each $h \in H$, let us multiply the constraints of S_h by corresponding scalars $\delta_i^h \geqslant 0$, $i \in Q_h$ and add them up to obtain the surrogate constraint

$$\sum_{j=1}^{n}\left\{\sum_{i \in Q_h} \delta_i^h a_{ij}^h\right\} x_j \geqslant \sum_{i \in Q_h} \delta_i^h,\ h \in H. \tag{4.2}$$

Further, assuming that not all δ_i^h are zero for $i \in Q_h$, (4.2) may be re-written as

$$\sum_{j=1}^{n}\left\{\sum_{i \in Q_h}\left[\frac{\delta_i^h}{\left[\sum_{p \in Q_h} \delta_p^h\right]}\right] a_{ij}^h\right\} x_j \geqslant 1,\ h \in H. \tag{4.3}$$

Finally, denoting $\delta_i^h \Big/ \sum_{p \in Q_h} \delta_p^h$ by λ_i^h for $i \in Q_h$, $h \in H$, we may write (4.3) as

$$\sum_{j=1}^{n}\left\{\sum_{i \in Q_h} \lambda_i^h a_{ij}^h\right\} x_j \geqslant 1 \text{ for each } h \in H \tag{4.4}$$

where,

$$\sum_{i \in Q_h} \lambda_i^h = 1 \text{ for each } h \in H,\ \lambda_i^h \geqslant 0 \text{ for } i \in Q_h,\ h \in H. \tag{4.5}$$

Observe that by surrogating the constraints of (4.1) using parameters λ_i^h, $i \in Q_h$, $h \in H$ satisfying (4.5), we have essentially represented DC2 as DC1 through (4.4). In other words, since $x \in S_h$ implies x satisfies (4.4) for each $h \in H$, then given λ_i^h, $i \in Q_h$, $h \in H$, DC2 implies that at least one of (4.4) must be satisfied. Now, whereas Theorem 1 would directly employ (4.2) to derive a cut, since we have normalized (4.2) to obtain (4.4), we know from the previous section that the optimal strategy is to derive a cut (3.8) using inequalities (4.4).

Now let us consider in turn the two criteria of Section 2.

4.1. Euclidean Distance-Based Criterion

Consider any selection of values for the parameters λ_i^h, $i \in Q_h$, $h \in H$ satisfying (4.5) and let the corresponding disjunction DC1 derived from DC2 be that at least one of (4.4) must hold. Then, Theorem 2 tells us through Equations (3.5), (3.10) that the euclidean distance criterion value for the resulting cut (3.8) is

$$\theta_e(\lambda) = 1 \Big/ \sqrt{\sum_{j=1}^{n} y_j^2} \tag{4.6}$$

where,

$$y_j = \max\{0, z_j\}, \ j = 1, \dots, n \tag{4.7}$$

and

$$z_j = \max_{h \in H} \left\{ \sum_{i \in Q_h} \lambda_i^h a_{ij}^h \right\}, \ j = 1, \dots, n. \tag{4.8}$$

Thus, the criterion of Section 2 seeks to

$$\text{maximize } \{\theta_e(\lambda): \lambda = (\lambda_i^h) \text{ satisfies (4.5)}\} \tag{4.9}$$

or equivalently, to

$$\text{minimize } \{ \sum_{j=1}^{n} y_j^2 : \ (4.5), \ (4.7), \ (4.8) \text{ are satisfied}\}. \tag{4.10}$$

It may be easily verified that the problem of (4.10) may be written as

$$\text{PD}_2: \quad \text{minimize } \sum_{j=1}^{n} y_j^2 \tag{4.11}$$

$$\text{subject to } y_j \geqslant \sum_{i \in Q_h} \lambda_i^h a_{ij}^h \text{ for each } h \in H \text{ for each } j = 1, \dots, n \tag{4.12}$$

$$\sum_{i \in Q_h} \lambda_i^h = 1 \text{ for each } h \in H \tag{4.13}$$

$$\lambda_i^h \geqslant 0 \ i \in Q_h, \ h \in H \tag{4.14}$$

Note that we have deleted the constraints $y_j \geqslant 0$, $j = 1, \dots, n$ since for any feasible λ_i^h, $i \in Q_h$, $h \in H$, there exists a dominant solution with nonnegative $y_j = j = 1, \dots, n$. This relaxation is simply a matter of convenience in our solution strategy.

Before proposing a solution procedure for Problem PD_2, let us make some pertinent remarks. Note that Problem PD_2 has the purpose of generating parameters λ_i^h, $i \in Q_h$, $h \in H$ which are to be used to obtain the surrogate constraints (4.4). Thereafter, the cut that we derive for the disjunction DC2 is the cut (3.8) obtained from the statement that at least one of (4.4) must hold. Hence, Problem PD_2 attempts to find values for λ_i^h, $i \in Q_h$, $h \in H$, such that this resulting cut achieves the euclidean distance criterion.

Problem PD_2 is a convex quadratic program for which the Kuhn-Tucker conditions are both necessary and sufficient. Several efficient simplex-based quadratic programming procedures are available to solve such a problem. However, these procedures require explicit handling of the potentially large number of constraints in Problem PD_2. On the other hand, the subgradient optimization procedure discussed below takes full advantage of the problem structure. We are first able to write out an almost complete solution to the Kuhn-Tucker system. We will refer to this as a *partial solution.* In case we are unable to either actually construct a complete solution or to assert that a feasible completion exists, then through the construction procedure itself, we have a subgradient direction available. Moreover, this latter direction is very likely to be a direction of ascent. We therefore propose to move in the negative of this direction and if necessary, project back onto the feasible region. These iterative steps are now repeated at this new point.

4.1.1 Kuhn-Tucker Systems for PD_2 and Its Implications

Letting u_j^h, $h \in H$, $j = 1, \ldots, n$ denote the lagrangian multipliers for constraints (4.12), t_h, $h \in H$ those for constraints (4.13), and w_i^h, $i \in Q_h$, $h \in H$ those for constraints (4.14), we may write the Kuhn-Tucker optimality conditions as

(4.15) $$\sum_{h \in H} u_j^h = 2y_j \quad j = 1, \ldots, n$$

(4.16) $$\sum_{j=1}^{n} u_j^h a_{ij}^h + t_h - w_i^h = 0 \text{ for each } i \in Q_h, \text{ and for each } h \in H$$

(4.17) $$u_j^h \left\{ \sum_{i \in Q_h} \lambda_i^h a_{ij}^h - y_j \right\} = 0 \text{ for each } j = 1, \ldots, n \text{ and each } h \in H$$

(4.18) $$\lambda_i^h w_i^h = 0 \text{ for } i \in Q_h, \ h \in H$$

(4.19) $$w_i^h \geqslant 0 \quad i \in Q_h, \ h \in H$$

(4.20) $$u_j^h \geqslant 0 \ j = 1, \ldots, n, \ h \in H.$$

Finally, Equations (4.12), (4.13), (4.14) must also hold. We will now consider the implications of the above conditions. This will enable us to construct at least a partial solution to these conditions, given particular values of λ_i^h, $i \in Q_h$, $h \in H$. First of all, note that Equations (4.7), (4.10) and (4.20) imply that

(4.21) $$y_j \geqslant 0 \quad \text{for each } j = 1, \ldots, n$$

(4.22) $$y_j = \max\left\{0, \sum_{i \in Q_h} \lambda_i^h a_{ij}^h, \ h \in H\right\} \text{ for } j = 1, \ldots, n.$$

Now, having determined values for y_j, $j = 1, \ldots, n$, let us define the sets

(4.23) $$H_j = \begin{cases} \{\varnothing\} \text{ if } y_j = 0 \\ \{h \in H: y_j = \sum_{i \in Q_h} \lambda_i^h a_{ij}^h > 0\} \end{cases} \quad \text{for } j = 1, \ldots, n.$$

Now, consider the determination of u_j^h, $h \in H$, $j = 1, \ldots, n$. Clearly, Equations (4.15), (4.17) and (4.20) along with the definition (4.23) imply that for each $j = 1, \ldots, n$

(4.24) $$u_j^h = 0 \text{ for } h \in H/H_j \text{ and that } \sum_{h \in H_j} u_j^h = 2y_j, \ u_j^h \geqslant 0 \text{ for each } h \in H_j.$$

Thus, for any $j \in \{1, \ldots, n\}$, if H_j is either empty or a singleton, the corresponding values for u_j^h, $h \in H$ are uniquely determined. Hence, we have a choice in selecting values for u_j^h, $h \in H_j$

only when $|H_j| \geqslant 2$ for any $j \in \{1, \ldots, n\}$. Next, multiplying (4.16) by λ_i^h and using (4.18), we obtain

$$(4.25) \qquad \sum_{j=1}^{n} \left[u_j^h \sum_{i \in Q_h} \{\lambda_i^h a_{ij}^h\} \right] + t_h \sum_{i \in Q_h} \lambda_i^h = 0 \text{ for each } h \in H.$$

Using Equations (4.13), (4.17), this gives us

$$(4.26) \qquad t_h = - \sum_{j=1}^{n} u_j^h y_j \text{ for each } h \in H.$$

Finally, Equations (4.16), (4.26) yield

$$(4.27) \qquad w_i^h = \sum_{j=1}^{n} u_j^h \, [a_{ij}^h - y_j] \text{ for each } i \in Q_h, \; h \in H.$$

Notice that once the variables u_j^h, $h \in H$, $j = 1, \ldots, n$ are fixed to satisfy (4.24), all the variables are uniquely determined. We now show that if the variables w_i^h, $i \in Q_h$, $h \in H$ so determined are nonnegative, we then have a Kuhn-Tucker solution. Since the objective function of PD_2 is convex and the constraints are linear, this solution is also optimal.

LEMMA 2: Let a primal feasible set of λ_i^h, $i \in Q_h$, $h \in H$ be given. Determine values for all variables y_j, u_j^h, t_h, w_i^h using Equations (4.22) through (4.27), selecting an arbitrary solution in the case described in Equation (4.24) if $|H_j| \geqslant 2$. If $w_i^h \geqslant 0$, $i \in Q_h$, $h \in H$, then λ_i^h, $i \in Q_h$, $h \in H$ solves Problem PD_2.

PROOF: By construction Equations (4.12), through (4.17), and (4.20) clearly hold. Thus, noting that in our problem the Kuhn-Tucker conditions are sufficient for optimality, all we need to show is that if $w = (w_i^h) \geqslant 0$ then (4.18) holds. But from (4.17) and (4.27) for any $h \in H$, we have,

$$\sum_{i \in Q_h} \lambda_i \, w_i^h = \sum_{i \in Q_h} \lambda_i^h \left\{ \sum_{j=1}^{n} u_j^h \, [a_{ij}^h - y_j] \right\} = \sum_{j=1}^{n} \left\{ u_j^h \left[\sum_{i \in Q_h} \lambda_i^h a_{ij}^h - y_j \right] \right\} = 0$$

for each $h \in H$. Thus, $\lambda_i^h \geqslant 0$, $w_i^h \geqslant 0$ $i \in Q_h$, $h \in H$ imply that (4.18) holds and the proof is complete.

The reader may note that in Section 4.1.4 we will propose another stronger sufficient condition for a set of variables λ_i^h, $i \in Q_h$, $h \in H$ to be optimal. The development of this condition is based on a subgradient optimization procedure discussed below.

4.1.2 Subgradient Optimization Scheme for Problem PD

For the purpose of this development, let us use (4.22) to rewrite Problem PD_2 as follows. First of all define

$$(4.28) \qquad \Lambda = \{\lambda = (\lambda_i^h) \colon \text{ constraints (4.13) and (4.14) are satisfied}\}$$

and let $f \colon \Lambda \to R$ be defined by

$$(4.29) \qquad f(\lambda) = \sum_{j=1}^{n} \left[\text{maximum} \left\{ 0, \sum_{i \in Q_h} \lambda_i^h a_{ij}^h, \; h \in H \right\} \right]^2.$$

Then, Problem PD_2 may be written as

$$\text{minimize } \{f(\lambda): \lambda \in \Lambda\}.$$

Note that for each $j = 1, \ldots, n$, $g_j(\lambda) = \max \{0, \sum_{i \in Q_h} \lambda_i^h a_{ij}^h, h \in H\}$ is convex and nonnegative. Thus, $[g_j(\lambda)]^2$ is convex and so $f(\lambda) = \sum_{j=1}^{n} [g_j(\lambda)]^2$ is also convex.

The main thrust of the proposed algorithm is as follows. Having a solution $\bar{\lambda}$ at any stage, we will attempt to construct a solution to the Kuhn-Tucker system using Equations (4.15) through (4.20). If we obtain nonnegative values $\bar{w}_i^h$ for the corresponding variables w_i^h, $i \in Q_h$, $h \in H$, then by Lemma 2 above, we terminate. Later in Section 4.1.7, we will also use another sufficient condition to check for termination. If we obtain no indication of optimality, we continue. Theorem 3 below established that in any case, the vector $w = \bar{w}$ constitutes a subgradient of $f(\cdot)$ at the current point $\bar{\lambda}$. Following Poljak [18,19], we hence take a suitable step in the negative subgradient direction and project back onto the feasible region Λ of Equation (4.28). This completes one iteration. Before presenting Theorem 3, consider the following definition.

DEFINITION 1: Let f: $\Lambda \rightarrow R$ be a convex function and let $\bar{\lambda} \in \Lambda \subset R^m$. Then $\xi \in R^m$ is a *subgradient* of $f(\cdot)$ at $\bar{\lambda}$ if

$$f(\lambda) \geqslant f(\bar{\lambda}) + \xi^t (\lambda - \bar{\lambda}) \text{ for each } \lambda \in \Lambda.$$

THEOREM 3: Let $\bar{\lambda}$ be a given point in Λ defined by (4.28) and let $\bar{w}$ be obtained from Equations (4.22) through (4.27), with an arbitrary selection of a solution to (4.24).

Then, $\bar{w}$ is a subgradient of $f(\cdot)$ at $\bar{\lambda}$, where f:$\Lambda \rightarrow R$ is defined in Equation (4.29).

PROOF. Let y and $\bar{y}$ be obtained through Equation (4.22) from $\lambda \in \Lambda$ and $\bar{\lambda} \in \Lambda$ respectively. Hence,

$$f(\lambda) = \sum_{j=1}^{n} y_j^2 \text{ and } f(\bar{\lambda}) = \sum_{j=1}^{n} \bar{y}_j^2.$$

Thus, from Definition 1, we need to show that

$$\text{(4.30)} \qquad \sum_{h \in H} \sum_{i \in Q_h} \bar{w}_i^h (\lambda_i^h - \bar{\lambda}_i^h) \leqslant \sum_{j=1}^{n} y_j^2 - \sum_{j=1}^{n} \bar{y}_j^2.$$

Noting from Equations (4.17), (4.27) that $\sum_{h \in H} \sum_{i \in Q_h} \bar{w}_i^h \bar{\lambda}_i^h = 0$, we have,

$$\sum_{h \in H} \sum_{i \in Q_h} \bar{w}_i^j (\lambda_i^h - \bar{\lambda}_i^h) = \sum_{h \in H} \sum_{i \in Q_h} \bar{w}_i^h \lambda_i^h = \sum_{h \in H} \sum_{i \in Q_h} \sum_{j=1}^{n} \bar{u}_i^h \lambda_i^h [a_{ij}^h - \bar{y}_j]$$

$$= \sum_{h \in H} \sum_{j=1}^{n} \bar{u}_i^h \left[\sum_{i \in Q_h} \lambda_i^h a_{ij}^h\right] - \sum_{h \in H} \sum_{j=1}^{n} \left[\bar{u}_i^h y_j \sum_{i \in Q_h} \lambda_i^h\right].$$

Using (4.13) and (4.15), this yields

$$\sum_{h \in H} \sum_{i \in Q_h} \bar{w}_i^h (\lambda_i^h - \bar{\lambda}_i^h) = \sum_{h \in H} \sum_{j=1}^{n} \bar{u}_i^h \left[\sum_{i \in Q_h} \lambda_i^h a_{ij}^h\right] - 2 \sum_{j=1}^{n} \bar{y}_j^2.$$

Combining this with (4.30), we need to show that

$$\sum_{j\in H}\sum_{j=1}^{n}\bar{u}_j^h\left[\sum_{i\in Q_h}\lambda_i^h a_{ij}^h\right] \leqslant \sum_{j=1}^{n} y_j^2 + \sum_{j=1}^{n}\bar{y}_j^2. \tag{4.31}$$

But Equations (4.15), (4.20), (4.22) imply that

$$\sum_{h\in H}\sum_{j=1}^{n}\bar{u}_j^h\left[\sum_{i\in Q_h}\lambda_j^h a_{ij}^h\right] \leqslant \sum_{h\in H}\sum_{j=1}^{n}\bar{u}_j^h y_j = 2\sum_{j=1}^{n} y_j\bar{y}_j \leqslant 2\|y\|\,\|\bar{y}\| \leqslant \|y\|^2 + \|\bar{y}\|^2$$

so that Equation (4.31) holds. This completes the proof.

Although, given $\bar{\lambda} \in \Lambda$, any solution to Equations (4.22) through (4.27) will yield a subgradient of $f(\cdot)$ at the current point $\bar{\lambda}$, we would like to generate, without expending much effort, a subgradient which is hopefully a direction of ascent. Hence, this would accelerate the cut generation process. Later in Section 4.1.6 we describe one such scheme to determine a suitable subgradient direction. For the present moment, let us assume that we have generated a subgradient $\bar{w}$ and have taken a suitable step size $\bar{\theta}$ in the direction $-\bar{w}$ as prescribed by the subgradient optimization scheme of Held, Wolfe, and Crowder [12]. Let

$$\bar{\bar{\lambda}} = \bar{\lambda} - \bar{\theta}\,\bar{w} \tag{4.32}$$

be the new point thus obtained. To complete the iteration, we must now project $\bar{\bar{\lambda}}$ into Λ, that is, we must determine a new $\bar{\lambda}$ according to

$$\bar{\lambda}_{new} \equiv P_\Lambda(\bar{\bar{\lambda}}) = \text{minimum}\,\{\|\lambda - \bar{\bar{\lambda}}\| : \lambda \in \Lambda\}. \tag{4.33}$$

The method of accomplishing this efficiently is presented in the next subsection.

4.1.3 Projection Scheme

For convenience, let us define the following linear manifold

$$M_h = \left\{\lambda_i^h,\ i\in Q_h:\ \sum_{i\in Q_h}\lambda_i^h = 1\right\},\ h\in H \tag{4.34}$$

and let $\bar{M}_h$ be the intersection of M_h with the nonnegative orthant, that is,

$$\bar{M}_h = \{\lambda_i^h,\ i\in Q_h:\ \sum_{i\in Q_h}\lambda_i^h = 1,\ \lambda_i^h \geqslant 0,\ i\in Q_h\}. \tag{4.35}$$

Note from Equation (4.28) that

$$\Lambda = \bar{M}_1 \times \ldots \times \bar{M}_{|H|}. \tag{4.36}$$

Now, given $\bar{\bar{\lambda}}$, we want to project it onto Λ, that is, determine $\bar{\lambda}_{new}$ from Equation (4.33). Towards this end, for any vector $\alpha = (\alpha_i,\ i\in I)$, where I is a suitable index set for the $|I|$ components of α, let $P(\alpha,I)$ denote the following problem:

$$P(\alpha,I):\qquad \text{minimize}\left\{\frac{1}{2}\sum_{i\in I}(\lambda_i - \alpha_i)^2 : \sum_{i\in I}\lambda_i = 1,\ \lambda_i \geqslant 0,\ i\in I\right\}. \tag{4.37}$$

Then to determine $\bar{\lambda}_{new}$, we need to find the solutions $(\bar{\lambda}_{new}^h)_i$, $i\in Q_h$ as projections onto $\bar{M}_h$ of $\bar{\bar{\lambda}}^h = (\bar{\bar{\lambda}}_i^h,\ i\in Q_h)$ through each of the $|H|$ separable Problems $P(\bar{\bar{\lambda}}^h, Q_h)$. Thus, henceforth in this section, we will consider only one such $h\in H$. Theorem 4 below is the basis of a finitely convergent iterative scheme to solve Problem $P(\bar{\bar{\lambda}}^h, Q_h)$.

THEOREM 4: Consider the solution of Problem $P(\beta^k, I_k)$, where $\beta^k = (\beta_i^k, i \in I_k)$, with $|I_k| \geq 1$. Define

(4.38) $$\rho_k = \left[1 - \sum_{i \in I_k} \beta_i^k\right] \Big/ |I_k|$$

and let

(4.39) $$\bar{\beta}^k = \beta^k + (\rho_k)\, l_k$$

where l_k denotes a vector of $|I_k|$ elements, each equal to unity. Further, define

(4.40) $$I_{k+1} = \{i \in I_k : \bar{\beta}_i^k > 0\}.$$

Finally, let β^{k+1} defined below be a subvector of $\bar{\beta}^k$,

(4.41) $$\beta^{k+1} = (\beta_i^{k+1},\ i \in I_{k+1})$$

where, $\beta_i^{k+1} = \bar{\beta}_i^k$, $i \in I_{k+1}$. Now suppose that $\hat{\beta}^{k+1}$ solves $P(\beta^{k+1}, I_{k+1})$.

(a) If $\bar{\beta}^k \geq 0$, then $\bar{\beta}^k$ solves $P(\beta^k, I_k)$.

(b) If $\bar{\beta}^k \ngeq 0$, then β solves $P(\beta^k, I_k)$, where β has components given by

(4.42) $$\beta_i = \begin{cases} \hat{\beta}_i^{k+1}, & \text{if } i \in I_{k+1} \text{ for each } i \in I_k. \\ 0 & \text{otherwise} \end{cases}$$

PROOF: For the sake of convenience, let $RP(\alpha, I)$ denote the problem obtained by relaxing the nonnegativity restrictions in $P(\alpha, I)$. That is, let

$$RP(\alpha, I): \qquad \text{minimize} \left\{\frac{1}{2} \sum_{i \in I} (\lambda_i - \alpha_i)^2 : \sum_{i \in I} \lambda_i = 1\right\}.$$

First of all, note from Equations (4.38), (4.39) that $\bar{\beta}^k$ solves $RP(\beta^k, I_k)$ since $\bar{\beta}^k$ is the projection of β^k onto the linear manifold

(4.43) $$\left\{\lambda = (\lambda_i, i \in I_k) : \sum_{i \in I_k} \lambda_i = 1\right\}$$

which is the feasible region of $RP(\beta^k, I_k)$. Thus, $\bar{\beta}^k \geq 0$ implies that $\bar{\beta}^k$ also solves $P(\beta^k, I_k)$. This proves part (a).

Next, suppose that $\bar{\beta}^k \ngeq 0$. Observe that β is feasible to $P(\beta^k, I_k)$ since from (4.42), we get $\beta \geq 0$ and $\sum_{i \in I_k} \beta_i = \sum_{i \in I_{k+1}} \hat{\beta}^{k+1} = 1$ as $\hat{\beta}^{k+1}$ solves $P(\beta^{k+1}, I_{k+1})$.

Now, consider any $\lambda = (\lambda_i, i \in I_k)$ feasible to $P(\beta^k, I_k)$. Then, by the Pythagorem Theorem, since $\bar{\beta}^k$ is the projection of β^k onto (4.43), we get

$$||\lambda - \beta^k||^2 = ||\lambda - \bar{\beta}^k||^2 + ||\bar{\beta}^k - \beta^k||^2.$$

Hence, the optimal solution to $P(\bar{\beta}^k, I_k)$ is also optimal to $P(\beta^k, I_k)$. Now, suppose that we can show that the optimal solution to Problem $P(\bar{\beta}^k, I_k)$ must satisfy

(4.44) $$\lambda_i = 0 \text{ for } \notin I_{k+1}.$$

Then, noting (4.41), (4.42), and using the hypothesis that $\hat{\beta}^{k+1}$ solves $P(\beta^{k+1}, I_{k+1})$, we will have established part (b). Hence, let us prove that (4.44) must hold. Towards this end, consider the following Kuhn-Tucker equations for Problem $P(\bar{\beta}^k, I_k)$ with t and w_i, $i \in I_k$ as the appropriate lagrangian multipliers:

(4.46) $\sum_{i \in I_k} \lambda_i = 1,\ \lambda_i \geqslant 0$ for each $i \in I_k$

(4.47) $(\lambda_i - \bar{\beta}_i^k) + t - w_i = 0$ and $w_i \geqslant 0$ for each $i \in I_k$

(4.48) $\lambda_i w_i = 0$ for each $i \in I_k$.

Now, since $\sum_{i \in I_k} \bar{\beta}_i^k = 1$, we get from (4.45), (4.46) that

$$t = \sum_{i \in I_k} w_i \Big/ |I_k| \geqslant 0.$$

But from (4.46), (4.47), and (4.48) we get for each $i \in I_k$,

$$0 = w_i \lambda_i = \lambda_i(\lambda_i + t - \bar{\beta}_i^k)$$

which implies that for each $i \in I_k$, we must have,

either $\lambda_i = 0$, whence from (4.46), $w_i = t - \bar{\beta}_i^k$ must be nonnegative

or $\lambda_i = \bar{\beta}_i^k - t$, whence from (4.46), $w_i = 0$.

In either case above, noting (4.45), if $\bar{\beta}_i^k \leqslant 0$, that is, if $i \notin I_{k+1}$, we must have $\lambda_i = 0$. This completes the proof.

Using Theorem 4, one may easily validate the following procedure for finding $\bar{\lambda}_{new}^h$ of Equation (4.33), given $\bar{\bar{\lambda}}^h$. This procedure has to be repeated separately for each $h \in H$.

Initialization

Set $k = 0$, $\beta^0 = \bar{\bar{\lambda}}^h$, $I_0 = Q_h$. Go to Step 1.

Step 1

Given β^k, I_k, determine ρ_k and $\bar{\beta}^k$ from (4.38), (4.39). If $\bar{\beta}^k \geqslant 0$, then terminate with $\bar{\lambda}_{new}^h$ having components given by

$$(\bar{\lambda}_{new}^h)_i = \begin{cases} \bar{\beta}_i^k \text{ if } i \in I_k \\ 0 \text{ otherwise.} \end{cases}$$

Otherwise, proceed to Step 2.

Step 2

Define I_{k+1}, β^{k+1} as in Equations (4.40), (4.41), increment k by one and return to Step 1.

Note that this procedure is finitely convergent as it results in a strictly decreasing, finite sequence $|I_k|$ satisfying $|I_k| \geqslant 1$ for each k, since $\sum_{i \in I_k} \bar{\beta}^k = 1$ for each k.

EXAMPLE: Suppose we want to project $\bar{\bar{\lambda}}^h = (-2,3,1,2)$ on to $\Lambda \subset R^4$. Then the above procedure yields the following results.

Initialization

$k = 0$, $\beta^0 = (-2, 3, 1, 2)$, $I_0 = \{1, 2, 3, 4\}$.

Step 1

$$\rho_0 = -3/4,\ \bar{\beta}^0 = \left[-\frac{11}{4},\ \frac{9}{4},\ \frac{1}{4},\ \frac{5}{4}\right]$$

Step 2

$$k = 1,\ I_1 = \{2,3,4\},\ \beta^1 = \left[\frac{9}{4},\ \frac{1}{4},\ \frac{5}{4}\right]$$

Step 1

$$\rho_1 = -\frac{11}{12},\ \bar{\beta}^1 = \left[\frac{4}{3},\ -\frac{2}{3},\ \frac{1}{3}\right]$$

Step 2

$$k = 2,\ I_2 = \{2,4\},\ \beta^2 = \left[\frac{4}{3},\ \frac{1}{3}\right]$$

Step 1

$$\rho_2 = -\frac{1}{3},\ \bar{\beta}^2 = (1,0) \geqslant 0$$

Thus, $\bar{\lambda}^h_{new} = (0,1,0,0)$.

4.1.4 A Second Sufficient Condition for Termination

As indicated earlier in Section 4.1.2, we will now derive a second sufficient condition on $\bar{w}$ for $\bar{\lambda}$ to solve PD_2. For this purpose, consider the following lemma:

LEMMA 3: Let $\bar{\lambda} \in \Lambda$ be given and suppose we obtain $\bar{w}$ using Equations (4.22) through (4.27). Let $\hat{w}$ solve the problem.

$$PR_h\text{: minimize} \left\{\frac{1}{2} \sum_{i \in Q_h} (\bar{w}_i^h - w_i^h)^2 : \sum_{i \in Q_h} w_i^h = 0,\ w_i^h \leqslant 0 \text{ for } i \in J_h\right\} \text{ for each } h \in H$$

where,

$$J_h = \{i \in Q_h : \bar{\lambda}_i^h = 0\},\ h \in H. \tag{4.49}$$

Then, if $\hat{w} = 0$, $\bar{\lambda}$ solves Problem PD_2.

PROOF. Since $\hat{w} = 0$ solves PR_h, $h \in H$, we have for each $h \in H$,

$$\sum_{i \in Q_h} (\bar{w}_i^h)^2 \leqslant \sum_{i \in Q_h} (\bar{w}_i^h - w_i^h)^2 \tag{4.50}$$

for all w_i^h, $i \in Q_h$ satisfying $\sum_{i \in Q_h} w_i^h = 0$, $w_i^h \leqslant 0$ for $i \in J_h$. Given any $\lambda \in \Lambda$ and given any $\mu > 0$ define,

$$w_i^h = (\bar{\lambda}_i^h - \lambda_i^h)/\mu,\ i \in Q_h,\ h \in H. \tag{4.51}$$

Then, $\sum_{i \in Q_h} w_i^h = 0$ for each $h \in H$ and since $\bar{\lambda}_i^h = 0$ for $i \in J_h$, $h \in H$, we get $w_i^h \leqslant 0$ for $i \in J_h$, $h \in H$. Thus, for any $\lambda \in \Lambda$, by substituting (4.51) into (4.50), we have,

(4.52) $$\mu^2 \sum_{i \in Q_h} (\bar{w}_i^h)^2 \leqslant \sum_{i \in Q_h} (\lambda_i^h - \bar{\lambda}_i^h + \mu \bar{w}_i^h)^2 \text{ for each } h \in H.$$

But Equation (4.52) implies that for each $h \in H$, $\lambda^h = \bar{\lambda}^h$ solves the problem

$$\text{minimize} \left\{ \sum_{i \in Q_h} [\lambda_i^h - (\bar{\lambda}_i^h - \mu \bar{w}_i^h)]^2 : \sum_{i \in Q_h} \lambda_i^h = 1,\ \lambda_i^h \geqslant 0\ i \in Q_h \right\} \text{ for each } h \in H.$$

In other words, the projection $P_\Lambda(\bar{\lambda} - \bar{w}\mu)$ of $(\bar{\lambda} - \bar{w}\mu)$ onto Λ is equal to $\bar{\lambda}$ for any $\mu = 0$.

In view of Poljak's result [18,19], since $\bar{w}$ is a subgradient of $f(\cdot)$ at $\bar{\lambda}$, then $\bar{\lambda}$ solves PD_2. This completes the proof.

Note that Lemma 3 above states that if the "closest" feasible direction $-w$ to $-\bar{w}$ is a zero vector, then $\bar{\lambda}$ solves PD_2. Based on this result, we derive through Lemma 4 below a second sufficient condition for $\bar{\lambda}$ to solve PD_2.

LEMMA 4: Suppose $w = 0$ solves Problems PR_h, $h \in H$ as in Lemma 3. Then for each $h \in H$, we must have

(4.53) (a) $\bar{w}_i^h = t_h$, a constant, for each $i \notin J_h$

(b) $\bar{w}_i^h \leqslant t_h$ for each $i \in J_h$

where J_h is given by Equation (4.49).

PROOF: Let us write the Kuhn-Tucker conditions for Problem PR_h, for any $h \in H$. We obtain

$$(w_i^h - \bar{w}_i^h) + t_h = 0 \text{ for } i \notin J_h$$

$$(w_i^h - \bar{w}_i^h) + t_h - u_i^h = 0 \text{ for } i \in J_h$$

$$u_i^h \geqslant 0,\ i \in J_h,\ u_i^h w_i^h = 0\ \ i \in J_h,\ t_h \text{ unrestricted}$$

$$\sum_{i \in Q_h} w_i^h = 0,\ w_i^h \geqslant 0 \text{ for } i \in J_h.$$

If $w = 0$ solves PR_h, $h \in H$, then since PR_h has a convex objective function and linear constraints, then there must exist a solution to

$$\bar{w}_i^h = t_h \text{ for each } i \notin J_h$$

and

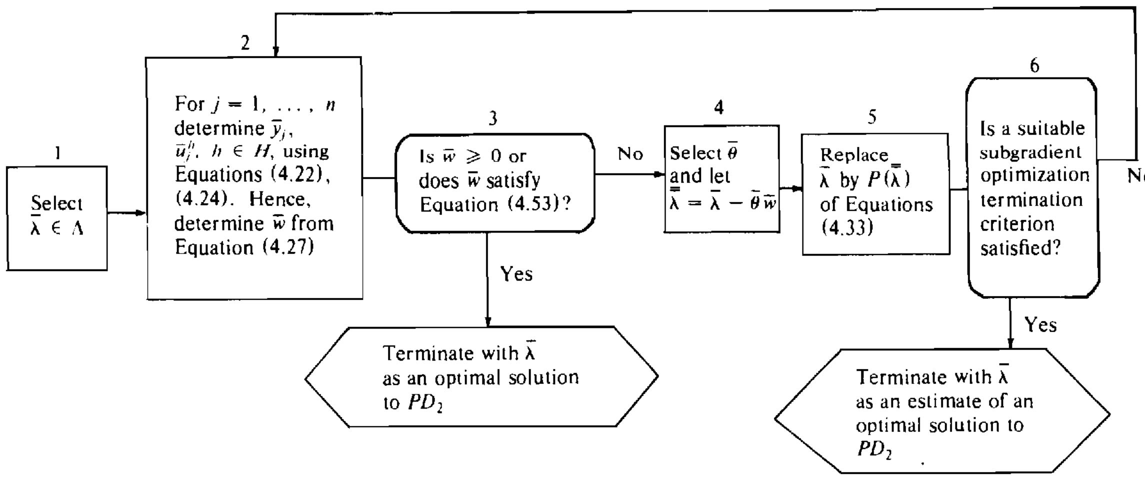
$$u_i^h = (t_h - \bar{w}_i^h) \geqslant 0 \text{ for each } i \in J_h.$$

This completes the proof.

Thus Equation (4.53) gives us another sufficient condition for $\bar{\lambda}$ to solve PD_2. We illustrate the use of this condition through an example in Section 4.1.7.

4.1.5 Schema of an Algorithm to Solve Problem PD_2

The procedure is depicted schematically below. In block 1, an arbitrary or preferably, a good heuristic solution $\bar{\lambda} \in \Lambda$ is sought. For example, one may use $\bar{\lambda}_i^h = 1/|Q_h|$ for each $i \in Q_h$, for $h \in H$. For blocks 4 and 6, we recommend the procedural steps proposed by Held, Wolfe and Crowder [12] for the subgradient optimization scheme.



4.1.6 Derivation of a Good Subgradient Direction

In our discussion in Section 4.1.1, we saw that given a $\lambda \in \Lambda$ of Equation (4.28), we were able to uniquely determine $\bar{y}_j$, $j = 1, \ldots, n$ through Equation (4.22). Thereafter, once we fixed values $\bar{u}_j^h$ for u_j^h, $j = 1, \ldots, n$, $h \in H$ satisfying Equation (4.24), we were able to uniquely determine values for the other variables in the Kuhn-Tucker System using Equations (4.26), (4.27). Moreover, the only choice in determining $\bar{u}_j^h$, $j = 1, \ldots, n$, $h \in H$ arose in case $|H_j| \geq 2$ for some $j \in \{1, \ldots, n\}$ in Equation (4.25). We also established that no matter what feasible values we selected for u_j^h, $j \in \{1, \ldots, n\}$, $h \in H$, the corresponding vector w obtained was a subgradient direction. In order to select the best such subgradient direction, we are interested in finding a vector $\bar{w}$ which has the smallest euclidean norm among all possible vectors corresponding to the given solution $\bar{\lambda} \in \Lambda$. However, this problem is not easy to solve. Moreover, since this step will merely be a subroutine at each iteration of the proposed scheme to solve PD_2, we will present a heuristic approach to this problem.

Towards this end, let us define for convenience, mutually exclusive but not uniquely determined sets $N_h, h \in H$ as follows:

(4.54) $$N_h \subset \{j \in \{1, \ldots, n\}: h \in H_j \text{ of Equation } (4.23)\}$$

(4.55) $$N_i \cap N_j = \{\phi\} \text{ for any } i, j \in H \text{ and } \bigcup_{h \in H} N_h = \{j \in \{1, \ldots, n\}: \bar{y}_j > 0\}.$$

In other words, we take each $j \in \{1, \ldots, n\}$ which has $\bar{y}_j > 0$, and assign it to some $h \in H_j$, that is, assign it to a set N_h, where $h \in H_j$. Having done this, we let

(4.56) $$\bar{u}_j^h = \begin{cases} 2\bar{y}_j & \text{if } j \in N_h \\ 0 & \text{otherwise} \end{cases} \quad \text{for each } j \in \{1, \ldots, n\},\ h \in H.$$

Note that Equation (4.56) yields values $\bar{u}_j^h$ for u_j^h, $j \in \{1, \ldots, n\}$, $h \in H$ which are feasible to (4.24). Hence, having defined sets N_h, $h \in H$ as in Equations (4.54), (4.55), we determine $\bar{u}_j^h$, $j \in \{1, \ldots, n\}$, $h \in H$ through (4.56) and hence $\bar{w}$ through (4.27).

Thus, the proposed heuristic scheme commences with a vector w obtained through an arbitrary selection of sets N_h, $h \in H$ satisfying Equations (4.54), (4.55). Thereafter, we attempt to improve (decrease) the value of $w^t w$ in the following manner. We consider in turn each $j \in \{1, \ldots, n\}$ which satisfies $|H_j| \geq 2$ and move it from its current set N_{h_j}, say, to another set

N_h with $h \in H_j$, $h \neq h_j$, if this results in a decrease $w'w$. If no such single movements result in a decrease in $w'w$, we terminate with the incumbent solution w as the sought subgradient direction. This procedure is illustrated in the example given below.

4.1.7 Illustrative Example

The intention of this subsection is to illustrate the scheme of the foregoing section for determining a good subgradient direction as well as the termination criterion of Section 4.1.4.

Thus, let $H = \{1,2\}$, $n = 3$, $|Q_1| = |Q_2| = 3$ and consider the constraint sets

$$S_1 = \left\{ \begin{array}{rl} x: & 2x_1 - 3x_2 + x_3 \geqslant 1 \\ & -x_1 + 2x_2 + 3x_3 \geqslant 1 \\ & 3x_1 - x_2 - x_3 \geqslant 1 \\ & x_1, x_2, x_3 \geqslant 0 \end{array} \right\} \text{ and } S_2 = \left\{ \begin{array}{rl} x: & 3x_1 - x_2 - x_3 \geqslant 1 \\ & 2x_1 + x_2 - 2x_3 \geqslant 1 \\ & -x_1 + 3x_2 + 3x_3 \geqslant 1 \\ & x_1, x_2, x_3 \geqslant 0 \end{array} \right\}$$

Further, suppose we are currently located at a point $\bar{\lambda}$ with

$$\bar{\lambda}_1^1 = 0,\ \bar{\lambda}_2^1 = 5/12,\ \bar{\lambda}_3^1 = 7/12;\ \bar{\lambda}_1^2 = 7/12,\ \bar{\lambda}_2^2 = 0,\ \bar{\lambda}_3^2 = 5/12.$$

Then the associated surrogate constraints are

$$\frac{4}{3}x_1 + \frac{1}{4}x_2 + \frac{2}{3}x_3 \geqslant 1 \text{ for } h = 1$$

(4.57)

$$\frac{4}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \geqslant 1 \text{ for } h = 2.$$

Using Equations (4.22), (4.25), we find

$$\bar{y}_1 = \frac{4}{3} \text{ with } H_1 = \{1,2\},\ \bar{y}_2 = \frac{2}{3} \text{ with } H_2 = \{2\} \text{ and } \bar{y}_3 = \frac{2}{3} \text{ with } H_3 = \{1,2\}.$$

Note that the possible combinations of N_1 and N_2 are as follows:

(i) $N_1 = \{1\}$, $N_2 = \{2,3\}$,

(ii) $N_1 = \{\phi\}$, $N_2 = \{1,2,3\}$,

(iii) $N_1 = \{1,3\}$, $N_2 = \{2\}$, and

(iv) $N_1 = \{3\}$, $N_2 = \{1,2\}$.

A total enumeration of the values of u obtained for these sets through (4.56) and the corresponding values for w are shown below.

		$u_j^h,\ j \in \{1,\ldots,n\}$						$w_i^h,\ i \in Q_h,\ h \in H$						
N_1	N_2	u_1^1	u_2^1	u_3^1	u_1^2	u_2^2	u_3^2	w_1^1	w_2^1	w_3^1	w_1^2	w_2^2	w_3^2	$w'w$
{1}	{2,3}	8/3	0	0	0	4/3	4/3	16/9	−56/9	40/9	−40/9	−28/9	56/9	129.78
{ϕ}	{1,2,3}	0	0	0	8/3	4/3	4/3	0	0	0	0	−4/3	0	1.78
{1,3}	{2}	8/3	0	4/3	0	4/3	0	20/9	−28/9	20/9	−20/9	4/9	28/9	34.37
{3}	{1,2}	0	0	4/3	8/3	4/3	0	−4/9	28/9	−20/9	20/9	20/9	−28/9	34.37

Thus, according to the proposed scheme, if we commence with $N_1 = \{1\}$, $N_2 = \{2,3\}$, then picking $j = 1$ which has $|H_j| = 2$, we can move $j = 1$ into N_2 since $2 \in H_1$. This leads to an improvement. As one can see from above, no further improvement is possible. In fact, the best solution shown above is accessible by the proposed scheme by all except the third case which is a "local optimal".

We now illustrate the sufficient termination condition of Section 4.1.4. The vector $\bar{w}$ obtained above is $(\overset{h=1}{0,0,0}\,|\,0,-4/3,0)$. Further the vector $\bar{\lambda}$ is $(\overset{h=1}{0,\ 5/12,\ 7/12}\,|\,\overset{h=2}{7/12,\ 0,\ 5/12})$. Thus, even though $\bar{w} \not\geq 0$, we see that the conditions (4.53) of Lemma 6 are satisfied for each $h \in H = \{1,2\}$ and thus the given $\bar{\lambda}$ solves PD_2.

The disjunctive cut (3.8) derived with this optimal solution $\bar{\lambda}$ is obtained through (4.57) as

$$\frac{4}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \geqslant 1. \tag{4.58}$$

It is interesting to compare this cut with that obtained through the parameter values $\bar{\lambda}_i^h = 1/|Q_h|$ for each $i \in Q_h$ as recommended by Balas [1,2]. This latter cut is

$$\frac{4}{3}x_1 + x_2 + x_3 \geqslant 1. \tag{4.59}$$

Observe that (4.58) uniformly dominates (4.59).

4.2 Maximizing the Rectilinear Distance Between the Origin and the Disjunctive Cut

In this section, we will briefly consider the case where one desires to use rectilinear instead of euclidean distances. Extending the developments of Sections 2, 3 and 4.1, one may easily see that the relevant problem is

$$\text{minimize}\,\{\underset{j \in \{1,\ldots,n\}}{\text{maximum}}\, y_j\text{: constraints (4.12), (4.13), (4.14) are satisfied}\}.$$

The reason why we consider this formulation is its intuitive appeal. To see this, note that the above problem is separable in $h \in H$ and may be rewritten as

$$PD_1\text{: minimize}\left\{\xi^h\text{: } \xi^h \geqslant \sum_{i \in Q_h} \lambda_i^h a_{ij}^h \text{ for each } j = 1, \ldots, n,\ \sum_{i \in Q_h} \lambda_i^h = 1,\ \lambda_i^h \geqslant 0 \text{ for } i \in Q_h,\ \xi^h \geqslant 0\right\} \text{ for each } h \in H.$$

Thus, for each $h \in H$, PD_1 seeks λ_i^h, $i \in Q_h$ such that the largest of the surrogate constraint coefficients is minimized. Once such surrogate constraints are obtained, the disjunctive cut (3.8) is derived using the principles of Section 3.

As far as the solution of Problem PD_1 is concerned, we merely remark that one may either solve it as a linear program or rewrite it as the minimization of a piecewise linear convex function subject to linear constraints and use a subgradient optimization technique.

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A FINITELY CONVERGENT PROCEDURE FOR FACIAL DISJUNCTIVE PROGRAMS

by

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A FINITELY CONVERGENT PROCEDURE FOR FACIAL DISJUNCTIVE PROGRAMS†

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and

C. M. Shetty**

Abstract: This paper addresses an important special class of disjunctive programs called facial disjunctive programs, examples of which include the zero-one linear integer programming problem and the linear complementarity problem. Balas has characterized some fundamental properties of such problems, one of which has been used by Jeroslow to obtain a finitely covergent procedure. This paper exploits another basic property of facial disjunctive programs in order to develop an alternative finitely convergent algorithm.

Key words: Facial Problem, Disjunctive Program, Cutting Plane Procedure

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A Finitely Convergent Procedure for Facial

Disjunctive Programs

1. Introduction

A linear disjunctive programming problem is concerned with the minimization of a linear objective function subject to a set of linear constraints with the added restriction that a given number of logical disjunctive statements must be satisfied. Each of these logical conditions is usually stated in terms of linear constraints. Mathematically, one may formulate such a problem as

DP: minimize cx

subject to

$$x \in X = \{x: Ax = b,\ x \geq 0\} \tag{1.1}$$

$$x \in D = \bigcap_{h \in H} \left[\bigcup_{i \in Q_h} S_i \right] \tag{1.2}$$

where

$$S_i = \{x: D^i x \geq d^i,\ x \geq 0\},\ i \in Q_h,\ h \in H \tag{1.3}$$

Here, c is a $(1 \times n)$ real vector, $x = (x_1, \ldots, x_n)$ is an $(n \times 1)$ vector of variables, X is assumed to be a non-empty and bounded polyhedral set and both H and Q_h, $h \in H$ are assumed to be index sets of finite cardinality. The disjunction (1.2), written above in the so-called conjunctive normal form [1,2], requires that for each $h \in H$, a feasible point x must belong to at least one of the sets S_i for $i \in Q_h$.

The basic thrust of solution procedures which have been proposed for Problem DP has been one of relaxation, wherein one relaxes the constraints

(1.2) initially, and then generates inequalities implied by (1.2) whenever the optimal solution to the relaxed problem (which is a linear program over that subset of X which is feasible to previously generated inequalities) is infeasible to (1.2). The inequalities implied by (1.2) are generated through the result stated below.

Theorem 1: Let the sets S_i, $i \in Q_h$ be as defined in Equation (1.3), and consider the disjunction

$$x \in \bigcup_{i \in Q_h} S_i \tag{1.4}$$

Then, for any choice of non-negative (row) vectors λ^i, $i \in Q_h$, the inequality

$$\left(\max_{i \in Q_h} \lambda^i D^i\right) x \geq \min_{i \in Q_h} \lambda^i d^i \tag{1.5}$$

is implied by (1.4), where $\max_{i \in Q_h} \lambda^i D^i$ denotes the point-wise (or component-wise) maximum of the vectors $\lambda^i D^i$ for $i \in Q_h$. Conversely, if each S_i, $i \in Q_h$ is consistent, then given any inequality $\sum_{j=1}^{n} \pi_j x_j \geq \pi_0$ implied by (1.4), there exist non-negative vectors λ^i, $i \in Q_h$ such that $\pi_0 \leq \min_{i \in Q_h} \lambda^i d^i$, and for each $j = 1,\ldots,n$, the j^{th} component of $\max_{i \in Q_h} \lambda^i D^i$ does not exceed π_j.

Proof: See [8].

The forward part of Theorem 1 is due to Balas [1,2] and the converse is due to Jeroslow [8]. However, this result has also been independently established by Glover [5,6] in a somewhat different problem setting. Incidentally, Blair and Jeroslow [4] also discuss the conditions under which

(1.5) yields all valid inequalities. Essentially, Theorem 1 asserts that based on the disjunction (1.4), an implied inequality, also called a valid inequality or a (valid) disjunctive cut, may be obtained by simply surrogating the rows of each S_i using non-negative multipliers and then performing the operation of (1.5) on the resulting $|Q_h|$ surrogate constraints. The converse points out that there always exist non-negative surrogate multiplier vectors λ^i such that the inequality (1.5) uniformly dominates on the non-negative orthant any given valid inequality.

We will now proceed to introduce a special class of disjunctive programs which is of interest to us, namely, facial disjunctive programs [3]. Thereafter, we will propose a finitely covergent algorithm for such problems, and finally, we will present an illustrative example.

2. Facial Disjunctive Programs (FDP)

A facial disjunctive program [3] (denoted FDP) is a special class of disjunctive programs (DP) in which each set S_i is comprised of a single constraint, viz,

$$S_i = \{x: d^i x \geq d_{i0}\} \quad i \in Q_h, \; h \in H \tag{2.1}$$

where d^i is a 1xn real vector and d_{i0} is a scalar, such that $X \cap S_i$ is a face of X for each $i \in Q_h$, $h \in H$. (Recall that for any convex set X, a non-empty subset F of X is called a (proper) face of X provided there exists a supporting hyperplane H of X such that $F = X \cap H$ [7]). Important cases of facial disjunctive programs include the zero-one linear integer programming problem and the linear complementarity problem. For the sake of illustration, consider a linear complementarity problem which involves orthogonality or complementarity constraints of the type $x_p x_q = 0$, $p, q \in \{1,\ldots,n\}$. Each such constraint constitutes a disjunction $h \in H$ with the corresponding set $Q_h = \{p,q\}$, and the constraints $d^p x \geq d_{p0}$, $d^q x \geq d_{q0}$ being respectively $-x_p \geq 0$ and $-x_q \geq 0$. Note that we do not necessarily require that $Q_i \cap Q_j = \phi$ for $i,j \in H$, $i \neq j$.

Several procedures can readily be developed for solving Problem FDP. Some of these may in fact be viable approaches in special instances. The first alternative would be to rank the extreme points of the set X with respect to the objective function cx till such time as a ranked extreme point satisfies the disjunction (1.2). Theorem 2 below validates this approach. A second alternative would be to write Problem FDP in a disjunctive normal form [1,2] and then solve the linear programs of minimizing cx subject to $x \in X$ and each of the resulting disjunctions. Trivially, the best solution to these linear programs would be optimal to FDP. Note that for the linear complementarity problem for example, this implies that each of the linear programs would enforce one of every pair of complementary variables equal to zero. Hence, this approach is viable, in general, only if there are very few disjunctive constraints in the disjunctive normal form.

Several other approaches are also available by noting that Problem DP in general, and Problem FDP in particular, is equivalent to the linear program [3]

$$\text{minimize} \quad \{cx: x \in \hat{Y} \equiv \text{conv}[X \cap D]\} \qquad (2.2)$$

where conv [S] denotes the closure of the convex hull of a set S. Now consider a relaxation strategy wherein constraints (1.2) are relaxed. Let $\bar{x}$ solve the resulting linear program. If $\bar{x} \in D$, then clearly $\bar{x}$ solves Problem DP. Otherwise, a cutting plane which is a facet of the set Y could theoretically be developed and the procedure repeated. Balas [3] discusses how one may perform this rather difficult task. Noting the equivalence of Problem DP and (2.2), the procedure is clearly finite.

Now in the case of Problem FDP some specializations are possible based on two important results established by Balas [3] concerning the set Y. The first provides a means for inductively constructing the set Y. Jeroslow's precedure [10] uses this result and at a point $\bar{x} \notin D$, specific conditions are laid on the type of cutting planes used and the manner of generating them. Essentially, finiteness is ensured since any given disjunction can be violated only finitely often.

The procedure we propose is also a relaxation strategy and uses the second result of Balas [3] stated as Theorem 2 below. It deals with the extreme points of the set Y.

Theorem 2: Let the set X be defined by Equation (1.1), and let Y be the convex hull of $X \cap D$, where the set D is given by Equations (1.2) and (2.1) such that the disjunctive program DP is facial. Then,

$$\text{vert } Y \subseteq \text{vert } X \tag{2.3}$$

where vert Y denotes the set of extreme points of the polyhedral set Y.

Proof: (Balas [3]).

The advantage of using property (2.3) along with the fact that Problem DP (and hence Problem FDP) is equivalent to Problem (2.2) is that, unlike

Jeroslow's procedure, our scheme places no restrictions on the type of cuts which are permissible. The reason being that we secure finiteness by relying instead on the type of points at which cuts are generated. Specifically, these points are required to be extreme faces [11] of the set X with respect to the set of cuts generated at any stage of the procedure. For the sake of completeness, we summarize below Majthay and Whinston's [11] discussion on extreme faces and their detection.

3. Extreme Faces and Their Detection

Let us assume that at a particular stage s cuts, $Gx \leq g$, have been generated in the space of the x-variables. Let

$$\Lambda = \{x \varepsilon R^n : \quad Gx + Ix_S = g, \quad x_S \geq 0\} \tag{3.1}$$

be the subset of R^n feasible to these cuts. Here, $x_S = (x_{n+1},\ldots,x_{n+s})$ denotes the vector of slack variables, and I is an identity matrix of size s. Further, let $N = \{1,\ldots,n\}$ denote the index set of the original x-variables, which we will call key variables. Also, let $K = \{n+1,\ldots n+s\}$ denote the index set of the slack variables for the s cuts, which we will call nonkey variables. For a set $Z \subseteq N$, let

$$F_Z = \{x \varepsilon X \quad : \quad x_j = 0 \quad \text{for } j \varepsilon Z\} \tag{3.2}$$

Note that all faces of X can be represented as F_Z for some suitable set Z. Finally, for any point $x \varepsilon F_Z$, let the zero components of x be denoted by

$$Z(x) = \{j \varepsilon N \quad : \quad x_j = 0\} \tag{3.3}$$

Definition [11]

Let F_Z be a face of X defined by some $Z \subseteq N$ such that $F_Z \cap \Lambda \neq \phi$. Then F_Z is an extreme face of X relative to Λ if for any two points $x^1, x^2 \in F_Z \cap \Lambda$, we have $Z(x^1) = Z(x^2)$.

In other words, an extreme face F_Z satisfies the property that $F_Z \cap \Lambda$ does not contain any point in a lower dimensional face of X. Examples of extreme faces of X relative to Λ are extreme points of X feasible to Λ, or an edge of X not disjoint with Λ but with neither of the two extreme points of X defining this edge being feasible to Λ.

Given a simplex tabular representation of an extreme point of $X \cap \Lambda$ at any stage, Majthay and Whinston [11] propose a simple procedure to find an extreme face of X relative to Λ. This procedure utilizes the following restricted basis entry rule:

"Only a nonkey variable x_j, $j \in K$, is eligible to enter the basis" (3.4)

Based on this, the method outlined below either finds an extreme face or indicates that no such face exists.

Step 1: Let x_γ denote the largest valued basic key variable in the current solution which has not yet been considered at a previous iteration. If no such variable exists, go to Step 3. Otherwise, proceed to Step 2.

Step 2: Solve the Problem P_γ: minimize $\{x_\gamma : x \in X \cap \Lambda\}$ as a linear program subject to the restricted basis entry rule (3.4). If the solution yields $x_\gamma = 0$ and x_γ is basic, pivot it out of the basis, if possible, by exchanging it with a nonkey, non-basic variable. Return to Step 1.

Step 3: If all key variables are basic, there is no extreme face of X relative to Λ. (That is, $X \cap \Lambda$ is contained in the interior of X). Otherwise, the current set Z of indices of nonbasic key variables defines, through Equation (3.2), an extreme face F_Z of X relative to Λ. In particular, if all nonbasic variables are key variables, then F_Z represents an extreme point of X.

Now, observe that from Theorem 2 the relaxation procedure for Problem FDP can be made finite if each cutting plane introduced deletes an extreme point of X. However, we will find it simpler to restrict our search to a larger set, namely, the extreme faces of X. Since extreme faces of X relative to some Λ are also faces of X, the number of such extreme faces of X (relative to all Λ's) is finite. Hence, a procedure which detects and deletes in a finite number of steps at least one extreme face per iteration is finitely convergent [11]. This is indeed the principal thrust of our scheme.

4. Proposed Algorithm for Facial Disjunctive Programs

The procedure we advocate is summarized in the flowchart of Figure 1 and is clearly finitely convergent. The flowchart uses two types of cuts to delete an extreme face depending upon whether F_Z is an extreme point $\bar{x} \notin D$ or not. The specific means of obtaining these cuts are developed in Sections 4.2 and 4.1 respectively. The flow chart also incorporates an objective function cut $cx \leq \hat{\nu}$ based on the current best value $\hat{\nu}$ of the objective function. The purpose of this cut is discussed later in this section.

The proposed procedure proceeds as follows. At any stage, given the set Λ of Equation (3.1), we solve the relaxed problem

$$\underline{P(\Lambda)}: \text{ minimize } \{cx: x \in X \cap \Lambda\} \tag{4.1}$$

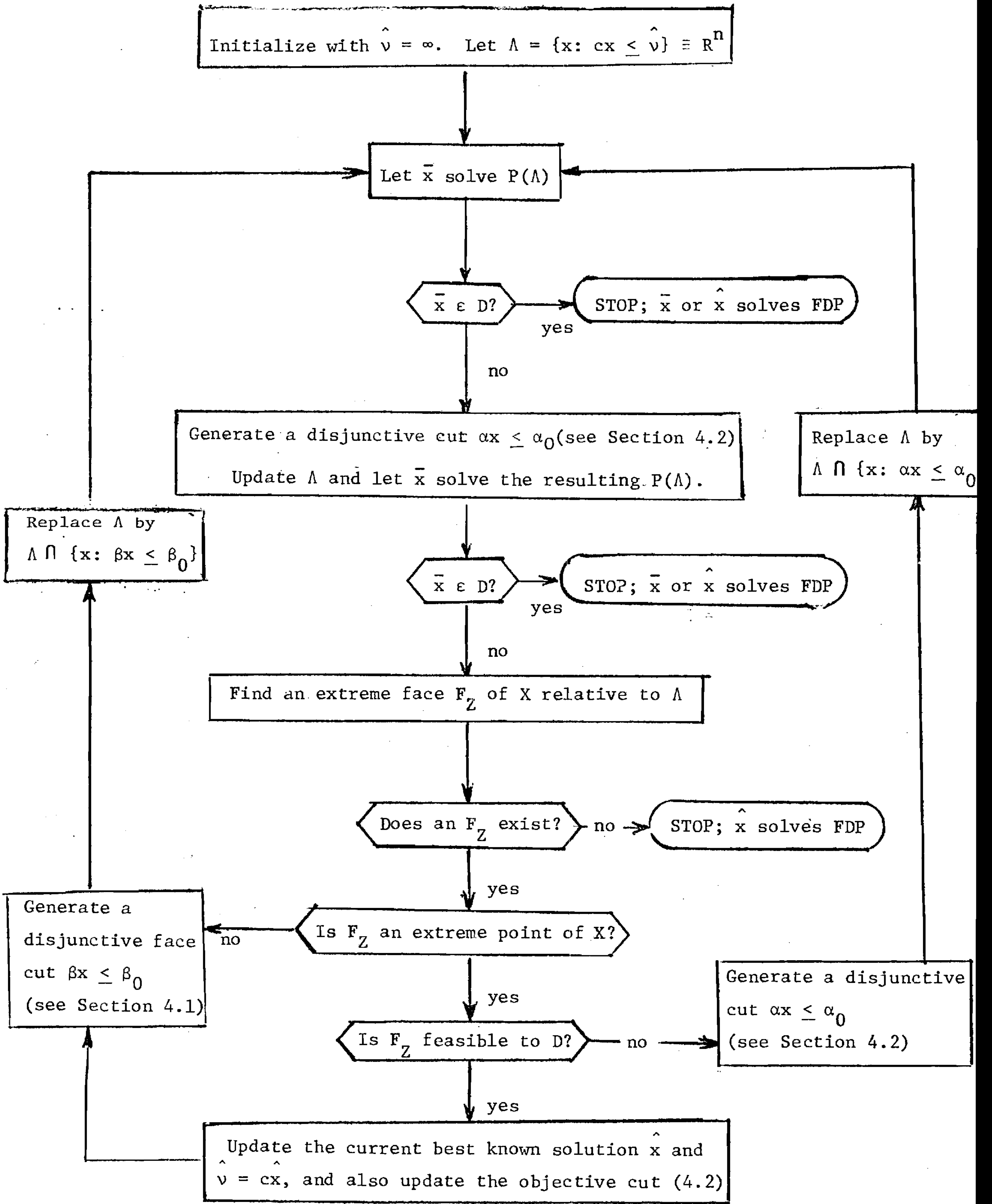

Figure 1. Flowchart for the Proposed Scheme

If an optimal solution $\bar{x}$ to this problem satisfies $\bar{x} \in D$ of Eqs. (1.2) and (2.1), then $\bar{x}$ clearly solves the problem of minimizing cx over the set $X \cap \Lambda \cap D$. However, since the set of cuts Λ we generate may also have been used to validly delete extreme points of X which have been accounted for, we terminate the solution procedure by picking the better of the two solutions, namely, $\bar{x}$ and the known incumbent solution $\hat{x}$, say. Otherwise, $\bar{x}$ violates at least one disjunction $h \in H$, and we generate an inequality or disjunctive cut implied by the most violated disjunction. The set Λ of Equation (3.1) is updated using this cut and $P(\Lambda)$ is resolved. Let $\bar{x}$ represent the new optimal solution. Again, if $\bar{x} \in D$, then either $\bar{x}$ or $\hat{x}$ solves Problem FDP. Now, if $\bar{x} \notin D$, then instead of generating another disjunctive cut, in order to ensure finiteness, we use the routine of Section 3 to find an extreme face F_Z of X relative to Λ. If no extreme face exists, then we terminate with the current best known solution as optimal to FDP since then, $X \cap \Lambda$ is contained in the interior of X. Otherwise, depending on the dimension of F_Z, we adopt one of the following two alternative routes. If F_Z is of dimension greater than zero, then a disjunctive face cut is developed (see 4.1) which deletes F_Z but no extreme point of X feasible to Λ. On the other hand, if F_Z represents an extreme point of X then we check if this extreme point is feasible to D. If it is, then we update the current best known solution, if necessary, and generate a disjunctive face cut which deletes only this particular extreme point of X. If the extreme point is infeasible to D, however, the usual disjunctive cut may be generated, which is typically stronger than the disjunctive face cut. In any case, after the appropriate cut has been generated and Λ has been updated, we say that an iteration has been completed. A new iteration is now commenced by solving Problem $P(\Lambda)$ of Equation (4.1).

Now, if the procedure does not terminate with the solution $\bar{x}$ to a relaxation $P(\Lambda)$ satisfying the disjunction (1.2), (2.1), then it is possible that a significant effort may be expended in detecting and cutting away extreme points of X feasible to D. To avoid this, we adopt an additional expedient of imposing the cut

$$cx \leq \hat{\nu} \tag{4.2}$$

based on the objective function, where $\hat{\nu}$ is the current best known objective value of Problem FDP. Hence, the right hand side of this cut is simply updated each time an improved solution is detected. Although this cut will not (locally) affect the solution of Problems $P(\Lambda)$, it will assist in confining the search to improving solutions during the extreme face finding routine. Again, this is essential because otherwise, the extreme face finding routine would simply concentrate on feasibility, regardless of objective function values. Hence, two main advantages accrue from the use of (4.2). One, we are able to use (4.2) to delete non-improving extreme points of X feasible to D, which hence need not be explicitly enumerated. Second, it is likely that we will detect an optimal solution to FDP early on in the process since the extreme face finding routine attempts to trace extreme points of X in the neighborhood of the relaxed problem solution. This is important if the procedure is prematurely terminated for large problems.

To complete the details on the implementation of the proposed algorithm, in the following subsection we will discuss the manner in which one may generate a disjunctive face cut $\beta x \leq \beta_0$ in case F_Z is either of dimension greater than zero, or F_Z is of dimension zero and is contained in D. Section 4.2 will deal with the generation of disjunctive cuts $\alpha x \leq \alpha_0$ in the other cases.

4.1. Disjunctive face cuts $\beta x < \beta_0$ [13].

Suppose we are at a stage where we have solved Problem $P(\Lambda)$ for some Λ and have obtained an optimal solution $\bar{x} \notin D$. Now, to find an extreme face of X relative to Λ we will be attempting to minimize each basic key variable subject to the current constraints and the restricted basis entry rule (3.4). Thus, suppose we have currently minimized a key variable x_r through Problem P_r of Step 2 of this routine and it has turned out that x_r is positive at optimality. Define

$$N_r = \{j \varepsilon N: x_j \text{ is non-basic at optimality of } P_r\} \tag{4.3}$$

$$K_r = \{j \varepsilon K: x_j \text{ is non-basic at optimality of } P_r\} \tag{4.4}$$

where N and K are the index sets of key and nonkey variables respectively. Let the canonical representation of x_r in terms of the non-basic variables x_j, $j \varepsilon N_r \cup K_r$ be

$$x_r + \sum_{j \varepsilon N_r} a_{rj}\, x_j + \sum_{j \varepsilon K_r} a_{rj}\, x_j = b_r \tag{4.5}$$

Hence, by assumption, $b_r > 0$. But observe that since the coefficients a_{rj}, $j \varepsilon N_r \cup K_r$ are reduced cost coefficients at the optimality of P_r, we must have

$$a_{rj} \leq 0 \quad \text{for } j \varepsilon K_r \tag{4.6}$$

since otherwise, x_j is a candidate to enter the basis. Of course, the restricted basis entry rule could result in the coefficients a_{rj}, $j\varepsilon N_r$ being of either sign.

In this manner, when the routine of Section 3 finally terminates with an extreme face F_Z of X (assuming one exists) relative to Λ, let the tableau represent an extreme point $x^0 = (x_1^0,\ldots,x_n^0)$ of $X \cap \Lambda$, with $x^0 \; \varepsilon \; F_Z$ where

$$Z = \{j\varepsilon N: x_j \text{ is non-basic in the tableau representing } x^0\} \tag{4.7}$$

Now, define

$$R = \{r: x_r^0 > 0\} \tag{4.8}$$

and

$$\bar{K} = \bigcup_{r\varepsilon R} K_r \tag{4.9}$$

Note that $N_r \subseteq Z$ for each $r\varepsilon R$, and that the canonical equation (4.5) is available for each $r\varepsilon R$. Hence, adding zero coefficients as required, we get

$$x_r + \sum_{j\varepsilon Z \cup \bar{K}} a_{rj} \, x_j = b_r \quad \text{for each } r\varepsilon R \tag{4.10}$$

We want to develop a cut which deletes the extreme face F_Z, but does not cut away any other extreme point of X feasible to Λ. This is accomplished by insuring that for any such point at least one of the x_r, $r\varepsilon R$ must be zero. That is, at least one of the inequalities $x_r \leq 0$, $r \; \varepsilon \; R$ must hold (in the presence of non-negativity restrictions). Using (4.10), this condition may be restated as requiring that at least one of the following constraint sets must be satisfied

$$\left\{ \sum_{j \varepsilon Z \cup \bar{K}} \left(\frac{a_{rj}}{b_r} \right) x_j \geq 1, \; x_j \geq 0 \quad \text{for } j \;\varepsilon\; Z \cup \bar{K} \right\} \text{ for } r\varepsilon R \qquad (4.11)$$

From Theorem 1, a valid cut is

$$\sum_{j \varepsilon Z \cup \bar{K}} \left[\max_{r \varepsilon R} \left(\frac{a_{rj}}{b_r} \right) \right] x_j \geq 1 \qquad (4.12)$$

Observe from (4.6) that the inequality (4.12) implies over the non-negative orthant that

$$\sum_{j \varepsilon Z} \left[\max_{r \varepsilon R} \left(\frac{a_{rj}}{b_r} \right) \right] x_j \geq 1$$

or that (4.12) deletes F_Z since any $x \varepsilon F_Z$ satisfies $x_j = 0$ for each $j \varepsilon Z$. The cut (4.12) is hence called a disjunctive face cut. Observe that when F_Z is an extreme point of X then each equation of the type (4.10) is obtained from the same tableau representing the solution x^0, and furthermore $N_r \equiv Z$, and $K_r = \emptyset$ for each $r \varepsilon R$.

4.2 Disjunctive Cuts $\alpha x \leq \alpha_0$

We will now discuss the generation of a disjunctive cut at a point infeasible to D. This point may either be $\bar{x}$, an optimal solution to some $P(\Lambda)$, or it may be an extreme face $F_Z \equiv x^0$ of dimension zero. In either case, letting $\tilde{x}$ represent $\bar{x}$ or x^0 as the case may be, we have

$$\sum_{j\in N} d_j^i \tilde{x}_j < d_{i0} \quad \text{for each } i\in Q_h, \text{ for some } h\in H.$$

Let $\hat{h}\in H$ be the most violated disjunction, that is,

$$\min_{i\in Q_{\hat{h}}} \left(d_{i0} - \sum_{j\in N} d_j^i \tilde{x}_j \right) = \max_{h\in H} \left\{ \min_{i\in Q_h} \left(d_{i0} - \sum_{j\in N} d_j^i \tilde{x}_j \right) \right\} > 0$$

According to the current tableau representing $\tilde{x}$, let us partition $(x_1,\ldots,x_n)$ as $(x_1,\ldots,x_n) \equiv (x_B,x_Z)$ where x_B and x_Z are respectively the key variables which are currently basic and non-basic. Accordingly, partition d^i as $d^i = (d_B^i, d_Z^i)$. Finally, let x_J represent the vector of slack variables (a subvector of x_S) which are currently non-basic. Of course, if $\tilde{x} \equiv x^0$, then $x_J \equiv \phi$. Then the current tableau representing $\tilde{x}$ expresses x_B in terms of x_Z and x_J as

$$x_B = f + Ex_Z + Fx_J \tag{4.13}$$

where f, E and F are appropriate vector and matrices. Using (4.13), we may write $d^i x \geq d_{i0}$, $i\in Q_{\hat{h}}$ as

$$d_B^i(f+Ex_Z+Fx_J) + d_Z^i x_Z \geq d_{i0},\ i\in Q_{\hat{h}}$$

$$\text{or } (d_B^i E + d_Z^i)x_Z + d_B^i F x_J \geq d_{i0} - d_B^i f,\ i\in Q_{\hat{h}}$$

Since the disjunction $x\in \bigcup_{i\in Q_{\hat{h}}} S_i$ is violated (with S_i defined by (2.1)), and since $\tilde{x}_Z = 0$, $\tilde{x}_J = 0$, it follows that $d_{i0} - d_B^i f > 0$ for each $i\in Q_{\hat{h}}$. Thus one may invoke the disjunction that at least one of the following constraint sets must be satisfied

$$S_i = \left\{x: \left(\frac{1}{d_{i0}-d_B^i f}\right)\left[\{d_B^i E+d_Z^i\}x_Z + d_B^i F x_J\right] \geq 1,\ x \geq 0\right\}\ i\varepsilon \hat{Q}_h \qquad (4.14)$$

From this, Theorem 1 yields the disjunctive cut

$$\left(\max_{i\varepsilon \hat{Q}_h} \left(\frac{1}{d_{i0}-d_B^i f}\right)\left[\{d_B^i E+d_Z^i\},\ d_B^i F\right]\right)\begin{pmatrix} x_Z \\ x_J \end{pmatrix} \geq 1 \qquad (4.15)$$

which clearly deletes $\tilde{x}$.

Now, reference [14] suggests a means for strengthening the disjunctive cut (4.15). To see how this is achieved, let us denote the non-basic variables $x_Z \cup x_J$ by x_T and accordingly re-write (4.14) as

$$S_i = \{x: \sum_{j\varepsilon T} a_{ij}x_j \geq 1,\ x \geq 0\} \qquad i\varepsilon \hat{Q}_h \qquad (4.16)$$

Note that in addition to the disjunction $x \; \varepsilon \bigcup_{i\varepsilon \hat{Q}_h} S_i$, a feasible x must also satisfy the constraint set $Wx_T \leq w$ corresponding to the current tableau. Hence, one may invoke instead a disjunction $x \; \varepsilon \bigcup_{i\varepsilon \hat{Q}_h} \bar{S}_i$ where

$$\bar{S}_i = S_i \cap \{x: Wx_T \leq w\} \qquad i\varepsilon \hat{Q}_h \qquad (4.17)$$

The improvement technique proposed in [14] essentially attempts to derive a cut, in terms of the non-basic variables x_j, $j\varepsilon T$, which is a support for the closure of the convex hull of $\bigcup_{i\varepsilon \hat{Q}_h} \bar{S}_i$. This is accomplished by commencing with the cut (4.15), say, and attempting to improve (decrease) as much as possible each cut coefficient one at a time, holding the other cut coefficients fixed. Theorem 1 essentially lays the foundation for improving a given cut in this manner. One need only formulate an appropriate linear program to determine non-negative surrogate multipliers for the constraints of each set $\bar{S}_i$, $i \varepsilon Q_{\hat{h}}$ so that a given cut coefficient is minimized in the resulting disjunctive cut, without worsening the other coefficients. Instead of formally restating this strategy, we illustrate it in the following section.

5. Illustrative Example

Consider the facial disjunctive program

FDP:

$$\left.\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 \\ \text{subject to} & x_2 + x_3 = 5 \\ & x_1 + x_4 = 8 \\ & x_1 + x_2 + x_5 = 10 \\ & x \geq 0 \end{array}\right\} \quad (5.1)$$

$$\text{and } x_1 \leq 0 \text{ or } x_2 \leq 0 \quad (5.2)$$

where (5.1) represents $x \varepsilon X$ and (5.2) represents $x \varepsilon D$. With $\Lambda \equiv R^n$, the solution to $P(\Lambda)$ is summarized in the following simplex tableau

	x_3	x_5	RHS
objective row	1	2	25
x_2	1	0	5
x_4	1	-1	3
x_1	-1	1	5

This solution violates (5.2). The cut (4.15) may be generated from the disjunctive statement

$$x \in \left\{x: \frac{x_3}{5} \geq 1,\ x \geq 0\right\} \text{ or } \left\{x: \frac{-x_3}{5} + \frac{x_5}{5} \geq 1,\ x \geq 0\right\} \quad (5.3)$$

whence, (4.15) is

$$\left(\frac{x_3}{5} + \frac{x_5}{5} \geq 1\right) \text{ or } \left(x_3 + x_5 - x_6 = 5,\ x_6 \geq 0\right) \quad (5.4)$$

Later, we will generate a stronger cut using the strategy outlined at the end of Section 4.2. Appending (5.4) to the above tableau and updating it, one obtains the following tableau.

	x_4	x_6	RHS
objective row	1/2	3/2	19
x_2	-1/2	1/2	1
x_5	-1/2	-1/2	1
x_1	1	0	8
x_3	1/2	-1/2	4

This solution is still infeasible to (5.2) and we hence need to find an extreme face F_Z. Using the routine of Section 3 and commencing by minimizing x_2 above, subject to (3.4), we obtain the following tableau

		x_4	x_2	RHS
objective row		2	-3	16
	x_6	-1	2	2
	x_5	-1	1	2
	x_1	1	0	8
	x_3	0	1	5

This tableau represents an extreme face F_Z of X of dimension zero, and moreover F_Z is contained in D. Thus, we need to generate a disjunctive face cut as in Section (4.1). Here, R = {1,3,5}, Z = {2,4}, $\bar{K} = \phi$. The cut (4.12) is

$$\max\left\{-\frac{1}{2}, \frac{1}{8}\right\} x_4 + \max\left\{\frac{1}{2}, \frac{1}{5}\right\} x_2 \geq 1$$

or

$$\left(\frac{x_4}{8} + \frac{x_2}{2} \geq 1\right) \equiv (x_1 - 4x_2) \leq 0 \qquad (5.7)$$

Furthermore, we update our incumbent solution $\hat{x}$ as $(\hat{x}_1, \hat{x}_2) \equiv (8,0)$ with $\hat{\nu} = 16$. The objective cut (4.2) is

$$2x_1 + 3x_2 \geq 16 \qquad (5.8)$$

Now, (5.7) and (5.8) are appended to either tableau (5.5) and (5.6) and one iteration is completed. Figure 2 below illustrates the current situation.

The shaded area represents the remaining feasible region. The point H is an optimal solution to the current problem $P(\Lambda)$. The next disjunctive cut is easily seen to be $x_1 \leq 0$ which renders Problem $P(\Lambda)$ infeasible.

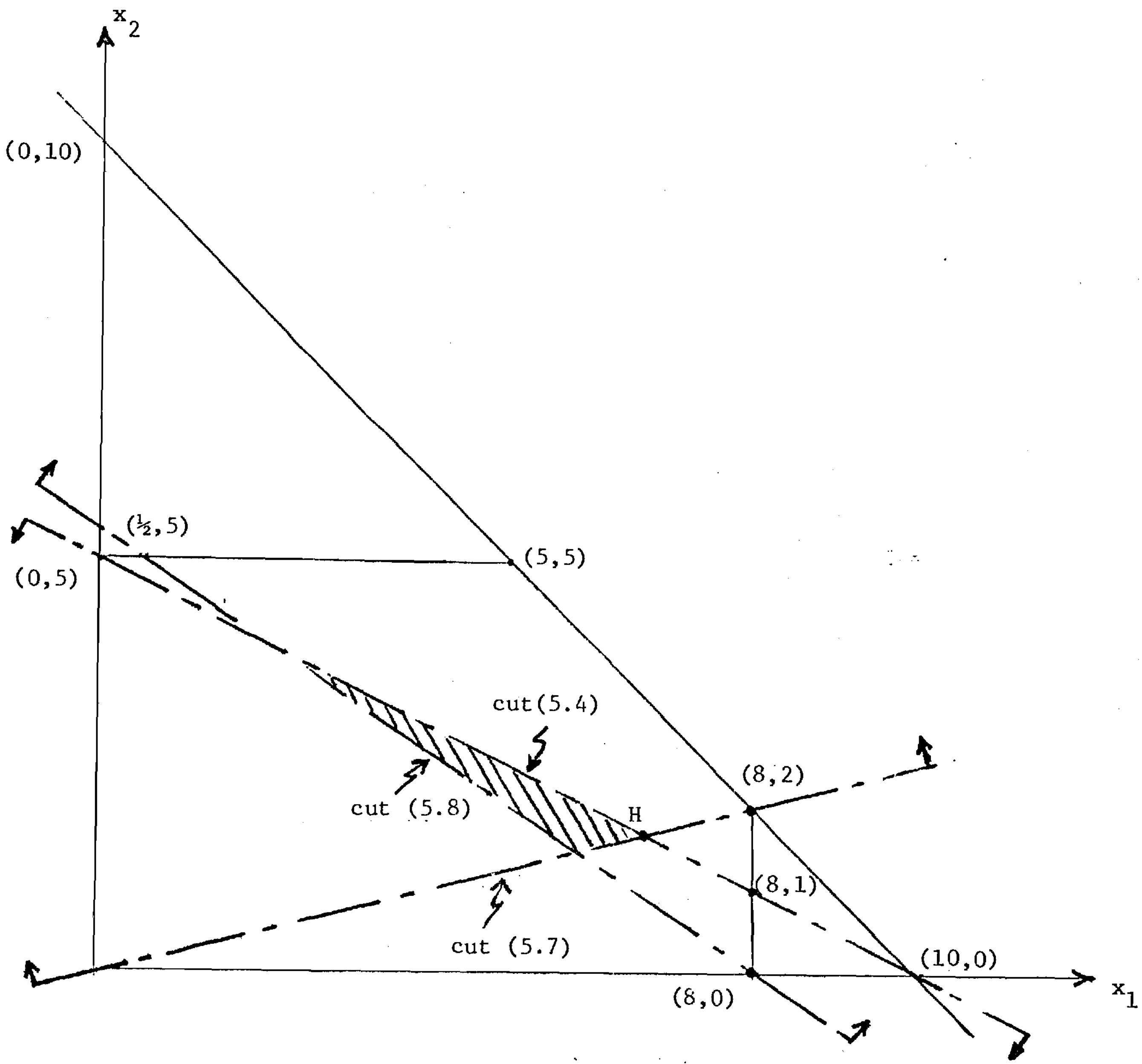

Figure 2. Illustrative Example

Thus $(\hat{x}_1,\hat{x}_2) \equiv (8,0)$ solves FDP. Note also that the region shaded in Figure 2 is contained in the interior of X and hence no extreme face of X exists. Furthermore, note that the cost cut (5.8) deletes the extreme points (0,5) and (0,0) of X which are feasible to (5.2), and hence saves the effort of having to explicitly enumerate these points.

Finally, let us illustrate how the disjunctive cut (5.4) may be strenthened using the ideas outlined in Section 4.2. Toward this end, note that $Wx_T \leq w$ may be taken to be the inequality $x_3 - x_5 \leq 3$ from the row x_4 of the first tableau, so that $\bar{S}_1$ and $\bar{S}_2$ of Equation (4.17) are

$$\bar{S}_1 = \left\{x: -\frac{x_3}{5} + \frac{x_5}{5} \geq 1,\ x_3 - x_5 \leq 3,\ x \geq 0\right\}$$

and

$$\bar{S}_2 = \left\{x: \frac{x_3}{5} \geq 1,\ x_3 - x_5 \leq 3,\ x \geq 0\right\}.$$

Now, it is easily verified that the coefficient of x_5 in (5.4) cannot be improved. Hence, let us attempt to decrease the coefficient of x_3 without worsening that of x_5. Letting (λ_1,λ_2) and (γ_1,γ_2) be respectively the surrogate multipliers for the sets $\bar{S}_1$and $\bar{S}_2$ yields the general disjunctive cut

$$\max\left\{-\frac{\lambda_1}{5} + \lambda_2,\ \frac{\gamma_1}{5} - \gamma_2\right\}x_3 + \max\left\{\frac{\lambda_1}{5} + \lambda_2, \gamma_2\right\}x_5$$

$$\geq \min\,\{\lambda_1 - 3\lambda_2, \gamma_1 - 3\gamma_2\} \tag{5.9}$$

Since this cut must dominate $\frac{x_3}{5} + \frac{x_5}{5} \geq 1$ while minimizing the coefficient of x_3, it is easily seen that the following linear program appropriately determines λ_1, λ_2, γ_1, and γ_2

$$\text{minimize} \quad \beta$$

$$\text{subject to}$$

$$\left(\begin{array}{l} \beta \geq -\frac{\lambda_1}{5} + \lambda_2 \\ \beta \geq \frac{\gamma_1}{5} - \gamma_2 \end{array}\right)$$

$$\left(\begin{array}{r} -\frac{\lambda_1}{5} + \lambda_2 \leq \frac{1}{5} \\ \gamma_2 \leq \frac{1}{5} \end{array}\right)$$

$$\left(\begin{array}{l} \lambda_1 - 3\lambda_2 \geq 1 \\ \gamma_1 - 3\gamma_2 \geq 1 \end{array}\right)$$

$$\lambda_1,\ \lambda_2,\ \gamma_1,\ \gamma_2 \geq 0.$$

An optimal solution to this problem is $\lambda_1 = 1$, $\lambda_2 = 0$, $\gamma_1 = \frac{40}{25}$, $\gamma_2 = \frac{5}{25}$, $\beta = \frac{3}{25}$ so that the cut (5.9) is

$$\frac{3}{25} x_3 + \frac{x_5}{5} \geq 1 \qquad (5.10)$$

In terms of (x_1,x_2), the cut (5.10) is

$$5x_1 + 8x_2 \leq 40$$

It is easily verified that when the cut (5.10) is appended to the first tableau, the resulting optimal solution yields $x_1 = 8$, $x_2 = 0$ and hence yields an optimal solution to Problem FDP.

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Disjunctive Programming, Polyhedral Annexation Techniques and Nondominated Disjunctive Cutting Planes

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Disjunctive Programming, Polyhedral Annexation Techniques and Nondominated Disjunctive Cutting Planes

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Abstract

This paper discusses some important aspects of disjunctive programs and establishes relevant connections between disjunctive programming methods and Glover's polyhedral annexation scheme. The effect of the formulation of disjunctions on the depth of cuts recovered therefrom is demonstrated. A procedure is presented for generating nondominated cuts. For a special class of problems it is compared with Glover's "sequential implementation" method in the context of polyhedral annexation.

Keywords: Disjunctive Programming, nondominated cuts, deep cutting planes, polyhedral annexation

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Disjunctive Programming, Polyhedral Annexation Techniques and Nondominated Disjunctive Cutting Planes

I. Introduction

The term disjunctive program is used to characterize problems which contain logical conditions stated as linear constraints. These logical conditions may in general be conjunctions, disjunctions, negations and implications. However, as discussed by Balas [1,2], an implication can be restated as a disjunction, and conjunctions and negations lead to (convex) polyhedral constraint sets. Problems with logical conditions which contain disjunctive statements are inherently much more difficult to solve. It is specifically this class of nonconvex problems which is referred to as disjunctive programs. Mathematically, the problem addressed may be stated as

$$\underline{\text{DP:}} \quad \text{minimize} \quad f(x)$$

$$\text{subject to} \quad x \in X \qquad (1.1)$$

$$x \in \bigcup_{h \in H} S_h \qquad (1.2)$$

where $f: R^n \to R$ is lower semicontinuous, X is a closed subset of R^n, and where

$$S_h = \{x: \; A^h x \geq b^h, \; x \geq 0\} \qquad \text{for each } h \in H, \; |H| < \infty \qquad (1.3)$$

Several well known problems including the generalized lattice point problem, the cardinality constrained problem, the extreme point optimization problem, the complimentarity problem and the mixed integer programming problem may be cast in the framework of Problem DP. One possible solution strategy for Problem DP evolves from the following straightforward result.

<u>Theorem 1.1</u> Consider Problem DP stated above and define problems

$$DP_h: \text{ minimize } \{f(x): \ x \varepsilon X \cap S_h\} \qquad \text{for each } h \varepsilon H \tag{1.4}$$

Let x^h solve DP_h. Then x^{h*} solves DP, where

$$f(x^{h*}) = \underset{h \varepsilon H}{\text{minimum}} \ \{f(x^h)\} \tag{1.5}$$

<u>Proof:</u> By contradiction, suppose x^* solves DP with $f(x^*) < f(x^{h*})$, and assume that $x^* \varepsilon S_{\hat{h}}$ for some $\hat{h} \varepsilon H$. Since x^* is feasible to $DP_{\hat{h}}$ and $x^{\hat{h}}$ solves $DP_{\hat{h}}$, we must have $f(x^*) \geq f(x^{\hat{h}}) \geq f(x^{h*})$, a contradiction. This completes the proof.

Essentially, Theorem 1.1 involves the solution of $|H|$ problems in order to recover an optimal solution to Problem DP. This may be a viable approach for some special problems for which the cardinality of H is not too large. For example, one may be considering a production planning problem in which each set S_h may be representing the restrictions on the process accruing from the implementation of production method $h \varepsilon H$. On the other hand, for zero-one linear integer programs for example, the application of Theorem 1.1 is tantamount to total enumeration and for a complementarity problem which requires, say, $u_j v_j = 0$ for $j = 1,\ldots,m$, one would need to solve 2^m problems to obtain an optimal solution. For such problems, one may adopt a relaxation strategy in which the constraints (1.2) are relaxed (except perhaps for the nonnegativity restrictions) and inequalities implied by (1.2) are iteratively generated as and when needed until Problem DP is solved through an equivalent representation of it in the neighborhood of an optimal solution. This paper specifically addressed itself to the technique of deriving suitable implied inequalities based on the constraint (1.2) for use in such a solution strategy for Problem DP.

The organization of this paper is as follows. First, we briefly discuss some well known and useful results on the derivation of inequalities implied by (1.2). Next, we demonstrate how the formulation of the disjunctive constraints (1.2) can effect the quality of such implied inequalities. Thereafter, we consider a special case in which X is polyhedral and each set S_h, $h \epsilon H$ is comprised of only a single constraint. For this situation, we establish connections between disjunctive programming methods and polyhedral annexation techniques [5] by discussing how the latter approach derives implied inequalities for the system (1.2). Finally, we present a scheme which may be easily implemented to obtain stronger implied inequalities than those available through a specific sequential polyhedral annexation technique of Glover [5]. This scheme may be easily extended to a broader class of problems in which each set S_h, $h \epsilon H$ contains more than one constraint.

2. Set of Implied Inequalities

Let us begin our discussion by presenting a fundamental result of disjunctive programming. The forward part of this result is due to Balas [1,2] and the converse due to Jeroslow [6]. This result has also been independently established by Glover in a somewhat different problem setting, the forward part appearing in the context of negative edge extension cutting planes [4, Theorem 1] and the converse appearing in the context of polyhedral annexation methods [5, Theorem 4.2]. Based on the disjunction (1.2), this result characterizes all implied inequalities which must be satisfied by any point feasible to (1.2). Such implied inequalities are also referred to as valid inequalities or valid disjunctive cuts or simply valid cuts. Notationally, for a set of vectors $\{v^h: h\varepsilon H\}$, where $v^h = \{v_1^h, \ldots, v_n^h\}$ for each $h\varepsilon H$, we will denote by $\sup_{h\varepsilon H}\{v^h\}$, the pointwise supremum $v = (v_1, \ldots, v_n)$ of the vectors v^h, $h\varepsilon H$, such that $v_j = \sup_{h\varepsilon H}\{v_j^h\}$ for $j = 1, \ldots, n$. Similarly, we define $\inf_{h\varepsilon H}\{v^h\}$. Further, throughout this paper, a superscript t will denote the matrix transpose operation. Now, consider the following well known result.

<u>Theorem 2.1 (Basic Disjunctive Cut Principle) - Balas [1,2], Glover [4,5], Jeroslow [6]</u>

Suppose that we are given linear inequality systems S_h, $h\varepsilon H$ of Equation (1.3), where $|H|$ may or may not be finite. Consider the disjunctive statement (1.2). Then, for any choice of nonnegative vectors λ^h, $h\varepsilon H$, the inequality

$$\left(\sup_{h\varepsilon H} (\lambda^h)^t A^h\right) x \geq \inf_{h\varepsilon H}\left((\lambda^h)^t b^h\right) \tag{2.1}$$

is a valid disjunctive cut. Furthermore, if every system S_h, $h\varepsilon H$ is consistent, and if $|H| < \infty$, then for any valid inequality $\sum_{j=1}^{n} \pi_j x_j \geq \pi_0$, there exist nonnegative vectors λ^h, $h\varepsilon H$ such that $\pi_0 \leq \inf_{h\varepsilon H} (\lambda^h)^t b^h$ and for $j = 1, \ldots, n$, the jth component of $\sup (\lambda^h)^t A^h$ does not exceed π_j.

This theorem essentially asserts that for each $h\varepsilon H$, one may surrogate the constraints of S_h through the use of nonnegative multipliers which form the components of a vector λ^h in order to reduce (1.2) into the weaker statement

$$x\varepsilon \bigcup_{h\varepsilon H} \hat{S}_h, \text{ where } \hat{S}_h = \{x: (\lambda^h)^t A^h x \geq (\lambda^h)^t b^h, \ x \geq 0\} \tag{2.2}$$

It is easy to see then that (2.2) implies (2.1). The strength of this Theorem lies in its (partial) converse which states that any valid inequality $\pi^t x \geq \pi_0$ can be uniformly dominated over the nonnegative orthant by an inequality of the type (2.1) obtained through some suitable surrogate multipliers. Of particular interest, therefore, is the choice of the nonnegative parameters λ^h, $h \epsilon H$. Balas [1] and Glover [5] have shown how different choices of λ^h, $h \epsilon H$, recover different cuts available in the literature. Further, note that the inequality (2.1) defines a closed convex set which contains $\bigcup_{h \epsilon H} S_h$. Thus, this set must also contain the set

$$S_c = \text{closure of the convex hull of } \bigcup_{h \epsilon H} S_h \tag{2.3}$$

Moreover, if $|H| < \infty$, then S_c is polyhedral and hence, a desirable or deep valid inequality would be one which supports S_c and preferably, is a facet of S_c. Indeed, Balas [3] has shown how one may generate with some difficulty cuts which contain as a subset the facets of S_c. On the other hand, through polyhedral annexation methods, Glover [5] has described a linear programming technique which generates supports of S_c. Recently, Sherali and Shetty [8] have shown how one may derive surrogate multipliers λ^h, $h \epsilon H$ to recover cuts which are the deepest according to some suitably defined criteria. These cuts always support S_c, and are facets of S_c in case each S_h, $h \epsilon H$ is comprised of only a single constraint. Since this latter special case is relevant to the present paper, we state below the prescribed choice of the parameters λ^h, $h \epsilon H$. It is interesting to note that this choice precisely generates the strengthened version of Glover's negative edge extension cut [see 5, Theorem 2].

Theorem 2.2 Consider a disjunction $x \epsilon \bigcup_{h \epsilon H} S_h$, where each S_h, $h \epsilon H$, is given by

$$S_h = \{x: \sum_{j=1}^{n} a_j^h x_j \geq b^h, \; x \geq 0\}, \; h\varepsilon H \tag{2.4}$$

Further, assume that these sets S_h, $h\varepsilon H$ are all consistent and are stated with respect to a point infeasible to the disjunction as the origin. That is, assume that $b^h > 0$, $h\varepsilon H$ and that for each $h\varepsilon H$, $a_j^h > 0$ for some $j\varepsilon\{1,\ldots,n\}$. Define

$$\lambda^h = \underset{j:\; a_j^h > 0}{\text{minimum}} \; \{b^h a_j^* / a_j^h\} \text{ for each } h\varepsilon H \tag{2.5}$$

where

$$a_j^* = \underset{h\varepsilon H}{\text{maximum}} \; \frac{a_j^h}{b^h}, \; j = 1,\ldots,n \tag{2.6}$$

Then the valid inequality

$$\sum_{j=1}^{n} \left(\underset{h\varepsilon H}{\max} \frac{\lambda^h a_j^h}{b^h} \right) x_j \geq 1 \tag{2.7}$$

is a facet of the closure of the convex hull of $\bigcup_{h\varepsilon H} S_h$.

<u>Proof</u>: See Sherali and Shetty [8]

We will now proceed to discuss how the formulation of a disjunction affects the depth of a cut which one may derive from it.

3. Formulating a Disjunction to Derive Deeper Cuts

In order to illustrate the concept involved in this issue, we will find it more expeditious to use a numerical example first, and then generalize the situation to the case which is of interest to us. For this purpose, we use the following example from a paper by Owen [7]

$$\text{maximize } \{z = 2x_1 + 3x_2 : \ x_1 + x_2 \leq 10, \ x_2 \leq 5, \ x_1 \leq 8, \ x_1, x_2 \geq 0 \\ x_1 x_2 = 0\}$$

Following the relaxation strategy, if one disregards the constraint $x_1x_2 = 0$ and solves the resulting linear program, then one will obtain the following optimal simplex tableau. Here, s_1, s_2, s_3 are respectively the slack variables for the three constraints given above.

	s_1	s_2	RHS
z	2	1	25
s_3	-1	1	3
x_1	1	-1	5
x_2	0	1	5

Now the condition $x_1x_2 = 0$ is violated and therefore, one may impose the disjunction that at least one of the following constraint sets must be satisfied

$$S_1 = \{x: \ x_1 \leq 0, \ x \geq 0\} \qquad\qquad S_2 = \{x: \ x_2 \leq 0, \ x \geq 0\}.$$

To invoke Theorem 2.2, we may rewrite these sets in terms of the nonbasic variables as

$$S_1 = \{(s_1,s_2): \ s_1 - s_2 \geq 5, \ s_1,s_2 \geq 0\} \text{ and } S_2: \{(s_1,s_2): \ s_2 \geq 5, \ s_1,s_2 \geq 0\} \quad (3.1)$$

The cut (2.7) is easily seen to be

$$\frac{s_1}{5} + \frac{s_2}{5} \geq 1 \qquad (3.2)$$

Let us now investigate whether the cut (3.2) can be improved. Figure 3.1(a) shows the region $S_1 \cup S_2$ as the cross hatched area and depicts the cut (3.2). As Theorem 2.2 indicates, this cut is a facet of S_c (Equation (2.3)). However, the region feasible to the original problem is shown as darkened lines in Figure 3.1 (b). Clearly the cut $5s_1 + 3s_2 \geq 25$ is valid and dominates the cut (3.2). In order to use the basic disjunctive cut principle to derive this inequiality, one has to reformulate the disjunction by imposing the nonnegativity restriction $s_3 = s_1 - s_2 + 3 \geq 0$ on each of the sets S_1 and S_2 given in (3.1). In other words one has to stipulate that $s_3 \geq 0$ as well as at least one of the sets S_1, S_2 must be satisfied. This leads to the following two sets, at least one of which must be satisfied.

$$S_1 = \{(s_1,s_2):\ s_1 - s_2 \geq 5,\ s_1 - s_2 \geq -3,\ s_1, s_2 \geq 0\} \quad \text{and} \quad S_2 = \{(s_1,s_2):\ s_2 \geq 5,\ s_1 - s_2 \geq -3,\ s_1, s_2 \geq 0\} \tag{3.3}$$

Using surrogate multipliers $\lambda^1 = (5,0)$ and $\lambda^2 = (8,5)$ one may obtain the cut (2.1) as $5s_1 + 3s_3 \geq 25$. In Section 5, we will demonstrate how this cut can be derived through (3.3) without explicitly resorting to the determination of the parameters λ^h, $h \varepsilon H$.

Insert Figure 3.1 here

Let us now generalize this concept to the situation of interest to us, namely the case where the objective function f is quasiconcave, and the set X of Equation (1.1) is polyhedral. In order to establish connections between disjunctive programming techniques and Glover's polyhedral annexation scheme [5], and to simplify the presentation, we will initially assume that each of the sets S_h, $h\varepsilon H$ is comprised of only a single constraint. Again, we will assume that a relaxation strategy is being adopted to solve Problem DP, so that currently, we have an extreme point optimal solution to the problem minimize $\{f(x): x\varepsilon X, x \geq 0\}$, which violates the disjunction $x\varepsilon \bigcup_{h\varepsilon H} S_h$. Here, we are assuming that the set X is comprised of the original linear constraints along with any valid inequalities which may have been generated over previous iterations. Accordingly, in terms of the current nonbasic variables, let the sets X and S_h, $h\varepsilon H$ be given by

$$X = \{x: \ Gx \leq g\} \equiv \{x: \ \sum_{j\varepsilon J} g_{ij}x_j \leq g_i \text{ for } i = 1,\ldots,m\} \quad (3.4)$$

$$S_h = \{x: \ \sum_{j\varepsilon J} a_j^h x_j \geq 1, \ x \geq 0\} . \qquad h\varepsilon H \quad (3.5)$$

where J is the index set of the nonbasic variables, so that $|J| = n$. For each set S_h, $h\varepsilon H$, we have normalized the single constraint by its respective right-hand-side which must be positive since the origin violates each such constraint. Now, in order to derive a valid inequality which deletes the origin, one may invoke the disjunction

$$x\varepsilon \bigcup_{h\varepsilon H} S_h \quad (3.6)$$

However, we propose to derive stronger cuts by invoking the alternate disjunction

$$x\varepsilon \bigcup_{h\varepsilon H} XS_h \quad (3.7)$$

where,

$$XS_h = \{x: \ Gx \leq g, \ \sum_{j\varepsilon J} a_j^h x_j \geq 1, \ x \geq 0\} \equiv X \bigcap S_h \quad (3.8)$$

Note that one may invoke other valid disjunctions between the extremes (3.6) and (3.7) by adding on a subset of the constraints of X to each of the sets S_h, $h \varepsilon H$. As one may guess, in the formulation of the disjunction, there is a tradeoff involved between the strength of the inequalities derived and the effort expended in generating these inequalities. Now, one viable approach is to commence with the disjunction (3.6) to obtain an initial cut, and then to sequentially add on constraints of X, attempting at each step to improve the current cut. This is basically the central point of the discussion of the following section.

4. Application of Glover's [5] Sequential Polyhedral Annexation Technique

In this section, we will first briefly discuss Glover's polyhedral annexation technique [5] as is relevant to the present exposition. We will then demonstrate how an algorithmic scheme called sequential polyhedral annexation by Glover [5], may be implemented to use the set X defined by (3.4) in order to improve the fundamental cut (2.7) available from the disjunction (3.6). We will also indicate some drawbacks of this method which lead us to proposing a variation of the scheme.

Let us begin our discussion by making the observation that a disjunction which stipulates that at least one of the sets S_h of Equation (3.5) must be satisfied is equivalent to the statement that the interior of the polyhedron

$$S_H = \{x: \sum_{j \varepsilon J} a_j^h x_j \leq 1, \quad \text{for each } h \varepsilon H, \; x \geq 0\} \tag{4.1}$$

contains no feasible points. Henceforth, for the sake of convenience, we will call a polyhedron <u>NFIP</u> if its interior contains no feasible points. Thus, Glover's polyhedral annexation procedure essentially does the following.

Given several NFIP polyhedra, the technique suitably annexes them to each other in order to derive a new NFIP polyhedron of the type (4.1). Then, based on the constraints of this polyhedron, a cut of the type (2.7) is generated. The annexation scheme is based on the following main result

Theorem 4.1 Let the polyhedra

$$S_P = \{x: \sum_j a_j^p x_j \leq b^p \text{ for each } p\varepsilon P,\ x \geq 0\} \tag{4.2}$$

and

$$S_Q = \{x: \sum_j a_j^q x_j \leq b^q \text{ for each } q\varepsilon Q,\ x \geq 0\} \tag{4.3}$$

be NFIP. Then, for any $k\varepsilon P$, and for any nonnegative parameters μ_{kq}, μ_q, $q\varepsilon Q$, the following polyhedron is NFIP:

$$\left.\begin{aligned} S_R = \{x: & \sum_j a_j^r x_j \leq b^r \text{ for each } r\varepsilon R,\ x \geq 0\} \\ \equiv \{x: & \sum_j a_j^p x_j \leq b^p \text{ for each } p\varepsilon P-\{k\} \\ & \sum_j (\mu_{kq} a_j^k + \mu_q a^q) x_j \leq (\mu_{kq} b^k + \mu_q b^q) \text{ for each } q\varepsilon Q \\ & x \geq 0\} \end{aligned}\right\} \tag{4.4}$$

Proof: See Glover [5]

In terms of the traditional disjunctive programming methods, Theorem 4.1 has the following interpretation. The condition that at least one of the constraint sets

$$S_p = \{x: \sum_j a_j^p x_j \geq b^p,\ x \geq 0\},\ p\varepsilon P \tag{4.5}$$

and at least one of the constraint sets

$$S_q = \{x: \sum_j a_j^q x_j \geq b^q,\ x \geq 0\},\ q\varepsilon Q \tag{4.6}$$

must be satisfied, implies the weaker condition that at least one of the following constraint sets must be satisfied for some $k\varepsilon P$

$$\begin{aligned} &S_p \text{ for } p\varepsilon P-\{k\}, \\ &S_{k,q} = \{x: \sum_j a_j^k x_k \geq b^k,\ \sum_j a_j^q x_j \geq b^q,\ x \geq 0\} \text{ for } q\varepsilon Q \end{aligned} \tag{4.7}$$

Given any set of nonnegative surrogate multipliers μ_{kq}, μ_q for the two constraints in each of the sets $S_{k,q}$, $q\varepsilon Q$, this in turn implies that at least one of the constraint sets

$$\begin{aligned} &S_p \text{ for } p\varepsilon P-\{k\}, \\ &S_{kq} = \{x: \sum_j (\mu_{kq} a_j^k + \mu_q a_j^q) x_j \geq (\mu_{kq} b^k + \mu_q b^q),\ x \geq 0\} \quad \text{for } q\varepsilon Q \end{aligned} \tag{4.8}$$

must be satisfied, or that S_R of Equation (4.4) must be NFIP.

Clearly, the choice of $k\varepsilon P$ for the purpose of annexation is crucial with regard to the strength of the inequality which may be derived from the disjunction (4.8). We will now discuss this choice in the context of the sequential polyhedral annexation scheme of Glover as applied to the concepts introduced in Section 3.

Thus, suppose one has derived the following cut (2.7) from the disjunction that at least one of the sets S_h, $h\varepsilon H$ of Equation (3.5) must be satisfied

$$\sum_{j\varepsilon J} \bar{\pi}_j x_j \geq 1 \tag{4.9}$$

The question addressed at this point is whether or not a given cut coefficient $\bar{\pi}_k$, $k\varepsilon J$ can be improved (decreased) without worsening (increasing) the other coefficients. (In the discussion below, the reader may note that the sets X, S_H, S_P and S_Q are defined by (3.4), (4.1), (4.2) and (4.3) respectively). The manner in which the sequential method proposes to accomplish this is to commence with the NFIP polyhedron S_H and annex constraints of X one at a time. During this annexation process, that constraint is chosen to be surrogated with the newly added constraints which is a "blocking hyperplane", i.e., forms a "block", for the k^{th} edge extension. That is the cut coefficient $\bar{\pi}_k$ is determined by that particular constraint through (2.7). In other words, the surrogation serves the purpose of attempting to rotate this blocking hyperplane so as to permit an improved edge intercept. Of course, if more than one constraint form a block for the k^{th} edge extension, then this process will have to be repeated for each of the blocking hyperplanes. Thus, starting with S_P equal to S_H, a set S_Q with $|Q| = 1$ is chosen to contain a single constraint of X. Let us assume that a constraint $k\varepsilon P$ of S_P forms a block for the k^{th} edge extension. Then, S_P and S_Q are annexed through nonnegative parameters μ_{kq} and μ_q as follows.

Note that since the origin is infeasible to each S_p, $p\varepsilon P$ of Equation (4.5), we may assume as before without loss of generality that $b^p = 1$, $p\varepsilon P$. To maintain consistency, we may also stipulate without loss of generality that the surrogation makes the right hand side of the constraint in S_{kq} of Equation (4.8) equal to unity, i.e., $\mu_{kq} + \mu_q b^q = 1$. Thus, under the restriction that the cut derived from the disjunction (4.8) improves the k^{th} edge intercept without worsening the other edge intercepts, we are searching for parameters μ_{kq}, μ_q satisfying

$$\mu_q \geq 0, \; \mu_{kq} = 1 - \mu_q b^q \geq 0 \tag{4.10}$$

$$\bar{\pi}_j \geq \mu_{kq} a_j^k + \mu_q a_j^q \qquad \text{for each } j\varepsilon J \tag{4.11}$$

One may easily deduce from this that the appropriate choice reduces to finding the largest $\mu_q \geq 0$ satisfying

$$\mu_q \leq \underset{j\varepsilon J}{\text{minimum}} \left\{ \frac{\bar{\pi}_j - a_j^k}{\left(a_j^q - a_j^k b^q\right)} : \; (a_j^q - a_j^k b^q) > 0 \right\} \tag{4.12}$$

and $$\mu_q b^q \leq 1$$

Then μ_{kq} is given through (4.10) and thus, the resulting NFIP polyhedron S_R of Equation (4.4) becomes the new polyhedron of the type (4.2). The cut (2.7) is updated, if necessary, with this new NFIP polyhedron (or disjunction) and the process is similarly repeated until the improvement of all edge intercepts have been attempted using all the constraints of X one at a time. Note that at each annexation, if the corresponding parameter μ_q obtained through (4.12) turns out to be zero, then this implies that $S_R \equiv S_P$ so that no improvement is possible with the current annexation.

Now, there is one principal drawback of this technique and that is, the final cut derived is dependent on the order in which one considers the constraints of X of Equation (3.4) to be used as sets S_Q of Equation (4.3). We illustrate this fact below through an example and then proceed to propose an alternative method.

Illustrative Example:

Let us modify the example of Section 3 by adding an additional constraint to the set X of Equation (3.4). Hence, let the sets of Equation (3.5) or (4.5) be

$$S_1 = \{(s_1,s_2): \frac{s_1}{5} - \frac{s_2}{5} \geq 1,\ s_1,s_2 \geq 0\},\ S_2 = \{(s_1,s_2): \frac{s_2}{5} \geq 1,\ s_1,s_2 \geq 0\} \quad (4.13)$$

and suppose X is given by

$$X = \{(s_1,s_2): -s_1 + s_2 \leq 3,\ -s_1 + 3s_2 \leq 12\} \quad (4.14)$$

The sets XS_1 and XS_2 of Equation (3.8) as well as the best cut available from the disjunction (3.7) are depicted in Figure 4.1.

Insert Figure 4.1 here

Now, the cut (2.7) available from the disjunction $(s_1,s_2)\varepsilon S_1 \bigcup S_2$ is $\frac{s_1}{5} + \frac{s_2}{5} \geq 1$. This cut passes through the points T and W of Figure 4.1. One can see that the extension corresponding to edge s_1 cannot be improved. Hence, let us attempt to improve the edge intercept corresponding to s_2 using Glover's sequential polyhedral annexation scheme. Towards this end, note that the constraint of S_2 represents the blocking hyperplane. Using the first constraint of X in the initial set S_Q of Equation (4.3), (with the inequality reversed) the relationships (4.12) yield

$$\mu_q \leq \text{minimum } \{\frac{\frac{1}{5} - 0}{1 - (0)(-3)},\quad \cdot\ \},\ -3\mu_q \leq 1,\ \mu_q \geq 0$$

The largest μ_q satisfying this is $\mu_q = \frac{1}{5}$, whence (4.10) gives $\mu_{kq} = 1 - (\frac{1}{5})(-3) = \frac{8}{5}$. Thus, the disjunction (4.8) is $(s_1,s_2)\varepsilon S_1 \bigcup S_{2q}$ where,

$$S_{2q} = \{(s_1,s_2): \frac{1}{5}s_1 + \frac{3}{25}s_2 \geq 1,\ s_1,s_2 \geq 0\} \equiv \text{New } S_2, \text{ say} \quad (4.15)$$

The cut (2.7) from this disjunction is

$$\frac{1}{5}s_1 + \frac{3}{25}s_2 \geq 1 \quad (4.16)$$

which passes through points U and W in Figure 4.1, and is also shown in Figure 3.1(b). Now let us repeat this by taking S_1 as in (4.13), S_2 as given by (4.15), the second constraint of X forming the set S_Q, and the constraint of S_2 representing the blocking hyperplane for the edge s_2 in the cut (4.16). The relationships (4.12) yield

$$\mu_q \leq \text{minimum} \left\{ \frac{\frac{1}{5} - \frac{1}{5}}{1 - (\frac{1}{5})(-12)}, \quad \bullet \right\}, \; -12\mu_q \leq 1, \; \mu_q \geq 0$$

which implies, that $\mu_q = 0$ or that no further improvement is possible.

In this example, if one had considered the constraints of X in the reverse order then one would have obtained the deepest cut as shown in Figure 4.1. However, the appropriate ordering of the constraints of X is a combinatorial problem. Furthermore, conceivably it may be possible in some instances that the best cut is not recoverable no matter in which order the constraints of X are considered.

The method we propose to employ in the next section considers all the constraints of X simultaneously, that is, examines the disjunction (3.7) itself in an attempt to improve edge intercepts one at a time, holding other edge intercepts fixed at each stage. This technique is easy to implement and directly yields the best cut coefficients, the corresponding appropriate surrogate multipliers being available, if required, as a set of optimal dual variables.

5. A Supporting Hyperplane Scheme for Improving Edge Extensions

Suppose as before that we are given sets S_h, $h \epsilon H$ defined by Equation (3.5) with the stipulation that at least one of these sets must be satisfied. We reemphasize here that we continue to assume that each set S_h has only one constraint merely for convenience. In addition, we are given a constraint set X (Equation (3.4)) which must also be satisfied by any feasible point. The disjunction under consideration is that $x \in \bigcup_{h \in H} XS_h$ (Equation (3.7)) where, as in Equation (3.8), $XS_h = X \cap S_h$, $h \in H$.

Thus, assume that currently, we have a cut of the form

$$\sum_{j \in J} \bar{\pi}_j x_j \geq 1 \tag{5.1}$$

which is valid for the disjunction (3.7). Note that initially, (5.1) may be taken as the cut (2.7) derived from the disjunction $x \in \bigcup_{h \in H} S_h$.

Now, consider a $k \in J$ and suppose that we are presently trying to improve the k^{th} edge intercept, that is, decrease $\bar{\pi}_k$. Towards this end, let us assume that we are able to solve for each $h \in H$

$\underline{P}_{kh}$: minimize π_{kh}

subject to

$$\pi_{kh} x_k + \sum_{\substack{j \in J \\ j \neq k}} \bar{\pi}_j x_j \geq 1 \text{ for each } x \in XS_h$$
$$\text{and } \pi_{kh} x_k + \sum_{\substack{j \in J \\ j \neq k}} \bar{\pi}_j x_j = 1 \text{ supports } XS_h \tag{5.}$$

Let

$$\bar{\pi}_k^* = \underset{h \in H}{\text{maximum}} \{\bar{\pi}_{kh}\} \tag{5.3}$$

where $\bar{\pi}_{kh}$ is the solution to problem P_{kh}. Now consider the cut

$$\bar{\pi}_k^* x_k + \sum_{\substack{j \in J \\ j \neq k}} \bar{\pi}_j x_j \geq 1 \tag{5.4}$$

Clearly, (5.4) is satisfied by each $x \in \bigcup_{h \in H} XS_h$, that is, (5.4) is a valid cut for the disjunction (3.7). Moreover, any inequality $\sum_{j \in J} \pi_j x_j \geq 1$ with $\pi_j = \bar{\pi}_j$ for $j \in J - \{k\}$ and $\pi_k < \bar{\pi}_k^*$ is not valid because it deletes a point $\hat{x}$ of $XS_{\hat{h}}$ at which the corresponding hyperplane $\bar{\pi}_{k\hat{h}} x_k + \sum_{\substack{j \in J \\ j \neq k}} \bar{\pi}_j x_j = 1$ supports $XS_{\hat{h}}$, where $\hat{h} \in H$ is an index for which equality holds in (5.3). To see this, it is sufficient to show that if $\bar{\pi}_{kh} > -\infty$ in (5.2), then a point of support referred to in (5.2) occurs at an $\bar{x}$ satisfying $\bar{x}_k > 0$. This is clearly so, for if not, then $\bar{\pi}_{kh}$ can be reduced still further. Thus, (5.4) gives the best intercept possible for the k^{th} edge when all other intercepts are held fixed. Hence, replacing $\bar{\pi}_k$ of (5.1) by $\bar{\pi}_k^*$, we would obtain a (possibly) new valid cut (5.1). This process may now be repeated for each edge in turn till no further improvement is possible. Of course, different cuts may be obtained by considering the edges in different orders, but each of these cuts cannot be uniformly dominated by any other cut.

We will now proceed to discuss the determination of $\bar{\pi}_{kh}$, the coefficient of x_k in the cutting plane under consideration, given through (5.2). The problem we formulate below to accomplish this, has the following motivation. Observe that the cut hyperplane is constrained to pass through (n-1) linearly independent points of the form $(0,\ldots,\frac{1}{\bar{\pi}_j},\ldots,0)$ for $j\varepsilon J-\{k\}$. In order to uniquely define the cutting plane, we need to identify a suitable point $\bar{x}$ which has $\bar{x}_k > 0$. Now, according to Equation (5.2), this cutting plane will need to support the set XS_h with each point of XS_h being feasible to it. Hence, in order to determine $\bar{\pi}_{kh}$, we may hold the intercepts on the axes $j\varepsilon J-\{k\}$ fixed and decrease the intercept on the k^{th} axis (increase π_{kh}) until the hyperplane merely supports XS_h at some point $\bar{x}$ with $\bar{x}_k > 0$. This problem is mathematically stated below. Theorem 5.1 later establishes that an optimal solution to this problem yields $\pi_k = \bar{\pi}_{kh}$

$\bar{P}_{kh}$ maximize π_k

$$\text{subject to } \pi_k x_k + \sum_{\substack{j\varepsilon J \\ j\neq k}} \bar{\pi}_j x_j = 1 \tag{5.5}$$

$$x\varepsilon XS_h \tag{5.6}$$

$$x_k > 0 \tag{5.7}$$

Note that π_k is unrestricted, in sign. Now using Equations (3.4), (3.5), (3.8) and solving for π_k through Equation (5.5), we may rewrite the above problem as

$$\text{maximize} \quad \frac{1}{x_k} - \sum_{\substack{j\varepsilon J \\ j\neq k}} \bar{\pi}_j \left(\frac{x_j}{x_k}\right)$$

$$\text{subject to} \sum_{j\varepsilon J} g_{ij}\left(\frac{x_j}{x_k}\right) \leq \frac{g_i}{x_k} \qquad \text{for } i = 1,\ldots,m$$

$$\sum_{j\varepsilon J} a_j^h\left(\frac{x_j}{x_k}\right) \geq \frac{1}{x_k}$$

$$\left(\frac{x_j}{x_k}\right) \geq 0, \; x_k > 0$$

Finally, letting

$$\xi = \left(\frac{1}{x_k}\right) \text{ and } y_j = \frac{x_j}{x_k} \qquad \text{for each } j\varepsilon J \tag{5.8}$$

we obtain the following linear programming problem in $|J| = n$ variables

$\underline{LP}_{kh}$: maximize $\quad z(\xi,y) = \xi - \sum_{\substack{j\varepsilon J \\ j\neq k}} \bar{\pi}_j y_j$

$$\text{subject to} \sum_{\substack{j\varepsilon J \\ j\neq k}} g_{ij}y_j - g_i\xi \leq - g_{ik} \qquad \text{for } i = 1,\ldots,m$$

$$\xi - \sum_{\substack{j\varepsilon J \\ j\neq k}} a_j^h y_j \leq a_k^h$$

$$\xi \geq 0, \; y_j \geq 0 \text{ for } j\varepsilon J - \{k\}$$

Consider the following result

<u>Theorem 5.1</u> If Problem LP_{kh} is feasible, then it has an optimal solution $\bar{\xi},\bar{y}_j$, $j\varepsilon J-\{k\}$ with $\bar{\xi} < \infty$. Moreover, the optimal solution values of Problems LP_{kh} and Problem P_{kh} (defined by (5.2)) are equal.

Proof: Note that the constraints of Problem LP_{kh} may be rewritten as $\sum_{j\varepsilon J} g_{ij}y_j - g_i\xi \leq 0$ for $i = 1,\ldots,m$; $-\sum_{j\varepsilon J} a_j^h y_j + \xi \leq 0$ and $y_k = 1$, with $\xi, y \geq 0$. Letting μ_i, $i = 1,\ldots,m$, γ and β_k be the respective dual variables associated with these constraints, the dual to Problem LP_{kh} may be written as

$\underline{DLP}_{kh}$: minimize β_k

$$\text{subject to } a_j^h\gamma - \sum_{i=1}^{m} g_{ij}\mu_i \leq \bar{\pi}_j \quad \text{for } j\varepsilon J - \{k\} \tag{5.9}$$

$$a_k^h\gamma - \sum_{i=1}^{m} g_{ik}\mu_i \leq \beta_k \tag{5.10}$$

$$\gamma - \sum_{i=1}^{m} g_i\mu_i \geq 1 \tag{5.11}$$

$$\gamma, \mu \geq 0$$

Letting $\bar{\beta}_k$ denote the minimum value of β_k, we will show that $\bar{\beta}_k = \bar{\pi}_{kh}$. We have from (5.2)

$$\bar{\pi}_{kh}x_k + \sum_{\substack{j\varepsilon J \\ j\neq k}} \bar{\pi}_j x_j \geq 1 \qquad \text{for each } x\varepsilon XS_h \tag{5.12}$$

$$\bar{\pi}_{kh}\hat{x}_k + \sum_{\substack{j\varepsilon J \\ j\neq k}} \bar{\pi}_j \hat{x}_j = 1 \qquad \text{for some } \hat{x}\varepsilon XS_h \tag{5.13}$$

Hence (5.12) is a valid cut for $H = \{h\}$ in (1.2), and Theorem 2.1 asserts the existence of $\hat{\gamma} \geq 0$, $\hat{\mu}_i \geq 0$ satisfying (5.9) through (5.11) for $\beta_k = \bar{\pi}_{kh}$. That is $\hat{\gamma}$, $\hat{\mu}_i$ and $\bar{\pi}_{kh}$ is feasible to DLP_{kh}. Thus LP_{kh} is bounded. Hence, $\bar{\beta}_k \leq \bar{\pi}_{kh} < \infty$. Now let $\bar{\gamma}$, $\bar{\mu}_i$ and $\bar{\beta}_k$ solve DLP_{kh}. Then, $x\varepsilon XS_h$ implies

$$\sum_{j\varepsilon J} [a_j^h\bar{\gamma} - \sum_{i=1}^{m} g_{ij}\bar{\mu}_i]x_j \geq [\bar{\gamma} - \sum_{i=1}^{m} g_i\bar{\mu}_i]$$

Then noting (5.9) through (5.11), we get

$$\bar{\beta}_k x_k + \sum_{\substack{j\varepsilon J \\ j\neq k}} \bar{\pi}_j x_j \geq 1 \qquad \text{for each } x\varepsilon XS_h \tag{5.14}$$

We have shown that $\bar{\beta}_k \leq \bar{\pi}_{kh}$. Now if $\bar{\beta}_k < \bar{\pi}_{kh}$ from (5.13)

$$\bar{\beta}_k \hat{x}_k + \sum_{\substack{j\varepsilon J \\ j\neq k}} \bar{\pi}_j \hat{x}_j < 1$$

contradicting (5.14). Hence $\bar{\beta}_k = \bar{\pi}_{kh}$.

Finally, since LP_{kh} is bounded, there exists an optimal extreme point solution $(\bar{\xi},\bar{y})$ with $\bar{\xi}$ finite. This completes the proof.

<u>Corollary</u> Let $\bar{\xi},\bar{y}_j$, $j\varepsilon J - \{k\}$ solve LP_{kh} with $\bar{\xi} < \infty$, and with $\bar{\beta}_k$ as the corresponding objective function value. Then, $\pi_k = \bar{\beta}_k$, $x_k = \frac{1}{\bar{\xi}}$ and $x_j = \frac{\bar{y}_j}{\bar{\xi}}$ for $j\varepsilon J-\{k\}$ solves P_{kh}.

It is easy to show that the following expedient for determining $\bar{\pi}_k^*$ of Equation (5.3) through the solutions of Problems LP_{kh}, $h\varepsilon H$ for a given $k\varepsilon J$, is a valid scheme.

<u>Step 1</u> Consider the cut (2.7) derived for the disjunction $x\varepsilon \bigcup_{h\varepsilon H} S_h$. Let $h\varepsilon H$ be a "blocking hyperplane" for the k^{th} edge as defined in Section 4.

<u>Step 2</u> Solve LP_{kh}. If LP_{kh} is infeasible, then select any $h\varepsilon H$ not considered thus far and repeat Step 2. If LP_{kh} is infeasible for each $h\varepsilon H$, then $x_k = 0$ for each $x\varepsilon \bigcup_{h\varepsilon H} XS_h$ and the variable x_k may be disregarded from the problem. Otherwise, obtain an optimal solution value $\bar{\pi}_{kh}$. If $\bar{\pi}_{kh} = \bar{\pi}_k$, terminate with $\bar{\pi}_k^* = \bar{\pi}_k$. Otherwise, select another $h\varepsilon H$ and repeat with the additional restriction that the objective value be greater than or equal to

that of the largest $\bar{\pi}_{kh}$ value found thus far. If all $h\varepsilon H$ have been considered, then $\bar{\pi}_k^*$ is given through Equation (5.3).

We remark at this point that the development of the present section may be easily extended to a broader class of problems in which each of the sets S_h, $h\varepsilon H$ may contain more than one constraint. This is accomplished by simply writing the second constraint of LP_{kh} for each of the constraints in S_h. Further, step 1 of the scheme discussed above may be started from an arbitrary h or from one determined heuristically.

We also draw the reader's attention to a "linear programming approach" of Glover [5] which may be used to determine a valid cut for the disjunction $x\varepsilon \bigcup_{h\varepsilon H} XS_h$. This formulation is similar in thrust to Problem DLP_{kh}. However, its motivation is to determine surrogate multipliers for simultaneously for each of the constraint sets XS_h, $h\varepsilon H$ so that the resulting disjunctive cut derived according to Theorem 2.1 supports the closure of the convex hull of the set $\{x: \ x\varepsilon \bigcup_{h\varepsilon H} XS_h\}$. Further, the objective function used to accomplish this minimizes $\sum_{j\varepsilon J} \alpha_j \pi_j$ where π_j, $j\varepsilon J$ are the disjunctive cut coefficients and α_j, $j\varepsilon J$ are any suitable choice of positive weights.

Finally, we would like to mention that a forthcoming paper will deal with the specialization and the computational aspects of applying the techniques developed in this paper to a general class of linear complementarity problems.

6. Illustrative Examples

Suppose that we are given

$$S_1 = \{x: \ -x_1 + x_2 \geq 1, \ x \geq 0\} \text{ and } S_2 = \{x: \ x_1 \geq 1, \ x \geq 0\}$$

Consider the following three examples of the set X. (i) $X = \{x: 2x_1 - 2x_2 \leq 1\}$

(ii) $X = \{x: 2x_1 - x_2 \leq 1\}$ and (iii) $X = \{x: 6x_1 - 2x_2 \leq 3\}$.

These three cases are depicted in Figure 6.1. Also shown in this figure are the respective best cuts available.

Insert Figure 6.1 here

Now, the cut (2.7) from the disjunction $x \varepsilon \bigcup_{h \varepsilon H} S_h$ is $x_1 + x_2 \geq 1$

Using the development of the previous section to improve the extension corresponding to the variable x_1 holding the other extensions fixed yields the following problems. Note that h = 2 is selected at Step 1 in each case.

	Example (i)	Example (ii)	Example (iii)
Problems LP_{kh} with k=1, h=2	maximize $\xi - y_2$	maximize $\xi - y_2$	maximize $\xi - y_2$
	subject to	subject to	subject to
	$2y_2 + \xi \geq 2$	$y_2 + \xi \geq 2$	$2y_2 + 3\xi \geq 6$
	$\xi \leq 1$	$\xi \leq 1$	$\xi \leq 1$
	$\xi, y_2 \geq 0$	$\xi, y_2 \geq 0$	$\xi, y_2 \geq 0$
Solution;			
value	1/2	0	-1/2
ξ	1	1	1
y_2	1/2	1	3/2

It may be easily seen that $\bar{\pi}_{11} \leq \bar{\pi}_{12}$ in each instance. Further $\bar{\pi}_2$ cannot be improved in each case. The corresponding best cuts are

Example (i)	$\frac{1}{2}x_1 + x_2 \geq 1$
Example (ii)	$x_2 \geq 1$
Example (iii)	$-\frac{1}{2}x_1 + x_2 \geq 1$

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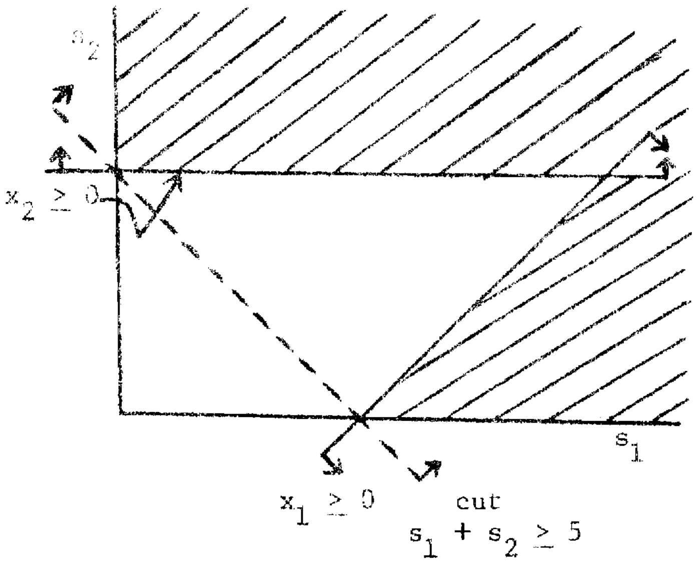
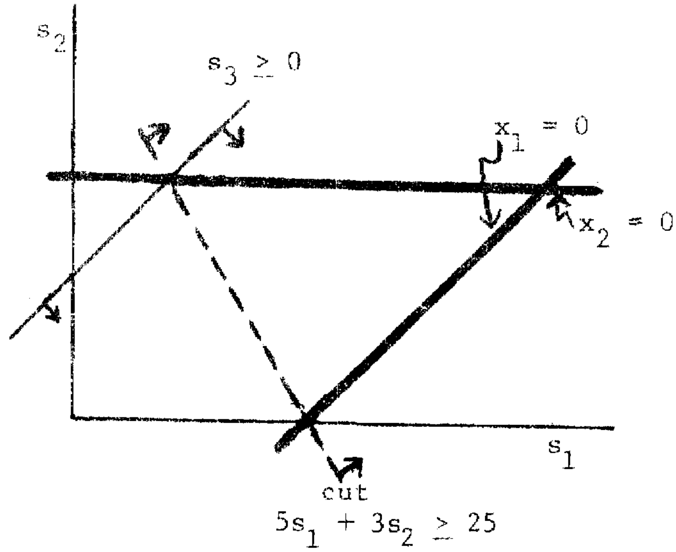


(a)



(b)

Figure 3.1

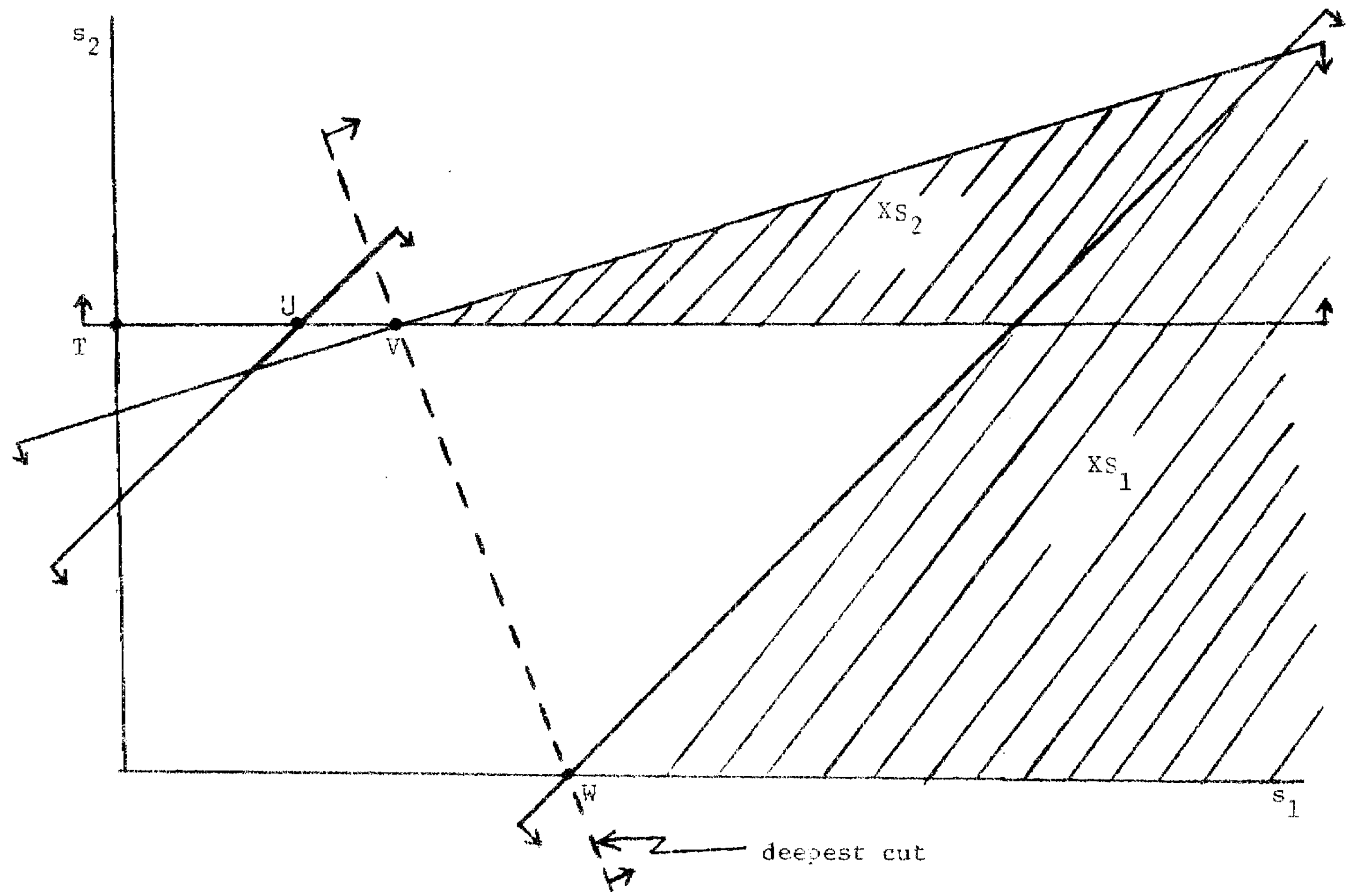
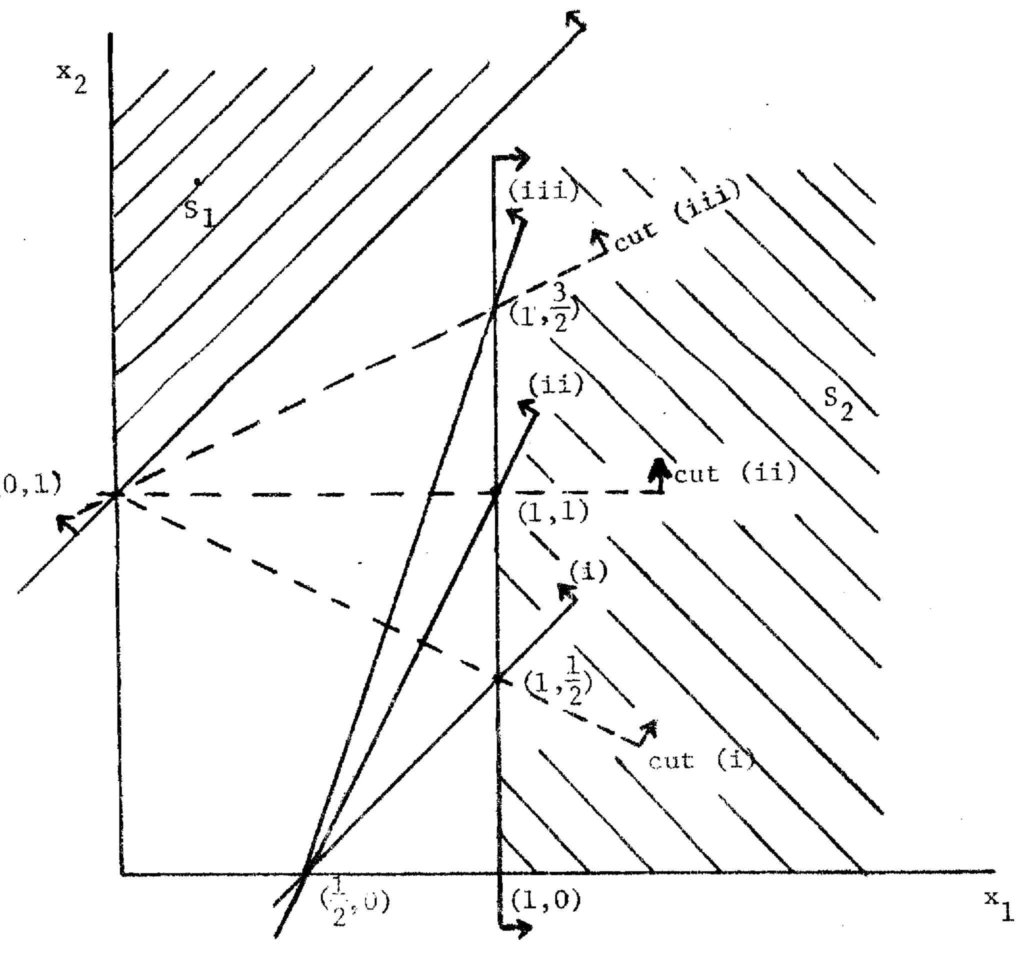


Figure 4.1



x_2
S_1
(iii)
cut (iii)
$(1,\frac{3}{2})$
(ii)
S_2
cut (ii)
(0,1)
(1,1)
(i)
$(1,\frac{1}{2})$
cut (i)
$(\frac{1}{2},0)$
(1,0)
x_1

Figure 6.1

E-24-610

SCHOOL OF INDUSTRIAL AND SYSTEMS ENGINEERING

Atlanta, Georgia 30332 (404) 894-2300

July 10, 1979

Mr. William Brogan
Systems Theory and Applications Program
Electrical Sciences and Analysis Section
Division of Engineering
National Science Foundation
Washington, DC 20550

Subject: NSF Grant No. ENG-77-23683

Dear Mr. Brogan:

A short progress report for the subject grant is attached. The discussion is limited to the overall progress of the research. If any further information is needed, please contact Dr. Shetty at 404-894-2315.

Sincerely,

C. M. Shetty
Principal Investigator
Professor, School of Industrial
& Systems Engineering

Duane Hutchison
Contract Administrator
Office of Contract Administration

CMS:gt

Enclosure

Progress Report on NSF Grant No. ENG 77-23683 on

"SOLUTION TO GENERALIZED LATTICE POINT AND RELATED PROBLEMS BY DISJUNCTIVE PROGRAMMING"

July 1979

The following is the annual progress report covering the period May 15, 1978 to May 14, 1979 for Grant No. ENG-77-23683. The research deals with Disjunctive programs, and the objective of the research is,

1. Study of deep cuts in Disjunctive Programming

2. Development of suitable cuts and solution procedures for GLP and related problems, and their computational testing.

3. Development of finitely convergent algorithms.

1. PARTICIPANTS

In addition to the principal investigator, two Ph.D. students namely Mr. Hanif D. Sherali and Mr. Chung-Chi Liu have worked on the project for the past year. While Mr. Sherali was a major participant, Mr. Liu was being initiated to the subject area. The major results thus far as listed below.

2. SUMMARY OF PROGRESS

Bilinear Programming

The principal investigator had worked on this area during the past 3-4 years, and had developed a convergent solution procedure which had earlier been published. However, this procedure was not finite. Using the results of Disjuncture Programming, a finite procedure has been developed, and is contained in a paper titled, "A Finitely Convergent Algorithm for Bilinear Programming Problems using the Polar Cut and Disjunctive Face Cuts." The paper has been accepted for publication with revisions in Mathematical Programming (see Appendix A).

Generation of Deep Cuts

The question of a definition of deep cuts was addressed first. It was proved that when each set in the disjunctive program consists of a single constraint, one can specify the deepest-cut (under any criterion). Using this powerful result, a subgradient optimization procedure was developed for the general problem. A paper summarizing these results, and titled, "On the Generation of Deep Disjunctive Cutting Planes" has been submitted to Naval Research Logistics Quarterly and is being reviewed. A paper on the same subject was also presented at the Joint ORSA-TIMS conference at New Orleans in May-June 1979.

Lecture Notes

To initiate new Ph.D. students a set of comprehensive notes were prepared. The notes synthesized several different approaches, and as such has appeal to researchers in this area. Among the publishers contacted, Springer-Verlag has expressed interest in its publication. The permission of NSF has been sought under our letter of June 28, 1979. (See Appendix B)

3. SUMMARY OF PROPOSED WORK

In our work thus far we attempted to develop deep cuts under suitable criteria. During the coming year, we would like to address the question of how to specify nondominated cuts. This is likely to yield simpler subproblems and, therefore, more efficient algorithms. The procedure will be tested on a subclass of Generalized Lattice Point Problems, namely the Complementarity Problem. At least two more papers are expected to be submitted to the refereed Journals. In addition, during the coming year, the two papers and the monograph already submitted for publication will also be revised and updated. We hope these will be in print in 1980. We also expect to present two papers at national technical conferences.

4. OTHER COMMENTS

Even though at the end of the second year, we expect at least 4 papers and a monograph to be published, much need to be achieved in this area. Two of the papers are strongly theory oriented and addresses the general problem. These can be specialized to various problems. A proposal for the renewal of the grant on this basis will be submitted soon. The monograph that is under revision should help significantly in bringing Ph.D. students to speed in this area as there are no other formal texts or courses being offered at the moment.

MATHEMATICAL PROGRAMMING

APPENDIX A

05 March 1979

Professor C.M. Shetty
School of Industrial & Systems Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

Dear Professor Shetty:

This letter concerns your joint paper with H.D. Sherali entitled "A finitely convergent algorithm for bilinear programming problems using polar cuts and disjunctive face cuts," submitted for publication in this Journal.

Please find enclosed a copy of the Associate Editor's recommendation together with three copies of Referees' Reports. In addition, you will also find enclosed two annotated copies of the manuscript itself.

In brief, we are interested in publishing a substantial revision of the manuscript. The advice of Referee #2 is particularly important.

Also, please find enclosed copies of Instructions to Authors. Would you be kind enough to closely follow the instructions contained therein.

I look forward to hearing from you.

Very sincerely yours,

Michel L. Balinski

MLB:jd

Enclosures: Associate Editor's Recommendation
Referees' Reports (3)
Annotated Manuscripts (2)
Instructions to Authors (2)

For authors

Associate Editor's Recommendation

on the paper

C.M. Shetty and H.D. Sherali: A Finitely Convergent Algorithm for Bilinear Programming Problems Using Polar Cuts and Disjunctive Face Cuts

The paper seems to make a considerable contribution of high technicality to the theory of bilinear programming, as the referees write in their reports, so that it deserves publication in Mathematical Programming.

However, before publication, the manuscript should be substantially revised and improved, with the comments from the referees, especially those from Referee #2, seriously considered.

SCHOOL OF INDUSTRIAL AND SYSTEMS ENGINEERING

Atlanta, Georgia 30332 (404) 894-2300

June 28, 1979

Ms. Charlotte Raymond
Head, Section II
MPE/STIA Branch
Division of Grants and Contracts
National Science Foundation
Washington, D.C. 20550

Subject: NSF Grant No. ENG77-23683

Dear Ms. Raymond:

The area of research of the subject grant, Disjunctive Programming, is a rather broad subject area, and related publications have appeared in literature dealing with discrete programming and nonconvex programming. To motivate Ph.D. students in this area, and also to consolidate results published thus far and to record some new fundamental results, a set of attached notes coauthored with Mr. H. Sherali were prepared.

It seems to us that the monograph will be of interest to the operations researchers, and we note from GPM 752 that the NSF Act "authorizes the foundation to publish or to arrange for the publication of scientific and technical information so as to further the full dissemination of information of scientific value consistent with the national interest". The principal investigator has identified the following three publishers who have a record of low cost and quick publication of research monographs.

1. Springer Verlag, New York, under the Lecture Notes in Economics and Mathematical Systems series. Publication under this series is on the basis of no royalty. Appendix A gives their publication Policy.

2. Addison Wesley author Advanced Book Program. Appendix B gives some background on publications in this series.

3. Wiley Interscience monographs.

Springer-Verlag have verbally indicated interest after a preliminary review. All three publishers have sent the manuscript for a formal review.

The purpose of this letter is to seek NSF approval of publication of the notes (in revised updated form) since it overlaps with research results under the NSF grant. No doubt an acknowledgement as in GPM 752.4 will be given.

Ms. Charlotte Raymond
Page Two

If any further information is needed, please contact Dr. Shetty at 404/894-2315 or Ms. McHan at 404/894-4819.

Sincerely,

C. M. Shetty
Principal Investigator
Professor, School of Industrial
and Systems Engineering

Addressee: In duplicate

xc: William Brogan, NSF/ENG

Nancy S. McHan
Contract Administrator
Office of Contract Administration

PROGRESS REPORT: N.I.L.E.C.J. GRANT #78-NI-AX-00

SCHOOL OF INDUSTRIAL AND SYSTEMS ENGINEERING

Atlanta, Georgia 30332 (404) 894-2300

12 March, 1979

E-24-669

Mr. Edwin Zedlewski
N.I.L.E.C.J.
633 Indiana Avenue, N. W.
Washington, D. C. 20531

Dear Ed:

The purpose of this letter is to formally summarize the progress to date on Grant No. 78-NI-AX-0040 titled "Stochastic Modeling and Analysis of Crime," through March 1979.

During the first 11 months of the grant, a major activity has been the development of the theory for multivariate time series modeling for space-time systems. The enclosed 560-page manuscript, titled "Spatial-Dynamic Modeling," fully documents this model-building procedure along with detailed numerical examples of its application. These models allow description of one random variable measured at n points in time over time. Some of the examples include: modeling the total number of arrests for assault in zones within Boston and the modeling of reported homicides in seven major cities.

Several software development activities for interactive graphical analysis and modeling of univariate modeling, shift detection and evaluative shift dynamics have been completed. These software packages are now to be adapted to interactive computer graphics capability. Expected delivery of the graphics equipment is now May 1979. These tasks will be rekindled at that time. Software for dynamic multiple interventions models has also been developed.

Until the graphics equipment arrives development activities on multivariate shift detection models will be initiated.

Sincerely,

Stuart Jay Deutsch
Professor

RECEIVED
MAR 14 1979
OFFICE OF CONTRACT ADMINISTRATION

SJD/vdb

xc: Al Becker

E24-670

NATIONAL SCIENCE FOUNDATION
Washington, D.C. 20550

FINAL PROJECT REPORT
NSF FORM 98A

PLEASE READ INSTRUCTIONS ON REVERSE BEFORE COMPLETING

PART I–PROJECT IDENTIFICATION INFORMATION

1. Institution and Address	2. NSF Program	3. NSF Award Number
Georgia Institute of Technology Atlanta, Georgia 30332	Systems Theory and Operations Research	ENG-77-23683
	4. Award Period From 5/15/78 To 10/31/80	5. Cumulative Award Amount $54,443

6. Project Title
Solution to the Generalized Lattice Point and Related Problems by Disjunctive Programming

PART II–SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

Several important classes of problems including generalized lattice point problems can be modeled as disjunctive programs. The purpose of this research is to explore general solution strategies for such problems. For this purpose, using certain criterion for measuring the depth of a cut we have shown how the deepest possible cuts could be generated. We also explore the interplay between model formulation and the depth of a cut. Using earlier results, we show how one may generate a nondominated cut. The scheme not only gives a deep cut but also yields a feasible point which is useful in approaches such as the Branch-and-Bound approach. Finally, we explore means by which finiteness of the solution procedure can be assured. Using a result due to Balas, we obtain a finitely convergent scheme for facial disjunctive programs. The scheme is tested on bilinear programs and computational results are promising. The research results are published in a monograph under Lecture Notes in Economics and Mathematical Systems, No. 181, Springer-Verlag, 1980.

PART III–TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

1. ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM: Check (✓)	Approx. Date
a. Abstracts of Theses	X				
b. Publication Citations		X			
c. Data on Scientific Collaborators		X			
d. Information on Inventions	X				
e. Technical Description of Project and Results		X			
f. Other *(specify)* Copies of publications		X			

2. Principal Investigator/Project Director Name *(Typed)*	3. Principal Investigator/Project Director Signature	4. Date
C. M. Shetty		3-3-81

NSF Form 98A (5-78) Supersedes All Previous Editions

Form Approved OMB No. 99R0013

Publications

1. Optimization With Disjunctive Constraints, Lecture Notes in Economics and Mathematical Systems, No. 181, Springer-Verlag, 1980.

2. "On the Generation of Deep Disjunctive Cutting Planes," Naval Research Logistics Quarterly, 22, pp. 453-475, 1980.

3. "A Finitely Convergent Algorithm for Bilinear Programming Problems Using Polar Cuts and Disjunctive Face Cuts," Mathematical Programming, 19, pp. 14-31, 1980.

4. "Nondominated Cuts for Disjunctive Programs and Polyhedral Annexation Methods," submitted to Discrete Applied Mathematics (see also report J-79-28, School of Industrial and Systems Engineering, Georgia Institute of Technology).

5. "A Finitely Convergent Procedure for Facial Disjunctive Programs," submitted to Discrete Applied Mathematics, (see also report J-80-19, School of Industrial and Systems Engineering, Georgia Institute of Technology).

List of Collaborators

1. Mr. H. D. Sherali, Ph. D. student, currently Assistant Professor at the Virginia Polytechnic Institute and State University.

E-24-670

SOLUTION TO THE GENERALIZED LATTICE POINT AND RELATED PROBLEMS BY DISJUNCTIVE PROGRAMMING

C. M. Shetty
Principal Investigator
NSF Research Grant No. ENG-77-23683

Final Technical Report covering the period May 15, 1978 to October 31, 1980.

1. Introduction

The research covered by this final report deals with Disjunctive Programming. This is a broad subject area and related publications have appeared in literature dealing with discrete programming and non-convex programming. The specific objectives of the research are

1. Specification and development of deep cuts for disjunctive programs.
2. Development of suitable cuts and solution procedures for certain disjunctive programs, and their computational testing.
3. Development of finitely convergent algorithms.

The research has resulted in the publications listed below.

a. Optimization with Disjunctive Constraints, Lecture Notes in Economics and Mathematical Systems, No. 181, Springer-Verlag, New York, September 1980.

   These notes were originally intended to summarize published results strongly related to the research topic to aid Ph.D. students interested in working on the research topic. In its final published form, it is a self contained document going far beyond the original objective. It brings together several apparently dissimilar approaches for handling programs with logical constraints. It also covers the research results obtained under the NSF Grant No. Eng-77-23683.

b. On the Generation of Deep Disjunctive Cutting Planes, Naval Research Logistics Quarterly, 27, 453-475, September 1980.

   This paper addresses the question of deriving deep cuts for general disjunctive programs. Based on the works of Balas, Glover, and Jeroslow it examines the set of valid inequalities or cuts which one may derive in this context, and defining reasonable criteria to measure depth of a cut we demonstrate how one may obtain the "deepest" cut. The analysis covers the case where each constraint set in the logical statement has only one constraint and is also extended for the case where each of these constraint sets may have more than one constraint.

c. "A Finitely Convergent Algorithm for Bilinear Programming Problems Using Polar Cuts and Disjunctive Face Cuts," Mathematical Programming, 19, 14-31, 1980.

This paper considers the bilinear program the application of which are well known and discussed elsewhere. Using the results of publication citied immediately above, an improved cutting plane procedure is developed. Computational results are presented which supports the use of disjunctive cuts. An interesting thought pursued in this study is that of developing finite schemes using the notion of "extreme faces" introduced by Majthay and Whinston [9].

d. Nondominated Cuts for Disjunctive Programs and Polyhedral Annexation Methods, Research Report J-79-28, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia, 1979 (submitted to Discrete Applied Mathematics).

In this paper, we consider the generation of nondominated cutting planes for linear disjunctive programs. We demonstrate how the formulation of a disjunction affects the strength of a cut derived from it, and show how one may generate a nondominated cut and the point at which it supports the convex hull of points feasible to the disjunction. A simpler computational variation is also suggested for a special case. In discussing this, we provide some insights into Glover's polyhedral annexation technique by demonstrating its relationships to disjunctive programming methods.

e. A Finitely Convergent Procedure for Facial Disjunctive Programs, Report J-80-19, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia, 1980 (submitted to Discrete Applied Mathematics)

This paper addresses an important special clsss of disjunctive programs called facial disjunctive programs, examples of which include the zero-one linear integer programming problem and the linear complementarity problem. Balas has characterized some fundamental properties of such problems, one of which has been used by Jeroslow to obtain a finitely convergent procedure. This paper exploits another basic property of facial disjunctive programs in order to develop an alternative finitely convergent algorithm.

Reprints/copies of the first three publications and copies of the last two papers cited above have been sent with the final project report (Form 98A).

Papers based on the research under this contract have been presented at

1. Joint ORSA/TIMS Conference, New Orleans, May 1979

2. Joint ORSA/TIMS Conference, Washington, D. C., November 1979

3. Joint ORSA/TIMS Conference, Denver, May 1980

The research results have been discussed in detail in the research monograph Optimization with Disjunctive Constraints published by Springer-Verlag under Lecture Notes in Economics and Mathematical Systems, No. 181, 1980, Hence, in this report we will focus only on some important results and their relevancy.

2. Deep Cuts and Their Generation

Consider a disjuctive program of the form

$$\text{DP:} \qquad \text{minimize} \quad f(x) \tag{1}$$

$$\text{subject to} \quad x \in X \tag{2}$$

$$x \in \bigcup_{h \in H} S_h \tag{3}$$

when X is a closed subset of the nonnegative orthant of E_n, and each S_h, $h \in H$ is of the form

$$S_h = \{x: A^h x \geq b^h,\ x \geq 0\}, \quad h \in H \tag{4}$$

for some finite index set $H = \{1,\ldots,\hat{h}\}$.

The basic disjunctive principle due to Balas [1,2], Glover [5] and Jeroslow [6] states that for any $\lambda^h \geq 0$, the inequality

$$\left[\sup_{h \in H} (\lambda^h)^t A^h\right] x \geq \inf_{h \in H} (\lambda^h)^t b^h \tag{5}$$

is a valid cut. Furthermore, if each S_h is nonempty then for any valid cut $\pi x \geq \pi_0$, there exists $\lambda^h \geq 0$ such that (5) uniformly dominates $\pi x \geq \pi_0$.

The questions that arises are

a. Can we select λ^h so that the resultant cut is in some sense strongest?

b. How can we characterize the strength of the cut so that the deepest cut coincides with the intuitive notions of a desirable cut?

The above questions are explored in references [10, 11]. From among the alternatives, the distance from the current point to the non-negative region feasible to the cut is shown to be preferable. Furthermore, if each S_h consists of a single inequality, then the deepest cut is provided by the following Theorem.

Theorem 1. Let

$$S_h = \{x: \sum_{j=1}^{n} a_{1j}^h x_j \geq b_1^h, \ x \geq 0\} \quad \text{for } h \varepsilon H$$

a. Then the euclidean as well as the rectilinear distance is maximized by letting $\lambda_1^h = 1/\hat{h}$ for each $h \varepsilon H$ to yield the cut

$$\sum_{j=1}^{n} a_{1j}^* x_j \geq 1 \quad \text{where } a_{1j}^* = \max_{h \varepsilon H} a_{1j}^h \quad \text{for } j = 1,\ldots,n \tag{6}$$

b. Define

$$\gamma_1^h = \underset{j:\ a_{1j}^h > 0}{\text{minimum}} \{a_{1j}^*/a_{1j}^h\} \quad h \varepsilon H$$

Then both the distance measures are maximized by letting

$$\lambda_1^h = \gamma_1^h \Big/ \sum_{p \varepsilon H} \gamma_1^p \qquad \text{for } h \varepsilon H$$

to yield the cut

$$\sum_{j=1}^{n} a_{1j}^{**} x_j \geq 1 \quad \text{where} \quad a_{1j}^{**} = \max_{h \varepsilon H} \; a_{1j}^h \, \gamma_1^h \quad \text{for } j = 1,\ldots,n \qquad (7)$$

c. Cut (7) uniformly dominates cut (6).

d. Cut (7) is a facet of $\bigcup_{h \varepsilon H} S_h$ so that it cannot be dominated.

Proof. See [11]

In the case where S_h consists of more than one constraint for some $h \varepsilon H$, a clear procedure for finding the multipliers is not available. In [11] a subgradient optimization procedure is suggested. But further research is needed to answer the question satisfactorily.

3. Deep Cuts and Model Formulation

Now consider a disjunctive program where the constraints (2) and (3) are explicitely stated in the form

$$X = \left\{x: \sum_{j=1}^{n} g_{ij} x_j \leq g_i \quad \text{for } i = 1,\ldots,m\right\} \qquad (8)$$

$$S_h = \left\{x: \sum_{j=1}^{n} a_{ij}^h x_j \geq b_i \quad \text{for } i = 1,\ldots,r_h\right\}, \quad h \varepsilon H \qquad (9)$$

In section 2 we discussed how one may attempt to get a support to the set $\bigcup S_h$ in order to obtain a deep valid cut. However, since a feasible point in X must also be contained in at least one of the set S_h, $h\varepsilon H$, we must have

$$x\varepsilon \bigcup_{h\varepsilon H} XS_h \quad \text{where} \quad XS_h = X \cap S_h. \tag{10}$$

It can easily be shown [10,13] that cuts generated by using the disjunction (10) can be potentially much deeper than that available through the disjuction $x\varepsilon \bigcup_{h\varepsilon H} S_h$. Such a cut is available by Theorem 2 below and the subsequent discussion.

Theorem 2

Consider the disjunction $x\varepsilon \bigcup_{x\varepsilon H} XS_h = \bigcup_{x\varepsilon H} \{X \cap S_h\}$ where X and S_h are as defined in (8) and (9). The $\sum_{j=1}^{n} \bar{\pi}_j x_j \geq 1$ is a nondominated cut deleting the current point $\bar{x}$ and supporting the set $x\varepsilon \bigcup XS_h$ if and only if there exists $(\bar{\pi},\ldots,\bar{\pi}_n)$ such that $\bar{\pi}_k$ solves LP_k below for $k = 1,\ldots,n$.

LP_k: minimize

subject to

$$\pi_k \geq \sum_{i=1}^{r_h} \gamma_i^h a_{ik}^h - \sum_{i=1}^{m} \mu_i^h g_{ik} \quad \text{for each } h\varepsilon H$$

$$\bar{\pi}_j \geq \sum_{i=1}^{r_h} \gamma_i^h a_{ij}^h - \sum_{i=1}^{m} \mu_i^h g_{ij} \quad \text{for each } j\neq k \text{ and each } h\varepsilon H$$

$$\sum_{i=1}^{r_h} \gamma_i^h b_i - \sum_{i=1}^{m} \mu_i^h g_i \geq 1 \quad \text{for each } h\varepsilon H$$

$$\gamma_i^h \geq 0 \quad i = 1,\ldots,r_h, \quad h\varepsilon H, \quad \text{and} \quad \mu_i^h \geq 0 \quad i = 1,\ldots,m \quad h\varepsilon H$$

Proof. See [13]

LP_k can be decomposed into several problem LP_{kh}, $h\varepsilon H$, defined below.

It is shown in [13] that $\bar{\pi}_k$ will then be given by $\bar{\pi}_k = \max_{h\varepsilon H} \bar{\pi}_{kh}$ where $\bar{\pi}_{kh}$ is obtained by solving the following:

$$\underline{LP_{kh}}: \quad \text{minimize} \quad \pi_{kh}$$

$$\text{subject to } \pi_{kh} \geq \sum_{i=1}^{r_h} \gamma_i^h a_{ik}^h - \sum_{i=1}^{m} \mu_i^h g_{ik}$$

$$\bar{\pi}_j \geq \sum_{i=1}^{r_h} \gamma_i^h a_{ij}^h - \sum_{i=1}^{m} \mu_i^h g_{ij} \quad \text{for each } j \neq k$$

$$\sum_{i=1}^{r_h} \gamma_i^h b_i - \sum_{i=1}^{m} \mu_i^h g_i \geq 1$$

$$\gamma^h, \mu^h \geq 0$$

Computationally, in order to generate a nondominated cut through the use of Theorem 2, one may begin with a valid cut $\sum_{j=1}^{n} \bar{\pi}_j x_j \geq 1$ obtained through some judicious choice of surrogate multipliers. Then, holding (n-1) of the coefficients fixed in turn, one may attempt to improve (decrease) the remaining coefficient, π_k say, through Problem LP_k. Thus, essentially, one would solve LP_{kh} for $h\varepsilon H$ and thereby compute the new $\bar{\pi}_k$. Of course, if for some Problem LP_{kh}, one obtains $\bar{\pi}_{kh}$ equal to the current $\bar{\pi}_k$, then clearly no further decrease in $\bar{\pi}_k$ is possible, and so no additional problems LP_{kh} need be solved.

It turns out that in the process of generating a cut in this fashion, one also obtains the point at which the cut supports the set $\bigcup XS_h$. Intuitively, if one considers the final cut obtained and holds (n-1) intercepts fixed while varying the last intercept, then the linear program LP_{kh} seeks to increase the intercept on this axis such that at least one point in XS_h remains feasible to the cut. This concept is formalized as Theorem 2.2 in [13]. The utility of such a point is that it is likely to be a good quality feasible solution to DP and hence may yield a strong upper bound on the problem. The likelihood of feasibility comes from the fact that it is a point in the closure of the convex hull of points feasible to the disjunction $x \in \bigcup_{h \in H} XS_h$, whereas the likelihood of it being a good quality solution stems from the fact that it is in the vicinity of the current super-optimal relaxation solution $\bar{x}$.

It turns out that a computational variation of the above approach for the special case where S_h consists of only a single constraint is precisely the sequential polyhedral annexation approch of Glover [5]. This is discussed in detail in [10] and [13].

4. Finitely Convergent Algorithms

In earlier sections we have discussed how one may generate valid cutting planes for disjunctive programs. However, in common with other cutting plane algorithms, the implementation of this scheme is likely to exhibit slow convergence. However, under certain special cases we can generate finitely convergent algorithms.

Consider the disjunctive program

$$\begin{aligned} \text{Minimize} \quad & cx \\ \text{subject to} \quad & x \in X = \{x: Ax = b,\ x \geq 0\} \\ & x \in D = \bigcap_{h \in H} \left[\bigcup_{i \in Q_h} S_i \right] \end{aligned}$$

where

$$S_i = \{x: d_i x \geq d_{i0}\}, \quad i \in Q_h,\ h \in H$$

Thus, we require that for each $h \in H$ a feasible point must belong to one of the half-spaces S_i for $i \in Q_h$. If $X \cap S_i$ is a face of X for each $i \in Q_h$,

$h \varepsilon H$, then the disjunctive program is called a <u>facial disjunctive program</u>. A linear complementary problem, for example, can be posed as a facial disjunctive program. Note that the problem under consideration can be rewritten as

$$\text{minimize} \quad cx \quad \text{subject to} \quad x \varepsilon Y = \text{conv}\,[X \cap D].$$

Let us inductively define

$$k_0 = X$$

$$k_h = \text{conv}\left[\bigcup_{i \varepsilon Q_h} k_{h-1} \cap \{x: d_i x \geq d_{io}\}\right] \quad \text{for } h = 1,2,\ldots,\hat{h}$$

where $\hat{h} = |H|$. Then, Balas [3] has shown that

Property 1: $k_{\hat{h}} = Y$

Property 2: Extreme points of Y is a subset of the extreme points of X.

Jeroslow [7] uses the first property to develop a finitely convergent algorithm for facial disjunctive programs. The method specifies conditions on the cutting planes used. On the other hand, a finitely convergent scheme can be specified using propery 2 [12]. In this case instead of conditions on the cutting planes, we generate cutting planes only at extreme faces of X relative to the cuts generated thus far. This concept is due to Majthay and Whinston [9]. We have implemented this scheme on bi-linear programs where the requirement that the optimum is at an extreme point leads to a disjunctive statement. Computational results [12] indicate that convergence is improved by the use of such disjunctive face cuts.

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14. Sherali, H. D. and C. M. Shetty, "A Finitely Convergent Procedure for Facial Disjunctive Programs," submitted to Discrete Applied Mathematics, see also Report J-80-19, School of Industrial and Systems Engineering, Georgia Institute of Technology, 1980.