# Theory of Aperiodic Solids

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*The PI is continuing his long standing program on the Theory of Aperiodic Solid. Presently 6 collaborations are running.*

**Reminder:** an aperiodic solid can be described as follows: let $\mathcal{L} \subset \mathbb{R}^d$ be the discrete set of equilibrium positions of its atoms in the real space in which the solid is setting. The Hull of $\mathcal{L}$ is obtained as the closure of the family $\{\mathcal{L} + a; a \in \mathbb{R}^d\}$ of translated of the solid. The topology used here is the following: a subsequence of discrete subsets converges if and only if their intersections with any bounded open set of $\mathbb{R}^d$ converges for the Hausdorff distance. If the initial set of atoms is *uniformly discrete*, the Hull is a compact space $\Omega$ endowed with an action of $\mathbb{R}^d$ by homeomorphisms. Each point in the Hull is itself a discrete subset of $\mathbb{R}^d$. This dynamical system has a *canonical transversal* made of points such that the corresponding discrete subset contains the origin in $\mathbb{R}^d$. This transversal is called the *atomic surface* by physicist studying quasicrystals. Such a Hull can be described through various points of view: as a dynamical system as previously, as a groupoid (of the dynamical system or of its transversal), as a tiling space, as a lamination (or foliated space). The crossed product $C^*$-algebra $\mathcal{C}(\Omega) \rtimes \mathbb{R}^d$ can be interpreted as the space of continuous functions on a noncommutative manifold called the *Noncommutative Brillouin Zone* (NCBZ) [1]. Any invariant measure on the Hull gives rise to a trace on the NCBZ and vice-versa, and the trace is positive if and only if the invariant measure is positive.

# 1 Results of 2005 and current work

## 1.1 The Hull of a Repetitive Finite Type Tiling:

**Collaboration:** *J.-M. Gambaudo, Math, Dijon, France & R. Benedetti, Math., Pisa, Italy.*
**Article:** *Spaces of Tilings, Finite Telescopic Approximations and Gap-Labeling.* [3]

This paper has been accepted for Publication in Communication in Mathematical Physics in 2003. It took until April 2005 to finish the revision because of a lot of details and technicalities that needed to be fixed. The paper should appear in 2006.

If the set of atoms is a repetitive Delone set of finite type, the present paper shows that it is associated with a repetitive tiling satisfying the finite pattern condition (the number of pattern of each size is finite modulo translation), the Hull can be constructed as an inverse limit of Branched Oriented Flat Riemannian manifolds (BOF). This construction allows to compute the longitudinal cohomology, the K-theory and the longitudinal homology of the corresponding NCBZ. It is proved that the set of invariant measures is given by the homology group in degree $d$. Criterion for unique ergodicity are given. This work is an extension to tilings of a previous work by Gambaudo and Martens [11] written in 2000 and likely to be published soon.

## 1.2 $K$-theory of the Hull and Box Decomposition:

**Collaboration:** *J. Savinien, Graduate School, Math., Georgia Tech.*

In 2005, J. Savinien has switched from continuing his study on phonons (see Section 2.1 below) to the problem of understanding the computation of the group of $K$-theory of the Hull of a repetitive finite type tiling. In an important '99 paper, Forrest and Hunton [9] have proved that the $K$-group of the $C^*$-algebra of a $\mathbb{Z}^d$-action on a Cantor set $X$ is isomorphic to the group cohomology of $\mathbb{Z}^d$ with values in the $K$-group of $X$, namely the space $\mathcal{C}(X, \mathbb{Z})$ of continuous function on $X$ with integer values. A claim made in [9] that these cohomology groups are torsion free was recently corrected and computation of such groups shows indeed that torsion is actually present in various quasicrystal on dimension 3 [10]. In principle the result of Forrest and Hunton also applies to all repetitive finite type tilings thanks to a theorem by Sadun and Williams [18] asserting that the Hull is homeomorphic to a bundle over a torus with fibers given by a Cantor set. However, this homeomorphism is not a conjugacy for the $\mathbb{R}^d$ action on the Hull. Moreover, the theorem of Forrest and Hunton is proved through a series of general arguments making the isomorphism quite implicit. Jean Savinien has taken the task of looking for a different point of view to describe this group.

In the case of a $\mathbb{Z}^d$-action on a Cantor set $X$, the Hull can be constructed as a *mapping torus* of the of $X$ by this action. In the tiling space construction the results given in the previous Section 1.1 shows that the mapping torus construction can be replaced by the so-called *box decomposition*. Namely a box is a subset of the Hull homeomorphic to a cartesian product $T \times U$ where $T$ is a clopen subset of the transversal, $U \subset \mathbb{R}^d$ is open and the homeomorphism is such that $\{t\} \times U$ corresponds to open subset of the leave through $t \in T$. A box decomposition is a finite disjoint family of such boxes, with closure covering the Hull. Box decompositions always exist in such tiling spaces. The mapping torus is a special case for which there is a box decomposition with a unique box. Using box decomposition, it is easy to built a spectral sequence converging to the $K$-theory (Atiyah-Hirzebruch's or Schochet's spectral sequence). However in the course of building the spectral sequence, it occurs that the combinatoric of the various groups entering in the construction is described through the cohomology of a complex of discrete groups, generalizing the Pimsner-Voiculescu exact sequence. The groups occurring in this complex are given by the space $C(\Xi_F, \mathbb{Z})$ where $F$ runs through the set of prototiles or their faces, $\Xi_F$ is $F$'s acceptance zone and is a clopen transversal to the lamination in the Hull and the differential is a combination of the usual simplicial cohomology, induced by the complex of faces of prototiles, and the translations in the Hull allowing to identify the various $\Xi_F$'s. In the case of a mapping torus, such complex was proposed by Pimsner. The goal is to prove that the cohomology of this complex gives the page two of the spectral sequence.

## 1.3 Transverse Dirac Operator:

**Collaboration:** *J. Pearson, Graduate School, Math., Georgia Tech., M. Benameur, Univ Metz, France*

This part of the project is to show that the singular foliated space defined by the Hull can be treated in a way similar to the Connes-Moscovici approach to foliated smooth manifolds. The PI has already defined a Dirac operator acting transversally to the orbits of $\mathbb{R}^d$ and defining a Fredholm module with respect to the crossed-product C*-algebra of the Hull (the so-called Noncommutative Brillouin zone, NCBZ). The PI expects that this is the first step toward describing the transverse fundamental class of the NCBZ. He has already found relations between parts of the dimension spectrum of this operator and various fractal exponents describing the complexity of the atomic distribution at large scale. One open question is whether there is a canonical Hopf algebra, like in Connes-Moscovici, describing the invariance under diffeomorphisms of the transverse structure.
This topics is now handled by John Pearson, a graduate student from Georgia Tech, who started in October 2004. He has now a general understanding of the Dirac operator on a metric Cantor set. With each metric Cantor set $X$, a canonical tree graph can be associated whose boundary is homeomorphic to $X$. This tree graph picture is illuminating in connection with the construction of the Dirac operator, because the representation used for building a spectral triple uses the $\ell^2$-space of the tree. However, John Pearson would like to prove a converse, namely to reconstruct the metric from the tree. Partial results

can be found in the literature but so far the reconstruction is not complete.

## 1.4 Homotopy of the Hull:

**Collaboration:** *I. Palmer Graduate School, Math., Georgia Tech.*

In a '00 paper, Gambaudo and Martens [11] have considered a $\mathbb{Z}$-action on a Cantor set and proposed to see the Hull as the inverse limit of graphs, which can be seen as the one-dimensional analog of branched oriented flat Riemannian manifolds used to describe the Hull of a tiling. In their study, they have shown that the homology gives a classification of invariant probability measures, a result that has been generalized to the tiling case (Section 1.1). However, they have also shown that the homotopy gives precise information upon the entropy of the dynamical system. In the tiling theory, this aspect has not been investigated yet. The goal of Ian Palmer Ph. D. Thesis is to fill this gap.

**Results obtained:** The first step being investigated is the construction of the covering space of a branched oriented flat Riemannian manifold $B$ of the type used to describe the Hull of a tiling. Such a covering space is a branched manifold obtained by picking a prototile, gluing on its side a copy of all the prototiles that are allowed to be glued by the local matching rules, without producing any loop or closed surface, then repeating the operation on each free side of the new tiles. Such a construction is the multidimensional analog of the construction of a tree graph. The result will be called a *garland* here. It is a branched oriented flat Riemanian manifold which is simply connected, and locally compact, but not compact. If $G$ denotes such a garland, a smooth sheet is an unbranched submanifold, obtained by picking a tile of $G$, then on each of the maximal face, picking one tile among the possible branches, and keep going forever to get a non compact, locally compact, simply connected, flat Riemannian manifold without boundary. However, as in the case of Riemann surfaces of a multivalued holomorphic function, such a manifold can have *helicoidal points* (a notion introduced by R. Williams). A smooth sheet is a tiling if and only if tit has no helicoidal points. The set of smooth sheets can be given a topology like the set of infinite paths of a tree graph. Namely two smooth sheet are close enough if they coincide on a large finite patch. Such a topology produces a Cantor set called the *boundary* of $G$ and denoted by $\partial G$. By adding the longitudinal structure of a sheet into the game, one gets a lamination $\mathfrak{L}(G)$ with leaves given by the smooth sheets of $G$ and transversal $\partial G$. Clearly any closed submanifold $C$ of $B$ can be lifted to $G$ in many ways and $G$ helps investigating the homotopy class of $C$. The first result obtained so far is that if $(B_n)_{n\in\mathbb{N}}$ is a family of BOF manifolds the inverse limit of which defines the Hull of a tiling, then the corresponding garlands $(G_n)_{n\in\mathbb{N}}$ satisfy $G_{n+1} \subset G_n$ so that $\mathfrak{L}(G_{n+1})$ can be seen also as a subset of $\mathfrak{L}(G_n)$ and the Hull of the original tiling can be recovered as the intersection of the $\mathfrak{L}(G_n)$'s.

## 1.5 The Mott Variable Range Hopping:

**Collaboration:** *Peter Hislop, U. Kentucky*

**Article** J. Bellissard, P. Hislop, "Smoothness of correlations in the Anderson model at strong disorder", Submitted to *Ann. H. Poincaré*, (8 Oct 2005), available on `http://www.math.gatech.edu/~jeanbel/publijb.html`

**Result:** In the paper submitted to *Annales Henri Poincaré* in the Fall 2005, the PI and Peter Hislop have shown that the the current-current correlation measure $dm(E, E')$ is *analytic* away from the diagonal $\Delta = \{(E, E')\,;\, E = E'\}$. The same is true for higher order current-current correlations functions. □

In 1968 Sir N. Mott [15] predicted that in strongly disordered systems, the low temperature conductivity should decay to zero as

$$\sigma \sim e^{-(T_M/T)^{1/(d+1)}}\,, \qquad\qquad \text{as } T \downarrow 0\,,$$

where $T_M$, the Mott temperature, is a constant depending upon the characteristic of the electron gas near the Fermi level: density of states and localization length. This behavior has been observed in many insulating materials in which the Anderson localization is realized. It has been updated in the eighties by Efros & Shklovskii [19] if the Coulomb interaction between electrons is unscreened, in which case the exponent in $1/2$ instead of $1/(d+1)$. Physicists believe that this Mott transport is responsible for the accuracy of the Hall plateaux in the Quantum Hall effect [16, 17]. The theoretical description of such result is so difficult that it takes a fair part of the book by Efros & Shklovskii [19] on semiconductors to be treated.

In a series of paper with D. Spehner [21, 22, 2] the PI has built models describing this phenomenon. More recently, Faggionato, Schulz-Baldes, Spehner [8], have shown that Mott's formula is a lower bound for the diffusion of a semiclassical stochastic model built on Mott's hypothesis (called the FSBS model below). However, such model has not been validated from first principle yet. Electrons are quantized degrees of freedom and their dynamics is given by a Schrödinger equation. In disordered solids, such as the conduction electrons in semiconductor at very low temperature, it is generally accepted that the Anderson model is a good approximation to describe this dynamics. To assess the validity of the axioms leading to the FSBS model, the PI, in collaboration with P. Hislop, has conjectured that the estimate (1) below should be true and is necessary to lead to Mott result. More precisely, let $H$ be the Anderson Hamiltonian on $\mathbb{Z}^d$. namely $H = \Delta + V$ where $\Delta$ is the discrete Laplacean, while $V = (V(x))_{x\in\mathbb{Z}^d}$ is a potential given by identically distributed independent real random variables, with covariance $\langle V(x)^2\rangle = W^2$. If $dm(E, E')$ represents the current-current correlation function of this model, if $I$ is a small interval around $E_F$ (the Fermi level), then there is $W_c$ large enough so that for $W > W_c$ there is $c > 0$ for which

$$\int_{I\times I} dm(E,E') \exp\left(c/|E-E'|^{1/(d+1)}\right) < \infty . \qquad \textbf{(conjecture)} \qquad (1)$$

As a consequence of this estimate, the Mott argument should lead to a rigorous proof of the Mott prediction. However, the PI and Hislop not only have failed to prove (1) but have strong evidence that instead $dm(E,E')$ vanishes like $|E-E'|^2 \ln^{\alpha}(|E-E'|)$ for some $\alpha$ related to the dimension $d$. A recent result by Klein, Lenoble, and Müller [12] goes into the same direction.

This raises the question of the validity of Anderson's model in accounting for Mott's prediction. Considering that there is an overwhelming set of experimental evidences validating Mott prediction, especially in semiconductors at low temperature but also in many insulators, the problem is to understand what mechanisms and what microscopic models are leading to it. The PI is currently working at it with P. Hislop

## 1.6 Kubo's formula:

**Collaboration:** *D. Spehner (Essen, Germany), R. Rebolledo (Santiago, Chile), M. Loss, E.Carlen, (School of Mathematics, Georgia Tech)*
The PI is continuing a longstanding program in dealing with the proof of Kubo's formula. This program was initiated in [5] in order to estimate the accuracy of the Quantum Hall Effect from first principle. Since then, the PI has produced several papers in collaboration with his former students, H. Schulz-Baldes, D. Spehner, on the one hand and more recently R. Rebolledo (see the review paper [2]).

**Results obtained since 2004:** During a visit at Santiago in March 2004, the PI and R. Rebolledo have developped the formalism needed to treat the dissipation dynamics for the electron gas (N-body problem) in an aperiodic solid. They proved that the generator of the dynamic $\mathfrak{L}$ (Lindblad Operator) has an infinite volume limit and defines a Markov semigroup preserving the C*-algebra of observable (and not only the von Neumann algebra). Moreover the PI has obtained the Green-Kubo formula for all transport cœfficients, including the thermal conductivity and the thermopower [4]. This formula requires the existence of an inverse for $\mathfrak{L}$. In trying to deal with this problem, the PI has recently (Fall 2004) found an amazing equivalence with a class of $XY$-model for quantum spins. This part is in preparation.

Presently the PI is writing a review paper on this subject. He is also extending the *information estimates* found by E. Carlen and Lieb [7] in the case of free fermions at infinite temperature, to the case of finite temperature in order to estimate the rate of convergence to equilibrium. Since this is an estimate on $e^{t\mathfrak{L}}$, there is a chance that, through a Laplace transform, tan estimate be available soon on the inverse

$$-\mathfrak{L}^{-1} = \int_0^{\infty} dt \, e^{t\mathfrak{L}}$$

## 1.7 Completely Positive Maps:

**Collaboration:** *William Green, School of Math., Georgia Tech.*

Will writing the paper on Kubo's formula (see Section 1.6) the PI came across the problem of characterizing the Markov semigroups in $C^*$-algebras. For the $C^*$-algebraof bounded operators on a Hilbert space, this problem has been solved by Lindblad [14] using the Stinespring [23] theorem and a theorem of Kraus [13] on completely positive maps. The Lindblad theorem is the noncommutative analog of the Levy-Khintchin theorem for stationary stochastic processes (the so-called Levy processes). Amazingly, Such a result has never been extended to other cases of $C^*$-algebras. Some extension to hyperfinite factors do exist, but it is likely that for non hyperfinite ones, topological obstructions, like rigidity, do exists. In collaboration with W. Green, form Georgia Tech, the PI has taken the task to consider the case of Markov semigroups on an UHF algebra and to see whether there is an extension of Lindbald's theorem. It seems that using a description of such algebras in terms of groupoid $C^*$-algebras a generalization of Lindblad's theorem, using the ideas used in the proof of the Levy-Khintchin theorem, can be obtained.

# 2 Previous topics:

## 2.1 Phonons in Aperiodic Solids:

**Collaboration:** *J. Savinien, Graduate School, Math., Georgia Tech.*

A new project started in May 2003 with a Ph D student, Jean SAVINIEN, at Georgia Tech: *Theory of phonons in aperiodic solids.* During the Spring 2004 semester this student is paid through the present NSF program.

*This project has been stopped for a while, due to difficulties that has not been overcome yet.*

Phonons are acoustic waves in solids due to vibrations of atoms about their equilibrium position. These waves are quantized and their frequencies constitute the "phonon spectrum" and are given by the spectrum of an operator $H$ that belongs to the C*-algebra of the Noncommutative Brillouin zone (see Reminder above). The corresponding Density of Vibrational States (DOVS) is obtained as the spectral measure with respect to a given extremal trace on the NCBZ (or equivalently to an invariant ergodic probability measure on the Hull). The first step in this program consists in controlling the low frequency limit: in this limit, the phonons see only the large scale features of the solid. They behave like solutions of the wave equation with a sound velocity depending upon the polarization. If the solid is macroscopically isotropic and homogeneous, the sound velocities of all transverse modes are identical, while the longitudinal mode may have a different velocity. These two velocities can be computed from the invariant probability measure and the explicit form of $H$.

**Results:** In 2003-2004, an homogeneisation technique has been used to get an estimate on the selfadjoint operator giving the distribution of vibration modes (the 1-phonon Hamiltonian) in the harmonic approximation. In particular the low frequency limit produces a dispersion relation with a sound velocity that can be computed from the microscopic data (average spring constants). During the last ten months in 2004-2005, Jean Savinien has extended this work to the case of an electron in the effective mass approximation (band edges).

Jean Savinien and the PI are working at using these estimates to get the diffusion exponent in this latter limit. If $r(t)$ represents the (average) distance spent by the electron within time $t$ then $r(t) \sim t^{\beta}$ where $\beta$ is the diffusion exponent. It is expected that, whenever the effective mass approximation applies, then $\beta \sim 1$ (ballistic motion). There are still some technical difficulties that have to be overcome in order to get a complete proof.

## 2.2 The Jacobi matrix of a Julia set:

**Collaboration:** *J. Geronimo, School Mathematics, Georgia Tech & P. Yuditskii, Kharkov, Ukrain*
**Reference:** Bellissard, J.; Geronimo, J.; Volberg, A.; Yuditskii, P. "Are they limit periodic?" *Complex analysis and dynamical systems II*, 43-53, *Contemp. Math.*, **382**, *Amer. Math. Soc.*, Providence, RI, (2005).

**History**: this problem was initiated in 1982 by the PI and became the source of a long-standing collaboration with J. Geronimo (Georgia Tech) since 1986. Recently, P. Yuditskii (Kharkov) joined this group after his contribution to this problem using the Ruelle thermodynamical formalism [20]. Since may 2003, the PI together with J. Geronimo and P. Yuditskii have revised this problem after some progress made by P. Yuditskii.

## 2.3 Magnetoresistance of Graphite Monolayer:

**Collaboration:** *C. Berger, W. de Heer, School of Physics, Georgia Tech*
**Proceedings APS March Meeting 2004:** *Evidence for 2D electron gas behavior in ultrathin epitaxial graphite on a SiC substrate* [6]

The group led by Walter de Heer in the School of Physics, Georgia Tech, has been involved in studying carbon nanotubes for several years so far. A recent move lead this group toward studying a flat version of these nanotubes: monolyaers of carbon graphite. It is technologically possible to produce such monolayers on surface of a Silicon-Carbide clean sample, by usual chemical treatments. While the Si-C substrate is an insulator with wide gap, the surface monolayer behaves like a semimetal. C. Berger has performed several measurements of the magnetoresistance in high magnetic field (larger than $1T$), at relatively low temperature (down to few $K$). The finding is surprising: there is a *gigantic variation of the magnetoresistance* (three orders of magnitude while the magnetic field varies from $b = 0$ to $B \approx 1T$ !). I have proposed them a possible mechanism for explaining their result that is described below. A preliminary announcement of this result has been posted for a poster session at the APS March meeting in March 2004 [6].

Building these monolayers has been actually more difficult than expected. In 2004, the group has spent all his time to improve the surface techniques in order to get flat monolayer without defects. Then, in the Fall, the unexpected publication by another group, increased the pressure to get fast more results, in particular in proving that transistor can be designed.

Since this last part of the work is more material science, the PI has not been able to collaborate on this issue. However, since early January 2005, Walt de Heer and his group have resumed the measurement of the magneto resistance and the collaboration should resume as well.

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**Reminder:** an aperiodic solid can be described as follows: let $\mathcal{L} \subset \mathbb{R}^d$ be the discrete set of equilibrium positions of its atoms in the real space in which the solid is setting. The Hull of $\mathcal{L}$ is obtained as the closure of the family $\{\mathcal{L} + a; a \in \mathbb{R}^d\}$ of translated of the solid. The topology used here is the following: a subsequence of discrete subsets converges if and only if their intersections with any bounded open set of $\mathbb{R}^d$ converges for the Hausdorff distance. If the initial set of atoms is *uniformly discrete*, the Hull is a compact space $\Omega$ endowed with an action of $\mathbb{R}^d$ by homeomorphisms. Each point in the Hull is itself a discrete subset of $\mathbb{R}^d$. This dynamical system has a *canonical transversal* made of points such that the corresponding discrete subset contains the origin in $\mathbb{R}^d$. This transversal is called the *atomic surface* by physicist studying quasicrystals. Such a Hull can be described through various points of view: as a dynamical system as previously, as a groupoid (of the dynamical system or of its transversal), as a tiling space, as a lamination (or foliated space). The crossed product $C^*$-algebra $\mathcal{C}(\Omega) \rtimes \mathbb{R}^d$ can be interpreted as the space of continuous functions on a noncommutative manifold called the *Noncommutative Brillouin Zone* (NCBZ) [3]. Any invariant measure on the Hull gives rise to a trace on the NCBZ and vice-versa, and the trace is positive if and only if the invariant measure is positive.

**PhD. students:** During the period 2003-2007, the PI has advised four PhD students, Jean Savinien, John Pearson, Ian Palmer and Michael Burkhart. Jean Savinien just submitted a paper in May 2007 [12], John Pearson is about to finish the writing of his first paper [13] hopefully before the end of August 2007. Both should defend their PhD Thesis in 2008. The other two start having preliminary results. The four of them have been occasionally supported by the present grant as R.A.. The four of them have contributed to the present program of research.

# 1 The Hull of a Repetitive Finite Type Tiling:

**Collaboration:** *J.-M. Gambaudo, Math, Dijon, France & R. Benedetti, Math., Pisa, Italy.*
**Article:** *Spaces of Tilings, Finite Telescopic Approximations and Gap-Labeling.* [5]

This paper has been accepted for Publication in Communication in Mathematical Physics in 2003. It took until April 2005 to finish the revision because of a lot of details and technicalities that needed to be fixed.

If the set of atoms is a repetitive Delone set of finite type, the present paper shows that it is associated with a repetitive tiling satisfying the finite pattern condition (the number of pattern of each size is finite modulo translation), the Hull can be constructed as an inverse limit of Branched Oriented Flat Riemannian manifolds (BOF). This construction allows to compute the longitudinal cohomology, the K-theory and the longitudinal homology of the corresponding NCBZ. It is proved that the set of invariant measures is given by the homology group in degree $d$. Criterion for unique ergodicity are given. This work is an extension of the result by Anderson and Putnam [1] on substitution tilings, and of a previous work by Gambaudo and Martens [20] to the case of tilings which are repetitive and with finite local complexity.

# 2 $K$-theory of the Hull and Box Decomposition:

**Collaboration:** *J. Savinien, Graduate School, Math., Georgia Tech.*

In 2005, J. Savinien has switched from continuing his study on phonons (see Section 5.3 below) to the problem of understanding the computation of the group of $K$-theory of the Hull of a repetitive finite type tiling. In an important '99 paper, Forrest and Hunton [18] have proved that the $K$-group of the $C^*$-algebra of a $\mathbb{Z}^d$-action on a Cantor set $X$ is isomorphic to the group cohomology of $\mathbb{Z}^d$ with values in the $K$-group of $X$, namely the space $C(X, \mathbb{Z})$ of continuous function on $X$ with integer values. A claim made in [18] that these cohomology groups are torsion free was recently corrected and computation of such groups shows indeed that torsion is actually present in various quasicrystal on dimension 3 [19]. In principle the result of Forrest and Hunton also applies to all repetitive finite type tilings thanks to a theorem by Sadun and Williams [29] asserting that the Hull is homeomorphic to a bundle over a torus with fibers given by a Cantor set. However, this homeomorphism is not a conjugacy for the $\mathbb{R}^d$ action on the Hull. Moreover, the theorem of Forrest and Hunton is proved through a series of general arguments making the isomorphism quite implicit. Jean Savinien has taken the task of looking for a different point of view to describe this group.

In the case of a $\mathbb{Z}^d$-action on a Cantor set $X$, the Hull can be constructed as a *mapping torus* of the of $X$ by this action. More generally, let $\mathcal{A}$ be a $C^*$-algebra endowed with a

$\mathbb{Z}^d$ action $\alpha : \mathbb{Z}^d \mapsto \mathrm{Aut}\mathcal{A}$. The Pimsner complex is defined as

$$K_*(\mathcal{A}) \otimes \Lambda\mathbb{Z}^d \xrightarrow{d_P} K_*(\mathcal{A}) \otimes \Lambda\mathbb{Z}^d\,, \qquad\qquad d_P = \sum_{i=1}^{d} (\alpha_{i*} - \mathbf{1}) \otimes e_i \wedge$$

where $\{e_1, \cdots, e_d\}$ is the canonical basis of $\mathbb{Z}^d$, $\alpha_i = \alpha_{e_i}$ is the restriction of $\alpha$ to the $i$-th component of $\mathbb{Z}^d$, whereas $x\wedge$ is the exterior multiplication by $x \in \mathbb{Z}^d$. Then in [12] the following result is proved as a preparation for the rest

**Theorem 1** *Let $\mathcal{A}$ be a $C^*$-algebra endowed with a $\mathbb{Z}^d$ action $\alpha$ by $*$-automorphisms. Then, there is a spectral sequence converging to the $K$-theory of $\mathcal{A} \rtimes \mathbb{Z}^d$*

$$E_2^{rs} \Rightarrow K_{r+s+d}(\mathcal{A} \rtimes_\alpha \mathbb{Z}^d)\,,$$

*with page-2 isomorphic to the cohomology of the Pimsner complex.*

It is worth noticing though, that such result is a special case of the Kasparov spectral sequence [21]. In the proof of this theorem, the crossed product $\mathcal{A} \rtimes \mathbb{Z}^d$ is replaced by the mapping torus crossed-product. Namely $M_\alpha\mathcal{A}$ is the $C^*$-algebra of continuous functions $f : [0,1]^d \mapsto \mathcal{A}$ such that, if $x \in [0,1]^d$ and $a \in \mathbb{Z}^d$ is such that $x + a \in [0,1]^d$, then $f(x + a) = \alpha_a(f(x))$. Here the hypercube $[0,1]^d$ can be replaced by any fundamental domain for the action of $\mathbb{Z}^d$ in $\mathbb{R}^d$. The latter plays the role of the universal cover of the classifying space of $\mathbb{Z}^d$. So that the hypercube could be replaced by the classifying space, which is homeomorphic to the $d$-torus $\mathbb{T}^d$. Then, a general result in $C^*$-algebra-theory shows that $\mathcal{A} \rtimes \mathbb{Z}^d$ is Morita equivalent to $M_\alpha\mathcal{A} \rtimes \mathbb{R}^d$ [28]. In particular they have the same $K$-theory. A theorem by Connes [15], called the Thom-Connes isomorphism, shows that the $K$-theory of $M_\alpha\mathcal{A} \rtimes \mathbb{R}^d$ is the same as the $K$-theory of $M_\alpha\mathcal{A}$ with a shift by $d$ (modulo 2) in the degree.

In the tiling space construction the results given in the previous Section 1 shows that the mapping torus construction can be replaced by the branched manifold $\mathcal{B}_0$ built by gluing together the elementary prototiles (equivalence classes of tiles modulo translations) along their faces, according to whether some of their representatives in the tiling are glued together this way. The algebra $\mathcal{A}$ is then replaced by $\mathcal{C}(\Xi)$, the space of continuous functions on the transversal. Then the Pimsner complex is replaced by the following: let the tiles in the tiling be given a compatible $\Delta$-complex decomposition. Then each simplex of this decomposition is punctured so that the set of punctures is a Delone set. Let $\Xi_\Delta$ be the corresponding transversal. This give the tiling the structure of a singular chain complex. If $\sigma$ is a $k$-cell and $\tau$ is a $k-1$ cell, both given with an orientation, let $[\sigma : \tau]$ be the incidence number (in $\{-1, 0, 1\}$) of the pair. Then the translation $x_{\sigma,\tau}$ from the puncture of $\tau$ to the puncture of $\sigma$ is well defined and depends only upon the corresponding prototile. This translation acts on $\Xi_\Delta$ as well. The operator $\theta_{\sigma,\tau} = \chi_\sigma \mathrm{T}^{x_{\sigma\tau}} \chi_\tau$, where $\chi_\sigma$ is the characteristic function, on $\Xi_\Delta$ of the acceptance zone of $\sigma$, represents such a translation. Then the Pimsner complex is defined as

$$C(\Xi_\Delta,\mathbb{Z}) \stackrel{d_P}{\to} C(\Xi_\Delta,\mathbb{Z})\,, \qquad\qquad d_P = \sum_{\sigma,\tau}[\sigma:\tau]\theta_{\sigma,\tau}\,.$$

As a result, J. Savinien and the PI have obtained the following theorem

**Theorem 2** *There is a spectral sequence that converges to the $K$-theory of the $C^*$-algebra of the hull $\Omega$*

$$E_2^{rs} \Rightarrow K_{r+s+d}\big(C(\Omega)\rtimes\mathbb{R}^d\big)\,,$$

*and whose page-2 is given by the cohomology of the Pimsner complex*

$$E_2^{rs} \simeq H_P^r\big(\mathcal{B}_0; K^s(\Xi_\Delta)\big)\,.$$

# 3 Transverse Dirac Operator:

**Collaboration:** *J. Pearson, Graduate School, Math., Georgia Tech.*

This part of the project is to show that the singular foliated space defined by the Hull can be treated in a way similar to the Connes-Moscovici approach to foliated smooth manifolds [16]. The main problem comes from the fact that, at least for tilings with finite local complexity, the transversal is completely disconnected, so that the tools of differential Geometry used in [16] cannot be used anymore. However, a differential structure can always be defined on such a space using the Connes notion of *spectral triple*. A spectral triple $(\mathcal{A},\mathcal{H},D)$ is a family where $\mathcal{A}$ is dense $*$-subalgebra of a $C^*$-algebra invariant by holomorphic functional calculus, $\mathcal{H}$ is a Hilbert space on which $\mathcal{A}$ is represented and $D$, called the *Dirac operator*, is a selfadjoint operator acting on $\mathcal{H}$ with compact resolvent, such that the commutator $[D,A]$ is bounded for $A\in\mathcal{A}$. Such a triple is *even* whenever, in addition, there is a bounded operator $\Gamma$ acting on $\mathcal{H}$, called *grading*, such that (i) $\Gamma=\Gamma^*=\Gamma^{-1}$, (ii) $[\Gamma,A]=0$ for $A\in\mathcal{A}$ and (iii) $\Gamma D + D\Gamma = 0$. Whenever $\mathcal{A}$ is abelian, namely when it is a dense $*$-subalgebra of $\mathcal{C}_0(K)$ where $K$ is a locally compact space, the following formula

$$\rho(x,y) = \sup\{|f(x)-f(y)|\,;\; f\in\mathcal{A}\,,\; \|[D,f]\|\leq 1\}$$

defines a distance on $K$, that will be called the *Connes metric.* Then a differential structure can be defined on $\mathcal{A}$ through

$$dA = [D,A]\,, \qquad\qquad A\in\mathcal{A}\,.$$

As a warm up towards such a program, the PI has proposed John Pearson to consider first the case of a metric Cantor set. In [13], J. Pearson and the PI have obtained the main part of the results. An abstract Cantor set is a completely disconnected compact space with no isolated point. Up to homeomorphism, such a space is unique and can be described as the triadic Cantor set. However, if a metric is added, there are many non isometric Cantor sets. So let $(C, d)$ be a metric Cantor set. Then Michon [25] has proposed to consider the following family of equivalence relation: for $\epsilon > 0$ two points $x, y$ in $C$ are $\epsilon$-connected whenever there is a finite family $x_0 = x, x_1, \cdots, x_n = y$ in $C$ such that $d(x_{k_1}, x_k) \leq \epsilon$ for $1 \leq k \leq n$. This relation is clearly an equivalence. The $\epsilon$-equivalence classes are called $\epsilon$-connected components. As $\epsilon$ decreases, there are values $\epsilon_0 > \epsilon_1 > \cdots > \epsilon_m > \cdots$ decreasing to zero as $m \to \infty$ such that the equivalence relation is constant if $\epsilon_{n-1} > \epsilon \leq \epsilon_n$ and changes otherwise. Then the metric $d$ will be called *saturated* if given $x, y \in C$, the distance $d(x, y)$ is equal to the diameter of the $\epsilon$-connected component of $x$ for the smallest possible $\epsilon$ such that $x$ and $y$ are $\epsilon$-connected. The previous decomposition provided $C$ with a tree graph, called the *Michon tree* as follows: the vertices at level $n$ are the connected components for the $\epsilon_n$-equivalence. Let $V_n$ be the set of such vertices. Then $V_0$ as only one point, namely $C$ itself. It will be called the *root* of the tree. A vertex $v \in V_n$ is linked to $v' \in V_{n+1}$ by and *edge*, if $v' \subset v$. The Michon tree $\mathcal{T}_{(C,d)}$ is such that any vertex $v$ has one grandchild having at least two sons. Then the limit set $\partial \mathcal{T}_{(C,d)}$ is defined as the set of infinite paths starting from the root and there is a natural topology on it, defined by the clopen sets $[v]$ namely the set of infinite path going through $v$. Then it is easy to check that $C$ is homeomorphic to $\partial \mathcal{T}_{(C,d)}$. The first result is

**Theorem 3 (Classifications of saturated ultrametrics)** *Let $T$ be a rooted tree such that every vertex has a nonzero but finite number of children. Let $V(T)$ denotes its vertex set. Then there is a one-to-one correspondence between saturated ultrametrics on $\partial T$ for which the metric topology agrees with the boundary topology and the set of functions $f : V(T) \to \mathbb{R}^+$ such that for $v, v' \in V(T)$ with $v \succeq v'$ the following axioms are satisfied:*
*(1) $f(v) \geq f(v')$.*
*(2) If $[v] = [v']$ then $f(v) = f(v')$.*
*(3) If $[v]$ has more than one point then $f(v) > 0$.*
*(4) For $v_0 v_1 \cdots \in \partial T, \lim_{n \to \infty} f(v_n) = 0$.*

**Corollary 1** *Let $(C, d)$ be a metric Cantor set with Michon's tree $T = T_{(C,d)}$. If $d$ is a saturated ultrametric, then $(C, d)$, is isometrically isomorphic to $\partial T$ where the distance on $T$ is associated with the function $f$ on $V(T)$ defined by $f(v) = \operatorname{diam}_d[v]$.*

Thanks to Michon's result and to Theorem 3, it is then sufficient to start from a rooted tree endowed with a saturated ultrametric $d$. Then a spectral triple is defined as follows: $\mathcal{A} = C_{\text{Lip}}(\partial T)$ is the space of Lipshitz continuous functions on $\partial T$, $\mathcal{H} = \ell^2(V(T)) \otimes \mathbb{C}^2$, and the Dirac operator is defined by

$$D \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} (v) = \frac{1}{\operatorname{diam}[v]} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} (v) , \qquad v \in V(T) , \psi_i \in \ell^2(V(T)) .$$

A grading operator can be defined, with the same notations, by

$$\Gamma \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} (v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} (v) .$$

The main difficulty comes from the representation of $\mathcal{A} = \mathcal{C}_{\text{Lip}}(\partial T)$. For indeed there are many possible representations. To describe them, a *choice* is a function $\tau : v \in V(T) \mapsto (\partial T)^{\times 2}$ such that $\tau(v) \in [v] \times [v]$ and that, if $\tau(v) = (x_v, y_v)$ then $d(x_v, y_v) = \text{diam}[v]$. Given such a choice the representation $\pi_\tau$ of $\mathcal{C}_{\text{Lip}}(\partial T)$ is defined, for $f \in \mathcal{C}_{\text{Lip}}(\partial T)$, $v \in V(T)$ and $\psi_i \in \ell^2(V(T))$, by

$$\pi_\tau(f) \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} (v) = \begin{bmatrix} f(x_v) & 0 \\ 0 & f(y_v) \end{bmatrix} \begin{bmatrix} \psi_1(v) \\ \psi_2(v) \end{bmatrix} .$$

It is elementary to check that each such representation is faithful. However they are all needed for recovering the metric, as shown by the following result [13]

**Theorem 4** *The Connes metric on $\partial T$ defined by*

$$\rho(x, y) = \sup\{|f(x) - f(y)| \,;\, f \in \mathcal{A}, \sup_\tau \|[D, \pi_\tau(f)]\| \leq 1\}$$

*coincides with the original metric $d$ on $\partial T$.*

The last result concerns the $\zeta$-function of $D$, defined by

$$\zeta(s) = \text{Tr}\,(|D|^{-s})$$

Since $\partial T$ is a metric space, the notion of (upper) *box dimension* is well defined (see [17] for the case of Cantor sets embedded in $\mathbb{R}^d$). If $N_\delta$ is the smallest number of clopen sets of diameter at most $\delta$ needed to cover $\partial T$, then

$$\overline{\dim}_B(\partial T) = \limsup_{\delta \downarrow 0} \frac{\ln N_\delta}{-\ln \delta}$$

**Theorem 5** *If $T$ is a rooted tree such that each vertex as at most $M$ children, for some $M \geq 2$. Let $d$ is a saturated ultrametric on $\partial T$ and let $D$ be the Dirac operator associated with $d$. Then $\zeta(s)$ converges for $\Re s > s_0$ where $s_0 = \overline{\dim}_B(\partial T)$ and diverges at $s = s_0$.*

# 4 Coherent transport in disordered media:

**Collaboration:** *Peter Hislop, U. Kentucky, Günter Stolz, U. Alabama Birmingham*
The coherent transport of electrons in an aperiodic solid concerns the dissipationless transport. In a series of previous work by the PI and its collaborators, published in the nineties, this transport is characterized by a *diffusion exponent* $\beta$ (see [4] for a review),

defined by $L(t) \sim t^{\beta}$ if $L(t)$ represents the distance covered by the electron during time $t$. The precise definition of $L(t)$ can be involved but has been thoroughly studied previously. One of the main consequence of such a qualitative analysis is that the conductivity satisfies an *anomalous Drude formula* namely

$$\sigma \sim \tau_{rel}^{2\beta-1}$$

where $\tau_{rel}$ denotes the typical *inelastic* relaxation time of the electron. As a consequence, since the relaxation times diverges at zero temperature, the systems behaves like a conductor if $\beta > 1/2$ and like an insulator if $\beta < 1/2$. It has been advocated that the strange behavior of the conductivity in quasicrystals is likely to be explained by a diffusion exponent closed to 1/3. The case $\beta = 1/2$ is likely to describe the regime of *weak localization* in disordered systems whenever the level statistic in the energy spectrum looks like the one for random matrices.

In any case, it has been shown (see [4] for a review) that if $H$ describes the one-electron (or one-hole) Hamiltonian and if $X = (X_1, \cdots, X_d)$ is the position operator then, the current operator is given by $J = \imath[H, X] = (J_1, \cdots, J_d)$ and describes the coherent part of the current. The *current-current correlation function* is defined through the following formula: if $f, g$ are continuous functions on the spectrum of $H$ vanishing at infinity then

$$\mathcal{T}_{\mathbb{P}}\left(J_i f(H) J_j g(H)\right) = \int_{\mathbb{R}^2} dm_{ij}(E, E')\, f(E)\, g(E') \,. \tag{1}$$

The symbol $\mathcal{T}_{\mathbb{P}}$ represents the *trace per unit volume* that is usually defined whenever the electron moves in a solid described by a set of atomic position with Hull $\Omega$ and $\mathbb{P}$ represents a Probability measure on $\Omega$ which is translation invariant and ergodic. Then $\mathcal{T}_{\mathbb{P}}$ defines a trace on the $C^*$-algebra of the Hull $\mathcal{C}(\Omega) \rtimes \mathbb{R}^d$. In most cases of interest, it is convenient to use the *tight binding representation* in which the electronic wave function lives on the set of atomic positions, a discrete subset of $\mathbb{R}^d$, in which case $H$ belongs to the $C^*$-algebra of the transversal $\Xi$ and $\mathbb{P}$ is replaced by the probability measure induced on the transversal by $\mathbb{P}$. Therefore the left-hand side of eq. (1) is well defined for a large class of aperiodic solids. The classical Riesz theorem then implies the existence of the measure $m_{ij}$ called the *current-current correlation function*. Not much is known about this measure apart that the matrix valued measure $m = (m_{ij})_{ij}$ is positive. It is usually unbounded unless the current $J$ is bounded, which happens in the tight binding representation. However it can be shown (see [4]) that the diffusion exponent is given by

$$\beta = \inf\{0 < \beta' \leq 1\,;\, \sum_i \int_{\mathbb{R}^2} \frac{dm_{ii}(E, E')}{|E - E'|^{2(1-\beta')}}\} \,.$$

This suggests that the diffusion exponent is related to the behavior of the current current correlation function near the diagonal. For models leading to strong localization, the *localization length* $\ell(E)$ at a given energy $E$ is given by

$$\int_\Delta dE\ \ell(E)^2 = \sum_i \int_{\Delta\times\mathbb{R}} \frac{dm_{ii}(E,E')}{|E-E'|^2}\,.$$

In a recent work by Klein, Lenoble, and Müller [22], it is strongly suggested that $dm(E,E')$ vanishes like $|E-E'|^2 \ln^\alpha(|E-E'|)$ for some $\alpha$ related to the dimension $d$ in such a case. In much the same way the $n$-point current correlation can be defined by

$$\mathcal{T}_\mathbb{P}\left(J_{i_1}f_1(H)J_{i_2}f_2(H)\cdots J_{i_n}f_n(H)\right) \;=\; \int_{\mathbb{R}^2} dm_i(E_1,\cdots,E_n)\ f_1(E_1)\cdots f_n(E_n)$$

In this case however it is unclear whether $m_i$ is a positive measure.

In front of this situation the PI has tried to convince the community of mathematical Physicists working in this direction to take the task of investigating the smoothness properties of the current-current correlation function. As a warm up he obtained, in collaboration with Peter Hislop, a partial result for the case of the Anderson model at strong disorder. The model is defined on $\ell^(\mathbb{Z}^d)$ as

$$H = \lambda\Delta + V$$

where $\Delta$ is the discrete Laplacean defined by $\Delta\psi(x) = \sum_{y;|y-x|=1}\psi(y)$ whereas $V$ is the operator of multiplication $V\psi(x) = V_x\psi(x)$ where the $V_x$'s are independent, identically distributed random variables such that $\mathbb{E}(v_x) = 0$ and $\mathbb{E}(V_x^2) = W^2$ is the disorder parameter. In the paper [10] the following Theorem was proved

**Theorem 6 (see [10])** *Assume that the common distribution of the $V_x$'s can be continued as a holomorphic function in a strip $\{z\in\mathbb{C}\,;\,|\Im z|<r\}$ and is integrable along any line $\mathbb{R}+\imath s$ uniformly for $|s|<r$.*
*Then there is $\lambda_0>0$ such that for $|\lambda|<\lambda_0$, the $n$-point current-current correlation functions are absolutely continuous measure with analytic density away from the diagonal.*

In 1996, Minami [26] proved that the level spacing distribution for the Anderson model at large disorder was given by a Poissonian statistics. The proof was based on an estimate that has surprised most experts in the field, in that it has been very difficult to extend it beyond the situation investigated by Minami. In particular, to these days nobody has succeeded to extend it to the case of an Anderson model on the continuum.

During a series of discussions with Peter Hislop and Günter Stolz, the PI found with them a new proof of this estimate [11], allowing to extend the result of Minami on Poissonian statistics to a much larger class of Hamiltonians. To explain this estimate, the Hamiltonian for which this estimate holds have the form $H = H_0 + V$ where, as before, $V$ is a random potential. But $H_0$ is simply a fixed selfadjoint bounded operator on $\ell^(\mathbb{Z}^d)$. Let $\Lambda$ be a finite subset of the lattice $\mathbb{Z}^d$ and let $P_\Lambda$ be the projection onto the finite dimensional space $\ell^2(\Lambda)$. Then $H_\Lambda = P_\Lambda H P_\Lambda$. Let now $\Delta\subset\Lambda$ and let

$$g_\Delta(z) \;=\; P_\Delta \frac{1}{H_\Lambda - z} P_\Delta \,.$$

Then

**Theorem 7 (see [11])** *Let $H = H_0 + V$ where $H_0$ is a bounded selfadjoint operator on $\ell^2(\mathbb{Z}^d)$ and $V = (V_x)_{x\in\mathbb{Z}^d}$ is a random potential made of independent identically distributed random variable. If $d\rho(V)$ be the common distribution of the $V_x$'s, then $d\rho$ is supposed to be absolutely continuous, namely $d\rho(V) = \rho(V)dV$ with bounded density.*
*If $\Im z > 0$ and if $\Delta$ is any subset of $\Lambda$ having $n$ points, the following inequality holds*

$$\mathbb{E}\left(\det\{\Im g_\Delta(z)\}\right) \leq \pi^n \|\rho\|_\infty^n \,, \qquad\qquad |\Delta| = n \,.$$

As a consequence the following $n$-level Wegner estimate has been proved. Let $J \subset \mathbb{R}$. Let $E_\Lambda(J)$ be the spectral projection of $H_\Lambda$ corresponding to eigenvalues in $J$

**Corollary 2 (see [11])** *For any $n \in \mathbb{N}$, interval $J \subset \mathbb{R}$ and any cube $\Lambda \subset \mathbb{Z}^d$*

$$\mathbb{P}(\mathrm{Tr}\, E_\Lambda(J) \geq n) \leq \frac{\pi^n}{n!} \|\rho\|_\infty^n |J|^n |\Lambda|^n .$$

Using these estimates the Minami argument showing that the level spacing distribution of the eigenvalues of $H$ is Poissonian can be extend to Hamiltonian of the form $H = H_0 + V$, provided $H_0$ satisfies

$$|\langle x|H_0|y\rangle| \;\leq\; C\, e^{-\eta|x-y|}$$

# 5 Other Topics

During the period 2003-2007, the PI has worked on several other projects related to his proposal, namely the Theory of aperiodic solids. Here is an account of this activity, which did not give rise to published papers so far.

## 5.1 Kubo's formula:

**Collaboration:** *D. Spehner (Essen, Germany), R. Rebolledo (Santiago, Chile), M. Loss, E.Carlen, (School of Mathematics, Georgia Tech)*
The PI is continuing a longstanding program in dealing with the proof of Kubo's formula. This program was initiated in [7] in order to estimate the accuracy of the Quantum Hall Effect from first principle. Since then, the PI has produced several papers in collaboration with his former students, H. Schulz-Baldes, D. Spehner, on the one hand and more recently R. Rebolledo (see the review paper [4]).

**Results obtained since 2004:** During a visit at Santiago in March 2004, the PI and R. Rebolledo have developped the formalism needed to treat the dissipation dynamics for the electron gas (N-body problem) in an aperiodic solid. They proved that the generator of the dynamic $\mathfrak{L}$ (Lindblad Operator) has an infinite volume limit and defines a Markov semigroup preserving the C*-algebra of observable (and not only the von Neumann algebra). Moreover the PI has obtained the Green-Kubo formula for all transport cœfficients, including the thermal conductivity and the thermopower [6]. This formula requires the existence of an inverse for $\mathfrak{L}$. In trying to deal with this problem, the PI has recently (Fall 2004) found an amazing equivalence with a class of $XY$-model for quantum spins. This part is in preparation.

Presently the PI intend to write a review paper on this subject. He also worked at extending the *information estimates* found by E. Carlen and Lieb [14] in the case of free fermions at infinite temperature, to the case of finite temperature in order to estimate the rate of convergence to equilibrium. Since this is an estimate on $e^{t\mathfrak{L}}$, there is a chance that, through a Laplace transform, an estimate be available soon on the inverse

$$-\mathfrak{L}^{-1} = \int_0^\infty dt\; e^{t\mathfrak{L}}$$

While working at the review paper on Kubo's formula the PI came across the problem of characterizing the Markov semigroups in $C^*$-algebras. For the $C^*$-algebraof bounded operators on a Hilbert space, this problem has been solved by Lindblad [24] using the Stinespring [32] theorem and a theorem of Kraus [23] on completely positive maps. The Lindblad theorem is the noncommutative analog of the Levy-Khintchin theorem for stationary stochastic processes (the so-called Levy processes). Amazingly, such a result has never been extended to other cases of $C^*$-algebras. Some extensions to hyperfinite factors do exist, but it is likely that for non hyperfinite ones, topological obstructions, like rigidity, do exist. The PI has taken the task to consider the case of Markov semigroups on an UHF algebra and to see whether there is an extension of Lindbald's theorem. It seems that using a description of such algebras in terms of groupoid $C^*$-algebras a generalization of Lindblad's theorem, using the ideas used in the proof of the Levy-Khintchin theorem, can be obtained. In addition, he succeeded to get various example of Noncommutative Levy's processes for which a Levy-Khintchine formula can be proved. This work is still under progress.

## 5.2 Groundstate of a Solid

**Collaboration:** *Ch.Radin U. Texas Austin*

The PI has started a program to understand better, from the point of view of Thermodynamics, what are the characteristic properties of the zero temperature states for an assembly of neutral atoms, interacting through a two-body potential. In a collaboration with Charles Radin he has obtained in 2006 several results that have not been written yet.

The assumption on the potential are the following: $V : x \in \mathbb{R}^d/\{0\} \mapsto \mathbb{R}$ will denote a continuous function with the following properties and will be called a *two-body potential.*

**(V1)** $V$ is *rotation invariant*, namely there is $\Phi : \mathbb{R}_+ \mapsto \mathbb{R}$ such that $V(x) = \Phi(|x|)$.

**(V2)** $V$ is *attractive* at long distance and *repulsive* at short distance, namely, there is $a > 0$ such that $\Phi$ is decreasing on $(0, a)$ and increasing on $(a, +\infty)$.

**(V3)** $V$ has *finite range*, namely $\int_{|x|\geq a} |V(x)| d^d x < \infty$.

**(V4)** $V$ is *strongly repulsive*, namely $\lim_{x\to 0} |x|^d V(x) = +\infty$.

**(V5)** $V$ is *stable*, namely, there exists $B \geq 0$ such that that for any $n \in \mathbb{N}_*$ and any family $(x_1, \cdots, x_n) \in (\mathbb{R}^d)^{\times n}$ then $\sum_{i<j} V(x_i - x_j) \geq -nB$.

**(V6)** $V$ is *smooth*, namely, away from the origin, $V \in C^{d+1}$ and $|(1 - \Delta)^{(d+1)/2} V|$ is integrable.

Under this set of assumption Ch. Radin and the PI have proved that the zero temperature limit of Gibbs measures exists and are concentrated on atomic configurations given by Delone sets. This work is still under progress.

### 5.3 Phonons in Aperiodic Solids:

**Collaboration:** *J. Savinien, Graduate School, Math., Georgia Tech.*

This project was the initial program of research of Jean Savinien Ph. D. thesis (see Section 2) which started in May 2003. *This project has been stopped, due to difficulties that has not been overcome yet. Since then J. Savinien has changed his direction of and recently published a paper. The PI did not continue along these lines by lack of time.*

Phonons are acoustic waves in solids due to vibrations of atoms about their equilibrium position. These waves are quantized and their frequencies constitute the "phonon spectrum" and are given by the spectrum of an operator $H$ that belongs to the C*-algebra of the Noncommutative Brillouin zone (see Reminder above). The corresponding Density of Vibrational States (DOVS) is obtained as the spectral measure with respect to a given extremal trace on the NCBZ (or equivalently to an invariant ergodic probability measure on the Hull). The first step in this program consists in controlling the low frequency limit: in this limit, the phonons see only the large scale features of the solid. They behave like solutions of the wave equation with a sound velocity depending upon the polarization. If the solid is macroscopically isotropic and homogeneous, the sound velocities of all transverse modes are identical, while the longitudinal mode may have a different velocity. These two velocities can be computed from the invariant probability measure and the explicit form of $H$.

**Results:** In 2003-2004, an homogeneisation technique has been used to get an estimate on the selfadjoint operator giving the distribution of vibration modes (the 1-phonon Hamiltonian) in the harmonic approximation. In particular the low frequency limit produces a dispersion relation with a sound velocity that can be computed from the microscopic data (average spring constants). During the last ten months in 2004-2005, Jean Savinien has extended this work to the case of an electron in the effective mass approximation (band edges).

## 5.4 The Jacobi matrix of a Julia set:

**Collaboration:** *J. Geronimo, School Mathematics, Georgia Tech & P. Yuditskii, Kharkov, Ukrain*
**Reference:** [9]

**History :** this problem was initiated in 1982 by the PI and became the source of a longstanding collaboration with J. Geronimo (Georgia Tech) since 1986. Recently, P. Yuditskii (Kharkov) joined this group after his contribution to this problem using the Ruelle thermodynamical formalism [31]. Since may 2003, the PI together with J. Geronimo and P. Yuditskii have revised this problem after some progress made by P. Yuditskii.

## 5.5 Magnetoresistance of graphene:

**Collaboration:** *C. Berger, W. de Heer, School of Physics, Georgia Tech*
**Proceedings APS March Meeting 2004:** *Evidence for 2D electron gas behavior in ultrathin epitaxial graphite on a SiC substrate* [8]

The group led by Walter de Heer in the School of Physics, Georgia Tech, has been involved in studying carbon nanotubes for several years so far. A recent move lead this group toward studying a flat version of these nanotubes: monolyaers of carbon graphite. It is technologically possible to produce such monolayers on surface of a Silicon-Carbide clean sample, by usual chemical treatments. While the Si-C substrate is an insulator with wide gap, the surface monolayer behaves like a semimetal. C. Berger has performed several measurements of the magnetoresistance in high magnetic field (larger than $1T$), at relatively low temperature (down to few $K$). The finding is surprising: there is a *gigantic variation of the magnetoresistance* (three orders of magnitude while the magnetic field varies from $b = 0$ to $B \approx 1T$ !). I have proposed them a possible mechanism for explaining their result that is described below. A preliminary announcement of this result has been posted for a poster session at the APS March meeting in March 2004 [8].

Since then the group of de Heer has refined the technics to produce high quality graphene. The properties of this material are just amazing. The electrons have a coherent length of the order of $10\mu m$. This is because the graphene is such a hard material (harder than diamond) that the phonon frequencies are too high to couple to low energy electrons. Thus the dissipation mechanisms are essentially negligible. In addition, the electrons behave like massless relativistic Majorana fermions, because the dispersion relation is linear in the module of the quasi momentum. Hence the effective Hamiltonian describing their kinetic energy is a 2D-Dirac operator. Due to the absence of dissipation, the magnetoresistance oscillations can be seen with an amazing amount of details, leading to difficulties in interpreting them.

Since this last part of the work is more material science, the PI has not been able to collaborate on this issue after 2005.

## 5.6 Entropy of Tilings and Homology of the Hull:

**Collaboration:** *I. Palmer Graduate School, Math., Georgia Tech.*

In a '00 paper, Gambaudo and Martens [20] have considered a $\mathbb{Z}$-action on a Cantor set and proposed to see the Hull as the inverse limit of graphs, which can be seen as the one-dimensional analog of branched oriented flat Riemannian manifolds used to describe the Hull of a tiling. In their study, they have shown that the homology gives a classification of invariant probability measures, a result that has been generalized to the tiling case (Section 1). However, they have also shown that the homotopy gives precise information upon the entropy of the dynamical system. In the tiling theory, this aspect has not been investigated yet. The goal of Ian Palmer Ph. D. Thesis is to fill this gap.

**Results obtained:** The configurational entropy is defined as follows: let $N(R)$ be the number of distinct patches of radius $R$. Then, if $B(R)$ denotes a ball of radius $R$

$$h_c = \limsup_{R\uparrow\infty} \frac{\ln N(R)}{vol B(R)}$$

Thanks to a recent result of M. Baake *et al.* [2], the configurational entropy of a tiling coming from a repetitive Delone with finite local complexity coincides with the topological entropy of the Hull, seen as a dynamical system with its $\mathbb{R}^d$-action.

I. Palmer and the PI have built a class of examples of tiling, coming from a repetitive Delone set with finite local complexity, having a positive configurational entropy. It is likely that among these examples, many are actually uniquely ergodic.

Shub [30] *conjectured* that if $M$ is a smooth compact manifold and $fM \mapsto M$ a smooth mapping, then the topological entropy $h(f)$ of $f$ satisfies

$$h(f) \geq \ln \mathrm{sp}(f_*)\,,$$

where $\mathrm{sp}(f_*)$ is the spectral radius of $f_* : H_*(M;\mathbb{R}) \mapsto H_*(M;\mathbb{R})$ in the homology space. This conjecture actually holds in various cases. For instance, if $f$ is continuous and either $M = \mathbb{T}^d$ and or $M = \mathbb{T}^d \times X$ where $X$ is a compact orientable manifold which is $\mathbb{R}$-homologically a sphere [27].

The PI conjectures that some similar inequality should hold for repetitive tiling with finite local complexity. Using the construction of the Hull through an inverse limit $\Omega = \lim_{\leftarrow}(\mathcal{B}_n, f_n)$ of branched oriented flat compact Riemannian manifolds, the following extension of the Shub conjecture should be correct

$$h_c \geq \limsup_{n\to\infty} \frac{\ln \mathrm{sp}(f_{n,*})}{vol(\mathcal{B}_n)}$$

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