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SPONSORED PROJECT INITIATION

Date: March 4, 1980

Project Title: Mechanics of Elastic-Plastic Shells

Project No: E-23-649

Project Director: Dr. Gerald A. Wempner

Sponsor: National Science Foundation

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Sponsor Contact Person (s):

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SPONSORED PROJECT TERMINATION SHEET

Date 4/14/83

Project Title: Mechanics of Elastic-Plastic Shells

Project No: E-23-649

Project Director: Dr. Gerald A. Wempner

Sponsor: National Science Foundation

Effective Termination Date: 11/30/82

Clearance of Accounting Charges: 11/30/82

Grant/Contract Closeout Actions Remaining:

- [ ] Final Invoice and Closing Documents
- [x] Final Fiscal Report (FCTR)
- [ ] Final Report of Inventions
- [ ] Govt. Property Inventory & Related Certificate
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- [ ] Other ______

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FINAL PROJECT REPORT
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PART I—PROJECT IDENTIFICATION INFORMATION

1. Institution and Address	2. NSF Program	3. NSF Award Number
Georgia Tech Research Institute 225 North Ave., N.W. Atlanta, Georgia 30332	Structural Mechanics 4. Award Period From 2/1/80 To 11/30/82	ENG 7918420 5. Cumulative Award Amount $121,466

6. Project Title

Mechanics of Elastic-Plastic Shells

PART II—SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

The primary objectives were concepts, theories and approximations for the rational and practical analyses of elastic-plastic shells. The general variational theorem of Hu-Washizu was used as a means to gain insight, but was found to provide effective approximations which retain all the attributes of finite elements derived by the theorem of minimum potential. Much versatility was achieved by the independent selection of all variables, displacements, strains and stresses. Very simple approximations were developed to provide generality and computational efficiencies. Problems with predominant bending and/or extensional deformations were accommodated with the same elements based upon tri-linear displacements and simplified linear approximations of strains and stresses. Piecewise constant approximations of the latter were shown to be suited to plastic deformation and to provide a rational mechanism for step-wise evolution of plasticity.

Two publications contain the basic concepts and methods:

A Simple and Efficient Approximations of Shell via Finite Quadrilateral Elements, J. Applied Mech., 104 (1982) pp. 115-120, Wempner, G., Talaslidis, D., and Hwang, C.-M.

A Simple Finite Element for Elastic-plastic Deformation of Shells, Comp. Meth. Applied Mech & Eng., 34 (1982) pp. 1051-1064 Talaslidis, D. and Wempner, G.

The simple elements are drawn from a modified potential. Consequently, the user can be assured of computational stability and convergence.

PART III—TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

1. ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM: Check (✓)	Approx. Date
a. Abstracts of Theses				✓	
b. Publication Citations		✓			
c. Data on Scientific Collaborators			✓		
d. Information on Inventions	✓				
e. Technical Description of Project and Results					
f. Other *(specify)*					

2. Principal Investigator/Project Director Name *(Typed)*	3. Principal Investigator/Project Director Signature	4. Date
Gerald A. Wempner	Gerald A. Wempner	3/29/83

E-23-647

"Annual Progress Rpt."
Submitted 12/19/80
with request for renewal/
Continued Support.

MECHANICS OF ELASTIC-PLASTIC SHELLS

A summary of progress on research sponsored by the National Science Foundation under grant ENG-CME-7918420.

Introduction

The mechanics of elastic shells had attained a remarkable status with the definitive works of Love in the past century (see Referencel). Numerous refinements during the ensuing years have resolved inconsistencies and have extended the theory to include nonlinear problems of finite rotations. Much of the dynamics and kinematics is applicable to inelastic shells, but the constitutive equations of inelastic shells remain in relatively primitive forms. The formulation of consistent, yet practical, theories and computational methods for elastoplastic shells is the objective of the present effort.

Two recent developments present alternative approaches to the problem: the direct formulation of Bieniek [2] applies classical plasticity, concepts of flow and hardening, to obtain a simple relation between the incremental stresses and strains of the shell (forces and moments versus strains and curvatures). The theory of Wempner [3] utilizes the relations for the three-dimensional body to derive the equations for the two-dimensional shell. Both viewpoints are currently under consideration to evolve a consistent theory which can be effectively implemented in practical computations.

Another facet of the entire problem is the development of a discrete model of the shell to effect the computations. Since the elastoplastic problem requires incremental procedures, as well as additional variables (e.g. stresses and hardening parameters), a simple model is particularly desirable. To this end, the most rudimentary, yet admissible, approximations are currently under examination.

Specific features of the current research toward the development of the theory and computational tools follow:

Elastoplasticity of the Shell

Two studies are currently in progress: The first is an effort to develop a theory for hardening materials. The intent is to incorporate essential features of the Bieniek theory [2] and also strain hardening according to the classical concepts of plasticity. A formulation has been programmed and is currently under examination.

The second effort is an attempt to develop a theory in terms of the stresses employed in the derived theory [3]. That theory involves two integrals (M_2, M_3) of the stress distribution, in addition to the customary force and moment (M_0, M_1). These higher moments are workless under the hypothesis of Kirchhoff-Love; that implies certain relations between the dissipation and residual elastic energy, associated with the distribution of stress after unloading. Hopefully, these observations and our knowledge of the initial and subsequent conditions will be sufficient to devise the requisite equations of evolution.

Discrete Model of the Shell

Our objective is the simplest discrete approximation which is particularly suited to the elastoplastic problems. A finite element must be simple enough to admit a fine gridwork in regions of elastoplastic behavior, to accommodate the transition and abrupt gradients which may arise in such regions. Moreover, the model is to be cast in the context of shell equations, in order to take full advantage of the established theories, analogies and simplifications. For example, a century of experience has established the validity of the Kirchhoff-Love hypothesis. Some specific features of our formulation follow:

The simplest continuous displacement of a quadrilateral finite element was included among "new concepts for finite elements of shells" [4]. The trilinear approximation imposes severe transverse shear strain which must be accommodated. This shear strain fluctuates from positive to negative values at alternate corners and vanishes at an intermediate point, in accordance with our observations of the mathematical and physical phenomenon. A straightforward formulation according to the principles of virtual work leads to unwarranted stiffness. The apparent difficulties can be circumvented by invoking a discrete counterpart of Kirchhoff's hypothesis, as proposed in the original work [4]. More recently, computers have circumvented the problem by including only a mean value of the shear energy, specifically by using only the central value which is nearly zero in thin shells. The latter device has been termed "reduced integration". In our current model, the earlier difficulty is entirely eliminated by a consistent formulation which briefly follows: A general functional including all primitive variables, stresses, strains, rotations, and displacements, provides all the differential equations and boundary conditions as the stationary (Euler) equations. These variables are approximated in the simplest forms: The stresses and strains, but not their derivatives appear; therefore, these are approximated by piece-wise constant forms. The rotations and displacements are approximated by trilinear forms to give full compatibility. With the resulting algebraic equations, the stresses and strains are eliminated to obtain the equations in displacements alone. The latter are governed by a symmetrical positive-definite matrix as the equations of the minimum potential. The formulation is consistent and shows great promise in terms of simplicity and accuracy. The procedure has been tested in simple cases and is now undergoing further computational testing.

An important innovation in our discrete formulation is the adherence to a natural system in the "assembly". The assembly is accomplished in the manner of an intrinsic theory of shells; the variables and equations are expressed in the directions of the normals and tangents to the surface. Consequently, the orientation, in a fixed global frame, enters only if a prescribed traction is so specified. Like the differential equations of the continuum theory, the algebraic equations contain no rotation, only the small relative rotations of contiguous elements. Specifically, the crucial terms in the nonlinear continuum theories and in buckling are products of forces and curvatures; in analogous fashion, the counterparts in the discrete approximation are forces and relative rotations.

Summary of Activities and Progress

The investigator was advised of the present grant in March, 1980, and work was initiated immediately. Funds and time were allocated for full-time research in the summer 1980 and efforts were initiated to recruit assistants. The recruitment efforts have led to prospective candidates who could effectively support the development during the coming year.

Certain facets of the discrete approximation have been explored jointly with Dr. D. Talaslidis of the Ruhr University, who has extensively examined the questions of "mixed models" which entail approximations of stresses and displacements. Plans for the coming year include a period of collaboration when Dr. Talaslidis works, as a post doctoral fellow, at the Georgia Institute of Technology.

Important concepts of the elastoplasticity have been set forth and employed in preliminary formulations: The direct theory of Bieniek [1] has been reformulated to include simple forms of hardening. A program has been prepared with the modified equations and computations are currently in progress to assess the validity of these modifications.

A consistent, simple, discrete approximation has been completely developed. The simplicity of the formulation serves to reduce storage requirements and computational expenses. Consequently, the formulation shows great promise as a practical tool for computations, especially in problems involving inelastic behavior. Several programs have been prepared to test the formulation in applications to simple problems. The numerical results from these tests have been remarkably good in view of the simplicity of the computations, the minimal storage, and ease of handling data. In general, the progress in the initial phases has been quite satisfactory. The continuation of support is expected to produce the complete computational machinery for practical problems of elastoplastic shells.

References

1. Wempner, G. A., "Mechanics of Shells in the Age of Computation," in "Structural Engineering and Structural Mechanics," (ed. K. Pister), Prentice-Hall, 1980.

2. Bieniek, M. P. and Funaro, J. R., "Elastic-Plastic Behavior of Plates and Shells," Tech. Rep. DNA 3954T, Weidlinger Associates, New York, 1976.

3. Wempner, G. A. and Hwang, C. M., "A Derived Theory of Elastic-Plastic Shells," Int. J. Solids Structures, Vol. 13, pp. 1123-1132, 1977.

4. Wempner, G., "New Concepts for Finite Elements of Shells," Zeit. f. Ang. Math. Mech., 1969.

COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING 34 (1982) 1051–1064
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A SIMPLE FINITE ELEMENT FOR ELASTIC-PLASTIC DEFORMATIONS OF SHELLS

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The quadrilateral element of this paper is especially suited to the behavior of elastoplastic shells. It combines the simplicity needed in nonlinear analyses with a competitive accuracy. The elemental approximation shares some common features with an earlier one applied to hookean shells [1], but incorporates an important modification which leads to further simplifications and provides a mechanism for the progressive yielding within an element. The element is derived via the Hu–Washizu functional, which admits independent approximations of all basic variables. The elimination of strains and stresses results in a displacement formulation with all the desired properties, a positive-definite matrix and convergence. Some properties of the elemental matrices are outlined and special considerations are given to plasticity. Numerical tests upon plates and shells demonstrate the efficiency of the element. Finally, some distinctions are drawn between our model and certain 'mixed' elements.

1. Introduction

The analysis of elastoplastic shells demands incremental procedures with the attendant costs of repetitious computations. The accurate description of local yielding also calls for a finer assembly of elements, additional storage and further expense. Consequently, a simple element is needed which can accommodate a broad class of practical problems in an expedient and economical way (see e.g., [2, 3]).

The quadrilateral element of a recent article [1] provides a simple and efficient approximation of elastic shells. Like that element, the present model is developed entirely from consistent mechanical concepts without recourse to ad hoc numerical schemes. The current element incorporates an important modification which achieves further simplifications and also provides a mechanism for the progressive yielding of elastoplastic elements.

The basis of the present formulation is the general functional of Hu–Washizu, which admits independent approximations of displacements, strains and stresses, and provides consistent relations between the discrete parameters. The strains and stresses have piecewise constant approximations which are just sufficient to accommodate the various deformational modes. The subsequent elimination of the strains and stresses leads to a final system which involves only displacements. This elimination requires no inversion of submatrices and very simple integrations. Since stresses are constant within subregions, homogeneous yielding occurs

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within such regions, which then provide a consistent mechanism for a stepwise progression of yielding through the entire element. Consequently, plastic deformations are more accurately represented by relatively fewer elements.

In the present paper two alternative approximations are described; both are consistent and offer the advantages previously cited. The tangent moduli of the stiffness matrix are given in Appendix A. The structure and the properties of the derived matrices reveal the advantages of the proposed element in the analysis of elastoplastic shells. Some distinctions are drawn between the present element and certain mixed elements based on the Hellinger–Reissner theorem. Finally, numerical tests upon plates and shells serve to demonstrate the effectiveness of this simple elemental approximation.

2. Basic features

A typical element of the shell is shown in Fig. 1; it is bounded by the top (S_+) and bottom (S_-) surfaces, and by the edges ($\theta^\alpha = a^\alpha \pm l^\alpha$). Also shown is an intermediate surface (S), the origin (O) of local normalized coordinates. $\xi^\alpha = (\theta^\alpha - a^\alpha)/l^\alpha$, $\xi^3 = \theta^3/h$, and a typical nodal point N on surface S.

The normal stress is again neglected ($S^{33} = 0$) so that extension of the normal line (ξ^3) is immaterial. Then the tri-linear approximation expresses the displacement in terms of the 4 nodal displacements ($\boldsymbol{W}_{KM}$) and rotations ($\boldsymbol{\phi}_{KM}$). Since the latter have but two components, the element has altogether twenty degrees of freedom. The form follows:

$$\boldsymbol{V} \doteq (\boldsymbol{W}_{KM} + \xi^3 \boldsymbol{\phi}_{KM})\mathscr{F}_{KM}(\xi^1, \xi^2), \tag{1a}$$

$$\mathscr{F}_{KM} \equiv \tfrac{1}{4}(1 + \xi^1_K \xi^1)(1 + \xi^2_M \xi^2). \tag{1b}$$

Here majuscules signify nodes, $\xi^\alpha_K = \pm 1$ are the nodal coordinates, and $\boldsymbol{W}_{KM} = \boldsymbol{V}(\xi^1_K, \xi^2_M, 0)$.

The formulation of the elemental approximations follows the identification of 20 modes, which are uncoupled in the rectangular element of a plate: 6 rigid motions and 8 homo-

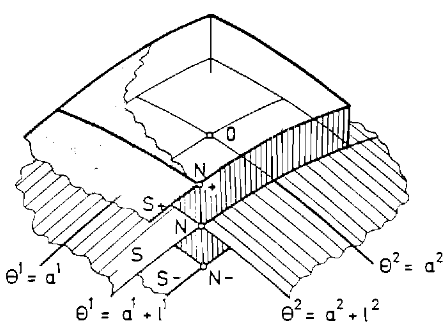


Fig. 1. Hexalateral shell element.

geneous strains are represented by the linear terms in the components of displacement:

$$w_\alpha \doteq \bar{w}_\alpha + \bar{\omega}_{\alpha\beta}\xi^\beta + \bar{\epsilon}_{\alpha\beta}\xi^\beta\,, \tag{2a}$$

$$w_3 \doteq \bar{w}_3 + \bar{w}_{3\alpha}\xi^\alpha\,, \tag{2b}$$

$$\varphi^\mu e_{\alpha\mu} \doteq \bar{\gamma}_\alpha - \bar{w}_{3\alpha} - \bar{\kappa}_{\alpha\beta}\xi^\beta\,. \tag{2c}$$

The coefficients $\bar{w}_i$, $\bar{w}_{3\alpha}$ and $\bar{\omega}_{12} = \bar{\omega}_{21}$, represent rigid motions. The coefficients $\bar{\epsilon}_{\alpha\beta}$, $\bar{\kappa}_{\alpha\beta}$ and $\bar{\gamma}_\alpha$, represent the homogeneous extensional, flexural and transverse shear strains, respectively. The 6 higher modes of deformation are 2 extensional modes, 2 flexural modes and 2 shear modes as depicted in Fig. 2. The 2 extensional modes correspond to the displacements:

$$\tilde{w}_\alpha = \tilde{\epsilon}_\alpha\xi^1\xi^2\,. \tag{3a}$$

The 2 flexural modes correspond to the rotations

$$\varphi^\mu e_{\alpha\mu} = -\tilde{\kappa}_\alpha\xi^1\xi^2\,. \tag{3b}$$

One shear mode is a torsion about the normal:

$$\phi^\alpha = -t\xi^\alpha\,. \tag{3c}$$

The final mode is a warping of the surface (S):

$$w_3 = \tilde{\gamma}\xi^1\xi^2\,. \tag{3d}$$

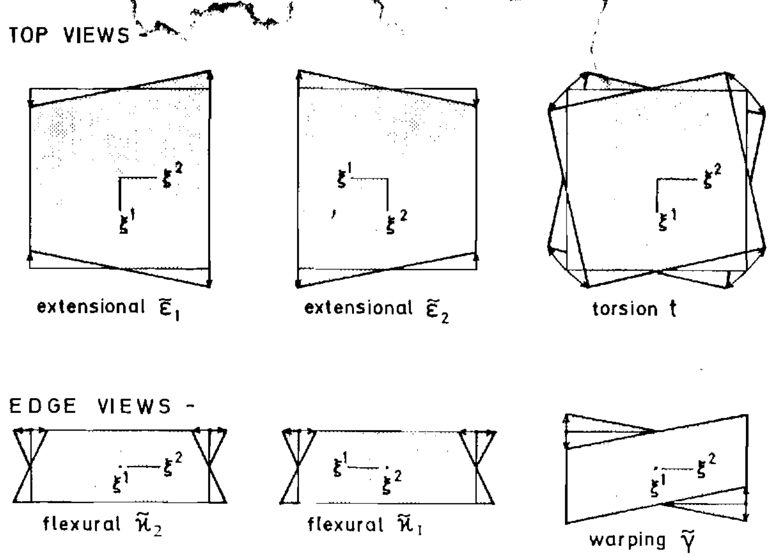


Fig. 2. Higher order deformational modes.

In the earlier element [1], the approximations of the strains consisted of 6 homogeneous strains, augmented by 6 linear terms which inhibit the higher deformational modes, (3a)–(3d), of Fig. 2. Now we introduce alternatives which are more effective in the elastic and elastoplastic element. For the sake of brevity and comparison we denote the extensional strains (of surface S) and flexural strains (changes of curvature) by $\epsilon^0_{\alpha\beta}$ and $\epsilon^1_{\alpha\beta}$; as before, transverse shear strain is denoted by γ_α. The forms of approximation follow:

$$\epsilon^n_{11} \doteq \bar{\epsilon}^n_{11} + \tilde{\epsilon}^n_{11} g(\xi^2)\,, \tag{4a}$$

$$\epsilon^n_{22} \doteq \bar{\epsilon}^n_{22} + \tilde{\epsilon}^n_{22} g(\xi^1)\,, \tag{4b}$$

$$\epsilon^n_{12} \doteq \bar{\epsilon}^n_{12} + \underline{\tilde{\epsilon}_{12} g(\xi^1)} + \underline{\tilde{\tilde{\epsilon}}_{12} g(\xi^2)}\,, \tag{4c}$$

$$\gamma_1 \doteq \bar{\gamma}_1 + \tilde{\gamma}_1 g(\xi^2)\,, \tag{5a}$$

$$\gamma_2 \doteq \bar{\gamma}_2 + \tilde{\gamma}_2 g(\xi^1)\,, \tag{5b}$$

$$n = 0, 1\,.$$

In the previous element [1] the function g was simply $g(\xi^\alpha) = \xi^\alpha$. Now, we set $g(\xi^\alpha) = H(\xi^\alpha)$, the unit step ($H = 0$, $\xi^\alpha < 0$; $H = 1$, $\xi^\alpha > 0$).

The reader can appreciate our choice of these distributions by examining one of the deformational modes, e.g., the first extensional mode of Fig. 2 which corresponds to the

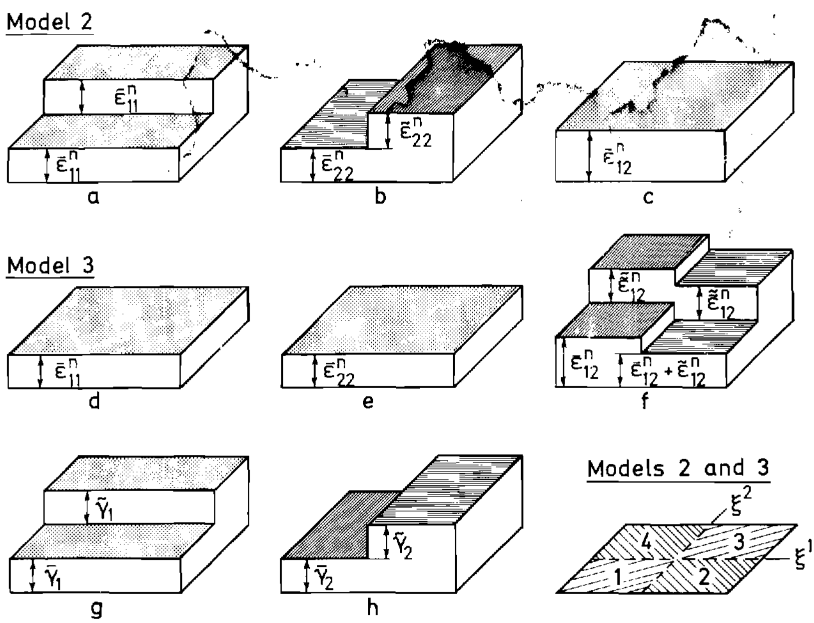


Fig. 3. Piecewise constant approximations of strains.

displacement $\tilde{w}_1 = \tilde{\epsilon}_1 \xi^1 \xi^2$ and the strains

$$\epsilon_{11} = \tilde{\epsilon}_1 \xi^2 , \qquad \epsilon_{12} = \tfrac{1}{2}\tilde{\epsilon}_1 \xi^1 .$$

The mean values of these strains vanish and, therefore, the initial (constant) terms of (4a) and (4c) cannot inhibit such deformation. The addition of the second term in (4a) *or* the second term of (4c) serves that purpose. Because the elements are inherently stiff, we retain only enough terms to suppress the higher modes. Fig. 3 illustrates the piecewise constant approximations of (4a)–(4c) and (5a), (5b). By retaining the additional terms ($\tilde{\epsilon}^n_{11}$, $\tilde{\epsilon}^n_{22}$) of (4a), (4b) and omitting the underlined terms in (4c), we obtain an elemental approximation which we call Model 2 (see Fig. 3). In an alternative approximation (Model 3 of Fig. 3), we omit the additional terms ($\tilde{\epsilon}^n_{11}$, $\tilde{\epsilon}^n_{22}$) of (4a), (4b) but retain the additional terms ($\tilde{\epsilon}^n_{12}$, $\tilde{\tilde{\epsilon}}^n_{12}$) of (4c). In both models the shear strains γ_α take the form shown in Figs. 3g, 3h. Subsequent results show the adequacy of either choice.

3. Variational formulation

Again the basis of the formulation is the general theorem of Hu–Washizu in the form:

$$H = \int_{-1}^{1}\int_{-1}^{1} \{U(\epsilon^n_{\alpha\beta}, \gamma_\alpha) - m^{\alpha\beta}_n[\epsilon^n_{\alpha\beta} - E^n_{\alpha\beta}(w^i, \phi^\alpha)] - q^\alpha[\gamma_\alpha - G_\alpha(w^i, \phi^\alpha)] - \boldsymbol{F}\cdot\boldsymbol{W}\}$$

$$\times A(\xi^1, \xi^2)\, d\xi^1\, d\xi^2 - \int_C \{\boldsymbol{N}\cdot\boldsymbol{W} + \boldsymbol{M}\cdot\boldsymbol{\phi}\} , \quad n = 0, 1 . \tag{6}$$

Here $A(\xi^1, \xi^2)$ denotes the metric of area on S, U the strain-energy density; $m^{\alpha\beta}_n$, q^α denote the stresses which are conjugate to $\epsilon^n_{\alpha\beta}$, γ_α, respectively. The external load $\boldsymbol{F}$ includes force applied upon the top (S_+), bottom (S_-) surface, and/or distributed through the shell; $\boldsymbol{N}$ and $\boldsymbol{M}$ are the force and couple upon the edge C of the element. The operators $E^n_{\alpha\beta}$ and G_α express the strains $\epsilon^n_{\alpha\beta}$ and γ_α in terms of the displacements w^i and rotations ϕ^α; the linear or nonlinear kinematical equations are the Euler conditions that the bracketed expressions vanish, $[\cdots] = 0$. For the incremental formulation of plasticity, the functional H is replaced by a quadratic functional, wherein each of the functions is replaced by the corresponding increment, e.g., the 'strain energy' is a quadratic form

$$U_2 = U_2(\dot{\epsilon}^n_{\alpha\beta}, \dot{\gamma}_\alpha) . \tag{7}$$

The latter presupposes the availability of a plasticity theory of the shell; alternatively, a theory of plasticity is applied to layers:

$$U_2 = h\int_{-1}^{+1} \tfrac{1}{2}\bar{C}^{ijkl}\dot{\gamma}_{ij}\dot{\gamma}_{kl}\mu(\xi^3)\, d\xi^3 . \tag{8}$$

The function $\mu(\xi^3)$ is the ratio of the volume metric to the area metric of the reference surface

(S). Here the strain components are expressed according to the approximations (4) and the basic kinematical hypothesis of the shell:

$$\dot{\gamma}_{\alpha\beta} \doteq \dot{\epsilon}^{0}_{\alpha\beta} - \xi^3 \dot{\epsilon}^{1}_{\alpha\beta}, \tag{9a}$$

$$\dot{\gamma}_{\alpha 3} \doteq \dot{\gamma}_{3\alpha} = \dot{\gamma}_{\alpha} \bar{g}(\xi^3). \tag{9b}$$

The latter can be taken in the form $\bar{g}(\xi) = 1 - \xi^2$ or simply $\bar{g}(\xi) = 1$, since the effects in thin shells are minimal. The integration through the thickness can be approximated with piecewise-constant approximations of the strains (and stresses) in a number of layers. According to classical plasticity each tangent modulus $\bar{C}^{ijkl}$ is the sum of the elastic modulus C^{ijkl} and a term $-B^{ijkl}$ which accounts for plastic flow: $\bar{C}^{ijkl} = C^{ijkl} - B^{ijkl}$. Specific forms are given in Appendix A, where the moduli are contained in the matrix $\bar{C}$ of (A.1). These are based upon the von Mises yield condition and associated flow, and include the effects of transverse shear stresses.

In keeping with the 'limitation theorem' of Fraeijs de Veubeke [4], each stress is approximated in the form of the conjugate strain:

$$m^{11}_{n} \doteq \bar{m}^{11}_{n} + \tilde{m}^{11}_{n} g(\xi^2), \tag{10a}$$

$$m^{22}_{n} \doteq \bar{m}^{22}_{n} + \tilde{m}^{22}_{n} g(\xi^1), \tag{10b}$$

$$m^{12}_{n} \doteq \bar{m}^{12}_{n} + \underline{\tilde{m}^{12}_{n} g(\xi^1)} + \underline{\tilde{\tilde{m}}^{12}_{n} g(\xi^2)}, \tag{10c}$$

$$q^1 \doteq \bar{q}^1 + \tilde{q}^1 g(\xi^2), \tag{11a}$$

$$q^2 \doteq \bar{q}^2 + \tilde{q}^2 g(\xi^1). \tag{11b}$$

Under the similar, piecewise constant, approximations of strain (4), (5) and stress (10), (11), those terms of the functional (6) which contain only stresses and strains require no integration. Let the 4 quadrants of the element be labeled as shown in Fig. 3. Then

$$\int_{-1}^{+1} \int_{-1}^{+1} U \cdot A(\xi^1, \xi^2)\, d\xi^1\, d\xi^2 = \sum_{N=1}^{4} A_N U(\epsilon^{0N}_{\alpha\beta}, \epsilon^{1N}_{\alpha\beta}, \gamma^{N}_{\alpha}). \tag{12}$$

Here the majuscule signifies the quadrant and the component within that quadrant is given in accordance with (4), (5). The reduction of the functional H of (6) to a function of the discrete nodal displacements, constant stresses and strains requires only the following integrals:

$$E^{nN}_{\alpha\beta} \equiv \frac{1}{A_N} \int_{A_N} E^{n}_{\alpha\beta}(w^i, \phi^\alpha) A(\xi^1, \xi^2)\, d\xi^1\, d\xi^2, \tag{13a}$$

$$G^{N}_{\alpha} \equiv \frac{1}{A_N} \int_{A_N} G_{\alpha}(w^i, \phi^\alpha) A(\xi^1, \xi^2)\, d\xi^1\, d\xi^2 \tag{13b}$$

where the operators $E^{n}_{\alpha\beta}$, G_α and the metric $A(\xi^1, \xi^2)$ depend upon the surface and choice of coordinates, and the functions w^i and ϕ^α are given by (1b). In some instances (rectangular or circular plates or cylindrical shells), the integrations are trivial. In other instances, numerical approximation is needed.

The discrete counterpart of the functional (6) is a function of the form

$$\bar{H} = \sum_{N=1}^{4} \{A_N U(\epsilon^{0N}_{\alpha\beta}, \epsilon^{1N}_{\alpha\beta}, \gamma^N_\alpha) - A_N m^{\alpha\beta}_{nN}[\epsilon^{nN}_{\alpha\beta} - E^{nN}_{\alpha\beta}(w^i_{KM}, \phi^\alpha_{KM})] - A_N q^\alpha_N[\gamma^N_\alpha - G^N_\alpha(w^i_{KM}, \phi^\alpha_{KM})]\} - F^{KM}_i w^i_{KM} - M^{KM}_\alpha \phi^\alpha_{KM} . \tag{14}$$

Now, the variations of nodal displacements w^i_{KM} and rotations ϕ^α_{KM} give the 20 equations of equilibrium:

$$\sum_{N=1}^{4} \left\{ m^{\alpha\beta}_{nN} \frac{\partial}{\partial w^i_{KM}} E^{nN}_{\alpha\beta}(w^i_{KM}, \phi^\alpha_{KM}) + q^\alpha_N \frac{\partial}{\partial w^i_{KM}} G^N_\alpha(w^i_{KM}, \phi^\alpha_{KM}) \right\} = F^{KM}_i , \tag{15a}$$

$$\sum_{N=1}^{4} \left\{ m^{\alpha\beta}_{nN} \frac{\partial}{\partial \phi^\alpha_{KM}} E^{nN}_{\alpha\beta}(w^i_{KM}, \phi^\alpha_{KM}) + q^\alpha_N \frac{\partial}{\partial \phi^\alpha_{KM}} G^N_\alpha(w^i_{KM}, \phi^\alpha_{KM}) \right\} = M^{KM}_\alpha . \tag{15b}$$

Variation of the independent parameters of strain $(\bar{\epsilon}^n_{\alpha\beta}, \tilde{\epsilon}^n_{\alpha\beta}, \bar{\gamma}_\alpha, \tilde{\gamma}_\alpha)$ give the stress–strain relations. By the piecewise constant approximations, homogeneous stresses are expressed in terms of the constant strains in the region of definition, e.g.,

$$\begin{aligned} m^{11}_{n(1+2)} &= \frac{1}{A_1 + A_2} \sum_{N=1}^{2} A_N \frac{\partial}{\partial \epsilon^n_{11}} U(\epsilon^{nN}_{\alpha\beta}, \gamma^N_\alpha), \quad \text{in } A_1 + A_2 , \\ m^{11}_{m(3+4)} &= \frac{1}{A_3 + A_4} \sum_{N=3}^{4} A_N \frac{\partial}{\partial \epsilon^n_{11}} U(\epsilon^{nN}_{\alpha\beta}, \gamma^N_\alpha), \quad \text{in } A_3 + A_4 , \\ &\vdots \end{aligned} \tag{16}$$

Variation of the independent parameters of stresses $(\bar{m}^{\alpha\beta}_n, \tilde{m}^{\alpha\beta}_n, \bar{q}^\alpha, \tilde{q}^\alpha)$ provides the expressions of strains in terms of nodal displacements; e.g.,

$$\begin{aligned} \epsilon^n_{11(1+2)} &= \frac{1}{A_1 + A_2} \sum_{N=1}^{2} A_N E^{nN}_{11}(w^i_{KM}, \phi^\alpha_{KM}), \quad \text{in } A_1 + A_2 , \\ &\vdots \end{aligned} \tag{17}$$

Eq. (15) can be formulated in matrix notation for Model 2 as follows:

$$(\boldsymbol{E}^0)^t \boldsymbol{M}_0 + (\boldsymbol{E}^1)^t \boldsymbol{M}_1 + \boldsymbol{G}^t \boldsymbol{Q} = \boldsymbol{P} \tag{18}$$

where $\boldsymbol{E}^n$, $\boldsymbol{G}$ are (5×20)- and (4×20) matrices, respectively. The elements of these matrices are obtained by evaluating the integrals on the right sides of (13). The elements of the vectors $\boldsymbol{M}_n$ and $\boldsymbol{Q}$ are given in Appendix A in (A.2) and (A.3), respectively. Relationships (16) take the form

$$\begin{bmatrix} \boldsymbol{M}_0 \\ \boldsymbol{M}_1 \\ \boldsymbol{Q} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & & \bar{\boldsymbol{A}} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{C}}_{00} & \bar{\boldsymbol{C}}_{01} & \bar{\boldsymbol{C}}_{03} \\ & \bar{\boldsymbol{C}}_{11} & \bar{\boldsymbol{C}}_{13} \\ \text{symm.} & & \bar{\boldsymbol{C}}_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}^0 \\ \boldsymbol{\epsilon}^1 \\ \boldsymbol{\gamma}^3 \end{bmatrix} \tag{19}$$

with

$$\boldsymbol{\epsilon}^n = \boldsymbol{A} \cdot \boldsymbol{E}^n \cdot \boldsymbol{u}\,, \tag{20a}$$

$$\boldsymbol{\gamma}^3 = \bar{\boldsymbol{A}} \cdot \boldsymbol{G} \cdot \boldsymbol{u}\,. \tag{20b}$$

The matrices $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ are diagonal matrices containing the areas of the subregions ((A.4) and (A.5)) and the elements of vector $\boldsymbol{u}$ are the nodal displacements $(w^i_{KM}, \phi^\alpha_{KM})$. The matrices $\bar{\boldsymbol{C}}_{ij}$ $(i, j = 0, 1, 3)$ are obtained from the matrix $\bar{\boldsymbol{C}}$ of (A.1) after multiplying by

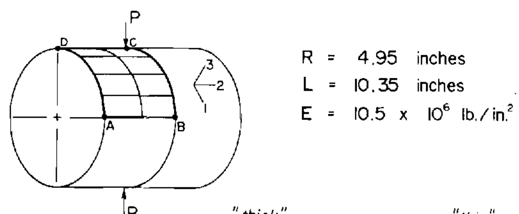
$$(\xi^3)^{i+j} \quad \text{for } i = 0, 1\,; \quad j = 0, 1\,, \qquad (\xi^3)^i \bar{g}(\xi^3) \quad \text{for } i = 0, 1\,; \quad j = 3\,,$$
$$\bar{g}(\xi^3) \cdot \bar{g}(\xi^3) \quad \text{for } i = j = 3\,,$$

and integrating through the thickness. Finally, the vectors $\boldsymbol{\epsilon}^n$, $\boldsymbol{\gamma}^3$ contain the strains, curvatures and shear strains of each subregion, respectively. The elements of these vectors are the strains of the shell and are conjugate to the elements of $\boldsymbol{M}_n$ and $\boldsymbol{Q}$. The direct substitution of the 14 strains (17) into the right sides of (16) and the direct substitution of the 14 stresses (16) into the 20 equations (15a), (15b) produces the stiffness matrix of the element. Observe that no inversions are required and the only integrations are those of (13). Moreover, the result is a positive definite symmetric matrix. The element is a model in terms of displacements, but less stiff than the model drawn from the usual potential, because only the essential strains (and energies) are retained (homogeneous strains are modified only enough to inhibit the higher modes).

4. Numerical examples

Some examples serve to demonstrate the efficiency and accuracy of the elemental approximation. First, we compare the results obtained with the earlier element (Model 1) in the hookean shell [1] with results obtained with the present element (Model 2 and Model 3). In other examples elastoplasticity is accommodated by piecewise constant stresses and strains in seven layers through the thickness. The incremental approximation of the nonlinear problem is implemented with a modified Newton–Raphson procedure, wherein the tangent stiffness is revised in the first iteration of each increment and maintained during subsequent iterations. Any imbalance in equilibrium is corrected at each step to inhibit drifting from accumulative errors. Also, the stresses are re-adjusted at the end of an iteration to avoid violations of the yield condition. Although the formulation provides for hardening, ideal elastoplasticity is used in the examples, since the latter imposes the more severe test of the elemental formulation and the computational procedures.

EXAMPLE 1 (Cylindrical shells under radial forces with free ends). Since the deformations of the 'pinched cylinder' are predominantly flexural, this problem has been used to test the bending behavior of the elements. The table of Fig. 4 shows a good agreement with the results obtained by all three models. As expected, the radial displacements of Model 1 are slightly less than the others. The table indicates monotonic convergence. The crude approximation of the four elements (32 d.o.f.) allows a reasonable approximation of the flexural behavior of the cylinder. Results are also given for the 'thin' shell ($R/t = 320$), which poses a more severe test. Though numerous finite elements give satisfactory results for the 'thick cylinder', only the more sophisticated provide satisfactory results for the thin problem. Our results indicate a rapid convergence from below.



		"thick" t = 0.094 in. P = 100 lb.			"thin" t = 0.0155 in. P = 0.100 lb.	
Mesh	D.O.F.	Model 1	Model 2	Model 3	Model 1	Model 2
1x4	32	.0976	.0980	—	.0214	.0214
1x8	64	.1078	.1083	—	.0236	.0237
2x8	105	.1089	.1092	.1156	.0237	.0237
2x16	209	.1117	.1120	.1188	.0243	.0243

RADIAL DEFLECTION AT C (in)

Fig. 4. Pinched cylinder—radial deflection under load.

EXAMPLE 2 (Cylindrical shell under radial forces with simply supported ends). If the ends of the cylinder are constrained against radial displacement, then membrane forces predominate. Accordingly, this circumstance poses a distinctly different test than the previous one.

Fig. 5a depicts an octant of the cylinder. Uniform meshes have been used, though a finer mesh in the vicinity of loading would produce better results. Fig. 5b shows two plots of the edge displacement, as obtained with different meshes.

Fig. 6 and Fig. 7 serve to exhibit the stepwise approximation of stresses according to Model 2. In each case, whether the mesh is finer (10×10) or coarser (6×6), each level represents an appropriate mean value within that element. Refinement of the mesh leads to a corresponding improvement in results, more accurate portrayal of the actual continuous distribution and, particularly, the peak stresses in regions of steep gradients, as occur at point C.

Fig. 8a indicates that the piecewise constant shear (m_0^{12}) and twist (m_1^{12}) also provide appropriate representations of these stresses.

A comparison of one stress (m_0^{22}) according to the Model 1 and Model 2 is depicted in Fig.

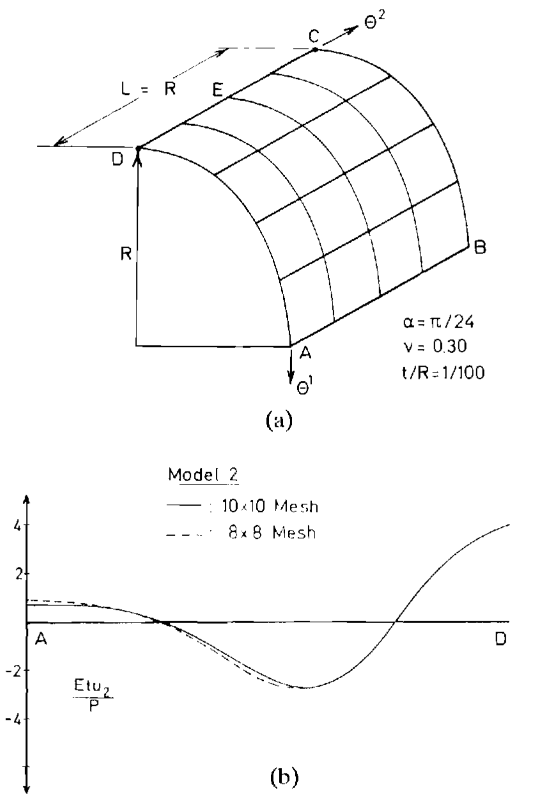


Fig. 5. Pinched cylinder with simply supported ends—axial displacement distribution.

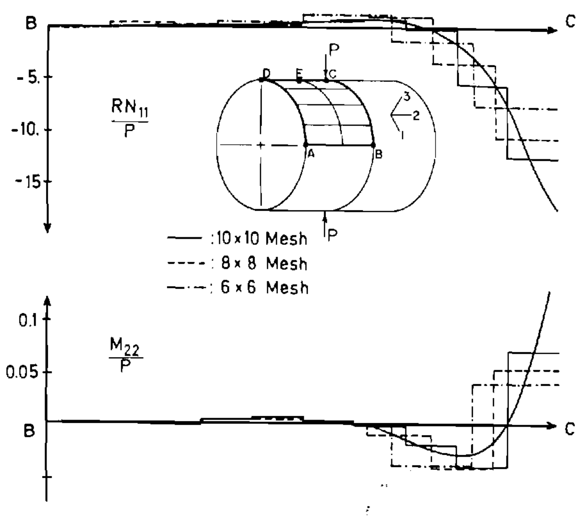


Fig. 6. Supported pinched cylinder—distribution of stress resultants.

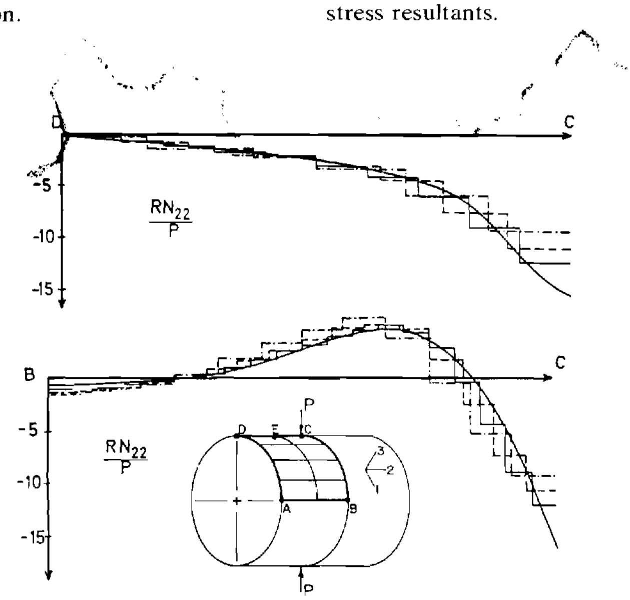


Fig. 7. Supported pinched cylinder—distribution of stress resultants.

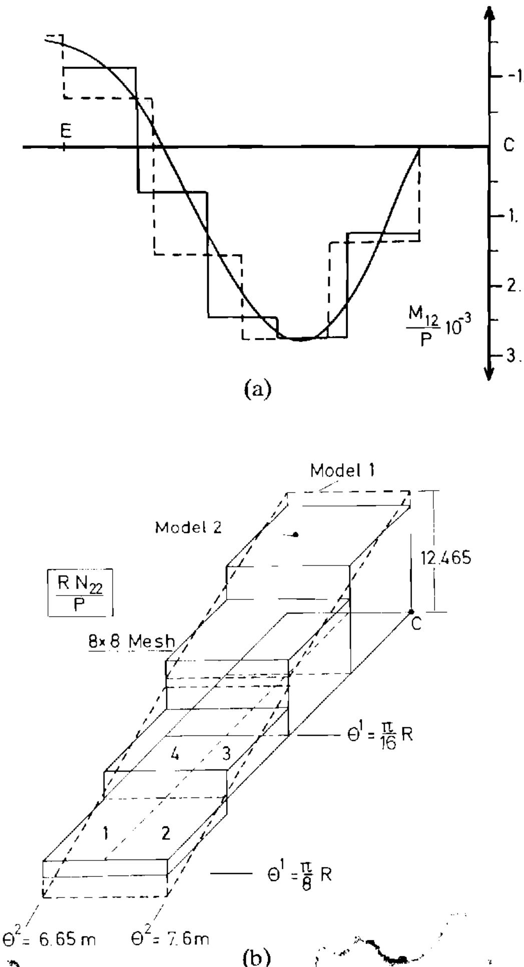


Fig. 8. Supported pinched cylinder—stress distributions.

8b. Again, the ramps of the former follow, but in smoother form, the constant steps of the latter.

EXAMPLE 3 (Analysis of an elastoplastic simply supported square plate). Fig. 9 displays the plot of load versus deflection at the center of a uniformly loaded plate. The limiting load compares favorably with the established bounds [6] and with other results in the literature [7, 8]. Despite the simplicity of our approximation (12 and 48 d.o.f.), values of the displacement are very satisfactory throughout the elastoplastic deformation. The plan view at an intermediate stage shows the partial yielding of elements.

EXAMPLE 4 (Analysis of an elastoplastic cylindrical roof). The cylindrical roof of Fig. 10 has been used for numerical studies by numerous investigators. Our procedures appear relatively insensitive to the size of increments, until the load approaches the limit. The accuracy of a

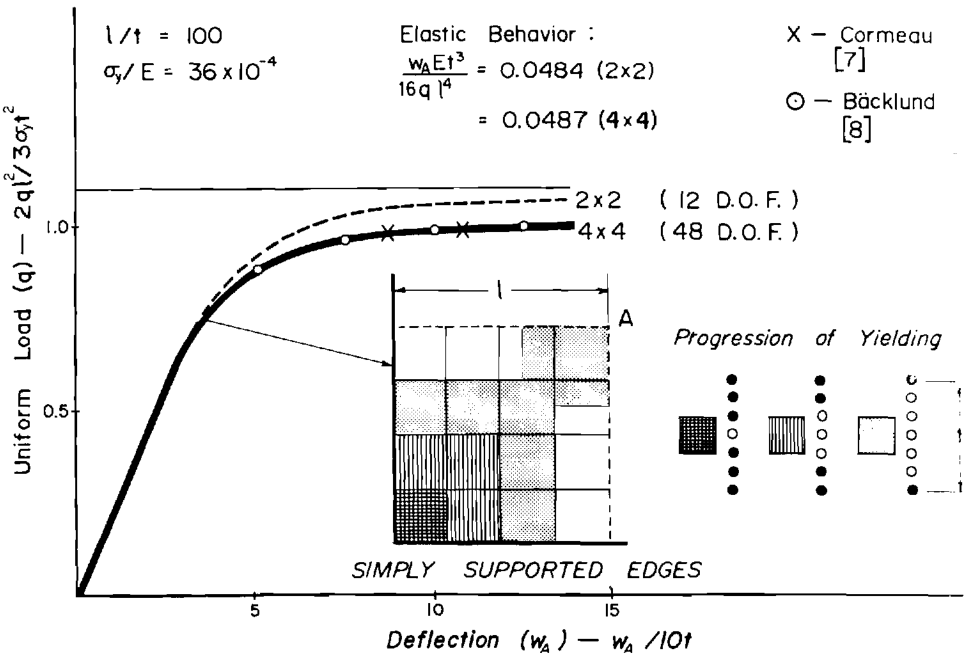


Fig. 9. Elastoplastic analysis of a simply supported plate.

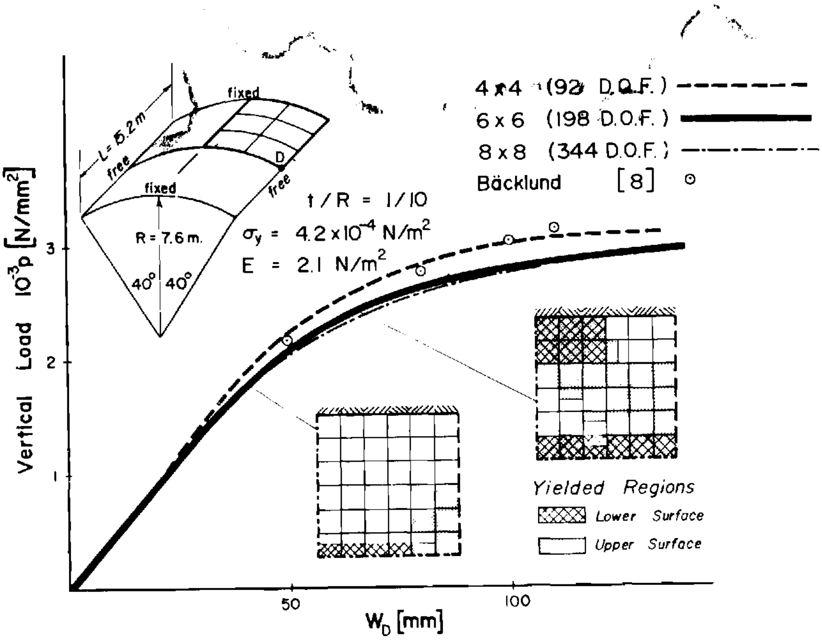


Fig. 10. Elastoplastic analysis of a cylindrical shell roof—progressive yielding.

coarse mesh (6×6) is exhibited in Fig. 10. The progress of yielding is depicted by the views at two different stages; partially yielded elements are seen in both states.

5. Conclusion

Some features of our approximation for elastoplastic deformations are noteworthy:

(1) The concept of piecewise constant stresses and strains within the element leads to the appropriate moduli in subregions (quadrants) of the element. This provides a consistent and effective mechanism for the progressive yielding of the entire element.

(2) The re-evaluation of the stiffness matrix at successive increments of deformation (and load) requires no additional integrations over the elemental area. In this respect, the element is simpler than comparable 'displacement' models.

(3) The evaluation of internal forces is also simple and, consequently, the correction of imbalances is readily achieved via the iterative process.

(4) The element has relatively few degrees-of-freedom and also has a mechanism to similate progressive yielding. These features permit efficient discrete approximations of shells: the systems of equations are smaller and the bandwidths narrower than those for many elements of higher order.

(5) In the elastic and especially in the elastoplastic case, the present element offers some advantages when compared to 'mixed' elements: the formulation does not require the inversion of stress–strain relations to obtain the stiffness matrix of the element. Also, no special procedures are needed (see e.g. [5]) to effect the solution of the equations, since the system is always positive definite.

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Appendix A

If the strains are arranged in a matrix $\boldsymbol{E} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}]^{\mathrm{t}}$, then the corresponding stresses are given by the product $\boldsymbol{S} = \bar{\boldsymbol{C}}\boldsymbol{E}$ where $\bar{\boldsymbol{C}}$ is a matrix of tangent moduli: $\bar{\boldsymbol{C}} = \boldsymbol{C} - \boldsymbol{B}$. The symmetric matrix $\bar{\boldsymbol{C}}$ has the following form:

$$\begin{bmatrix} \bar{C}^{1111} & \bar{C}^{1122} & 2\bar{C}^{1112} & 2\bar{C}^{1113} & 2\bar{C}^{1123} \\ & \bar{C}^{2222} & 2\bar{C}^{2212} & 2\bar{C}^{2213} & 2\bar{C}^{2223} \\ & \text{symm.} & 2\bar{C}^{1212} & 2\bar{C}^{1213} & 2\bar{C}^{1223} \\ & & & 2\bar{C}^{1313} & 2\bar{C}^{1323} \\ & & & & 2\bar{C}^{2323} \end{bmatrix}. \tag{A.1}$$

The elements $\bar{C}^{\alpha i\gamma j}$ are given by the formula $\bar{C}^{\alpha i\gamma j} = C^{\alpha i\gamma j} - B^{\alpha i\gamma j}$ where $C^{\alpha i\gamma j}$ are the elastic moduli of plane stress:

$$C^{\alpha\beta\gamma\eta} \equiv \frac{E}{2(1+\nu)}\left(\delta^{\alpha\gamma}\delta^{\beta\eta} + \delta^{\alpha\eta}\delta^{\beta\gamma} + \frac{2\nu}{1-\nu}\delta^{\alpha\beta}\delta^{\gamma\eta}\right), \qquad C^{\alpha 3\beta 3} = C^{3\alpha\beta 3} \equiv \frac{E}{2(1+\nu)}\delta^{\alpha\beta}.$$

The elements $B^{\alpha i\gamma j}$ depend on the moduli and deviatoric components of stress:

$$B^{\alpha\beta\gamma\eta} \equiv S_{\mu\phi}S_{\kappa\delta}C^{\mu\phi\gamma\eta}C^{\kappa\delta\alpha\beta}/D, \qquad B^{\alpha\beta\gamma 3} \equiv 2S_{3\phi}S_{\eta\mu}C^{3\phi 3\gamma}C^{\alpha\beta\eta\mu}/D,$$

$$B^{\alpha 3\gamma 3} = B^{3\alpha\gamma 3} \equiv 4S_{\eta 3}S_{\mu 3}C^{\alpha 3\eta 3}C^{\mu 3\gamma 3}/D, \qquad D \equiv G_p + S_{\alpha\beta}S_{\gamma\eta}C^{\alpha\beta\gamma\eta} + 4S_{\alpha 3}S_{\beta 3}C^{\alpha 3\beta 3}.$$

The scalar G_p is a hardening parameter which depends on the properties of the material and history. The structures for the matrices $\boldsymbol{M}_n$, $\boldsymbol{Q}$, $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ in (19) are indicated below.

$$(\boldsymbol{M}_0)^t = [n^{11}_{1+2} \quad n^{11}_{3+4} \quad n^{22}_{1+4} \quad n^{22}_{2+3} \quad n^{12}_{1+2+3+4}], \tag{A.2}$$

$$n^{11}_{1+2} = \bar{n}^{11}, \qquad n^{11}_{3+4} = \bar{n}^{11} + \tilde{n}^{11}, \qquad n^{12}_{1+2+3+4} = \bar{n}^{12}, \ldots .$$

$$\boldsymbol{Q}^t = [q^1_{1+2} \quad q^1_{3+4} \quad q^2_{1+4} \quad q^2_{2+3}], \qquad q^1_{1+2} = \bar{q}^1, \qquad q^1_{3+4} = \bar{q}^1 + \tilde{q}^1, \tag{A.3}$$

$$\boldsymbol{A} = \begin{bmatrix} \bar{\boldsymbol{A}} & \boldsymbol{0} \\ \boldsymbol{0} & 1\Big/\sum_{N=1}^{4} A_N \end{bmatrix}, \tag{A.4}$$

$$\bar{\boldsymbol{A}} = \begin{bmatrix} 1/(A_1 + A_2) & 0 \\ 0 & 1/(A_3 + A_4) \end{bmatrix}. \tag{A.5}$$

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A Simple and Efficient Approximation of Shells via Finite Quadrilateral Elements

Introduction

The approximation of shells via finite elements has attracted the attention of many proponents of such methods. Peculiar difficulties can be traced to the special attributes of a shell, so thin that flexural and extensional deformations predominate while transverse stress and shear strain are incidental. In short, the hypothesis of Kirchhoff-Love is indeed an accurate basis for the overall description of the continuous body. Consequently, early attempts to develop elemental approximations were founded on the Kirchhoff-Love hypothesis and were necessarily complicated: The simplest polynomial interpolation, which preserves the requisite smoothness, assigns 16 degrees of freedom to each quadrilateral element of a plate; at least eight more are needed to accommodate the extensional deformations of a plate or shell. Moreover, the hermitian interpolation requires a polynomial of sixth degree. The literature abounds with alternative elements. Many provide good approximations to certain problems but are unreliable or inaccurate in other circumstances. Others are quite accurate but too elaborate and uneconomical for the broad spectrum of practical problems.

Our approximation by quadrilateral elements is very simple but retains the essential attributes of the shell, including effects of transverse shear. Moreover, the discrete approximation is drawn in a consistent manner from the general theory of the continuum and the mechanical behavior of the finite element, without recourse to special manipulations or computational procedures. The formulation is ultimately expressed in terms of displacements and rotations only, and exhibits the desirable attributes of such systems. Additionally, the stresses can be readily computed in accordance with consistent equations which are derived from the variational procedure. Consequently, the model is adaptable to general-purpose programs and is generally applicable to shells, thick or thin.

Basic Features

The essential step toward simpler efficient elements requires a departure from the earlier tradition of strict adherence to the Kirchhoff-Love hypothesis. Such relaxation of the kinematical basis was proposed by Wempner [1] and implemented by Kross [2] in a rudimentary form. The main feature of the element is simple trilinear approximation of the displacement. The attendant features, apparent difficulties, and alternative means of accommodation are readily displayed by the example of simple flexure: The rectangular beam of Fig. 1 lies initially along the x axis. Under the action of end couples, the homogeneous beam is deformed such that the axis forms a circular arc. The *best* bilinear approximation of the elemental displacement produces identical trapezoids. If ξ^1, ξ^2 are local coordinates emanating from the midpoint of the Nth element, ϵ_{11} and γ_{12} the extensional and shear strains according to the elementary definitions, then

$$\frac{\gamma_{12}}{\epsilon_{11}} = \frac{\xi^1}{\xi^2}$$

In a beam (plate or shell) the thickness h is small in comparison to lengths along the axis (surface). The length l of an effective element must be relatively large $l/h > 1$; it follows that the shear energy in our *best* approximation is much larger than the extensional energy

$$\frac{\iint \gamma_{12}{}^2 d\xi^1 d\xi^2}{\iint \epsilon_{11}{}^2 d\xi^1 d\xi^2} = \frac{l^2}{h^2} >> 1$$

This does not mean that our approximation fails; in the limit ($l \rightarrow 0$, $\gamma_{12} \rightarrow 0$) the approximation approaches the exact solution. It means that the unrefined approximation is impractical and some alterations are needed.

The first proposal to modify the bilinear approximation and overcome the apparent difficulty was based on the Kirchhoff-Love hypothesis. Recognizing the validity of the hypothesis in the limit, Wempner [1] proposed a discrete

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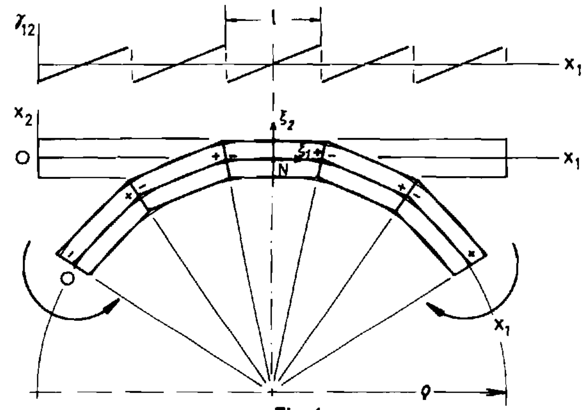


Fig. 1

counterpart, requiring the normality of an intermediate normal, insuring convergence to the Kirchhoff-Love theory and then suppressing the energy of shear at the onset. Alternative schemes are also possible to enforce a discrete Kirchhoff-Love counterpart and suppress the spurious shear energy [3, 4].

More recently the trilinear approximation has re-emerged as a means of approximating shells [5, 6]. The aforementioned shear strain and associated energy has been suppressed by computational schemes known as "reduced" or "selective integration" [7]. The basis of the scheme and the explanation are again evident in the simple example: Observe that the shear strain of Fig. 1 oscillates in the manner of sawteeth about the mean, which is zero in simple bending, but otherwise small in most practical circumstances. The "reduced integration" of the shear energy employs an intermediate value (usually the median) to approximate the integral. In the spirit of the mean-value theorem, that approximation is indeed much better than the exact integral, e.g., in pure bending of the hookean beam, the exact shear energy exceeds the flexural energy by the factor l^2/h^2, whereas the "reduced integration" provides the exact (zero) value.

Some attributes of the simplest approximation emerge: The simplest element has a trilinear approximation of displacement. Transverse shear strain can be accommodated in terms of elemental mean values. From the practical viewpoint, the discrete approximation should be formulated in terms of displacements and characterized by a symmetrical positive-definite matrix. Moreover, our goal should be achieved through consistent approximations which are systematically derived from a proven theory of the continuous shell. To achieve the objective, we turn to the general theorem of Hu-Washizu [8, 9] wherein the functional admits independent approximations of all stresses, strains, and displacements. We cast the functional in the two-dimensional space of the shell with the accepted "plane-stress" assumption ($S^{33}=0$) and a relaxed version of the Kirchhoff-Love hypothesis, which admits independent turning of the normal. In this context, we found the elemental description upon the simplest admissible approximations of the two-dimensional fields: bilinear approximations of the displacement and rotation, and piecewise constant approximations of the stresses and strains. The homogeneous stresses and strains are augmented by specific terms which are introduced to accommodate deformational modes of higher order. All are ultimately eliminated to achieve the desired formulation in terms of displacements. The essential details follow:

Kinematic Features

A typical hexilateral element is depicted in Fig. 2; it is bounded by top (S_+) and bottom (S_-) surfaces, and edges ($\theta^\alpha = a^\alpha \pm l^\alpha$). Also shown is an intermediate reference surface (S), the origin (O) of local normalized coordinates, $\xi^\alpha = (\theta^\alpha - a^\alpha)/l^\alpha$, $\xi^3 = \theta^3/h$), and nodal points (N_-, N, N_+).

Viewed as a three-dimensional finite element, the trilinear approximation of displacement $\mathbf{V}$ has the form

$$\mathbf{V} \doteq (\mathbf{W}_{NM} + \xi^3 \boldsymbol{\phi}_{NM})\, \mathfrak{f}_{NM}(\xi^1, \xi^2) \tag{1a}$$

$$\mathfrak{f}_{NM} \equiv \frac{1}{4}(1+\xi_N{}^1\xi^1)(1+\xi_M{}^2\xi^2) \tag{1b}$$

where majuscule suffices signify the nodal values:

$$\mathbf{W}_{NM} \equiv \mathbf{V}(\xi_N{}^1, \xi_M{}^2, 0)$$

In accordance with the first approximation, the transverse normal stress ($S^{33}=0$) and the associated work is neglected; then the small extension of the normal is also neglected, so that

$$\dot{\boldsymbol{\phi}}_{NM} = \dot{\boldsymbol{\Phi}}_{NM} \times \mathbf{N}$$

where $\mathbf{N}$ is the unit vector along the θ^3 line; $\dot{\boldsymbol{\Phi}}_{NM}$ is the spin of that line. In the linear or incremental formulation, the relation applies as well to the small rotation as to the spin. The important feature is the reduction of freedom: Our element has 20 degrees of freedom that are, most naturally, the displacements $\mathbf{W}_{NM}$ of the nodes N on the reference surface (S) and the two components of the rotation $\boldsymbol{\phi}_{NM}$.

The effective implementation of our simple approximation follows the identification of 20 modes, which are all uncoupled in the degenerate case of the plate: six degrees of freedom correspond to rigid motions and eight modes represent the homogeneous strains (three mean extensional strains $\bar{\epsilon}_{\alpha\beta}$, three mean flexural strains $\bar{\kappa}_{\alpha\beta}$, and two mean shear strains $\bar{\gamma}_\alpha$). In the degenerate case, these 14 modes are expressed in the following displacements:

$$W_\alpha \doteq \bar{W}_\alpha + \bar{\omega}_{\alpha\beta}\xi^\beta + \bar{\epsilon}_{\alpha\beta}\xi^\beta \tag{2a}$$

$$W_3 \doteq \bar{W}_3 + \bar{W}_{3\alpha}\xi^\alpha \tag{2b}$$

$$\varphi^\mu e_{\alpha\mu} \doteq \bar{\gamma}_\alpha - \bar{W}_{3\alpha} - \bar{\kappa}_{\alpha\beta}\xi^\beta \tag{2c}$$

Here, the six constants, $\bar{W}_i$, $\bar{\omega}_{\beta\alpha} = \bar{\omega}_{\alpha\beta}$, and $\bar{W}_{3\alpha}$ represent the six degrees of small rigid motion (see reference [10], p. 62). The eight constants, $\bar{\epsilon}_{\alpha\beta}$, $\bar{\kappa}_{\alpha\beta}$, and $\bar{\gamma}_\alpha$, represent the homogeneous extensional, flexural and transverse shear strains.

Evidently, six additional modes are present: two extensional modes are illustrated in Fig. 3 and correspond to the bilinear terms:

$$\bar{\bar{W}}_\alpha = \bar{\bar{\epsilon}}_\alpha \xi^1 \xi^2 \tag{3a}$$

Similarly, two flexural modes are depicted in the edge views and correspond to the bilinear terms in rotation:

$$\varphi^\mu e_{\alpha\mu} = -\bar{\bar{\kappa}}_\alpha \xi^1 \xi^2 \tag{3b}$$

Two additional modes cause transverse shear strain. One is the torsional mode shown in the top view of Fig. 3 and associated with the linear rotation:

$$\phi^\alpha = -t\xi^\alpha \tag{3c}$$

The consequences are shear strains:

$$\gamma_1 = -t\xi^2, \qquad \gamma_2 = t\xi^1$$

The final mode is a warping of the middle surface, shown in the edge view of Fig. 3 and given by the displacement:

$$W_3 = \bar{\bar{\gamma}}\xi^1\xi^2 \tag{3d}$$

At a glance, it may appear that the warping mode (3*d*) imposes unwarranted shear strains and a "locking" of higher order: The strains of warping (3*d*) follow:

$$\gamma_1 = \bar{\bar{\gamma}}\xi^2, \qquad \gamma_2 = \bar{\bar{\gamma}}\xi^1$$

However, our approximation of the rotation admits the simple form:

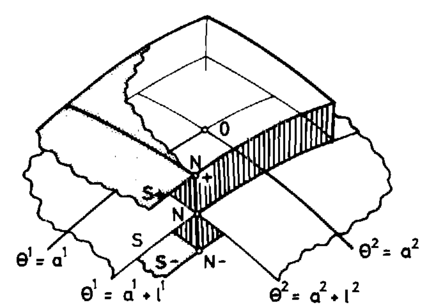


Fig. 2

$$\phi_1 = \bar{\tilde{\gamma}}\xi^1, \qquad \phi_2 = -\bar{\tilde{\gamma}}\xi^2$$

According to the usual linear equations, the combination produces the strains:

$$\gamma_1 = \gamma_2 = 0$$

$$\kappa_{12} = \frac{1}{2}[\phi_{1,1} - \phi_{2,2}] = \bar{\tilde{\gamma}}$$

The warping alone produces shear energy $0(G\bar{\tilde{\gamma}}hl^3)$ whereas the natural combination of warping and rotation produces twisting energy $0(G\bar{\tilde{\gamma}}h^3l)$ in accordance with the theory of Kirchhoff-Love. The latter statement is confirmed by the computation for the twisting of a thin plate.

In summary, the 20 modes of our element are the six rigid motions, the eight homogeneous states of strain ($\bar{\epsilon}_{\alpha\beta}$, $\bar{\kappa}_{\alpha\beta}$, $\bar{\gamma}_\alpha$) and the six additional modes ($\bar{\tilde{\epsilon}}_\alpha$, $\bar{\tilde{\kappa}}_\alpha$, t, and $\bar{\tilde{\gamma}}$). If the origin of the elemental coordinates (ξ^1,ξ^2) is at the center of a rectangular element, then the deformational modes (3*a*–*d*) contribute nothing to the mean values of the strains, i.e., $\bar{\epsilon}_{\alpha\beta}$, $\bar{\kappa}_{\alpha\beta}$, $\bar{\gamma}_\alpha$. It follows that our approximations of the strains (and stresses) must be augmented by six terms of higher order to provide the necessary resistance against the six modes (3*a*–*d*), which are otherwise uninhibited. Higher approximations of the extensional and flexural strains follow:

$$\epsilon_{11} \doteq \bar{\epsilon}_{11} + \bar{\tilde{\epsilon}}_{11}\xi^2 \tag{4a}$$

$$\epsilon_{22} \doteq \bar{\epsilon}_{22} + \bar{\tilde{\epsilon}}_{22}\xi^1 \tag{4b}$$

$$\epsilon_{12} \doteq \bar{\epsilon}_{12} + \underline{\bar{\tilde{\epsilon}}_{12}\xi^1 + \bar{\tilde{\epsilon}}_{21}\xi^2} \tag{4c}$$

$$\kappa_{11} \doteq \bar{\kappa}_{11} + \bar{\tilde{\kappa}}_{11}\xi^2 \tag{5a}$$

$$\kappa_{22} \doteq \bar{\kappa}_{22} + \bar{\tilde{\kappa}}_{22}\xi^1 \tag{5b}$$

$$\kappa_{12} \doteq \bar{u}_{12} + \underline{\bar{\tilde{\kappa}}_{12}\xi^1 + \bar{\tilde{\eta}}_{21}\xi^2} \tag{5c}$$

The additional terms of (4*a*–*c*) and (5*a*–*c*) stem from the extensional and flexural modes of Fig. 2. In our subsequent formulation, we require only four terms that are enough to inhibit these four higher modes: It is enough to retain the additional extensional terms ($\bar{\tilde{\epsilon}}_{11}$, $\bar{\tilde{\epsilon}}_{22}$) of (4*a*,*b*) *or* the additional terms ($\bar{\tilde{\epsilon}}_{12}$, $\bar{\tilde{\epsilon}}_{21}$) of (4*c*); likewise, it suffices to retain the flexural terms ($\bar{\tilde{\kappa}}_{11}$, $\bar{\tilde{\kappa}}_{22}$) of (5*a*, *b*) *or* the terms ($\bar{\tilde{\kappa}}_{12}$, $\bar{\tilde{\kappa}}_{21}$) of (5*c*). Higher approximations of the shear strains that correspond to the torsional *and* warping modes follow:

$$\gamma_1 \doteq \bar{\gamma}_1 + \bar{\tilde{\gamma}}_1\xi^2 \tag{6a}$$

$$\gamma_2 \doteq \bar{\gamma}_2 + \bar{\tilde{\kappa}}_2\xi^1 \tag{6b}$$

Variational Formulation

The basis of the elemental approximation is the variational theorem of Hu-Washizu [8,9]. The Hu-Washizu functional of all stresses, strains, and displacements in the element assumes the following form:

$$H = \int_{-1}^{+1}\int_{-1}^{+1} \{U(\epsilon_{\alpha\beta}, \kappa_{\alpha\beta}, \gamma_\alpha)$$
$$-n^{\alpha\beta}[\epsilon_{\alpha\beta} - E_{\alpha\beta}(W_i)]$$
$$-m^{\alpha\beta}[\kappa_{\alpha\beta} - K_{\alpha\beta}(\phi_\alpha, W_i)]$$

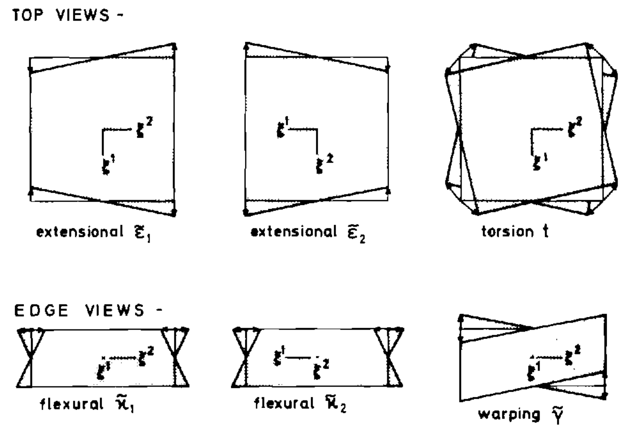


Fig. 3

$$-q^\alpha[\gamma_\alpha - G_\alpha(\phi_\alpha, W_i)]$$
$$-\mathbf{F}\cdot\mathbf{W}\}A d\xi_1 d\xi_2$$
$$-\int_c \{\mathbf{N}\cdot\mathbf{W} + \mathbf{M}\cdot\mathbf{\Phi}\}d\xi \tag{7}$$

Here the function $A(\xi^1, \xi^2)$ is the metric of the area, U the strain-energy density; $n^{\alpha\beta}$, $m^{\alpha\beta}$, q^α are the stresses that are conjugate to the strains $\epsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$, γ_α, respectively. The applied load **F** acts on the surface; force **N** and couple **M** act on the edge C of the element. The operators $E_{\alpha\beta}$, $K_{\alpha\beta}$, and G_α express the strains $\epsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$, and γ_α in terms of the displacements W_i and rotations ϕ^α, i.e., the kinematical equations, linear or nonlinear, are the conditions that the bracketed terms vanish [ ] = 0.

A crucial step in the formulation of our finite element, is the introduction of the simplest admissible approximations of each field. Accordingly, the displacement is expressed by the bilinear form (1). The approximations of the strains and stresses are also to be as simple as possible, yet consistent with established criteria: The so-called "limitation theorem" of Fraeijs de Veubeke [11] asserts that no advantages accrue from an approximation of a higher-order than one consistent with related fields. The patch test of Irons [12] requires that the element accommodate homogeneous states of strain. These criteria support our proposed approximations (4)–(6) of the strains. The eight constants represent homogeneous strains and the additional terms are only enough to accommodate the six modes of Fig. 3, which otherwise possess no energy. The approximations of the stresses follow the forms of the conjugate strains:

$$n^{11} \doteq \bar{n}^{11} + \bar{\tilde{n}}^{11}\xi^2, \text{--}, \text{--}; \tag{8a,b,c}$$

$$m^{11} \doteq \bar{m}^{11} + \bar{\tilde{m}}^{11}\xi^2, \text{--}, \text{--}; \tag{9a,b,c}$$

$$q^1 \doteq \bar{q}^1 + \bar{\tilde{q}}^1\xi^2, \text{--}; \tag{10a,b}$$

It is important to note that the supplementary terms (marked by the tilda ~) are higher order; specifically, if the gradient of the stress (or strain) is finite, then the supplementary term vanishes in the limit ($l^\alpha \to 0$).

In the rectangular element of a plate or cylindrical shell, the constants in the approximations (4-10) are orthogonal to the linear functions. In a nonrectangular element, the approximations must also account for the metric $A(\xi^1,\xi^2)$. Then the discrete counterpart of the functional (7) is a function of the form:

$$\bar{H} = \bar{U}(\bar{\epsilon}_{\alpha\beta}, \bar{\kappa}_{\alpha\beta}, \bar{\gamma}_\alpha, \bar{\tilde{\epsilon}}_{\alpha\beta}, \bar{\tilde{\kappa}}_{\alpha\beta}, \bar{\tilde{\gamma}}_\alpha)$$
$$-\bar{n}^{\alpha\beta}[\bar{\epsilon}_{\alpha\beta} - \bar{E}_{\alpha\beta}(W_{NM})]$$
$$-\bar{m}^{\alpha\beta}[\bar{\kappa}_{\alpha\beta} - \bar{K}_{\alpha\beta}(\phi^\alpha_{NM}, W^i_{NM})]$$
$$-\frac{1}{3}\bar{\tilde{n}}^{\alpha\beta}[\bar{\tilde{\epsilon}}_{\alpha\beta} - \bar{\tilde{E}}_{\alpha\beta}(W_{NM})]$$

$$
\begin{aligned}
&-\frac{1}{3}\,\bar{\bar{m}}^{\alpha\beta}[\bar{\bar{\kappa}}_{\alpha\beta}-\bar{\bar{K}}_{\alpha\beta}(\phi^{\alpha}_{NM},W^{i}_{NM})]\\
&-\frac{1}{3}\,\bar{\bar{q}}^{\alpha}[\bar{\bar{\gamma}}_{\alpha}-\bar{\bar{G}}_{\alpha}(\phi^{\alpha}_{NM},W^{i}_{NM})]\\
&-F^{NM}_{i}W^{i}_{NM}-M^{NM}_{\alpha}\phi^{\alpha}_{NM} \qquad (11)
\end{aligned}
$$

Here the factor 1/3 is the result of integrating the products $(\xi^{\alpha})^2$. The function (11) is to be stationary with respect to each of the variables. The results consist of

(*a*) 20 equilibrium equations in terms of the stresses ($\bar{n}^{\alpha\beta}$, $\bar{m}^{\alpha\beta}$, $\bar{q}^{\alpha}$, $\bar{\bar{n}}^{\alpha\beta}$, $\bar{\bar{m}}^{\alpha\beta}$, $\bar{\bar{q}}^{\alpha}$), nodal forces (F_i^{NM}), and couples ($M_{\alpha}{}^{NM}$).

(*b*) 14 stress-strain relations. Because of the orthogonality of the functions, the equations of the isotropic hookean element are uncoupled. Specifically,

$$
\begin{aligned}
\bar{n}^{\alpha\beta}&=C^{\alpha\beta\gamma\eta}\,\bar{\epsilon}_{\gamma\eta}\\
\bar{m}^{\alpha\beta}&=\frac{h^2}{12}\,C^{\alpha\beta\gamma\eta}\,\bar{\kappa}_{\gamma\eta}\\
\bar{q}^{\alpha}&=C^{\alpha\beta}\,\bar{\gamma}_{\beta} \qquad (12b)
\end{aligned}
$$

The coefficients are the usual moduli of the continuous theory (see Ref. [10], p. 590).

(c) 14 strain-displacement equations that are also uncoupled. Specifically,

$$
\begin{aligned}
\bar{\epsilon}_{\alpha\beta}&=\bar{E}_{\alpha\beta}(W_{NM})\\
\bar{\kappa}_{\alpha\beta}&=\bar{K}_{\alpha\beta}(\phi^{\alpha}_{NM},W^{i}_{NM})\\
\bar{\gamma}_{\alpha}&=\bar{G}_{\alpha\beta}(\phi^{\alpha}_{NM},W^{i}_{NM}). \qquad (12c)
\end{aligned}
$$

The 14 equations (12*c*) can be substituted into the 14 equations (12*b*) which can be substituted into the 20 equations (12*a*) in order to obtain the elemental approximation in terms of *displacements* only.

Some features of the approximation are noteworthy:

(1) The higher-order terms (=) vanish in the limit. The remaining constitutive equations are those written explicitly under (*b*) and these approach the equations of the continuous shell. Likewise, the remaining kinematical equations are these written under (12*c*) and these difference equations approach the differential equations of the continuous shell.

(2) Five equilibrium conditions of the finite element emerge from the virtual work upon the three displacements and two rotations at each node. Like the continuum theory, rotation about the normal imposes a condition of symmetry upon the stresses ($\bar{n}^{\alpha\beta}=\bar{n}^{\alpha\beta}$), which is inherent in our elemental approximation.

(3) The evaluation of all coefficients requires only the integration of simple functions. All can be done *exactly*, or by any accurate numerical procedure, which might be expedient to the user. Moreover, the formation of the elemental matrix requires *only direct substitutions*. Consequently, the formulation is exceedingly simple and economical.

(4) The effects of extensional and flexural behavior are represented by comparable approximations. As our examples indicate, the formulation serves as well whether extensional or flexural effects predominate.

(5) The final result of our formulation is a system of equations in terms of displacements alone. That system has the most desirable attributes of a displacement formulation: Specifically, the matrix is symmetric and positive definite.

The omission of higher-order terms reduces the internal energy, so that the element is less stiff than the element derived via minimum potential and displacements alone. Provided only that the gradients remain finite, the effects of these higher-order terms diminish as the element shrinks and, therefore, convergence is assured.

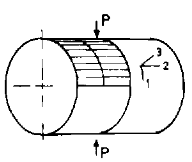
Some examples serve to demonstrate the effectiveness of this simple elemental approximation.

Example 1: Cylindrical Shell Under Radial Forces With Free Ends. The "pinched cylinder" of Fig. 4 has been used frequently for numerical testing of approximations. The radial displacement at the load is given in the table according to various authors. The deformation is predominantly flexural; indeed, the largest calculated value [13] is only 5 percent greater than that based on inextensionality [14].

For most purposes, the measure of an approximation is the number of degrees of freedom (and equations) required to obtain satisfactory results. In this respect, our simple model excels. The models of Ashwell and Sabir [15] and Dawe [19] give comparable results with fewer degrees of freedom, 20 and 30, respectively. However, the former is an element, which is especially tailored to the cylindrical shell; the latter is formed with a higher-order triangular element, which has 54 degrees of freedom based on a quintic polynomial.

It is noteworthy that our elemental matrix possesses six, and only six, *distinct* zero eigenvalues, which confirm an accurate representation of the rigid modes.

Example 2: Cylindrical Shell Under Radial Forces With Simply Supported Ends. If the ends of the "pinched



R = 4.95 in , t = 0.094 in
L = 10.35 in , P = 100 lb.
E = 10.5×10^6 lb. in.^{2}
( ): denotes degrees of freedom

Mesh in actant	Cantin [13]	Ashwell, Sabir [15]	Thomas, Gallagher [16]	Bogner, et al [17]	Cantin, Clough [18]	Dawe [19]	Present
1×1		0.104 (20)				0.106 (30)	
1×4		0.111 (50)	0.111 (67)	0.109 (120)			0.098 (32)
1×6							0.105 (48)
1×7					0.099 (96)		
1×8			0.112 (131)				0.108 (64)
1×9					0.106 (120)		
2×2	0.093 (54)	0.110 (45)				0.112 (96)	
2×4							0.099 (53)
2×6							0.106 (79)
2×9					0.107 (180)		
4×4	0.113 (150)	0.113 (125)				0.114 (374)	
6×6	0.114 (294)	0.114 (245)					

Fig. 4 Radial deflection under load in inches

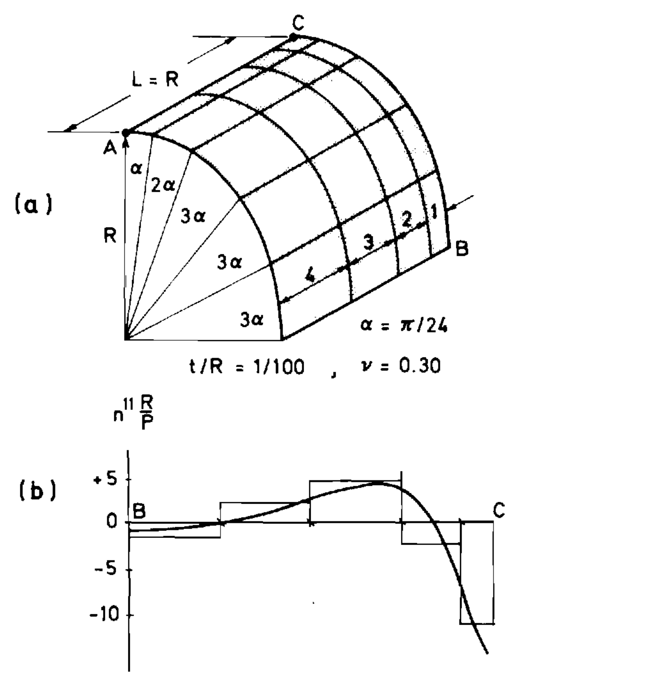


Fig. 5

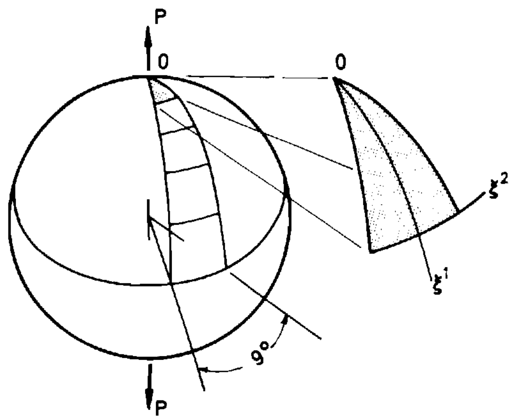


Fig. 6

cylinder" are supported by an inextensible diaphragm, then the flexure is suppressed and membrane actions prevail. Computations were made with the mesh illustrated in the octant of Figure 5(*a*), which provides 104 degrees of freedom. The radial displacement W_3 at the load (C) and axial displacement W_2 at the edge (A) are comparable to the results of Dawe [19] and Lukasiewicz [20], as follows:

	[19]	Present	[20]
EtW_3/P(at C)	164	152	155
EtW_2/P (at A)	4.11	4.06	-
Degrees of Freedom	(366)	(104)	-

The number (366) of degrees of freedom is our count according to the information given by Dawe [19]. The results given by Lukasiewicz were obtained with a few terms of a series solution; the number 155 for W_3 is estimated from his graphical presentation [20, p. 397].

The plot of Figure 5(*b*) serves to exhibit the piecewise representation of the stress n^{11}. The smooth curve is taken from the series solution, as displayed by Edwards and Webster [21].

Example 3: Spherical Shell Under Radial Forces. The spherical shell under diametrically opposed forces serves as an example in which the elements are nonrectangular. Moreover, an element at the pole degenerates to a triangle as depicted in the auxiliary view at 0 in Fig. 6. The consistent approximation of displacement and rotation in the triangular element has the form:

$$W = \bar{W} + W_1 \xi^1 + W_{12} \xi^1 \xi^2$$

Each field is determined by the three nodal values and each is compatible with the approximation of adjoining elements. The triangular element has 15 degrees of freedom, six rigid modes, eight homogeneous strains, and one warping mode which requires one additional term ($\bar{\bar{\gamma}}_2 \xi^1$) in the approximation of shear:

$$\gamma_2 = \doteq \bar{\gamma}_2 + \bar{\bar{\gamma}}_2 \xi^1$$

Here the terms of the approximations are not orthogonal in the element, since $A(\xi^1) = \sin \pi \xi^1 / 2n$, where n is the number of elements in a segment of the hemisphere, as shown. Consequently, the linear approximations introduce coupling between the mean values ($^-$) and supplementary terms ($\sim$). Otherwise, the formulation is unaffected.

Once again, the elemental matrix exhibits six *distinct* zero eigenvalues corresponding to the rigid modes.

In the numerical example, $t/R = 50$, $\nu = 0.30$. The displacement W at the pole is comparable to the result obtained by Koiter [22]:

$ETW/P = 21.2$	Koiter [22]
$\doteq 20.7$	15 Elements (87)
$\doteq 20.5$	5 Elements (27)

Here the shear modulus was amplified by a factor $0(10^3)$ to suppress the effect of transverse shear. The 15 elements were equally spaced along the meridian, whereas the five elements were unequally spaced in segments of 5 deg, 5 deg, 5 deg, 30 deg, and 45 deg to take advantage of the local deformational pattern. For purposes of testing, two-dimensional elements were employed with a breadth of 9 deg. Again, the parenthetical numbers are degrees of freedom, but these do not reflect the capabilities of the approximation, since the numbers are reduced to (44) and (14), respectively, if the symmetry is invoked at the outset and the problem is treated as one-dimensional.

Other examples of plates and arches also attest to the abilities of this elementary approximation to accommodate flexure.

Conclusion

From a practical viewpoint, the important attributes of this discrete approximation are the simplicity of the formulation and the economy of implementation. The latter is achieved without significant sacrifices in accuracy or reliability. Indeed, the comparable approximations of extensional and flexural deformations produce an element that accommodates both actions with facility. The intermediate formulations of the kinematic and constitutive equations facilitate the treatment of interactions with supporting structures and/or adjoining media.

From a theoretical viewpoint, the discrete model is derived in a consistent manner from the continuum. Each approximation is taken as simply as possible and founded entirely on mechanical and geometrical concepts. Finally, the algebraic equations of the discrete model are analogous to the corresponding equations of the continuous shell and, indeed, reproduce their counterparts in the limit.

Specific features (1)–(5), are enumerated in the "formulation."

Some distinctions between the present element and certain "mixed" elements are noteworthy: The present approximation is founded on an elaborate form (7) of *potential energy*, whereas most "mixed" elements are based on the functional of Hellinger-Reissner [23, 24], an elaboration of *complementary energy*. Our stiffness matrix has the properties of one derived from potential energy. No inversion is needed if the element is rectangular; otherwise, coupling between the constant ($-$) and higher ($=$) stresses and strains requires, at most, the inversion of 3×3 matrices. The elemental matrix of the "mixed" element requires an inversion, since the complementary formulations express displacements in terms of stresses.

The extension to finite deformations poses no inherent difficulties, but requires additional algebra and storage. The functional of Hu-Washizu [9] is generally available in terms of the Cauchy-Green components of strain and Kirchhoff-Trefftz components of stress (see reference [10] p. 443); alternatively, a precise version is also expressible in terms of engineering components of strain and Jaumann components of stress [25]. By adhering to a local convected (Lagrangian) system, the essential nonlinear terms in the discrete formulation are products of forces and relative nodal rotations, analogous to the products of forces and curvatures in the continuum theories of thin shells. Then, rigid rotations appear only where loads, surface or edge tractions, are prescribed in a fixed (Eulerian) system. Such extensions are currently under development, but beyond the scope of this initial presentation.

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