Sequential item pricing for unlimited supply

Maria-Florina Balcan[1] and Florin Constantin[1]

College of Computing, Georgia Institute of Technology
{ninamf,florin}@cc.gatech.edu

Abstract. A common challenge faced by sellers is to explore and exploit, in limited time, buyers' willingness to pay. We propose distribution-free revenue maximization techniques for a seller with unlimited supply of n item types facing m myopic buyers present for $k < n$ days.
We highlight the power of linear uniform prices (i.e. equal item prices) for buyer valuations with *hereditary maximizers* (HM), e.g. multi-unit or gross substitutes valuations. In particular, we present a non-increasing, randomized, schedule of k linear uniform prices with expected revenue within a $\log_k(mn)$ factor of optimal given HM valuations. Our result improves by a $\log k$ factor [2]'s approximation for general valuations and one-shot prices, and does not go through even for submodular valuations. We also consider *influences* (in the form of allocative externalities) between buyers with HM valuations. We provide a rather general model of positive influence of other buyers' ownership of some items on a buyer's valuation. For affine, submodular externalities, we present an influence-and-exploit [8] marketing strategy based on our algorithm for private HM valuations. This strategy preserves our $\log_k(mn)$ approximation factor, despite an affine increase (due to externalities) in the optimum revenue.

1 Introduction

Consider a seller sequentially offering items for sale to myopic buyers, i.e. buyers that do not take into account possible future price reductions. At first glance, the seller appears to be able to obtain good revenue by simply setting high prices initially and then lowering them. That is however not always the case, due to complementarities among items, as illustrated in the following example.

Consider three movies in digital format (e.g. DVD, Blu-Ray, or available for download): a very good science-fiction one and an animation movie and a drama, both of slightly lower quality. A typical family prefers the science-fiction one to any of the other two, but the animation and drama (for variety) over any other pair. The family does not strategize about price schedules. If a greedy movie retailer starts with high prices and reduces them afterwards (on all movies) then, despite good revenue on the science-fiction movie, it loses the opportunity of more revenue by selling the other two instead.

Inspired by similar optimization problems faced by sellers in other domains, we study revenue maximization problems via dynamic pricing over k days for a seller with n item types and with very low marginal cost (typically termed "unlimited supply" in the literature [2, 6]) for an additional copy of any item.

We assume that the m buyers (present throughout the k days) are myopic: that is, each buyer chooses her preferred (i.e. utility-maximizing) bundle in each day without reasoning about future price reductions. In modeling such buyers' dynamic preferences, we use a simple, cumulative, assumption on their additional value for a set of items given the items they already own. To ensure consistency among preferred bundles in consecutive days[1], we assume that buyers' private valuations have *hereditary maximizers* (HM), a new class of valuations that we introduce. In particular, multi-unit (not necessarily concave) valuations and gross substitutes valuations (a classical model in economics, see e.g. [4, 7]) have hereditary maximizers. While the HM property for gross substitutes was already established in [3], we provide a simpler proof based on a fundamental graph-theoretic fact. Our example fails the HM consistency property, since the science-fiction movie is preferred to each of the other two, but the pair of the other two is preferred to either pair containing the science-fiction movie.

We provide competitive algorithms that price all items equally (i.e. that use *linear uniform* prices). This model involves the least price discrimination possible under sequential pricing: no buyer or item is favored over another. In fact, some online movie retailers (e.g. iTunes) have very limited variability in prices – iTunes offers only two prices, with the lower one for older movies. A $O(\log mn)$ approximation to the optimal revenue is known [2] for such prices if all buyers (with any valuations) are impatient (i.e. $k = 1$). Our main contribution is an algorithm with an improved approximation ratio $O(\frac{\log mn}{\log k})$ for general $k < n$.

In the second part of the paper, we allow a buyer's valuation for a set of items to depend on the others' ownership of copies of (possibly different) items. In economic language, others exert an allocative[2] externality on a buyer's value. Movie delivery services (e.g. Netflix) exploit such effects, allowing users to befriend each other and to observe which movies they watched (and their rating).

We introduce a (distribution-free) model of externalities and extend our algorithm for private valuations. Our model departs from existing [1, 8] revenue maximization problems in the presence of externalities in two aspects, that, in our view, allow for more generality. First, there is more than one item type for sale, which requires a new language for expressing externalities. We introduce such a language that extends [1, 8]; it allows a buyer to express a positive, affine influence by others that is monotone and submodular in their bundles. Second, we assume only certain properties of the valuation functions, as opposed to values drawn from a (known) distribution. We obtain an algorithm with the same approximation ratio as without externalities, despite an affine increase in the optimum. Our algorithm is an influence-and-exploit (IE) strategy, introduced by Hartline et al. [8]. In an IE strategy, a set S of buyers is given some items for free and then other customers are charged a price that exploits other owners' (a superset of S) influence on the items' value.

Related work Balcan et al. [2] present structural results for one-shot pricing and demand with arbitrary valuations and unlimited supply. They show that an

[1] Private valuations, as opposed to preferred bundles, cannot change between days.

[2] As opposed to *informational* or *financial* externalities [9], where a valuation depends on others' information (e.g. signals of the quality of the item) or on their payments.

exponentially scaled random price has revenue that approximates in expectation the optimum social welfare to a logarithmic factor (in the number of items).

Fiat and Wingarten [6] show that revenue-maximizing envy-free prices can be found in polynomial time for single-minded buyers and unlimited supply.

Jehiel and Moldovanu [9] find that many classical results no longer hold when externalities (allocative or informational) are present in auctions.

While externalities are a natural and well-studied phenomenon in social networks [10], the corresponding revenue maximization problem is introduced by Hartline et al. [8], who investigate the approximation properties of single-item distribution-based influence-and-exploit marketing strategies. Akhlaghpour et al. [1] present approximation and inapproximability results for the same problem, but additionally precluding the seller from using price discrimination amongst buyers. In fact, our algorithm builds upon a scheme in [1].

Paper structure. We introduce notation and review static pricing results in Sec. 2, defining a dynamic model in Sec. 2.1. In Sec. 3 we impose a restriction on valuations permitting good sequential revenue, fact established in Sec. 4. Finally, in Sec. 5, we model externalities, where a buyer's valuation depends on items owned by others, and extend the Sec. 4 revenue approximation.

2 Preliminaries

We consider m customers (present for k days) with quasilinear utilities buying bundles of n item types. We start with a generic customer with arbitrary valuation (the same in any day) $v : 2^{1..n} \to \mathbb{R}$. We denote valuation maxima by H. We only use the simplest form of pricing, with no item or buyer discrimination.

Definition 1 *A price vector* $\mathbf{p} \in \mathbb{R}^n$ *is* linear uniform[3] *if* $p_j = p, \forall j = 1..n$.

Given a price vector, a customer buys a preferred (utility-maximizing) bundle.

Definition 2 *For price vector* $\mathbf{p} \in \mathbb{R}^n$*, the* demand correspondence *[7]* $\mathcal{D}_v(\mathbf{p})$ *of valuation* v *is the set of utility-maximizing bundles at prices* $\mathbf{p}$*:*

$$\mathcal{D}_v(\mathbf{p}) = \mathrm{argmax}_{S \subseteq 1..n} \{v(S) - \textstyle\sum_{j \in S} p_j\} \tag{1}$$

For linear uniform price $\mathbf{p} = p \cdot \mathbf{1}$*, let* $\mathcal{D}_v(p) = \mathcal{D}_v(\mathbf{p})$ *and* $F_v(p) = \min_{S \in \mathcal{D}(p \cdot \mathbf{1})} |S|$ *be the least number of items in a bundle demanded (by valuation*[4] v*) at prices* p*.*

For any valuation, a higher price cannot increase the least quantity bought.

Lemma 1 *(Balcan et al. [2]) For arbitrary valuation* v *and* $p > p'$*,* $F(p) \leq F(p')$*.*

v's *demand curve* is a step function given by $(p_l, F(p_l))_{l=0..n_v+1}$ (with $n_v \leq n$) where threshold prices $0 = p_0 < p_1 < .. < p_{n_v} \leq p_{n_v+1} = H$ satisfy $F(p_l) = F(p) > F(p_{l+1}), \forall p \in [p_l, p_{l+1}), \forall l = 0..n_v$. The area A_F under the demand curve is $\sum_{l=1}^{n_v} p_l (F(p_l) - F(p_{l+1}))$.

[3] Different (non-uniform) item prices are also (e.g. [4]) called linear prices.

[4] Except for Sec. 5, v will be clear from context and omitted from $\mathcal{D}$ and F.

Lemma 2 *[2]* $\mathrm{A}_F = H = \max_{S\subseteq 1..n} v(S)$*, i.e. v's maximum willingness to pay.*

Our goal is revenue maximization via (possibly randomized) price sequences decided ahead of time (but only revealed gradually to buyers).

Definition 3 *A pricing scheme $\mathcal{P}$ is a sequence of $k' \leq k$ prices $p'_1 \geq \ldots \geq p'_{k'}$. The part of* A_F covered *by $\mathcal{P}$ is $\sum_{l=1}^{k'} p'_l(F(p'_l) - F(p'_{l-1}))$ where $F(p'_0) = 0$.* $\mathrm{Rev}_{\mathcal{P}}(v_1, \ldots, v_m)$ *denotes $\mathcal{P}$'s revenue (in expectation if $\mathcal{P}$ is randomized), given valuations $v_1, \ldots, v_m$ and the least favorable tie-breaking decisions by buyers.*

For instance, prices $r_1 > r_2$ cover a $F(r_1)r_1 + (F(r_2) - F(r_1))r_2$ part of A_F, i.e. the area of the union of two rectangles with opposite corners $(0,0)$ and $(r_i, F(r_i))$.

A standard [2, 8] revenue benchmark is customers' total willingness to pay.

Definition 4 *A (possibly randomized) pricing scheme $\mathcal{P}$ is a c-*revenue approximation *($c \geq 1$) if $c \cdot \mathbb{E}[\mathrm{Rev}_{\mathcal{P}}(v_1, \ldots, v_m)] \geq \sum_{i\in 1..m} \max_{S\subseteq 1..n} v_i(S)$ for all valuations $v_1 \ldots v_m$, where the expectation is taken over $\mathcal{P}$'s random choices.*

We will typically use prices of the form

$$q_l = H/2^{l-1} \text{ for } l \geq 1$$

$(q_l, F(q_l))_{l=1..1+\log H}$ comprise the q-*demand curve*: $F(q_{l+1}) \geq F(p) = F(q_l), \forall p \in (q_{l+1}, q_l)$. In contrast to the p-demand curve, $F(q_l)$ is weakly increasing in l. Static ($k=1$) scheme $\mathrm{R{\small ANDOM}}_D^H$ selects price $q_l = H/2^{l-1}$ with l uniform in $1..D$.

Lemma 3 *[2] For $H = \max_{i\in 1..m, S\subseteq 1..n} v_i(S)$, $\mathrm{R{\small ANDOM}}^H_{\log(2mn)}$ is a*[5] *$4\log(mn)$-revenue approximation.*

For $m=1$, Lemma 3 follows from

$$H \leq 4\sum_{l=1}^{\log(2n)} q_l F(q_l) = 4\sum_{l=1}^{\log(2n)} (q_{l-1} - q_l) F(q_l) \qquad \text{Eq. (1) in [2]}$$

Given k days, we improve (Sec. 4) Lemma 3's $\Omega(\log mn)$ factor by a $\log k$ factor.

2.1 Sequential pricing

We define customer behavior when present for more than one day. We assume the seller offers linear uniform prices $r^d \in \mathbb{R}_+$ in day $d = 1..k$, with $r^1 > \ldots > r^k$.

Assume that before day d the customer buys (disjoint) sets $S_1, \ldots, S_{d-1}$. We model the customer as *forward-myopic*, i.e. for $S \subseteq 1..n \setminus (S_1 \cup \cdots \cup S_{d-1})$,

$$u_{d,\ldots,1}(S_1 \ldots S_{d-1}, S, r^1 \ldots r^d) = v(S_1 \cup \ldots \cup S_{d-1} \cup S) - (\textstyle\sum_{l=1}^{d-1} r^l |S_l|) - r^d |S| \quad (2)$$

i.e. a customer does not anticipate price drops but does take into account past purchases (accumulating items) and payments to decide a utility-maximizing bundle to buy today. In this model, a customer buys nothing in a day where the price increases, hence our focus on decreasing price sequences: the seller starts with a high price and then gradually reveals discounts, a common retail practice.

[5] Unless otherwise specified, all logarithms in this paper are base 2.

The maximum utility after day d cannot be negative (easily shown by induction on d), since the customer always has the option of not buying anything. Thus, any sequence of prices defines an individually rational mechanism.

Denote the collection of preferred bundles outside $S_1 \cup \cdots \cup S_{d-1}$ at $r^d \cdot \mathbf{1}$ by

$$\mathcal{D}_v^{S_1,\ldots,S_{d-1}}(r^1 \ldots r^d) = \operatorname*{argmax}_{S \subseteq 1..n \setminus (S_1 \cup \cdots \cup S_{d-1})} u_{d,\ldots,1}(S_1, \ldots, S_{d-1}, S, r^1 \ldots r^d)$$

We now provide a submodular valuation consistent with the example in the introduction and find that the revenue from a high, followed by a low, price may be lower than that of the best single price. We also see that while the revenue from a single price p amounts to at least the part of A_F covered by p, in general one cannot obtain an analogous statement for a price sequence.

Example 1 *Let a be the science-fiction movie, and b, c be the animation and drama. Define a valuation v by $v(a) = 3, v(b) = v(c) = 2.1, v(a, b) = v(a, c) = 3.8, v(b, c) = v(a, b, c) = 4.2$. For $r^1 = 1.5$, $\mathcal{D}(r^1) = \{\{a\}\}$ and for $r^2 = 1, \mathcal{D}(r^2) = \{\{b, c\}\}$. Neither b or c is worth \$1 given a: $\mathcal{D}^{\{a\}}(r^1, r^2) = \{\emptyset\}$. Less revenue (\$1.5) is obtained from offering r^1 followed by r^2 than from r^2 alone (\$2).*

Example 1 can be extended to m different items by singling out one item and valuing any set with at least 3 items at the highest value among its size 2 subsets. No analogous example exists if v treats items as identical. We define a more general valuation class (containing gross substitutes) with good sequential revenue.

3 Valuations with hereditary maximizers

We henceforth impose on valuations a novel, sequential consistency property: for any value-maximizing (when restricted to size j and base S_0) bundle S_j, one can add an item to S_j and obtain an analogous maximizer for size $j + 1$.
We introduce notation $\mathcal{M}^v_{S_0,j}$ for valuation maximizers (given base S_0) of size j.

Definition 5 *For a valuation v, subset S_0 and size $j \leq n - |S_0|$, let*

$$\mathcal{M}^v_{S_0,j} = \operatorname{argmax}_{|S|=j, S\cap S_0=\emptyset} \ v(S \cup S_0) - v(S_0)$$

Clearly, a size j set (if any) preferred at a linear uniform price cannot have a lower value than another size j set.

Lemma 4 *For all prices r and sizes j, $\mathcal{D}(r) \cap \{|S| = j\}$ is either empty or $\mathcal{M}^v_{\emptyset,j}$.*

Definition 6 *For a bundle S_0, valuation v has S_0-*hereditary maximizers *if*

$$\forall j, \forall S_j \in \mathcal{M}^v_{S_0,j}, \exists S_{j+1} \in \mathcal{M}^v_{S_0,j+1} \text{ with } S_j \subset S_{j+1} \tag{HM}$$

$$\text{implying } \forall j' > j, \forall S_j \in \mathcal{M}^v_{S_0,j}, \exists S_{j'} \in \mathcal{M}^v_{S_0,j'} \text{ with } S_j \subset S_{j'} \tag{HM*}$$

For $S_0 = \emptyset$, we simply say that v has hereditary maximizers (HM).

That is, given any size j subset S_j maximizing marginal value over S_0, one can add an element (outside S_0) to S_j and obtain a size $j+1$ subset maximizing marginal value over S_0. We omit v from $\mathcal{M}_j$ when clear from context.

Lemma 5 *A multi-unit valuation has S_0-hereditary maximizers for any set S_0.*

Proof. The identity of items does not matter for a multi-unit valuation v: thus $\mathcal{M}^v_{S_0,j}$ is the collection of all sets of size j outside S_0, for all j and S_0.

Further discussion and examples of HM valuations are deferred to sections 3.1 and C, where this class of valuations and that of gross substitutes are compared.

Except for Theorem 2, we henceforth focus on $\emptyset$-hereditary maximizers only.

We proceed with a quantity guarantee for HM valuations and linear uniform prices: no fewer items are sold for price sequence $r^1, \ldots, r^d$ (regardless of which preferred bundles are bought) than in the worst-case for r^d alone, i.e. $F(r^d)$.

Theorem 1 *Fix an HM valuation v, a day $d \leq k$ and item prices $r^1 > \cdots > r^d$.*
Let $S_\delta \in \mathcal{D}^{S_1,\ldots,S_{\delta-1}}(r^1 \ldots r^\delta)$ preferred at r^δ given that $S_1, \ldots, S_{\delta-1}$ were sequentially bought at $r^1, \ldots, r^{\delta-1}$, $\forall\, \delta = 1..d$. Then $\sum_{\delta=1}^{d} |S_\delta| \geq F(r^d)$.

Proof. We treat the case $d = 2$; the claims for general d follow similarly.

Let $S_1 \in \mathcal{D}(r^1)$ and assume $|S_1| < F(r^2)$ (otherwise the claim is immediate). Let $S_2 \in \mathcal{D}^{S_1}(r^1, r^2)$. By Lemma 4, $S_1 \in \mathcal{M}_{\emptyset,|S_1|}$. As $F(r^2) > |S_1|$, by (HM*), $\exists S'_2 \in \mathcal{M}_{\emptyset,F(r^2)}$ with $S_1 \subset S'_2$. As $\mathcal{M}_{\emptyset,F(r^2)} \cap \mathcal{D}(r^2) \neq \emptyset$, by Lemma 4, $S'_2 \in \mathcal{D}(r^2)$.

We establish a structural property: $S_2 \cup S_1 \in \mathcal{D}(r^2)$. Informally, set S_1 preferred at higher price r^1 can serve as base to create sets preferred at lower price r^2 alone via joining sets preferred sequentially at r^2 after buying S_1. S_2 is preferred at r^2 given base $S_1 \subset S'_2$ and then $S_2 \cup S_1$ must be preferred at r^2 alone.

Let $u_S = v(S \cup S_1) - r^1|S_1| - r^2|S|$ be the utility from buying $S \subseteq 1..n \setminus S_1$ at r^2 after buying S_1 at r^1. Note that $u_{S'_2 \setminus S_1} - u_{S_2} = (v(S'_2) - r^2|S'_2|) - (v(S_2 \cup S_1) - r^2|S_2 \cup S_1|) \leq 0$ as $S_2 \in \mathcal{D}^{S_1}(r^1, r^2)$. If $u_{S'_2 \setminus S_1} < u_{S_2}$ then $S_2 \cup S_1$ is preferred to S'_2 at r^2, contradicting $S'_2 \in \mathcal{D}(r^2)$. Thus $u_{S'_2 \setminus S_1} = u_{S_2}$ implying $S_2 \cup S_1 \in \mathcal{D}(r^2)$.

We use Theorem 1 in Sec. 4 and 5 to lower bound sequential revenue.

We obtain a much weaker statement (proof deferred to Appendix B) than Theorem 1 for submodular valuations, dichotomy reinforced by Example 1.

Proposition 1 *For a* submodular *valuation v, if $S_1 \in \mathcal{D}(r^1), S_2 \in \mathcal{D}^{S_1}(r^1, r^2)$ and $S'_2 \in \mathcal{D}(r^2)$ then $S_2 \not\supset S'_2 \setminus S_1$ (note that equality is allowed).*

3.1 Gross substitutes

A valuation satisfies the *gross substitutes* [7] condition if raising prices on some items does not reduce the demand on the other items. More formally,

Definition 7 *A valuation v is* gross substitutes *(GS) if for any price vectors*[6] *$\mathbf{p}' \geq \mathbf{p}$, and any $A \in \mathcal{D}(\mathbf{p})$ there exists $A' \in \mathcal{D}(\mathbf{p}')$ with $A' \supseteq \{i \in A : p_i = p'_i\}$*

Remarkably [7], for any set of GS buyers, there exists a Walrasian (or competitive) equilibrium with one-shot item (possibly non-uniform) prices, i.e. a set of such prices at which buyers' preferred bundles form a partition of all items.

[6] We compare price vectors $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^n$ component-wise: $\mathbf{p}' \geq \mathbf{p} \iff p'_j \geq p_j\, \forall\, j = 1..n$.

Concave multi-unit demand valuations and unit demand valuations are GS. We review a more general GS valuation class and give an HM extension in Sec. C.

Denote v's *conditioning* on set S (measuring marginal value over S) by $v^S(A) = v(S \cup A) - v(S)$, $\forall\, A \subseteq 1..n \backslash S$. There is a more direct, valuation-based (as opposed to price-based as in Def. 7) characterization of GS valuations.

Proposition 2 *[11] v is gross substitutes if and only if v is submodular*[7] *and*

$$\forall \text{ items } a, b, c, \text{set } S, v^S(ab) + v^S(c) \le \max\{v^S(ac) + v^S(b), v^S(bc) + v^S(a)\} \quad (3)$$

i.e. no unique maximizer among $v^S(ab) + v^S(c), v^S(ac) + v^S(b), v^S(bc) + v^S(a)$.

Theorem 2 is established[8] by Bertelsen [3]. Starting, like [3], from Eq. (3), we provide in Appendix A a simpler proof via a basic graph-theoretic fact.

Theorem 2 *A GS valuation v has S_0-hereditary maximizers for any bundle S_0.*

4 Revenue approximation for independent HM valuations

We now exploit Theorem 1's quantity guarantees for revenue maximization.

Algorithm 1 below leads (proved in Theorem 3) to a $O(\frac{\log(mn)}{\log k})$-revenue approximation. It returns a decreasing list of $q_{\cdot}$ prices whose *indices* lie within an interval (initially $[1, \log(2mn)]$) that is reduced by at least half at each iteration. Each execution of $\mathbb{A}(\underline{l}, \overline{l}, \cdot)$ adds three prices ($q_{\frac{l+\underline{l}}{2}}, q_l, q_{\frac{l+\overline{l}}{2}}$ with l uniform in $[\underline{l}, \overline{l}]$) to the list and recurses on the four subintervals determined by them, $\underline{l}$, and $\overline{l}$.

Algorithm 1 $\mathbb{A}(\underline{l}, \overline{l}, M')$ returns a list of $4^{M'} - 1$ prices of the form $q_l = H/2^{l-1}$ (with $l \in [\underline{l}, \overline{l}]$) in decreasing order.

---

if $M' \le 0$ **or** $\underline{l} > \overline{l}$ **then return** $\emptyset$
Choose l uniformly at random between $\underline{l}$ and $\overline{l}$
return $\mathbb{A}(\frac{l+\overline{l}}{2} + 1, \overline{l}, M' - 1) \cup \{q_{\frac{l+\overline{l}}{2}}\} \cup \mathbb{A}(l + 1, \frac{l+\overline{l}}{2} - 1, M' - 1) \cup \{q_l\} \cup \mathbb{A}(\frac{l+\underline{l}}{2} + 1, l - 1, M' - 1) \cup \{q_{\frac{l+\underline{l}}{2}}\} \cup \mathbb{A}(\underline{l}, \frac{l+\underline{l}}{2} - 1, M' - 1)$

We obtain the main result of this section, the desired quality of approximation of Algorithm 1's randomized price schedule. In particular we prove, leveraging the consistency property (Theorem 1) and Lemma 6 below, that a pricing scheme $\mathcal{P}$'s revenue amounts to at least the part of A_F covered by $\mathcal{P}$. We relate the bundles bought at each price separately and the bundles bought sequentially.

---

[7] A GS valuation may not be monotone, i.e. may not satisfy free disposal.

[8] Bertelsen provides a stronger result: for a GS valuation v, the collection of sets $\mathcal{M}^v_{S_0,j}$ for all j is a *greedoid*, i.e. a collection $\mathcal{F}$ of subsets of $1..n$ that is accessible (i.e. $\forall\, S \in \mathcal{F}, \exists x \in S$ with $S \backslash \{x\} \in \mathcal{F}$) and satisfies the augmentation property ($\forall\, S, S' \in \mathcal{F}$ with $|S| < |S'|$, $\exists x \in S' \backslash S$ with $S \cup \{x\} \in \mathcal{F}$). Clearly, the maximum of an HM valuation without ties can be found efficiently via a greedy algorithm – Bertelsen provides such an algorithm for a GS valuation that can also handle ties.

Theorem 3 *The expected revenue from offering to m forward-myopic customers with HM valuations with maxima $H_1 \ldots H_m$ (denote $H = \max_{i \in 1..m} H_i$) the k prices output by $\mathbb{A}(1, \log(2mn), \log_4(k+1))$ for H is $\Omega(\frac{\log k}{\log(mn)}) \sum_{i \in 1..m} H_i$.*

Proof. We proceed with one customer; linearity of expectation yields the claim.

Let $r^1 \geq \ldots \geq r^k$ be prices output by $\mathbb{A}(1, \log(2mn), \log_4(k+1))$ for H. Let set $S'_d \in \mathcal{D}^{S'_1, \ldots, S'_{d-1}}(r^1 \ldots r^d)$ be bought in day d. By Theorem 1, $\sum_{i=1}^{d} |S'_i| \geq F(r^d)$.

Via Lemma 6 below with $d_0 = 1, d = k, q^\delta = r^\delta$ and $x^\delta = F(r^\delta)$, revenue is at least $\sum_{\delta=1}^{d} r^\delta (F(r^\delta) - F(r^{\delta-1}))$, i.e. the area covered by Algorithm 1 with prices $r^1, \ldots, r^d$. The quality of approximation follows via Theorem 4 below. □

The proof structure follows that of Lemma 2.5 in [1], but with key differences: we assume that there is more than one item for sale and that the seller has no knowledge other than H (as opposed to distributional) of the valuation. Our approximation holds in expectation as opposed to [1]'s deterministic bound.

Theorem 4 $\mathbb{A}(1, \log(2mn), \log_4(k+1))$ *covers an $\Omega(\frac{\log k}{\log(mn)})$ fraction of* A_F.

Proof. We will aim to cover instead the area H' under the q-demand curve corresponding to q prices of at least $H/(2mn)$, which is, by standard facts (e.g. [2]), at least a constant fraction of H.

Let $k = 4^M - 1$. Denote by $A_{M'}$ the area newly covered in Algorithm 1 at iteration M', i.e. in all calls of $\mathbb{A}(\underline{l}, \overline{l}, M + 1 - M')$.

Following [1], we show by induction on M (clearly true for $M = 0$) that

$$\mathbb{E}_{1..M}[\textstyle\sum_{M'=1}^{M} A_{M'}] \geq \frac{M}{\log mn} H' \tag{4}$$

Consider a piece P of the q-demand curve, delimited by prices $\underline{q} = 2^{\underline{l}}$ and $\overline{q} = 2^{\overline{l}}$, corresponding to a call of $\mathbb{A}(\underline{l}, \overline{l}, M+1-M')$. Since $\mathbb{A}$ recurses on subintervals determined by midpoints, P's range of q-prices is bounded: $\overline{q}/\underline{q} \leq 2mn/2^{M'}$. By a standard result (see e.g. [1], Lemma 2.4 or [2], Theorem 9) $\textsc{Random}^{\overline{q}}_{\overline{l}-\underline{l}}$ covers, in expectation, at least an a fraction of P's area, with $a = \log(2mn/2^{M'}) = \log(mn) - (M'-1)$.

Iteration M covers at least a $\log mn - (M-1)$ fraction of each piece i.e. in all

$$\mathbb{E}_{1..M}[A_M] = \mathbb{E}_{1..M-1}[\mathbb{E}[A_M | A_1, \ldots, A_{M-1}]] \geq \frac{\mathbb{E}[H' - \sum_{M'=1}^{M-1} A_{M'}]}{\log mn - (M-1)}$$

$$\begin{aligned}
\mathbb{E}_{1..M}[\textstyle\sum_{M'=1}^{M} A_{M'}] &\geq \frac{\mathbb{E}_{1..M-1}[H']}{\log mn - (M-1)} + (1 - \frac{1}{\log mn - (M-1)}) \mathbb{E}_{1..M-1}[\textstyle\sum_{M'=1}^{M-1} A_{M'}] \\
&\geq \frac{H'}{\log mn - (M-1)} + \left(1 - \frac{1}{\log mn - (M-1)}\right) \frac{M-1}{\log mn} H' \qquad (5) \\
&= \frac{H'}{\log mn - (M-1)} \left(1 + \frac{(M-1)(\log mn - M)}{\log mn}\right) \\
&= \frac{H'}{\log mn - (M-1)} \frac{M \log mn - M(M-1)}{\log mn} = \frac{M}{\log mn} H'
\end{aligned}$$

where Eq. (5) follows from the inductive hypothesis. □

Finally, we lower bound revenue given a lower bound on total quantities bought.

Lemma 6 *If, at prices $q^{d_0} > \ldots > q^d$, at least x^δ items ($x^d \geq x^{d-1} \geq \ldots \geq x^{d_0} \geq x^{d_0-1} = 0$) are sold in total up to each day $\delta = d_0..d$ (e.g. at least x^{d_0+1} items in days d_0 and d_0+1 together) then the revenue is at least $\sum_{\delta=d_0}^{d} q^\delta(x^\delta - x^{\delta-1})$.*

Proof. The lowest revenue is obtained when *exactly* x^δ items are sold in day $\delta = d_0..d$ and (as prices are decreasing) when as few items as possible are sold in early days, i.e. $x^\delta - x^{\delta-1}$ items are sold in day $\delta = d_0..d$, yielding the claim. $\square$

As in [2], approximation factors can be improved to $\Omega(\frac{\log k}{\log n})$ instead of $\Omega(\frac{\log k}{\log mn})$ if $H_i = H$, $\forall\, i \in 1..m$ by using $\mathbb{A}(1, \log(2n), \log_4(k+1))$.

We will henceforth allow a buyer's valuation (for any bundle) to be influenced by the bundles owned by others (but not by others' respective values).

5 Positive allocative externalities

We now investigate revenue maximization in the presence of positive externalities, i.e. a buyer's valuation being increased by other buyers' ownership of certain items. Such influences can be subjective, e.g. resulting from peer pressure, or objective, e.g. resulting from ownership of a certain social network application. We define a new influence model via a predicate $\mathcal{I} : 1..m \to \{\texttt{false}, \texttt{true}\}$ such that $\mathcal{I}(i_0)$ only depends on seller's assignment of items to buyer i_0, e.g.

- $\mathcal{I}(i_0) = \texttt{true}$ iff buyer i_0 owns all (or, instead, at least two) items
- $\mathcal{I}(i_0) = \texttt{true}$ iff buyer i_0 owns his preferred bundle at current prices

Let $\mathcal{I}_d$ be the buyers i_0 satisfying $\mathcal{I}(i_0)$ before day d. $\mathcal{I}$ is *monotone* if $\mathcal{I}_d \subseteq \mathcal{I}_{d+1}$.

We model the valuation in day d of a buyer i as a linear mapping (depending on d only through its argument $\mathcal{I}_d \setminus \{i\}$) of i's base value

$$v_i^d(S|1..m\setminus\{i\}) = (a_i(\mathcal{I}_d\setminus\{i\})v_i(S)) \oplus b_i(\mathcal{I}_d\setminus\{i\}),\ \forall \text{ set } S \subseteq 1..n \qquad (6)$$

where $\alpha v_i(S) \oplus \beta = \{\alpha v_i(S) \text{ if } S = \emptyset \text{ and } \alpha v_i(S) + \beta \text{ if } S \neq \emptyset\}$ for $\alpha, \beta \in \mathbb{R}$.

Thus, $a_i(I)$ and $b_i(I)$ measure the multiplicative and additive influences that a buyer set I (satisfying $\mathcal{I}$) have on buyer i. Say i's value for any DVD of a TV series doubles as soon as one other friend (in a set F_i) has the entire series (the predicate $\mathcal{I}$) and is then constant. Then $a_i(I) = 2 \Longleftrightarrow |I \cap F_i| \geq 1$ and $b_i(I) = 0$.

Without any influence, a valuation reduces to the base value: $a_i(\emptyset) = 1, b_i(\emptyset) = 0$. Assume a_i and b_i are non-negative, monotone and submodular[9]. Also assume that a_i, b_i, v_i are bounded: $\max_{I \subseteq 1..m\setminus\{i\}} a_i(I) = a_i(1..m\setminus\{i\}) = H^a$, $\max_I b_i(I) = b_i(1..m\setminus\{i\}) = H^b$, $\max_{S \subseteq 1..n} v_i(S) = H_i$ with $\max_{i \in 1..m} H_i = H$.

Our influence model is a distribution-free extension of single-item models [1, 8]. It does not require or preclude symmetry, anonymity or a neighbor graph.

[9] Submodularity (non-increasing marginal influence) is a common assumption on externalities [1, 8]. Positive, monotone externalities are an instance of "herd mentality".

For $a_i = 1, b_i = 0$ we recover the model before this section. Buyers are still forward-myopic and do not strategize about which items to buy today so that other buyers' values increase, thus increasing their own value etc.

The additive increase is excluded in Eq. (6) for $S = \emptyset$ for two reasons. First, the intuitive requirement $v_i^d(\emptyset|\cdot) = 0$. Second, if b_i is much larger than $a_i \max_S v_i(S)$ then a multiplicative revenue approximation guarantee is impossible: prices close to b_i are needed, rendering $\emptyset$ the preferred set, i.e. zero revenue.

With positive externalities, a natural revenue maximization approach [8] is to provide certain items for free to some buyers and then charge others accordingly.

Definition 8 *The influence-and-exploit IE_k marketing strategy for $k \geq 2$ satisfies $\mathcal{I}$ (at no cost) for each buyer with probability 0.5, in day 1. Let A_1 be the set of buyers chosen in day 1: $\mathbb{E}[|A_1|] = 0.5m$. Independently of A_1, a price schedule $r^1 > r^2 > \cdots > r^{k-1}$ is constructed in $\mathbb{A}(1, \log(2mn), \log_4(k+1))$ for $H + \frac{H^b/3}{H^a/3}$.*

Each buyer $i \notin A_1$ is offered uniform item price $H^a/3 \cdot r^{d-1}$ in day $d \geq 2$. Let $\varphi_a^d(i) = a_i(\mathcal{I}_d \backslash \{i\}), \varphi_b^d(i) = b_i(\mathcal{I}_d \backslash \{i\})$ be random variables for the influence on buyer $i \in 1..m \backslash A_1$ just before day $d \in 1..k$ in IE_k. Thus $\varphi_a^1(i) = 1, \varphi_b^1(i) = 0$.

This section's main result (proved at its end), is that Theorem 3's approximation factor carries over to externalities, despite the affine increase in the optimum.

Theorem 5 *The IE_k strategy obtains an $O(\frac{\log mn}{\log k})$-revenue approximation to the optimal marketing strategy for a monotone $\mathcal{I}$ over IE_k and HM base valuations.*

The price schedule $r^1 > \ldots > r^{k-1}$ offers (by Theorem 3) a $\frac{\log mn}{\log k}$-revenue approximation given buyers' base valuations (translated by $\frac{H^b/3}{H^a/3}$). We establish that the influence of other buyers (an affine mapping of a buyer's value in each day) does not result in fewer items being bought in the worst-case.

By Lemma 2.1 in [8] (see Appendix D), as a_i, b_i are submodular and monotone and $\mathcal{I}$ is monotone, the *expected* influence on a buyer i after the day 1 give-away in IE_k is at least half the maximum influence.

Lemma 7 $\forall\, i \in 1..m$, $\mathbb{E}[\varphi_a^2(i)] \geq 0.5H^a$ *and* $\mathbb{E}[\varphi_b^2(i)] \geq 0.5H^b$.

We use this bound on the expected influence experienced by any buyer to lower bound the probability of being significantly influenced

Lemma 8 $\mathbb{P}[\varphi_a^2(i) \geq H^a/3] \geq \frac{1}{4}$ *and* $\mathbb{P}[\varphi_b^2(i) \geq H^b/3] \geq \frac{1}{4}$.

Proof. We only provide the proof for $\varphi_a^2(i)$ – the one for $\varphi_b^2(i)$ is similar.

Let $x = \mathbb{P}[\varphi_a^2(i) \geq H^a/3]$. We have $\mathbb{E}[\varphi_a^2(i)] \leq xH^a + (1-x)H^a/3$.

The claim follows via simple algebra from $\mathbb{E}[\varphi_a^2(i)] \geq \frac{H^a}{2}$ (Lemma 7).

As utilities are quasilinear, the lowest quantity bought can only decrease if the price is scaled by $\alpha \geq 1$, but stays constant if the valuation is also scaled.

Lemma 9 *For $\alpha \geq 1$, $\mathcal{D}_{\alpha v}(p) = \mathcal{D}_v(\frac{p}{\alpha})$ and $F_v(p) \leq F_{\alpha v \oplus x}(p), \forall x \geq 0$.*

We defer the proof of the inequalities concerning $F(p)$ to Appendix D.

Lemma 10 below parallels Theorem 1, offering a guarantee on the least number of items bought by a buyer outside A_1 from day 2 up to a given day d.

Lemma 10 *Fix buyer $i \in 1..m$ with $\varphi_a^2(i) \geq H^a/3, \varphi_b^2(i) \geq H^b/3$. Consider a price schedule $r^1 > \ldots > r^{k-1}$ as in Def. 8 and fix day $d \in 2..k$. Assume that $\mathcal{I}$ is monotone and each buyer's base valuation has hereditary maximizers.*

Let set $S_d \in \mathcal{D}^{S_2,\ldots,S_{d-1}}_{\varphi_a^d(i)v_i \oplus \varphi_b^d(i)}(H^a/3 \cdot r^{1..d-1})$ be preferred in day d in IE_k given (influenced) valuation $\varphi_a^d(i)v_i \oplus \varphi_b^d(i)$ and previously bought bundles $S_2,\ldots,S_{d-1}$:

$$S_d \in \operatorname*{argmax}_{S \cap \bigcup_{\delta=2}^{d-1} S_\delta = \emptyset} \left\{ \varphi_a^d(i) v_i(S \cup \bigcup_{\delta=2}^{d-1} S_\delta) \oplus \varphi_b^d(i) - H^a/3 \cdot r^{d-1}|S| - \sum_{\delta=2}^{d-1} H^a/3 \cdot r^{\delta-1}|S_\delta| \right\}$$

Then $\sum_{\delta=2}^{d} |S_\delta| \geq F_{v_i \oplus \frac{H^b/3}{H^a/3}}(r^{d-1})$.

Proof. Assume wlog that the customer makes the first purchase in day $2 : S_2 \neq \emptyset$.

By Lemma 9 ($\alpha = \frac{\varphi_a^2(i)}{H^a/3} \geq 1$ and $x = \frac{\varphi_b^2(i)}{H^a/3}$), $S_2 \in \mathcal{D}_{\varphi_a^2(i)v_i \oplus \varphi_b^2(i)}\left(H^a/3 \cdot r^1\right) = \mathcal{D}_{\frac{\varphi_a^2(i)}{H^a/3} v_i \oplus \frac{\varphi_b^2(i)}{H^a/3}}(r^1)$ and $|S_2| \geq F_{v_i \oplus \frac{\varphi_b^2(i)}{H^a/3}}(r^1) \geq F_{v_i \oplus \frac{H^b/3}{H^a/3}}(r^1)$. Lemma 9 ($x = \frac{\varphi_b^2(i) - H^b/3}{H^a/3}$) implies the second inequality (holding with equality unless $F_{v_i \oplus \frac{H^b/3}{H^a/3}}(r^1) = 0$).

In day $d > 2$, $\bigcup_{\delta=2}^{d-1} S_\delta \neq \emptyset$; thus $\varphi_b^d(i)$ is added to $\varphi_a^d(i)v_i(S \cup \bigcup_{\delta=2}^{d-1} S_\delta)$ in the argmax. We get $S_d \in \mathcal{D}^{S_2,\ldots,S_{d-1}}_{v_i \oplus \frac{\varphi_b^d(i)}{\varphi_a^d(i)}}\left(\frac{H^a/3}{\varphi_a^d(i)} r^1, \ldots, \frac{H^a/3}{\varphi_a^d(i)} r^{d-2}, \frac{H^a/3}{\varphi_a^d(i)} r^{d-1}\right)$ which equals $\mathcal{D}^{S_2,\ldots,S_{d-1}}_{v_i \oplus \frac{H^b/3}{H^a/3}}\left(\frac{H^a/3}{\varphi_a^2(i)} r^1, \ldots, \frac{H^a/3}{\varphi_a^{d-1}(i)} r^{d-2}, \frac{H^a/3}{\varphi_a^d(i)} r^{d-1}\right)$ since the current preferred set (see Eq. (2)) is invariant to additions of scalars and to modifications of earlier prices (but not to current price, i.e. $\frac{H^a/3}{\varphi_a^d(i)} r^{d-1}$) given at least one earlier purchase.

Clearly, since v_i has hereditary maximizers, so does $v_i \oplus x$. Theorem 1 for prices $\frac{H^a/3}{\varphi_a^2(i)} r^1, \ldots, \frac{H^a/3}{\varphi_a^{d-1}(i)} r^{d-2}, \frac{H^a/3}{\varphi_a^d(i)} r^{d-1}$ yields $\sum_{\delta=2}^{d} |S_\delta| \geq F_{v_i \oplus \frac{H^b/3}{H^a/3}}(\frac{H^a/3}{\varphi_a^d(i)} r^{d-1})$.

As $\mathcal{I}$, a_i and b_i are monotone, $\varphi_a^d(i) \geq \varphi_a^2(i) \geq H^a/3$.

By Lemma 9 for $\alpha = \frac{\varphi_a^d(i)}{H^a/3} \geq 1$, $F_{v_i \oplus \frac{H^b/3}{H^a/3}}(\frac{H^a/3}{\varphi_a^d(i)} r^{d-1}) \geq F_{v_i \oplus \frac{H^b/3}{H^a/3}}(r^{d-1})$.

Finally, we prove Theorem 5, i.e. IE_k's $O(\frac{\log mn}{\log k})$-revenue approximation

Proof. By Lemma 8, $\varphi_a^2(i) \geq H^a/3$ for a constant fraction of buyers outside A_1. By Lemma 10, for any day $d = 2..k$, each such buyer $i \in 1..m \backslash A_1$ buys at least $F_{v_i \oplus \frac{H^b/3}{H^a/3}}(r^{d-1})$ items in total in days $2..d$ at prices $H^a/3 \cdot r^1, .., H^a/3 \cdot r^{d-1}$.

For $q^\delta = H^a/3 \cdot r^{\delta-1}$ and $x^\delta = F_{v_i \oplus \frac{H^b/3}{H^a/3}}(r^{\delta-1})$ for $\delta = 2..k$ in Lemma 6 we get that buyer i pays at least

$$\sum_{\delta=2}^{k} H^a/3 \cdot r^{\delta-1}(F_{v_i \oplus \frac{H^b/3}{H^a/3}}(r^\delta) - F_{v_i \oplus \frac{H^b/3}{H^a/3}}(r^{\delta-1})) \geq H^a/3 \cdot \Omega(\frac{\log k}{\log mn})(H_i + \frac{H^b/3}{H^a/3})$$

by Theorem 3. The approximation ratio follows after noting that the optimal marketing strategy can yield revenue at most $\sum_{i \in 1..m}(H_i H^a + H^b)$.

6 Conclusions and future directions

We have studied revenue maximization with equal item prices. We have presented a sequential pricing scheme that improves the revenue approximation factor of an existing one-shot pricing scheme. We have also considered a setting with positive, non-anonymous, allocative externalities and presented a simple marketing strategy preserving the approximation factor without externalities, despite an increase in revenue available due to influences among buyers. Several open directions look promising to us.

Hereditary maximizers guarantees consistency of bundles bought sequentially. We deem it of interest to find a less restrictive assumption, related to sequential revenue instead, that still allows revenue approximation guarantees.

Within a day, buyers have no profitable manipulations since prices are uniform. An appealing additional consideration is that of the incentive properties of our scheme for buyers whose value for an acquired item changes between days.

The widespread presence of externalities in applications leaves many exciting open questions, both practical and theoretical, notably in multiple-item settings.
Acknowledgments. We thank Avrim Blum for detailed comments on an earlier draft of this paper, Mark Braverman and Yuanchuan Lien for helpful discussions and Daniel Lehmann for providing us with a copy of [3].

References

1. H. Akhlaghpour, M. Ghodsi, N. Haghpanah, H. Mahini, V. Mirrokni, and A. Nikzad. Optimal iterative pricing over social networks, 2009. Fifth Workshop on Ad Auctions.
2. M.-F. Balcan, A. Blum, and Y. Mansour. Item pricing for revenue maximization. In *EC '08: Proceedings of the 9th ACM conference on Electronic Commerce*, pages 50–59, 2008.
3. A. Bertelsen. Substitutes valuations and $M^\natural$-concavity. Master's thesis, Hebrew University of Jerusalem, 2004.
4. S. Bikhchandani and J. M. Ostroy. Ascending price vickrey auctions. *Games and Economic Behavior*, 55(2):215–241, May 2006.
5. M. Bing, D. Lehmann, and P. Milgrom. Presentation and structure of substitutes valuations. In *EC '04: Proceedings of the 5th ACM conference on Electronic Commerce*, 2004.
6. A. Fiat and A. Wingarten. Envy, multi envy, and revenue maximization. In *Workshop on Internet and Network Economics*, 2009.
7. F. Gul and E. Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 87(1):95–124, 1999.
8. J. Hartline, V. Mirrokni, and M. Sundararajan. Optimal marketing over social networks. In *Conference on the World Wide Web (WWW08)*, 2008.
9. P. Jehiel and B. Moldovanu. Allocative and informational externalities in auctions and related mechanisms. In R. Blundell, W. K. Newey, and T. Persson, editors, *Advances in economics and econometrics: theory and applications*. 2006.
10. J. Kleinberg. *Cascading behavior in networks: algorithmic and economic issues.* Cambridge University Press, 2007.
11. Y. Lien and J. Yan. On the gross substitutes condition. Working paper, Jul 2007.

A Alternate proof of Theorem 2

Proof. We fix base bundle S_0 and drop it from $\mathcal{M}_j$ for notational simplicity. Suppose towards a contradiction that for some $S_j \in \mathcal{M}_j$ no $S_{j+1} \in \mathcal{M}_{j+1}$ contains it and choose S_{j+1} with lowest $|S_j \setminus S_{j+1}| + |S_{j+1} \setminus S_j|$.

Let $v_\cap^B(S) = v^{(S_j \cap S_{j+1}) \cup B}(S)$ (simply $v_\cap$ if $B = \emptyset$).

Assume that

$$\max_{x \in S_j \setminus S_{j+1}} v_\cap(\{x\}) \geq \max_{x \in S_{j+1} \setminus S_j} v_\cap(\{x\}) \tag{7}$$

and let $c_1 \in \operatorname{argmax}_{x \in S_j \setminus S_{j+1}} v_\cap(\{x\})$.

We prove inductively that for any L distinct items $a^1 ... a^L \in S_{j+1} \setminus S_j$,

$$\exists l \in 1..L \text{ with } v_\cap(a^1, \ldots, a^L) \leq v_\cap(a^{-l}, c_1) \tag{8}$$

where a^{-l} denotes $a^1 ... a^{l-1} a^{l+1} ... a^L$. The theorem follows for $L_{21} = |S_{j+1} \setminus S_j|$: $v_\cap(S)$ is maximized by $S_{j+1} \setminus S_j$ among sets of size L_{21}. By Eq. (8), $(S_{j+1} \setminus \{a^l\}) \cup \{c_1\} \in \mathcal{M}_{j+1}$ and has strictly fewer elements in the symmetric difference with S_j than S_{j+1}, contradicting the choice of sets S_j and S_{j+1}.

As a base case $L = 1$, Eq. (8) holds by choice of c_1 as $l = 1$.

Assume that Eq. (8) holds for $L - 1$, and suppose it fails for $L \geq 2$.

We define a directed bipartite graph $G_{L-1, c_1 L-2}$ with vertices a^{-l} for $l = 1..L$ in one partition (that we call $\mathcal{P}_{L-1}$) and $c_1 a^{-h,l}$ for $1 \leq l < h \leq L$ in the other partition (that we call $\mathcal{P}_{c_1 L-2}$). Directed edge $a \rightarrow b$ in $G_{L-1, c_1 L-2}$ represents $v_\cap(a) \leq v_\cap(b)$, with strict inequality if $a \in \mathcal{P}_{c_1 L-2}$ and $b \in \mathcal{P}_{L-1}$.

We show that in $G_{L-1, c_1 L-2}$ each vertex has at least one outgoing edge, i.e. there exists a cycle of inequalities (at least half of them strict), contradiction. This claim holds for vertices in $\mathcal{P}_{L-1}$ by the inductive hypothesis.

Fix $1 \leq h < l \leq L$ and $c_1 a^{-h,l}$. The failure of Eq. (8) for L requires

$$v_\cap(a^1, ..., a^L) > \max\{v_\cap(a^{-l}, c_1), v_\cap(a^{-h}, c_1)\} \text{ i.e.} \tag{9}$$

$$v_\cap^{a^{-h,l}}(a^h, a^l) > \max\{v_\cap^{a^{-h,l}}(a^h, c_1), v_\cap^{a^{-h,l}}(a^l, c_1)\} \text{ implying, via Eq. (3)} \tag{10}$$

$$v_\cap^{a^{-h,l}}(c_1) < \max\{v_\cap^{a^{-h,l}}(a^h), v_\cap^{a^{-h,l}}(a^l)\} \text{ i.e.} \tag{11}$$

$$v_\cap(c_1 a^{-h,l}) < \max\{v_\cap(a^{-l}), v_\cap(a^{-h})\} \tag{12}$$

exhibiting one outgoing edge from $c_1 a^{-h,l}$, i.e. to a^{-l} or a^{-h}.

If Eq. (7) did not hold, then one can show as above, that for any L distinct items $b^1 ... b^L \in S_j \setminus S_{j+1}$, and $c_2 \in \operatorname{argmax}_{x \in S_{j+1} \setminus S_j} v_\cap(\{x\})$

$$\exists l \in 1..L \text{ with } v_\cap(b^1, \ldots, b^L) < v_\cap(b^{-l}, c_2) \tag{13}$$

For $L = |S_j \setminus S_{j+1}|$, this contradicts $S_j \in \mathcal{M}_j$.

A.1 Other structural properties

We discuss extensions to sets of the characterizing Eq. (3) of GS valuations.

Proposition 3 *Eq.* (3) *holds for a GS v and any* sets a,b, *item c and set S.*

Proof. Suppose towards a contradiction that for some sets a, b (assume wlog that $|a| \leq |b|$), item c and set S Eq. (3) were false and choose such b with minimum cardinality: note that $b \neq \emptyset$.

Let $y \in b$ and let $b' = b \setminus y$. By choice of b,

$$v^{Sy}(ab') + v^{Sy}(c) \leq \max\{v^{Sy}(ac) + v^{Sy}(b'), v^{Sy}(b'c) + v^{Sy}(a)\} \tag{14}$$

$v^{Sy}(ab') + v^{Sy}(c) \leq v^{Sy}(ac) + v^{Sy}(b')$ would contradict v's submodularity as

$$v^{Sy}(ac) - v^{Sy}(c) \geq v^{Sy}(ab') - v^{Sy}(b') > v^{S}(ac) - v^{S}(c) \tag{15}$$

Thus[10] $v^{Sy}(b'c) + v^{Sy}(a) \geq v^{Sy}(ab') + v^{Sy}(c) > v^{Sy}(ac) + v^{Sy}(b')$. If $b' = \emptyset$ then v's submodularity would be violated. Thus $|b| \geq 2$ and

$$v^{S}(c) - v^{S}(a) > v^{S}(bc) - v^{S}(ab) \geq v^{Sy}(c) - v^{Sy}(a) \text{ i.e.} \tag{16}$$

$$v^{S}(ay) + v^{S}(c) > v^{S}(cy) + v^{S}(a), \text{ implying} \tag{17}$$

$$v^{S}(ay) + v^{S}(c) = v^{S}(ac) + v^{S}(y) \tag{18}$$

where the equality holds since y, c are items and b was chosen of minimum cardinality among sets a, b with $|a| \geq |b|$ violating Eq. (3). We have

$$v^{S}(ab) - v^{S}(b) > v^{S}(ac) - v^{S}(c) = v^{S}(ay) - v^{S}(y) \tag{19}$$

violating v's submodularity ($y \in b$), i.e. marginal decreasing value of a.

Note that unless a or b are also items, Prop. 3 does not imply equality of the two highest quantities in Eq. (3).

Eq. (3) does not hold for sets in general. For example, for a concave multi-unit valuation it does not hold for $a = 1, b = 1, c = 2$ copies of an item. Eq. (3) does not hold even if a, b, c have the same cardinality. Consider 4 items of type X and 2 of type Y, with valuation $v(\alpha X + \beta Y) = \min(\alpha, 2)$, clearly GS. However, $v(X_1Y_1X_2Y_2) + v(X_3X_4) = 4 > 3 = v(X_iY_iX_3X_4) + v(X_{3-i}Y_{3-i})$, for $i = 1, 2$. One can show however that $v(123) + v(4) < \max\{v(234) + v(1), v(124) + v(3), v(134) + v(2)\}$.

B Proof of Proposition 1

By optimality of S_2 after having bought S_1,

$$v(S_1 \cup S_2) - r^1|S_1| - r^2|S_2| \geq v(S_1 \cup S) - r^1|S_1| - r^2|S|, \forall S \subseteq \{1..n\} \setminus S_1$$

$$\text{i.e. } v(S_1 \cup S_2) - r^2|S_2| \geq v(S_1 \cup (S_2' \setminus S_1)) - r^2|S_2' \setminus S_1| \text{ for } S = S_2' \setminus S_1$$

[10] Cannot assume equality of LHS with RHS2 in Eq. (14) unless a is also an item.

By moving $r^2|S_2'\setminus S_1|$ to the left-hand side and subtracting $r^2|S_2'|$ we get,

$$v(S_1 \cup S_2) - r^2(|S_2| - |S_2'\setminus S_1| + |S_2'|) \geq v(S_1 \cup S_2') - r^2|S_2'|. \tag{20}$$

By optimality of S_2' for price r^2 and $S_2 \cap (S_2' \cap S_1) = \emptyset$ (as $S_2 \cap S_1 = \emptyset$),

$$v(S_2') - r^2|S_2'| \geq v(S_2 \cup (S_2' \cap S_1)) - r^2(|S_2| + |S_2' \cap S_1|) \tag{21}$$

Adding Eqs. (20) and (21) and canceling r^2 terms ($|S_2'| - |S_2'\setminus S_1| = |S_2' \cap S_1|$),

$$v(S_1 \cup S_2) + v(S_2') > v(S_1 \cup S_2') + v(S_2 \cup (S_2' \cap S_1)) \text{ i.e.} \tag{22}$$

$$v(S_2 \cup S_1) - v(S_2 \cup (S_2' \cap S_1)) > v((S_2'\setminus S_1) \cup S_1) - v(S_2') \tag{23}$$

$S_2 \not\supset S_2'\setminus S_1$ follows from $(S_2'\setminus S_1) \cup (S_2' \cap S_1) = S_2'$ and v's submodularity.

C Hierarchies and a class of HM, but not GS, valuations

We now define a set-based hierarchy: an example in the movie-based setting is {comedies, romantic comedies, dramas}. In this particular set system, larger (with respect to inclusion) sets are higher in the hierarchy.

Definition 9 *Let $n' \in 1..n$. A collection $\mathcal{H}$ of subsets of $1..n'$ is an n'-*hierarchy *if for any two sets H_1, H_2 in $\mathcal{H}$, $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$ or $H_1 \cap H_2 = \emptyset$. Any n'-hierarchy is an $n'+1$-hierarchy; an n-hierarchy is simply called a hierarchy.*

A hierarchy is also known as a *laminar family*.

Hierarchies are closely related to GS valuations. Bing et al. [5] define two non-disjoint GS classes based on hierarchies, including the following.

Definition 10 *Given a n'-hierarchy $\mathcal{H}$, an* $\mathrm{S}_{n'}$-presentation *is a valuation with*

$$v_{\mathcal{H}}(S) = \sum_{H \in \mathcal{H}} c_H(|H \cap S|), \forall S \subseteq 1..n \tag{24}$$

where c_H is concave and increasing for all $H \in \mathcal{H}$.

That is, $v_{\mathcal{H}}$ is additively separable into concave, subset-dependent symmetric (among common elements of S and H) functions c_H. For example, a subset of movies may have a value increasing linearly in the number of dramas and comedies and additionally at a sub-linear rate in the number of romantic comedies.

As shown by [5, 11],

Lemma 11 *Any S-presentation is GS, i.e. it satisfies Eq.* (3).

By Theorem 2 we get that S-presentations are HM.

We introduce a generalization of S-presentations that preserves the HM property, but in general not the GS property. In our movie setting, the items $n'+1, \ldots, n$ that are singled out can, for example, be seasons of TV shows, which have elements of each genre.

Definition 11 *Given a n'-hierarchy $\mathcal{H}$, an* $\overline{\mathrm{S}}_{n'}$-presentation *is a valuation with*

$$v_{\mathcal{H}}(S) = \sum_{H\in\mathcal{H}} \left\{ c_H(|H\cap S|) + c_H\left(|H\cap (S\cup\{n'+1,\ldots,n\})|\right)\right\}, \forall\, S\subseteq 1..n \quad (25)$$

where c_H is concave and increasing for all $H\in\mathcal{H}$.

Any S-presentation corresponding to an n'-hierarchy $\mathcal{H}$ is an $\overline{\mathrm{S}}_n$-presentation.

Any such valuation is HM, as stated in Prop. 4 below, but it may not be GS, as shown in Example 2 below.

Proposition 4 *Any $\overline{\mathrm{S}}_{n'}$-presentation is HM.*

Proof. A's conditioning (marginal value) on S is $v^S_{\mathcal{H}}(A) = v_{\mathcal{H}}(S\cup A) - v_{\mathcal{H}}(S)$.

It is enough to show that for any items $a, b, c,$ set S:

$$v^S_{\mathcal{H}}(a) < \max\{v^S_{\mathcal{H}}(b), v^S_{\mathcal{H}}(c)\} \;\text{ or }\; v^S_{\mathcal{H}}(bc) \le \max\{v^S_{\mathcal{H}}(ac), v^S_{\mathcal{H}}(ab)\} \text{ i.e.} \quad (26)$$

$$v_{\mathcal{H}}(S\cup a) < \max\{v_{\mathcal{H}}(S\cup b), v_{\mathcal{H}}(S\cup c)\} \text{ or } v_{\mathcal{H}}(S\cup bc) \le \max\{v_{\mathcal{H}}(S\cup ac), v_{\mathcal{H}}(S\cup ab)\} \quad (27)$$

Note that Eq. (3) implies Eq. (26): if the latter did not hold, then, by adding its two alternatives, Eq. (3) would be contradicted.

Any $H\in\mathcal{H}$ with $H\cap\{a,b,c\}=\emptyset$ can be dropped ($\mathcal{H}$ remains a hierarchy) since it has no marginal contribution in any quantity in Eq. (26).

If $a, b, c\in n'+1,\ldots,n$ then Eq. (26) holds: a, b, c are treated equally by $v_{\mathcal{H}}$.

If $a, b, c\notin n'+1,\ldots,n$ then Eq. (26) holds for $v_{\mathcal{H}}$ via Eq. (3):

$$v^S_{\mathcal{H}}(A) = 2\sum_{H\in\mathcal{H}} \left\{c_H(|H\cap(S\cup A)|) - c_H(|H\cap S|)\right\}, \forall\, A\subseteq\{a,b,c\}$$

which is the marginal value of an S-presentation defined as $2\sum_{H\in\mathcal{H}} c_H(|H\cap S|)$.

Suppose $|\{a,b,c\}\cap\{n'+1,\ldots,n\}| = 2$. If the first alternative in Eq. (26) did not hold (i.e. $v^S_{\mathcal{H}}(a) \ge \max\{v^S_{\mathcal{H}}(b), v^S_{\mathcal{H}}(c)\}$) then $\{a,b,c\}\cap\{n'+1,\ldots,n\} = \{b,c\}$. But then for any $H\in\mathcal{H}$ (H can only contain a out of $\{a,b,c\}$), $c_H(|H\cap(S\cup bc)|) = c_H(|H\cap S|) \le c_H(|H\cap(S\cup ab)|) = c_H(|H\cap(S\cup ac)|) \in \{c_H(|H\cap S|), c_H(1+|H\cap S|)\}$. Thus Eq. (26)'s second alternative must hold in this case.

We are left with the case $|\{a,b,c\}\cap\{n'+1,\ldots,n\}| = 1$. If $\{a,b,c\}\cap\{n'+1,\ldots,n\} = \{a\}$ then $c_H(|H\cap(S\cup bc)|) \le c_H(|H\cap(S\cup ab)|) = c_H(|H\cap(S\cup ac)|)$. We can assume wlog that $\{a,b,c\}\cap\{n'+1,\ldots,n\} = \{c\}$.

Let $\eta_H = |H\cap S|$, $\forall\, H\in\mathcal{H}$. For $A\subseteq\{a,b,c\}$, consider

$$\text{valuation } \sigma_A(S) = \textstyle\sum_{H\cap\{a,b\}=A\setminus\{c\}} c_H(|H\cap(S\cup A)|) \text{ thus} \quad (28)$$

$$v_{\mathcal{H}}(S) = \sigma_{ab}(S) + \sigma_a(S) + \sigma_b(S) + \sigma_{abc}(S) + \sigma_{ac}(S) + \sigma_{bc}(S)$$

$$\tau_A : 0..n\to\mathbb{R}, \tau_A(x) = \textstyle\sum_{H\cap\{a,b\}=A\setminus\{c\}} c_H(x+\eta_H) \quad (29)$$

Assume $c_H(\eta_H) = 0$ i.e. $\tau_A(0) = 0$ by subtracting $c_H(\eta_H)$ from any c_H. Thus

$v_{\mathcal{H}}(S \cup a) = 2\tau_{ab}(1)+2\tau_a(1)$	$v_{\mathcal{H}}(S \cup bc) = \tau_{ab}(1)+\tau_b(1)+\tau_{ab}(2)+\tau_a(1)+\tau_b(2)$
$v_{\mathcal{H}}(S \cup b) = 2\tau_{ab}(1)+2\tau_b(1)$	$v_{\mathcal{H}}(S \cup ac) = \tau_{ab}(1)+\tau_a(1)+\tau_{ab}(2)+\tau_a(2)+\tau_b(1)$
$v_{\mathcal{H}}(S \cup c) = \tau_{ab}(1)+\tau_a(1)+\tau_b(1)$	$v_{\mathcal{H}}(S \cup ab) = 2\tau_{ab}(2)+2\tau_a(2)+2\tau_b(2)$

Note that $2v_{\mathcal{H}}(S \cup c) + 2\tau_{ab}(1) = v_{\mathcal{H}}(S \cup a) + v_{\mathcal{H}}(S \cup b)$ and thus c cannot be favored analogously to a in Eq. (26). Since the argument so far is symmetric in a and b (but not in c!), it is enough to ensure Eq. (26) for a. We have

$$v_{\mathcal{H}}(S \cup a) \geq v_{\mathcal{H}}(S \cup b) \iff \tau_a(1) \geq \tau_b(1) \tag{30}$$

$$v_{\mathcal{H}}(S \cup a) \geq v_{\mathcal{H}}(S \cup c) \iff \tau_a(1) + \tau_{ab}(1) \geq \tau_b(1) \tag{31}$$

$$v_{\mathcal{H}}(S \cup bc) > v_{\mathcal{H}}(S \cup ac) \iff \tau_b(2) > \tau_a(2) \tag{32}$$

$$v_{\mathcal{H}}(S \cup bc) > v_{\mathcal{H}}(S \cup ab) \iff \tau_{ab}(1) + \tau_b(2) > \tau_{ab}(2) + \tau_a(1) + \tau_b(1) \tag{33}$$

Suppose Eq. (26) did not hold. Then Eq. (30) must hold (and thus Eq. (31)) and imply in Eq. (33): $\tau_{ab}(1) + \tau_b(2) > \tau_{ab}(2) + 2\tau_b(1)$ i.e. $0 \geq \tau_{ab}(1) - \tau_{ab}(2) > 2\tau_b(1) - \tau_b(2)$, contradicting τ_b's concavity (also recall $\tau_b(0) = 0$).

The inclusion of GS in the class of $\overline{\mathrm{S}}_{n'}$-presentations is strict.

Example 2 *Consider the following $\overline{\mathrm{S}}_2$-presentation for $n' = 2, n = 3$ with $\mathcal{H} = \{\{a,b\},\{a\},\{b\}\}$: $c_{ab}(1) = c_a(1) = 1, c_{ab}(2) = c_a(2) = 1.5, v_b(1) = 1, v_b(2) = 1.8$. Then $v_{\mathcal{H}}(a) + v_{\mathcal{H}}(bc) > \max\{v_{\mathcal{H}}(b) + v_{\mathcal{H}}(ac), v_{\mathcal{H}}(c) + v_{\mathcal{H}}(ab)\}$, i.e. Eq. (3) is violated and $v_{\mathcal{H}}$ is not GS.*

D Lemmas from Section 5

Lemma 12 *[8] Consider a monotone submodular function $f : 1..m \to \mathbb{R}$. Consider random set I by choosing each buyer in $1..m$ independently with probability at least p. Then $\mathbb{E}[f(I)] \geq p \cdot f(I)$.*

Proof (of Lemma 9). Assume $F_v(p) > 0$ (otherwise second statement trivially holds) and let $S \in \mathcal{D}_v(p)$ be a minimal preferred set (by v) at price p: $|S| = F_v(p)$.

We claim that $F_{\alpha v \oplus x}(p) > 0$; otherwise $\emptyset \in \mathcal{D}_{\alpha v \oplus x}(p)$ implying $0 \geq \alpha v(S) + x - p|S| > v(S) - p|S|$, contradicting $S \in \mathcal{D}_v(p)$. Let $S_\alpha \in \mathcal{D}_{\alpha v \oplus x}(p)$ be a minimal preferred set (by $\alpha v \oplus x$) at price p: $|S_\alpha| = F_{\alpha v \oplus x}$.

Suppose towards a contradiction $F_v(p) > F_{\alpha v \oplus x}(p)$, implying $S_\alpha \notin \mathcal{D}_v(p)$. Thus, at price p, S is strictly preferred by v to S_α i.e.

$$v(S) - pF_v(p) > v(S_\alpha) - pF_{\alpha v \oplus x}(p) \text{ i.e.} \tag{34}$$

$$v(S) - v(S_\alpha) > pF_v(p) - pF_{\alpha v \oplus x}(p) > 0 \text{ implying} \tag{35}$$

$$\alpha v(S) - \alpha v(S_\alpha) > pF_v(p) - pF_{\alpha v \oplus x}(p) \text{ i.e.} \tag{36}$$

$$\alpha v(S) + x - pF_v(p) > \alpha v(S_\alpha) + x - pF_{\alpha v \oplus x}(p) \tag{37}$$

i.e. S is strictly preferred to S_α by $\alpha v \oplus x$ at p, contradicting $S_\alpha \in \mathcal{D}_{\alpha v \oplus x}(p)$.