Robust Control of Linear Time-Invariant Plants Using Periodic Compensation

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Abstract—**This paper considers the use and design of linear periodic time-varying controllers for the feedback control of linear time-invariant discrete-time plants. We will show that for a large class of robustness problems, periodic compensators are superior to time-invariant ones. We will give explicit design techniques which can be easily implemented. In the context of periodic controllers, we also consider the strong and simultaneous stabilization problems. Finally, we show that for the problem of weighted sensitivity minimization for linear time-invariant plants, time-varying controllers offer no advantage over the time-invariant ones.**

I. INTRODUCTION

IN this paper we study the use and design of linear periodic time-varying controllers for feedback control of finite-dimensional linear time-invariant (LTI) plants. We will show that time-varying controllers are superior to time-invariant ones for a large class of control problems. We shall be particularly concerned with the key questions of robust stabilization and sensitivity minimization, and shall stress explicit design techniques.

Throughout most of this work, we deal with periodically varying discrete-time plants. A frequency domain approach for discrete-time periodic time-varying systems appears in the work of Davis [3], Jury and Mullin [11], Meyer and Burrus [15]. However, we have been particularly influenced by the book of Sz. Nagy-Foias [18] (see especially ch. 5) whose ideas lead to a "categorical" equivalence between periodic discrete-time systems and certain kinds of LTI systems. Essentially, an m-input, p-output, N-periodic discrete-time system can be treated as an mN-input, pN-output, LTI discrete-time system. Many of the results which we obtain in this paper are derived using this LTI system representation for periodic time-varying systems together with some recent results of Khargonekar and Tannenbaum [12].

In [12], the authors have studied and solved certain kinds of robust feedback system design problems. In particular, in the context of LTI compensators, these authors show that for a discrete-time plant $P(z)$ having both zeros and poles outside the closed unit disk, the maximal attainable gain margin is bounded. Indeed, they also derive explicit formulas for the maximal attainable gain margin in terms of unstable poles and zeros. However, we will show in this paper that by using time-varying controllers, it is possible in several interesting cases to significantly improve gain and phase margins for a discrete-time LTI plant $P(z)$. The time-varying controllers we use are periodic with period less than or equal to the dimension plus one of the plant. In point of fact in most cases 2-periodic controllers suffice. Moreover, these controllers can be explicitly computed.

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The basic reason why we may expect improvement in robustness via the use of periodic compensation is seen through the representation of periodic systems as LTI systems (see Section II). Briefly, given an LTI $p \times m$ plant $P(z)$, we can regard $P(z)$ as defining an N-periodic system, and represent it by a $pN \times mN$ transfer matrix (Section II). This transformation (for N sufficiently large) has the effect of *removing blocking zeros*, and based on the work of [12] (see Section III for details), allows us to construct periodic controllers which in many cases drastically improve the robustness of the feedback system.

In this paper we will also examine the problems of simultaneous stabilization and stabilization with a stable controller. Youla *et al.* [22] have proved the beautiful result that a continuous-time (respectively, discrete-time) LTI plant can be stabilized by a stable LTI controller if and only if a certain interlacing property involving the real right half plane (respectively, complement of the unit disk) poles and blocking zeros of the plant is satisfied. We show that any LTI plant can be stabilized by a stable periodic time-varying controller. The problem of simultaneous stabilization for LTI plants is the following. Given n LTI plants $P_1(z)$, $P_2(z)$, $\cdots$, $P_n(z)$, find (if possible) one controller that stabilizes each of the plants. This problem has been studied by [20], [17], [10] with the restriction that the controller be time-invariant. It will be seen that by using periodic time-varying controllers, it is possible to stabilize any finite collection of discrete-time time-invariant plants, and in point of fact, with a stable controller.

Recently, Zames and Francis [24], have formulated and solved the important problem of weighted sensitivity minimization. (See, also, [9], [2], [4], and the references cited therein.) Khargonekar and Tannenbaum [12] have shown that certain robust system design problems (e.g., gain/phase margin optimization problems, robust stabilization problem of Kimura [13]) and the sensitivity minimization problem are equivalent (in a precise mathematical sense), if one considers LTI controllers. We show in this paper that the minimal sensitivity cannot be improved by the use of arbitrary (not necessarily periodic) time-varying feedback. This is in contrast with such robustness properties as gain/phase margins, which in certain cases can be very significantly improved using periodic time-varying controllers. Thus, if one considers the more general class of time-varying controllers, the problems of robustness and sensitivity minimization appear to be dichotomous.

The use of time-varying controllers for the control of time-invariant plants has been known to be quite useful in some instances. For example, Anderson and Moore [1] and Wang [21] have shown that time-invariant plants with unstable decentralized fixed modes cannot be stabilized using decentralized time-invariant controllers, and yet can be stabilized by decentralized time-varying controllers. Our results provide further evidence that the use of time-varying controllers can be advantageous in many other control problems.

II. TRANSFER FUNCTIONS FOR PERIODIC SYSTEMS

In this section we briefly discuss the basic elements of a transfer function theory for periodic, discrete-time, linear, time-varying systems. Similar theories have been developed in the system

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theory literature in [3], [11], [15], and the references cited therein. However, our explicit technique is strongly motivated by [18, ch. 5].

Here it will be shown that to an N-periodic linear $p \times m$ discrete-time system f, one can associate a $pN \times mN$ discrete-time LTI system $\tilde{f}$. Conversely, any such LTI system defines a periodic time-varying system. Moreover, this correspondence is an isomorphism of systems in the sense that both the algebraic and analytic properties of systems are preserved. In particular, f is stable if and only if $\tilde{f}$ is stable, and in this case the operator norm of f is equal to the operator norm of $\tilde{f}$. This construction allows us to use the methods from LTI design theory to design periodic controllers for LTI plants.

Let L denote the set of all one sided sequences of real numbers, and let $\tilde{L} = \mathbb{R}[[z^{-1}]]$ be the set of all formal power series in the indeterminate z^{-1} with real coefficients. As is well known, L and $\tilde{L}$ are isomorphic, the isomorphism being defined in the usual way

$$\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots) \mapsto \alpha(z) = \sum_{i=0}^{\infty} \alpha_i z^{-i};$$

$\alpha(z)$ is the usual z-transform of α. Let $\Lambda: \tilde{L} \to \tilde{L}$ denote the *right shift operator*

$$\Lambda(\alpha(z)) = z^{-1}\alpha(z).$$

Consider now the isomorphism

$$W: L \to L^N : \alpha = (\alpha_0, \alpha_1, \cdots) \mapsto \begin{bmatrix} (\alpha_0, \alpha_N, \alpha_{2N}, \cdots) \\ (\alpha_1, \alpha_{N+1}, \alpha_{2N-1}, \cdots) \\ \vdots \\ (\alpha_{N-1}, \alpha_{2N-1}, \cdots) \end{bmatrix}. \tag{2.1}$$

In the transform domain, $W(\alpha)$ has the representation $[\alpha_0(z), \alpha_1(z), \cdots, \alpha_{N-1}(z)]'$ where $\alpha_i(z) := \sum_{j=0}^{\infty} \alpha_{Nj+i} z^{-j}$. It is very easy to verify that

$$W\Lambda^N = \Lambda W. \tag{2.2}$$

Also observe that

$$W^{-1}([\alpha_0(z), \alpha_1(z), \cdots, \alpha_{N-1}(z)]') = \sum_{i=0}^{N-1} z^{-i}\alpha_i(z^N).$$

Let $f: L^m \to L^p$ be an $\mathbb{R}$-linear causal time-varying input/output operator. The I/O map f is *N-periodic* if and only if

$$f\Lambda^N = \Lambda^N f,$$

i.e., the map f commutes with the Nth power of right shift. The I/O map f induces via W a map $\tilde{f}: L^{mN} \to L^{pN}$ defined by the following commutative diagram:

$$\begin{array}{ccc} L^{mN} & \xrightarrow{\tilde{f}} & L^{pN} \\ W \uparrow & & \uparrow W. \\ L^m & \xrightarrow[f]{} & L^p \end{array}$$

Thus, $\tilde{f} = WfW^{-1}$, and

$$\tilde{f}\Lambda = WfW^{-1}\Lambda = Wf\Lambda^N W^{-1} = W\Lambda^N f W^{-1} = \Lambda W f W^{-1} = \Lambda \tilde{f}.$$

We see that $\tilde{f}$ *commutes with the shift operator* Λ, *and hence defines a discrete-time LTI system.*

It is interesting and useful to reformulate the above constructions using matrix representations. For $q > 0$, consider the linear space L^q with the natural basis (consisting of elements v_i of the form:

$$v_{ni+j} = (0, 0, \underset{(n-1)}{\cdots}, 0, e_j, 0, \cdots),$$
$$j = 1, 2, \cdots, 1, \; n = 0, 1, \cdots \tag{2.3}$$

where e_j is the jth column of the $q \times q$ identity matrix). With respect to such natural bases, let M_f be the matrix representation of a causal linear input/output map $f: L^m \to L^p$. Now it is not difficult to see that f is N-periodic if and only if the matrix M_f has the following block Toeplitz structure:

$$M_f = \begin{bmatrix} M_0 & 0 & 0 & \\ M_2 & M_0 & 0 & \cdots \\ M_2 & M_1 & M_0 & \cdots \end{bmatrix} \tag{2.4}$$

where each M_i *is* $pN \times mN$, and further M_0 is lower triangular. Indeed, the block Toeplitz structure in M_f corresponds to the fact that $\tilde{f}$ commutes with the shift operator. Also note that the matrix representations of f and $\tilde{f}$ (with respect to the corresponding natural bases) are the same. We can now associate the transfer matrix

$$T_f(z) = \sum_{i=0}^{\infty} M_i z^{-i} \tag{2.5}$$

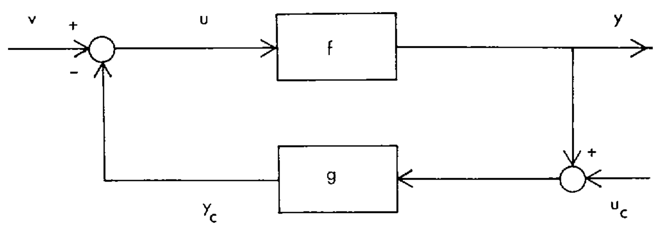
with the periodic input/output map f. Note that $T_f(z)$ is the transfer matrix of the map $\tilde{f}$. We summarize the above discussion in the following.

Proposition 2.6: Given an m-input, p-output casual linear N-periodic input/output map f, one can canonically associate a $pN \times mN$ *transfer matrix* $T_f(z)$ *given by (2.5). Conversely, given any* $pN \times mN$ *proper transfer matrix* $T(z)$ *such that* $T(\infty)$ *is lower triangular, we can find a unique N-periodic casual linear input/output map* $f: L^m \to L^p$ *such that* $T(z) = T_f(z)$.

This one-to-one correspondence between periodic linear time-varying systems and the "larger" LTI systems is natural from a system theoretic point of view in the sense that it preserves both the analytic and algebraic properties of systems. In particular, all the standard formulas for interconnection of systems hold. For example, the transfer function of the LTI system associated with the cascade connection fg of two periodic systems f and g is given by

$$T_{fg} = T_f T_g.$$

We also have the following result on the feedback interconnection of systems:



Lemma 2.7: Let f and g be periodic linear, discrete-time, time-varying systems of compatible dimensions. Consider the feedback system shown above.

a) *The feedback system is also a periodic linear, time-varying system and the transfer function of the associated LTI system is*

$$T_{yv} = (I + T_f T_g)^{-1} T_f.$$

b) *The system f is input/output stable if and only if* T_f *is a stable transfer function matrix.*

c) *The feedback system is internally stable if and only if T_g internally stabilizes T_f.*

We omit the proof as it is quite easy.

Let f be a stable periodic linear system. Then we may regard f, in the usual way, as a bounded linear operator

$$f:(h^2)^m \rightarrow (h^2)^p$$

where h^2 is the Hilbert space of square summable one-sided sequences of real numbers. (Clearly, h^2 is a subspace of L.) In this case, we define $\|f\|$ to be the operator of norm of f, i.e.,

$$\|f\| = \sup\{\|fu\|_2 : u \text{ in } (h^2)^m,\ \|u\|_2 = 1\}.$$

Also, we may restrict the isomorphism W to h^2 to get the isomorphism

$$w: h^2 \rightarrow (h^2)^N.$$

It is a simple but important fact that W is an *isometry*, i.e., $\|Wx\|_2 = \|x\|_2$. Therefore, we have

$$\|\tilde{f}\| = \|WfW^{-1}\| = \|f\|.$$

Now $\tilde{f}:(h^2)^{mN} \rightarrow (h^2)^{pN}$ is a shift-invariant operator. It is well known (see [16]) that the operator norm of $\tilde{f}$ is given by the H^∞ norm of the transfer function matrix T_f of $\tilde{f}$. In other words,

$$\|\tilde{f}\| = \|T_f\|_\infty := \sup\{\bar{\sigma}(T_f(e^{j\omega})) : \omega \text{ in } [0, 2\pi]\}$$

where $\bar{\sigma}(M)$ denotes the largest singular value of M. Let us summarize the above discussion by the following.

Lemma 2.8: Let f be an N-periodic, stable, linear, discrete-time, time-varying system. Then

$$\|f\| = \|T_f\|_\infty.$$

Lemmas 2.7 and 2.8 are the key results that allow us to use the methods from LTI design theory for the design of periodic compensators for periodic plants. More precisely, if f is a periodic time-varying plant, we first compute its associated transfer matrix $T_f(z)$ via (2.5). We then design a controller $C(z)$ for the plant $T_f(z)$ using LTI design methods. *We should emphasize that $C(z)$ must be such that $C(\infty)$ is lower triangular. This condition ensures that $C(z)$ corresponds to a causal periodic linear time-varying system.*

We close this section with the computation of the transfer function of an LTI system viewed as an N-periodic system. Following the above ideas, this is done explicitly as follows. Let $P(z)$ be the transfer matrix of an LTI discrete-time system. Then we can write (uniquely)

$$P(z) = P_1(z^N) + z^{-1}P_2(z^N) + \cdots + z^{-(N-1)}P_N(z^N).$$

If we regard $P(z)$ as defining an N-periodic system, and "lift" $P(z)$ via W, an easy computation shows that

$$\tilde{P}(z) := WP(z)W^{-1} = \begin{bmatrix} P_1(z) & z^{-1}P_N(z) & z^{-1}P_{N-1}(z) & \cdots & z^{-1}P_2(z) \\ P_2(z) & P_1(z) & z^{-1}P_N(z) & \cdots & z^{-1}P_3 \\ P_3(z) & P_2(z) & P_1(z) & \cdots\cdots & z^{-1}P_4(z) \\ \vdots & \vdots & \vdots & & \vdots \\ P_N(z) & P_{N-1}(z) & P_{N-2}(z) & \cdots & P_1(z) \end{bmatrix}. \tag{2.9}$$

(Note that $\tilde{P}(z)$ has a block Toeplitz matrix structure.)

It is also fairly straightforward to derive a formula for $\tilde{P}(z)$ in terms of a state-space realization of $P(z)$. Indeed, let

$$P(z) = C(zI - A)^{-1}B + D. \tag{2.10}$$

Then

$$\tilde{P}(z) = \begin{bmatrix} D & 0 & 0 & 0 \\ CB & D & 0 & 0 \\ CAB & CB & D & 0 \\ CA^{N-2}B & CA^{N-3}B & & D \end{bmatrix} + \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^{N-1} \end{bmatrix} (zI - A^N)^{-1}\{A^{N-1}B \cdots AB\ B\}. \tag{2.11}$$

Note that if (2.10) represents a minimal realization of $P(z)$, then (2.11) also gives a minimal realization of $\tilde{P}(z)$. In particular, λ is a pole of $P(z)$ if and only if λ^N is a pole of $\tilde{P}(z)$.

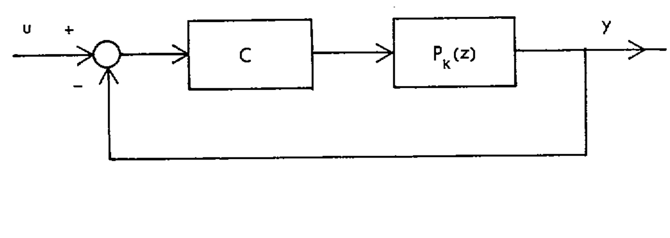
Now in this paper we take an input/output point of view. Thus, internal stability in Lemma 2.7-c is in the usual sense of stability of feedback systems as in [26]. As far as implementation is concerned, it is not difficult to show from our representation of a periodic discrete-time system as an LTI system that a given periodic input/output map with a rational LTI representation (2.5) can always be realized by a stabilizable and detectable periodic discrete-time system. Hence, internal stability in 2.7-c implies internal stability in the state-space sense as long as the plant and the compensator are implemented using stabilizable and detectable realizations.

Finally, using again the ideas of Sz.Nagy-Foias [18, ch. 5], one can associate, in a natural way, an LTI system to a periodic time-varying system in the continuous-time case as well. However, here the input and output spaces of the LTI equivalent are in general *infinite-dimensional Hilbert spaces*. Since in this paper we are primarily interested in certain issues concerning robust stabilization, this construction, as of now, does not seem to be very useful.

III. Periodic Controllers for Robust Stabilization of LTI Systems

In this section we consider the use of periodic compensation in order to improve the optimal gain or phase margin for LTI discrete-time plants. In [19] and [12] it is shown that for nonminimum phase unstable plants, the maximal obtainable gain margin (via LTI compensators) is finite and depends only on the unstable poles and zeros of the plant. Here we will show that for bicausal (i.e., proper but *not* strictly proper) SISO LTI discrete-time plants, the optimal gain margin is infinite. For strictly proper plants, we will show that it is often possible to obtain significant improvement in the maximal obtainable gain margin by using periodic feedback. In addition, we also consider similar problems for complex parameter variations.

In order to be more explicit, we must first review the basic setup from [12]. Let $P_k(z)$ be a parameterized family of LTI discrete-time SISO plants, where the parameter k takes values in some compact set K. Then we want to design a linear controller (possibly time-varying) such that for each k in K, the following closed-loop system is internally asymptotically stable:



In this section, we will be considering families of SISO plants of the following form:

$$P_k(z)=kP_o(z)$$

where $P_o(z)$ is the fixed nominal plant and k is a parameter taking values in some compact subset K of the complex plant $\mathbb{C}$. The choice of K determines the kind of plant uncertainty for which a robust controller is to be designed. We are then interested in solving the following.

Basic Problem 3.2: Given a nominal plant $P_o(z)$ and a compact subset K in $\mathbb{C}$, find (if possible) a controller C (perhaps time-varying) such that for each k in K, the feedback system shown in (3.1) *is internally asymptotically stable.*

In this paper, we will consider the following four possibilities for K (for more details, see [12]).

i) $K=[a, b]$, $0<a<1<b$. This is the gain margin problem. (Note that the gain margin is 20 log (b/a) dB.)

ii) $K=\{e^{j\phi}: \phi \text{ in } [-\theta_1, \theta_2], 0\leqslant\theta_i\leqslant\pi/2, i=1,2\}$. This is the phase margin problem.

iii) $K=\{ke^{j\phi}: k \text{ in } [a,b], \phi \text{ in } [-\theta_1,\theta_2], \text{ where } 0<a<1<b, 0\leqslant\theta_i\leqslant\pi/2, i=1,2\}$. This is the gain phase margin problem.

iv) $K=$ a compact region in $\mathbb{C}$ containing 1 as an interior point. This is the problem of complex parameter variations. (3.2)

A plant $P_o(z)$ is called *bicausal* if $P_o(\infty)$ is invertible. We begin with the following key result.

Theorem 3.3: Let $P_o(z)$ be a SISO LTI *bicausal plant. Suppose the unstable poles of $P_o(z)$ are distinct. Then for any K in $\mathbb{C}$ as in* (3.2-i)–(3.2-iv) *there always exists a periodic controller C such that the feedback system in* (3.1) *is internally asymptotically stable for each k in K. Moreover, generically the controller C may be taken to be* 2*-periodic.*

Note that Theorem 3.3 implies that for plants satisfying the above hypotheses, one can obtain any desired gain margin by suitable design of a periodic controller.

Proof: Following the method of Section II, we will consider $P_o(z)$ as an N-periodic, linear time-varying discrete-time plant. As in Section II, write

$$P_o(z)=P_1(z^N)+z^{-1}P_2(z^N)+\cdots+z^{-(N-1)}P_N(z^N).$$

Suppose z_o is a complex number with $|z_o|\geqslant 1$ such that

$$P_i(z_o)=0, \qquad i=1,2,\cdots,N.$$

Let

$$\lambda_i=\sqrt[N]{z_o}\,e^{(2\pi j)(i/N)}, \qquad i=0,1,2,\cdots,N-1.$$

where $\sqrt[N]{z_o}$ is some fixed Nth root of z_o.

Then

$$P_o(\lambda_i)=P_1(z_o)+\lambda_i^{-1}P_2(z_o)+\cdots+\lambda_i^{-(N-1)}P_N(z_o)=0.$$

Thus, each λ_i, $0\leqslant i\leqslant N-1$ is a zero of $P_o(z)$. Moreover, the λ_i's are distinct. Since $P_o(z)$ has only finitely many zeros, we can certainly find N sufficiently large (actually, $N>$ the number of unstable zeros suffices) such that there does not exist z_o with $|z_o|\geqslant 1$ and $P_i(z_o)=0$, $1\leqslant i\leqslant N$. Further, we can choose this N such that the unstable poles of $\tilde{P}_o(z)$ in (2.9) are distinct. To see this, first note from (2.11) that μ_i is an unstable pole of $P_o(z)$ only if μ_i^N is an unstable pole of $\tilde{P}_o(z)$. Let $\mu_1, \mu_2, \cdots, \mu_m$ be the unstable poles of $P_o(z)$. By hypothesis, $\mu_i\neq\mu_j$, for $i\neq j$. Now we need to choose N such that

$$\mu_i^N\neq\mu_j^N, \qquad i\neq j.$$

We claim that for each $i\neq j$, there is at most one prime number q_{ij} such that for any prime number $q\neq q_{ij}$

$$\mu_i^q\neq\mu_j^q, \qquad q\neq q_{ij}.$$

If not, for some $i\neq j$, there exist two prime numbers q, r such $q\neq r$ and

$$\mu_i^q=\mu_j^q, \qquad \mu_i^r=\mu_j^r.$$

As q, r are coprime integers, there exist integers α, β such that $q\alpha+r\beta=1$. Hence,

$$\mu_i=\mu_i^{q\alpha+r\beta}=\mu_i^{q\alpha}\mu_i^{r\beta}=\mu_j^{q\alpha}\mu_j^{r\beta}=\mu_j^{q\alpha+r\beta}=\mu_j$$

which is a contradiction. Therefore, our claim is valid. Thus, there is a finite set of prime numbers q_{ij} such that for any prime number $q\neq q_{ij}$, $i, j=1,2,\cdots,m$, we have

$$\mu_i^q\neq\mu_j^q,\ i\neq j, \qquad i,j=1,2,\cdots,m.$$

We conclude that we can find a positive integer N such that there does not exist z_o with $|z_o|\geqslant 1$ and $P_i(z_o)=0$, $i=1,2,\cdots,N$, and such that the unstable poles of $\tilde{P}_o(z)$ are distinct.

Generically, $N=2$ will work. For $N=2$, $P_1(z)$ and $P_2(z)$ have a common zero outside the unit disk if and only if $P_o(z)$ and $P_o(-z)$ have a common zero outside the unit disk. However, generically $P_o(z)$ and $P_o(-z)$ will have no common zeros. Simarily, for $N=2$, $\tilde{P}_o(z)$ will have distinct poles since

$$\mu_i^2=\mu_j^2$$

for some $i\neq j$ implies that $\mu_i=-\mu_j$, which is again nongeneric.

For notational simplicity, we will suppose that $N=2$, i.e., if we write

$$P_o(z)=P_1(z^2)+z^{-1}P_2(z^2)$$

then $P_1(z)$, $P_2(z)$ have no common zeros in the complement of the closed unit disk and the unstable poles of $\tilde{P}_o(z)$ are distinct. (The proof for general N is identical.) Thus, we represent $kP_o(z)$ as a 2-periodic plant. Following the recipe in Section II

$$k\tilde{P}_o(z)=k\begin{bmatrix} P_1(z) & z^{-1}P_2(z)\\ P_2(z) & P_1(z)\end{bmatrix}.$$

We are required to find a proper compensator $\hat{C}(z)$ of the form

$$\hat{C}(z)=\begin{bmatrix} C_1(z) & z^{-1}C_2(z)\\ C_3(z) & C_4(z)\end{bmatrix}$$

such that $\hat{C}(z)$ internally stabilizes $kP_o(z)$ for each k in K. (Any $\hat{C}(z)$ in the above form will correspond to a causal 2-periodic linear, time-varying compensator C.) Set

$$U(z):=\begin{bmatrix} z^{-1} & 0\\ 0 & 1\end{bmatrix}. \tag{3.4}$$

Then

$$k\tilde{P}_o(z)U(z)=k\begin{bmatrix} z^{-1}P_1(z) & z^{-1}P_2(z)\\ z^{-1}P_2(z) & P_1(z)\end{bmatrix}.$$

Since by hypothesis, $P_o(z)$ is proper but not strictly proper, $P_1(\infty)\neq 0$. Further, $P_1(z)$ and $P_2(z)$ have no common zeros in $|z|\geqslant 1$. Hence, $\tilde{P}_o(z)U(z)$ has no blocking zeros in $|z|\geqslant 1$ including infinity. We now follow the argument of [12, Theorem 3.1]. Indeed, consider the Smith–McMillan form of $\tilde{P}_o(z)U(z)$ over the ring R of stable proper rational functions. Then there exist unimodular matrices $V_1(z)$, $V_2(z)$ over R such that

$$kV_1\tilde{P}_oUV_2=k\begin{bmatrix} a_1(z) & 0 \\ 0 & a_2(z)\end{bmatrix}$$

where a_1 is minimum phase and a_2 is asymptotically stable. Consequently, from the results of [19], [12], [5], and [14], there exist proper rational functions $b_i(z)$, $i = 1, 2$ such that $b_i(z)$ internally stabilizes $ka_i(z)$, for all k in K, $i = 1, 2$. [K can be any region as in (3.2-i)–(3.2-iv).] Note that as $a_2(z)$ is already stable, we can choose $b_2 = 0$. In any event, define

$$\tilde{C}(z)=V_2(z)\begin{bmatrix} b_1(z) & 0 \\ 0 & b_2(z)\end{bmatrix}V_1(z).$$

Then it is straightforward to check that $\tilde{C}(z)$ internally stabilizes $k\tilde{P}_o(z)U(z)$, for each k in K. But now set

$$\hat{C}(z)=U(z)\tilde{C}(z).$$

Then $\hat{C}(z)$ internally stabilizes $k\tilde{P}_o(z)$ for each k in K, and since $\hat{C}(\infty)$ is *lower triangular*, $\hat{C}(z)$ corresponds to a 2-periodic time-varying linear discrete-time controller C. It now follows from Lemma 2.7 that C internally stabilizes the LTI family of plants $kP_o(z)$, for each k in K.

Example 3.5: We would like to illustrate our result with an explicit example. Let the nominal plant be $P_o(z) = (z - 2)/(z - 3)$. From the results of [19] and [12], it follows that *one can find an LTI compensator $C(z)$ which stabilizes the family of plants $kP_o(z)$ for each k in $[a, b]$ if and only if $(b/a) < 2.25$*. In other words, the maximal attainable gain margin for the plant $P_o(z)$ by a suitable design of an LTI compensator is 7 dB. We will now design a periodic linear time-varying controller C which internally stabilizes the family $kP_o(z)$ for each k in $[0.5, 5]$. (The results of [19], [12] clearly imply that one cannot solve this problem using an LTI compensator.)

We follow the procedure indicated in the proof of Theorem 3.3. First note that $N = 2$ works since $P_o(z)$ has only one zero. Now write

$$P_o(z)=\frac{z-2}{z-3}=\frac{z^2-6}{z^2-9}+z^{-1}\left(\frac{z^2}{z^2-9}\right).$$

Thus, $P_1(z) = (z - 6)/(z - 9)$, $P_2(z) = z/(z - 9)$. Representing $P_o(z)$ as a 2-periodic plant, we get

$$\tilde{P}_o(z)=\begin{bmatrix} \frac{z-6}{z-9} & \frac{1}{z-9} \\ \frac{z}{z-9} & \frac{z-6}{z-9}\end{bmatrix}.$$

Multiply $\tilde{P}_o$ on the right by $U(z)$ of (3.4) to get

$$\tilde{P}_o(z)U(z)=\begin{bmatrix} \frac{z-6}{z(z-9)} & \frac{1}{z-9} \\ \frac{1}{z-9} & \frac{z-6}{z-9}\end{bmatrix}.$$

We now need to find unimodular stable proper rational matrices $V_1(z)$ and $V_2(z)$ to bring $\tilde{P}_o(z)U(z)$ to its Smith–McMillan form. Some routine calculations give

$$V_1(z)=\begin{bmatrix} 0 & 1 \\ 1 & \frac{36-7z}{z^2}\end{bmatrix},\quad V_2(z)=\begin{bmatrix} 6 & \frac{z-6}{z} \\ 1 & \frac{-1}{z}\end{bmatrix}.$$

Indeed,

$$V_1(z)\tilde{P}_oU(z)(z)V_2(z)=\begin{bmatrix} \frac{z}{z-9} & 0 \\ 0 & \frac{z-4}{z^2}\end{bmatrix}.$$

Now let us design a compensator $b_1(z)$ which stabilizes $kz/(z - 9)$ for each k in $[0.5, 5]$. It is easy to check that $b_1(z) = 18$ is one such compensator. As $(z - 4)/z^2$ is already stable, we may choose $b_2(z) = 0$. Now the final compensator is

$$\hat{C}(z)=U(z)V_2(z)\begin{bmatrix} b_1(z) & 0 \\ 0 & b_2(z)\end{bmatrix}V_1(z)=\begin{bmatrix} 0 & 108z^{-1} \\ 0 & 18\end{bmatrix}.$$

As $\hat{C}(\infty)$ is lower triangular, $\hat{C}(z)$ represents a 2-periodic linear time-varying controller C. Indeed, the input/output equation for C is given by

$$C: (f_0, f_1, f_2, \cdots)=(0, 18f_1, 108f_1, 18f_3, 108f_3, \cdots).$$

Obviously, the controller C may be implemented via the linear periodic one-dimensional system

$$x_c(k+1)=u_c(k)$$

$$y_c(k)=h(k)x(k)+d(k)u_c(k)$$

where x_c, y_c, u_c are the state, output, and input of the controller C, respectively, and

$$h(k)=0 \quad \text{for } k \text{ even}$$

$$h(k)=108 \quad \text{for } k \text{ odd}$$

$$d(k)=0 \quad \text{for } k \text{ odd}$$

$$d(k)=18 \quad \text{for } k \text{ even.}$$

This controller C then stabilizes the family of plants $k(z - 2)/(z - 3)$ for each k in $[0.5, 5]$. Thus, whereas one *could not have solved* this robust synthesis problem using an LTI compensator, a relatively simple periodic control scheme allows us to solve the problem. This example illustrates the power of periodic control for robust synthesis problems.

Remarks on the Strictly Proper Case 3.5: Given the fact that we can arbitrarily improve robustness for LTI discrete-time bicausal plants by using periodic compensation, we would like to discuss here a design procedure which could improve robustness in the strictly proper case as well.

First, we want to describe precisely why the proof of Theorem 3.3 fails for strictly proper plants. Indeed, given a strictly proper nominal $P_o(z)$, from the argument of Theorem 3.3, we can always represent it as an N-periodic system $\tilde{P}_o(z)$ which has no blocking zeros outside the unit disk including ∞. (Note that if the power series in z^{-1} for $P_o(z)$ starts with z^{-l} then, for $N \geq l$, $P_l(z)$ in (29) is bicausal, and therefore $P_o(z)$ does not have a blocking zero at ∞ for $N \geq l$.) Hence, from [12, Theorem 3.1], we can always find an $N \times N$ matrix $\hat{C}(z)$ to guarantee a given gain margin. However, recall from Section II, that in order for $\hat{C}(z)$ to correspond to a *causal* N-periodic controller, $\hat{C}(\infty)$ must be lower triangular. In order to ensure this in the bicausal case, we used the trick in the proof of Theorem 3.3 of multiplying $\tilde{P}_o(z)$ by $U(z)$. The reader can easily check that such a trick fails in the strictly proper case. Apparently, any attempt to obtain a causal C leads to an unstable blocking zero at ∞. We believe that this is not just a technical problem, but a fundamental difference between bicausal and strictly proper plants taken in the context of robust stabilization. *However, the general method we have discussed can be used to improve the gain (or phase) margin.* We would like to explicitly outline this design procedure now.

i) Represent $P_o(z)$ as an N-periodic plant $\tilde{P}_o(z)$ with no blocking zeros. From the proof of Theorem 3.3 this can always be done, and generically we can take $N = 2$.

ii) Multiply $\tilde{P}_o(z)$ on the right by the $N \times N$ matrix

$$U(z)=\text{diag }(z^{-1}, z^{-1}, \cdots, z^{-1}, 1). \tag{3.6}$$

Note that $\tilde{P}_o(z)U(z)$ has no blocking zeros except at $z = \infty$.

iii) We now need to find a compensator $\hat{C}(z)$ which stabilizes $k\tilde{P}_o(z)U(z)$ for all k in K. Since $\tilde{P}_o(z)U(z)$ has an unstable blocking zero at ∞, a complete solution to the basic problem (3.2) is unknown at the moment. However, the problem may be approached as follows. Via unimodular matrices (with entries in the ring R of stable proper rational functions) bring $\tilde{P}_o(z)U(z)$ into a diagonal form. (We do not necessarily take the Smith–McMillan form.) Thus, find unimodular matrices $V_1(z)$, $V_2(z)$ such that

$$V_1(z)\tilde{P}_o(z)U(z)V_2(z) = \text{diag}\,(a_1, a_2, \cdots, a_N). \quad (3.7)$$

Now each a_i has an unstable zero at ∞. Further, we may assume that the only unstable zero of a_1 is at ∞. Also, if we assume that $P_o(z)$ has distinct unstable poles, then we can force the a_i's to have any distribution of the unstable poles of $\tilde{P}_o(z)$.

iv) Construct (using the results of [19] and [12]) compensators $b_i(z)$ to stabilize $ka_i(z)$ for each k in K, $i = 1, 2, \cdots, N$. Since $a_1(z)$ has only *one* unstable zero at ∞, it can be expected that the amount of plant uncertainty as given by K for which stabilizing compensators b_i's can be designed is larger than those which can be handled by LTI compensators. This is especially true if $P_o(z)$ has several unstable zeros. Finally, define

$$\hat{C}(z) := U(z)V_2(z)\ \text{diag}\,(b_1(z), b_2(z), \cdots, b_N(z))V_1(z).$$

Then $\hat{C}(z)$ corresponds to a periodic linear time-varying compensator. In (3.7) there are only finitely many possibilities for the poles of the a_i's, and one could take the one which will maximize the robustness.

Example 3.8: We would like to illustrate the above procedure with a simple example. Consider the SISO strictly proper plant

$$P_o(z) = \frac{z-3}{(z-2)(3z-5)}.$$

In [19] and [12] it is shown that one can find an LTI stabilizing compensator $C(z)$ which stabilizes $kP_o(z)$ for each k in $[a, b]$ if and only if

$$\frac{b}{a} < 1.116.$$

In other words, the *maximal* obtainable gain margin for $P_o(z)$ using any LTI compensator is 0.953 dB! As $P_o(z)$ is strictly proper, we cannot apply Theorem 3.3 directly. However, let us apply the design approach outlined above. First, let us write (with $N = 2$),

$$P_o(z) = P_1(z^2) + z^{-1}P_2(z^2) = \frac{2z^2-30}{(z^2-4)(9z^2-25)} + z^{-1}\frac{3z^4-23z^2}{(z^2-4)(9z^2-25)}.$$

Hence,

$$\tilde{P}_o(z) = \frac{1}{(z-4)(9z-25)}\begin{bmatrix} 2z-30 & 3z-23 \\ 3z^2-23z & 2z-30 \end{bmatrix}.$$

Multiply $\tilde{P}_o(z)$ on the right by $U(z)$

$$\tilde{P}_o(z)U(z) = \frac{1}{(z-4)(9z-25)}\begin{bmatrix} 2-30z^{-1} & 3z-23 \\ 3z-23 & 2z-30 \end{bmatrix}.$$

We now find unimodular matrices $V_1(z)$ and $V_2(z)$ over the ring of stable proper rational functions such that

$$V_1(z)\tilde{P}_o(z)U(z)V_2(z) = \begin{bmatrix} a_1(z) & 0 \\ 0 & a_2(z) \end{bmatrix}$$

where

$$a_1 = \frac{1}{z-4},\ a_2 = \frac{z-9}{z(9z-25)}.$$

Now using the results of [19] and [12] it follows that we can find proper compensators $b_1(z)$ and $b_2(z)$ which stabilize $ka_1(z)$ and $ka_2(z)$, respectively, for each k in $[a, b]$ if and only if

$$\frac{b}{a} < 1.45.$$

If a, b satisfy the above inequality, then we can find appropriate compensators $b_1(z)$, $b_2(z)$ which stabilize $ka_1(z)$ and $ka_2(z)$, respectively, for each k in $[a, b]$. Now define

$$\hat{C}(z) = U(z)V_2(z)\ \text{diag}\ (b_1(z), b_2(z))V_1(z).$$

Then $\hat{C}(\infty)$ is lower triangular and $\hat{C}(z)$ corresponds to a time-varying periodic system C. This compensator C stabilizes $kP_o(z)$ for each k in $[a, b]$. Thus, we can find a 2-periodic compensator C to stabilize $kP_o(z)$ for each k in $[a, b]$ if

$$\frac{b}{a} < 1.45.$$

We conclude that *we can obtain a gain margin using a 2-periodic compensator of at least* 3.25 dB. This shows that in this example we can obtain an *improvement in the maximal obtainable gain margin by at least a factor of* 3 by using periodic feedback instead of LTI feedback.

IV. Strong and Simultaneous Stabilization

In the previous section we have shown that it is possible to improve maximal attainable stability margins for linear discrete-time LTI plants by using periodic time-varying controllers. This improvement is due to the fact that we are able to "remove" blocking zeros by viewing LTI plants as being N periodic, and then applying the methods of [12]. We again exploit this fact to demonstrate the advantage of using periodic feedback for the strong and simultaneous stabilization problems.

The strong stabilization problem is: given an LTI plant $P(z)$, find (if possible) an asymptotically stable controller which internally stabilizes the plant. If the controller is restricted to be time-invariant, Youla *et al.* [22] have proved that $P(z)$ can be stabilized using a stable controller if and only if a certain interlacing property, involving real blocking zeros and poles of $P(z)$ lying outside the open unit disk, is satisfied. We show in this section that *any* LTI plant can be internally stabilized by using a *periodic, stable,* time-varying controller.

We also study the problem of simultaneous stabilization using time-varying controllers. In this connection, we show that any finite collection of discrete-time LTI plants can be simultaneously stabilized by a periodic time-varying controller. Moreover, this controller can be guaranteed to be internally stable.

We begin with the following.

Theorem 4.1: Let $P(z)$ be the (proper) transfer matrix of a $p \times m$ discrete-time LTI plant. Then there exists a periodic time-varying asymptotically stable controller that internally stabilizes the plant.

Proof: Let n be the McMillan degree of $P(z)$. From the proof of Theorem 3.3, it follows that we can find an integer N, $1 \leqslant N \leqslant n+1$ such that if we view the system $P(z)$ as an N-periodic discrete-time, time-varying system, then the associated $pN \times mN$ transfer function matrix $\tilde{P}(z)$ has no blocking zeros in the complement of the open unit disk. Consider the (fictitious) plant

$$G(z) = \tilde{P}(z)\ \text{diag}\ (z^{-1}, z^{-1}, \cdots, z^{-1}, 1).$$

Clearly, $G(z)$ has no blocking zeros in the complement of the open unit disk except possibly at $z = \infty$. But now the interlacing property of real blocking zeros and poles of $G(z)$ in the complement of unit disk holds trivially. (See [22, p. 166].) Hence, it follows from [22] that there exists a proper controller $H(z)$ that internally stabilizes $G(z)$. Define

$$\hat{C}(z) = \text{diag}\,(z^{-1}, z^{-1}, \cdots, z^{-1}, 1)H(z).$$

Notice that $\hat{C}(\infty)$ is lower triangular. Consequently, $\hat{C}(z)$ corresponds to an N-periodic, linear, discrete-time, time-varying system C. Further, C is stable since $H(z)$ as well z^{-1} is stable. Finally, by Lemma 2.7, C internally stabilizes $P(z)$. This completes the proof.

Alternate Proof: Let n be the dimension of $P(z)$. Let $C(z)$ be any deadbeat controller for P and let $\hat{\Sigma}_c = (\hat{F}_c, \hat{G}_c, \hat{H}_c, \hat{J}_c)$ be a canonical realization of $C(z)$. Let n_c be the dimension of $\hat{\Sigma}_c$. The fact that $C(z)$ is a deadbeat controller means that starting from any initial state x_o in $\mathbb{R}^{n+n_c}$ of the feedback system at the time t_o the closed-loop system state $x(k) = 0$, for $k \geq n + n_c + t_o$. Define an $(n + n_c + 1)$-periodic time-varying system $\Sigma_c = (F_c, G_c, H_c, J_c)$

$$G_c = \hat{G}_c, \; H_c = \hat{H}_c, \; J_c = \hat{J}_c,$$

$$F_c(k) = \begin{cases} \hat{F}_c & 0 \leq k \leq n + n_c - 1 \\ 0 & k = n + n_c. \end{cases}$$

Clearly Σ_c is asymptotically stable and it internally stabilizes the plant $P(z)$.

It is possible to generalize the alternate proof of Theorem 4.1 to show that any LTI *continuous-time* plant can be internally stabilized by an asymptotically stable periodic controller. We omit the details for the sake of brevity.

Let us now turn to the problem of simultaneous stabilization. Let $\{P_i(z): i = 1, 2, \cdots, k\}$ be a finite collection of $p \times m$ discrete-time LTI plants. The problem of simultaneous stabilization is to find (if possible) a controller Σ_c which stabilizes each of the plants $P_i(z)$, $i = 1, 2, \cdots, k$. If one restricts Σ_c to be an LTI system, this is a rather difficult problem in its complete generality and has been studied by [17], [20], and [10]. Using periodic controllers, however, we can easily obtain the following.

Theorem 4.2: Let $P_i(z)$, $i = 1, 2, \cdots, k$ be any collection of $p \times m$ discrete-time LTI plants. Then there exists an asymptotically stable periodic time-varying controller Σ^c that internally stabilizes each $P_i(z)$, $i = 1, 2, \cdots, k$.

Proof: For $i = 1, 2, \cdots, k$, let $C_i(z)$ be a deadbeat controller for $P_i(z)$. Let $\Sigma_i^c = (A_i^c, B_i^c, C_i^c, D_i^c)$ be a canonical realization of $C_i(z)$. Without loss of generality, we may assume that each Σ_i^c has the same dimension n_c. Let n_i be the dimension of $P_i(z)$. Let

$$N_l = 1 + l n_c + \sum_{j=1}^{l} n_j, \qquad l = 1, 2, \cdots, k.$$

Define the N_k-periodic system $\Sigma^c = (A^c, B^c, C^c, D^c)$ by

$$\Sigma^c(t) := \begin{cases} (A_j^c, B_j^c, C_j^c, D_j^c), & \text{if } N_{j-1} \leq t \leq N_j, \\ (0, \; 0, \; 0, \; 0), & \text{if } t = N_k. \end{cases}$$

Clearly, Σ^c is asymptotically stable since $A^c(N_k) = 0$. Also, Σ^c internally stabilizes $P_i(z)$ because within N_k steps, Σ^c switches to Σ_i^c and drives the state of the closed-loop system to zero state. [Recall that $C_i(z)$ is a deadbeat controller for $P_i(z)$.] This completes the proof.

Remark 4.3: It is possible to give a proof of Theorem 4.2 using the representation of an LTI system as an N-periodic system. However, this proof using the results of [20] turns out to be rather long and complicated. We omit this alternate proof for the sake of brevity.

Remark 4.4: Following the reasoning used in the proof of Theorem 3.3, it is easy to show that in Theorem 4.1, generically, one can find an asymptotically stable 2-periodic stabilizing controller. In Theorem 4.2, it is possible to show that generically for a pair of plants, one can obtain a 2-periodic simultaneously stabilizing controller. However, for a K-tuple of plants, we have not been able to obtain a similar conclusion.

V. Sensitivity Minimization

We have seen in Section III that via time-varying periodic compensation, one can in certain cases improve the gain margin for a linear time-invariant system. In [12], it is shown that for time-invariant SISO plants, the gain margin optimization problem and the sensitivity minimization problem of [24] and [9] are equivalent (in a certain precise sense) provided we use linear time-invariant compensation. Given the results of Section III, one might expect that one could perhaps improve the minimum sensitivity by using time-varying compensators. In this section, however, we show that in contrast to the gain margin problem, the minimum weighted sensitivity for a linear time-invariant plant is attained when the compensator is also time-invariant. Thus, the use of time-varying controllers in the context of sensitivity minimization offers no advantage.

We should mention that in a recent paper, Feintuch and Francis [8] consider the sensitivity minimization problem for a time-varying plant and derive the existence of an optimal controller. This controller will, of course, in general be time-varying.

In order to state and prove our result, we will first need some notation. We are essentially following the setup in [9], [7], [8]. Let, as before, h^2 denote the Hilbert space of square summable sequences $\{h_i : i \geq 0\}$. Let H^2 denote the "z-transform" of h^2, i.e., $H^2 := \{\sum_{i=0}^{\infty} h_i z^{-i} : \{h_i\} \text{ is in } h^2\}$. It is a standard fact from Fourier analysis (see, e.g., [16]) that H^2 can be identified with the space of analytic functions in the complement of the closed unit disk $\bar{D}$ with square integrable boundary values, and with this induced Hilbert space structure, H^2 is isomorphic to h^2. (For all the relevant details, see [16], [6].) Note that stability of a time-invariant system corresponds to analyticity of the transfer function in the complement of the unit disk.

Throughout this section, $P(z)$ will denote the $p \times m$ transfer matrix of an LTI discrete-time plant with no poles or zeros on the unit circle $\{z : |z| = 1\}$. Consider now left and right coprime stable proper factorizations of $P(z)$

$$P(z) = A(z)B(z)^{-1} = \tilde{B}(z)^{-1}\tilde{A}(z)$$

with the corresponding Bezout identities

$$X(z)A(z) + Y(z)B(z) = I$$

$$\tilde{A}(z)\tilde{X}(z) + \tilde{B}(z)\tilde{Y}(z) = I.$$

(Here all the matrices are stable and proper.) From Zames [23], it follows that the set of all internally stabilizing, linear, time-varying, causal compensators for P is given by

$$\{C = (\tilde{X} + BZ)(\tilde{Y} - AZ)^{-1} : Z$$

$$: (H^2)^p \rightarrow (H^2)^m \text{ is a bounded linear causal operator}\}. \quad (5.1)$$

It is important to emphasize here that Z is *not necessarily a multiplication operator*. Indeed, the class of all *LTI compensators* for P is obtained by letting Z vary in the space $(RH^\infty)^{m \times p}$, where RH^∞ denotes the space of bounded real rational functions on the complement of the unit disk. Thus, the "multiplication" in the above expression (5.1) for C should be thought of as composition of operators.

Let $C = (\tilde{X} + BZ)(\tilde{Y} - AZ)^{-1}$ be a stabilizing compensator for P. Let W_1 and W_2 be unimodular matrices in $(RH^\infty)^{p \times p}$. Following Zames [23] we define the *weighted sensitivity operator*

$$S := W_2(I + PC)^{-1}W_1 : (H^2)^p \rightarrow (H^2)^p.$$

A simple computation shows that

$$S = W_2(\tilde{Y} - AZ)\tilde{B}W_1.$$

Set

$$\nu := \inf \{\|S\| : C \text{ stabilizing, causal}\}$$
$$= \inf \{\|S\| : Z \text{ bounded, causal, } Z : (H^2)^p \to (H^2)^m\}.$$

Moreover, set

$$\nu_1 := \inf \{\|S\| : C \text{ stabilizing, causal, and LTI}\}$$
$$= \inf \{\|S\|_\infty : Z \in (RH^\infty)^{m\times p}\}.$$

Now in [9], [7] it is shown that ν_1 is in point of fact obtained for some linear time-invariant optimal controller C. Clearly $\nu \leqslant \nu_1$, and so it is natural to conjecture that minimal sensitivity could be improved by taking the compensator C to be time-varying (we are after all minimizing over a much larger class of operators). However, this is not the case. Indeed we have the following.

Theorem 5.2: $\nu = \nu_1$. That is, the minimal sensitivity for an LTI plant cannot be improved by taking time-varying compensators.

Proof: We follow essentially the same method as in [9], [24], and [7]. Using the above notation, our assumption that $P(z)$ has no poles on the unit circle $\{|z| = 1\}$, implies that $\tilde{B}$ is nonsingular on the unit circle. Therefore (see, e.g., [7])

$$\tilde{B}W_1 = (\tilde{B}W_1)_o(\tilde{B}W_1)_i$$

where $(\tilde{B}W_1)_o$ is an outer factor of $\tilde{B}W_1$, and $(\tilde{B}W_1)_i$ is an inner factor of $\tilde{B}W_1$. Then

$$\|S\| = \|W_2(\tilde{Y} - AZ)(\tilde{B}W_1)_o(\tilde{B}W_1)_i\|$$
$$= \|W_2(\tilde{Y} - AZ)(\tilde{B}W_1)_o\|.$$

Set $T = W_2\tilde{Y}(\tilde{B}W_1)_o$, $U = W_2A$, $V = Z(\tilde{B}W_1)_o$. Since $(\tilde{B}W_1)_o$ is unimodular, we have that

$$\nu = \inf \{\|S\| : C \text{ stabilizing, causal}\}$$
$$= \inf \{\|T - UV\| : V : (H^2)^p \to (H^2)^m \text{ bounded, causal}\}.$$

In order to complete the proof we must show that ν is obtained when $V \in (RH^\infty)^{m\times p}$.

Set $Q := U(H^2)^m$, and let $Q^\perp$ denote the orthogonal complement of Q in $(H^2)^m$. Let Π_Q and $\Pi_{Q^\perp}$ denote the projection operators from $(H^2)^m$ onto Q and $Q^\perp$, respectively. Then

$$T - UV = \Pi_{Q^\perp}(T - UV) + \Pi_Q(T - UV)$$
$$= \Pi_{Q^\perp}T + \Pi_Q(T - UV).$$

Consequently, $\|T - UV\| \geqslant \|\Pi_{Q^\perp}T\|$ for all V causal, bounded. But by the work of [9], there exists $V_o \in (RH^\infty)^{m\times p}$ such that

$$\|T - UV_o\| = \|T - UV_o\|_\infty$$
$$= \|\Pi_{Q^\perp}T\|.$$

This clearly completes the proof.

Remark 5.3: Theorem 5.2 together with the results of Section II, show that for an N-periodic discrete-time plant P, an optimal compensator C (in the sense of minimizing sensitivity) will also be N-periodic. In order to see this, just represent P as an LTI plant, and thus by Theorem 5.2 an optimal compensator C will also be LTI (of the same input and output dimensions). Hence, C will represent an N-periodic system. The problem is however, because of (2.5), in order to ensure that C corresponds to a *causal* N-periodic system, we must have that $C(\infty)$ be lower triangular. At present, we do not see how this can be ensured, and so we leave this as an open problem.

Remark 5.4: Results analogous to Theorem 5.2 have been independently and simultaneously obtained by Feintuch and Francis [25].

VI. Conclusions

In this paper we have seen how the use of periodic time-varying controllers can be used to improve robustness, and completely solve the strong and simultaneous stabilization problems for discrete-time linear time-invariant systems. Moreover, we have indicated how such controllers can be implemented by using a very simple and natural design procedure.

As we have noted, if we restrict ourselves to linear time-invariant compensators, then there are strict bounds on the maximal obtainable gain margin in case the plant has unstable poles and zeros. But by implementing simple feedback designs involving periodic compensation, it is possible to guarantee arbitrarily large gain and phase margins for bicausal plants, and to improve these margins for strictly proper plants. (However, the optimal design for the strictly proper case using periodic controllers is still not solved, and remains an important research area.) Moreover, while the solution of the strong stabilization problem using LTI controllers is determined by a certain interlacing property of the real unstable poles and zeros of the plant, again by using periodic compensation, the problem becomes trivially solvable.

In contrast to all of this, for the question of sensitivity minimization of LTI plants, time-varying controllers offer no advantage. An interesting open problem is whether one can minimize the sensitivity of a periodic plant by using a causal periodic compensator.

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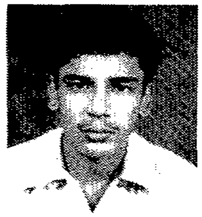
Pramod P. Khargonekar (S'81-M'81), for a photograph and biography, see p. 1013 of the October 1985 issue of this TRANSACTIONS.



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