Technical Notes and Correspondence

Stabilization of Time-Delay Systems Using Finite-Dimensional Compensators

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Abstract—**For linear time-invariant systems with one or more noncommensurate time delays, necessary and sufficient conditions are given for the existence of a finite-dimensional stabilizing feedback compensator. In particular, it is shown that a stabilizable time-delay system can always be stabilized using a finite-dimensional compensator. The problem of explicitly constructing finite-dimensional stabilizing compensators is also considered.**

I. Stabilization of Systems with Delays

In this note we consider the problem of stabilizing a linear time-invariant continuous-time system with q noncommensurate time delays $h_1, h_2, \cdots, h_q$. The systems we shall study are given by a state representation of the form

$$\frac{dx(t)}{dt} = (F(d_{h_1}, d_{h_2}, \cdots, d_{h_q})x)(t) + (G(d_{h_1}, d_{h_2}, \cdots, d_{h_q})u)(t)$$
$$y((t) = (H(d_{h_1}, d_{h_2}, \cdots, d_{h_g})x)(t) + (J(d_{h_1}, d_{h_2}, \cdots, d_{h_q})u)(t) \quad (1.1)$$

where the m-vector $u(t)$ is the input at time t, the n-vector $x(t)$ is the instantaneous state at time t, the p-vector $y(t)$ is the output at time t, and $F(d_{h_1}, \cdots, d_{h_q})$, $G(d_{h_1}, \cdots, d_{h_q})$, $H(d_{h_1}, \cdots, d_{h_q})$, $J(d_{h_1}, \cdots, d_{h_q})$ are matrices whose entries are polynomials in the delay operators $d_{h_1}, \cdots, d_{h_q}$ with coefficients in the reals $\mathbb{R}$. (Here $(d_{h_i}^r f)(t) = f(t - rh_i)$ for any positive integer r.)

With the system (1.1), we shall associate the quadruple $(F(z), G(z), H(z), J(z))$, where $z = (z_1, z_2, \cdots, z_q)$ and $F(z), G(z), H(z), J(z)$ are the coefficient matrices in (1.1) with d_{h_i} replaced by z_i. Conversely, any quadruple $(F(z), G(z), H(z), J(z))$ of matrices over the ring $\mathbb{R}[z]$ of polynomials in the z_i defines a time-delay system of the form (1.1) in the obvious way. We shall always assume that $J(z) = 0$ for the given system (1.1), and we shall denote this system by the triple $(F(z), G(z), H(z))$.

A fundamental problem in the control of systems with delays is determining whether or not there is an (output) feedback system $(A(z), B(z), C(z), D(z))$ over the polynomial ring $\mathbb{R}[z]$ or over the reals $\mathbb{R}$ (the finite-dimensional case) such that the closed-loop system consisting of the given system $(F(z), G(z), H(z))$ and the feedback system is internally asymptotically stable. If such a feedback system exists, we say that $(F(z), G(z), H(z))$ is *regulable*.

Several individuals have worked on the problem of feedback stabilization of systems with delays. Much of this past work has centered on the commensurate-delay case ($q = 1$) with delays in control only, delays in state only, or delays in both control and state (the case of interest here). For results on the case $q = 1$ and $q \geq 1$, we refer the reader to Pandolfi [1], Sontag [2], Morse [3], Manitius and Olbrot [4], Kamen [5], [6], Khargonekar and Sontag [7], Byrnes *et al.* [8], Kamen *et al.* [9], Spong and Tarn [10], Schumacher [11], Emre and Knowles [12], and Nett *et al.* [13].

In the paper of Kamen *et al.* [9], it is shown [Theorem (3.5)] that there is a feedback system $(A(z), B(z), C(z), D(z))$ over the polynomial ring $\mathbb{R}[z]$ such that the resulting closed-loop system is "*pointwise stable*" (which implies stability for all nonnegative values of the delays $h_1, h_2, \cdots, h_q$) if and only if

$$\text{rank } [sI - F(z) \vdots G(z)] = n \text{ and rank} \begin{bmatrix} H(z) \\ \cdots \\ sI - F(z) \end{bmatrix} = n \quad (1.2)$$

for all $(s, z) \in \bar{H} \times \bar{\Delta}^q$, where $\bar{H}$ is the closed right-half plane ($\bar{H} = \{s \in \mathbb{C}: \operatorname{Re} s \geq 0\}$, where $\mathbb{C}$ = field of complex numbers) and $\bar{\Delta}^q$ is the q-fold Cartesian product of the closed unit disk $\bar{\Delta} = \{z \in \mathbb{C}: |z| \leq 1\}$.

In Emre and Knowles [12], the authors use the fact that the proof of Theorem (3.5) in Kamen *et al.* [9] may be modified and extended to yield the following much stronger result.

Theorem 1.3: A system $(F(z), G(z), H(z))$ over $\mathbb{R}[z]$ where $z = (z_1, z_2, \cdots, z_q)$ is regulable (stabilizable) by a feedback system $(A(z), B(z), C(z), D(z))$ over $\mathbb{R}[z]$ if and only if

$$\text{rank } [sI - F(z) \vdots G(z)] = n \text{ and rank} \begin{bmatrix} H(z) \\ \cdots \\ sI - F(z) \end{bmatrix} = n \quad (1.4)$$

for all $s \in \bar{H}$ and $z = (e^{-h_1 s}, e^{-h_2 s}, \cdots, e^{-h_q s})$.

The first part of the stabilizability condition (1.4) originated in the work of Pandolfi [1], although Pandolfi's framework allowed for the presence of *distributed delays* in the system matrix F (with no delays in G), and Pandolfi's stabilizability result does not guarantee that there is a feedback system $(A(z), B(z), C(z), D(z))$ over $\mathbb{R}[z]$ whenever $F(z)$ is over $\mathbb{R}[z]$.

Condition (1.4) is clearly much weaker than condition (1.2) which ensures *stabilizability independent of delay*. However, it is interesting to note that condition (1.2) is weaker than the requirement that the given system admit a state representation which is *split* (i.e., reachable and coreachable in the ring-theoretic sense).

Emre and Knowles [12] comment that the notion of stabilizability independent of delay is too strong. However, as noted above this condition is weaker than the split condition, and the latter is known to be a generic condition in many situations (e.g., when the number of inputs and the number of outputs exceeds the number q of noncommensurate delays—see Lee and Olbrot [14]). Hence, we feel that the criticism in Emre and Knowles [12] of the notion of stabilizability independent of delay must be weighed against the need for the existence of a split state representation. Finally, it should also be pointed out that condition (1.2) can be checked by forming the resultants associated with the $n \times n$ minors of the matrices in (1.2). On the other hand, for $q > 1$ we are not aware of any general implementable procedure for checking (1.4). For results on the case $q = 1$ in checking (1.4), see Kamen *et al.* [18].

II. Stabilization Using Finite-Dimensional Compensators

The proof of Theorem (3.5) in Kamen *et al.* [9] can be extended to yield a result stronger than Theorem 1.3. In particular, we have the following new result.

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Theorem 2.1: A system $(F(z), G(z), H(z))$ *over* $\mathbb{R}[z]$ *where* $z = (z_1, z_2, \cdots, z_q)$ *is regulable by a finite-dimensional feedback system* (A, B, C, D) *(i.e., the matrices* A, B, C, D *are over the reals* $\mathbb{R}$*) if and only if condition (1.4) is satisfied.*

By Theorem 2.1, we have the surprising result that if a time-delay system over $\mathbb{R}[z]$ can be stabilized by a time-delay system, then it can be stabilized by a finite-dimensional system. This result is of obvious practical interest since it implies that stabilizable time-delay systems can *always* be stabilized without having to implement time delays.

In the literature there are sufficient conditions guaranteeing the existence of finite-dimensional stabilizing compensators for infinite-dimensional systems, such as systems with time delays (for example, see Schumacher [11] and Nett *et al.* [13]). However, we are not aware of any existing conditions which are equivalent to the necessary and sufficient condition given by (1.4). We should note that the application of the results in Schumacher [11] and Nett *et al.* [13] to systems with time delays is based on the representation of the system in terms of an infinite-dimensional state space, rather than the operator-ring representation given by (1.1). An advantage of the operator-ring approach is that necessary and sufficient conditions for stabilizability [i.e., (1.4)] are given directly in terms of the coefficient matrices $F(z)$, $G(z)$, $H(z)$ of the system representation. Hence, as far as the issue of stabilizability is concerned, it is not necessary to describe the system in terms of an infinite-dimensional state-space model (given by a differential equation in a Banach or Hilbert space).

We will now give an explicit, very simple proof of Theorem 2.1. In the commensurate-delay case, our proof leads to a systematic procedure for designing stabilizing feedback compensators. First, we need some notation.

Let H (respectively, $\bar{H}$) denote the open (closed) right-half plane. Let $\Gamma_r(\tilde{H})$ denote the algebra of functions holomorphic in H and continuous on the boundary of $\tilde{H} = \bar{H} \cup \{\infty\}$, and which have real coefficients (i.e., any power-series expansion about a real point has real coefficients). We then have the following standard lemma (Edwards [15], Mergelyan [16]).

Lemma 2.2: Let $\varphi: \tilde{H} \to \bar{\Delta}$ *be any conformal equivalence. Then any* $f \in \Gamma_r(\tilde{H})$ *may be uniformly approximated on* $\tilde{H}$ *by polynomials of the form* $a_n\varphi(s)^n + \cdots + a_1\varphi(s) + a_0\varphi(s)$*, where* $a_0, a_1, \cdots, a_n \in \mathbb{R}$*. In other words, for any* $f \in \Gamma_r(\tilde{H})$ *and any* $\epsilon > 0$*, there is a polynomial* $\pi_\epsilon(s) = a_n\varphi(s)^n + \cdots + a_1\varphi(s) + a_0\varphi(s)$ *such that*

$$\sup_{s\in \tilde{H}} |f(s) - \pi_\epsilon(s)| = \sup_{s \in \bar{H}} |f(s) - \pi_\epsilon(s)| < \epsilon.$$

Proof: Follows directly from Edwards [15], Mergelyan [16], or Rudin [17]. □

Remark 2.3: For simplicity, in the following development we will choose the conformal equivalence in Lemma 2.2 to be $\varphi(s) = (s-1)/(s+1)$.

Remark 2.4: It should be noted that there are rather simple algorithms for the approximation of functions in $\Gamma_r(\tilde{H})$. Indeed, by Lemma 2.2, it is enough to see how to approximate by polynomials a function g belonging to $\Gamma_r(\bar{\Delta})$, the disk algebra of functions holomorphic on the open unit disk Δ and continuous on the unit circle $\bar{T}$, and which have real coefficients. A standard method (Edwards [15]) for doing this is to construct the Fourier series

$$\sum_{n=0}^{\infty} c_n e^{jn\omega}, \ \omega \in [0, 2\pi]$$

associated with $g(e^{j\omega})$. (The coefficients c_n are all zero for $n < 0$ since g is holomorphic in the open unit disk Δ.) Now for each nonnegative integer N, define the polynomial

$$S_N(z) = \sum_{n \le N} c_n z^n, \ z \in \bar{\Delta}$$

and let $P_N(z)$ denote the Nth Cesáro sum given by

$$P_N(z) = \frac{S_0(z) + S_1(z) + \cdots + S_N(z)}{N+1}.$$

Then $P_N(z) \to g(z)$ uniformly on $\bar{\Delta}$. Hence, approximation of functions in $\Gamma_r(\tilde{H})$ reduces to computing Fourier coefficients.

We can now show how Theorem 2.1 follows from Theorem 3.5 in Kamen *et al.* [9].

Proof of Theorem 2.1: By Theorem 1.3, regulability of the given system $(F(z), G(z), H(z))$ implies that condition (1.4) must be satisfied. Conversely, suppose that the first part of (1.4) holds. Setting $F = F(z)$, $G = G(z)$, we certainly have that the matrices $(sI - F)/(s+1)$, $G/(s+1)$ have entries in $\Gamma_r(\tilde{H})$. Then by the first part of condition (1.4), it is easy to show (see the proof of Theorem (3.5) in Kamen *et al.* [9]) that there exist matrices $\tilde{P}$, $\tilde{Q}$ with entries in $\Gamma_r(\tilde{H})$ such that

$$((sI-F)/(s+1))\tilde{Q} + (G/(s+1))\tilde{P} = I \tag{2.5}$$

where I is the $n \times n$ identity matrix. By Lemma 2.2, we can uniformly approximate in $\tilde{H}$ the matrices $\tilde{P}$ and $\tilde{Q}$ by matrices whose entries are polynomials in $(s-1)/(s+1)$ with real coefficients; that is, we can find polynomial matrices P, Q in $(s-1)/(s+1)$ with real coefficients such that

$$((sI-F)/(s+1))Q + (G/(s+1))P = :A \tag{2.6}$$

is arbitrarily close to I. We claim that Q is bicausal; that is, Q has inverse Q^{-1} which is also a proper rational matrix in s. Indeed, write

$$Q = \frac{1}{(s+1)^q} [s^\mu Q_\mu + s^{\mu-1} Q_{\mu-1} + \cdots + Q_0]$$

where the Q_i are constant matrices. Clearly, $((G/(s+1))P)(\infty) = 0$ and $((sI-F)/(s+1))(\infty) = I$. Therefore, $Q(\infty) = A(\infty)$ must be close to I. Hence, $\mu = q$, Q_μ is invertible, and so Q is bicausal. It follows that det A can be written as

$$\det A = \frac{b_\nu s^\nu + \text{lower-order terms}}{(s+1)^\nu}, \quad \text{for some positive integer } \nu$$

and where b_ν is a nonzero constant. Further, as A is close to I in $\tilde{H}$, det $A \neq 0$ in $\tilde{H}$. Hence, PQ^{-1} defines a finite-dimensional proper stabilizing compensator. To finish the proof, just note that we can dualize the entire preceding argument, so that the second part of condition (1.4) implies the analogous result for observers. □

From the above proof, it is clear that in order to make our procedure constructive, we need a technique for computing Bezout-type identities of the form (2.5) over the algebra $\Gamma_r(\tilde{H})$. In the commensurate-delay case, such a technique has been developed by the authors (see Kamen *et al.* [18]). We will give the key elements of this technique below, and then in the next section we apply it to the stabilization of a time-delay system for which the construction of a stabilizing finite-dimensional compensator was unsolved (until this work).

For each complex number s_0, define $\theta_{s_0}(s) = (1 - z\exp(hs_0))/(s - s_0)$, where h is a fixed positive real number and $z = e^{-hs}$. The complex function $\theta_{s_0}(s)$ is the transfer function of a *distributed delay*. Let $\mathbb{C}(s, z)$ (respectively, $\mathbb{R}(s, z)$) denote the field of rational functions in s and z with coefficients in $\mathbb{C}$ (respectively, $\mathbb{R}$). Let $R_{\mathbb{C}}$ denote the subring of $\mathbb{C}(s, z)$ generated by z, $\{1/(z - z_0): |z_0| > 1\}$, $\{d^i/ds^i \theta_{s_0}: s_0 \in \mathbb{C}, i \ge 0\}$. Set $R = \mathbb{R}(s, z) \cap R_{\mathbb{C}}$ and let $R(s)$ denote the ring of rational functions in s with coefficients in R. Then if condition (1.4) holds (with $q = 1$), we have a constructive procedure (see Kamen *et al.* [18]) for computing matrices $\tilde{P}$ and $\tilde{Q}$ with entries in $R(s) \cap \Gamma_r(\tilde{H})$ such that the Bezout-type identity (2.5) is satisfied. Using the approximation procedure described in Remark 2.4, we can then approximate $\tilde{P}$ and $\tilde{Q}$ by polynomial matrices in $(s-1)/(s+1)$, which results in a finite-dimensional stabilizing compensator. The procedure is illustrated by the second example in the next section.

III. Examples

Our first example is taken from Schumacher [11].

Example 3.1: Consider the delay system given by

$$\dot{x}_1(t) = -(\pi/2)x_1(t-1) + x_2(t)$$
$$\dot{x}_2(t) = u(t)$$
$$y(t) = x_1(t).$$

Here

$$F(z)=\begin{bmatrix} -(\pi/2)z & 1 \\ 0 & 0 \end{bmatrix},\ G=\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \text{and}\ H=[1\ \ 0].$$

We have

$$[sI-F(z):G(z)]=\begin{bmatrix} s+(\pi/2)z & -1 & 0 \\ 0 & s & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} H(z) \\ \cdots \\ sI-F(z) \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ s+(\pi/2)z & -1 \\ 0 & s \end{bmatrix}.$$

Both of these matrices have rank 2 for all $s \in \mathbb{C}$ and all $z \in \mathbb{C}$, and thus the system is split. From existing results on systems over polynomial rings (see Morse [3]), it follows that for any stable polynomial $\alpha(s, z)$ of degree three in s (and where now $z = e^{-s}$), there is a feedback system over $\mathbb{R}[z]$ such that the closed-loop system has characteristic polynomial $\alpha(s, z)$. Using the transfer-function approach, we shall compute a stabilizing compensator which yields a closed-loop characteristic polynomial $\alpha(s, z)$ given by

$$\alpha(s, z)=(s+3+(\pi/2)z)(s^2+7s+4). \tag{3.2}$$

Clearly, $\alpha(s, z)$ is stable (all zeros are in the open left-half plane). Now the transfer function $W(s)$ of the system is given by

$$W(s)=\frac{1}{s(s+(\pi/2)z)}.$$

With $\alpha(s, z)$ given by (3.2), we have

$$s(s+(\pi/2)z)(s+10)+(25-(3\pi/2)z)s+(3+\pi/2)z)4=\alpha(s, z). \tag{3.3}$$

Then selecting the feedback compensator to have transfer function $C(s)$ given by

$$C(s)=\frac{(25-(3\pi/2z)s+(3+\pi/2)z)4}{s+10} \tag{3.4}$$

from (3.3) we have that $\alpha(s, z)$ is the characteristic polynomial of the resulting closed-loop system. Thus, the compensator defined by (3.4) is stabilizing. To compute a finite-dimensional compensator, we can attempt to approximate the delays in $C(s)$. Taking the simplest approximation ($z = e^{-s} = 1$), we have

$$C(s)=\frac{(25-(3\pi/2))s+(3+(\pi/2))4}{s+10}. \tag{3.5}$$

Using the Nyquist test, we found that the *first-order finite-dimensional compensator with transfer function C(s) given by (3.5) stabilizes the system for all values of delay.* In contrast, the finite-dimensional stabilizing compensator computed by Schumacher [11] has order 3.

Example 3.6: Consider the time-delay system given by

$$\dot{x}(t)=ax(t)+u(t-h)$$
$$y(t)=x(t) \tag{3.7}$$

where a is a positive real number. Here the time delay h is a delay in control. Since $a > 0$, the system is clearly unstable. Checking condition (1.4), we have

$$\text{rank}\ [s-a\ \ z]=1\ \text{for all}\ s\in\mathbb{C},\ \text{where}\ z=e^{-hs}$$

and

$$\text{rank}\begin{bmatrix} 1 \\ s-a \end{bmatrix}=1\ \text{for all}\ s\in\mathbb{C}.$$

Thus, by Theorem 2.1, there is a finite-dimensional stabilizing compensator. In fact, there is a finite-dimensional stabilizing compensator no matter how large a and h are! It is easy to find a stabilizing finite-dimensional compensator using "classical" techniques when $h \le 1$ and $a \le 1.8$, but for larger values of h or a, for example, $h = 1$ and $a = 2$, the construction of a finite-dimensional stabilizing compensator appears to be a nontrivial problem. We were not able to solve this using ad hoc techniques, such as replacing the delay by a Padé approximation, or by using lead compensation. We shall apply our procedure described in the previous section. The first step is to compute $\tilde{P}$, $\tilde{Q}$ over $R(s) \cap \Gamma_r(\tilde{H})$ such that

$$\frac{s-2}{s+1}\tilde{Q}+\frac{z}{s+1}\tilde{P}=1. \tag{3.8}$$

Using the procedure in Kamen *et al.* [18], we have

$$\tilde{Q}=\frac{s+4+9\theta(s)}{s+1}\ \text{and}\ \tilde{P}=\frac{9(e^2)}{s+1}$$

where $\theta(s) = (1 - e^2z)/(s - 2)$ is the transfer-function of a distributed delay (here $z = e^{-s}$). Then if we take the transfer function $C(s)$ of the compensator to be

$$C(s)=\frac{9(e^2)}{s+4+9\theta(s)} \tag{3.9}$$

the characteristic polynomial of the resulting closed-loop system is $(s + 1)^2$, and thus the compensator with transfer function $C(s)$ given by (3.9) is stabilizing. The input/output differential equation for this compensator is given by

$$\frac{dw(t)}{dt}+4w(t)+9\int_{t-h}^{t}\theta(t-\lambda)w(\lambda)d\lambda=9(e^2)y(t) \tag{3.10}$$

where $\theta(t)$ is the inverse Laplace transform of $\theta(s)$. The equation (3.10) can be implemented by using a finite-time numerical integration package to realize the distributed delay [the third term on the left side of (3.10)]. We should note that for the given system (3.7), Manitius and Olbrot [4] also obtain a stabilizing compensator with distributed delays, except that they consider input and state feedback, whereas we use state feedback only. Now suppose that we want a finite-dimensional stabilizing compensator for the given system (3.7). As noted in the previous section, we can construct a finite-dimensional compensator by first approximating $\tilde{P}$, $\tilde{Q}$ in (3.8) by polynomials in $(s - 1)/(s + 1)$. Since $\tilde{P}$ is already rational, there is no need to approximate it. To approximate $\tilde{Q}$, define

$$f(z)=\left.\frac{\theta(s)}{s+1}\right|_{s=\frac{1-z}{1+z}}.$$

Let $c_0, c_1, \cdots$, denote the Fourier coefficients associated with $f(e^{j\omega})$. The c_i can be computed with reasonable accuracy using the fast Fourier transform. So there exists excellent software for calculating the c_i. Now define the "Cesáro sum"

$$H_n(s)=\frac{s+4}{s+1}+9\sum_{i=0}^{n}\left(\frac{n-i+1}{n+1}\right)c_i\left(\frac{s-1}{s+1}\right)^i$$

Then $H_n \to \tilde{Q}$ as $n \to \infty$ uniformly on $\tilde{H}$. Defining

$$C_n(s)=\frac{9(e^2)}{(s+1)H_n(s)}$$

we have that $C_n(s)$ is a stabilizing compensator for n suitably large. Using a computer program for the Nyquist test (supplied to us by M. Taylor), we found that $C_n(s)$ is stabilizing for $n = 11$, so we have a stabilizing compensator of order 11. Using a third-order Padé approximation of $\theta(s)$, we obtained a fourth-order stabilizing compensator. The relationship between the order of the stabilizing compensator and the approximation technique that is used is left for future work.

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A Note on the Location of the Roots of a Polynomial

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Abstract—**Two new theorems and two corollaries are presented which give sufficient conditions for a polynomial to have all its roots inside the unit circle. These results unify and extend certain earlier stability tests for discrete time systems. The significance of the new results is illustrated by a couple of examples.**

I. Introduction

A linear, time-invariant discrete time system is asymptotically stable if all roots of the system's characteristic polynomial

$$N(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_n \tag{1}$$

with

$$c_0 = 1, \quad c_i \in \mathbb{R}, \qquad i = 0, 1, 2, \cdots, n$$

are located inside the unit circle $|z| < 1$. Besides the well-known comprehensive stability tests by Jury and others [1]-[5], there exist in the literature certain sufficient stability conditions, which we will call the monotony and the dominance conditions. Both conditions are based on a simple test of the coefficients c_i of $N(z)$ and can be applied with ease even in the case of high-order polynomials.

This correspondence provides a new theorem and a corollary which lead to a so-called monotony/dominance condition. This criterion contains the monotony and the dominance conditions as special cases. As a second result we present a new theorem and a corollary that permit stability tests of a characteristic equation with perturbed coefficients.

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II. Known Sufficient Conditions

For reference purposes we shall repeat the following well-known theorems.

Theorem 1 (Monotony Condition): If the coefficients c_i of polynomial (1) satisfy the inequality

$$1 = c_0 > c_1 > c_2 > \cdots > c_n > 0 \tag{2}$$

then all roots of $N(z)$ lie inside the unit circle.

Theorem 2 (Dominance Condition): If the coefficients c_i of (1) satisfy the inequality

$$1 = c_0 > \sum_{i=1}^{n} |c_i| \tag{3}$$

then all the roots or $N(z)$ lie inside the unit circle.

For a proof of both theorems see, for example, [3].

III. New Sufficient Conditions

Our first result is as follows.

Theorem 3: If the coefficients c_i of (1) satisfy the following conditions:

i) $$c_0 > c_1 > c_2 > \cdots > c_k > 0, \qquad 0 \leqslant k \leqslant n \tag{4}$$

ii) $$|c_{k+1} - \sigma c_k| + |c_{k+2} - \sigma c_{k+1}| + \cdots + |c_n - \sigma c_{n-1}| + |\sigma c_n| < (1-\sigma)(c_0 + c_1 + \cdots + c_{k-1}) + c_k \tag{5}$$

with

$$\sigma = \begin{cases} 0, & k = 0 \\ \max\ (c_1/c_0,\ c_2/c_1,\ \cdots,\ c_k/c_{k-1}), & k > 0 \end{cases} \tag{6}$$

then all roots of $N(z)$ lie inside the unit circle.

Proof: We formulate a new polynomial

$$N_1(z) = (z - \sigma) \cdot N(z) = c_0 z^{n+1} + (c_1 - \sigma c_0) z^n + (c_2 - \sigma c_1) z^{n-1} + \cdots - \sigma c_n. \tag{7}$$

According to Theorem 2, all roots of $N_1(z)$ lie inside the unit circle, if

$$|c_0| > |c_1 - \sigma c_0| + |c_2 - \sigma c_1| + \cdots + |\sigma c_n|. \tag{8}$$

We obtain from (6) that

$$c_i - \sigma c_{i-1} \leqslant 0, \qquad i = 1, 2, \cdots, k. \tag{9}$$

With this result and (4) we can rewrite (8) as

$$c_0 > -(c_1 - \sigma c_0) - (c_2 - \sigma c_1) - \cdots - (c_k - \sigma_{k-1}) + |c_{k-1} - \sigma c_k| + \cdots + |\sigma c_n| \tag{10}$$

or

$$|c_{k+1} - \sigma c_k| + \cdots + |\sigma c_n| < (1-\sigma)(c_0 + c_1 + c_2 + \cdots + c_{k-1}) + c_k.$$

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