

GABOR AND WAVELET ANALYSIS WITH APPLICATIONS TO SCHATTEN CLASS INTEGRAL OPERATORS

A Thesis
Presented to
The Academic Faculty

by

Shannon Renee Smith Bishop

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Mathematics

Georgia Institute of Technology
May 2010

**GABOR AND WAVELET ANALYSIS WITH
APPLICATIONS TO SCHATTEN CLASS INTEGRAL
OPERATORS**

Approved by:

Professor Chris Heil, Advisor
School of Mathematics
Georgia Institute of Technology

Professor Bill Green
School of Mathematics
Georgia Institute of Technology

Professor Michael Lacey
School of Mathematics
Georgia Institute of Technology

Professor Doron Lubinsky
School of Mathematics
Georgia Institute of Technology

Professor Elliot Moore
Department of Electrical and
Computer Engineering
Georgia Institute of Technology

Date Approved: March 11, 2010

ACKNOWLEDGEMENTS

I am indebted to a number of people for their assistance with this thesis. I thank you.

To my advisor, Chris Heil: for his ideas and encouragement.

To the faculty, staff and graduate students at Georgia Tech, especially Alfred Andrew, Christy Dalton, Luca Dieci, Justin Filoleta, Celia Grams, Bill Green, Klara Grodzinsky, Matt Hanes, Evans Harrell, Karen Hinds, Mitch Keller, Michael Lacey, Lew Lefton, Jan Lewis, Doron Lubinsky, Chris Malbrue, Sharon McDowell, Elliot Moore, Annette Rohrs, Genola Turner and Chris Weinberg: for doing what they do so well.

To my husband, Daniel Bishop: for his understanding.

To my parents, Rick and Kathleen Smith: for their unwavering support.

To my brother, Derek Smith: for the 30-year bet, which has been a surprising source of motivation.

To my in-laws, Phillip and Brenda Bishop: for their encouragement.

To my cat, Waffles: for keeping my lap warm.

To God: for allowing me this unique experience.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS	iii
SUMMARY	vii
I INTRODUCTION AND BACKGROUND	1
1.1 Introduction	1
1.2 Background	6
1.2.1 Weights and Mixed Norm Spaces	6
1.2.2 Transforms	8
1.2.3 Frames	11
1.2.4 Operators	12
II DENSITY COMPARISON FOR SEPARABLE WAVELET FRAMES	15
2.1 Introduction	15
2.2 Definitions and preliminary lemmas	18
2.2.1 Affine Group	18
2.2.2 General Density	19
2.2.3 Affine Density	20
2.2.4 Fourier Frames	25
2.2.5 Wavelet Frames	25
2.3 Main result	28
2.4 Comparison Theorem for wavelet frames	32
2.5 Comparison Theorem for wavelet frame sequences	35
III AFFINE PSEUDODIFFERENTIAL OPERATORS	40
3.1 Introduction	40
3.1.1 Relationship to Pseudodifferential Operators	40
3.1.2 Summary of Results	41
3.2 Definitions and preliminary lemmas	44
3.2.1 Integral operator composition	44

3.2.2	Singular integral operators	45
3.2.3	The relationship between kernel and symbol	45
3.3	A Schatten Class Result for Integral Operators	46
3.4	New Kernel and Symbol Classes	51
3.4.1	Kernel Spaces	52
3.4.2	Symbol Spaces	55
3.5	Schatten Class Affine Pseudodifferential Operators	59
3.6	Affine Pseudodifferential Operators as Calderon-Zygmund Operators	65
IV	MIXED MODULATION SPACES AND PSEUDODIFFERENTIAL OP- ERATORS	69
4.1	Introduction	69
4.2	Definitions and preliminary lemmas	72
4.2.1	Gabor Frames and Wilson Bases	72
4.2.2	The relationship between kernel and symbols	73
4.3	Mixed Modulation Spaces	76
4.3.1	The Inversion Formula for Mixed Modulation Space	78
4.3.2	Mixed Modulation Spaces as Banach Spaces	81
4.3.3	A larger window class	83
4.3.4	Banach Frames	86
4.3.5	Mixed Modulation Embeddings	91
4.3.6	Wilson Bases	92
4.4	Bounded Pseudodifferential Operators	96
4.5	Pseudodifferential Operators and Schatten classes	99
4.5.1	New Kernel and Symbol classes	99
4.5.2	Relationship between old and new kernel and symbol classes	104
V	SCHATTEN CLASS FOURIER INTEGRAL OPERATORS	111
5.1	Introduction	111
5.2	A Schatten class result for Fourier Integral Operators	113
5.3	Pointwise Multiplication in the Mixed Modulation Spaces	116

5.4 Sharp Time-Frequency Conditions on the Symbol of a Fourier Integral Operator	127
APPENDIX A FOURIER FRAME LEMMA	132
REFERENCES	136

SUMMARY

This thesis addresses four topics in the area of applied harmonic analysis. First, we show that the affine densities of separable wavelet frames affect the frame properties. In particular, we describe a new relationship between the affine densities, frame bounds and weighted admissibility constants of the mother wavelets of pairs of separable wavelet frames. This result is also extended to wavelet frame sequences. Second, we consider affine pseudodifferential operators, generalizations of pseudodifferential operators that model wideband wireless communication channels. We find two classes of Banach spaces, characterized by wavelet and ridgelet transforms, so that inclusion of the kernel and symbol in appropriate spaces ensures the operator is Schatten p -class. Third, we examine the Schatten class properties of pseudodifferential operators. Using Gabor frame techniques, we show that if the kernel of a pseudodifferential operator lies in a particular mixed modulation space, then the operator is Schatten p -class. This result improves existing theorems and is sharp in the sense that larger mixed modulation spaces yield operators that are not Schatten class. The implications of this result for the Kohn-Nirenberg symbol of a pseudodifferential operator are also described. Lastly, Fourier integral operators are analyzed with Gabor frame techniques. We show that, given a certain smoothness in the phase function of a Fourier integral operator, the inclusion of the symbol in appropriate mixed modulation spaces is sufficient to guarantee that the operator is Schatten p -class.

CHAPTER I

INTRODUCTION AND BACKGROUND

1.1 Introduction

Decomposition and reconstruction are ideas fundamental to harmonic analysis and signal processing. The Fourier transform, arguably the bedrock of these fields, would not be so interesting if we could not decompose a distribution f into frequencies $\widehat{f}(w)$ and reconstruct f again by

$$f(x) = \int \widehat{f}(w)e^{2\pi ix \cdot w} dw.$$

Other transforms also possess the powerful properties of decomposition and reconstruction. In particular, by the wavelet transform and the Gabor transform functions are decomposed into time-scale and time-frequency data, respectively, and are so characterized by this data that they can be reconstructed from it. There are discrete transforms controlled by sequences $\{f_x\}_{x \in X}$ which admit dual systems $\{\widetilde{f}_x\}_{x \in X}$ so that decompositions

$$f = \sum_{x \in X} \langle f, \widetilde{f}_x \rangle f_x = \sum_{x \in X} \langle f, f_x \rangle \widetilde{f}_x, \quad (1)$$

hold for all f in appropriate function spaces. In general, the combined action of decomposition and reconstruction is called a resolution of the identity.

This thesis contains new insight into the applications of resolutions of the identity. It explores how these resolutions of the identity can capture information about integral operators, that is, operators of the form

$$Af(t) = \int k(t, y)f(y) dy.$$

A resolution of the identity for either the elements of the domain or codomain of A gives a resolution of the identity of the operator, and through this decomposition and

reconstruction, properties of the operator are more apparent. Specifically, suppose that $\{\psi_x\}_{x \in X}$ is some collection of functions generating a resolution of the identity for $L^2(\mathbb{R}^d)$, that is

$$\langle f, g \rangle = C \int_X \langle f, \psi_x \rangle \langle \psi_x, g \rangle d\mu(x) \quad \text{for all } f, g \in L^2(\mathbb{R}^d).$$

Then the mixed norm of the slices of k , i.e. $k_y(t) = k(t, y)$, decomposed with the resolution of the identity determines whether the integral operator with kernel k is Schatten p -class. Specifically, we show that if

$$\left(\int_X \left(\int_{\mathbb{R}^d} |\langle k_y, \psi_x \rangle|^2 dy \right)^{\frac{p}{2}} d\mu(x) \right)^{\frac{1}{p}} < \infty$$

and $p \in [1, 2]$, then the integral operator with kernel k is Schatten p -class. This result is stated precisely as Theorem 3.3.2.

Theorem 3.3.2 is powerful because of its generality. It is a result applicable to all integral operators and resolutions of the identity. However, while integral operators are a broad class of operators, many interesting integral operators are not naturally expressed in the form $Af(t) = \int k(t, y)f(y) dy$. Pseudodifferential operators, for instance, are integral operators that are superpositions of time-frequency shifts and these operators are often specified not by their kernels but by their symbols, functions controlling the “amount” of each time-frequency shift present in the operator. Similarly, affine pseudodifferential operators are superpositions of time-scale shifts. These operators are determined by their symbols, which describe the amount of each time-scale shift in the operator. Fourier integral operators are integral operators determined by both a symbol function and a phase function.

Since pseudodifferential operators, affine pseudodifferential operators, and Fourier integral operators are naturally formulated in terms of symbol functions, it is desirable to characterize the properties of these operators by characterizing the properties of their symbols. It is clear that by relating the symbol of one of these operators to the kernel of the integral operator, Theorem 3.3.2 can be used to describe some

property of the symbol that ensures the operator is Schatten p -class. However, the meaning of this property depends on the resolution of the identity. This thesis shows that for pseudodifferential operators, affine pseudodifferential operators, and Fourier integral operators, analyzing the symbol with the “correct” resolution of the identity yields natural and meaningful conditions on the symbol that ensure the corresponding operator is Schatten class.

Because of the multipath and Doppler effects, a wideband wireless communication channel can be modeled as a superposition of time-scale shifts, i.e. as an affine pseudodifferential operator. Affine pseudodifferential operators have been relatively unstudied until recently, with [5], [29] and [68] the only mathematical publications on the topic. However, their application to wireless communications ensures that they are of interest to mathematicians and engineers alike. As affine pseudodifferential operators are superpositions of time-scale shifts, it is natural to analyze these operators with a time-scale resolution of the identity. In Chapter 3, Theorem 3.3.2 is used with the wavelet resolution of the identity to find new conditions on the kernel and symbol of an affine pseudodifferential operator that ensure the operator is Schatten p -class. These conditions on the symbol are equivalent to inclusion in a Banach space characterized by a mixed norm on the ridgelet transform, a transform which captures directional time-scale data about the symbol. This chapter also describes smoothness and decay conditions on the Radon transform of the symbol that imply the given operator is Schatten class or Calderon-Zygmund.

Pseudodifferential operators are superpositions of time-frequency shifts. Because the Doppler effect for narrowband wireless communications is best modeled not as a change in scale but as a shift in frequency, pseudodifferential operators model narrowband wireless communications. In Chapter 4, Theorem 3.3.2 is used with a Gabor resolution of the identity, a natural choice for analyzing pseudodifferential operators. The resulting mixed norm is a time-frequency decay condition on the kernel itself.

We show that this condition holds for kernels belonging to certain Banach spaces that we call mixed modulation spaces. These spaces are natural generalizations of the traditional modulation spaces, and in Chapter 4, we show that many of the interesting properties of traditional modulation spaces also hold for mixed modulation spaces. Furthermore, by exploiting the relationship between the Gabor transforms of the kernel and Kohn-Nirenberg symbol, we show that inclusion of this symbol in an appropriate mixed modulation space guarantees the corresponding pseudodifferential operator is Schatten class.

Fourier integral operators arise naturally in the study of hyperbolic differential equations because they give approximate solutions to certain partial differential equations. Although Fourier integral operators are more complex than pseudodifferential operators and affine pseudodifferential operators because they are controlled by a symbol and a phase function, we can focus on the influencing properties of the symbol when the phase function is smooth. This is the approach taken in Chapter 5. Like pseudodifferential operators, Fourier integral operators act on the time-frequency content of functions. In Chapter 5 we prove that the mixed modulation spaces are the natural symbol spaces for describing Fourier integral operators. In particular, we show that if a Fourier integral operator has a sufficiently smooth phase function and a symbol belonging to an appropriate mixed modulation space, then the operator is Schatten class.

Although our analysis of pseudodifferential, affine pseudodifferential and Fourier integral operators begins with the idea in Theorem 3.3.2, our results are not merely direct applications of this theorem. Rather we use the idea of Theorem 3.3.2 with the unique properties of each of these types of operators to develop our Schatten class analysis. As a consequence, each of the Schatten class results in Chapters 3, 4 and 5 has a flavor different from that of Theorem 3.3.2 and different from one another.

Furthermore, the Schatten class results in Chapters 3, 4 and 5 augment the knowledge of pseudodifferential, affine pseudodifferential and Fourier integral operators found in the literature. In particular, our result for affine pseudodifferential operators in Chapter 3 is the first Schatten class result for affine pseudodifferential operators. Although time-frequency analysis is an oft-used tool to study pseudodifferential operators, the approach in Chapter 4 is new and yields new symbol spaces of Schatten class pseudodifferential operators undiscovered by previous results. Furthermore, the kernel results for pseudodifferential operators in Chapter 4 improve upon existing kernel theorems and are sharp. The results for Schatten class Fourier integral operators in Chapter 5 are not directly comparable to previously known results. However, through natural isomorphisms, the largest symbol classes in the literature embed into the symbol classes described in Chapter 5. In addition, several of the results in this chapter are sharp. The relationships between the results in this thesis and related results in the literature are described in greater detail in each chapter.

At their heart, Chapters 3, 4 and 5 depend on the idea of resolution of the identity. In Chapter 2, we explore the behavior of a specific discrete resolution of the identity of the form (1). Specifically, we compare pairs of wavelet frames of the form $\{\sigma(u, v)f\}_{u \in U, v \in V}$ and $\{\sigma(s, t)g\}_{s \in S, t \in T}$ (see Chapter 3 for a precise definition). Our main result is a Homogeneous Approximation Property for separable wavelet frames that allows us to delineate relationships between the densities of U, V, S and T , the admissibility constants of f, g and the frame bounds of the sequences $\{\sigma(u, v)f\}_{u \in U, v \in V}$ and $\{\sigma(s, t)g\}_{s \in S, t \in T}$. This result is interesting because it parallels known results for other common types of resolutions of the identities, namely LCA frames, and because it gives insight into the density properties of wavelet frames. In particular, unlike LCA frames, wavelet frames are known not to exhibit a Nyquist density. Our main result shows that wavelet frames fail to have a Nyquist density because density depends on both frame bounds and admissibility.

1.2 Background

In this section we give precise definitions and properties of the topics fundamental to the main ideas of the thesis.

Given sets S, X such that $S \subset X$, we define $\chi_S : X \rightarrow \mathbb{R}$ by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

We let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of functions of d real variables.

Definition 1.2.1. *Suppose $f, g : X \rightarrow [0, \infty)$. Then f and g are equivalent, written $f \equiv g$, if there exists $C \in (0, \infty)$ such that*

$$\frac{g(x)}{C} \leq f(x) \leq Cg(x) \quad \forall x \in X.$$

1.2.1 Weights and Mixed Norm Spaces

1.2.1.1 Weight functions

Definition 1.2.2. *A locally integrable function $v : \mathbb{R}^d \rightarrow (0, \infty)$ is called a weight function. A weight function $v : \mathbb{R}^d \rightarrow (0, \infty)$ is submultiplicative if*

$$v(z_1 + z_2) \leq v(z_1)v(z_2) \quad \text{for all } z_1, z_2 \in \mathbb{R}^d.$$

A weight function v has polynomial growth if there are $C, s \geq 0$ such that $v(z) \leq C(1 + |z|)^s$ for all $z \in \mathbb{R}^d$.

For each $s \geq 0$, the function $v_s(z) = (1 + |z|)^s$ is a submultiplicative weight function with polynomial growth. Notice that v_s is equivalent to the weight $(1 + |z|^2)^{\frac{s}{2}}$. We will use $(1 + |z|^2)^{\frac{s}{2}}$ and $(1 + |z|)^s$ interchangeably as weights on mixed norm spaces.

Definition 1.2.3. *Suppose $w : \mathbb{R}^d \rightarrow (0, \infty)$ is a weight function and $v : \mathbb{R}^d \rightarrow (0, \infty)$ is submultiplicative. If there is a constant C such that*

$$w(z_1 + z_2) \leq Cv(z_1)w(z_2) \quad \text{for all } z_1, z_2 \in \mathbb{R}^d,$$

then we call w a v -moderate weight.

We will assume throughout this thesis that $v : \mathbb{R}^d \rightarrow (0, \infty)$ is a submultiplicative weight function of polynomial growth symmetric in each coordinate, i.e. $v(x_1, \dots, -x_i, \dots, x_d) = v(x_1, \dots, x_i, \dots, x_d)$ for each $i = 1, 2, \dots, d$. We also assume throughout that w is a v -moderate weight.

1.2.1.2 Mixed norm spaces

Definition 1.2.4. Given measure spaces (X_i, μ_i) and indices $p_i \in [1, \infty]$ for $i = 1, 2, \dots, d$ and given weight function $w : X_1 \times X_2 \times \dots \times X_d \rightarrow (0, \infty)$, we let

$$L_w^{p_1, p_2, \dots, p_d}(X_1, X_2, \dots, X_d, \mu_1, \mu_2, \dots, \mu_d)$$

consist of all measurable functions $F : X_1 \times X_2 \times \dots \times X_d \rightarrow \mathbb{C}$ for which the following norm is finite:

$$\begin{aligned} & \|F\|_{L_w^{p_1, p_2, \dots, p_d}} \\ &= \left(\int_{X_d} \dots \left(\int_{X_1} |F(x_1, \dots, x_d) w(x_1, \dots, x_d)|^{p_1} d\mu_1(x_1) \right)^{\frac{p_2}{p_1}} \dots d\mu_d(x_d) \right)^{\frac{1}{p_d}}, \end{aligned}$$

with the usual modifications for those indices p_i which equal ∞ .

If the measures μ_i for all $i = 1, 2, \dots, d$ are clear from context we simply write $L_w^{p_1, p_2, \dots, p_d}(X_1, X_2, \dots, X_d)$. If $w = 1$ we write

$$L^{p_1, p_2, \dots, p_d}(X_1, X_2, \dots, X_d, \mu_1, \mu_2, \dots, \mu_d).$$

If $X_i = \mathbb{R}$ and μ_i is Lebesgue measure on \mathbb{R} for all $i = 1, 2, \dots, d$, then we simply write $L_w^{p_1, p_2, \dots, p_d}$. If each X_i is countable and μ_i is counting measure on X_i we simply write $\ell_w^{p_1, p_2, \dots, p_d}(X_1, X_2, \dots, X_d)$.

Unless otherwise noted, we assume that the measure associated to any subset of \mathbb{R} is Lebesgue measure and the measure associated to any countable set is counting measure.

The mixed norm spaces $L_w^{p_1, p_2, \dots, p_d}(X_1, X_2, \dots, X_d, \mu_1, \mu_2, \dots, \mu_{2d})$ are generalizations of the classical spaces L^p , and the proof that L^p is a Banach space can be extended to the mixed norm spaces (see [7]).

A Wiener amalgam norm is a type of mixed norm that measures local boundedness with global decay.

Definition 1.2.5. *Suppose $p_1, \dots, p_d \in [1, \infty]$. Define a norm by*

$$\|f\|_{W(L_w^{p_1, p_2, \dots, p_d})} = \left\| \left\{ \|f\chi_{[0,1]^{d+n}}\|_\infty \right\}_{n \in \mathbb{Z}^d} \right\|_{\ell_w^{p_1, p_2, \dots, p_d}}.$$

The Wiener space $W(L_w^{p_1, p_2, \dots, p_d})$ is the set of functions for which this norm is finite.

In the case that $p_1 = p_2 = \dots = p_d = 1$ we write $W(L_w^1(\mathbb{R}^d))$ instead of $W(L_w^{p_1, p_2, \dots, p_d})$, i.e.

$$\|f\|_{W(L_w^1(\mathbb{R}^d))} = \sum_{n \in \mathbb{Z}^d} \|f\chi_{[0,1]^{d+n}}\|_\infty w(n).$$

For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ with $\alpha_1, \dots, \alpha_d \in (0, \infty)$, we can define an equivalent norm on $W(L_w^{p_1, p_2, \dots, p_d})$ by

$$\|f\| = \left\| \left\{ \|f\chi_{\alpha \cdot [0,1]^{d+n}}\|_\infty \right\}_{n \in \mathbb{Z}^d} \right\|_{\ell_w^{p_1, p_2, \dots, p_d}}.$$

The following lemma, a generalization of Theorem 11.1.5 in [33], is a convolution relation for the Wiener amalgam spaces.

Lemma 1.2.6. *There is some $C \in (0, \infty)$ so that for all $F \in W(L_w^{p_1, p_2, \dots, p_d}), G \in W(L_v^1(\mathbb{R}^d))$ we have*

$$\|F * G\|_{L_w^{p_1, p_2, \dots, p_d}} \leq C \|F\|_{W(L_w^{p_1, p_2, \dots, p_d})} \|G\|_{W(L_v^1(\mathbb{R}^d))}.$$

1.2.2 Transforms

1.2.2.1 Continuous Wavelet Transform

Definition 1.2.7. *The continuous wavelet transform of $h \in L^2(\mathbb{R})$ with respect to $\psi \in L^2(\mathbb{R})$ is*

$$W_\psi h(a, b) = \int_{\mathbb{R}} h(t) |a|^{-\frac{1}{2}} \overline{\psi\left(\frac{t}{a} - b\right)} dt = \langle h, D_a T_b \psi \rangle, \quad (a, b) \in \mathbb{R}^+ \times \mathbb{R},$$

where D_a denotes the dilation $D_a f(t) = |a|^{-\frac{1}{2}} f(\frac{t}{a})$ and T_b denotes the translation $T_b f(t) = f(t - b)$. A function $\psi \in L^2(\mathbb{R})$ is admissible if

$$C_\psi = \int_{\mathbb{R}} |\hat{\psi}(w)|^2 \frac{dw}{|w|} < \infty.$$

If ψ is admissible then C_ψ is called the admissibility constant of ψ . If ψ is admissible and furthermore

$$\int_{(0,\infty)} |\hat{\psi}(w)|^2 \frac{dw}{|w|} = \int_{(-\infty,0)} |\hat{\psi}(w)|^2 \frac{dw}{|w|},$$

then inversion formula

$$h(t) = C_\psi^{-1} \iint_{\mathbb{R}^+ \times \mathbb{R}} W_\psi h(a, b) D_a T_b \psi(t) \frac{da}{a} db \quad (2)$$

holds weakly for all $h \in L^2(\mathbb{R})$.

The value of $W_\psi f(a, b)$ is a measure of the time-scale localization of f at position b and the scale a . See [25] for more information regarding wavelets.

1.2.2.2 Gabor Transform

Suppose $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is measurable. For $x, \xi \in \mathbb{R}^d$ define the translation operator T_x and modulation operator M_ξ by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i t \cdot \xi} f(t),$$

and define the time-frequency shift $\pi_{(x,\xi)}$ by $\pi_{(x,\xi)} = M_\xi T_x$.

Definition 1.2.8. Fix $\phi \in \mathcal{S}(\mathbb{R}^d)$. Given $f \in \mathcal{S}'(\mathbb{R}^d)$, the Gabor transform of f with respect to ϕ is

$$V_\phi f(x, \xi) = \int_{\mathbb{R}^d} f(t) \overline{\phi(t - x)} e^{-2\pi i \xi \cdot t} dt = \langle f, M_\xi T_x \phi \rangle, \quad x, \xi \in \mathbb{R}^d.$$

The function ϕ is called the window function of the Gabor transform.

The value of $V_\phi f(x, \xi)$ gives information about the time-frequency content of f around x in time and ξ in frequency. See [33] for background and information about the Gabor transform.

Both the wavelet and Gabor transforms arise from unitary representations of locally compact groups, namely the affine group and the Heisenberg group, respectively. The general properties of transforms determined by unitary representations are described in [39].

Fix $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $p, q \in [1, \infty]$. For $f \in \mathcal{S}'(\mathbb{R}^d)$, define

$$\|f\|_{M_w^{p,q}(\mathbb{R}^d)} = \|V_\phi f\|_{L_w^{p_1, p_2, \dots, p_{2d}}},$$

where $p = p_1 = p_2 = \dots = p_d$ and $q = p_{d+1} = p_{d+2} = \dots = p_{2d}$. Let

$$M_w^{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M_w^{p,q}(\mathbb{R}^d)} < \infty \right\}.$$

Each $M_w^{p,q}(\mathbb{R}^d)$ is a *modulation space*. For $w = 1$ we write $M_w^{p,q}(\mathbb{R}^d) = M^{p,q}(\mathbb{R}^d)$.

The modulation space $M_w^{p,q}(\mathbb{R}^d)$ consists of functions with a particular time-frequency decay controlled by the parameters p, q and weight w . See [33] for an overview of modulation spaces and time-frequency analysis.

In particular we have the following inclusion relationship between the modulation space $M^{1,1}(\mathbb{R}^d)$ and the Wiener space $W(L^1(\mathbb{R}^d))$ (see Proposition 12.1.4 in [33]).

Lemma 1.2.9. *If $\phi \in M^{1,1}(\mathbb{R}^d)$, then $\phi \in W(L^1(\mathbb{R}^d))$.*

1.2.2.3 The Radon Transform

We let S^1 denote the unit sphere in \mathbb{R}^2 . It will be useful to equate S^1 with $[0, 2\pi)$. Hence, for each $\theta \in S^1$, let $\phi(\theta)$ denote the unique number in $[0, 2\pi)$ such that $\theta = (\cos \phi(\theta), \sin \phi(\theta))$.

Definition 1.2.10. *Let $\ell(\theta, s) = \{x \in \mathbb{R}^2 : x \cdot \theta = s\}$ and let $dx_{\ell(\theta, s)}$ denote the one-dimensional Lebesgue measure on the set $\ell(\theta, s)$. The Radon transform of $\mathcal{L} \in L^1(\mathbb{R}^2)$ is given by*

$$R_\theta \mathcal{L}(s) = R\mathcal{L}(\theta, s) = \int_{\{x \in \mathbb{R}^2 : x \cdot \theta = s\}} \mathcal{L}(x) dx_{\ell(\theta, s)}, \quad \text{for all } (\theta, s) \in S^1 \times \mathbb{R}.$$

1.2.2.4 The Ridgelet Transform

There are a number of ways to generalize the wavelet transform on $L^2(\mathbb{R})$ to analyze functions in $L^2(\mathbb{R}^d)$ (see [25]). However these wavelet transforms are best used in analyzing pointwise characteristics of functions and are not suitable for detecting higher-dimensional singularities. In contrast, the ridgelet transform was developed in [11] and [13] to analyze the behavior of functions on \mathbb{R}^2 over lines.

Definition 1.2.11. *Suppose $\psi \in \mathcal{S}(\mathbb{R})$ is admissible. Then the ridgelet transform of $\mathcal{L} \in L^1(\mathbb{R}^2)$ is*

$$\mathcal{R}(\mathcal{L})(a, b, \theta) = \langle R_\theta \mathcal{L}, T_b D_a \psi \rangle \quad \forall \theta \in S^1, a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}.$$

1.2.3 Frames

Definition 1.2.12. *A frame for a Hilbert space H is a sequence of elements $\{\phi_x\}_{x \in X}$ in H such that there are $A, B > 0$ with*

$$A \|f\|^2 \leq \sum_{x \in X} |\langle f, \phi_x \rangle|^2 \leq B \|f\|^2$$

for all $f \in H$. In this case A, B are frame bounds. If we can take $A = B$ then $\{\phi_x\}_{x \in X}$ is a tight frame. A tight frame is Parseval if we can choose $A = B = 1$.

Frames give nonorthogonal expansions of elements of H in terms of the frame elements, and these expansions are stable but usually redundant. If $\{\phi_x\}_{x \in X}$ is a frame for H , there is a dual sequence $\{\tilde{\phi}_x\}_{x \in X} \subset H$ such that

$$f = \sum_{x \in X} \langle f, \phi_x \rangle \tilde{\phi}_x = \sum_{x \in X} \langle f, \tilde{\phi}_x \rangle \phi_x$$

for all $f \in H$, and the sequence $\{\tilde{\phi}_x\}_{x \in X}$ can be chosen to be a frame for H . In particular, if $\{\phi_x\}_{x \in X}$ is a tight frame for H with frame bound B , we have

$$f = B^{-1} \sum_{x \in X} \langle f, \phi_x \rangle \phi_x \quad \forall f \in H.$$

The frame operator of $\{\phi_x\}_{x \in X}$ is the self-adjoint bounded invertible operator

$$Sf = \sum_{x \in X} \langle f, \phi_x \rangle \phi_x \quad \forall f \in H.$$

See [14] for general background on frames.

Definition 1.2.13. A Bessel sequence for a Hilbert space H is a sequence of elements $\{\phi_x\}_{x \in X}$ in H such that there is $B > 0$ with

$$\sum_{x \in X} |\langle f, \phi_x \rangle|^2 \leq B \|f\|^2$$

for all $f \in H$. In this case B is the Bessel bound.

Definition 1.2.14. A sequence $\{\phi_x\}_{x \in X}$ satisfying

$$\forall h \in \overline{\text{span}}\{\phi_x\}_{x \in X}, \quad A \|h\|^2 \leq \sum_{x \in X} |\langle h, \phi_x \rangle|^2 \leq B \|h\|^2$$

is called a frame sequence. Equivalently, $\{\phi_x\}_{x \in X}$ is a frame for its closed span.

The best-known frames, frame sequences and Bessel sequences for function spaces are *coherent state frames* of the form $\{\sigma(x)f\}_{x \in X}$ where σ is a unitary representation of a locally compact group G on H and X is some collection of points in G . In particular, wavelet frames and Gabor frames for $L^2(\mathbb{R})$ have this form, as do Fourier frames for $L^2(I)$ where I is a compact interval.

1.2.4 Operators

1.2.4.1 Schatten class operators

Definition 1.2.15. Fix $1 \leq p < \infty$. Suppose H is a Hilbert space and $A : H \rightarrow H$ is a linear operator. We say A is Schatten p -class and write $A \in \mathcal{I}_p(H)$ if

$$\|A\|_{\mathcal{I}_p} = \sup \left(\sum_{n \in \mathbb{N}} |\langle Af_n, g_n \rangle|^p \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all pairs of orthonormal sequences $\{f_n\}_{n \in \mathbb{N}}, \{g_n\}_{n \in \mathbb{N}}$ in H .

Equivalently, an operator is Schatten p -class if its singular values constitute an ℓ^p sequence. Consequently, trace-class operators are exactly the Schatten 1-class operators and Hilbert-Schmidt operators are the Schatten 2-class operators. For $p = \infty$, we define Schatten p -class operators to be bounded operators.

1.2.4.2 Integral operators

Definition 1.2.16. *An operator A of the form*

$$Af(t) = \int_{\mathbb{R}^d} k(t, y)f(y) dy \quad \text{for all } t \in \mathbb{R}^d$$

is an integral operator, defined for all f for which these integrals converge. The function k is the kernel of A . Throughout the paper we write $k(t, y) = k_y(t)$.

1.2.4.3 Pseudodifferential Operators

Definition 1.2.17. *A pseudodifferential operator with Kohn-Nirenberg symbol τ is an operator having the form*

$$K_\tau f(t) = \iint_{\mathbb{R}^{2d}} \hat{\tau}(\xi, x) M_\xi T_{-x} f(t) dx d\xi.$$

A pseudodifferential operator with Weyl symbol σ is an operator having the form

$$L_\sigma f(t) = \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\xi, x) e^{-\pi i \xi \cdot x} T_{-x} M_\xi f(t) dx d\xi.$$

A pseudodifferential operator acting on a function f is a superposition of time-frequency shifts of f . Every suitable pseudodifferential operator K_τ can be also realized as an operator L_σ and in this case we have $\hat{\tau}(\xi, x) = e^{\pi i x \cdot \xi} \hat{\sigma}(\xi, x)$. Similarly, suitable K_τ and L_σ can be realized as integral operators.

1.2.4.4 Fourier Integral Operators

Definition 1.2.18. *A Fourier integral operator is one of the form*

$$Af(x) = \iint a(x, y, \xi) f(y) e^{i\varphi(x, y, \xi)} dy d\xi.$$

In this case, a is called the symbol of the operator A and φ is called the phase function.

Throughout this thesis, we assume the phase functions of Fourier integral operators are real-valued.

Like a pseudodifferential operator, a Fourier integral operator changes the time-frequency content of a function. In particular a pseudodifferential operator with Kohn-Nirenberg symbol τ is a Fourier integral operator with symbol $a(x, y, \xi) = \tau(x, \xi)$ and phase $\varphi(x, y, \xi) = 2\pi x \cdot \xi - 2\pi y \cdot \xi$. Suitable Fourier integral operators can be realized as integral operators.

1.2.4.5 Affine Pseudodifferential Operators

Definition 1.2.19. *An affine pseudodifferential operator with symbol \mathcal{L} is an operator having the form*

$$Af(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{L}(a, b) \frac{1}{a} f\left(\frac{t-b}{a}\right) da db.$$

Note that the operators that we call affine pseudodifferential operators have also been called “wideband channels” in the literature because these operators model the Doppler and multipath effects of wireless communications. Also the definitions of affine pseudodifferential operators and wideband channels vary in the literature (see [4], [75], [68] and [29]). In particular, Definition 1.2.19 is different from these sources in that dilations are L^1 normalized, not L^2 normalized.

CHAPTER II

DENSITY COMPARISON FOR SEPARABLE WAVELET FRAMES

2.1 *Introduction*

The best-known frames for function spaces are coherent state frames of the form $\{\sigma(x)f\}_{x \in X}$ where σ is a unitary representation of a locally compact group G and X is some collection of points in G . The density of X in G , which is in some sense the “average” number of points of X in a subset of G with unit measure, influences the properties of the frame. In the case that G is a locally compact abelian (LCA) group, much is known about the relationship between the frame properties of $\{\sigma(x)f\}_{x \in X}$ and the density of X . In particular, X must have density larger than some fixed “critical density” or Nyquist density in order for $\{\sigma(x)f\}_{x \in X}$ to be a frame. This critical Beurling density phenomenon underlies the classic Nyquist-Shannon Sampling Theorem and the work of Landau, both of which characterize frames of exponentials for $L^2(I)$ (see [55], [62], [53]). The Heisenberg group is “almost abelian” in some sense, and the Nyquist density properties of arbitrary Gabor frames were derived by Ramanathan and Steger in [58] (see [42] for an exposition of the history of density theorems for Gabor frames as well as extensive references). These critical density results were extended to arbitrary LCA groups in [2]. The Homogeneous Approximation Property (HAP), originally developed in [58], is a powerful tool for analyzing frames. As demonstrated in [2] and [37], it is the HAP for LCA frames that gives rise to the critical density that these frames obey. The HAP for LCA frames also gives rise to a “comparison theorem”: if $\{\sigma(x)f\}_{x \in X}$ is a frame with bounds A, B and

$\{\sigma(y)g\}_{y \in Y}$ is a frame with bounds E, F then

$$\frac{A \|g\|^2}{F \|f\|^2} \leq \frac{D(X, p, c)}{D(Y, p, c)} \leq \frac{B \|g\|^2}{E \|f\|^2} \quad (3)$$

(see Theorem 7 in [2]).

If σ is a unitary representation of a locally compact *non-abelian* group, then a frame $\{\sigma(x)f\}_{x \in X}$ need not demonstrate a critical density phenomenon. In particular, wavelet frames are well-known for not having a critical density. For any $a > 1$, $b \neq 0$ there is some ψ so that $\{a^{-\frac{m}{2}}\psi(\frac{x}{a^m} - bn)\}_{m, n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, which implies that for any positive number d , there is a wavelet frame for $L^2(\mathbb{R})$ with density d (see [24]). This fact still holds when we consider ψ having some fixed admissibility coefficient (see [24]), and in the case that $\{a^{-\frac{m}{2}}\psi(\frac{x}{a^m} - bn)\}_{m, n \in \mathbb{Z}}$ is a Riesz basis, $\{a^{-\frac{m}{2}}\psi(\frac{x}{a^m} - \beta n)\}_{m, n \in \mathbb{Z}}$ is still a Riesz basis for all β near b (see [1]). In light of these facts, it is surprising that wavelet frames do satisfy a homogeneous approximation property. In [44], the authors prove a HAP for wavelet frames, and for suitable wavelet frames $\{\sigma(x)f\}_{x \in X}$ and $\{\sigma(y)g\}_{y \in Y}$, the HAP gives one-sided density estimates: for each $\varepsilon > 0$, there is some $R(g, \varepsilon)$ so that

$$\frac{1 - \varepsilon}{e^{R(g, \varepsilon)}} \leq \frac{D(X, p, c)}{D(Y, p, c)}. \quad (4)$$

However the HAP cannot imply a critical density or a two-sided estimate like (3). These results are generalized to arbitrary locally compact groups in [34], although the results are qualitative in nature, in contrast to the very precise results known for LCA frames.

In this chapter we will compare separable wavelet frames of the form

$$\{\sigma(u, v)f\}_{u \in U, v \in V} \quad \text{and} \quad \{\sigma(s, t)g\}_{s \in S, t \in T}.$$

Since the best-known wavelet frames have this form, these results are applicable to a broad class of familiar wavelets as well as certain more general irregular wavelet systems. The main result in this section is a HAP for separable wavelet frames

that is both more powerful than the usual HAP in some sense but less powerful in another. This HAP allows us to delineate relationships between the densities of U, V, S and T , the admissibility constants of f, g and the frame bounds of the sequences $\{\sigma(u, v)f\}_{u \in U, v \in V}$ and $\{\sigma(s, t)g\}_{s \in S, t \in T}$. As a consequence, we obtain a comparison theorem for separable wavelet frames analogous to (3). Our comparison theorem is interesting because it shows a new similarity between wavelet frames and LCA frames. Both LCA frames and certain wavelet frames have a HAP and have a two-sided comparison theorem. Yet LCA frames have a critical density, while wavelet frames do not.

Separable wavelet frames allow us to independently analyze the translation and dilation parameters comprising the frame. Our main result concerns the dilation indices. For suitable $U, S \subset \mathbb{R}^+$ and suitable $f, g \in L^2(\mathbb{R})$ we show that

$$0 = \lim_{M \rightarrow \infty} \frac{1}{2M} \left(\sum_{s \in S \cap a_M [e^{-M}, e^M]} \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right. \\ \left. - \sum_{u \in U \cap a_M [e^{-M}, e^M]} \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right)$$

for all sequences $\{a_M\}_{M \in \mathbb{N}} \subset \mathbb{R}^+$. For separable wavelet frames whose translations form a Fourier frame, this result is a type of HAP on \mathbb{R}^+ because it ensures that functions are well-approximated by finitely many dilations and infinitely many translations. However, it is in fact more powerful than the usual HAP because it ensures simultaneous approximation by $\{\sigma(u, v)f\}_{u \in U, v \in V}$ and $\{\sigma(s, t)g\}_{s \in S, t \in T}$.

As a consequence of our HAP, we obtain a comparison theorem for the densities of two wavelet frames. In particular, if $\{\sigma(u, v)f\}_{(u, v) \in U \times V}$, $\{\sigma(s, t)g\}_{(s, t) \in S \times T}$ are frames for $L^2(\mathbb{R})$ with frame bounds A, B and E, F , respectively then

$$\frac{A C_g}{F C_f} \leq \frac{D(U \times V, c, p)}{D(S \times T, c, p)} \leq \frac{B C_g}{E C_f}$$

for all suitable $f, g \in L^2(\mathbb{R})$, $U, S \subset \mathbb{R}^+$ and $V, T \subset \mathbb{R}$, where C_f, C_g are the admissibility constants of f, g .

The remainder of this chapter is organized into five sections. The first contains definitions and preliminary lemmas necessary to prove key theorems. Section 2.3 contains the main result and its proof. The applications of the main result to wavelet frames are explored in 2.4. These results are extended to certain wavelet frame sequences in 2.5.

2.2 Definitions and preliminary lemmas

2.2.1 Affine Group

Definition 2.2.1. *Assume G is a locally compact group. Let $\mathcal{U}(L^2(\mathbb{R}^d))$ denote the set of unitary operators on $L^2(\mathbb{R}^d)$. A unitary representation of G is a homomorphism $\pi : G \rightarrow \mathcal{U}(L^2(\mathbb{R}^d))$ that is continuous with respect to the strong operator topology on $L^2(\mathbb{R}^d)$.*

Definition 2.2.2. *Let G be a locally compact group. The left Haar measure on G is the unique nonzero Radon measure μ on G which satisfies $\mu(xE) = \mu(E)$ for all $x \in G$ and all Borel $E \subset G$.*

The book [31] explains the theory the unitary representations of locally compact groups.

Definition 2.2.3. *The affine group \mathbb{A} is the set $\mathbb{R}^+ \times \mathbb{R}$ with multiplication*

$$(a, b)(x, y) = \left(ax, y + \frac{b}{x} \right).$$

For $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$, we let $\sigma(a, b)$ denote the operator $D_a T_b$, where D_a denotes the dilation $D_a f(t) = |a|^{-\frac{1}{2}} f(\frac{t}{a})$ and T_b denotes the translation $T_b f(t) = f(t - b)$.

It is known that σ is a unitary representation of the affine group on $L^2(\mathbb{R})$. We let μ denote the left Haar measure of the affine group on $L^2(\mathbb{R})$; that is, $d\mu(a, b) = \frac{da}{a} db$.

It is worth noting that the affine group is sometimes defined with the larger set $\mathbb{R} \setminus \{0\} \times \mathbb{R}$ or with multiplication

$$(u, v)(x, y) = (ux, v + uy).$$

When the affine group is defined with this alternate multiplication the left Haar measure is $\frac{du}{u^2} dv$ and $\sigma'(u, v) = T_v D_u$ is a unitary representation. See [51] for a comparison of these different definitions of the affine group.

2.2.2 General Density

The density of X in G is in some sense the “average” number of points of X in a subset of G with unit measure.

Definition 2.2.4. *Let G be a locally compact group with left Haar measure μ , and let $\{Q_M\}_{M \in \mathbb{N}} \subset G$ be a sequence of compact sets satisfying $Q_M \subset Q_{M+1}$ for all $M \in \mathbb{N}$ and $\cup Q_M = G$. Let X be any collection of points in G . For any free ultrafilter p and each sequence $c = \{c_M\}_{M \in \mathbb{N}} \subset G$, we define the density of X with respect to p and c to be*

$$D_G(X, p, c) = p\text{-}\lim \frac{|X \cap c_M Q_M|}{\mu(Q_M)}.$$

The upper density of X is

$$D_G^+(X) = \limsup_{M \rightarrow \infty} \sup_{g \in G} \frac{|X \cap g Q_M|}{\mu(Q_M)}$$

while the lower density of X is

$$D_G^-(X) = \liminf_{M \rightarrow \infty} \inf_{g \in G} \frac{|X \cap g Q_M|}{\mu(Q_M)},$$

where $c_M Q_M, g Q_M$ denote left multiplication by c_M, g , respectively.

The properties of free ultrafilters are described in the appendix of [3]. It is a fact that every free ultrafilter limit of a sequence is an accumulation point of the sequence. So for each free ultrafilter p and each sequence $c = \{c_M\}_{M \in \mathbb{N}} \subset G$, we have

$$D_G^-(X) \leq D_G(X, p, c) \leq D_G^+(X).$$

Furthermore, there are p, c so that $D_G^+(X) = D(X, p, c)$. Similarly there exist p, c so that $D_G^-(X) = D(X, p, c)$.

In general, if there are p, c so that $D_G(X, p, c) = \infty$ then no $\{\sigma(x)f\}_{x \in X}$ will be a frame. To avoid such sets we make the following definition.

Definition 2.2.5. *Suppose G is a locally compact group and X is a collection of points in G . If for any compact $U \subset G$, there is some finite K so that*

$$\left\| \sum_{x \in X} \chi_{xU} \right\|_{\infty} \leq K$$

then X is relatively separated.

2.2.3 Affine Density

We will consider affine density with respect to the choice of sets $\{Q_M\}_{M \in \mathbb{N}}$ given by $Q_M = [e^{-M}, e^M] \times [-M, M]$. Henceforth $D_{\mathbb{A}}(X, p, c)$, $D_{\mathbb{A}}^+(X)$ and $D_{\mathbb{A}}^-(X)$ are defined as in Definition 2.2.4 with respect to this particular choice of Q_M . The set Q_M is a rectangle in \mathbb{A} centered at $(1, 0)$, and $\mu(Q_M) = 4M^2$.

The following lemma ensures that relatively separated sets in the affine group have finite density (see Lemma 3.1 in [69] for proof).

Lemma 2.2.6. *If X is a relatively separated set in \mathbb{A} , then there is some finite K so that*

$$D_{\mathbb{A}}(X, p, c) \leq K$$

for all free ultrafilters p and all sequences $c = \{c_M\}_{M \in \mathbb{N}} \subset \mathbb{A}$. In particular, $D_{\mathbb{A}}^+(X) < \infty$.

2.2.3.1 Density in \mathbb{R}^+, \mathbb{R}

In addition to density of sets in \mathbb{A} , it will be useful to measure the densities of subsets of \mathbb{R}^+ and \mathbb{R} . We fix $I_M = [e^{-M}, e^M]$. Following Definition 2.2.4, for $S \subset \mathbb{R}^+$ and $a = \{a_M\} \subset \mathbb{R}^+$ we set

$$D_{\mathbb{R}^+}(S, p, a) = p\text{-lim} \frac{|S \cap [a_M e^{-M}, a_M e^M]|}{2M} = p\text{-lim} \frac{|S \cap a_M I_M|}{2M},$$

$$D_{\mathbb{R}^+}^+(S) = \limsup_{M \rightarrow \infty} \sup_{r \in \mathbb{R}^+} \frac{|S \cap rI_M|}{2M},$$

and

$$D_{\mathbb{R}^+}^-(S) = \liminf_{M \rightarrow \infty} \inf_{r \in \mathbb{R}^+} \frac{|S \cap rI_M|}{2M}.$$

For $T \subset \mathbb{R}$ and $b = \{b_M\} \subset \mathbb{R}$ we set

$$D_{\mathbb{R}}(T, p, b) = p\text{-lim} \frac{|T \cap (b_M + [-M, M])|}{2M},$$

$$D_{\mathbb{R}}^+(T) = \limsup_{M \rightarrow \infty} \sup_{x \in \mathbb{R}} \frac{|T \cap (x + [-M, M])|}{2M},$$

and

$$D_{\mathbb{R}}^-(T) = \liminf_{M \rightarrow \infty} \inf_{x \in \mathbb{R}} \frac{|T \cap (x + [-M, M])|}{2M}.$$

Relatively separated sets in both \mathbb{R}^+ and \mathbb{R} have finite density. In the next two lemmas, we prove density-like results for relatively separated sets in \mathbb{R}^+ and \mathbb{R} . These results are needed for the proof of the main theorem.

Lemma 2.2.7. *Suppose $S \subset \mathbb{R}^+$ is relatively separated. Then there is $C \in [0, \infty)$ such that*

$$(a) |S \cap rI_M| \leq 2CM \quad \forall M \in \mathbb{N}, r \in \mathbb{R}^+ \text{ and}$$

$$(b) |S \cap r(I_{M+N} \setminus I_M)| \leq 2CN \quad \forall M, N \in \mathbb{N}, r \in \mathbb{R}^+.$$

Proof. First we prove (a). Consider the compact set $I_1 = [e^{-1}, e]$. By definition, there is some $C_1 \in [0, \infty)$ such that

$$\left\| \sum_{s \in S} \chi_{sI_1} \right\|_{\infty} \leq C_1.$$

Thus for each $j \in \mathbb{Z}$ we have $\sum_{s \in S} \chi_{sI_1}(re^j) \leq C_1$. Notice that

$$\begin{aligned} \sum_{s \in S} \chi_{sI_1}(re^j) &= |\{s \in S : se^{-1} \leq re^j \leq se\}| \\ &= |\{s \in S : re^{j-1} \leq s \leq re^{j+1}\}|. \end{aligned}$$

Thus if $M = 1$ we have

$$|S \cap rI_M| = \sum_{s \in S} \chi_{sI_1}(r) \leq C_1.$$

For $M > 1$ we have

$$S \cap rI_M \subset \bigcup_{j=-M+1}^{M-1} \{s \in S : re^{j-1} \leq s \leq re^{j+1}\},$$

which means

$$\begin{aligned} |S \cap rI_M| &\leq \sum_{j=-M+1}^{M-1} |\{s \in S : re^{j-1} \leq s \leq re^{j+1}\}| \\ &= \sum_{j=-M+1}^{M-1} \sum_{s \in S} \chi_{sI_1}(re^j) \\ &\leq \sum_{j=-M+1}^{M-1} C_1 \\ &\leq C_1(2M - 1). \end{aligned}$$

Choosing $C = C_1$ gives (a).

We will show that (b) is also satisfied for $C = C_1$. Notice that

$$\begin{aligned} &S \cap r(I_{M+N} \setminus I_M) \\ &= \{s \in S : re^{-M-N} \leq s < re^{-M} \text{ or } re^M < s \leq re^{M+N}\} \\ &\subset \bigcup_{j=-M-N+1}^{-M-1} \{s \in S : re^{j-1} \leq s \leq re^{j+1}\} \cup \bigcup_{j=M+1}^{M+N-1} \{s \in S : re^{j-1} \leq s \leq re^{j+1}\}. \end{aligned}$$

Thus

$$\begin{aligned} &|S \cap r(I_{M+N} \setminus I_M)| \\ &\leq \sum_{j=-M-N+1}^{-M-1} \sum_{s \in S} \chi_{sI_1}(re^j) + \sum_{j=M+1}^{M+N-1} \sum_{s \in S} \chi_{sI_1}(re^j) \\ &\leq \sum_{n=-M-N+1}^{-M-1} C_1 + \sum_{n=M+1}^{M+N-1} C_1 \\ &\leq 2C_1(N - 1) \\ &\leq 2CN. \end{aligned}$$

□

Lemma 2.2.8. *Suppose $T \subset \mathbb{R}$ is relatively separated. Then there is $C \in [0, \infty)$ such that*

$$(a) |T \cap (x + [-M, M])| \leq 2CM \quad \forall M \in \mathbb{N}, x \in \mathbb{R} \text{ and}$$

$$(b) |T \cap (x + [-M - N, M + N] \setminus [-M, M])| \leq 2CN \quad \forall M, N \in \mathbb{N}, x \in \mathbb{R}.$$

Proof. Since T is relatively separated, there is some $C_1 < \infty$ such that

$$\left\| \sum_{t \in T} \chi_{t+[-1,1]} \right\|_{\infty} \leq C_1.$$

Choose $C = C_1$. Notice that

$$\sum_{t \in T} \chi_{t+[-1,1]}(x+n) = |\{t \in T : t \in x + [n-1, n+1]\}|.$$

Thus

$$\begin{aligned} |T \cap x + [-M, M]| &\leq \sum_{n=-M+1}^{M-1} |\{t \in T : t \in x + [n-1, n+1]\}| \\ &= \sum_{n=-M+1}^{M-1} \sum_{t \in T} \chi_{t+[-1,1]}(x+n) \\ &\leq \sum_{n=-M+1}^{M-1} C_1 \\ &\leq C_1(2M-1) \\ &\leq 2C_1M. \end{aligned}$$

Also,

$$\begin{aligned} &|T \cap x + [-M - N, M + N] \setminus [-M, M]| \\ &\leq \sum_{n=-M-N+1}^{-M-1} |\{t \in T : t \in x + [n-1, n+1]\}| \\ &\quad + \sum_{n=M+1}^{M+N-1} |\{t \in T : t \in x + [n-1, n+1]\}| \\ &= \sum_{n=-M-N+1}^{-M-1} \sum_{t \in T} \chi_{t+[-1,1]}(x+n) + \sum_{n=M+1}^{M+N-1} \sum_{t \in T} \chi_{t+[-1,1]}(x+n) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=-M-N+1}^{-M-1} C_1 + \sum_{n=M+1}^{M+N-1} C_1 \\
&\leq 2C_1(N-1) \\
&\leq 2CN.
\end{aligned}$$

□

The following lemma relates density in \mathbb{A} to density in \mathbb{R}^+ and \mathbb{R} .

Lemma 2.2.9. *Suppose $S \subset \mathbb{R}^+$ and $T \subset \mathbb{R}$. For any sequence $\{c_M\} = \{(a_M, b_M)\} \subset \mathbb{A}$ and any free ultrafilter p we have*

$$D_{\mathbb{R}}^-(T) D_{\mathbb{R}^+}(S, a, p) \leq D_{\mathbb{A}}(S \times T, c, p) \leq D_{\mathbb{R}}^+(T) D_{\mathbb{R}^+}(S, a, p)$$

Proof. Notice that $(s, t) \in c_M Q_M$ if and only if $s = a_M x$ for some $x \in [e^{-M}, e^M]$ and $t = y + \frac{a_M b_M}{s}$ for some $y \in [-M, M]$. That is $(s, t) \in c_M Q_M$ if and only if $s \in U \cap a_M I_M$ and $t \in \frac{a_M b_M}{s} + [-M, M]$. Thus

$$|S \times T \cap c_M Q_M| \leq |S \cap a_M [e^{-M}, e^M]| \cdot \sup_{x \in \mathbb{R}} |T \cap x + [-M, M]|.$$

Using the product preservation property of free ultrafilters, we see

$$\begin{aligned}
&p\text{-lim} \frac{|S \times T \cap c_M Q_M|}{\mu(Q_M)} \\
&\leq \left(p\text{-lim} \frac{|S \cap a_M [e^{-M}, e^M]|}{2M} \right) \cdot \left(p\text{-lim sup}_{x \in \mathbb{R}} \frac{|T \cap x + [-M, M]|}{2M} \right) \\
&= D_{\mathbb{R}^+}(S, a, p) \cdot \left(p\text{-lim sup}_{x \in \mathbb{R}} \frac{|T \cap x + [-M, M]|}{2M} \right).
\end{aligned}$$

Since $p\text{-lim sup}_{x \in \mathbb{R}} \frac{|T \cap x + [-M, M]|}{2M}$ is an accumulation point of the sequence

$$\left\{ \sup_{x \in \mathbb{R}} \frac{|T \cap x + [-M, M]|}{2M} \right\}_{M \in \mathbb{N}},$$

we must have

$$\begin{aligned}
p\text{-lim sup}_{x \in \mathbb{R}} \frac{|T \cap x + [-M, M]|}{2M} &\leq \limsup_{M \rightarrow \infty} \sup_{x \in \mathbb{R}} \frac{|T \cap x + [-M, M]|}{2M} \\
&= D_{\mathbb{R}}^+(T).
\end{aligned}$$

The other inequality is proven similarly. □

2.2.4 Fourier Frames

Definition 2.2.10. We say that $\mathcal{E}(T) = \{e^{2\pi itx}\}_{t \in T}$ is a Fourier frame if there is some r so that $\mathcal{E}(T)$ is a frame for $L^2[-r, r]$.

It is well known that $\left\{r^{-\frac{1}{2}}e^{2\pi i\frac{n}{r}x}\right\}_{n \in \mathbb{Z}}$ is a frame (and in fact an orthonormal basis) for $L^2[-\frac{r}{2}, \frac{r}{2}]$. The following theorem, from [78], shows that Fourier frames are stable under ℓ^∞ perturbations.

Theorem 2.2.11. Suppose I is a compact interval and $\mathcal{E}(T) = \{e^{2\pi itx}\}_{t \in T}$ is a Fourier frame for $L^2(I)$. Then there is some $\epsilon > 0$ so that if $\{s_t\}_{t \in T}$ satisfies $\sup_{t \in T} |t - s_t| < \epsilon$ then $\{e^{2\pi is_t x}\}_{t \in T}$ is also a Fourier frame for $L^2(I)$.

The density of a Fourier frame is determined by the frame bounds.

Lemma 2.2.12. Suppose $V \subset \mathbb{R}$ and $\mathcal{E}(V) = \{e^{2\pi i v w}\}_{v \in V}$ is a frame for $L^2[-r, r]$ with bounds A, B . Then

$$A \leq D_{\mathbb{R}}^-(V) \leq D_{\mathbb{R}}^+(V) \leq B.$$

Proof. This holds by Theorem 7 in [2]. The exact details are given in the appendix. □

2.2.5 Wavelet Frames

A wavelet frame for $L^2(\mathbb{R})$ with frame bounds A, B is a sequence $\{\sigma(x)f\}_{x \in X}$, where $f \in L^2(\mathbb{R})$ and $X \subset \mathbb{A}$, satisfying

$$\forall h \in L^2(\mathbb{R}), \quad A \|h\|^2 \leq \sum_{x \in X} |W_f h(x)|^2 \leq B \|h\|^2.$$

A separable wavelet frame is one of the form $\{\sigma(s, t)g\}_{(s,t) \in S \times T}$. Separable wavelet frames of the form $\{\sigma(a^m, bn)g\}_{m,n \in \mathbb{Z}}$ have been studied extensively and used widely in applications. If $\mathcal{E}(T)$ is a Fourier frame then the frame and Bessel sequence properties of a sequence of the form $\{\sigma(s, t)g\}_{(s,t) \in S \times T}$ are largely determined by the behavior of the function $\sum_{s \in S} |\hat{g}(sx)|^2$. For this reason, we make the following definition.

Definition 2.2.13. Let $S \subset \mathbb{R}^+$. We say that g is Chui-Shi bounded with respect to S if there is some finite K such that

$$\sum_{s \in S} |\hat{g}(sx)|^2 \leq K \text{ a.e.}$$

It was proved in [15] that for regular wavelet frames $\{\sigma(a^m, bn)g\}$, the function $\sum_m |\hat{g}(a^m x)|^2$ is bounded almost everywhere. This was generalized in [77] to the following theorem.

Theorem 2.2.14. Suppose $\{\sigma(u, v)f\}_{(u,v) \in U \times V}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B and $\mathcal{E}(V) = \{e^{2\pi i vw}\}_{v \in V}$ is a frame for $L^2[-r_V, r_V]$ with bounds A_V, B_V . Then

$$\frac{A}{B_V} \leq \sum_{u \in U} |\hat{f}(uw)|^2 \leq \frac{B}{A_V} \text{ a.e.}$$

Definition 2.2.15. Given a free ultrafilter p , sequence $c = \{c_M\}_{M \in \mathbb{N}} = \{(a_M, b_M)\}_{M \in \mathbb{N}} \subset \mathbb{A}$ and admissible f, g generating wavelet Bessel sequences $\mathcal{G} = \{\sigma(s, t)g\}_{(s,t) \in S \times T}$ and $\mathcal{F} = \{\sigma(u, v)f\}_{(u,v) \in U \times V}$, we define the relative admissibility measure of \mathcal{F} with respect to \mathcal{G} to be

$$\mu_{\mathcal{F}, \mathcal{G}}(p, c) = p\text{-lim} \frac{1}{|U \cap a_M I_M|} \sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|}.$$

If g is Chui-Shi bounded with respect to S , then $\mu_{\mathcal{F}, \mathcal{G}}(p, c)$ is a type of average admissibility constant for f .

In this section we develop results that allow us to estimate sums of the form $\sum_{u \in U \cap a_M I_M^c} \sum_{s \in S} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|}$.

Definition 2.2.16. Suppose $f, g \in L^2(\mathbb{R})$. We say that f, g are a localized pair if

$$\int_{[0, \infty)} \left(\sup_{c \in [ye^{-1}, ye]} \int |\hat{g}(x)|^2 |\hat{f}(cx)|^2 \frac{dx}{|x|} \right) \frac{dy}{y} < \infty.$$

Notice that

$$\int_{[0, \infty)} \sup_{c \in [ye^{-1}, ye]} \int |\hat{g}(x)|^2 |\hat{f}(cx)|^2 \frac{dx}{|x|} \frac{dy}{y} = \int_{[0, \infty)} \sup_{c \in [ye^{-1}, ye]} \int |\hat{g}(cx)|^2 |\hat{f}(x)|^2 \frac{dx}{|x|} \frac{dy}{y}$$

so that localization is a symmetric relation.

The following lemma gives a class of functions that form a localized pair with any admissible wavelet. A generalization of this proof technique shows that any function in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ whose Fourier transform is supported in $[-\Omega_1, -\Omega_0] \cup [\Omega_0, \Omega_1]$ forms a localized pair with any admissible wavelet.

Lemma 2.2.17. *Fix $a > 1$. Every admissible function f forms a localized pair with the function g whose Fourier transform is $\hat{g} = \chi_{[-1, -a^{-1}] \cup [a^{-1}, 1]}$.*

Proof. Fix an admissible function f . We have $\hat{g} = \chi_{[-1, -a^{-1}] \cup [a^{-1}, 1]}$. Then

$$\int_{[0, \infty)} |\hat{g}(x)|^2 |\hat{f}(cx)|^2 \frac{dx}{|x|} = \int_{[ca^{-1}, ca]} |\hat{f}(x)|^2 \frac{dx}{|x|}.$$

For $e^m \leq y \leq e^{m+1}$ and $c \in [ye^{-1}, ye]$ we have $[ca^{-1}, ca] \subset [a^{-1}e^{m-1}, ae^{m+2}]$. Choose $k > 0$ so that $a^2 \leq e^k$. Then $[a^{-1}e^{m-1}, ae^{m+2}] \subset [a^{-1}e^{m-1}, a^{-1}e^{m+k+2}]$. We have

$$\begin{aligned} & \int_{[0, \infty)} \sup_{c \in [ye^{-1}, ye]} \int_{[0, \infty)} |\hat{g}(x)|^2 |\hat{f}(cx)|^2 \frac{dx}{|x|} \frac{dy}{y} \\ &= \sum_{m \in \mathbb{Z}} \int_{[e^m, e^{m+1})} \sup_{c \in [ye^{-1}, ye]} \int_{[ca^{-1}, ca]} |\hat{f}(x)|^2 \frac{dx}{|x|} \frac{dy}{y} \\ &\leq \sum_{m \in \mathbb{Z}} \int_{[e^m, e^{m+1})} \int_{[a^{-1}e^{m-1}, ae^{m+2}]} |\hat{f}(x)|^2 \frac{dx}{|x|} \frac{dy}{y} \\ &= \sum_{m \in \mathbb{Z}} \int_{[a^{-1}e^{m-1}, ae^{m+2}]} |\hat{f}(x)|^2 \frac{dx}{|x|} \\ &\leq \sum_{m \in \mathbb{Z}} \int_{[a^{-1}e^{m-1}, a^{-1}e^{m+k+2}]} |\hat{f}(x)|^2 \frac{dx}{|x|} \\ &\leq (k+3)C_f. \end{aligned}$$

Similar estimates hold for

$$\int_{(-\infty, 0]} |\hat{g}(x)|^2 |\hat{f}(cx)|^2 \frac{dx}{|x|}. \quad \square$$

The following result is a special case of Lemma 1 in [34].

Lemma 2.2.18. *Suppose that $R \subset \mathbb{R}^+$ is relatively separated. If f, g are a localized pair then there is some finite K independent of f, g and R so that*

$$\sum_{r \in R \cap I_M^C} \int |\hat{g}(x)|^2 |\hat{f}(rx)|^2 \frac{dx}{|x|} \leq K \int_{I_{M-1}^C} \sup_{r \in [ye^{-1}, ye]} \int |\hat{g}(x)|^2 |\hat{f}(rx)|^2 \frac{dx}{|x|} \frac{dy}{y}$$

for all $M > 1$.

2.3 Main result

We begin by showing that for suitable f, g , certain average admissibility constants of f, g are proportional. We need not have wavelet frames to derive this result.

Theorem 2.3.1. *Suppose that U and S are relatively separated in \mathbb{R}^+ and $f, g \in L^2(\mathbb{R})$ are admissible, form a localized pair, and are Chui-Shi bounded with respect to U, S , respectively. Then for any sequence $\{a_M\} \subset \mathbb{R}^+$, we have*

$$0 = \lim_{M \rightarrow \infty} \frac{1}{2M} \left(\sum_{s \in S \cap a_M I_M} \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} - \sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right)$$

Proof. Fix $\varepsilon > 0$. Since f, g are a localized pair, we can choose $M_\varepsilon \in \mathbb{N}$ so that

$$\int_{I_{M_\varepsilon-1}^C} \sup_{r \in [ye^{-1}, ye]} \int |\hat{g}(x)|^2 |\hat{f}(rx)|^2 \frac{dx}{|x|} \frac{dy}{y} < \varepsilon$$

and

$$\int_{I_{M_\varepsilon-1}^C} \sup_{r \in [ye^{-1}, ye]} \int |\hat{g}(rx)|^2 |\hat{f}(x)|^2 \frac{dx}{|x|} \frac{dy}{y} < \varepsilon.$$

By Lemma 2.2.18, we can choose $K_1 < \infty$ so that for all $M > 1$ we have

$$\sum_{u \in U \cap I_M^C} \int |\hat{g}(x)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \leq K_1 \int_{I_{M_\varepsilon-1}^C} \sup_{r \in [ye^{-1}, ye]} \int |\hat{g}(x)|^2 |\hat{f}(rx)|^2 \frac{dx}{|x|} \frac{dy}{y}$$

and

$$\sum_{s \in S \cap I_M^C} \int |\hat{g}(sx)|^2 |\hat{f}(x)|^2 \frac{dx}{|x|} \leq K_1 \int_{I_{M_\varepsilon-1}^C} \sup_{r \in [ye^{-1}, ye]} \int |\hat{g}(rx)|^2 |\hat{f}(x)|^2 \frac{dx}{|x|} \frac{dy}{y}.$$

Since f, g are Chui-Shi bounded with respect to U, S , we can choose $K_2 < \infty$ so that

$$\sum_{s \in S} |\hat{g}(sw)|^2 < K_2 \text{ a.e.}$$

and

$$\sum_{u \in U} |\hat{f}(uw)|^2 < K_2 \text{ a.e.}$$

By Lemma 2.2.7, since U and S are relatively separated in \mathbb{R}^+ , we can choose $K_3 < \infty$ so that for all $M \in \mathbb{N}$ and $r \in \mathbb{R}^+$ we have

$$|S \cap rI_M| \leq 2K_3M, \quad |S \cap r(I_{M+M_\epsilon} \setminus I_M)| \leq 2K_3M_\epsilon,$$

$$|U \cap rI_M| \leq 2K_3M \quad \text{and} \quad |U \cap r(I_{M+M_\epsilon} \setminus I_M)| \leq 2K_3M_\epsilon.$$

Write

$$\begin{aligned} & \sum_{s \in S \cap a_M I_M} \sum_{u \in U} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} - \sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\ &= \sum_{s \in S \cap a_M I_M} \sum_{u \in U \cap a_M I_M^C} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\ & \quad - \sum_{u \in U \cap a_M I_M} \sum_{s \in S \cap a_M I_M^C} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\ &= \sum_{s \in S \cap a_M I_M} \sum_{u \in U \cap a_M I_{M+M_\epsilon}^C} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\ & \quad + \sum_{s \in S \cap a_M I_M} \sum_{u \in U \cap a_M(I_{M+M_\epsilon} \setminus I_M)} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\ & \quad - \sum_{u \in U \cap a_M I_M} \sum_{s \in S \cap a_M I_{M+M_\epsilon}^C} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\ & \quad - \sum_{u \in U \cap a_M I_M} \sum_{s \in S \cap a_M(I_{M+M_\epsilon} \setminus I_M)} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\ &= T_1 + T_2 - T_3 - T_4 \end{aligned}$$

We can estimate T_1 by noting that for $s \in S \cap a_M I_M$ and $u \in U \cap a_M I_{M+M_\epsilon}^C$, we

have $\frac{s}{u} \in I_{M_\varepsilon}^C$. Using this fact along with Lemma 2.2.18 we have

$$\begin{aligned}
|T_1| &= \sum_{s \in S \cap a_M I_M} \sum_{u \in U \cap a_M I_{M+M_\varepsilon}^C} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\
&\leq |S \cap a_M I_M| \sup_{s \in a_M I_M} \sum_{u \in U \cap a_M I_{M+M_\varepsilon}^C} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\
&\leq |S \cap a_M I_M| \sup_{s \in a_M I_M} \sum_{u \in U \cap a_M I_{M+M_\varepsilon}^C} \int |\hat{g}(w)|^2 |\hat{f}\left(\frac{u}{s}w\right)|^2 \frac{dw}{|w|} \\
&\leq |S \cap a_M I_M| \sup_{s \in a_M I_M} \sum_{r \in s^{-1}U \cap a_M I_{M_\varepsilon}^C} \int |\hat{g}(w)|^2 |\hat{f}(rw)|^2 \frac{dw}{|w|} \\
&\leq |S \cap a_M I_M| \sup_{s \in a_M I_M} K_1 \int_{I_{M_\varepsilon-1}^C} \sup_{r \in [ye^{-1}, ye]} \int |\hat{g}(x)|^2 |\hat{f}(rx)|^2 \frac{dx}{|x|} \frac{dy}{y} \\
&= |S \cap a_M I_M| K_1 \int_{I_{M_\varepsilon-1}^C} \sup_{r \in [ye^{-1}, ye]} \int |\hat{g}(x)|^2 |\hat{f}(rx)|^2 \frac{dx}{|x|} \frac{dy}{y} \\
&\leq 2MK_1 K_3 \varepsilon.
\end{aligned}$$

We estimate T_2 by

$$\begin{aligned}
|T_2| &= \sum_{s \in S \cap a_M I_M} \sum_{u \in U \cap a_M (I_{M+M_\varepsilon} \setminus I_M)} \int |\hat{g}(sx)|^2 |\hat{f}(ux)|^2 \frac{dx}{|x|} \\
&\leq \sum_{u \in U \cap a_M (I_{M+M_\varepsilon} \setminus I_M)} K_2 \int |\hat{f}(ux)|^2 \frac{dx}{|x|} \\
&= \sum_{u \in U \cap a_M (I_{M+M_\varepsilon} \setminus I_M)} K_2 C_f \\
&\leq 2K_2 K_3 C_f M_\varepsilon.
\end{aligned}$$

Similarly, we can show

$$|T_3| \leq 2MK_1 K_3 \varepsilon$$

and

$$|T_4| \leq 2K_2 K_3 C_g M_\varepsilon.$$

Thus

$$\begin{aligned}
& \frac{1}{2M} \left| \sum_{s \in S \cap a_M I_M} \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} - \sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right| \\
& \leq \frac{|T_1| + |T_2| + |T_3| + |T_4|}{2M} \\
& \leq 4K_1 K_3 \varepsilon + 2K_2 K_3 (C_g + C_f) \frac{M_\varepsilon}{M} \\
& \rightarrow 4K_1 K_3 \varepsilon \quad \text{as } M \rightarrow \infty.
\end{aligned}$$

Since ε is arbitrary, the result follows. \square

For separable wavelet frames and frame sequences in $L^2(\mathbb{R})$, Theorem 2.3.1 can be restated as a useful relationship between the relative admissibility measure of a frame and the density of its dilation parameters.

Corollary 2.3.2. *Suppose U, S are relatively separated in \mathbb{R}^+ and $f, g \in L^2(\mathbb{R})$ are admissible, form a localized pair, and are Chui-Shi bounded with respect to U, S , respectively. Let $\mathcal{G} = \{\sigma(s, t)g\}_{(s,t) \in S \times T}$ and $\mathcal{F} = \{\sigma(u, v)f\}_{(u,v) \in U \times V}$. Then for any sequence $a = \{a_M\} \subset \mathbb{R}^+$,*

$$\mu_{\mathcal{F}, \mathcal{G}}(p, c) \cdot D_{\mathbb{R}^+}(U, p, a) = \mu_{\mathcal{G}, \mathcal{F}}(p, c) \cdot D_{\mathbb{R}^+}(S, p, a),$$

where $c = \{(a_M, b_M)\} \subset \mathbb{A}$ for any sequence $\{b_M\} \subset \mathbb{R}$.

Proof. Notice that

$$\begin{aligned}
& \mu_{\mathcal{F}, \mathcal{G}}(p, c) \cdot D_{\mathbb{R}^+}(U, p, a) - \mu_{\mathcal{G}, \mathcal{F}}(p, c) \cdot D_{\mathbb{R}^+}(S, p, a) \\
& = \left(p\text{-lim} \frac{|\{U \cap a_M I_M\}|}{2M} \right) \\
& \quad \times \left(p\text{-lim} \frac{1}{|\{U \cap a_M I_M\}|} \sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right) \\
& - \left(p\text{-lim} \frac{|\{S \cap a_M I_M\}|}{2M} \right) \\
& \quad \times \left(p\text{-lim} \frac{1}{|\{S \cap a_M I_M\}|} \sum_{s \in S \cap a_M I_M} \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right)
\end{aligned}$$

$$\begin{aligned}
&= \left(p\text{-}\lim \frac{|\{U \cap a_M I_M\}|}{2M} \frac{1}{|\{U \cap a_M I_M\}|} \sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right) \\
&- \left(p\text{-}\lim \frac{|\{S \cap a_M I_M\}|}{2M} \frac{1}{|\{S \cap a_M I_M\}|} \sum_{s \in S \cap a_M I_M} \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right) \\
&= p\text{-}\lim \frac{1}{2M} \left(\sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right. \\
&\quad \left. - \sum_{s \in S \cap a_M I_M} \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right).
\end{aligned}$$

By the previous theorem,

$$\begin{aligned}
0 &= \lim_{M \rightarrow \infty} \frac{1}{2M} \left(\sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right. \\
&\quad \left. - \sum_{s \in S \cap a_M I_M} \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right).
\end{aligned}$$

Since every free ultrafilter limit is an accumulation point, we have

$$\begin{aligned}
0 &= p\text{-}\lim \frac{1}{2M} \left(\sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right. \\
&\quad \left. - \sum_{s \in S \cap a_M I_M} \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \right).
\end{aligned}$$

Hence

$$\mu_{\mathcal{F}, \mathcal{G}}(p, c) \cdot D_{\mathbb{R}^+}(U, p, a) - \mu_{\mathcal{G}, \mathcal{F}}(p, c) \cdot D_{\mathbb{R}^+}(S, p, a) = 0 \quad \square$$

2.4 Comparison Theorem for wavelet frames

In this section, we apply Theorem 2.3.1 and Corollary 2.3.2 to wavelet frames for $L^2(\mathbb{R})$ to derive a comparison theorem.

Theorem 2.4.1. *Suppose the following conditions hold.*

(a) $f, g \in L^2(\mathbb{R})$ are admissible and form a localized pair.

(b) U and S are relatively separated in \mathbb{R}^+ .

(c) $V \subset \mathbb{R}$ and $\mathcal{E}(V) = \{e^{2\pi i v w}\}_{v \in V}$ is a frame for $L^2[-r_V, r_V]$ with bounds A_V, B_V .

(d) $T \subset \mathbb{R}$ and $\mathcal{E}(T) = \{e^{2\pi i t w}\}_{t \in T}$ is a frame for $L^2[-r_T, r_T]$ with frame bounds E_T, F_T .

(e) $\mathcal{F} = \{\sigma(u, v)f\}_{(u, v) \in U \times V}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B .

(f) $\mathcal{G} = \{\sigma(s, t)g\}_{(s, t) \in S \times T}$ is a frame for $L^2(\mathbb{R})$ with frame bounds E, F .

Then for any free ultrafilter p and any sequence $a = \{a_M\} \subset \mathbb{R}^+$

$$\frac{A E_T C_g}{F B_V C_f} \leq \frac{D_{\mathbb{R}^+}(U, a, p)}{D_{\mathbb{R}^+}(S, a, p)} \leq \frac{B F_T C_g}{E A_V C_f}$$

Proof. By Theorem 2.2.14, we have

$$\frac{A}{B_V} \leq \sum_{u \in U} |\hat{f}(uw)|^2 \leq \frac{B}{A_V} \text{ a.e. } w \in \mathbb{R}.$$

Hence

$$\frac{A}{B_V} C_g \leq \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} \leq \frac{B}{A_V} C_g,$$

which implies

$$\frac{A}{B_V} C_g \leq \mu_{\mathcal{G}, \mathcal{F}}(p, c) \leq \frac{B}{A_V} C_g \tag{5}$$

for all p and $c = \{(a_M, b_M)\} \subset \mathbb{A}$. Similarly,

$$\frac{E}{F_T} C_f \leq \mu_{\mathcal{F}, \mathcal{G}}(p, c) \leq \frac{F}{E_T} C_f \tag{6}$$

for all p and $c = \{(a_M, b_M)\} \subset \mathbb{A}$. Finally, by Corollary 2.3.2 we have

$$\frac{D_{\mathbb{R}^+}(U, a, p)}{D_{\mathbb{R}^+}(S, a, p)} = \frac{\mu_{\mathcal{G}, \mathcal{F}}(p, c)}{\mu_{\mathcal{F}, \mathcal{G}}(p, c)}. \tag{7}$$

Combining estimates (5), (6) and (7) proves the theorem. \square

Corollary 2.4.2. *If $\mathcal{F} = \{\sigma(u, v)f\}_{(u,v) \in U \times V}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B and $\mathcal{E}(V) = \{e^{2\pi i v w}\}_{w \in V}$ a frame for $L^2[-r_V, r_V]$ with bounds A_V, B_V , then for any free ultrafilter p and any sequence $a = \{a_M\} \subset \mathbb{R}^+$ we have*

$$\frac{2A}{B_V C_f} \leq D_{\mathbb{R}^+}(U, a, p) \leq \frac{2B}{A_V C_f}.$$

Proof. Letting $\hat{g} = \chi_{[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]}$ and $\mathcal{G} = \{\sigma(2^m, n)g\}_{m, n \in \mathbb{Z}}$ we obtain an orthonormal basis for $L^2(\mathbb{R})$ with $C_g = 2 \ln 2$. Notice $\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$. We have

$$D_{\mathbb{R}^+}(\{2^m\}_{m \in \mathbb{Z}}, a, p) = \frac{1}{\ln 2}$$

for all p, a . Since Lemma 2.2.17 ensures that f, g are a localized pair, the result follows from Theorem 2.4.1. \square

We can use Theorem 2.4.1 to draw conclusions about the affine density of wavelet frames.

Theorem 2.4.3. *Suppose that the hypotheses of Theorem 2.4.1 hold. Then for any sequence $c = \{c_M\} \subset \mathbb{A}$ and free ultrafilter p we have*

$$\frac{A A_V E_T C_g}{F B_V F_T C_f} \leq \frac{D_{\mathbb{A}}(U \times V, c, p)}{D_{\mathbb{A}}(S \times T, c, p)} \leq \frac{B B_V F_T C_g}{E A_V E_T C_f}$$

Proof. Write $\{c_M\} = \{(a_M, b_M)\}$. By Theorem 2.4.1, we have

$$\frac{A E_T C_g}{F B_V C_f} \leq \frac{D_{\mathbb{R}^+}(U, a, p)}{D_{\mathbb{R}^+}(S, a, p)} \leq \frac{B F_T C_g}{E A_V C_f},$$

where $a = \{a_M\}$. By Lemma 2.2.12, we obtain

$$A_V \leq D_{\mathbb{R}}^-(V) \leq D_{\mathbb{R}}^+(V) \leq B_V$$

and

$$E_T \leq D_{\mathbb{R}}^-(T) \leq D_{\mathbb{R}}^+(T) \leq F_T.$$

Combining these estimates with Lemma 2.2.9 proves the result. \square

We recover the main theorem in [50] as a corollary to Theorem 2.4.3.

Corollary 2.4.4. *If $\mathcal{F} = \{\sigma(u, v)f\}_{(u,v) \in U \times V}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B and $\mathcal{E}(V) = \{e^{2\pi i v w}\}_{v \in V}$ is a frame for $L^2[-r_V, r_V]$ with bounds A_V, B_V , then for any free ultrafilter p and any sequence $c = \{c_M\} \subset \mathbb{A}$ we have*

$$\frac{2A A_V}{B_V C_f} \leq D_{\mathbb{A}}(U \times V, c, p) \leq \frac{2B B_V}{A_V C_f}.$$

Proof. Let $\hat{g} = \chi_{[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]}$ and $\mathcal{G} = \{\sigma(2^m, n)g\}_{m, n \in \mathbb{Z}}$ so \mathcal{G} is an orthonormal basis for $L^2(\mathbb{R})$ with $C_g = 2 \ln 2$. We have

$$D_{\mathbb{A}}(\{(2^m, n)\}_{m, n \in \mathbb{Z}}, c, p) = \frac{1}{\ln 2}$$

for all p, c . Since Lemma 2.2.17 ensures that f, g are a localized pair, the result follows from Theorem 2.4.3. \square

2.5 Comparison Theorem for wavelet frame sequences

It may appear that the crux of the proofs of Theorems 2.4.1 and 2.4.3 is the estimates

$$\frac{A}{B_V} \leq \sum_{u \in U} |\hat{f}(uw)|^2 \leq \frac{B}{A_V} \quad \text{a.e. } w \in \mathbb{R} \quad (8)$$

and

$$\frac{E}{F_T} \leq \sum_{s \in S} |\hat{g}(sw)|^2 \leq \frac{F}{E_T} \quad \text{a.e. } w \in \mathbb{R}, \quad (9)$$

which are guaranteed by [77] when \mathcal{F}, \mathcal{G} are frames for $L^2(\mathbb{R})$. However, this is not true. We can adapt our above approach to obtain similar comparison results for certain separable wavelet frame sequences for which the inequalities (8) and (9) need not hold.

Define an operator Δ by

$$(\Delta h)^\wedge(w) = \frac{\hat{h}(w)}{\sqrt{|w|}}.$$

A function h is admissible if and only if $h \in L^2(\mathbb{R})$ and $\Delta h \in L^2(\mathbb{R})$. The admissibility constant of h is $C_h = \|\Delta h\|_2^2$.

Theorem 2.5.1. *Suppose the following conditions hold.*

- (a) U and S are relatively separated in \mathbb{R}^+ .
- (b) $f, g \in L^2(\mathbb{R})$ are admissible, form a localized pair, are Chui-Shi bounded with respect to U, S , respectively, and \hat{f}, \hat{g} have compact support.
- (c) $V \subset \mathbb{R}$ and $\mathcal{E}(V) = \{e^{2\pi i v w}\}_{w \in V}$ is a frame for $L^2(\text{supp } \hat{f})$ with bounds A_V, B_V .
- (d) $T \subset \mathbb{R}$ and $\mathcal{E}(T) = \{e^{2\pi i t w}\}_{w \in T}$ is a frame for $L^2(\text{supp } \hat{g})$ with bounds E_T, F_T .
- (e) $S_{\mathcal{F}}$ is the frame operator for sequence $\mathcal{F} = \{\sigma(u, v)f\}_{(u,v) \in U \times V}$.
- (f) $S_{\mathcal{G}}$ is the frame operator for sequence $\mathcal{G} = \{\sigma(s, t)g\}_{(s,t) \in S \times T}$.

Then there exist constants $\alpha_{s,t} \in [B_V^{-1}, A_V^{-1}]$ and $\lambda_{u,v} \in [F_T^{-1}, E_T^{-1}]$ so that

$$0 = \lim_{M \rightarrow \infty} \frac{1}{2M} \left(\sum_{s \in S \cap a_M I_M} \alpha_{s,t} \langle \sigma(s, t) \Delta g, S_{\mathcal{F}} \sigma(s, t) \Delta g \rangle - \sum_{u \in U \cap a_M I_M} \lambda_{u,v} \langle \sigma(u, v) \Delta f, S_{\mathcal{G}} \sigma(u, v) \Delta f \rangle \right)$$

for any sequence $a = \{a_M\} \subset \mathbb{R}^+$.

Proof. Note that

$$\begin{aligned} & \langle \sigma(s, t) \Delta g, S_{\mathcal{F}} \sigma(s, t) \Delta g \rangle \\ &= \sum_{(u,v) \in U \times V} |\langle \sigma(s, t) \Delta g, \sigma(u, v) f \rangle|^2 \\ &= \sum_{(u,v) \in U \times V} \left| \left\langle \sigma \left(\frac{s}{u}, t \right) \Delta g, \sigma(1, v) f \right\rangle \right|^2 \\ &= \sum_{(u,v) \in U \times V} \left| \int |x|^{-\frac{1}{2}} \hat{g} \left(\frac{sx}{u} \right) e^{2\pi i t s x u^{-1}} \overline{\hat{f}(x)} e^{-2\pi i v x} dx \right|^2 \end{aligned}$$

$$\leq B_V \sum_{u \in U} \int \left| \hat{g} \left(\frac{sw}{u} \right) \right|^2 |\hat{f}(w)|^2 \frac{dw}{|w|}, \quad (10)$$

where estimate (10) comes from the fact $\mathcal{E}(V)$ is a frame for $L^2(\text{supp } \hat{f})$ with bounds A_V, B_V . Similarly,

$$\langle \sigma(s, t) \Delta g, S_{\mathcal{F}} \sigma(s, t) \Delta g \rangle \geq A_V \sum_{u \in U} \int \left| \hat{g} \left(\frac{sw}{u} \right) \right|^2 |\hat{f}(w)|^2 \frac{dw}{|w|}.$$

Choose $\alpha_{s,t} \in [B_V^{-1}, A_V^{-1}]$ so that

$$\alpha_{s,t} \langle \sigma(s, t) \Delta g, S_{\mathcal{F}} \sigma(s, t) \Delta g \rangle = \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|}$$

and choose $\lambda_{u,v} \in [F_T^{-1}, E_T^{-1}]$ so that

$$\lambda_{u,v} \langle \sigma(u, v) \Delta f, S_{\mathcal{G}} \sigma(u, v) \Delta f \rangle = \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|}.$$

Then

$$\begin{aligned} & \sum_{s \in S \cap a_M I_M} \alpha_{s,t} \langle \sigma(s, t) \Delta g, S_{\mathcal{F}} \sigma(s, t) \Delta g \rangle - \sum_{u \in U \cap a_M I_M} \lambda_{u,v} \langle \sigma(u, v) \Delta f, S_{\mathcal{G}} \sigma(u, v) \Delta f \rangle \\ &= \sum_{s \in S \cap a_M I_M} \sum_{u \in U} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|} - \sum_{u \in U \cap a_M I_M} \sum_{s \in S} \int |\hat{g}(sw)|^2 |\hat{f}(uw)|^2 \frac{dw}{|w|}. \end{aligned}$$

The technique used to prove Theorem 2.3.1 can be used to complete the proof. \square

We can think of

$$p\text{-lim} \frac{1}{|U \cap a_M I_M|} \sum_{u \in U \cap a_M I_M} \lambda_{u,v} \langle \sigma(u, v) \Delta f, S_{\mathcal{G}} \sigma(u, v) \Delta f \rangle$$

as a value similar to $\mu_{\mathcal{F}, \mathcal{G}}(p, c)$. With this understanding, Theorem 2.5.1 is analogous to Theorem 2.3.1.

Theorem 2.5.2. *Suppose the following conditions hold.*

(a) U and S are relatively separated in \mathbb{R}^+ .

(b) $f, g \in L^2(\mathbb{R})$ are admissible, form a localized pair, are Chui-Shi bounded with respect to U, S , respectively, and \hat{f}, \hat{g} have compact support.

(c) $V \subset \mathbb{R}$ and $\mathcal{E}(V) = \{e^{2\pi i v w}\}_{v \in V}$ is a frame for $L^2(\text{supp } \hat{f})$ with bounds A_V, B_V .

(d) $T \subset \mathbb{R}$ and $\mathcal{E}(T) = \{e^{2\pi i t w}\}_{t \in T}$ is a frame for $L^2(\text{supp } \hat{g})$ with bounds E_T, F_T .

(e) $\mathcal{F} = \{\sigma(u, v)f\}_{(u,v) \in U \times V}$ and $\mathcal{G} = \{\sigma(s, t)g\}_{(s,t) \in S \times T}$ are frames for some common subspace of $L^2(\mathbb{R})$ with frame bounds A, B and E, F , respectively.

Then for any free ultrafilter p and any sequence $a = \{a_M\} \subset \mathbb{R}^+$, we have

$$\frac{A E_T C_g}{F B_V C_f} \leq \frac{D_{\mathbb{R}^+}(U, a, p)}{D_{\mathbb{R}^+}(S, a, p)} \leq \frac{B F_T C_g}{E A_V C_f}.$$

Proof. Let $\{\alpha_{s,t}\}_{(s,t) \in S \times T} \subset [B_V^{-1}, A_V^{-1}]$, $\{\lambda_{u,v}\}_{(u,v) \in U \times V} \subset [F_T^{-1}, E_T^{-1}]$ be defined as in the proof of Theorem 2.5.1 and let $S_{\mathcal{F}}, S_{\mathcal{G}}$ be the frame operators for \mathcal{F}, \mathcal{G} respectively.

Since $A \leq S_{\mathcal{F}} \leq B$ and $\alpha_{s,t} \in [B_V^{-1}, A_V^{-1}]$, we see

$$\alpha_{s,t} \langle \sigma(s, t)\Delta g, S_{\mathcal{F}}\sigma(s, t)\Delta g \rangle = \alpha_{s,t} \left\| S_{\mathcal{F}}^{\frac{1}{2}}\Delta g \right\|^2 \in \left[\frac{A C_g}{B_V}, \frac{B C_g}{A_V} \right].$$

Therefore

$$p\text{-lim} \frac{1}{2M} \left(\sum_{s \in S \cap a_M I_M} \alpha_{s,t} \langle \sigma(s, t)\Delta g, S_{\mathcal{F}}\sigma(s, t)\Delta g \rangle \right) \in \left[\frac{A C_g}{B_T}, \frac{B C_g}{A_T} \right] \cdot D_{\mathbb{R}^+}(S, a, p).$$

Similarly,

$$\lambda_{u,v} \langle \sigma(u, v)f, S_{\mathcal{G}}\sigma(u, v)f \rangle = \lambda_{u,v} \left\| S_{\mathcal{G}}^{\frac{1}{2}}\Delta f \right\|^2 \in \left[\frac{E C_f}{F_T}, \frac{F C_f}{E_T} \right],$$

which implies

$$p\text{-lim} \frac{1}{2M} \left(\sum_{u \in U \cap a_M I_M} \lambda_{u,v} \langle \sigma(u, v)\Delta f, S_{\mathcal{G}}\sigma(u, v)\Delta f \rangle \right) \in \left[\frac{E C_f}{F_T}, \frac{F C_f}{E_T} \right] \cdot D_{\mathbb{R}^+}(U, a, p).$$

Hence from Theorem 2.5.1, we obtain

$$\frac{A E_T C_g}{F B_V C_f} \leq \frac{D_{\mathbb{R}^+}(U, a, p)}{D_{\mathbb{R}^+}(S, a, p)} \leq \frac{B F_T C_g}{E A_V C_f}. \quad \square$$

Corollary 2.5.3. *Suppose the hypotheses of Theorem 2.5.2 hold. Then for any free ultrafilter p and any sequence $c = \{c_M\} \subset \mathbb{A}$ we have*

$$\frac{A A_V E_T C_g}{F B_V F_T C_f} \leq \frac{D(U \times V, c, p)}{D(S \times T, c, p)} \leq \frac{B B_V F_T C_g}{E A_V E_T C_f}.$$

Proof. Corollary 2.5.3 follows from Theorem 2.5.2 for the same reasons that Theorem 2.4.3 follows from Theorem 2.4.1. □

CHAPTER III

AFFINE PSEUDODIFFERENTIAL OPERATORS

3.1 Introduction

In this chapter, we investigate the Schatten class properties of affine pseudodifferential operators. An affine pseudodifferential operator is a superposition of translation and dilation operators. More precisely, an affine pseudodifferential operator is one of the form

$$Af(t) = \iint_{\mathbb{R}^2} \mathcal{L}(a, b) \frac{1}{a} f\left(\frac{t-b}{a}\right) da db,$$

Affine pseudodifferential operators arise naturally in the study of wideband mobile communications, as noted in [4], [29], [38], [68] and [75]. Due to the multipath effect, a signal is received via a wireless communications channel as a superposition of delays of the transmitted signal. If the transmitter or receiver are moving, then the Doppler effect implies that the signal received is a superposition of rescalings of the signal transmitted. Hence, the received signal consists of superpositions of time-scale shifts of the transmitted signal f of the form $f\left(\frac{t-b}{a}\right)$. The quantity $\mathcal{L}(a, b)$ represents the “amount” of the transmitted signal, distorted by scale-shift amount a and delay amount b , comprising the received signal.

3.1.1 Relationship to Pseudodifferential Operators

Affine pseudodifferential operators are so-named because they are analogous to the more widely-studied pseudodifferential operators. Just as an affine pseudodifferential operator is a superposition of time-scale shifts, a pseudodifferential operator is a superposition of time-frequency shifts. Pseudodifferential operators have appeared widely in the literature of physics, signal processing and differential equations. In

particular, since the Doppler effect for narrowband wireless communications is closely modeled not as a change in scale but a shift in frequency, pseudodifferential operators are models for narrowband wireless communications (see [38] and [68]).

Because of the role of pseudodifferential operators in partial differential equations, the smoothness of the Weyl and Kohn-Nirenberg symbols of a pseudodifferential operator has traditionally been used to characterize properties of the operator, with the Hörmander symbol classes playing key roles. More recently, the continuity and Schatten class properties of pseudodifferential operators have been well-described by time-frequency analysis. In particular the modulation spaces $M_w^{p,q}(\mathbb{R}^d)$, which are Banach spaces characterized by time-frequency shifts, have been useful symbol spaces for studying continuity and Schatten class properties of pseudodifferential operators. Using the Gabor transform, elements in these spaces can be decomposed into a superposition of time-frequency shifts, and this Gabor decomposition of the symbol of a pseudodifferential operator can be used to characterize the properties of the operator. Results of this type appear in [21], [35], [41], [52], [71] and [73], while modulation spaces appear implicitly in [23], [45], [47], [59] and [64]. See [33] for an overview of modulation spaces and time-frequency analysis of pseudodifferential operators.

3.1.2 Summary of Results

3.1.2.1 Schatten class integral operators

Both affine pseudodifferential operators and pseudodifferential operators are types of integral operators. In this paper we develop a technique for analyzing the kernel of an integral operator to determine its Schatten-class properties. To obtain our main result, we analyze the “slices” of the kernel of an integral operator using a resolution of the identity. If these decomposed slices have a certain decay, then the operator is Schatten p -class. As a special case, we obtain the following theorem.

Theorem 3.1.1. *Suppose X is a locally compact group and σ is an irreducible unitary*

representation of X on $\mathcal{U}(L^2(\mathbb{R}^d))$ with left Haar measure μ . Assume A is an integral operator with kernel k and let $k(t, y) = k_y(t)$. Then there exists $\psi \in L^2(\mathbb{R}^d)$ such that if

$$\left(\int_X \left(\int_{\mathbb{R}} |\langle k_y, \sigma(x)\psi \rangle|^2 dy \right)^{\frac{p}{2}} d\mu(x) \right)^{\frac{1}{p}} < \infty, \quad (11)$$

for $p \in [1, 2]$, then A is Schatten p -class on $L^2(\mathbb{R}^d)$.

Notice that the integral in (11) is a mixed norm. The idea of using mixed norm spaces to classify the Schatten class properties of an integral operator is not new. In [60] and [56], it is shown that if the kernel of an integral operator belongs to an appropriate mixed norm space, then the operator is Schatten class. However, Theorem 3.1.1 is distinct from these older results. In particular, the mixed norm in (11) is not a mixed norm on the kernel k . Instead, it is a mixed norm on a transformation of k given by $(Zk)(x, y) = \langle k_y, \sigma(x)\psi \rangle$, arising from analyzing the slices of the kernel with the resolution of the identity determined by $\{\sigma(x)\psi\}_{x \in X}$.

Theorem 3.1.1 is a general result that is applicable to all integral operators including pseudodifferential operators, affine pseudodifferential operators and Fourier integral operators. The implications of this theorem for pseudodifferential operators and Fourier integral operators will be examined in Chapters 4 and 5.

3.1.2.2 Kernel and Symbol classes

The success of time-frequency analysis in characterizing pseudodifferential operators suggests that time-scale analysis may be useful in analyzing affine pseudodifferential operators. A direct application of Theorem 3.1.1 to affine pseudodifferential operators yields a slice-wavelet condition on the kernel which ensures the operator is Schatten class. Furthermore, because of the relationship between the kernel and symbol of an affine pseudodifferential operator, Theorem 3.1.1 gives rise to conditions on the ridgelet transform of the symbol which ensure certain spectral properties of the operator. The importance of the ridgelet transform of the symbol of a Schatten class

affine pseudodifferential operator is surprising to us because it is not analogous to the symbol results for pseudodifferential operators and Fourier integral operators in Chapters 4 and 5.

The wavelet and ridgelet conditions on the kernel and symbol, respectively, of an affine pseudodifferential operator give rise to families of spaces, $S_{q,p}$ and $R_{q,p}^s$ (defined precisely in Section 3.4), useful for characterizing the Schatten class properties of affine pseudodifferential operators. In particular, we obtain the following theorem.

Theorem 3.1.2. *Suppose A is an affine pseudodifferential operator with kernel k and symbol \mathcal{L} .*

- (a) *If $k \in S_{2,p}$ for some $p \in [1, 2]$, then $A \in \mathcal{I}_p(L^2(\mathbb{R}))$.*
- (b) *If $\mathcal{L} \in T_{2,p}^1$ for some $p \in [1, 2]$, then $A \in \mathcal{I}_p(L^2(\mathbb{R}))$.*

We will show that the spaces $S_{2,p}$ and $T_{2,p}^1$ are Banach spaces and Banach algebras under operations corresponding to composition of affine pseudodifferential operators. Furthermore, we find smoothness and decay conditions on the kernel and Radon transform of the symbol of an affine pseudodifferential operator that ensure the kernel and symbol lie in $S_{2,p}$ and $T_{2,p}^1$, respectively. Interestingly, these types of conditions also imply that the corresponding affine pseudodifferential operator is a Calderon-Zygmund operator.

The chapter is organized as follows. Definitions and basic lemmas are given in Section 3.2. In Section 3.3, we develop a Schatten class result for the kernel of an arbitrary integral operator. In Section 3.4, we describe new function classes that will be useful for characterizing Schatten class affine pseudodifferential operators. In Section 3.5, we state the main result and prove that these new function classes are nonempty. In Section 3.6, we find conditions on the Radon transform of the symbol which ensure the operator is Calderon-Zygmund.

3.2 Definitions and preliminary lemmas

3.2.1 Integral operator composition

This chapter concerns integral operators and affine pseudodifferential operators in particular. We introduce two operations to describe the effects of operator composition on these operators.

Definition 3.2.1. *Suppose $k_1, k_2 : \mathbb{R}^{2d} \rightarrow \mathbb{C}$. Define $k = k_1 \sharp k_2$ by*

$$k(t, y) = \int_{\mathbb{R}^d} k_1(t, x) k_2(x, y) dx,$$

for all $(t, y) \in \mathbb{R}^{2d}$ for which this integral converges.

Definition 3.2.2. *For $\mathcal{L}_1, \mathcal{L}_2 : \mathbb{R}^2$ define the affine convolution of $\mathcal{L}_1, \mathcal{L}_2$ by*

$$\mathcal{L}_1 \circledast \mathcal{L}_2(a, b) = \int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} \mathcal{L}_1(u, v) \mathcal{L}_2\left(\frac{a}{u}, \frac{b-v}{u}\right) \frac{du}{u^2} dv$$

for all $(a, b) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$ for which this integral converges.

We note that this definition of affine convolution agrees with the more general definition of convolution on locally compact groups (see [30] for background) for the affine group multiplication given by $(u, v)(x, y) = (ux, v + uy)$. However, this fact is not relevant to the analysis in this chapter.

The composition of two integral operators is an integral operator and the composition of two affine pseudodifferential operators is again an affine pseudodifferential operator. The following lemma, which is proved by direct computation, describes how new kernels and symbols are obtained through operator composition.

Lemma 3.2.3. *Suppose A_1, A_2 are affine pseudodifferential operators with symbols, $\mathcal{L}_1, \mathcal{L}_2$, respectively and kernels k_1, k_2 , respectively. Then $A_1 \circ A_2$ is an affine pseudodifferential operator with symbol $\mathcal{L}_1 \circledast \mathcal{L}_2$ and kernel $k_1 \sharp k_2$.*

3.2.2 Singular integral operators

An integral operator whose kernel is singular along its diagonal is called a singular integral operator. The following theorem concerning singular integral operators comes from [26].

Theorem 3.2.4. *Suppose T is an integral operator with kernel k such that $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is bounded. If there is some finite C such that*

$$\sup_{t,t'} \int_{|t-t'| < \frac{1}{2}|t-y|} |k(t,y) - k(t',y)| dy \leq C$$

and

$$\sup_{y,y'} \int_{|y-y'| < \frac{1}{2}|t-y|} |k(t,y) - k(t,y')| dt \leq C,$$

then $T : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ is bounded for all $p \in (1, \infty)$. In this case, T is called a (generalized) Calderon-Zygmund operator

See [26], [32] and [67] for background on Calderon-Zygmund operators.

3.2.3 The relationship between kernel and symbol

Recall that the Radon transform of $\mathcal{L} \in L^1(\mathbb{R}^2)$ is given by

$$R_\theta \mathcal{L}(s) = R\mathcal{L}(\theta, s) = \int_{\{x \in \mathbb{R}^2 : x \cdot \theta = s\}} \mathcal{L}(x) dx_{\ell(\theta,s)}, \quad \text{for all } (\theta, s) \in S^1 \times \mathbb{R},$$

where $dx_{\ell(\theta,s)}$ denotes the one-dimensional Lebesgue measure on the set $\ell(\theta, s) = \{x \in \mathbb{R}^2 : x \cdot \theta = s\}$. The next lemma describes a well-known property of the Radon transform. See [57] for the proof.

Lemma 3.2.5. *For each $\theta \in S^1$ we have $\|R_\theta \mathcal{L}\|_{L^1(\mathbb{R})} \leq \|\mathcal{L}\|_{L^1(\mathbb{R}^2)}$.*

We also recall that for admissible $\psi \in \mathcal{S}(\mathbb{R})$, the ridgelet transform of $\mathcal{L} \in L^1(\mathbb{R}^2)$ is

$$\mathcal{R}(\mathcal{L})(a, b, \theta) = \langle R_\theta \mathcal{L}, T_b D_a \psi \rangle = (R_\theta \mathcal{L} * D_{-a} \bar{\psi})(b) \quad \forall \theta \in S^1, a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}.$$

Comparing Definitions 1.2.19 and 1.2.16, we see that each affine pseudodifferential operator is also an integral operator, and the kernel of the affine pseudodifferential operator with symbol \mathcal{L} is

$$k(t, y) = \int_{\mathbb{R}} \mathcal{L}(a, t - ay) da.$$

This shows that the kernel and symbol of an affine pseudodifferential operator are related via the Radon transform. It is this relationship, stated precisely in the following lemma, that will allow us to use kernel conditions of Schatten class integral operators to draw conclusions about the symbols of affine pseudodifferential operators. Direct computation gives the following result.

Lemma 3.2.6. *Suppose A is an affine pseudodifferential operator with symbol \mathcal{L} and kernel k . Then $k = R\mathcal{L} \circ O$ where $O : \mathbb{R}^2 \rightarrow S^1 \times \mathbb{R}$ is given by $O(t, y) = \left(\left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}} \right), \frac{t}{\sqrt{y^2+1}} \right)$.*

Since the kernel of an affine pseudodifferential operator is closely related to the Radon transform of the symbol, the wavelet transform of the kernel corresponds to the ridgelet transform of the symbol. The exact relationship is given in the next lemma, which is proved directly using Lemma 3.2.6

Lemma 3.2.7. *Suppose A is an affine pseudodifferential operator with kernel k and symbol \mathcal{L} . Then*

$$\langle k_y, T_v D_u \psi \rangle = (y^2 + 1)^{\frac{1}{4}} \mathcal{R}(\mathcal{L}) \left(\frac{u}{\sqrt{y^2 + 1}}, \frac{v}{\sqrt{y^2 + 1}}, \left(\frac{y}{\sqrt{y^2 + 1}}, \frac{1}{\sqrt{y^2 + 1}} \right) \right).$$

3.3 A Schatten Class Result for Integral Operators

In this section, we develop a general Schatten class result for integral operators. Although the result (Theorem 3.3.2) is elementary, it does not seem to be in the literature. The crux of the proof lies in the following lemma.

Lemma 3.3.1. *Assume $\{f_j\}_{j \in \mathbb{N}}, \{g_j\}_{j \in \mathbb{N}}$ are orthonormal sequences in $L^2(\mathbb{R}^d)$. Suppose $\{\psi_x\}_{x \in X}$ is some collection of functions in $L^2(\mathbb{R}^d)$ with $B = \sup_{x \in X} \|\psi_x\|_{L^2(\mathbb{R}^d)}^2 < \infty$ and suppose that (X, μ) is a measure space satisfying*

$$\langle f, g \rangle = K_\psi^{-1} \int_X \langle f, \psi_x \rangle \langle \psi_x, g \rangle d\mu(x), \quad \text{for all } f, g \in L^2(\mathbb{R}^d).$$

For $G \in L^{2,p}(\mathbb{R}^d, X)$ define

$$T(G) = \left\{ \int_X \langle f_j, G(\cdot, x) \rangle \langle \psi_x, g_j \rangle d\mu(x) \right\}_{j \in \mathbb{N}}.$$

Then for all $p \in [1, 2]$, $T : L^{2,p}(\mathbb{R}^d, X) \longrightarrow \ell^p(\mathbb{N})$ is bounded with $\|T\| \leq B^{\frac{1}{p}-\frac{1}{2}} K_\psi^{1-\frac{1}{p}}$.

Proof. By Tonelli's Theorem and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|T(G)\|_{\ell^1} &= \sum_{j \in \mathbb{N}} \left| \int_X \langle f_j, G(\cdot, x) \rangle \langle \psi_x, g_j \rangle d\mu(x) \right| \\ &\leq \int_X \sum_{j \in \mathbb{N}} |\langle f_j, G(\cdot, x) \rangle| |\langle \psi_x, g_j \rangle| d\mu(x) \\ &\leq \int_X \left(\sum_{j \in \mathbb{N}} |\langle f_j, G(\cdot, x) \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{N}} |\langle \psi_x, g_j \rangle|^2 \right)^{\frac{1}{2}} d\mu(x) \\ &\leq \int_X \|G(\cdot, x)\|_{L^2(\mathbb{R}^d)} \|\psi_x\|_{L^2(\mathbb{R}^d)} d\mu(x) \\ &\leq B^{\frac{1}{2}} \int_X \|G(\cdot, x)\|_{L^2(\mathbb{R}^d)} d\mu(x) \\ &= B^{\frac{1}{2}} \|G\|_{L^{2,1}(\mathbb{R}^d, X)} \end{aligned}$$

and

$$\begin{aligned} \|T(G)\|_{\ell^2} &= \left(\sum_{j \in \mathbb{N}} \left| \int_X \langle f_j, G(\cdot, x) \rangle \langle \psi_x, g_j \rangle d\mu(x) \right|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j \in \mathbb{N}} \left(\int_X |\langle f_j, G(\cdot, x) \rangle|^2 d\mu(x) \right) \left(\int_X |\langle \psi_x, g_j \rangle|^2 d\mu(x) \right) \right)^{\frac{1}{2}} \\ &= \left(\sum_{j \in \mathbb{N}} \left(\int_X |\langle f_j, G(\cdot, x) \rangle|^2 d\mu(x) \right) K_\psi \|g_j\|_{L^2(\mathbb{R}^d)}^2 \right)^{\frac{1}{2}} \\ &= K_\psi^{\frac{1}{2}} \left(\int_X \sum_{j \in \mathbb{N}} |\langle f_j, G(\cdot, x) \rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq K_\psi^{\frac{1}{2}} \left(\int_X \|G(\cdot, x)\|_{L^2(\mathbb{R}^d)}^2 d\mu(x) \right)^{\frac{1}{2}} \\
&= K_\psi^{\frac{1}{2}} \|G\|_{L^{2,2}(\mathbb{R}^d, X)}.
\end{aligned}$$

Hence the theorem holds for $p = 1$ and $p = 2$. The Riesz-Thorin Interpolation Theorem (see [8]) gives the result for $p \in (1, 2)$. \square

The next theorem gives sufficient conditions on the kernel of an integral operator so that the operator is Schatten p -class when $p \in [1, 2]$. Notice that part (b) of the theorem shows that the analogous conditions are not sufficient for $p \in (2, \infty]$.

Theorem 3.3.2. *Suppose $\{\psi_x\}_{x \in X}$ is some collection of functions in $L^2(\mathbb{R}^d)$ with $B = \sup_{x \in X} \|\psi_x\|_{L^2(\mathbb{R}^d)}^2 < \infty$ and suppose that (X, μ) is a measure space satisfying*

$$\langle f, g \rangle = K_\psi^{-1} \int_X \langle f, \psi_x \rangle \langle \psi_x, g \rangle d\mu(x) \quad \text{for all } f, g \in L^2(\mathbb{R}^d).$$

Assume A is an integral operator with kernel k .

(a) *If $p \in [1, 2]$ then*

$$\|A\|_{\mathcal{I}_p(L^2(\mathbb{R}^d))} \leq B^{\frac{1}{p} - \frac{1}{2}} K_\psi^{-\frac{1}{p}} \left\| \left(\int_{\mathbb{R}^d} |\langle k_y, \psi_x \rangle|^2 dy \right)^{\frac{1}{2}} \right\|_{L^p(X, \mu)}.$$

(b) *If $p \in [2, \infty]$ then*

$$\left\| \left(\int_{\mathbb{R}^d} |\langle k_y, \psi_x \rangle|^2 dy \right)^{\frac{1}{2}} \right\|_{L^p(X, \mu)} \leq B^{\frac{1}{2} - \frac{1}{p}} K_\psi^{1 - \frac{1}{p}} \|A\|_{\mathcal{I}_p(L^2(\mathbb{R}^d))}.$$

Proof. Suppose $p \in [1, 2]$ and $\{f_j\}_{j \in \mathbb{N}}, \{g_j\}_{j \in \mathbb{N}}$ are orthonormal sequences in $L^2(\mathbb{R}^d)$.

Let $G(y, x) = \langle \psi_x, k_y \rangle$. Notice that

$$\langle Af_j, \psi_x \rangle = \iint f_j(y) k(t, y) \overline{\psi_x(t)} dy dt = \langle f_j, G(\cdot, x) \rangle.$$

Using the previous lemma, we have

$$\left(\sum_{j \in \mathbb{N}} |\langle Af_j, g_j \rangle|^p \right)^{\frac{1}{p}} = K_\psi^{-1} \left(\sum_{j \in \mathbb{N}} \left| \int_X \langle Af_j, \psi_x \rangle \langle \psi_x, g_j \rangle d\mu(x) \right|^p \right)^{\frac{1}{p}}$$

$$\begin{aligned}
&= K_\psi^{-1} \left(\sum_{j \in \mathbb{N}} \left| \int_X \langle f_j, G(\cdot, x) \rangle \langle \psi_x, g_j \rangle d\mu(x) \right|^p \right)^{\frac{1}{p}} \\
&\leq K_\psi^{-1} B^{\frac{1}{p}-\frac{1}{2}} K_\psi^{1-\frac{1}{p}} \|G\|_{L^2, p(\mathbb{R}^d, X)} \\
&= B^{\frac{1}{p}-\frac{1}{2}} K_\psi^{-\frac{1}{p}} \left(\int_X \left(\int_{\mathbb{R}^d} |\langle k_y, \psi_x \rangle|^2 dy \right)^{\frac{p}{2}} d\mu(x) \right)^{\frac{1}{p}}.
\end{aligned}$$

Taking the supremum over all orthonormal sequences gives (a).

Now we prove (b). In the case $p = 2$ we have

$$\begin{aligned}
\left\| \left(\int_{\mathbb{R}^d} |\langle k_y, \psi_x \rangle|^2 dy \right)^{\frac{1}{2}} \right\|_{L^p(X, \mu)} &= \left(\int_X \int_{\mathbb{R}^d} |\langle k_y, \psi_x \rangle|^2 dy d\mu(x) \right)^{\frac{1}{2}} \\
&= K_\psi^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \|k_y\|_{L^2(\mathbb{R}^d)}^2 dy \right)^{\frac{1}{2}} \\
&= K_\psi^{\frac{1}{2}} \|k\|_{L^2(\mathbb{R}^{2d})} \\
&= K_\psi^{\frac{1}{2}} \|A\|_{\mathcal{I}_2(L^2(\mathbb{R}^d))}
\end{aligned}$$

Consider the case $p = \infty$. We see that $\langle k_y, \psi_x \rangle = \overline{A^* \phi_x(y)}$. Hence

$$\begin{aligned}
\left\| \left(\int_{\mathbb{R}^d} |\langle k_y, \psi_x \rangle|^2 dy \right)^{\frac{1}{2}} \right\|_{L^p(X, \mu)} &= \sup_{x \in X} \left(\int_{\mathbb{R}^d} |\langle k_y, \psi_x \rangle|^2 dy \right)^{\frac{1}{2}} \\
&= \sup_{x \in X} \|A^* \psi_x\|_{L^2(\mathbb{R}^d)} \\
&\leq \sup_{x \in X} \|A^*\| \|\psi_x\|_{L^2(\mathbb{R}^d)} \\
&\leq B^{\frac{1}{2}} \|A\|_{\mathcal{I}_\infty(L^2(\mathbb{R}^d))}.
\end{aligned}$$

The case $p \in (2, \infty)$ now follows by interpolation. □

The conditions assumed in Lemma 3.3.1 and Theorem 3.3.2 are valid for two common types of resolution of the identity, namely frames and irreducible unitary representations.

Example 3.3.3. *Suppose X is a locally compact group and σ is an irreducible unitary representation of X on $\mathcal{U}(L^2(\mathbb{R}^d))$ with left Haar measure μ . Then for some $\psi \in$*

$L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2(\mathbb{R}^d)} = 1$, there is $K_\psi \in (0, \infty)$ with

$$\langle f, g \rangle = K_\psi^{-1} \int_X \langle f, \psi_x \rangle \langle \psi_x, g \rangle d\mu(x) \quad \text{for all } f, g \in L^2(\mathbb{R}^d).$$

Thus the conditions of Theorem 3.3.2 are satisfied with $B = 1$.

Example 3.3.4. Suppose $\{\psi_n\}_{n \in \Lambda}$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound B .

Then

$$f = B^{-1} \sum_{n \in \Lambda} \langle f, \psi_n \rangle \psi_n \quad \forall f \in L^2(\mathbb{R}^d).$$

Hence

$$\langle f, g \rangle = B^{-1} \sum_{n \in \Lambda} \langle f, \psi_n \rangle \langle \psi_n, g \rangle \quad \forall f, g \in L^2(\mathbb{R}^d),$$

which in turn implies $\|\psi_n\|^2 \leq B$ for all $n \in \Lambda$. Thus we see the conditions of Theorem 3.3.2 are satisfied with $K_\psi = B$ and μ equal to counting measure on Λ .

As a consequence of these examples, we obtain following corollaries as special cases of Theorem 3.3.2.

Corollary 3.3.5. Suppose X is a locally compact group and σ is an irreducible unitary representation of X on $\mathcal{U}(L^2(\mathbb{R}^d))$ with left Haar measure μ . Assume A is an integral operator with kernel k . Then for some $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2(\mathbb{R}^d)} = 1$, there is $K_\psi \in (0, \infty)$ depending only on the group X , the representation σ and the function ψ with

$$\|A\|_{\mathcal{I}_p(L^2(\mathbb{R}^d))} \leq K_\psi^{-\frac{1}{p}} \left(\int_X \left(\int_{\mathbb{R}^d} |\langle k_y, \sigma(x)\psi \rangle|^2 dy \right)^{\frac{p}{2}} d\mu(x) \right)^{\frac{1}{p}} \quad \forall p \in [1, 2].$$

Corollary 3.3.6. Suppose $\{\psi_n\}_{n \in \Lambda}$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound B . Assume A is an integral operator with kernel k . Then

$$\|A\|_{\mathcal{I}_p(L^2(\mathbb{R}^d))} \leq B^{-\frac{1}{2}} \left(\sum_{n \in \Lambda} \left(\int_{\mathbb{R}^d} |\langle k_y, \psi_n \rangle|^2 dy \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \quad \forall p \in [1, 2].$$

Corollary 3.3.7. *Suppose $p \in [1, 2]$ and $\{\psi_m\}_{m \in \Lambda}$ is a Parseval frame for $L^2(\mathbb{R}^d)$. If A is an integral operator with kernel k and*

$$\left(\sum_{n \in \Lambda} \left(\sum_{m \in \Lambda} |\langle k, \psi_n \otimes \psi_m \rangle|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} < \infty,$$

then $A \in \mathcal{I}_p(L^2(\mathbb{R}^d))$.

Proof. Since $\{\psi_m\}_{m \in \Lambda}$ is a Parseval frame for $L^2(\mathbb{R}^d)$, Lemma 3.2 in [45] implies $\{\psi_n \otimes \psi_m\}_{m, n \in \Lambda}$ is a Parseval frame for $L^2(\mathbb{R}^{2d})$.

By Corollary 3.3.6, we have

$$\|A\|_{\mathcal{I}_p(L^2(\mathbb{R}^d))} \leq \left(\sum_{n \in \Lambda} \left(\int_{\mathbb{R}^d} |\langle k_y, \psi_n \rangle|^2 \right)^{\frac{p}{2}} dy \right)^{\frac{1}{p}}.$$

Letting $F_n(y) = \langle k_y, \psi_n \rangle$, we see that

$$\begin{aligned} \|A\|_{\mathcal{I}_p(L^2(\mathbb{R}^d))} &\leq \left(\sum_{n \in \Lambda} \|F_n\|_{L^2(\mathbb{R}^d)}^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{n \in \Lambda} \left(\sum_{m \in \Lambda} |\langle F_n, \psi_m \rangle|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &= \left(\sum_{n \in \Lambda} \left(\sum_{m \in \Lambda} |\langle k, \psi_n \otimes \psi_m \rangle|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &< \infty \end{aligned} \tag{12}$$

where (12) comes from the fact $\{\psi_m\}_{m \in \Lambda}$ is a Parseval frame for $L^2(\mathbb{R}^d)$. \square

3.4 New Kernel and Symbol Classes

Corollary 3.3.5 points to new kernel classes useful in identifying Schatten class integral operators. These kernel spaces also give rise to symbol classes for Schatten class affine pseudodifferential operators. In this section, we define these spaces and examine their properties.

3.4.1 Kernel Spaces

Definition 3.4.1. Let $\psi \in \mathcal{S}(\mathbb{R})$ be an admissible function with $C_\psi = \|\psi\|_{L^2(\mathbb{R})} = 1$.

For $k \in \mathcal{S}'(\mathbb{R}^2)$ define

$$\|k\|_{S_{q,p}} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}-\{0\}} \left(\int_{\mathbb{R}} |\langle k_y, T_v D_u \psi \rangle|^q dy \right)^{\frac{p}{q}} \frac{du}{u^2} dv \right)^{\frac{1}{p}},$$

with the usual modifications when p or q is ∞ . Let

$$S_{q,p} = \left\{ k \in \mathcal{S}'(\mathbb{R}^2) : \|k\|_{S_{q,p}} < \infty \right\}.$$

Theorem 3.4.2. (a) For each $p \in [1, 2]$, $S_{2,p}$ is a normed linear space.

(b) For each $p \in [1, 2]$, we have $\|k\|_{L^2(\mathbb{R}^2)} \leq \|k\|_{S_{2,p}}$.

(c) For each $1 \leq p \leq q \leq 2$, we have $\|k\|_{S_{2,q}} \leq \|k\|_{S_{2,p}}$.

(d) If $p \in [1, 2]$, then $S_{2,p}$ is a Banach space under the norm $\|\cdot\|_{S_{2,p}}$.

Proof. First we prove (b). Note that equation (2) implies that $\|k\|_{L^2(\mathbb{R}^2)} = \|k\|_{S_{2,2}}$.

For $p \in [1, 2)$, let $\frac{1}{q} = \frac{p}{2}$. Then $q \in (1, 2]$ and the dual index of q is $q' = \frac{2}{2-p} \in [2, \infty)$.

We have

$$\begin{aligned} \|k\|_{L^2(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} |\langle k_y, T_v D_u \psi \rangle|^2 dy \frac{du}{u^2} dv \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}-\{0\}} \|k_y\|_{L^2(\mathbb{R})}^{2-p} |\langle k_y, T_v D_u \psi \rangle|^p \frac{du}{u^2} dv dy \\ &\leq \left(\int_{\mathbb{R}} \|k_y\|_{L^2(\mathbb{R})}^{q'(2-p)} dy \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}-\{0\}} |\langle k_y, T_v D_u \psi \rangle|^p \frac{du}{u^2} dv \right)^q dy \right)^{\frac{1}{q}} \\ &= \|k\|_{L^2(\mathbb{R}^2)}^{2-p} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}-\{0\}} |\langle k_y, T_v D_u \psi \rangle|^p \frac{du}{u^2} dv \right)^{\frac{2}{p}} dy \right)^{\frac{p}{2}} \\ &\leq \|k\|_{L^2(\mathbb{R}^2)}^{2-p} \int_{\mathbb{R}} \int_{\mathbb{R}-\{0\}} \left(\int_{\mathbb{R}} |\langle k_y, T_v D_u \psi \rangle|^2 dy \right)^{\frac{p}{2}} \frac{du}{u^2} dv \\ &\leq \|k\|_{L^2(\mathbb{R}^2)}^{2-p} \|k\|_{S_{2,p}}^p, \end{aligned} \tag{13}$$

where (13) results from Minkowski's integral inequality. Hence $\|k\|_{L^2(\mathbb{R}^2)}^p \leq \|k\|_{S_{2,p}}^p$, from which (b) follows.

Now we prove (a). Routine calculations show that $\|\cdot\|_{S_{2,p}}$ is a seminorm. If $\|k\|_{S_{2,p}} = 0$ then (b) implies $\|k\|_{L^2(\mathbb{R}^2)} = 0$. Hence $k = 0$ and $\|\cdot\|_{S_{2,p}}$ is in fact a norm.

To prove (c), we let $G_{u,v}(y) = \langle T_v D_u \psi, k_y \rangle$ and suppose $1 \leq p \leq q \leq 2$. Since $\|G_{u,v}\|_{L^2(\mathbb{R})} \leq \|k\|_{L^2(\mathbb{R}^2)}$ for all $(u, v) \in \mathbb{R}^+ \times \mathbb{R}$, we have

$$\begin{aligned} \|k\|_{S_{2,q}}^q &= \iint \|G_{u,v}\|_{L^2(\mathbb{R})}^q \frac{du}{u^2} dv \\ &\leq \iint \|k\|_{L^2(\mathbb{R}^2)}^{q-p} \|G_{u,v}\|_{L^2(\mathbb{R})}^p \frac{du}{u^2} dv \\ &= \|k\|_{L^2(\mathbb{R}^2)}^{q-p} \|k\|_{S_{2,p}}^p \\ &= \|k\|_{L^2(\mathbb{R}^2)}^{\frac{q-p}{2}} \|k\|_{L^2(\mathbb{R}^2)}^{\frac{q-p}{2}} \|k\|_{S_{2,p}}^p \\ &\leq \|k\|_{S_{2,q}}^{\frac{q-p}{2}} \|k\|_{S_{2,p}}^{\frac{q-p}{2}} \|k\|_{S_{2,p}}^p. \end{aligned}$$

Hence $\|k\|_{S_{2,q}}^{q-\frac{q-p}{2}} \leq \|k\|_{S_{2,p}}^{p+\frac{q-p}{2}}$. But $q - \frac{q-p}{2} = \frac{q}{2} + \frac{p}{2} = p + \frac{q-p}{2}$, so that $\|k\|_{S_{2,q}} \leq \|k\|_{S_{2,p}}$.

Now we prove (d). Suppose $\{k_m\}_{m \in \mathbb{N}}$ is a Cauchy sequence in the $\|\cdot\|_{S_{2,p}}$ norm. Since $\|k_m - k_n\|_{L^2(\mathbb{R}^2)} \leq \|k_m - k_n\|_{S_{2,p}}$, it follows that $\{k_m\}_{m \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^2)$. So there is some $k \in L^2(\mathbb{R}^2)$ such that $k_m \rightarrow k$ in $L^2(\mathbb{R}^2)$. In particular, we have $(k_m)_y \rightarrow k_y$ in $L^2(\mathbb{R})$ for a.e. $y \in \mathbb{R}$.

Define a linear isometry $H : S_{2,p} \rightarrow L^{2,p,p}(\mathbb{R}, \mathbb{R} \setminus \{0\}, \mathbb{R})$ by

$$H(f)(y, u, v) = |u|^{-\frac{2}{p}} \langle f_y, T_v D_u \psi \rangle.$$

It follows that $\{H(k_m)\}_{m \in \mathbb{N}}$ is a Cauchy sequence in $L^{2,p,p}(\mathbb{R}, \mathbb{R} \setminus \{0\}, \mathbb{R})$. Since $L^{2,p,p}(\mathbb{R}, \mathbb{R} \setminus \{0\}, \mathbb{R})$ is a Banach space, there is some $g \in L^{2,p,p}(\mathbb{R}, \mathbb{R} \setminus \{0\}, \mathbb{R})$ so that $\|g - H(k_m)\|_{L^{2,p,p}} \rightarrow 0$ as $m \rightarrow \infty$. Hence $\{H(k_m)\}_{m \in \mathbb{N}}$ converges to g in measure so that there is a subsequence $\{H(k_{m_j})\}_{j \in \mathbb{N}}$ of $\{H(k_m)\}_{m \in \mathbb{N}}$ with $\lim_j H(k_{m_j})(y, u, v) = g(y, u, v)$ for a.e. $(y, u, v) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}$. But for almost all $(y, u, v) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}$ we have

$$\lim_{m \rightarrow \infty} H(k_m)(y, u, v) = \lim_{m \rightarrow \infty} |u|^{-\frac{2}{p}} \langle (k_m)_y, T_v D_u \psi \rangle$$

$$\begin{aligned}
&= |u|^{-\frac{2}{p}} \langle k_y, T_v D_u \psi \rangle \\
&= H(k)(y, u, v).
\end{aligned}$$

Hence $H(k)(y, u, v) = g(y, u, v)$ a.e. and

$$\lim_{m \rightarrow \infty} \|k - k_m\|_{S_{2,p}} = \lim_{m \rightarrow \infty} \|g - H(k_m)\|_{L^{2,p,p}(\mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R})} = 0. \quad \square$$

The next two results show that the kernel operation corresponding to affine pseudodifferential operator composition is well-behaved in $S_{2,p}$.

Proposition 3.4.3. (a) For each $p \in [1, 2]$ we have $\|k_1 \# k_2\|_{S_{2,p}} \leq \|k_1\|_{S_{2,p}} \|k_2\|_{L^2(\mathbb{R}^2)}$.

$$(b) \|k_1 \# k_2\|_{L^2(\mathbb{R}^2)} \leq \|k_1\|_{L^2(\mathbb{R}^2)} \|k_2\|_{L^2(\mathbb{R}^2)}.$$

$$(c) \text{ For each } p \in [1, 2] \text{ we have } \|k_1 \# k_2\|_{S_{2,p}} \leq \|k_1\|_{S_{2,p}} \|k_2\|_{S_{2,p}}.$$

Proof. Let $k = k_1 \# k_2$ and assume $p \in [1, 2]$. Then

$$\begin{aligned}
\langle k_y, T_v D_u \psi \rangle &= \int_{\mathbb{R}} k(t, y) \overline{T_v D_u \psi(t)} dt \\
&= \iint_{\mathbb{R}^2} k_1(t, x) k_2(x, y) \overline{T_v D_u \psi(t)} dx dt \\
&= \int_{\mathbb{R}} \langle (k_1)_x, T_v D_u \psi \rangle k_2(x, y) dx.
\end{aligned}$$

By Cauchy-Schwarz we have

$$|\langle k_y, T_v D_u \psi \rangle|^2 \leq \left(\int_{\mathbb{R}} |\langle (k_1)_x, T_v D_u \psi \rangle|^2 dx \right) \left(\int_{\mathbb{R}} |k_2(x, y)|^2 dx \right).$$

Therefore

$$\int |\langle k_y, T_v D_u \psi \rangle|^2 dy \leq \left(\int_{\mathbb{R}} |\langle (k_1)_x, T_v D_u \psi \rangle|^2 dx \right) \|k_2\|_{L^2(\mathbb{R}^2)}^2.$$

Thus

$$\|k\|_{S_{2,p}} = \left(\iint \left(\int_{\mathbb{R}} |\langle k_y, T_v D_u \psi \rangle|^2 dy \right)^{\frac{p}{2}} \frac{du}{u^2} dv \right)^{\frac{1}{p}}$$

$$\begin{aligned}
&\leq \left(\iint \left(\int_{\mathbb{R}} |\langle (k_1)_x, T_v D_u \psi \rangle|^2 dx \right)^{\frac{p}{2}} \|k_2\|_{L^2(\mathbb{R}^2)}^p \frac{du}{u^2} dv \right)^{\frac{1}{p}} \\
&= \|k_1\|_{S_{2,p}} \|k_2\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Using the fact that $\|f\|_{L^2(\mathbb{R}^2)} = \|f\|_{S_{2,2}}$, we see that (b) follows from (a). Statement (c) follows from (a) and Theorem 3.4.2(b). \square

Corollary 3.4.4. *For $p \in [1, 2]$, $S_{2,p}$ is a Banach algebra and a left ideal in $L^2(\mathbb{R}^2)$ under \sharp .*

3.4.2 Symbol Spaces

In the remainder of this section we seek to define spaces useful for categorizing symbols of affine pseudodifferential operators and to identify their important properties.

Definition 3.4.5. *Define $Q : \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\} \times \mathbb{R} \times S^1$ by*

$$Q(y, u, v) = \left(\frac{u}{\sqrt{y^2 + 1}}, \frac{v}{\sqrt{y^2 + 1}}, \left(\frac{y}{\sqrt{y^2 + 1}}, \frac{1}{\sqrt{y^2 + 1}} \right) \right).$$

Define

$$\|\mathcal{L}\|_{T_{q,p}^s} = \|\mathcal{L}\|_{L^1(\mathbb{R}^2)} + \left(\int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} \left(\int_{\mathbb{R}} |\mathcal{R}(\mathcal{L})(Q(y, u, v))|^q v_s(y) dy \right)^{\frac{p}{q}} \frac{du}{u^2} dv \right)^{\frac{1}{p}},$$

with the usual modifications when p or q is ∞ . Let

$$T_{q,p}^s = \left\{ \mathcal{L} \in L^1(\mathbb{R}^2) : \|\mathcal{L}\|_{T_{q,p}^s} < \infty \right\}.$$

Because the ridgelet transform is

$$\mathcal{R}(\mathcal{L})(a, b, \theta) = \langle R_\theta \mathcal{L}, T_b D_a \psi \rangle = (R_\theta \mathcal{L} * D_{-a} \bar{\psi})(b) \quad \forall \theta \in S^1, a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R},$$

the norm in Definition 3.4.5 depends implicitly on the choice of ψ . Also notice that if $k = R\mathcal{L} \circ O \in S_{q,p}$, then Lemma 3.2.7 implies that

$$\|k\|_{S_{q,p}} \equiv \left(\int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} \left(\int_{\mathbb{R}} |\mathcal{R}(\mathcal{L})(Q(y, u, v))|^q [v_1(y)]^{\frac{q}{2}} dy \right)^{\frac{p}{q}} \frac{du}{u^2} dv \right)^{\frac{1}{p}}.$$

Lemma 3.4.6. For each $p, q \in [1, \infty]$ and each $s \in \mathbb{R}$, $T_{q,p}^s$ is a normed linear space.

Proof. Since the ridgelet transform is linear, it follows that $\|\cdot\|_{T_{q,p}^s}$ is a seminorm. If $\|\mathcal{L}\|_{T_{q,p}^s} = 0$ then $\|\mathcal{L}\|_{L^1} = 0$ so that $\mathcal{L} = 0$. \square

In order to show that $T_{q,p}^s$ is a Banach space, we need the following lemma.

Lemma 3.4.7. Suppose $\lim_{n \rightarrow \infty} \|\mathcal{L} - \mathcal{L}_n\|_{L^1(\mathbb{R}^2)} = 0$. Then for a.e. $(y, u, v) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}$ we have $\mathcal{R}(\mathcal{L})(Q(y, u, v)) = \lim_{n \rightarrow \infty} \mathcal{R}(\mathcal{L}_n)(Q(y, u, v))$.

Proof. By definition we have

$$\mathcal{R}(\mathcal{L} - \mathcal{L}_n)(Q(y, u, v)) = \left(R\left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}}\right) (\mathcal{L} - \mathcal{L}_n) * D_{\frac{-u}{\sqrt{y^2+1}}} \bar{\psi} \right) \left(\frac{v}{\sqrt{y^2+1}} \right).$$

Using Lemma 3.2.5, we have for all $y \in \mathbb{R}$, $u \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} & \left\| R\left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}}\right) (\mathcal{L} - \mathcal{L}_n) * D_{\frac{-u}{\sqrt{y^2+1}}} \bar{\psi} \right\|_{L^\infty(\mathbb{R})} \\ & \leq \left\| R\left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}}\right) (\mathcal{L} - \mathcal{L}_n) \right\|_{L^1(\mathbb{R})} \left\| D_{\frac{-u}{\sqrt{y^2+1}}} \bar{\psi} \right\|_{L^\infty(\mathbb{R})} \\ & \leq \|\mathcal{L} - \mathcal{L}_n\|_{L^1(\mathbb{R}^2)} \left\| D_{\frac{-u}{\sqrt{y^2+1}}} \psi \right\|_{L^\infty(\mathbb{R})} \\ & = \left| \frac{u}{\sqrt{y^2+1}} \right|^{-\frac{1}{2}} \|\mathcal{L} - \mathcal{L}_n\|_{L^1(\mathbb{R}^2)} \|\psi\|_{L^\infty(\mathbb{R})} \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence for almost every $(y, u, v) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}$ we have

$$\lim_{n \rightarrow \infty} \left| \left(R\left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}}\right) (\mathcal{L} - \mathcal{L}_n) * D_{\frac{-u}{\sqrt{y^2+1}}} \bar{\psi} \right) \left(\frac{v}{\sqrt{y^2+1}} \right) \right| = 0.$$

The result follows. \square

Theorem 3.4.8. For each $p, q \in [1, \infty]$ and each $s \in \mathbb{R}$, $T_{q,p}^s$ is a Banach space.

Proof. Suppose $\{\mathcal{L}_n\}$ is Cauchy in $T_{q,p}^s$. Then $\{\mathcal{L}_n\}$ is Cauchy in $L^1(\mathbb{R}^2)$. Hence there is some $\mathcal{L} \in L^1(\mathbb{R}^2)$ with $\|\mathcal{L} - \mathcal{L}_n\|_{L^1} \rightarrow 0$ as $n \rightarrow \infty$. By the previous lemma, we see that $\mathcal{R}(\mathcal{L})(Q(y, u, v)) = \lim_{n \rightarrow \infty} \mathcal{R}(\mathcal{L}_n)(Q(y, u, v))$ for a.e. $(y, u, v) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}$.

Let $w(u) = \frac{1}{u^2}$. Since $\{\mathcal{L}_n\}$ is Cauchy in $T_{q,p}^s$, we see that $\{\mathcal{R}(\mathcal{L}_n) \circ Q\}$ is Cauchy in the space $L_{v_s \otimes w \otimes v_0}^{q,p,p}(\mathbb{R}, \mathbb{R} \setminus \{0\}, \mathbb{R})$, which is a Banach space. Therefore there is some $H \in L_{v_s \otimes w \otimes v_0}^{q,p,p}(\mathbb{R}, \mathbb{R} \setminus \{0\}, \mathbb{R})$ so that $\|H - \mathcal{R}(\mathcal{L}_n) \circ Q\|_{L_{v_s \otimes w \otimes v_0}^{q,p,p}(\mathbb{R}, \mathbb{R} \setminus \{0\}, \mathbb{R})} \rightarrow 0$ as $n \rightarrow \infty$. Hence $\mathcal{R}(\mathcal{L}_n) \circ Q$ converges to H in measure, implying that there is some subsequence $\{\mathcal{R}(\mathcal{L}_{n_j}) \circ Q\}_j$ of $\{\mathcal{R}(\mathcal{L}_n) \circ Q\}_n$ so that $\lim_{j \rightarrow \infty} \mathcal{R}(\mathcal{L}_{n_j})(Q(y, u, v)) = H(y, u, v)$ for almost every $(y, u, v) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}$. Thus $\mathcal{R}(\mathcal{L})(Q(y, u, v)) = H(y, u, v)$ for a.e. $(y, u, v) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}$ and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|\mathcal{L} - \mathcal{L}_n\|_{T_{q,p}^s} \\ &= \lim_{n \rightarrow \infty} \|\mathcal{L} - \mathcal{L}_n\|_{L^1(\mathbb{R}^2)} + \lim_{n \rightarrow \infty} \|\mathcal{R}(\mathcal{L}) \circ Q - \mathcal{R}(\mathcal{L}_n) \circ Q\|_{L_{v_s, w, v_0}^{q,p,p}(\mathbb{R}, \mathbb{R} \setminus \{0\}, \mathbb{R})} \\ &= \lim_{n \rightarrow \infty} \|\mathcal{L} - \mathcal{L}_n\|_{L^1(\mathbb{R}^2)} + \lim_{n \rightarrow \infty} \|H - \mathcal{R}(\mathcal{L}_n) \circ Q\|_{L_{v_s, w, v_0}^{q,p,p}(\mathbb{R}, \mathbb{R} \setminus \{0\}, \mathbb{R})} \\ &= 0. \end{aligned} \quad \square$$

Now we show that certain $T_{q,p}^s$ spaces are well-behaved with respect to affine convolution. We need the following lemma.

Lemma 3.4.9. *If $\mathcal{L}_1, \mathcal{L}_2 \in L^1(\mathbb{R}^2)$, then $\mathcal{L}_1 \circledast \mathcal{L}_2 \in L^1(\mathbb{R}^2)$ with*

$$\|\mathcal{L}_1 \circledast \mathcal{L}_2\|_{L^1(\mathbb{R}^2)} \leq \|\mathcal{L}_1\|_{L^1(\mathbb{R}^2)} \|\mathcal{L}_2\|_{L^1(\mathbb{R}^2)}.$$

Proof.

$$\begin{aligned} \|\mathcal{L}_1 \circledast \mathcal{L}_2\|_{L^1(\mathbb{R}^2)} &= \int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} \left| \int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} \mathcal{L}_1(u, v) \mathcal{L}_2\left(\frac{a}{u}, \frac{b-v}{u}\right) \frac{du}{u^2} dv \right| da db \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} |\mathcal{L}_1(u, v)| \left(\int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} \left| \mathcal{L}_2\left(\frac{a}{u}, \frac{b-v}{u}\right) \right| da db \right) \frac{du}{u^2} dv \\ &= \int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} |\mathcal{L}_1(u, v)| \left(\int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} |\mathcal{L}_2(c, d)| u^2 dc dd \right) \frac{du}{u^2} dv \\ &= \|\mathcal{L}_1\|_{L^1(\mathbb{R}^2)} \|\mathcal{L}_2\|_{L^1(\mathbb{R}^2)}. \end{aligned} \quad \square$$

If μ is the left Haar measure of the affine group with multiplication $(u, v)(x, y) = (ux, v + uy)$ then we have $\|\mathcal{L}_1 \circledast \mathcal{L}_2\|_{L^1(\mathbb{R}^2, \mu)} \leq \|\mathcal{L}_1\|_{L^1(\mathbb{R}^2, \mu)} \|\mathcal{L}_2\|_{L^1(\mathbb{R}^2, \mu)}$, by the theory of convolution on locally compact groups (see [30]). Lemma 3.4.9 is different from this result because Lebesgue measure is used rather than the Haar measure.

Proposition 3.4.10. *For each $p \in [1, 2]$ we have*

$$(a) \quad \|\mathcal{L}_1 \circledast \mathcal{L}_2\|_{T_{2,p}^1} \leq \|\mathcal{L}_1\|_{T_{2,p}^1} \|\mathcal{L}_2\|_{T_{2,2}^1} \text{ and}$$

$$(b) \quad \|\mathcal{L}_1 \circledast \mathcal{L}_2\|_{T_{2,p}^1} \leq \|\mathcal{L}_1\|_{T_{2,p}^1} \|\mathcal{L}_2\|_{T_{2,p}^1}.$$

Proof. Let $K(\mathcal{L}) = R\mathcal{L} \circ O$ for all $\mathcal{L} \in L^1(\mathbb{R}^2)$. Then the mapping K is linear with $\|\mathcal{L}\|_{T_{2,p}^1} = \|\mathcal{L}\|_{L^1} + \|K(\mathcal{L})\|_{S_{2,p}}$. Furthermore, $K(\mathcal{L}_1 \circledast \mathcal{L}_2) = K(\mathcal{L}_1) \# K(\mathcal{L}_2)$ as the following computation shows.

$$\begin{aligned} K(\mathcal{L}_1 \circledast \mathcal{L}_2)(t, y) &= R(\mathcal{L}_1 \circledast \mathcal{L}_2)(O(t, y)) \\ &= \int (\mathcal{L}_1 \circledast \mathcal{L}_2)(z_1, t - z_1 y) dz_1 \\ &= \iiint \mathcal{L}_1(u, v) \mathcal{L}_2\left(\frac{z_1}{u}, \frac{t - z_1 y - v}{u}\right) \frac{du}{u^2} dv dz_1 \\ &= \iiint \mathcal{L}_1(u, v) \mathcal{L}_2\left(z_2, \frac{t - v}{u} - z_2 y\right) \frac{du}{u} dv dz_2 \\ &= \iiint \mathcal{L}_1(u, t - ux) \mathcal{L}_2(z_2, x - z_2 y) du dx dz_2 \\ &= \int R\mathcal{L}_1(O(t, x)) R\mathcal{L}_2(O(x, y)) dx \\ &= K(\mathcal{L}_1) \# K(\mathcal{L}_2)(t, y). \end{aligned}$$

Using Proposition 3.4.3(a) and Lemma 3.4.9 we have

$$\begin{aligned} \|\mathcal{L}_1 \circledast \mathcal{L}_2\|_{T_{2,p}^1} &= \|\mathcal{L}_1 \circledast \mathcal{L}_2\|_{L^1} + \|K(\mathcal{L}_1 \circledast \mathcal{L}_2)\|_{S_{2,p}} \\ &\leq \|\mathcal{L}_1\|_{L^1} \|\mathcal{L}_2\|_{L^1} + \|K(\mathcal{L}_1) \# K(\mathcal{L}_2)\|_{S_{2,p}} \\ &\leq \|\mathcal{L}_1\|_{L^1} \|\mathcal{L}_2\|_{L^1} + \|K(\mathcal{L}_1)\|_{S_{2,p}} \|K(\mathcal{L}_2)\|_{S_{2,2}} \\ &\leq \left(\|\mathcal{L}_1\|_{L^1} + \|K(\mathcal{L}_1)\|_{S_{2,p}} \right) \left(\|\mathcal{L}_2\|_{L^1} + \|K(\mathcal{L}_2)\|_{S_{2,2}} \right) \end{aligned}$$

$$= \|\mathcal{L}_1\|_{T_{2,p}^1} \|\mathcal{L}_2\|_{T_{2,2}^1}.$$

Statement (b) is proved similarly. □

Corollary 3.4.11. *For each $p \in [1, 2]$, $T_{2,p}^1$ is a Banach algebra and a left ideal in $T_{2,2}^1$ under affine convolution.*

3.5 Schatten Class Affine Pseudodifferential Operators

In this section, we draw connections between the spaces developed in the previous section and the Schatten class results of Section 3.3. In particular, we obtain the following theorem.

Theorem 3.5.1. *Suppose A is an affine pseudodifferential operator with kernel k and symbol \mathcal{L} , and suppose $p \in [1, 2]$. Then there is a $C \in (0, \infty)$ such that the following statements hold.*

(a) *If $k \in S_{2,p}$, then $A \in \mathcal{I}_p(L^2(\mathbb{R}))$ and $\|A\|_{\mathcal{I}_p(L^2(\mathbb{R}))} \leq C \|k\|_{S_{2,p}}$.*

(b) *If $\mathcal{L} \in T_{2,p}^1$, then $A \in \mathcal{I}_p(L^2(\mathbb{R}))$ and $\|A\|_{\mathcal{I}_p(L^2(\mathbb{R}))} \leq C \|\mathcal{L}\|_{T_{2,p}^1}$.*

Proof. Statement (a) follows immediately from Theorem 3.3.2(a) and Proposition 2.4.1 in [25], which states that for any admissible $\psi \in L^2(\mathbb{R})$ we have

$$\langle f, g \rangle = C_\psi^{-1} \iint_{\mathbb{R}^2} \langle f, T_v D_u \psi \rangle \langle T_v D_u \psi, g \rangle \frac{du}{u^2} dv \quad \forall f, g \in L^2(\mathbb{R}).$$

By Lemma 3.2.7, we have $\|k\|_{S_{2,p}} \leq \|\mathcal{L}\|_{T_{2,p}^1}$. Thus statement (a) implies statement (b). □

In light of the previous theorem, it is desirable to know which functions belong to $S_{2,p}$ and $T_{2,q}^1$. In the remainder of this section, we describe smoothness and decay conditions which guarantee inclusion in these spaces. The following lemma, adapted from the techniques in [46], will be useful.

Lemma 3.5.2. *Suppose $f, g : \mathbb{R} \rightarrow \mathbb{C}$ satisfy*

$$|f(t)| \leq C_f(1+t^2)^{-\frac{\gamma}{2}} \quad \text{and} \quad |g(t)| \leq C_g(1+t^2)^{-\frac{\gamma}{2}} \quad \text{for a.e. } t \in \mathbb{R},$$

for some $\gamma > 1$. Then there is a constant C_γ , independent of f, g , so that

$$|f * D_u g(v)| \leq \begin{cases} C_f C_g C_\gamma |u|^{\gamma-\frac{1}{2}} (1+v^2)^{-\frac{\gamma}{2}} & \text{if } |u| \geq 1, \\ C_f C_g C_\gamma |u|^{-\frac{1}{2}} (1+v^2)^{-\frac{\gamma}{2}} & \text{if } 0 < |u| < 1. \end{cases}$$

Proof. Let $w_{-\gamma}(t) = (1+t^2)^{-\frac{\gamma}{2}}$. By Lemma 11.0.1 in [46], since $\gamma > 1$ there is some C_γ so that $(w_{-\gamma} * w_{-\gamma})(t) \leq C_\gamma w_{-\gamma}(t)$ for all $t \in \mathbb{R}$.

Notice that for $|u| \geq 1$, we have

$$\left(1 + \frac{t^2}{u^2}\right) = |u|^{-2} (u^2 + t^2) \geq |u|^{-2} (1 + t^2).$$

Thus for $|u| \geq 1$ we have

$$\left(1 + \frac{t^2}{u^2}\right)^{-\frac{\gamma}{2}} \leq |u|^\gamma (1+t^2)^{-\frac{\gamma}{2}}.$$

If $0 < |u| < 1$, then $\frac{t^2}{u^2} > t^2$, which implies

$$\left(1 + \frac{t^2}{u^2}\right) > (1+t^2).$$

Thus for $0 < |u| < 1$, we have

$$\left(1 + \frac{t^2}{u^2}\right)^{-\frac{\gamma}{2}} < (1+t^2)^{-\frac{\gamma}{2}}.$$

Therefore

$$\begin{aligned} D_u w_{-\gamma}(t) &= |u|^{-\frac{1}{2}} \left(1 + \frac{t^2}{u^2}\right)^{-\frac{\gamma}{2}} \\ &\leq \begin{cases} |u|^{\gamma-\frac{1}{2}} (1+t^2)^{-\frac{\gamma}{2}} & \text{if } |u| \geq 1 \\ |u|^{-\frac{1}{2}} (1+t^2)^{-\frac{\gamma}{2}} & \text{if } 0 < |u| < 1 \end{cases} \end{aligned}$$

Hence

$$|(f * D_u g)(v)| \leq C_f C_g (w_{-\gamma} * D_u w_{-\gamma})(v)$$

$$\begin{aligned} &\leq \begin{cases} C_f C_g |u|^{\gamma-\frac{1}{2}} (w_{-\gamma} * w_{-\gamma})(v) & \text{if } |u| \geq 1 \\ C_f C_g |u|^{-\frac{1}{2}} (w_{-\gamma} * w_{-\gamma})(v) & \text{if } 0 < |u| < 1 \end{cases} \\ &\leq \begin{cases} C_f C_g C_\gamma |u|^{\gamma-\frac{1}{2}} (1+v^2)^{-\frac{\gamma}{2}} & \text{if } |u| \geq 1 \\ C_f C_g C_\gamma |u|^{-\frac{1}{2}} (1+v^2)^{-\frac{\gamma}{2}} & \text{if } 0 < |u| < 1. \quad \square \end{cases} \end{aligned}$$

Recall that the definitions of $S_{q,p}$ and $T_{q,p}^s$ depend on the choice of admissible function $\psi \in \mathcal{S}(\mathbb{R})$. If ψ possesses additional “nice” wavelet characteristics, made precise in the following definition, we can better analyze $S_{2,p}$ and $T_{2,p}^1$.

Definition 3.5.3. *Let*

$$B_1 = \{\psi \in \mathcal{S}(\mathbb{R}) : \psi \text{ is admissible and } \psi = \Psi' \text{ for some } \Psi \in \mathcal{S}(\mathbb{R})\}.$$

The Mexican hat wavelet $\psi(t) = (1-t^2)e^{-\frac{t^2}{2}}$ is in B_1 .

Theorem 3.5.4. *Suppose $\psi \in B_1$, $\alpha > 3$ and $\beta > \frac{1}{4}$. If*

$$|k(t, y)| \leq C (1+y^2)^{-\beta} (1+t^2)^{-\frac{\alpha}{2}} \text{ for a.e. } t, y \in \mathbb{R},$$

and there exists a $C \in (0, \infty)$ such that almost every k_y has a derivative satisfying

$$|k'_y(t)| \leq C (1+y^2)^{-\beta} (1+t^2)^{-\frac{\alpha}{2}} \text{ for all } t \in \mathbb{R},$$

then $k \in S_{2,p}$ for all $p \in [1, 2]$. In particular $\mathcal{S}(\mathbb{R}^2) \subset S_{2,p}$ for $p \in [1, 2]$.

Proof. Since $\psi \in B_1$, we have $\psi = \Psi'$ for some $\Psi \in \mathcal{S}(\mathbb{R})$. Without loss of generality, we assume

$$|\psi(t)| \leq C (1+t^2)^{-\frac{\alpha}{2}} \quad \text{and} \quad |\Psi(t)| \leq C (1+t^2)^{-\frac{\alpha}{2}} \quad \text{for all } t \in \mathbb{R}.$$

For $p \in [1, 2)$ write

$$\begin{aligned} \|k\|_{S_{2,p}}^p &= \int_{|v|<1} \int_{|u|\geq 1} \left(\int_{\mathbb{R}} |\langle k_y, T_v D_u \psi \rangle|^2 dy \right)^{\frac{p}{2}} \frac{du}{u^2} dv \\ &\quad + \int_{|v|<1} \int_{|u|<1} \left(\int_{\mathbb{R}} |\langle k_y, T_v D_u \psi \rangle|^2 dy \right)^{\frac{p}{2}} \frac{du}{u^2} dv \end{aligned}$$

$$\begin{aligned}
& + \int_{|v| \geq 1} \int_{|u| \geq 1} \left(\int_{\mathbb{R}} |\langle k_y, T_v D_u \psi \rangle|^2 dy \right)^{\frac{p}{2}} \frac{du}{u^2} dv \\
& + \int_{|v| \geq 1} \int_{|u| < 1} \left(\int_{\mathbb{R}} |\langle k_y, T_v D_u \psi \rangle|^2 dy \right)^{\frac{p}{2}} \frac{du}{u^2} dv \\
& = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

To estimate I_1 we note that

$$\begin{aligned}
|\langle k_y, T_v D_u \psi \rangle| & = |k_y * D_{-u} \bar{\psi}(v)| \\
& \leq \|k_y\|_{L^2(\mathbb{R})} \|D_u \psi\|_{L^2(\mathbb{R})} \\
& = \|\psi\|_{L^2(\mathbb{R})} \left(\int |k(t, y)|^2 dt \right)^{\frac{1}{2}} \\
& \leq \|\psi\|_{L^2(\mathbb{R})} \left(\int \frac{C^2}{(1+y^2)^{2\beta} (1+t^2)^\alpha} dt \right)^{\frac{1}{2}} \\
& \leq \|\psi\|_{L^2(\mathbb{R})} C (1+y^2)^{-\beta} \left(\int_{\mathbb{R}} \frac{1}{(1+t^2)^\alpha} dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$I_1 \leq C^p \|\psi\|_{L^2(\mathbb{R})}^p \left(\int_{\mathbb{R}} \frac{1}{(1+t^2)^\alpha} dt \right)^{\frac{p}{2}} \left(\int_{|v| < 1} \int_{|u| \geq 1} \left(\int_{\mathbb{R}} \frac{1}{(1+y^2)^{2\beta}} dy \right)^{\frac{p}{2}} \frac{du}{u^2} dv \right),$$

and this quantity is finite since $\beta > \frac{1}{4}$ and $\alpha > 1$.

To estimate I_2 , we use Theorem A.1 in [43]. By the proof of the this theorem, there is some C_2 satisfying

$$|\langle k_y, T_v D_u \psi \rangle| \leq C_2 |u|^{\frac{3}{2}} \|k'_y\|_{\infty} \quad \forall y \in \mathbb{R}, u \in \mathbb{R} \setminus \{0\}, v \in \mathbb{R}. \quad (14)$$

Hence

$$\begin{aligned}
I_2 & \leq C_2^p \left(\int_{|v| < 1} \int_{|u| < 1} |u|^{\frac{3p}{2}-2} du dv \right) \left(\int_{\mathbb{R}} \|k'_y\|_{\infty}^2 dy \right)^{\frac{p}{2}} \\
& \leq C^p C_2^p \left(\int_{|v| < 1} \int_{|u| < 1} |u|^{\frac{3p}{2}-2} du dv \right) \left(\int_{\mathbb{R}} \frac{1}{(1+y^2)^{2\beta}} dy \right)^{\frac{p}{2}} \\
& < \infty,
\end{aligned}$$

as $\beta > \frac{1}{4}$ and $\frac{3p}{2} - 2 > -1$.

As in the proof of Lemma 3.5.2, let $w_{-\gamma}(t) = (1 + t^2)^{-\frac{\gamma}{2}}$ for all $\gamma \in \mathbb{R}, t \in \mathbb{R}$. Notice that for any $\gamma \leq \alpha$ we have $w_{-\alpha}(t) \leq w_{-\gamma}(t)$ for all $t \in \mathbb{R}$. Hence for all $1 < \gamma \leq \alpha$ we have $|k_y(t)| \leq C(1 + y^2)^{-\beta} w_{-\gamma}(t)$ and $|\psi(t)| \leq Cw_{-\gamma}(t)$. If $1 < \gamma \leq \alpha$ then by Lemma 3.5.2 there is some C_γ satisfying the following inequality for all $|u| \geq 1$:

$$|\langle k_y, T_v D_u \psi \rangle| = |k_y * D_u \psi(v)| \leq C_\gamma C^2 (1 + y^2)^{-\beta} |u|^{\gamma - \frac{1}{2}} (1 + v^2)^{-\frac{\gamma}{2}}. \quad (15)$$

Choose $\gamma_o \in (1, \frac{1}{p} + \frac{1}{2})$. By (15), we have

$$\begin{aligned} I_3 &\leq C_{\gamma_o}^p C^{2p} \left(\int_{|v| \geq 1} \frac{1}{(1 + v^2)^{\frac{2\alpha p}{2}}} dv \right) \left(\int_{|u| \geq 1} |u|^{p(\gamma_o - \frac{1}{2}) - 2} du \right) \left(\int_{\mathbb{R}} \frac{1}{(1 + y^2)^{2\beta}} dy \right)^{\frac{p}{2}} \\ &< \infty, \end{aligned}$$

since $\beta > \frac{1}{4}$ and $1 < \gamma_o < \frac{1}{p} + \frac{1}{2}$.

To estimate I_4 , we use integration by parts to obtain

$$|\langle k_y, T_v D_u \psi \rangle| = |u| |\langle k'_y, T_v D_u \Psi \rangle|.$$

By Lemma 3.5.2, there is some C_α satisfying the following inequality for all $0 < |u| < 1$:

$$|\langle k_y, T_v D_u \psi \rangle| = |u| |\langle k'_y, T_v D_u \Psi \rangle| \leq C_\alpha C^2 |u|^{\frac{1}{2}} (1 + y^2)^{-\beta} (1 + v^2)^{-\frac{\alpha}{2}} \quad (16)$$

Using (14) and (16) we obtain the following estimates for $0 < |u| < 1$:

$$\begin{aligned} |\langle k_y, T_v D_u \psi \rangle| &= |\langle k_y, T_v D_u \psi \rangle|^{\frac{2}{3}} |\langle k_y, T_v D_u \psi \rangle|^{\frac{1}{3}} \\ &\leq C_2^{\frac{1}{2}} |u| \frac{C^{\frac{2}{3}}}{(1 + y^2)^{\frac{2\beta}{3}}} |\langle k_y, T_v D_u \psi \rangle|^{\frac{1}{3}} \\ &\leq C_2^{\frac{1}{2}} |u| \frac{C^{\frac{2}{3}}}{(1 + y^2)^{\frac{2\beta}{3}}} \frac{C_\alpha^{\frac{1}{3}} C^{\frac{4}{3}}}{(1 + y^2)^{\frac{\beta}{3}}} |u|^{\frac{1}{6}} \frac{1}{(1 + v^2)^{\frac{\alpha}{6}}}. \end{aligned}$$

Hence

$$I_4 \leq C^{2p} C_2^{\frac{p}{2}} C_\alpha^{\frac{p}{3}} \left(\int_{|v| \geq 1} \frac{1}{(1 + v^2)^{\frac{\alpha p}{6}}} dv \right) \left(\int_{|u| < 1} |u|^{p(\frac{7}{6}) - 2} du \right) \left(\int_{\mathbb{R}} \frac{1}{(1 + y^2)^{2\beta}} dy \right)^{\frac{p}{2}},$$

and this quantity is finite since $\alpha > 3$ and $\beta > \frac{1}{4}$.

Since $S_{2,2} = L^2(\mathbb{R}^2)$ direct computation gives the result when $p = 2$. \square

Recall that for $\theta \in S^1$, $\phi(\theta)$ denotes the unique number in $[0, 2\pi)$ such that $\theta = (\cos \phi(\theta), \sin \phi(\theta))$. In particular, $\cos \phi(\theta) = \theta_1$ and $\sin \phi(\theta) = \theta_2$ when $\theta = (\theta_1, \theta_2) \in S^1$.

Theorem 3.5.5. *Suppose $\psi \in B_1$ and $\mathcal{L} \in L^1(\mathbb{R}^2)$ satisfies*

$$|R\mathcal{L}_\theta(s)| = |R\mathcal{L}(\theta, s)| \leq \frac{C |\sin \phi(\theta)|^{2\beta+\alpha}}{(1+s^2)^{\frac{\alpha}{2}}} \quad \text{for a.e. } (\theta, s) \in S^1 \times \mathbb{R}, \quad (17)$$

and

$$|R\mathcal{L}'_\theta(s)| \leq \frac{C |\sin \phi(\theta)|^{2\beta+\alpha-1}}{(1+s^2)^{\frac{\alpha}{2}}} \quad \text{for a.e. } (\theta, s) \in S^1 \times \mathbb{R} \quad (18)$$

for some $\beta > \frac{1}{4}$, $\alpha > 3$. Then $\mathcal{L} \in T_{2,p}^1$ for $p \in [1, 2]$. In particular, $T_{2,p}^1$ is nontrivial for $p \in [1, 2]$.

Proof. Let $k = R\mathcal{L} \circ O$. Then by (17) we have

$$\begin{aligned} |k(t, y)| &= \left| R\mathcal{L} \left(\left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}} \right), \frac{t}{\sqrt{1+y^2}} \right) \right| \\ &\leq \frac{C}{\left(\sqrt{y^2+1} \right)^{2\beta+\alpha} \left(1 + \frac{t^2}{1+y^2} \right)^{\frac{\alpha}{2}}} \\ &= \frac{C}{(1+y^2)^\beta} \frac{1}{(1+y^2+t^2)^{\frac{\alpha}{2}}} \\ &\leq \frac{C}{(1+y^2)^\beta} \frac{1}{(1+t^2)^{\frac{\alpha}{2}}} \end{aligned}$$

and by (18) we have

$$\begin{aligned} |k'_y(t)| &= \frac{1}{\sqrt{1+y^2}} \left| R\mathcal{L}' \left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}} \right) \left(\frac{t}{\sqrt{1+y^2}} \right) \right| \\ &\leq \frac{C}{\sqrt{1+y^2}} \frac{1}{\left(\sqrt{y^2+1} \right)^{2\beta+\alpha-1} \left(1 + \frac{t^2}{1+y^2} \right)^{\frac{\alpha}{2}}} \\ &= \frac{C}{(1+y^2)^\beta} \frac{1}{(1+y^2+t^2)^{\frac{\alpha}{2}}} \end{aligned}$$

$$\leq \frac{C}{(1+y^2)^\beta} \frac{1}{(1+t^2)^{\frac{\alpha}{2}}}.$$

By Theorem 3.5.4, $k \in S_{2,p}$. Hence $\mathcal{L} \in T_{2,p}^1$.

To complete the proof of the theorem, it suffices to show that there is some $\mathcal{L} \in L^1(\mathbb{R}^2) \setminus \{0\}$ satisfying (17) and (18). Fix $m, n \in \mathbb{N}$ with $n \geq \beta$ and $m \geq \alpha$. Choose $f \in \mathcal{S}(\mathbb{R})$ with f even and $\int_{\mathbb{R}} f(s) ds = 0$. Set $F(\theta, s) = \sin^{2n+m}(2\phi(\theta))f(s)$. Then $F \in \mathcal{S}(S^1 \times \mathbb{R})$, F is even and F satisfies

$$|F_\theta(s)| = |F(\theta, s)| \leq \frac{C |\sin \phi(\theta)|^{2\beta+\alpha}}{(1+s^2)^{\frac{\alpha}{2}}}$$

$$|F'_\theta(s)| \leq \frac{C |\sin \phi(\theta)|^{2\beta+\alpha-1}}{(1+s^2)^{\frac{\alpha}{2}}}$$

for some $C > 0$. Furthermore $\int_{\mathbb{R}} F(\theta, s) ds = 0$ for all $\theta \in S^1$. By Theorem 7.7 in [65] there is $\mathcal{L} \in C^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ such that $R\mathcal{L} = F$. \square

Notice that the integral of a function \mathcal{L} satisfying (17) over almost any horizontal line must be zero.

3.6 Affine Pseudodifferential Operators as Calderon-Zygmund Operators

The conditions on the Radon transform of \mathcal{L} in Theorem 3.5.5 are almost enough to imply that the affine pseudodifferential operator with symbol \mathcal{L} is Calderon-Zygmund. In this section we find sufficient conditions for an affine pseudodifferential operator to be a Calderon-Zygmund operator.

Throughout the paper we have defined $\phi(\theta) \in [0, 2\pi)$ by $\theta = (\cos \phi(\theta), \sin \phi(\theta))$. Similarly, we can define $\theta : \mathbb{R} \rightarrow S^1$ by $\theta(\phi) = (\cos \phi, \sin \phi)$.

Theorem 3.6.1. *Suppose A is an affine pseudodifferential operator with kernel k and symbol \mathcal{L} .*

(a) *If $A : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is bounded and k satisfies*

$$|k(t, y)| \leq \frac{C}{|t - y|^\delta},$$

$$|k(t, y) - k(t', y)| \leq \frac{C |t - t'|^\delta}{|t - y|^{\delta+1}} \quad \text{if } |t - t'| < \frac{1}{2} |t - y|,$$

and

$$|k(t, y) - k(t, y')| \leq \frac{C |y - y'|^\delta}{|t - y|^{\delta+1}} \quad \text{if } |y - y'| < \frac{1}{2} |t - y|$$

for some $\delta > 0$, then A is a Calderon-Zygmund operator.

(b) If \mathcal{L} satisfies

$$\begin{aligned} |R\mathcal{L}(\theta, s)| &\leq \frac{C |\sin \phi(\theta)|^\beta}{(1 + |s|)}, \\ |R_\theta \mathcal{L}'(s)| &\leq \frac{C |\sin \phi(\theta)|^2}{(1 + |s|)^3}, \end{aligned}$$

and

$$\left| \left(\frac{d}{d\phi} R\mathcal{L} \right) (\theta(\phi), s) \right| \leq \frac{C}{(1 + |s|)^2},$$

for some $\beta > \frac{3}{2}$, then A is a Calderon-Zygmund operator.

Furthermore, if either (a) or (b) holds, then $A : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ is bounded for all $1 < p < \infty$.

Proof. Statement (a) is a direct consequence of Theorem 5.10 in [26].

Suppose (b) holds. Then

$$\begin{aligned} |k(t, y)| &= \left| R_{\left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}}\right)} \mathcal{L} \left(\frac{t}{\sqrt{y^2+1}} \right) \right| \\ &\leq \frac{C}{\sqrt{y^2+1}^\beta} \frac{1}{\left(1 + \frac{|t|}{\sqrt{y^2+1}}\right)} \\ &= \frac{C}{(y^2+1)^{\frac{\beta-1}{2}}} \frac{1}{\sqrt{y^2+1} + |t|} \\ &\leq \frac{C}{(y^2+1)^{\frac{\beta-1}{2}}} \frac{1}{(1 + |t|)}, \end{aligned}$$

which implies $k \in L^2(\mathbb{R}^2)$. Hence $A : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is bounded.

Also

$$|k(t, y)| = \left| R_{\left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}}\right)} \mathcal{L} \left(\frac{t}{\sqrt{y^2+1}} \right) \right|$$

$$\begin{aligned}
&\leq \frac{C}{\sqrt{y^2+1}} \frac{1}{\left(1 + \frac{|t|}{\sqrt{y^2+1}}\right)} \\
&= \frac{C}{\sqrt{y^2+1} + |t|} \\
&\leq \frac{C}{|y-t|}.
\end{aligned}$$

Fix $t, y \in \mathbb{R}$ and assume $|t-t'| < \frac{1}{2}|t-y|$. By the Intermediate Value Theorem, we have

$$|k(t, y) - k(t', y)| = \left| \left(\frac{\partial}{\partial t} k \right) (t_0, y) \right| |t-t'|$$

for some t_0 between t and t' . But

$$\begin{aligned}
\left| \left(\frac{\partial}{\partial t} k \right) (t_0, y) \right| &= (y^2+1)^{-\frac{1}{2}} \left| R_{\left(\frac{y}{\sqrt{y^2+1}}, \frac{1}{\sqrt{y^2+1}}\right)} \mathcal{L}' \left(\frac{t_0}{\sqrt{y^2+1}} \right) \right| \\
&\leq \frac{C}{\left(\sqrt{y^2+1} + |t_0|\right)^2}.
\end{aligned}$$

Because t_0 is between t and t' and $|t-t'| < \frac{1}{2}|t-y|$, we must have $|t_0-y| \geq \frac{1}{2}|t-y|$.

Hence

$$\begin{aligned}
|k(t, y) - k(t', y)| &= \left| \left(\frac{\partial}{\partial t} k \right) (t_0, y) \right| |t-t'| \\
&\leq \frac{C|t-t'|}{\left(\sqrt{y^2+1} + |t_0|\right)^2} \\
&\leq \frac{C|t-t'|}{|y-t_0|^2} \\
&\leq \frac{4C|t-t'|}{|y-t|^2},
\end{aligned} \tag{19}$$

where (19) holds because $|y-t_0| \leq |y| + |t_0| < \sqrt{y^2+1} + |t_0|$.

Now assume $|y-y'| < \frac{1}{2}|t-y|$ and consider $|k(t, y) - k(t, y')|$. By the Intermediate Value Theorem, there is some y_0 between y and y' so that $|k(t, y) - k(t, y')| = \left| \left(\frac{\partial}{\partial y} k \right) (t, y_0) \right| |y-y'|$. By Lemma 3.2.6,

$$k(t, y) = R\mathcal{L}(\theta(\phi), s),$$

where

$$\cos \phi = \frac{y}{\sqrt{y^2 + 1}}, \quad \sin \phi = \frac{1}{\sqrt{y^2 + 1}}, \quad \text{and} \quad s = \frac{t}{\sqrt{y^2 + 1}}.$$

By the chain rule we have

$$\begin{aligned} \left(\frac{\partial}{\partial y} k \right) (t, y) &= \left(\frac{\partial}{\partial \phi} R\mathcal{L} \right) (\theta(\phi), s) \cdot \frac{\partial \phi}{\partial y} + \left(\frac{\partial}{\partial s} R\mathcal{L} \right) (\theta(\phi), s) \cdot \frac{\partial s}{\partial y} \\ &= \left(\frac{\partial}{\partial \phi} R\mathcal{L} \right) (\theta(\phi), s) \cdot \frac{-y}{|y|} \cdot \frac{1}{y^2 + 1} + \left(\frac{\partial}{\partial s} R\mathcal{L} \right) (\theta(\phi), s) \cdot \frac{-ty}{\sqrt{y^2 + 1}}. \end{aligned}$$

Thus

$$\begin{aligned} \left| \left(\frac{\partial}{\partial y} k \right) (t, y_0) \right| &= \left| \left(\frac{\partial}{\partial \phi} R\mathcal{L} \right) \left(\left(\frac{y_0}{\sqrt{y_0^2 + 1}}, \frac{1}{\sqrt{y_0^2 + 1}} \right), \frac{t}{\sqrt{y_0^2 + 1}} \right) \cdot \frac{-y_0}{|y_0|} \cdot \frac{-1}{y_0^2 + 1} \right. \\ &\quad \left. + \left(R \left(\frac{y_0}{\sqrt{y_0^2 + 1}}, \frac{1}{\sqrt{y_0^2 + 1}} \right) \mathcal{L} \right)' \left(\frac{t}{\sqrt{y_0^2 + 1}} \right) \cdot \frac{-ty_0}{(y_0^2 + 1)^{\frac{3}{2}}} \right| \\ &\leq \frac{C}{y_0^2 + 1} \cdot \frac{1}{\left(1 + \frac{|t|}{\sqrt{y_0^2 + 1}} \right)^2} + \frac{C|ty_0|}{(y_0^2 + 1)^{\frac{3}{2}}} \cdot \frac{1}{(y_0^2 + 1)^{\frac{1}{2}}} \cdot \frac{1}{\left(1 + \frac{|t|}{\sqrt{y_0^2 + 1}} \right)^3} \\ &= \frac{C}{\left(\sqrt{y_0^2 + 1} + |t| \right)^2} + \frac{C|t|}{\sqrt{y_0^2 + 1} + |t|} \cdot \frac{|y_0|}{\sqrt{y_0^2 + 1}} \cdot \frac{1}{\left(\sqrt{y_0^2 + 1} + |t| \right)^2} \\ &\leq \frac{2C}{\left(\sqrt{y_0^2 + 1} + |t| \right)^2} \end{aligned}$$

Because $|y - y'| < \frac{1}{2}|t - y|$ and y_0 between is y and y' , we have $|y_0 - t| \geq \frac{1}{2}|y - t|$.

Hence

$$\begin{aligned} |k(t, y) - k(t, y')| &= \left| \left(\frac{d}{dy} k \right) (t, y_0) \right| |y - y'| \\ &\leq \frac{2C |y - y'|}{\left(\sqrt{y_0^2 + 1} + |t| \right)^2} \\ &\leq \frac{8C |y - y'|}{|y - t|^2} \end{aligned}$$

Thus k satisfies (a) for $\delta = 1$. □

CHAPTER IV

MIXED MODULATION SPACES AND PSEUDODIFFERENTIAL OPERATORS

4.1 *Introduction*

Integral operators arise naturally in many areas of mathematics and science. Pseudodifferential operators, which are a particular type of integral operator, have appeared widely in the literature of physics, signal processing and differential equations. An overview of pseudodifferential operators is given in Chapter 14 of [33], while more detailed expositions are found in [30], [48], and [67]. Because of the role of pseudodifferential operators in partial differential equations, the smoothness of the Weyl and Kohn-Nirenberg symbols of a pseudodifferential operator has traditionally been used to characterize properties of the operator, with the Hörmander symbol classes playing key roles.

More recently, pseudodifferential operators have been studied from a time-frequency perspective. Every pseudodifferential operator is a superposition of time-frequency shifts, and the properties of pseudodifferential operators have been well-described by time-frequency analysis. Results with this flavor appear in [22], [72] and [76]. In particular the classical modulation spaces $M_w^{p,q}(\mathbb{R}^d)$, which are Banach spaces characterized by time-frequency shifts and mixed norms, have been useful symbol spaces for studying continuity and Schatten class properties of pseudodifferential operators. (See [66] for applications of mixed norms in other areas of harmonic analysis.) Using Gabor frames, elements in these spaces can be decomposed into a superposition of time-frequency shifts, and this Gabor frame decomposition of the symbol of a pseudodifferential operator can be used to characterize the properties of

the operator. In particular, the following two theorems, from [41] and [33], respectively, can be proven with Gabor decomposition techniques.

Theorem 4.1.1. *Suppose A is a pseudodifferential operator with Kohn-Nirenberg symbol τ , Weyl symbol σ and kernel k . If one of τ, σ, k lies in $M_{v_s}^{2,2}(\mathbb{R}^{2d})$ with $s > \frac{d(2-p)}{p}$ and $s \geq 0$, then $A \in \mathcal{I}_p(L^2(\mathbb{R}^d))$.*

Theorem 4.1.2. *Suppose A is a pseudodifferential operator with Kohn-Nirenberg symbol τ and Weyl symbol σ . If one of σ, τ belongs to $M^{\infty,1}(\mathbb{R}^{2d})$, then*

$$A : M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)$$

is bounded for all $p, q \in [1, \infty]$.

Both of these theorems generalize results in [35]. Other modulation space results for pseudodifferential operators appear in [21], [52], [71] and [73], while modulation spaces appear implicitly in [45], [64], [23], [47] and [59].

In this chapter we develop a technique for analyzing the kernel of an integral operator which generalizes existing time-frequency analysis techniques of pseudodifferential operators and yields new classes of non-smooth Kohn-Nirenberg symbols which ensure that a given pseudodifferential operator is Schatten p -class. To obtain the main result of this chapter, we use Corollary 3.3.7 to analyze the kernel of an integral operator with a frame. In particular, analyzing the kernel as in Corollary 3.3.7 with a Gabor frame gives a time-frequency condition on the kernel which ensures the operator is Schatten p -class. We show that this condition holds for kernels belonging to certain Banach spaces $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ that we call mixed modulation spaces, which are natural generalizations of the traditional modulation spaces $M_w^{p,q}(\mathbb{R}^d)$. In this chapter we show that many of the interesting properties of traditional modulation spaces also hold for mixed modulation spaces. Furthermore, inclusion of the Kohn-Nirenberg symbol in an appropriate mixed modulation space ensures the corresponding operator is Schatten p -class. The relationship between mixed modulation

spaces and the kernels and Kohn-Nirenberg symbols of Schatten p -class operators is summarized in the following theorem.

Theorem 4.1.3. *Let A be a pseudodifferential operator with kernel k and Kohn-Nirenberg symbol τ . Assume $p \in [1, 2]$ and set $2 = p_1 = \cdots = p_{2d}$, $p = p_{2d+1} = \cdots = p_{4d}$, $2 = q_1 = \cdots = q_d$ and $p = q_{d+1} = \cdots = q_{4d}$. For suitable c, c' , if $k \in M(c)^{p_1, p_2, \dots, p_{4d}}$ or $\tau \in M(c')^{q_1, q_2, \dots, q_{4d}}$, then A is Schatten p -class on $L^2(\mathbb{R}^d)$.*

The strongest known Schatten class result for pseudodifferential operators obtained by time-frequency analysis is Theorem 4.1.1. Although the crux of both Theorem 4.1.3 and Theorem 4.1.1 is time-frequency analysis with Gabor frames, our Theorem 4.1.3 is obtained by analyzing the *slices* of the kernel with a Gabor frame, thus permitting a finer control on the properties of the kernel. As a result, we can show that Theorem 4.1.3 is stronger than Theorem 4.1.1 for kernels, in the sense that the mixed modulation space described by Theorem 4.1.3 strictly contains the space $M_{v_s}^{2,2}(\mathbb{R}^{2d})$. In fact, we show that Theorem 4.1.3 is sharp for kernels in the sense that larger mixed modulation spaces contain kernels of pseudodifferential operators that are not Schatten p -class. We also show that Theorem 4.1.3 gives a new class of Kohn-Nirenberg symbols of Schatten class operators distinct from the Kohn-Nirenberg symbol class described by Theorem 4.1.1.

The remainder of the chapter is organized as follows. Section 4.2 contains definitions and basic lemmas. In Section 4.3, the definition of mixed modulation spaces $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ is given and the properties of these spaces are developed. In Section 4.4, we show how the mixed modulation spaces can be used to generalize boundedness results for pseudodifferential operators. In Section 4.5, we apply the results of Section 4.3 and Corollary 3.3.6 to pseudodifferential operators and compare our results with Theorem 4.1.1.

4.2 Definitions and preliminary lemmas

In order to characterize the time-frequency properties of kernels and symbols of pseudodifferential operators, we need more information about frames and bases of time-frequency shifts, as well as the relationships between the kernels and symbols.

4.2.1 Gabor Frames and Wilson Bases

Definition 4.2.1. A Gabor frame for $L^2(\mathbb{R}^d)$ is a sequence $\{M_\xi T_x \phi\}_{(x,\xi) \in \Lambda}$ that is a frame for $L^2(\mathbb{R}^d)$.

There are tight Gabor frame for $L^2(\mathbb{R}^d)$ whose generator ϕ is a nice function, e.g., $\phi \in C_c^\infty(\mathbb{R}^d)$. However, the different statements of the Balian-Low Theorem show that the elements of a Gabor frame which offers unique expansions (i.e. a Gabor Riesz basis) necessarily have poor time-frequency localization. See [33] for examples and properties of Gabor frames.

Wilson bases are orthonormal bases similar to Gabor Riesz bases in that they allow for unique, discrete expansions of the elements of $L^2(\mathbb{R}^d)$ in terms of time-frequency “molecules.” However, in contrast with Gabor Riesz bases, the elements of a Wilson basis may be well-localized in time and frequency.

For each $k \in \mathbb{Z}^d, n \in (\mathbb{Z}^+)^d$ let

$$\Psi_{k,n}(t) = \psi_{k_1,n_1}(t_1) \psi_{k_2,n_2}(t_2) \cdots \psi_{k_d,n_d}(t_d),$$

where

$$\psi_{k_i,n_i}(t_i) = \begin{cases} T_{k_i} \psi(t_i), & \text{if } n_i = 0, \\ \frac{1}{\sqrt{2}} T_{\frac{k_i}{2}} \left(M_{n_i} + (-1)^{k_i+n_i} M_{-n_i} \right) \psi(t_i), & \text{if } n_i > 0. \end{cases}$$

For suitable $\psi \in L^2(\mathbb{R})$, the sequence $\{\Psi_{k,n}\}_{k \in \mathbb{Z}^d, n \in (\mathbb{Z}^+)^d}$ constitutes an orthonormal basis for $L^2(\mathbb{R}^d)$. In this case we call $\{\Psi_{k,n}\}_{k \in \mathbb{Z}^d, n \in (\mathbb{Z}^+)^d}$ the *Wilson basis* generated by ψ (see [33] for details).

4.2.2 The relationship between kernel and symbols

Recall that the pseudodifferential operator with Kohn-Nirenberg symbol τ is

$$K_\tau f(t) = \iint_{\mathbb{R}^{2d}} \hat{\tau}(\xi, x) M_\xi T_{-x} f(t) dx d\xi$$

and the pseudodifferential operator with Weyl symbol σ is

$$L_\sigma f(t) = \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\xi, x) e^{-\pi i \xi \cdot x} T_{-x} M_\xi f(t) dx d\xi.$$

Every suitable pseudodifferential operator K_τ can be also realized as an operator L_σ and in this case we have $\hat{\tau}(\xi, x) = e^{\pi i x \cdot \xi} \hat{\sigma}(\xi, x)$. Similarly, suitable K_τ and L_σ can be realized as integral operators. In particular, if we let \mathcal{F}_2 denote the partial Fourier transform on the last d variables of a function of $2d$ variables, i.e.

$$(\mathcal{F}_2 F)(x, w) = \int_{\mathbb{R}^d} F(x, y) e^{2\pi i y \cdot w} dy \quad \text{for all } x, w \in \mathbb{R}^d,$$

then K_τ is an integral operator with kernel $k = \mathcal{F}_2^{-1} \tau \circ N$, where $N(x, y) = (x, x - y)$ for $x, y \in \mathbb{R}^d$, and L_σ is an integral operator with kernel $k = \mathcal{F}_2^{-1} \sigma \circ M$, where $M(x, y) = (\frac{x+y}{2}, x - y)$ for $x, y \in \mathbb{R}^d$.

Lemma 4.2.2. *Suppose $f \in \mathcal{S}(\mathbb{R}^d)$ and $\Phi \in \mathcal{S}(\mathbb{R}^d)$.*

- (a) $\langle f \circ N^{-1}, M_{(c,d)} T_{(a,b)} \Phi \rangle = \langle f, M_{(c+d,-d)} T_{(a,a-b)} (\Phi \circ N) \rangle$
- (b) $\langle f \circ N, M_{(c,d)} T_{(a,b)} \Phi \rangle = \langle f, M_{(c+d,-d)} T_{(a,a-b)} (\Phi \circ N^{-1}) \rangle$
- (c) $\langle f \circ M^{-1}, M_{(c,d)} T_{(a,b)} \Phi \rangle = \langle f, M_{(\frac{c}{2}+d, \frac{c}{2}-d)} T_{(a+\frac{b}{2}, a-\frac{b}{2})} (\Phi \circ M) \rangle$
- (d) $\langle f \circ M, M_{(c,d)} T_{(a,b)} \Phi \rangle = \langle f, M_{(c+d, \frac{c-d}{2})} T_{(\frac{a+b}{2}, a-b)} (\Phi \circ M^{-1}) \rangle$
- (e) $|\langle \mathcal{F}_2 f, M_{(c,d)} T_{(a,b)} \Phi \rangle| = |\langle f, M_{(c,b)} T_{(a,-d)} (\mathcal{F}_2^{-1} \Phi) \rangle|$
- (f) $|\langle \mathcal{F}_2^{-1} f, M_{(c,d)} T_{(a,b)} \Phi \rangle| = |\langle f, M_{(c,-b)} T_{(a,d)} (\mathcal{F}_2 \Phi) \rangle|$

Proof. It is easy to show that $N^{-1} = N$. Hence (a) and (b) are equivalent. To prove them we have

$$\begin{aligned}
& \langle f \circ N, M_{(c,d)}T_{(a,b)}\Phi \rangle \\
&= \iint f(s, s-t) e^{-2\pi i c \cdot s} e^{-2\pi i d \cdot t} \overline{\Phi(s-a, t-b)} \, ds \, dt \\
&= \iint f(s, u) e^{-2\pi i c \cdot s} e^{-2\pi i d \cdot (s-u)} \overline{\Phi(s-a, s-u-b)} \, ds \, du \\
&= \iint f(s, u) e^{-2\pi i c \cdot s} e^{-2\pi i d \cdot (s-u)} \overline{\Phi(s-a, (s-a) - (u - (-b+a)))} \, ds \, du \\
&= \iint f(s, u) e^{-2\pi i(c+d) \cdot s} e^{2\pi i d \cdot u} \overline{T_{(a,a-b)}(\Phi \circ N)(s, u)} \, ds \, du \\
&= \langle f, M_{(c+d,-d)}T_{(a,a-b)}(\Phi \circ N) \rangle
\end{aligned}$$

Notice that $M^{-1}(x, y) = (x + \frac{y}{2}, x - \frac{y}{2})$. Hence we can prove (c) by

$$\begin{aligned}
& \langle f \circ M^{-1}, M_{(c,d)}T_{(a,b)}\Phi \rangle \\
&= \iint f\left(s + \frac{t}{2}, s - \frac{t}{2}\right) e^{-2\pi i c \cdot s} e^{-2\pi i d \cdot t} \overline{\Phi(s-a, t-b)} \, ds \, dt \\
&= \iint f(x, x-t) e^{-2\pi i c \cdot (x-\frac{t}{2})} e^{-2\pi i d \cdot t} \overline{\Phi\left(x - \frac{t}{2} - a, t-b\right)} \, dx \, dt \\
&= \iint f(x, y) e^{-2\pi i c \cdot (x-\frac{x-y}{2})} e^{-2\pi i d \cdot (x-y)} \overline{\Phi\left(x - \frac{x-y}{2} - a, x-y-b\right)} \, dx \, dy \\
&= \iint f(x, y) e^{-2\pi i(\frac{c}{2}+d) \cdot x} e^{-2\pi i(\frac{c}{2}-d) \cdot y} \overline{\Phi\left(\frac{x+y}{2} - a, x-y-b\right)} \, dx \, dy \\
&= \iint f(x, y) e^{-2\pi i(\frac{c}{2}+d) \cdot x} e^{-2\pi i(\frac{c}{2}-d) \cdot y} \\
&\quad \times \overline{\Phi\left(\frac{x - (a + \frac{b}{2}) + y - (a - \frac{b}{2})}{2}, x - \left(a + \frac{b}{2}\right) - \left(y - \left(a - \frac{b}{2}\right)\right)\right)} \, dx \, dy \\
&= \langle f, M_{(\frac{c}{2}+d, \frac{c}{2}-d)}T_{(a+\frac{b}{2}, a-\frac{b}{2})}(\Phi \circ M) \rangle.
\end{aligned}$$

Also, we have

$$\begin{aligned}
& \langle f \circ M, M_{(c,d)}T_{(a,b)}\Phi \rangle \\
&= \iint f\left(\frac{s+t}{2}, s-t\right) e^{-2\pi i c \cdot s} e^{-2\pi i d \cdot t} \overline{\Phi(s-a, t-b)} \, ds \, dt
\end{aligned}$$

$$\begin{aligned}
&= 2 \iint f(x, 2x - 2t) e^{-2\pi i c \cdot (2x - t)} e^{-2\pi i d \cdot t} \overline{\Phi(2x - t - a, t - b)} dx dt \\
&= \iint f(x, y) e^{-2\pi i c \cdot (2x - (x - \frac{y}{2}))} e^{-2\pi i d \cdot (x - \frac{y}{2})} \overline{\Phi\left(2x - \left(x - \frac{y}{2}\right) - a, x - \frac{y}{2} - b\right)} dx dy \\
&= \iint f(x, y) e^{-2\pi i x \cdot (c+d)} e^{-2\pi i y \cdot (\frac{c}{2} - \frac{d}{2})} \overline{\Phi\left(x + \frac{y}{2} - a, x - \frac{y}{2} - b\right)} dx dy \\
&= \iint f(x, y) e^{-2\pi i x \cdot (c+d)} e^{-2\pi i y \cdot (\frac{c}{2} - \frac{d}{2})} \\
&\quad \times \overline{\Phi\left(x - \frac{a+b}{2} + \frac{y - (a-b)}{2}, x - \frac{a+b}{2} - \frac{y - (a-b)}{2}\right)} dx dy \\
&= \langle f, M_{(c+d, \frac{c}{2} - \frac{d}{2})} T_{(\frac{a+b}{2}, a-b)}(\Phi \circ M^{-1}) \rangle,
\end{aligned}$$

proving (d). To prove (e), we note

$$\begin{aligned}
\langle \mathcal{F}_2 f, M_{(c,d)} T_{(a,b)} \Phi \rangle &= \langle f, \mathcal{F}_2^{-1} (M_{(c,d)} T_{(a,b)} \Phi) \rangle \\
&= e^{2\pi i b \cdot d} \langle f, M_{(c,b)} T_{(a,-d)} (\mathcal{F}_2^{-1} \Phi) \rangle
\end{aligned}$$

and (f) is proved similarly. \square

Corollary 4.2.3. *Let A be a pseudodifferential operator with kernel k , Weyl symbol σ and Kohn-Nirenberg symbol τ .*

$$\begin{aligned}
(a) \quad & \left| \langle k, M_{(c,d)} T_{(a,b)} \Phi \rangle \right| = \left| \langle \tau, M_{(c+d, b-a)} T_{(a,-d)} \mathcal{F}_2(\Phi \circ N^{-1}) \rangle \right| \\
(b) \quad & \left| \langle k, M_{(c,d)} T_{(a,b)} \Phi \rangle \right| = \left| \langle \sigma, M_{(c+d, b-a)} T_{(\frac{a+b}{2}, \frac{c-d}{2})} \mathcal{F}_2(\Phi \circ M^{-1}) \rangle \right| \\
(c) \quad & \left| \langle \sigma, M_{(c,d)} T_{(a,b)} \Phi \rangle \right| = \left| \langle \tau, M_{(c,d)} T_{(a-\frac{d}{2}, b-\frac{c}{2})} \mathcal{F}_2(\mathcal{F}_2^{-1} \Phi \circ M \circ N^{-1}) \rangle \right|.
\end{aligned}$$

Proof. Using the previous lemma we have

$$\begin{aligned}
\left| \langle k, M_{(c,d)} T_{(a,b)} \Phi \rangle \right| &= \left| \langle \mathcal{F}_2^{-1} \tau \circ N, M_{(c,d)} T_{(a,b)} \Phi \rangle \right| \\
&= \left| \langle \mathcal{F}_2^{-1} \tau, M_{(c+d, -d)} T_{(a, a-b)}(\Phi \circ N^{-1}) \rangle \right| \\
&= \left| \langle \tau, M_{(c+d, b-a)} T_{(a, -d)} \mathcal{F}_2(\Phi \circ N^{-1}) \rangle \right|
\end{aligned}$$

and

$$\left| \langle k, M_{(c,d)} T_{(a,b)} \Phi \rangle \right| = \left| \langle \mathcal{F}_2^{-1} \sigma \circ M, M_{(c,d)} T_{(a,b)} \Phi \rangle \right|$$

$$\begin{aligned}
&= \left| \langle \mathcal{F}_2^{-1} \sigma, M_{(c+d, \frac{c-d}{2})} T_{(\frac{a+b}{2}, a-b)} (\Phi \circ M^{-1}) \rangle \right| \\
&= \left| \langle \sigma, M_{(c+d, b-a)} T_{(\frac{a+b}{2}, \frac{c-d}{2})} \mathcal{F}_2 (\Phi \circ M^{-1}) \rangle \right|
\end{aligned}$$

and

$$\begin{aligned}
\left| \langle \sigma, M_{(c,d)} T_{(a,b)} \Phi \rangle \right| &= \left| \langle \mathcal{F}_2 (k \circ M^{-1}), M_{(c,d)} T_{(a,b)} \Phi \rangle \right| \\
&= \left| \langle \mathcal{F}_2 (\mathcal{F}_2^{-1} \tau \circ N \circ M^{-1}), M_{(c,d)} T_{(a,b)} \Phi \rangle \right| \\
&= \left| \langle \mathcal{F}_2^{-1} \tau \circ N \circ M^{-1}, M_{(c,b)} T_{(a,-d)} \mathcal{F}_2^{-1} \Phi \rangle \right| \\
&= \left| \langle \mathcal{F}_2^{-1} \tau \circ N, M_{(\frac{c}{2}+b, \frac{c}{2}-b)} T_{(a-\frac{d}{2}, a+\frac{d}{2})} \mathcal{F}_2^{-1} \Phi \circ M \rangle \right| \\
&= \left| \langle \mathcal{F}_2^{-1} \tau, M_{(c, b-\frac{c}{2})} T_{(a-\frac{d}{2}, -d)} \mathcal{F}_2^{-1} \Phi \circ M \circ N^{-1} \rangle \right| \\
&= \left| \langle \tau, M_{(c,d)} T_{(a-\frac{d}{2}, b-\frac{c}{2})} \mathcal{F}_2^{-1} (\mathcal{F}_2^{-1} \Phi \circ M \circ N^{-1}) \rangle \right| \quad \square
\end{aligned}$$

Notice that Corollary 4.2.3 is different than the relationship between the kernel and Kohn-Nirenberg symbol of an operator given on page 263 of [9]. Corollary 4.2.3 is in fact the correct relationship between the kernel and Kohn-Nirenberg symbol of an operator.

4.3 Mixed Modulation Spaces

In this section we introduce a generalization of the modulation spaces $M_w^{p,q}(\mathbb{R}^d)$.

Recall that the Gabor transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ is $V_\phi f(x, \xi) = \langle f, M_\xi T_x \phi \rangle \forall x, \xi \in \mathbb{R}^d$, where $\phi \in \mathcal{S}(\mathbb{R}^d)$ is fixed. Also recall that $v_s(z) = (1 + |z|)^s$. We will assume throughout this chapter that $v : \mathbb{R}^{2d} \rightarrow (0, \infty)$ is a submultiplicative weight function of polynomial growth symmetric in each coordinate, i.e.

$$v(x_1, \dots, -x_i, \dots, x_{2d}) = v(x_1, \dots, x_i, \dots, x_{2d})$$

for each $i = 1, 2, \dots, 2d$. We also assume that w is a v -moderate weight and c is a permutation of $\{1, 2, \dots, 2d\}$. To simplify some notation, we identify c with the bijection $\mathbf{c} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ given by $\mathbf{c}(x_1, \dots, x_{2d}) = (x_{c(1)}, \dots, x_{c(2d)})$.

Lemma 4.3.1. *Suppose w is a v -moderate weight. Then so is $\frac{1}{w}$.*

Proof. Since w is a v -moderate, there exists C such that for all $z_1, z_2 \in \mathbb{R}^{2d}$ we have $w(z_1 + z_2) \leq Cv(z_1)w(z_2)$. Then for any $z_1, z_2 \in \mathbb{R}^{2d}$, we have

$$w(z_2) = w(-z_1 + z_1 + z_2) \leq Cv(-z_1)w(z_1 + z_2) = Cv(z_1)w(z_1 + z_2),$$

which implies

$$\frac{1}{w(z_1 + z_2)} \leq \frac{Cv(z_1)}{w(z_2)}. \quad \square$$

Definition 4.3.2. *Suppose $\phi \in \mathcal{S}(\mathbb{R}^d)$ and c is a permutation of $\{1, 2, \dots, 2d\}$ corresponding to the map \mathbf{c} . Let $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ be the mixed modulation space consisting of all $f \in \mathcal{S}'(\mathbb{R}^d)$ for which*

$$\|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} = \|V_\phi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} < \infty.$$

When $w = 1$ we write $M(c)_w^{p_1, p_2, \dots, p_{2d}} = M(c)^{p_1, p_2, \dots, p_{2d}}$.

The most interesting properties of modulation spaces carry over to the mixed modulation spaces. What follows is an adaptation of the properties of modulation spaces that are presented in [33].

Lemma 4.3.3. (a) *If c is the identity permutation and $p = p_1 = p_2 = \dots = p_d$ and*

$$q = p_{d+1} = \dots = p_{2d} \text{ then } M(c)_w^{p_1, p_2, \dots, p_{2d}} = M_w^{p, q}(\mathbb{R}^d).$$

(b) *If $p = p_1 = p_2 = \dots = p_d = p_{d+1} = \dots = p_{2d}$ then $M(c)_{v_s}^{p_1, p_2, \dots, p_{2d}} = M_{v_s}^{p, p}(\mathbb{R}^d)$*

for any permutation c .

Proof. Both statements follow directly from the definition of mixed modulation spaces. □

Lemma 4.3.3(a) shows that the mixed modulation spaces are indeed generalizations of the modulation spaces. It is shown in [41] that $M_{v_s}^{p, p}(\mathbb{R}^d)$ is invariant under the Fourier transform. Lemma 4.3.3(b) can be viewed as a generalization of this fact to the mixed modulation spaces.

4.3.1 The Inversion Formula for Mixed Modulation Space

It will be useful to consider the formal adjoint of $f \longrightarrow V_\phi f \circ \mathbf{c}$, given by Γ_ϕ in the following definition.

Definition 4.3.4. *Suppose c is a permutation of $\{1, 2, \dots, 2d\}$. For each $x \in \mathbb{R}^{2d}$ let $\pi_x = M_{(x_{d+1}, \dots, x_{2d})} T_{(x_1, \dots, x_d)}$. For measurable $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$ define an operator Γ_ψ by*

$$\Gamma_\psi F(t) = \int_{\mathbb{R}^{2d}} F(x) \pi_{\mathbf{c}(x)} \psi(t) \, dx,$$

where the integral is interpreted in the weak sense.

Lemma 4.3.5. *Suppose c is a permutation of $\{1, 2, \dots, 2d\}$ associated to \mathbf{c} and suppose $F \in \mathcal{S}(\mathbb{R}^{2d})$. Then $F \circ \mathbf{c} \in \mathcal{S}(\mathbb{R}^{2d})$.*

Lemma 4.3.6. *Suppose $\psi \in \mathcal{S}(\mathbb{R}^d)$ is given. Then $\Gamma_\psi : L_w^{p_1, p_2, \dots, p_{2d}} \longrightarrow M(c)_w^{p_1, p_2, \dots, p_{2d}}$ is a bounded linear map satisfying*

$$\|\Gamma_\psi F\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \leq \|F\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\psi \psi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})}.$$

Proof. We adapt the proof of Proposition 11.3.2(a) in [33]. Clearly Γ_ψ is linear. Choose $F \in L_w^{p_1, p_2, \dots, p_{2d}}$. First we must show $\Gamma_\psi F$ is a tempered distribution. Choose $\gamma \in \mathcal{S}(\mathbb{R}^d)$. We have

$$\begin{aligned} |\langle \Gamma_\psi F, \gamma \rangle| &= \left| \int F(x) \langle \pi_{\mathbf{c}(x)} \psi, \gamma \rangle \, dx \right| \\ &= |\langle F, V_\psi \gamma \circ \mathbf{c} \rangle| \\ &\leq \|F\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\psi \gamma \circ \mathbf{c}\|_{L_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}} \\ &\leq \|F\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\psi \gamma(\mathbf{c}(x)) (1 + |\mathbf{c}(x)|)^n\|_\infty \|(1 + |\mathbf{c}(x)|)^{-n}\|_{L_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}} \\ &= \|F\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\psi \gamma(x) (1 + |x|)^n\|_\infty \|(1 + |x|)^{-n}\|_{L_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}}. \end{aligned}$$

This value is finite for n sufficiently large. Using Corollary 11.2.6 in [33], $\Gamma_\psi F$ is in fact a tempered distribution.

Consequently, $V_\phi \Gamma_\psi F$ is a well-defined, continuous function and

$$\begin{aligned}
|V_\phi \Gamma_\psi F(\mathbf{c}(x))| &= |\langle \Gamma_\psi F, \pi_{\mathbf{c}(x)} \phi \rangle| \\
&= \left| \int F(y) \langle \pi_{\mathbf{c}(y)} \psi, \pi_{\mathbf{c}(x)} \phi \rangle dy \right| \\
&\leq \int |F(y)| |\langle \psi, \pi_{\mathbf{c}(x)-\mathbf{c}(y)} \phi \rangle| dy \\
&= \int |F(y)| |\langle \psi, \pi_{\mathbf{c}(x-y)} \phi \rangle| dy \\
&= (|F| * |V_\phi \psi \circ \mathbf{c}|)(x).
\end{aligned}$$

Thus

$$\begin{aligned}
\|\Gamma_\psi F\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} &= \|V_\phi \Gamma_\psi F \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \\
&\leq \| |F| * |V_\phi \psi \circ \mathbf{c}| \|_{L_w^{p_1, p_2, \dots, p_{2d}}} \\
&\leq \|F\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\phi \psi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})}.
\end{aligned}$$

Since $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, Theorem 11.2.5 in [33] implies $V_\phi \psi \in \mathcal{S}(\mathbb{R}^{2d})$. Therefore $\|V_\phi \psi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})}$ is finite, and we obtain the desired boundedness of Γ_ψ . \square

Theorem 4.3.7. *Suppose $\psi \in \mathcal{S}(\mathbb{R}^d)$. For any $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$, we have*

$$\Gamma_\psi (V_\phi f \circ \mathbf{c}) = \langle \psi, \phi \rangle f.$$

Proof. We adapt the proof of Proposition 11.3.2(b) in [33]. By Corollary 11.2.7 in [33] we have for all $f \in \mathcal{S}'(\mathbb{R}^d)$ that $f = \frac{1}{\langle \psi, \phi \rangle} \int V_\phi f(x) \pi_x \psi dx$. Hence for all $f \in \mathcal{S}'(\mathbb{R}^d)$, we have

$$f = \frac{1}{\langle \psi, \phi \rangle} \int V_\phi f(\mathbf{c}(x)) \pi_{\mathbf{c}(x)} \psi dx = \frac{1}{\langle \psi, \phi \rangle} \Gamma_\psi (V_\phi f \circ \mathbf{c}).$$

This equality is valid in $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ because Lemma 4.3.6 ensures that $\Gamma_\psi (V_\phi f \circ \mathbf{c}) \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$. \square

Corollary 4.3.8. *Suppose $\psi, \gamma \in \mathcal{S}(\mathbb{R}^d)$ and $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$ are given. Then there exists some constant C independent of f, ψ and γ satisfying*

$$(a) \|V_\gamma f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \leq C \|V_\phi \gamma \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}},$$

$$(b) \|V_\gamma f \circ \mathbf{c}\|_{W(L_w^{p_1, p_2, \dots, p_{2d}})} \leq C \|V_\phi \gamma \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}}, \text{ and}$$

$$(c) \|V_\gamma \psi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \leq C \|V_\phi \psi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \|V_\phi \gamma \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})}$$

Proof. Notice that for any $\gamma, \psi \in \mathcal{S}(\mathbb{R}^d)$, Theorem 11.2.5 in [33] and Lemma 4.3.5 imply $V_\gamma \psi \circ \mathbf{c} \in \mathcal{S}(\mathbb{R}^{2d})$, so $\|V_\gamma \psi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))}$ is finite.

Fix $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$. By the previous theorem we have $f = \frac{1}{\langle \phi, \phi \rangle} \Gamma_\phi (V_\phi f \circ \mathbf{c})$.

Thus

$$\begin{aligned} |V_\gamma f(\mathbf{c}(x))| &= |\langle f, \pi_{\mathbf{c}(x)} \gamma \rangle| \\ &= \frac{1}{|\langle \phi, \phi \rangle|} |\langle \Gamma_\phi (V_\phi f \circ \mathbf{c}), \pi_{\mathbf{c}(x)} \gamma \rangle| \\ &= \frac{1}{|\langle \phi, \phi \rangle|} \left| \int V_\phi f(\mathbf{c}(y)) \langle \pi_{\mathbf{c}(y)} \phi, \pi_{\mathbf{c}(x)} \gamma \rangle \, dy \right| \\ &\leq \frac{1}{|\langle \phi, \phi \rangle|} \int |V_\phi f(\mathbf{c}(y))| |\langle \phi, \pi_{\mathbf{c}(x-y)} \gamma \rangle| \, dy \\ &\leq \frac{1}{|\langle \phi, \phi \rangle|} \int |V_\phi f(\mathbf{c}(y))| |V_\gamma \phi(\mathbf{c}(x-y))| \, dy \\ &= \frac{1}{|\langle \phi, \phi \rangle|} (|V_\phi f \circ \mathbf{c}| * |V_\gamma \phi \circ \mathbf{c}|)(x). \end{aligned} \tag{20}$$

Since for all $x \in \mathbb{R}^{2d}$ we have $v(-x) = v(x)$ and

$$|V_\gamma \phi(x)| = |\langle \phi, \pi_{\mathbf{c}(x)} \gamma \rangle| = |\langle \gamma, \pi_{\mathbf{c}(-x)} \phi \rangle| = |V_\phi \gamma(-x)|,$$

it follows that $\|V_\gamma \phi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} = \|V_\phi \gamma \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})}$. By Young's inequality we have

$$\begin{aligned} \|V_\gamma f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} &\leq |\langle \phi, \phi \rangle|^{-1} \|V_\phi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\gamma \phi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \\ &= |\langle \phi, \phi \rangle|^{-1} \|V_\phi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\phi \gamma \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})}. \end{aligned} \tag{21}$$

By Lemma 1.2.6 and (20), there exists some constant C_1 independent of f, γ, ϕ such that

$$\|V_\gamma f \circ \mathbf{c}\|_{W(L_w^{p_1, p_2, \dots, p_{2d}})} \leq |\langle \phi, \phi \rangle|^{-1} \| |V_\phi f \circ \mathbf{c}| * |V_\gamma \phi \circ \mathbf{c}| \|_{W(L_w^{p_1, p_2, \dots, p_{2d}})}$$

$$\begin{aligned}
&\leq C_1 |\langle \phi, \phi \rangle|^{-1} \|V_\phi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\gamma \phi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \\
&= C_1 |\langle \phi, \phi \rangle|^{-1} \|V_\phi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\phi \gamma \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))}, \quad (22)
\end{aligned}$$

where the last equality follows from the fact $v(-x) = v(x) \forall x \in \mathbb{R}^{2d}$.

Note that by (22) we have

$$\|V_\gamma \psi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \leq C_1 |\langle \phi, \phi \rangle|^{-1} \|\psi\|_{M(c)_v^{1, \dots, 1}} \|V_\phi \gamma \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))}.$$

Applying (22) again to $\|V_\phi \gamma \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))}$ gives

$$\begin{aligned}
&\|V_\gamma \psi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \\
&\leq C_1 |\langle \phi, \phi \rangle|^{-1} \|\psi\|_{M(c)_v^{1, \dots, 1}} \|V_\phi \gamma \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \\
&\leq C_1^2 |\langle \phi, \phi \rangle|^{-2} \|\psi\|_{M(c)_v^{1, \dots, 1}} \|V_\phi \gamma \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \|V_\phi \phi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \\
&\leq C_1^2 |\langle \phi, \phi \rangle|^{-2} \|V_\phi \psi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \|V_\phi \gamma \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \|V_\phi \phi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))}. \quad (23)
\end{aligned}$$

Examining the inequalities (21), (22) and (23), we see that the theorem is satisfied for

$$C \geq \max \left\{ |\langle \phi, \phi \rangle|^{-1}, C_1 |\langle \phi, \phi \rangle|^{-1}, C_1^2 |\langle \phi, \phi \rangle|^{-2} \|V_\phi \phi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \right\}. \quad \square$$

4.3.2 Mixed Modulation Spaces as Banach Spaces

In this section we show that the mixed modulation spaces are Banach spaces and we compute their duals.

Corollary 4.3.9. *For any $p_1, p_2, \dots, p_{2d} \in [1, \infty]$, $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ is a Banach space.*

Proof. Routine calculations show that $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ is a normed linear space. Suppose $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy in $M(c)_w^{p_1, p_2, \dots, p_{2d}}$. Then $\{V_\phi f_n \circ \mathbf{c}\}_{n \in \mathbb{N}}$ is Cauchy in $L_w^{p_1, p_2, \dots, p_{2d}}$. Since $L_w^{p_1, p_2, \dots, p_{2d}}$ is a Banach space, there is some $F \in L_w^{p_1, p_2, \dots, p_{2d}}$ with

$$\lim_{n \rightarrow \infty} \|V_\phi f_n \circ \mathbf{c} - F\|_{L_w^{p_1, p_2, \dots, p_{2d}}} = 0.$$

Let $f = \|\phi\|^{-2} \Gamma_\phi F$. Then $f \in L_w^{p_1, p_2, \dots, p_{2d}}$ and

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} = \lim_{n \rightarrow \infty} \|\phi\|^{-2} \|\Gamma_\phi (V_\phi f_n \circ \mathbf{c}) - \Gamma_\phi F\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}}$$

$$\begin{aligned}
&\leq \lim_{n \rightarrow \infty} \|\phi\|^{-2} \|\Gamma_\phi\| \|V_\phi f_n \circ \mathbf{c} - F\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \\
&= 0.
\end{aligned}$$

Therefore $\{f_n\}_{n \in \mathbb{N}}$ is convergent in $M(c)_w^{p_1, p_2, \dots, p_{2d}}$. \square

Theorem 4.3.10. *If $p_1, p_2, \dots, p_{2d} \in [1, \infty)$ then $M(c)_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}$ is the dual space of $M(c)_w^{p_1, p_2, \dots, p_{2d}}$, where each $p'_i \in [1, \infty]$ satisfies $\frac{1}{p_i} + \frac{1}{p'_i} = 1$.*

Proof. Note that $f \rightarrow V_\phi f \circ \mathbf{c}$ is the adjoint of Γ_ϕ . Suppose $g \in M(c)_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}$. We have

$$\begin{aligned}
|\langle f, g \rangle| &= \|\phi\|^{-2} |\langle \Gamma_\phi(V_\phi f \circ \mathbf{c}), g \rangle| \\
&= \left| \int V_\phi f(\mathbf{c}(x)) \overline{V_\phi g(\mathbf{c}(x))} dx \right| \\
&\leq \|V_\phi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\phi g \circ \mathbf{c}\|_{L_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}} \\
&= \|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \|g\|_{M(c)_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}}
\end{aligned}$$

so that g induces a bounded linear functional on $M(c)_w^{p_1, p_2, \dots, p_{2d}}$.

Now suppose $\alpha \in (M(c)_w^{p_1, p_2, \dots, p_{2d}})^*$. Because $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ is a Banach space, the space

$$V = \{F \in L_w^{p_1, p_2, \dots, p_{2d}} : F = V_\phi f \circ \mathbf{c} \text{ for some } f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}\}$$

is closed and isometrically isomorphic to $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ via Γ_ϕ^* . Hence α induces a functional on V . By the Hahn-Banach Theorem, α extends to a functional on $L_w^{p_1, p_2, \dots, p_{2d}}$. Hence there is $G \in L_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}$ with

$$\alpha(f) = \langle \Gamma_\phi^* f, G \rangle \quad \forall f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}.$$

Let $g = \Gamma_\phi G$. Then $g \in M(c)_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}$ by Lemma 4.3.6 and

$$\alpha(f) = \langle \Gamma_\phi^* f, G \rangle = \langle f, \Gamma_\phi G \rangle = \langle f, g \rangle.$$

Hence $(M(c)_w^{p_1, p_2, \dots, p_{2d}})^* = M(c)_{\frac{1}{w}}^{p'_1, p'_2, \dots, p'_{2d}}$. \square

4.3.3 A larger window class

Many of the mixed modulation spaces results for windows in $\mathcal{S}(\mathbb{R}^d)$ to also hold for windows in $M(c)_v^{1,1,\dots,1}$. We focus on these type results in this section. First a technical lemma is needed.

Lemma 4.3.11. $\mathcal{S}(\mathbb{R}^d)$ is dense in $M(c)_w^{p_1,p_2,\dots,p_{2d}}$ for all $p_1, p_2, \dots, p_{2d} \in [1, \infty)$.

Proof. First, suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Since v has polynomial growth, so does w and there is some $s \geq 0$ with $\|w\|_{L_{v^{-s}}^{p_1,p_2,\dots,p_{2d}}} < \infty$. Since $f \in \mathcal{S}(\mathbb{R}^d)$, Theorem 11.2.5 in [33] and Lemma 4.3.5 imply $V_\phi f \circ \mathbf{c} \in \mathcal{S}(\mathbb{R}^{2d})$, so $\|V_\phi f \circ \mathbf{c}\|_{L_{v_s}^\infty(\mathbb{R}^{2d})} < \infty$. Hence

$$\begin{aligned} \|f\|_{M(c)_w^{p_1,p_2,\dots,p_{2d}}} &= \|V_\phi f \circ \mathbf{c}\|_{L_w^{p_1,p_2,\dots,p_{2d}}} \\ &\leq \|V_\phi f \circ \mathbf{c}\|_{L_{v_s}^\infty(\mathbb{R}^{2d})} \|w\|_{L_{v^{-s}}^{p_1,p_2,\dots,p_{2d}}} \\ &< \infty. \end{aligned}$$

Thus $f \in M(c)_w^{p_1,p_2,\dots,p_{2d}}$, so $\mathcal{S}(\mathbb{R}^d) \subset M(c)_w^{p_1,p_2,\dots,p_{2d}}$.

Now suppose $f \in M(c)_w^{p_1,p_2,\dots,p_{2d}}$. Let $F_n = V_\phi f \cdot \chi_{[-n,n]^{2d}}$. By Proposition 11.2.4 in [33],

$$f_n = \|\phi\|_{L^2}^{-2} \int F_n(x) \pi_x \phi \, dx = \|\phi\|_{L^2}^{-2} \int F_n(\mathbf{c}(x)) \pi_{\mathbf{c}(x)} \phi \, dx = \|\phi\|_{L^2}^{-2} \Gamma_\phi(F_n \circ \mathbf{c}) \in \mathcal{S}(\mathbb{R}^d).$$

Notice that by Lemma 4.3.6(b), we have $V_\phi f \circ \mathbf{c} \in W(L_w^{p_1,\dots,p_{2d}})$, which implies

$$\lim_{n \rightarrow \infty} \left\| (V_\phi f \circ \mathbf{c}) \cdot \chi_{([-n,n]^{2d})^c} \right\|_{L_w^{p_1,p_2,\dots,p_{2d}}} = 0.$$

Also by Lemma 4.3.6,

$$\begin{aligned} \|f - f_n\|_{M(c)_w^{p_1,p_2,\dots,p_{2d}}} &= \|\phi\|_{L^2}^{-2} \|\Gamma_\phi(V_\phi f \circ \mathbf{c}) - \Gamma_\phi(F_n \circ \mathbf{c})\|_{M(c)_w^{p_1,p_2,\dots,p_{2d}}} \\ &\leq \|\phi\|_{L^2}^{-2} \|V_\phi f \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \left\| (V_\phi f \circ \mathbf{c}) \cdot \chi_{([-n,n]^{2d})^c} \right\|_{L_w^{p_1,p_2,\dots,p_{2d}}} \\ &\rightarrow 0. \end{aligned}$$

Hence $\mathcal{S}(\mathbb{R}^d)$ is dense in $M(c)_w^{p_1,p_2,\dots,p_{2d}}$. □

Lemma 4.3.12. *Suppose $\psi, \gamma \in M(c)_v^{1,1,\dots,1}$.*

(a) *For all $p_1, \dots, p_{2d} \in [1, \infty]$, $f \longrightarrow V_\gamma f \circ \mathbf{c}$ is a bounded operator from $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ to $L^{p_1, p_2, \dots, p_{2d}}$ and there is some constant C independent of f and γ such that*

$$\|V_\gamma f \circ \mathbf{c}\|_{L^{p_1, p_2, \dots, p_{2d}}} \leq C \|\gamma\|_{M(c)_v^{1,1,\dots,1}} \|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}}.$$

(b) *For all $p_1, \dots, p_{2d} \in [1, \infty]$, $F \longrightarrow \Gamma_\psi F$ is a bounded operator from $L^{p_1, p_2, \dots, p_{2d}}$ to $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ with*

$$\|\Gamma_\psi F\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \leq \|F\|_{L^{p_1, p_2, \dots, p_{2d}}} \|\psi\|_{M(c)_v^{1,1,\dots,1}}.$$

Proof. First we prove (a). Fix $p_1, \dots, p_{2d} \in [1, \infty]$ and let $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$. By Theorem 4.3.7 we have $f = \frac{1}{\langle \phi, \phi \rangle} \Gamma_\phi (V_\phi f \circ \mathbf{c})$. Thus

$$\begin{aligned} |V_\gamma f(\mathbf{c}(x))| &= |\langle f, \pi_{\mathbf{c}(x)} \gamma \rangle| \\ &= \frac{1}{|\langle \phi, \phi \rangle|} |\langle \Gamma_\phi (V_\phi f \circ \mathbf{c}), \pi_{\mathbf{c}(x)} \gamma \rangle| \\ &= \frac{1}{|\langle \phi, \phi \rangle|} \left| \int V_\phi f(\mathbf{c}(y)) \langle \pi_{\mathbf{c}(y)} \phi, \pi_{\mathbf{c}(x)} \gamma \rangle dy \right| \\ &\leq \frac{1}{|\langle \phi, \phi \rangle|} \int |V_\phi f(\mathbf{c}(y))| |\langle \phi, \pi_{\mathbf{c}(x-y)} \gamma \rangle| dy \\ &\leq \frac{1}{|\langle \phi, \phi \rangle|} \int |V_\phi f(\mathbf{c}(y))| |V_\gamma \phi(\mathbf{c}(x-y))| dy \\ &= \frac{1}{|\langle \phi, \phi \rangle|} (|V_\phi f \circ \mathbf{c}| * |V_\gamma \phi \circ \mathbf{c}|)(x). \end{aligned}$$

Thus by Young's inequality we have

$$\begin{aligned} \|V_\gamma f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} &\leq |\langle \phi, \phi \rangle|^{-1} \|V_\phi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\gamma \phi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \\ &= |\langle \phi, \phi \rangle|^{-1} \|V_\phi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \|V_\phi \gamma \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \\ &= |\langle \phi, \phi \rangle|^{-1} \|\gamma\|_{M(c)_v^{1,1,\dots,1}} \|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}}. \end{aligned}$$

Now we prove (b). Fix $p_1, \dots, p_{2d} \in [1, \infty]$. By Lemma 4.3.1, $\frac{1}{w}$ is v -moderate. Thus by (a), $f \rightarrow V_\psi f \circ \mathbf{c}$ is a bounded operator from $M(c)_w^{p'_1, p'_2, \dots, p'_{2d}}$ to $L^{\frac{1}{w}}_{p'_1, p'_2, \dots, p'_{2d}}$.

Since Γ_ψ is the adjoint of this operator, we see $F \longrightarrow \Gamma_\psi F$ is a bounded operator from $L^{p_1, p_2, \dots, p_{2d}}$ to $M(c)_w^{p_1, p_2, \dots, p_{2d}}$.

Fix $F \in L^{p_1, p_2, \dots, p_{2d}}$. By Lemma 4.3.6 we see that $\tau \longrightarrow \Gamma_\tau F$ is a linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ satisfying

$$\|\Gamma_\tau F\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \leq \|F\|_{L^{p_1, p_2, \dots, p_{2d}}} \|V_\phi \tau \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})}.$$

Since Lemma 4.3.11 shows $\mathcal{S}(\mathbb{R}^d)$ is dense in $M(c)_v^{1, 1, \dots, 1}$, this operator extends to a bounded linear operator from $M(c)_v^{1, 1, \dots, 1}$ to $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ with

$$\|\Gamma_\psi F\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \leq \|F\|_{L^{p_1, p_2, \dots, p_{2d}}} \|\psi\|_{M(c)_v^{1, 1, \dots, 1}}. \quad \square$$

Lemma 4.3.13. *Suppose $\psi, \gamma \in \mathcal{S}(\mathbb{R}^d)$. For any $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$, we have $\Gamma_\psi(V_\gamma f \circ \mathbf{c}) = \langle \psi, \gamma \rangle f$.*

Proof. By Corollary 11.2.7 in [33] we have for all $f \in \mathcal{S}'(\mathbb{R}^d)$ that

$$f = \frac{1}{\langle \psi, \gamma \rangle} \int V_\gamma f(x) \pi_x \psi \, dx. \text{ Hence for all } f \in \mathcal{S}'(\mathbb{R}^d), \text{ we have}$$

$$f = \frac{1}{\langle \psi, \gamma \rangle} \int V_\gamma f(\mathbf{c}(x)) \pi_{\mathbf{c}(x)} \psi \, dx = \frac{1}{\langle \psi, \gamma \rangle} \Gamma_\psi(V_\gamma f \circ \mathbf{c}).$$

This is an equality of distributions, but it is also valid in $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ because Lemmas 4.3.11 and 4.3.12 ensure that $\Gamma_\psi(V_\gamma f \circ \mathbf{c}) \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$. \square

Theorem 4.3.14. *Suppose $\psi, \gamma \in M(c)_v^{1, \dots, 1}$ are given.*

(a) *For any $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$, we have $\Gamma_\psi(V_\gamma f \circ \mathbf{c}) = \langle \psi, \gamma \rangle f$.*

(b) *$\|f\| = \|V_\psi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}}$ is an equivalent norm on $M(c)_w^{p_1, p_2, \dots, p_{2d}}$.*

Proof. First we prove (a). Choose $\{\psi_n\}, \{\gamma_n\} \subset \mathcal{S}(\mathbb{R}^d)$ with $\|\psi - \psi_n\|_{M(c)_v^{1, 1, \dots, 1}} \rightarrow 0$ and $\|\gamma - \gamma_n\|_{M(c)_v^{1, 1, \dots, 1}} \rightarrow 0$. Fix $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$. Using Lemma 4.3.13 we have

$$\begin{aligned} & \|f - \langle \psi, \gamma \rangle^{-1} \Gamma_\psi(V_\gamma f \circ \mathbf{c})\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \\ &= \|\langle \psi_n, \gamma_n \rangle^{-1} \Gamma_{\psi_n}(V_{\gamma_n} f \circ \mathbf{c}) - \langle \psi, \gamma \rangle^{-1} \Gamma_\psi(V_\gamma f \circ \mathbf{c})\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \end{aligned}$$

$$\begin{aligned}
&\leq |\langle \psi_n, \gamma_n \rangle^{-1} - \langle \psi, \gamma \rangle^{-1}| \|\Gamma_\psi (V_\gamma f \circ \mathbf{c})\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \\
&\quad + |\langle \psi_n, \gamma_n \rangle|^{-1} \|\Gamma_\psi (V_{\gamma - \gamma_n} f \circ \mathbf{c})\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \\
&\quad + |\langle \psi_n, \gamma_n \rangle|^{-1} \|\Gamma_{\psi - \psi_n} (V_{\gamma_n} f \circ \mathbf{c})\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}}
\end{aligned}$$

Using Lemma 4.3.12, we see that each term in this sum can be made arbitrarily small, proving (a).

To prove (b), fix $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$. Then by part (a) and Lemma 4.3.12, we have

$$\begin{aligned}
\|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} &= |\langle \phi, \psi \rangle|^{-1} \|\Gamma_\phi (V_\psi f \circ \mathbf{c})\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \\
&\leq |\langle \phi, \psi \rangle|^{-1} \|\phi\|_{M(c)_v^{1, \dots, 1}} \|V_\psi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \\
&= |\langle \phi, \psi \rangle|^{-1} \|\phi\|_{M(c)_v^{1, \dots, 1}} \|f\| \\
&= |\langle \phi, \psi \rangle|^{-1} \|\phi\|_{M(c)_v^{1, \dots, 1}} \|V_\psi f \circ \mathbf{c}\|_{L_w^{p_1, p_2, \dots, p_{2d}}} \\
&\leq |\langle \phi, \psi \rangle|^{-1} \|\phi\|_{M(c)_v^{1, \dots, 1}} C \|\psi\|_{M(c)_v^{1, 1, \dots, 1}} \|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \quad \square
\end{aligned}$$

Theorem 4.3.14(b) states that the definition of the mixed modulation spaces is independent of the choice of $\phi \in \mathcal{S}(\mathbb{R}^d)$, with different ϕ giving equivalent norms. Furthermore, this fact also holds for ϕ in the larger space $M(c)_v^{1, \dots, 1}$. Theorem 4.3.14(a) states that for Gabor window functions in $M(c)_v^{1, \dots, 1}$, there is an inversion formula valid on each $M(c)_w^{p_1, p_2, \dots, p_{2d}}$.

4.3.4 Banach Frames

Lemma 4.3.15. *If $\psi, \gamma \in M(c)_v^{1, 1, \dots, 1}$, then $V_\gamma \psi \circ \mathbf{c} \in W(L_v^1(\mathbb{R}^{2d}))$ and there exists C independent of ψ, γ with*

Proof. If $\psi, \gamma \in \mathcal{S}(\mathbb{R}^d)$ then Corollary 4.3.8(c) implies

$$\|V_\gamma \psi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \leq C \|\psi\|_{M(c)_v^{1, \dots, 1}} \|\gamma\|_{M(c)_v^{1, \dots, 1}},$$

for some C independent of ψ, γ . By the density of $\mathcal{S}(\mathbb{R}^d)$ in $M(c)_v^{1, 1, \dots, 1}$, this result extends to $\psi, \gamma \in M(c)_v^{1, 1, \dots, 1}$. \square

Proposition 4.3.16. *Suppose $\psi \in M(c)_v^{1,1,\dots,1}$. Then the analysis operator $C_\psi : M(c)_w^{p_1,\dots,p_{2d}} \longrightarrow \ell_w^{p_1,\dots,p_{2d}}$ defined by*

$$C_\psi f = \left\{ \langle f, \pi_{\mathbf{c}(\alpha \cdot n)} \psi \rangle \right\}_{n \in \mathbb{Z}^{2d}}$$

is bounded for all $p_1, \dots, p_{2d} \in [1, \infty]$ and all $\alpha = (\alpha_1, \dots, \alpha_{2d})$ with $\alpha_1, \dots, \alpha_{2d} \in (0, \infty)$.

Proof. First notice that we can define an equivalent norm on $W(L_w^{p_1,p_2,\dots,p_{2d}})$ by

$$\|F\| = \left\| \left\{ \|F \chi_{\alpha \cdot [0,1]^{2d} + \alpha \cdot n}\|_\infty \right\}_{n \in \mathbb{Z}^{2d}} \right\|_{\ell_w^{p_1,p_2,\dots,p_{2d}}}.$$

Hence, there exists finite K such that

$$\left\| \left\{ \|F \chi_{\alpha \cdot [0,1]^{2d} + \alpha \cdot n}\|_\infty \right\}_{n \in \mathbb{Z}^{2d}} \right\|_{\ell_w^{p_1,p_2,\dots,p_{2d}}} \leq K \|F\|_{W(L_w^{p_1,p_2,\dots,p_{2d}})}$$

for all $F \in W(L_w^{p_1,p_2,\dots,p_{2d}})$.

If $\psi \in \mathcal{S}(\mathbb{R}^d)$, then by Corollary 4.3.8(b) and (c) there is some C independent of ψ, f such that for all $f \in M(c)_w^{p_1,\dots,p_{2d}}$ we have

$$\begin{aligned} \|C_\psi f\|_{\ell_w^{p_1,\dots,p_{2d}}} &= \left\| \left\{ V_\psi f \circ \mathbf{c}|_{\alpha \cdot n} \right\}_{n \in \mathbb{Z}^{2d}} \right\|_{\ell_w^{p_1,\dots,p_{2d}}} \\ &\leq \left\| \left\{ \|(V_\psi f \circ \mathbf{c}) \cdot \chi_{\alpha \cdot [0,1]^{2d} + \alpha \cdot n}\|_\infty \right\}_{n \in \mathbb{Z}^{2d}} \right\|_{\ell_w^{p_1,p_2,\dots,p_d}} \\ &\leq K \|V_\psi f \circ \mathbf{c}\|_{W(L_w^{p_1,\dots,p_{2d}})} \\ &\leq KC \|V_\phi \psi \circ \mathbf{c}\|_{W(L_v^1(\mathbb{R}^{2d}))} \|f\|_{M(c)_w^{p_1,\dots,p_{2d}}} \\ &\leq KC^2 \|V_\phi \phi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \|V_\phi \psi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \|f\|_{M(c)_w^{p_1,\dots,p_{2d}}} \\ &\leq KC^2 \|V_\phi \phi \circ \mathbf{c}\|_{L_v^1(\mathbb{R}^{2d})} \|\psi\|_{M(c)_v^{1,\dots,1}} \|f\|_{M(c)_w^{p_1,\dots,p_{2d}}}. \end{aligned} \quad (24)$$

By the density of $\mathcal{S}(\mathbb{R}^d)$ in $M(c)_v^{1,1,\dots,1}$, the inequality (24) extends to all $\psi \in M(c)_v^{1,1,\dots,1}$. \square

Proposition 4.3.17. *Suppose $\psi \in M(c)_v^{1,1,\dots,1}$. Then the Gabor synthesis operator $D_\psi : \ell_w^{p_1,\dots,p_{2d}} \longrightarrow M(c)_w^{p_1,\dots,p_{2d}}$ defined by*

$$D_\psi d = \sum_{n \in \mathbb{Z}^{2d}} d_n \pi_{\mathbf{c}(\alpha \cdot n)} \psi$$

is bounded for all $p_1, \dots, p_{2d} \in [1, \infty]$ and all $\alpha = (\alpha_1, \dots, \alpha_{2d})$ with $\alpha_1, \dots, \alpha_{2d} \in (0, \infty)$.

Proof. Choose K such that

$$\|F\|_{W(L_w^{p_1, p_2, \dots, p_{2d}})} \leq K \left\| \left\{ \|F\chi_{\alpha \cdot [0,1]^{2d} + \alpha \cdot n}\|_\infty \right\}_{n \in \mathbb{Z}^{2d}} \right\|_{\ell_w^{p_1, p_2, \dots, p_{2d}}}$$

for all $F \in W(L_w^{p_1, p_2, \dots, p_{2d}})$.

Since $\psi \in M(c)_v^{1,1, \dots, 1}$ we have $V_\phi \psi \circ \mathbf{c} \in W(L_v^1(\mathbb{R}^{2d}))$. Consequently, there is a sequence $a \in \ell_v^1(\mathbb{Z}^{2d})$ with $a_m = \|(V_\psi f \circ \mathbf{c}) \cdot \chi_{\alpha \cdot [0,1]^{2d} + \alpha \cdot m}\|_\infty$ and

$$\|V_\phi \psi \circ \mathbf{c}\|_{W(L_v^1)} = \sum_{m \in \mathbb{Z}^{2d}} |a_m| v(\alpha \cdot m).$$

Thus

$$|V_\phi \psi(\mathbf{c}(x))| \leq \sum_{m \in \mathbb{Z}^{2d}} |a_m| T_{\alpha \cdot m} \chi_{\alpha \cdot [0,1]^{2d}}(x) \text{ a.e.}$$

It follows that

$$\begin{aligned} |V_\phi(D_\psi d)(\mathbf{c}(x))| &= \left| \sum_{n \in \mathbb{Z}^{2d}} d_n \langle \pi_{\mathbf{c}(\alpha \cdot n)} \psi, \pi_{\mathbf{c}(x)} \phi \rangle \right| \\ &\leq \sum_{n \in \mathbb{Z}^{2d}} |d_n| |V_\phi \psi(\mathbf{c}(x - \alpha \cdot n))| \\ &\leq \sum_{n \in \mathbb{Z}^{2d}} \sum_{m \in \mathbb{Z}^{2d}} |d_n| |a_m| T_{\alpha \cdot m} \chi_{\alpha \cdot [0,1]^{2d}}(x - \alpha \cdot n) \\ &= \sum_{n \in \mathbb{Z}^{2d}} \sum_{m \in \mathbb{Z}^{2d}} |d_n| |a_m| T_{\alpha \cdot (m+n)} \chi_{\alpha \cdot [0,1]^{2d}}(x) \\ &= \sum_{n \in \mathbb{Z}^{2d}} \sum_{m \in \mathbb{Z}^{2d}} |d_n| |a_{m-n}| T_{\alpha \cdot m} \chi_{\alpha \cdot [0,1]^{2d}}(x) \\ &= \sum_{m \in \mathbb{Z}^{2d}} (|d| * |a|)_m T_{\alpha \cdot m} \chi_{\alpha \cdot [0,1]^{2d}}(x). \end{aligned}$$

Thus

$$\begin{aligned} \|D_\psi d\|_{M(c)_w^{p_1, \dots, p_{2d}}} &= \|V_\phi(D_\psi d) \circ \mathbf{c}\|_{L_w^{p_1, \dots, p_{2d}}} \\ &\leq \left\| \sum_{m \in \mathbb{Z}^{2d}} (|d| * |a|)_m T_{\alpha \cdot m} \chi_{\alpha \cdot [0,1]^{2d}}(x) \right\|_{L_w^{p_1, \dots, p_{2d}}} \end{aligned}$$

$$\begin{aligned}
&\leq C \left\| \sum_{m \in \mathbb{Z}^{2d}} (|d| * |a|)_m T_{\alpha \cdot m} \chi_{\alpha \cdot [0,1]^{2d}}(x) \right\|_{W(L_w^{p_1, \dots, p_{2d}})} \\
&\leq CK \| |d| * |a| \|_{\ell_w^{p_1, \dots, p_{2d}}} \\
&\leq CK \|d\|_{\ell_w^{p_1, \dots, p_{2d}}} \|a\|_{\ell_v^1},
\end{aligned}$$

where C is some constant depending only on the Lebesgue measure of $\alpha \cdot [0, 1]^{2d}$. \square

Corollary 4.3.18. Fix $\alpha = (\alpha_1, \dots, \alpha_{2d})$ with $\alpha_1, \dots, \alpha_{2d} \in (0, \infty)$. Suppose $\psi \in M(c)_v^{1,1, \dots, 1}$. If $d \in \ell_w^{p_1, \dots, p_{2d}}$ then the sum

$$D_\psi d = \sum_{n \in \mathbb{Z}^{2d}} d_n \pi_{\mathbf{c}(\alpha \cdot n)} \psi$$

converges unconditionally in $M(c)_w^{p_1, \dots, p_{2d}}$ for all $p_1, \dots, p_{2d} \in [1, \infty)$ and converges weak* unconditionally in $M(c)_{\frac{1}{v}}^{\infty, \dots, \infty}$.

Proof. First we assume $p_1, \dots, p_{2d} \in [1, \infty)$. Fix $d \in \ell_w^{p_1, \dots, p_{2d}}$ and $\varepsilon > 0$. Choose a finite set $S_0 \subset \mathbb{Z}^{2d}$ so that $\|d - d\chi_{S_0}\|_{\ell_w^{p_1, \dots, p_{2d}}} < \varepsilon$. Then for each $S_0 \subset S \subset \mathbb{Z}^{2d}$ we have

$$\begin{aligned}
\left\| D_\psi d - \sum_{n \in S} d_n \pi_{\mathbf{c}(\alpha \cdot n)} \psi \right\|_{M(c)_w^{p_1, \dots, p_{2d}}} &= \|D_\psi (d - d\chi_S)\| \\
&\leq \|D_\psi\| \|d - d\chi_S\|_{\ell_w^{p_1, \dots, p_{2d}}} \\
&< \|D_\psi\| \varepsilon.
\end{aligned}$$

Hence $\sum_{n \in \mathbb{Z}^{2d}} d_n \pi_{\mathbf{c}(\alpha \cdot n)} \psi$ converges unconditionally to $D_\psi d$.

More generally, assume $p_1, \dots, p_{2d} \in [1, \infty]$. Fix $f \in M(c)_v^{1, \dots, 1}$ and let $\varepsilon > 0$. Then $C_\psi f \in \ell_v^{1, \dots, 1}$ and there is a finite set $S_0 \subset \mathbb{Z}^{2d}$ so that

$$\|C_\psi f - (C_\psi f) \chi_{S_0}\|_{\ell_v^{1, \dots, 1}(\mathbb{Z}^{2d})} < \varepsilon.$$

Thus for each $S_0 \subset S \subset \mathbb{Z}^{2d}$

$$\left| \left\langle D_\psi d - \sum_{n \in S} d_n \pi_{\mathbf{c}(\alpha \cdot n)} \psi, f \right\rangle \right| = |\langle D_\psi (d - d\chi_S) f \rangle|$$

$$\begin{aligned}
&= |\langle d(1 - \chi_S), C_\psi f \rangle| \\
&= |\langle d, C_\psi f(1 - \chi_S) \rangle| \\
&\leq \|d\|_{\ell_{\frac{1}{v}}^\infty} \|C_\psi f - (C_\psi f)\chi_S\|_{\ell_v^{1, \dots, 1}} \\
&\leq \|d\|_{\ell_{\frac{1}{v}}^\infty} \|C_\psi f - (C_\psi f)\chi_{S_0}\|_{\ell_v^{1, \dots, 1}} \\
&< \|d\|_{\ell_{\frac{1}{v}}^\infty} \varepsilon
\end{aligned}$$

It follows that $\sum_{n \in \mathbb{Z}^{2d}} d_n \pi_{\mathbf{c}(\alpha \cdot n)} \psi$ converges weak* unconditionally to $D_\psi d$ in $M(c)_{\frac{1}{v}}^\infty$. \square

The next theorem states that if the window function is nice then a Gabor frame for $L^2(\mathbb{R}^d)$ gives bounded decompositions for all mixed modulation spaces.

Theorem 4.3.19. *Fix $\beta > 0$. Suppose $p_1, p_2, \dots, p_{2d} \in [1, \infty]$ and $\psi \in M(c)_v^{1, \dots, 1}$. Further suppose that $\{\pi_{\beta n} \psi\}_{n \in \mathbb{Z}^{2d}}$ is a frame for $L^2(\mathbb{R}^d)$ with dual frame $\{\pi_{\beta n} \gamma\}_{n \in \mathbb{Z}^{2d}}$. Then*

(a) *$\{\pi_{\beta n} \psi\}_{n \in \mathbb{Z}^{2d}}$ is a Banach frame for $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ and there exist $0 < A \leq B < \infty$ independent of p_1, p_2, \dots, p_{2d} such that*

$$A \|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}} \leq \left\| V_\psi f \circ \mathbf{c} \Big|_{\beta \mathbb{Z}^{2d}} \right\|_{\ell_w^{p_1, p_2, \dots, p_{2d}}} \leq B \|f\|_{M(c)_w^{p_1, p_2, \dots, p_{2d}}},$$

for all $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$.

(b) *If $p_1, p_2, \dots, p_{2d} \in [1, \infty)$ then*

$$f = \sum_{m \in \mathbb{Z}^{2d}} \langle f, \pi_{\beta m} \psi \rangle \pi_{\beta m} \gamma = \sum_{m \in \mathbb{Z}^{2d}} \langle f, \pi_{\beta m} \gamma \rangle \pi_{\beta m} \psi$$

for all $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$ with unconditional convergence in $M(c)_w^{p_1, p_2, \dots, p_{2d}}$.

(c) *If $p_1, p_2, \dots, p_{2d} \in [1, \infty]$ then*

$$f = \sum_{m \in \mathbb{Z}^{2d}} \langle f, \pi_{\beta m} \psi \rangle \pi_{\beta m} \gamma = \sum_{m \in \mathbb{Z}^{2d}} \langle f, \pi_{\beta m} \gamma \rangle \pi_{\beta m} \psi$$

for all $f \in M(c)_w^{p_1, p_2, \dots, p_{2d}}$ with weak* convergence in $M(c)_{\frac{1}{v}}^{\infty, \dots, \infty}$.

Proof. Set $\alpha = (\beta, \beta, \dots, \beta) \in \mathbb{R}^{2d}$.

Since $\{\pi_{\beta n} \psi\}_{n \in \mathbb{Z}^{2d}}$ is a frame for $L^2(\mathbb{R}^d)$ with dual frame $\{\pi_{\beta n} \gamma\}_{n \in \mathbb{Z}^{2d}}$, we have

$$f = \sum_{m \in \mathbb{Z}^{2d}} \langle f, \pi_{\beta m} \psi \rangle \pi_{\beta m} \gamma = D_\gamma C_\psi f \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

By Proposition 4.3.17, Corollary 4.3.18, and Lemma 4.3.11, this equality extends to each mixed modulation space $M(c)_w^{p_1, \dots, p_{2d}}$, with unconditional convergence if $p_1, p_2, \dots, p_{2d} \in [1, \infty)$ and weak* convergence in $M(c)_w^{\infty, \dots, \infty}$. Arguing similarly with for D_ψ and C_γ completes the proof of (b) and (c).

To prove (a), we note that

$$\begin{aligned} \|f\|_{M(c)_w^{p_1, \dots, p_{2d}}} &= \|D_\gamma C_\psi f\|_{M(c)_w^{p_1, \dots, p_{2d}}} \\ &\leq \|D_\gamma\| \|C_\psi f\|_{\ell_w^{p_1, \dots, p_{2d}}} \\ &\leq \|D_\gamma\| \|C_\psi\| \|f\|_{M(c)_w^{p_1, \dots, p_{2d}}}, \end{aligned}$$

and

$$\left\| V_\psi f \circ \mathbf{c} \Big|_{\beta \mathbb{Z}^{2d}} \right\|_{\ell_w^{p_1, \dots, p_{2d}}} = \|C_\psi f\|_{\ell_w^{p_1, \dots, p_{2d}}}.$$

Letting $B = \|C_\psi\|$ and $A = \|D_\gamma\|^{-1}$ gives (a). □

4.3.5 Mixed Modulation Embeddings

Theorem 4.3.19 can be used to prove embeddings among the mixed modulation spaces.

Lemma 4.3.20. *If $s \geq t$ and $p_i, r_i \in [1, \infty]$ with $p_i \leq r_i$ for all $i = 1, 2, \dots, 2d$ then*

$$M(c)_{v_s}^{p_1, p_2, \dots, p_{2d}} \subset M(c)_{v_t}^{r_1, r_2, \dots, r_{2d}}.$$

Proof. Since $v_t(x) \leq v_s(x)$, we see $M(c)_{v_s}^{r_1, r_2, \dots, r_{2d}} \subset M(c)_{v_t}^{r_1, r_2, \dots, r_{2d}}$ from the definition of mixed modulation spaces. Hence it suffices to prove $M(c)_{v_s}^{p_1, \dots, p_{2d}} \subset M(c)_{v_s}^{r_1, r_2, \dots, r_{2d}}$.

By Theorem 12.2.2 in [33], $\ell_{v_s}^{p_1, \dots, p_{2d}} \subset \ell_{v_s}^{r_1, r_2, \dots, r_{2d}}$ with

$$\|d\|_{\ell_{v_s}^{r_1, r_2, \dots, r_{2d}}} \leq \|d\|_{\ell_{v_s}^{p_1, \dots, p_{2d}}}$$

for all $d \in \ell_{v_s}^{p_1, \dots, p_{2d}}$. Choose $\psi \in M(c)_{v_s}^{1, \dots, 1}$ and $\beta > 0$ so that $\{\pi_{\beta n} \psi\}_{n \in \mathbb{Z}^{2d}}$ is a frame for $L^2(\mathbb{R}^d)$. By Theorem 4.3.19(a) we have for any $f \in M(c)_{v_s}^{p_1, \dots, p_{2d}}$ that

$$\begin{aligned} \|f\|_{M(c)_{v_s}^{r_1, r_2, \dots, r_{2d}}} &\equiv \left\| V_\psi f \circ \mathbf{c} \Big|_{\beta \mathbb{Z}^{2d}} \right\|_{\ell_{v_s}^{r_1, \dots, r_{2d}}} \\ &\leq \left\| V_\psi f \circ \mathbf{c} \Big|_{\beta \mathbb{Z}^{2d}} \right\|_{\ell_{v_s}^{p_1, \dots, p_{2d}}} \\ &\equiv \|f\|_{M(c)_{v_s}^{p_1, \dots, p_{2d}}} \end{aligned} \quad \square$$

4.3.6 Wilson Bases

Let $X_1 = X_2 = \dots = X_d = \mathbb{Z}$ and $X_{d+1} = X_{d+2} = \dots = X_{2d} = \mathbb{Z}^+$.

Let

$$\tilde{C}_\psi f = \left\{ \langle f, \Psi_{n_{c(1)}, \dots, n_{c(2d)}} \rangle \right\}_{n_1 \in X_{c-1(1)}, n_2 \in X_{c-1(2)}, \dots, n_{2d} \in X_{c-1(2d)}}$$

be the Wilson basis analysis operator. The formal adjoint of \tilde{C}_ψ is

$$\tilde{D}_\psi \lambda = \sum_{n_1 \in X_{c-1(1)}, \dots, n_{2d} \in X_{c-1(2d)}} \lambda_{n_1, \dots, n_{2d}} \Psi_{n_{c(1)}, \dots, n_{c(2d)}}.$$

For submultiplicative weights $v : \mathbb{R}^{2d} \rightarrow (0, \infty)$ define a weight v' as follows. For each $t \in \mathbb{R}$ let $v'(t) = \max\{v(t, 0, \dots, 0), v(0, t, 0, \dots, 0), \dots, v(0, \dots, 0, t)\}$.

Proposition 4.3.21. *Assume $p_1, \dots, p_{2d} \in [1, \infty)$. If $\psi \in M_{v' \otimes v'}^{1,1}(\mathbb{R})$ then*

$$\tilde{C}_\psi : M(c)_v^{p_1, \dots, p_{2d}} \rightarrow \ell_v^{p_1, \dots, p_{2d}}(X_{c-1(1)}, \dots, X_{c-1(2d)})$$

and

$$\tilde{D}_\psi : \ell_v^{p_1, \dots, p_{2d}}(X_{c-1(1)}, \dots, X_{c-1(2d)}) \rightarrow M(c)_v^{p_1, \dots, p_{2d}}$$

are bounded linear operators.

Proof. Since $\tilde{C}_\psi, \tilde{D}_\psi$ are adjoints, it suffices to prove that

$$\tilde{D}_\psi : \ell_v^{p_1, \dots, p_{2d}}(X_{c-1(1)}, \dots, X_{c-1(2d)}) \rightarrow M(c)_v^{p_1, \dots, p_{2d}}$$

is bounded. Let $\Psi(t) = \Psi_{0,0}(t) = \psi(t_1)\psi(t_2)\cdots\psi(t_d)$. Since $\psi \in M_{v' \otimes v'}^{1,1}(\mathbb{R})$ we see $\Psi \in M_{v' \otimes v' \otimes \dots \otimes v' \otimes v'}^{1,1}(\mathbb{R}^d)$. Because $v(t) \leq v'(t_1)\cdots v'(t_{2d})$ we have $M_{v' \otimes v' \otimes \dots \otimes v' \otimes v'}^{1,1}(\mathbb{R}^d) = M(c)_{v' \otimes v' \otimes \dots \otimes v'}^{1,1, \dots, 1} \subset M(c)_v^{1,1, \dots, 1}$. Hence $\Psi \in M(c)_v^{1,1, \dots, 1}$.

Let $\{\Lambda_1, \Lambda_2, \dots, \Lambda_{2d}\}$ be the set of $d \times d$ diagonal matrices with diagonal entries in $\{0, 1\}$ and let I denote the $d \times d$ identity matrix. Let M_i be the $2d \times 2d$ block diagonal matrix with first entry $I - \frac{1}{2}\Lambda_i$ and second entry I . Let

$$S_i = \{(n_{c(1)}, \dots, n_{c(2d)}) \in \mathbb{Z}^d \times (\mathbb{Z}^+)^d : (\Lambda_i - I)(n_{c(d+1)}, \dots, n_{c(2d)}) = 0\}.$$

Then $\{S_1, \dots, S_{2d}\}$ partitions $\mathbb{Z}^d \times (\mathbb{Z}^+)^d$ according to the presence and position of zeros of the last d coordinates. If $(n_{c(1)}, \dots, n_{c(2d)}) \in S_i$, then we have

$$\Psi_{n_{c(1)}, n_{c(2)}, \dots, n_{c(2d)}} = \sum_{\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d} c_{i, \epsilon} \pi_{M_i(n_{c(1)}, \dots, n_{c(d)}, \epsilon_1 n_{c(d+1)}, \dots, \epsilon_d n_{c(2d)})} \Psi,$$

where $c_{i, \epsilon}$ is a scalar satisfying $|c_{i, \epsilon}| = 2^{-d + \frac{\text{trace}(\Lambda_i)}{2}}$.

Fix $\lambda \in \ell_v^{p_1, \dots, p_{2d}}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(2d)})$. Then

$$\begin{aligned} & \tilde{D}_\phi \lambda \\ &= \sum_{i=1}^{2d} \sum_{(n_{c(1)}, \dots, n_{c(2d)}) \in S_i} \lambda_{n_1, \dots, n_{2d}} \Psi_{n_{c(1)}, \dots, n_{c(2d)}} \\ &= \sum_{i=1}^{2d} \sum_{(n_{c(1)}, \dots, n_{c(2d)}) \in S_i} \sum_{\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d} \lambda_{n_1, \dots, n_{2d}} c_{i, \epsilon} \pi_{M_i(n_{c(1)}, \dots, n_{c(d)}, \epsilon_1 n_{c(d+1)}, \dots, \epsilon_d n_{c(2d)})} \Psi \end{aligned}$$

For each $1 \leq i \leq 2d$ and $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d$, define a sequence $\tilde{\lambda}_{i, \epsilon}$ by

$$\begin{aligned} & (\tilde{\lambda}_{i, \epsilon})_{n_1, \dots, n_d, \beta_1 n_{d+1}, \dots, \beta_d n_{2d}} \\ &= \begin{cases} \lambda_{n_1, \dots, n_{2d}} c_{i, \epsilon} & \text{if } (n_{c(1)}, \dots, n_{c(2d)}) \in S_i \text{ and } (\beta_1, \dots, \beta_d) = (\epsilon_1, \dots, \epsilon_d), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $(n_1, \dots, n_{2d}) \in \mathbb{Z}^d \times (\mathbb{Z}^+)^d$ and $(\beta_1, \dots, \beta_d) \in \{-1, 1\}^d$. Then

$$\left\| \tilde{\lambda}_{i, \epsilon} \right\|_{\ell_v^{p_1, \dots, p_{2d}}} \leq 2^{-d + \frac{\text{trace}(\Lambda_i)}{2}} \|\lambda\|_{\ell_v^{p_1, \dots, p_{2d}}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(2d)})}$$

for all i, ϵ and we have

$$\tilde{D}_\phi \lambda$$

$$\begin{aligned}
&= \sum_{i=1}^{2^d} \sum_{\epsilon=(\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d} \sum_{n_1, \dots, n_{2d} \in \mathbb{Z}} (\tilde{\lambda}_{i, \epsilon})_{n_1, \dots, n_d, n_{d+1}, \dots, n_{2d}} \pi_{M_i(n_{c(1)}, \dots, n_{c(d)}, n_{c(d+1)}, \dots, n_{c(2d)})} \Psi \\
&= \sum_{i=1}^{2^d} \sum_{\epsilon=(\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d} \sum_{n \in \mathbb{Z}^{2d}} (\tilde{\lambda}_{i, \epsilon})_n \pi_{M_i c(n)} \Psi.
\end{aligned}$$

But

$$\pi_{M_i c(n)} \Psi = \pi_{c(\alpha(i).n)} \Psi$$

for $\alpha(i) = (\alpha(i)_1, \alpha(i)_2, \dots, \alpha(i)_{2d})$ where $\alpha(i)_{d+1} = \dots = \alpha(i)_{2d} = 1$ and $\alpha(i)_1, \dots, \alpha(i)_d \in \{1, 2\}$.

By Corollary 4.3.18, we have

$$\tilde{D}_\psi \lambda = \sum_{i=1}^{2^d} \sum_{\epsilon=(\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d} D_{(i, \epsilon), \psi} \tilde{\lambda}_{i, \epsilon}$$

and each $D_{(i, \epsilon), \psi} : \ell_w^{p_1, \dots, p_{2d}} \rightarrow M(c)_w^{p_1, \dots, p_{2d}}$ is a bounded linear operator. Thus

$$\begin{aligned}
&\left\| \tilde{D}_\psi \lambda \right\|_{M(c)_w^{p_1, \dots, p_{2d}}} \\
&\leq \sum_{i=1}^{2^d} \sum_{\epsilon=(\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d} \left\| D_{(i, \epsilon), \psi} \tilde{\lambda}_{i, \epsilon} \right\|_{M(c)_w^{p_1, \dots, p_{2d}}} \\
&\leq \sum_{i=1}^{2^d} \sum_{\epsilon=(\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d} \left\| D_{(i, \epsilon), \psi} \right\| 2^{-d + \frac{\text{trace}(\Lambda_i)}{2}} \|\lambda\|_{\ell_v^{p_1, \dots, p_{2d}}(X_{c-1(1)}, \dots, X_{c-1(2d)})} \\
&= \left(\sum_{i=1}^{2^d} \sum_{\epsilon=(\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d} \left\| D_{(i, \epsilon), \psi} \right\| 2^{-d + \frac{\text{trace}(\Lambda_i)}{2}} \right) \|\lambda\|_{\ell_v^{p_1, \dots, p_{2d}}(X_{c-1(1)}, \dots, X_{c-1(2d)})}. \quad \square
\end{aligned}$$

Corollary 4.3.22. *Assume $p_1, \dots, p_{2d} \in [1, \infty)$. If $\psi \in M_{v' \otimes v'}^{1,1}(\mathbb{R})$ then*

$$\tilde{D}_\psi d = \sum_{n_1 \in X_{c-1(1)}, \dots, n_{2d} \in X_{c-1(2d)}} d_{n_1, \dots, n_{2d}} \Psi_{n_{c(1)}, \dots, n_{c(2d)}}$$

converges unconditionally in $M(c)_v^{p_1, \dots, p_{2d}}$.

Proof. The proof is similar to that of Corollary 4.3.18. □

The following theorem states that Wilson bases are bases for the mixed modulation spaces.

Theorem 4.3.23. *Let $v : \mathbb{R}^{2d} \rightarrow (0, \infty)$ be a weight and w a v -moderate weight. For each $t \in \mathbb{R}$ define $v'(t) = \max\{v(t, 0, \dots, 0), v(0, t, 0, \dots, 0), \dots, v(0, \dots, 0, t)\}$. Assume $\psi \in M_{v' \otimes v'}^{1,1}(\mathbb{R})$ generates an orthonormal Wilson basis $\{\Psi_{k,n}\}_{n \in (\mathbb{Z}^+)^d, k \in \mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$. Then $\{\Psi_{k,n}\}_{n \in (\mathbb{Z}^+)^d, k \in \mathbb{Z}^d}$ is an unconditional basis for $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ for each $p_1, p_2, \dots, p_{2d} \in [1, \infty)$.*

Proof. Since $\{\Psi_{k,n}\}_{k \in \mathbb{Z}^d, n \in (\mathbb{Z}^+)^d}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, we have $f = \tilde{D}_\psi \tilde{C}_\psi f$ for all $f \in \mathcal{S}(\mathbb{R}^d)$. By the density of $\mathcal{S}(\mathbb{R}^d)$ in $M(c)_w^{p_1, \dots, p_{2d}}$ and the boundedness of $\tilde{D}_\psi, \tilde{C}_\psi$ we have $f = \tilde{D}_\psi \tilde{C}_\psi f$ for all $f \in M(c)_w^{p_1, \dots, p_{2d}}$.

Hence, given $f \in M(c)_w^{p_1, \dots, p_{2d}}$ and $\varepsilon > 0$, we can choose a finite set S_0 so that $\|\tilde{C}_\psi f - (\tilde{C}_\psi f) \chi_{S_0}\|_{\ell_v^{p_1, \dots, p_{2d}}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(2d)})} < \varepsilon$. Then

$$\left\| f - \sum_{(n,k) \in S_0} \langle f, \Phi_{n,k} \rangle \Phi_{n,k} \right\|_{M(c)_w^{p_1, \dots, p_{2d}}} < \|\tilde{D}_\psi\| \varepsilon.$$

It follows that $\{\Psi_{k,n}\}_{k \in \mathbb{Z}^d, n \in (\mathbb{Z}^+)^d}$ is complete in $M(c)_w^{p_1, \dots, p_{2d}}$.

Also, if $\lambda \in \ell_v^{p_1, \dots, p_{2d}}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(2d)})$, then $\lambda = \tilde{C}_\psi \tilde{D}_\psi \lambda$ and for any $S_0 \subset S \subset \mathbb{Z}^d \times (\mathbb{Z}^+)^d$ we have

$$\begin{aligned} \left\| \sum_{(k,n) \in S} \mu_{k,n} \lambda_{k,n} \Psi_{k,n} \right\|_{M(c)_w^{p_1, \dots, p_{2d}}} &= \|\tilde{D}_\psi\| \|\mu \lambda \chi_S\|_{\ell_v^{p_1, \dots, p_{2d}}} \\ &\leq \|\tilde{D}_\psi\| \|\mu\|_{\ell^\infty} \|\lambda \chi_S\|_{\ell_v^{p_1, \dots, p_{2d}}} \\ &\leq \|\tilde{D}_\psi\| \|\mu\|_{\ell^\infty} \|\tilde{C}_\psi\| \|\tilde{D}_\psi(\lambda \chi_S)\|_{M(c)_w^{p_1, \dots, p_{2d}}} \\ &= \|\tilde{D}_\psi\| \|\mu\|_{\ell^\infty} \|\tilde{C}_\psi\| \left\| \sum_{(k,n) \in S} \lambda_{k,n} \Psi_{k,n} \right\|_{M(c)_w^{p_1, \dots, p_{2d}}}. \end{aligned}$$

The result follows. \square

Corollary 4.3.24. *Assume $\psi \in M_{v' \otimes v'}^{1,1}(\mathbb{R})$ generates an orthonormal Wilson basis. Let $X_1 = X_2 = \dots = X_d = \mathbb{Z}$ and $X_{d+1} = X_{d+2} = \dots = X_{2d} = \mathbb{Z}^+$. Then $M(c)_w^{p_1, p_2, \dots, p_{2d}}$ is isomorphic to $\ell_w^{p_1, p_2, \dots, p_{2d}}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(2d)})$ via the map \tilde{C}_ψ .*

4.4 Bounded Pseudodifferential Operators

The proof technique used for Theorem 4.1.2 can be extended to results involving mixed modulation spaces defined by the following types of permutations.

Definition 4.4.1. A switch permutation c is one that satisfies

- (a) c maps $\{2d + 1, 2d + 2, \dots, 3d, 3d + 1, 3d + 2, \dots, 4d\}$ to $\{1, 2, \dots, 2d\}$ bijectively and
- (b) c maps $\{1, 2, \dots, d, d + 1, d + 2, \dots, 2d\}$ to $\{2d + 1, \dots, 4d\}$ bijectively.

A first slice permutation c is one that satisfies

- (a) c maps $\{1, 2, \dots, d, 2d + 1, 2d + 2, \dots, 3d\}$ to $\{1, 2, \dots, 2d\}$ bijectively and
- (b) c maps $\{d + 1, d + 2, \dots, 2d, 3d + 1, 3d + 2, \dots, 4d\}$ to $\{2d + 1, \dots, 4d\}$ bijectively.

A second slice permutation c is one that satisfies

- (a) c maps $\{d + 1, d + 2, \dots, 2d, 3d + 1, 3d + 2, \dots, 4d\}$ to $\{1, 2, \dots, 2d\}$ bijectively and
- (b) c maps $\{1, 2, \dots, d, 2d + 1, 2d + 2, \dots, 3d\}$ to $\{2d + 1, \dots, 4d\}$ bijectively.

Theorem 4.4.2. Suppose A is a pseudodifferential operator with Weyl symbol σ . Suppose c_s is a switch permutation, s_1 is a first-slice permutation and s_2 is a second-slice permutation. Let $q_1 = \dots = q_{2d} = \infty$ and $q_{2d+1} = \dots = q_{4d} = 1$. If

- (a) $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$,
- (b) $\sigma \in M(c_s)^{q_1, q_2, \dots, q_{4d}}$,
- (c) $\sigma \in M(s_1)^{q_1, q_2, \dots, q_{4d}}$, or
- (d) $\sigma \in M(s_2)^{q_1, q_2, \dots, q_{4d}}$,

then $A : M(c)^{p_1, \dots, p_{2d}} \rightarrow M(c)^{p_1, \dots, p_{2d}}$ is bounded for all $p_1, \dots, p_{2d} \in (1, \infty)$ and all permutations c .

Proof. By Proposition 14.3.3 in [33] we can write $\langle L_\sigma f, g \rangle = \langle \sigma, W(g, f) \rangle$ for all $f, g \in \mathcal{S}(\mathbb{R}^d)$, where W denotes the Wigner distribution. Also, by Lemma 14.5.1 in [33] we have

$$|V_{W(\phi, \phi)}(W(g, f))(z_1, z_2, \zeta_1, \zeta_2)| = \left| V_\phi f \left(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2} \right) \right| \left| V_\phi g \left(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2} \right) \right|$$

for all $z_1, z_2, \zeta_1, \zeta_2 \in \mathbb{R}^d$. Hence for $\phi \in L^2(\mathbb{R}^d)$ with $\|\phi\|_{L^2} = 1$ we have for all $f, g \in \mathcal{S}(\mathbb{R}^d)$ that

$$\begin{aligned} & |\langle L_\sigma f, g \rangle| \\ & \leq \iiint \int |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| \\ & \quad \left| V_\phi f \left(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2} \right) \right| \left| V_\phi g \left(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2} \right) \right| dz_1 dz_2 d\zeta_1 d\zeta_2. \end{aligned} \quad (25)$$

To prove the sufficiency of (a), notice that (25) satisfies

$$\begin{aligned} & |\langle L_\sigma f, g \rangle| \\ & \leq \iint \sup_{z_1, z_2 \in \mathbb{R}^d} |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| d\zeta_1 d\zeta_2 \\ & \quad \times \sup_{\zeta_1, \zeta_2 \in \mathbb{R}^d} \int \left| V_\phi f \left(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2} \right) \right| \left| V_\phi g \left(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2} \right) \right| dz_1 dz_2 \\ & = \sup_{\zeta_1, \zeta_2 \in \mathbb{R}^d} \|f\|_{M(c)^{p_1, \dots, p_{2d}}} \|g\|_{M(c)^{p'_1, \dots, p'_{2d}}} \iint \sup_{z_1, z_2 \in \mathbb{R}^d} |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| d\zeta_1 d\zeta_2 \\ & \leq \|f\|_{M(c)^{p_1, \dots, p_{2d}}} \|g\|_{M(c)^{p'_1, \dots, p'_{2d}}} \iint \sup_{z_1, z_2 \in \mathbb{R}^d} |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| d\zeta_1 d\zeta_2 \\ & \equiv \|\sigma\|_{M^{\infty, 1}} \|f\|_{M(c)^{p_1, \dots, p_{2d}}} \|g\|_{M(c)^{p'_1, \dots, p'_{2d}}}. \end{aligned}$$

Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $M(c)^{p_1, \dots, p_{2d}}$ and $M(c)^{p'_1, \dots, p'_{2d}}$, (a) implies that

$A : M(c)^{p_1, \dots, p_{2d}} \rightarrow M(c)^{p_1, \dots, p_{2d}}$ is bounded.

Suppose (b) holds. Then (25) implies that

$$|\langle L_\sigma f, g \rangle|$$

$$\begin{aligned}
&\leq \int \sup_{\zeta_1, \zeta_2 \in \mathbb{R}^d} |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| \, dz_1 \, dz_2 \\
&\quad \times \sup_{z_1, z_2 \in \mathbb{R}^d} \int \left| V_\phi f \left(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2} \right) \right| \left| V_\phi g \left(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2} \right) \right| \, d\zeta_1 \, d\zeta_2 \\
&\leq 2^{2d} \sup_{z_1, z_2 \in \mathbb{R}^d} \|f\|_{M(c)^{p_1, \dots, p_{2d}}} \|g\|_{M(c)^{p'_1, \dots, p'_{2d}}} \int \sup_{\zeta_1, \zeta_2 \in \mathbb{R}^d} |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| \, dz_1 \, dz_2 \\
&\equiv 2^{2d} \|\sigma\|_{M(c_s)^{\infty, 1}} \|f\|_{M(c)^{p_1, \dots, p_{2d}}} \|g\|_{M(c)^{p'_1, \dots, p'_{2d}}}.
\end{aligned}$$

Again the density of $\mathcal{S}(\mathbb{R}^d)$ in the mixed modulation spaces implies $A : M(c)^{p_1, \dots, p_{2d}} \rightarrow M(c)^{p_1, \dots, p_{2d}}$ is bounded.

Suppose (c) holds. By (25) we have

$$\begin{aligned}
&|\langle L_\sigma f, g \rangle| \\
&\leq \iint \sup_{z_1, \zeta_1 \in \mathbb{R}^d} |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| \, dz_2 \, d\zeta_2 \\
&\quad \times \sup_{z_2, \zeta_2 \in \mathbb{R}^{2d}} \iint \left| V_\phi f \left(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2} \right) \right| \left| V_\phi g \left(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2} \right) \right| \, dz_1 \, d\zeta_1 \\
&\leq 2^d \sup_{z_2, \zeta_2 \in \mathbb{R}^{2d}} \|f\|_{M(c)^{p_1, \dots, p_{2d}}} \|g\|_{M(c)^{p'_1, \dots, p'_{2d}}} \iint \sup_{z_1, \zeta_1 \in \mathbb{R}^d} |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| \, dz_2 \, d\zeta_2 \\
&\equiv 2^d \|\sigma\|_{M(s_1)^{\infty, 1}} \|f\|_{M(c)^{p_1, \dots, p_{2d}}} \|g\|_{M(c)^{p'_1, \dots, p'_{2d}}}.
\end{aligned}$$

Again the density of $\mathcal{S}(\mathbb{R}^d)$ in the mixed modulation spaces implies $A : M(c)^{p_1, \dots, p_{2d}} \rightarrow M(c)^{p_1, \dots, p_{2d}}$ is bounded.

Assume (d) holds. Then from (25), we have

$$\begin{aligned}
&|\langle L_\sigma f, g \rangle| \\
&\leq \iint \sup_{z_2, \zeta_2 \in \mathbb{R}^d} |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| \, dz_1 \, d\zeta_1 \\
&\quad \times \sup_{z_1, \zeta_1 \in \mathbb{R}^{2d}} \iint \left| V_\phi f \left(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2} \right) \right| \left| V_\phi g \left(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2} \right) \right| \, dz_2 \, d\zeta_2 \\
&\leq 2^d \sup_{z_1, \zeta_1 \in \mathbb{R}^{2d}} \|f\|_{M(c)^{p_1, \dots, p_{2d}}} \|g\|_{M(c)^{p'_1, \dots, p'_{2d}}} \iint \sup_{z_2, \zeta_2 \in \mathbb{R}^d} |V_{W(\phi, \phi)} \sigma(z_1, z_2, \zeta_1, \zeta_2)| \, dz_1 \, d\zeta_1 \\
&\equiv 2^d \|\sigma\|_{M(s_2)^{\infty, 1}} \|f\|_{M(c)^{p_1, \dots, p_{2d}}} \|g\|_{M(c)^{p'_1, \dots, p'_{2d}}}.
\end{aligned}$$

Again the density of $\mathcal{S}(\mathbb{R}^d)$ in the mixed modulation spaces implies $A : M(c)^{p_1, \dots, p_{2d}} \rightarrow M(c)^{p_1, \dots, p_{2d}}$ is bounded. \square

4.5 Pseudodifferential Operators and Schatten classes

4.5.1 New Kernel and Symbol classes

In this section we will use Theorem 3.3.2 and its corollaries to find conditions on the kernel and Kohn-Nirenberg symbol of a pseudodifferential operator that guarantee the operator is Schatten p -class.

Theorem 4.5.1. *Let c be a slice permutation and $2 = p_1 = p_2 = \dots = p_{2d}$ and $p = p_{2d+1} = \dots = p_{4d}$. If $p \in [1, 2]$, $k \in M(c)^{p_1, \dots, p_{4d}}$ and A is an integral operator with kernel k , then $A \in \mathcal{I}_p(L^2(\mathbb{R}^d))$.*

Proof. We first prove the theorem in the case c is a second slice permutation. Let $\{\pi_{\alpha m} \phi\}_{m \in \mathbb{Z}^{2d}} = \{\phi_m\}_{m \in \mathbb{Z}^{2d}}$ be a Parseval Gabor frame for $L^2(\mathbb{R}^d)$ with $\phi \in M^{1,1}(\mathbb{R}^d)$ and let $\Phi(t, y) = \phi(t)\phi(y)$. Then $\Phi \in M^{1,1}(\mathbb{R}^{2d})$.

By the proof of Corollary 3.3.7, we have

$$\|A\|_{\mathcal{I}_p(L^2(\mathbb{R}^d))} \leq \left(\sum_{n \in \mathbb{Z}^{2d}} \left(\sum_{m \in \mathbb{Z}^{2d}} |\langle k, \phi_n \otimes \phi_m \rangle|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

For $m_1, m_2, n_1, n_2 \in \mathbb{Z}^d$, with $m = (m_1, m_2)$ and $n = (n_1, n_2)$, we have

$$\langle k, \phi_n \otimes \phi_m \rangle = V_{\Phi} k(\alpha n_1, \alpha m_1, \alpha n_2, \alpha m_2).$$

Since c is a second slice permutation, we see that

$$\begin{aligned} & \left(\sum_{n \in \mathbb{Z}^{2d}} \left(\sum_{m \in \mathbb{Z}^{2d}} |\langle k, \phi_n \otimes \phi_m \rangle|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &= \left(\sum_{n_1, n_2 \in \mathbb{Z}^d} \left(\sum_{m_1, m_2 \in \mathbb{Z}^d} |V_{\Phi} k(\alpha n_1, \alpha m_1, \alpha n_2, \alpha m_2)|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &\leq B \|k\|_{M(c)^{p_1, p_2, \dots, p_{4d}}}, \end{aligned}$$

where B is the constant ensured by Theorem 4.3.19(a). Hence if $k \in M(c)^{p_1, \dots, p_{4d}}$, then $A \in \mathcal{I}_p(L^2(\mathbb{R}^d))$.

Now suppose c is a first slice permutation and $k \in M(c)^{p_1, \dots, p_{4d}}$. Let \tilde{k} be the kernel of A^* . Then $\tilde{k} \in M(c')^{p_1, \dots, p_{4d}}$, where c' is the second slice permutation given by

$$\begin{aligned} c'(1) &= c(d+1), c'(2) = c(d+2), \dots, c'(d) = c(2d), \\ c'(d+1) &= c(1), c'(d+2) = c(2), \dots, c'(2d) = c(d), \\ c'(2d+1) &= c(3d+1), c'(2d+2) = c(3d+2), \dots, c'(3d) = c(4d), \end{aligned}$$

and

$$c'(3d+1) = c(2d+1), c'(3d+2) = c(2d+2), \dots, c'(4d) = c(3d).$$

Hence, $A^* \in \mathcal{I}_p(L^2(\mathbb{R}^d))$. But $\|A\|_{\mathcal{I}_p} = \|A^*\|_{\mathcal{I}_p}$. \square

By Lemma 4.3.20, increasing any one of the exponent parameters p_1, \dots, p_{4d} or decreasing the weight parameter s yields a mixed modulation space larger than $M(c)_{v_s}^{p_1, p_2, \dots, p_{4d}}$. The next theorem shows Theorem 4.5.1 is sharp in the following sense: increasing the exponent parameters or decreasing the weight parameter of the mixed modulation space in Theorem 4.5.1 gives a larger mixed modulation space, but integral operators with kernels in this larger space need not be Schatten class.

Theorem 4.5.2. *Assume $s \leq 0$, $q_1, \dots, q_{2d} \in [2, \infty]$, $q_{2d+1}, \dots, q_{4d} \in [p, \infty]$ and c is a slice permutation. Assume at least one of the following is true:*

- (a) $s < 0$.
- (b) At least one of q_1, \dots, q_{2d} is larger than 2.
- (c) At least one of q_{2d+1}, \dots, q_{4d} is larger than p .

If $1 \leq p \leq 2$ then there are integral operators with kernels in $M(c)_{v_s}^{q_1, q_2, \dots, q_{4d}}$ that are not in $\mathcal{I}_p(L^2(\mathbb{R}^d))$.

Proof. To avoid complicated notation, we prove the theorem only for the permutation

$$c(1) = d + 1, c(2) = d + 2, \dots, c(d) = 2d,$$

$$c(d + 1) = 3d + 1, \dots, c(2d) = 4d,$$

$$c(2d + 1) = 1, \dots, c(3d) = d$$

and

$$c(3d + 1) = 2d + 1, \dots, c(4d) = 3d.$$

The result is proven similarly for other slice permutations.

In the case that (a) or (b) holds, we can adapt some of the arguments in [36] to complete the proof. In particular, if $k(t, y) = k_1(t)k_2(y)$ is the kernel of an integral operator A , then $Af = \langle f, \overline{k_2} \rangle k_1$. Hence if $k_1 \notin L^2(\mathbb{R}^d)$, then A does not map into $L^2(\mathbb{R}^d)$, and if $k_2 \notin L^2(\mathbb{R}^d)$, then $A : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is not bounded. Let c' be the permutation of $\{1, 2, \dots, 2d\}$ with

$$c'(1) = d + 1, \dots, c'(d) = 2d$$

and

$$c'(d + 1) = 1, \dots, c'(2d) = d.$$

If (a) holds, choose $k_1 \in M_{v_s}^{2,2}(\mathbb{R}^d) \setminus L^2(\mathbb{R}^d)$ and $k_2 \in M^{p,p}(\mathbb{R}^d)$. If (b) holds, choose $k_1 \in M(c')^{q_1, \dots, q_{2d}} \setminus L^2(\mathbb{R}^d)$ and $k_2 \in M(c')^{q_{2d+1}, \dots, q_{4d}}$. In either case $k(t, y) = k_1(t)k_2(y) \in M(c)_{v_s}^{q_1, q_2, \dots, q_{4d}}$, but the integral operator with kernel k is not a bounded operator on $L^2(\mathbb{R}^d)$.

Hence we assume (c) is true. Choose

$$\lambda \in \ell^{q_{2d+1}, \dots, q_{3d}, q_{3d+1}, \dots, q_{4d}}((\mathbb{Z}^+)^d, \mathbb{Z}^d) \setminus \ell^{p,p}((\mathbb{Z}^+)^d, \mathbb{Z}^d).$$

Assume $\{\psi_{j,l}\}_{j \in \mathbb{Z}^d, l \in (\mathbb{Z}^+)^d}$ is a Wilson basis for $L^2(\mathbb{R}^d)$ generated by $\psi \in M^{1,1}(\mathbb{R})$.

Then

$$\{\Psi_{(j_1, j_2), (l_1, l_2)}\}_{j_1, j_2 \in \mathbb{Z}^d, l_1, l_2 \in (\mathbb{Z}^+)^d} = \{\psi_{j_1, l_1} \otimes \psi_{j_2, l_2}\}_{j_1, j_2 \in \mathbb{Z}^d, l_1, l_2 \in (\mathbb{Z}^+)^d}$$

is a Wilson basis for $L^2(\mathbb{R}^{2d})$ generated by $\psi \in M^{1,1}(\mathbb{R})$. Set

$$k(t, y) = \sum_{j \in \mathbb{Z}^d} \sum_{l \in (\mathbb{Z}^+)^d} \lambda_{l,j} \psi_{j,l}(t) \psi_{j,l}(y).$$

Then

$$\begin{aligned} & \Psi_{(n_{c(1)}, n_{c(2)}, \dots, n_{c(2d)}, (n_{c(2d+1)}, \dots, n_{c(4d)})} \\ &= \Psi_{(n_{d+1}, \dots, n_{2d}, n_{3d+1}, \dots, n_{4d}), (n_1, \dots, n_d, n_{2d+1}, \dots, n_{3d})} \\ &= \psi_{(n_{d+1}, \dots, n_{2d}), (n_1, \dots, n_d)} \otimes \psi_{(n_{3d+1}, \dots, n_{4d}), (n_{2d+1}, \dots, n_{3d})}. \end{aligned}$$

By Corollary 4.3.24

$$\begin{aligned} & \|k\|_{M(c)_{v_s}^{q_1, q_2, \dots, q_{4d}}} \\ & \equiv \left(\sum_{n_{4d} \in X_{c^{-1}(4d)}} \dots \left(\sum_{n_1 \in X_{c^{-1}(1)}} \left| \langle k, \Psi_{(n_{c(1)}, \dots, n_{c(2d)}, (n_{c(2d+1)}, \dots, n_{c(4d)})} \rangle \right|^{q_1} \right)^{\frac{q_2}{q_1}} \dots \right)^{\frac{1}{q_{4d}}} \\ & = \left(\sum_{n_{4d} \in \mathbb{Z}} \left(\dots \left(\sum_{n_{2d+1} \in \mathbb{Z}^+} \left| \lambda_{(n_{2d+1}, \dots, n_{3d}), (n_{3d+1}, \dots, n_{4d})} \right|^{q_{2d+1}} \right)^{\frac{q_{2d+2}}{q_{2d+1}}} \dots \right)^{\frac{q_{4d}}{q_{4d-1}}} \right)^{\frac{1}{q_{4d}}} \\ & = \|\lambda\|_{\ell^{q_{2d+1}, \dots, q_{3d}, q_{3d+1}, \dots, q_{4d}}((\mathbb{Z}^+)^d, \mathbb{Z}^d)} \end{aligned}$$

so $k \in M(c)_{v_s}^{q_1, q_2, \dots, q_{4d}} \subset M(c)^{q_1, q_2, \dots, q_{4d}}$. The pseudodifferential operator A with kernel k has singular values equal to the elements of the sequence λ . Hence $A \notin \mathcal{I}_p(L^2(\mathbb{R}^d))$. \square

Notice that the proof of the previous theorem shows that Theorem 4.5.1 does not hold for $p > 2$. That is, if $p > 2$ and $k \in M(c)^{2, 2, \dots, 2, p, \dots, p}$, the corresponding integral operator may not even be bounded on $L^2(\mathbb{R}^d)$.

We can extend Theorem 4.5.1 to conditions on the symbol of a pseudodifferential operator.

Theorem 4.5.3. *Assume $1 \leq p \leq 2$ and c_1, c_2 are permutations on \mathbb{R}^{4d} satisfying the following conditions.*

(a) c_1 maps $\{2d+1, 2d+2, \dots, 3d\}$ to $\{1, 2, \dots, d\}$ and maps $\{1, 2, \dots, 2d, 3d+1, 3d+2, \dots, 4d\}$ to $\{d+1, d+2, \dots, 4d\}$.

(b) c_2 maps $\{3d+1, 3d+2, \dots, 4d\}$ to $\{1, 2, \dots, d\}$ and maps $\{1, 2, \dots, 3d\}$ to $\{d+1, d+2, \dots, 4d\}$.

Suppose A is a pseudodifferential operator with Kohn-Nirenberg symbol τ . Let $2 = p_1 = \dots = p_{2d}$ and $p = p_{2d+1} = \dots = p_{4d}$. If $\tau \in M(c_1)^{p_1, p_2, \dots, p_{4d}}$ or $\tau \in M(c_2)^{p_1, p_2, \dots, p_{4d}}$ then $A \in \mathcal{I}_p(L^2(\mathbb{R}^d))$.

Proof. Let k be the kernel of A , and let $2 = q_1 = \dots = q_d$ and $p = q_{d+1} = \dots = q_{4d}$. Let s_1 be a first slice permutation and s_2 be a second slice permutation. By Lemma 4.3.20 and its proof we have for any permutation c , $M(c)^{p_1, \dots, p_{4d}} \subset M(c)^{q_1, \dots, q_{4d}}$ and $\|\tau\|_{M(c)^{q_1, \dots, q_{4d}}} \leq C \|\tau\|_{M(c)^{p_1, \dots, p_{4d}}}$ for some finite C . Hence, using Corollary 4.2.3, we have

$$\begin{aligned} \|k\|_{M(s_1)^{q_1, \dots, q_{4d}}} &= \left(\iint \left(\iint |V_{\Phi} k(a, b, c, d)|^2 da dc \right)^{\frac{p}{2}} db dd \right)^{\frac{1}{p}} \\ &= \left(\iint \left(\iint |V_{\mathcal{F}_2(\Phi \circ N)} \tau(a, -d, c+d, b-a)|^2 da dc \right)^{\frac{p}{2}} db dd \right)^{\frac{1}{p}} \\ &\leq C \left(\iiint \left(\int |V_{\mathcal{F}_2(\Phi \circ N)} \tau(a, -d, c+d, b-a)|^2 dc \right)^{\frac{p}{2}} da db dd \right)^{\frac{1}{p}} \\ &= C \left(\iiint \left(\int |V_{\mathcal{F}_2(\Phi \circ N)} \tau(a, d, c, b)|^2 dc \right)^{\frac{p}{2}} da db dd \right)^{\frac{1}{p}} \\ &\equiv \|\tau\|_{M(c_1)^{p_1, \dots, p_{4d}}} \end{aligned}$$

and

$$\begin{aligned} \|k\|_{M(s_2)^{q_1, \dots, q_{4d}}} &= \left(\iint \left(\iint |V_{\Phi} k(a, b, c, d)|^2 db dd \right)^{\frac{p}{2}} da dc \right)^{\frac{1}{p}} \\ &= \left(\iint \left(\iint |V_{\mathcal{F}_2(\Phi \circ N)} \tau(a, -d, c+d, b-a)|^2 db dd \right)^{\frac{p}{2}} da dc \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\iiint \left(\int |V_{\mathcal{F}_2(\Phi \circ N)} \tau(a, -d, c+d, b-a)|^2 db \right)^{\frac{p}{2}} da dc dd \right)^{\frac{1}{p}} \\
&= C \left(\iiint \left(\int |V_{\mathcal{F}_2(\Phi \circ N)} \tau(a, d, c, b)|^2 db \right)^{\frac{p}{2}} da dc dd \right)^{\frac{1}{p}} \\
&\equiv \|\tau\|_{M(c_2)^{p_1, \dots, p_{4d}}} .
\end{aligned}$$

The result now follows from Theorem 4.5.1. \square

4.5.2 Relationship between old and new kernel and symbol classes

In this section, we explain the relationship between Theorems 4.5.1 and 4.5.3 and the previously known results for Schatten class pseudodifferential operators. The most powerful previously known result for Schatten class pseudodifferential operators is Theorem 4.1.1. We will show that Theorem 4.5.1 is stronger than Theorem 4.1.1 as a kernel result. We will also show that Theorem 4.5.3 is neither stronger nor weaker than Theorem 4.1.1 as a Kohn-Nirenberg symbol result. Rather, it represents a distinct condition on the Kohn-Nirenberg symbol that ensures the corresponding operator is Schatten class.

Lemma 4.5.4. *If $s > \frac{d(2-p)}{p}$ and $s \geq 0$, then $\ell_{v_s}^{2,2}(\mathbb{Z}^{2d}, \mathbb{Z}^{2d}) \subsetneq \ell^{2,p}(\mathbb{Z}^{2d}, \mathbb{Z}^{2d})$ and there exists finite C such that*

$$\|c\|_{\ell^{2,p}} \leq C \|c\|_{\ell_{v_s}^{2,2}} \quad \forall c \in \ell_{v_s}^{2,2}(\mathbb{Z}^{2d}, \mathbb{Z}^{2d}).$$

Proof. In the case that $p \geq 2$ we have $\ell_{v_s}^{2,2} \subsetneq \ell^{2,2} \subsetneq \ell^{2,p}$ trivially.

Suppose $p \in [1, 2)$. Let $q = \frac{2}{p}$ so that $q \in (1, 2]$. Let $q' = \frac{2}{2-p}$ be the dual index of q . Then

$$\begin{aligned}
\|c\|_{\ell^{2,p}} &= \left(\sum_{k \in \mathbb{Z}^{2d}} \left(\sum_{j \in \mathbb{Z}^{2d}} |c_{j,k}|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\
&= \left(\sum_{k \in \mathbb{Z}^{2d}} \left(\sum_{j \in \mathbb{Z}^{2d}} |c_{j,k}|^2 \right)^{\frac{p}{2}} \frac{(1+|k|)^{sp}}{(1+|k|)^{sp}} \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{k \in \mathbb{Z}^{2d}} \left(\sum_{j \in \mathbb{Z}^{2d}} |c_{j,k}|^2 \right)^{\frac{pq}{2}} (1 + |k|)^{spq} \right)^{\frac{1}{pq}} \left(\sum_{k \in \mathbb{Z}^{2d}} \frac{1}{(1 + |k|)^{spq'}} \right)^{\frac{1}{pq'}} \\
&= \left(\sum_{k \in \mathbb{Z}^{2d}} \sum_{j \in \mathbb{Z}^{2d}} |c_{j,k}|^2 (1 + |k|)^{2s} \right)^{\frac{1}{2}} \left(\sum_{k \in \mathbb{Z}^{2d}} \frac{1}{(1 + |k|)^{\frac{2sp}{2-p}}} \right)^{\frac{2-p}{2p}} \\
&\leq \left(\sum_{k \in \mathbb{Z}^{2d}} \sum_{j \in \mathbb{Z}^{2d}} |c_{j,k}|^2 (1 + |(j, k)|)^{2s} \right)^{\frac{1}{2}} \left(\sum_{k \in \mathbb{Z}^{2d}} \frac{1}{(1 + |k|)^{\frac{2sp}{2-p}}} \right)^{\frac{2-p}{2p}} \\
&= \|c\|_{\ell_{vs}^{2,2}} \left(\sum_{k \in \mathbb{Z}^{2d}} \frac{1}{(1 + |k|)^{\frac{2sp}{2-p}}} \right)^{\frac{2-p}{2p}} \\
&< \infty,
\end{aligned}$$

as $s > \frac{d(2-p)}{p}$ implies $\frac{2sp}{2-p} > 2d$.

All that remains to be shown is that $\ell_{vs}^{2,2}(\mathbb{Z}^{2d}, \mathbb{Z}^{2d}) \neq \ell^{2,p}(\mathbb{Z}^{2d}, \mathbb{Z}^{2d})$. Since $s > \frac{d(2-p)}{p}$, we can choose $q \in \left(\frac{2d}{p}, d + s\right]$. Choose $f \in \mathcal{S}(\mathbb{R}^{2d})$ and set

$$c_{j,k} = \frac{f(j)}{(1 + |k|)^q} \quad \forall j, k \in \mathbb{Z}^{2d}.$$

Then

$$\begin{aligned}
\|c\|_{\ell_{vs}^{2,2}} &= \left(\sum_{j \in \mathbb{Z}^{2d}} \sum_{k \in \mathbb{Z}^{2d}} \frac{|f(j)|^2 (1 + |(j, k)|)^{2s}}{(1 + |k|)^{2q}} \right)^{\frac{1}{2}} \\
&\geq \left(\sum_{j \in \mathbb{Z}^{2d}} \sum_{k \in \mathbb{Z}^{2d}} \frac{|f(j)|^2}{(1 + |k|)^{2(q-s)}} \right)^{\frac{1}{2}} \\
&= \infty,
\end{aligned}$$

as $q \leq d + s$ implies $2(q - s) \leq 2d$. But

$$\begin{aligned}
\|c\|_{\ell^{2,p}} &= \left(\sum_{k \in \mathbb{Z}^{2d}} \frac{1}{(1 + |k|)^{qp}} \right)^{\frac{1}{p}} \left(\sum_{j \in \mathbb{Z}^{2d}} |f(j)|^2 \right)^{\frac{1}{2}} \\
&< \infty
\end{aligned}$$

as $\frac{2d}{p} < q$ implies $2d < qp$. □

Proposition 4.5.5. *Let c be a slice permutation and let $2 = p_1 = \cdots = p_{2d}$, $p = p_{2d+1} = \cdots = p_{4d}$. If $s > \frac{d(2-p)}{p}$ with $s \geq 0$, then $M_{v_s}^{2,2}(\mathbb{R}^{2d}) \subsetneq M(c)^{p_1, p_2, \dots, p_{4d}}$.*

Proof. Let $X_1 = X_2 = \cdots = X_{2d} = \mathbb{Z}$ and $X_{2d+1} = X_{2d+2} = \cdots = X_{4d} = \mathbb{Z}^+$. Since $M_{v_s}^{2,2}(\mathbb{R}^{2d}) = M(c)_{v_s}^{2,2, \dots, 2}$, Corollary 4.3.24 implies that we can find a map S so that

$$S : M_{v_s}^{2,2}(\mathbb{R}^{2d}) \rightarrow \ell_{v_s}^{2,2, \dots, 2}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(4d)})$$

and

$$S : M(c)^{p_1, p_2, \dots, p_{4d}} \rightarrow \ell^{p_1, p_2, \dots, p_{4d}}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(4d)})$$

are both isomorphisms. In particular we can choose $S = \tilde{C}_\psi$ for appropriate ψ .

Furthermore, since c is a slice permutation, we see that there is a map T by which $\ell_{v_s}^{2,2, \dots, 2}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(4d)})$ is isomorphic to $\ell_{v_s}^{2,2}(\mathbb{Z}^{2d}, \mathbb{Z}^{2d})$ and $\ell^{p_1, p_2, \dots, p_{4d}}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(4d)})$ is isomorphic to $\ell^{2,p}(\mathbb{Z}^{2d}, \mathbb{Z}^{2d})$. Hence, using the previous lemma, we obtain the following diagram.

$$\begin{array}{ccc} M_{v_s}^{2,2}(\mathbb{R}^{2d}) & & M(c)^{p_1, p_2, \dots, p_{4d}} \\ \downarrow S & & \downarrow S \\ \ell_{v_s}^{2,2, \dots, 2}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(4d)}) & & \ell^{p_1, p_2, \dots, p_{4d}}(X_{c^{-1}(1)}, \dots, X_{c^{-1}(4d)}) \\ \downarrow T & & \downarrow T \\ \ell_{v_s}^{2,2}(\mathbb{Z}^{2d}, \mathbb{Z}^{2d}) & \hookrightarrow & \ell^{2,p}(\mathbb{Z}^{2d}, \mathbb{Z}^{2d}). \end{array}$$

Since S, T are isomorphisms, the result follows. □

The next five results are intended to give context to Theorem 4.5.3.

Lemma 4.5.6. *If $s > \frac{3}{2} \frac{d(2-p)}{p}$ or $s = 0$ then*

$$\ell_{v_s}^{2,2,2,2}(\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d) \subsetneq \ell^{2,p,p,p}(\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d)$$

and there exists finite C such that

$$\|c\|_{\ell^{2,p,p,p}} \leq C \|c\|_{\ell_{v_s}^{2,2,2,2}} \quad \forall c \in \ell_{v_s}^{2,2,2,2}(\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d).$$

Proof. In the case that $p \geq 2$ or $s = 0$ we have $\ell_{v_s}^{2,2,2,2} \subset \ell^{2,2,2,2} \subsetneq \ell^{2,p,p,p}$ trivially.

Otherwise, let $q = \frac{2}{p}$. Then $q \geq 1$ and its dual index is $q' = \frac{2}{2-p}$. We have

$$\begin{aligned}
\|c\|_{\ell^{2,p,p,p}} &= \left(\sum_{k,l,m} \left(\sum_j |c_{j,k,l,m}|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\
&= \left(\sum_{k,l,m} \left(\sum_j |c_{j,k,l,m}|^2 \right)^{\frac{p}{2}} \frac{(1 + |(k,l,m)|)^{sp}}{(1 + |(k,l,m)|)^{sp}} \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{k,l,m} \left(\sum_j |c_{j,k,l,m}|^2 \right)^{\frac{pq}{2}} (1 + |(k,l,m)|)^{spq} \right)^{\frac{1}{qp}} \left(\sum_{k,l,m} \frac{1}{(1 + |(k,l,m)|)^{spq'}} \right)^{\frac{1}{pq'}} \\
&= \left(\sum_{j,k,l,m} |c_{j,k,l,m}|^2 (1 + |(k,l,m)|)^{2s} \right)^{\frac{1}{2}} \left(\sum_{k,l,m} \frac{1}{(1 + |(k,l,m)|)^{\frac{2sp}{2-p}}} \right)^{\frac{2-p}{2p}} \\
&\leq \|c\|_{\ell_{v_s}^{2,2,2,2}} \left(\sum_{k,l,m} \frac{1}{(1 + |(k,l,m)|)^{\frac{2sp}{2-p}}} \right)^{\frac{2-p}{2p}}
\end{aligned}$$

and this sum converges since $s > \frac{3}{2} \frac{d(2-p)}{p}$ implies $\frac{2sp}{2-p} > 3d$.

Furthermore since $s > \frac{3}{2} \frac{d(2-p)}{p}$, we can choose $q \in \left(\frac{3d}{p}, \frac{3d}{2} + s \right]$. Choose $f \in \mathcal{S}(\mathbb{R}^d)$ and set

$$c_{j,k,l,m} = \frac{f(j)}{(1 + |(k,l,m)|)^q} \quad \forall j, k, l, m \in \mathbb{Z}^d.$$

Then $c \in \ell^{2,p,p,p}(\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d) \setminus \ell_{v_s}^{2,2,2,2}(\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d)$. \square

Proposition 4.5.7. *Let c be a permutation on \mathbb{R}^{4d} satisfying one of the following conditions.*

(a) c maps $\{2d+1, 2d+2, \dots, 3d\}$ to $\{1, 2, \dots, d\}$ and maps

$\{1, 2, \dots, 2d, 3d+1, 3d+2, \dots, 4d\}$ to $\{d+1, d+2, \dots, 4d\}$.

(b) c maps $\{3d+1, 3d+2, \dots, 4d\}$ to $\{1, 2, \dots, d\}$ and maps $\{1, 2, \dots, 3d\}$ to

$\{d+1, d+2, \dots, 4d\}$.

Let $2 = p_1 = \dots = p_d$, $p = p_{d+1} = \dots = p_{4d}$. If $s > \frac{3}{2} \frac{d(2-p)}{p}$ or $s = 0$, then

$M_{v_s}^{2,2}(\mathbb{R}^{2d}) \subsetneq M(c)^{p_1, p_2, \dots, p_{4d}}$.

Proof. Using Lemma 4.5.6, this proposition can be proven like Proposition 4.5.5. \square

Lemma 4.5.8. *If $s > 0$ then*

$$\ell^{2,p,p,p}(\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d) \setminus \ell_{v_s}^{2,2,2,2}(\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d)$$

is nonempty.

Proof. Choose $q \in (\frac{d}{2}, \frac{d}{2} + s)$ and choose $f \in \mathcal{S}(\mathbb{R}^{3d})$. Set

$$c_{j,k,l,m} = \frac{1}{(1+|j|)^q} f(k, l, m) \quad \forall j, k, l, m \in \mathbb{Z}^d.$$

Then

$$\begin{aligned} \|c\|_{\ell_s^{2,2,2,2}} &= \left(\sum_{j,k,l,m \in \mathbb{Z}^d} \frac{(1+|j| + |(k,l,m)|)^{2s}}{(1+|j|)^{2q}} |f(k,l,m)|^2 \right)^{\frac{1}{2}} \\ &\geq \left(\sum_{j,k,l,m \in \mathbb{Z}^d} \frac{|f(k,l,m)|^2}{(1+|j|)^{2(q-s)}} \right)^{\frac{1}{2}} \\ &= \infty \end{aligned}$$

as $2(q-s) \in (d-s, d)$. Also

$$\begin{aligned} \|c\|_{\ell^{2,p,p,p}} &= \left(\sum_{k,l,m \in \mathbb{Z}^d} \left(\sum_{j \in \mathbb{Z}^d} \frac{|f(k,l,m)|^2}{(1+|j|)^{2q}} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &= \left(\sum_{k,l,m \in \mathbb{Z}^d} |f(k,l,m)|^p \right)^{\frac{1}{p}} \left(\sum_{j \in \mathbb{Z}^d} \frac{1}{(1+|j|)^{2q}} \right)^{\frac{1}{2}} \\ &< \infty, \end{aligned}$$

as $q > \frac{d}{2}$ and $f \in \mathcal{S}(\mathbb{R}^{3d})$. \square

Lemma 4.5.9. *If $\frac{3}{2} \frac{d(2-p)}{p} > s > 0$, then*

$$\ell_{v_s}^{2,2,2,2}(\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d) \setminus \ell^{2,p,p,p}(\mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d, \mathbb{Z}^d)$$

is nonempty.

Proof. Since $\frac{3d(2-p)}{2p} > s > 0$, we can choose $q \in \left(\frac{3d}{2} + s, \frac{3d}{p}\right]$. Choose $f \in \mathcal{S}(\mathbb{R}^d)$.

Set

$$c_{j,k,l,m} = f(j) \frac{1}{(1 + |(k, l, m)|)^q} \quad \forall j, k, l, m \in \mathbb{Z}^d.$$

Then

$$\begin{aligned} \|c\|_{\ell_s^{2,2,2,2}} &= \left(\sum_{j,k,l,m \in \mathbb{Z}^d} \frac{(1 + |j| + |(k, l, m)|)^{2s}}{(1 + |(k, l, m)|)^{2q}} |f(j)|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j,k,l,m \in \mathbb{Z}^d} \frac{(1 + |j|)^{2s} (1 + |(k, l, m)|)^{2s}}{(1 + |(k, l, m)|)^{2q}} |f(j)|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j \in \mathbb{Z}^d} (1 + |j|)^{2s} |f(j)|^2 \right)^{\frac{1}{2}} \left(\sum_{k,l,m \in \mathbb{Z}^d} \frac{1}{(1 + |(k, l, m)|)^{2(q-s)}} \right)^{\frac{1}{2}} \\ &< \infty \end{aligned}$$

as $q > \frac{3d}{2} + s$ and $f \in \mathcal{S}(\mathbb{R}^d)$. On the other hand

$$\begin{aligned} \|c\|_{\ell^{2,p,p,p}} &= \left(\sum_{k,l,m \in \mathbb{Z}^d} \frac{1}{(1 + |(k, l, m)|)^{pq}} \right)^{\frac{1}{p}} \left(\sum_{j \in \mathbb{Z}^d} |f(j)|^2 \right)^{\frac{1}{2}} \\ &= \infty \end{aligned}$$

as $q \leq \frac{3d}{p}$. □

Proposition 4.5.10. *Let c be a permutation on $\{1, 2, \dots, 4d\}$ satisfying one of the following conditions.*

(a) c maps $\{2d + 1, 2d + 2, \dots, 3d\}$ to $\{1, 2, \dots, d\}$ and maps

$\{1, 2, \dots, 2d, 3d + 1, 3d + 2, \dots, 4d\}$ to $\{d + 1, d + 2, \dots, 4d\}$.

(b) c maps $\{3d + 1, 3d + 2, \dots, 4d\}$ to $\{1, 2, \dots, d\}$ and maps $\{1, 2, \dots, 3d\}$ to

$\{d + 1, d + 2, \dots, 4d\}$.

Let $2 = p_1 = \dots = p_d$ and $p = p_{d+1} = \dots = p_{4d}$. If $\frac{3d(2-p)}{2p} > s > 0$ then neither one of $M_{v_s}^{2,2}(\mathbb{R}^{2d})$, $M(c)^{p_1, p_2, \dots, p_{4d}}$ contains the other.

Proof. Using the previous two lemmas, this proposition can be proven like Proposition 4.5.5. □

CHAPTER V

SCHATTEN CLASS FOURIER INTEGRAL OPERATORS

5.1 *Introduction*

Fourier integral operators, which arise in the study of hyperbolic differential equations (see [74]), are operators of the form

$$Af(x) = \iint a(x, y, \xi) f(y) e^{i\varphi(x, y, \xi)} dy d\xi. \quad (26)$$

For the Fourier integral operator in (26), a is the symbol and φ is the real-valued function called the phase function of A . The properties of Fourier integral operators with smooth symbols and phase functions have been studied extensively. In particular the boundedness properties of such operators are well-known (see [61] and the references therein). More recently, in [12] and [40], it was shown that the curvelet and shearlet representations of a Fourier integral operator with smooth symbol and phase function are sparse. Much less is known about Fourier integral operators with non-smooth symbols.

Both pseudodifferential operators and Fourier integral operators with smooth phase functions act on the time-frequency content of functions, although the time-frequency action of a Fourier integral operator is much more general and less explicit than the action of a pseudodifferential operator. However, this action still suggests that time-frequency analysis may play an important role in understanding Fourier integral operators with non-smooth symbols. Indeed, recent results confirm this intuition. In [10] it was shown that inclusion of the symbol of a Fourier integral operator with smooth phase in Sjöstrand's class implies boundedness of the operator on $L^2(\mathbb{R}^d)$. In [19] and [20], the authors use time-frequency analysis to obtain boundedness on certain modulation spaces for a particular type of Fourier integral operator. More

generally, in [16] and [17], the authors prove Schatten p -class membership for Fourier integral operators with sufficiently smooth phase functions whose symbols belong to $M^{p,1}(\mathbb{R}^{3d})$. Note that while Fourier integral operators generalize pseudodifferential operators, pseudodifferential operator analysis techniques do not appear to generalize to Fourier integral operators. The results in [10], [16], [17], [19] and [20] are proved with new Gabor frame techniques.

In this chapter, we use time-frequency techniques to prove that if the symbol of a Fourier integral operator belongs to the mixed modulation space $M(c)^{2,\dots,2,p,\dots,p,1,\infty}$ or $M(c)^{\infty,2,\dots,2,p,\dots,p,1}$ for appropriate permutations c and if the phase function is sufficiently smooth, then the operator is Schatten p -class for $p \in [1, 2]$. Although these results are not directly comparable to previously known Schatten class results for Fourier integral operators, such as those in [16] and [17], they seem stronger in the sense that $M(c)^{2,\dots,2,p,\dots,p,1,\infty}$, $M(c)^{\infty,2,\dots,2,p,\dots,p,1}$ are isomorphic to $\ell^{2,\dots,2,p,\dots,p,1,\infty}$ and $\ell^{\infty,2,\dots,2,p,\dots,p,1}$, respectively, while $M^{p,1}$ is isomorphic to $\ell^{p,\dots,p,1,\dots,1}$ and $\ell^{p,\dots,p,1,\dots,1} \subsetneq \ell^{2,\dots,2,p,\dots,p,1,\infty}$ and $\ell^{p,\dots,p,1,\dots,1} \subsetneq \ell^{\infty,2,\dots,2,p,\dots,p,1}$. Furthermore, our main results are sharp in the sense that larger mixed modulation spaces contain symbols of Fourier integral operators that are not Schatten p -class.

The remainder of this chapter is organized as follows. In Section 5.2, we prove a time-frequency condition on the product of the symbol and phase of a Fourier integral operator that ensures the operator is Schatten class. In Section 5.3, we prove mixed modulation space embeddings for products. Finally, in Section 5.4, we use the results of the previous two sections to give mixed modulation space conditions on the symbol of a Fourier integral operator that ensure the operator is Schatten class and prove the sharpness of these results.

5.2 A Schatten class result for Fourier Integral Operators

The mixed modulation spaces defined in Chapter IV depend on a permutation of the variables of the Gabor transform. For Fourier integral operators (FIOs), we will be interested in permutations c of $\{1, 2, \dots, 6d\}$ satisfying the following definition.

Definition 5.2.1. *A first FIO slice permutation c is a permutation of $\{1, 2, \dots, 6d\}$ such that*

- (a) c maps $\{1, 2, \dots, d, 3d + 1, 3d + 2, \dots, 4d\}$ to $\{1, 2, \dots, 2d\}$,
- (b) c maps $\{d + 1, d + 2, \dots, 2d, 4d + 1, 4d + 2, \dots, 5d\}$ to $\{2d + 1, 2d + 2, \dots, 4d\}$,
- (c) c maps $\{2d + 1, 2d + 2, \dots, 3d\}$ to $\{4d + 1, 4d + 2, \dots, 5d\}$, and
- (d) c maps $\{5d + 1, 5d + 2, \dots, 6d\}$ to $\{5d + 1, 5d + 2, \dots, 6d\}$.

A second FIO slice permutation c is a permutation of $\{1, 2, \dots, 6d\}$ such that

- (a) c maps $\{d + 1, d + 2, \dots, 2d, 4d + 1, 4d + 2, \dots, 5d\}$ to $\{1, 2, \dots, 2d\}$,
- (b) c maps $\{1, 2, \dots, d, 3d + 1, 3d + 2, \dots, 4d\}$ to $\{2d + 1, 2d + 2, \dots, 4d\}$,
- (c) c maps $\{2d + 1, 2d + 2, \dots, 3d\}$ to $\{4d + 1, 4d + 2, \dots, 5d\}$, and
- (d) c maps $\{5d + 1, 5d + 2, \dots, 6d\}$ to $\{5d + 1, 5d + 2, \dots, 6d\}$.

These FIO slice permutations relate to the slice analysis in Theorem 3.3.2 and can be used to analyze Fourier integral operators.

Theorem 5.2.2. *Suppose $p \in [1, 2]$ and c is a FIO slice permutation. Let $p_1 = p_2 = \dots = p_{2d} = 2$, $p_{2d+1} = p_{2d+2} = \dots = p_{4d} = p$, $p_{4d+1} = p_{4d+2} = \dots = p_{5d} = 1$ and $p_{5d+1} = p_{5d+2} = \dots = p_{6d} = \infty$. If A is a Fourier integral operator with symbol a and phase function φ and $ae^{i\varphi} \in M(c)^{p_1, p_2, \dots, p_{6d}}$, then $A \in \mathcal{I}_p(L^2(\mathbb{R}^d))$.*

Proof. We prove the result in the case c is a second FIO slice permutation. The case that c is a first FIO slice permutation can be proven similarly.

Let $\{f_n\}_{n \in \mathbb{N}}$, $\{g_n\}_{n \in \mathbb{N}}$ be arbitrary orthonormal sequences in $L^2(\mathbb{R}^d)$ and let $\{M_{\alpha k_2} T_{\alpha k_1} \phi\}_{k_1, k_2 \in \mathbb{Z}^d}$ be a Parseval Gabor frame for $L^2(\mathbb{R}^d)$ with $\phi \in M^{1,1}(\mathbb{R}^d)$. Choose C such that

$$\left\| \left\{ \left\| \hat{\phi} \chi_{\alpha[0,1]^d + \alpha n} \right\|_{\infty} \right\}_{n \in \mathbb{Z}^d} \right\|_{\ell^1(\mathbb{Z}^d)} \leq C \left\| \hat{\phi} \right\|_{W(L^1(\mathbb{R}^d))}.$$

We have

$$\langle Af_n, g_n \rangle = \langle ae^{i\varphi}, g_n \otimes \overline{f_n} \otimes 1 \rangle.$$

Since $1 \in M^{\infty,1}(\mathbb{R}^d)$, we have

$$1 = \sum_{k_1, k_2 \in \mathbb{Z}^d} \langle 1, M_{\alpha k_2} T_{\alpha k_1} \phi \rangle M_{\alpha k_2} T_{\alpha k_1} \phi \quad \text{weakly.}$$

Thus

$$\begin{aligned} \langle Af_n, g_n \rangle &= \sum_{k_1, k_2 \in \mathbb{Z}^d} \overline{\langle 1, M_{\alpha k_2} T_{\alpha k_1} \phi \rangle} \langle ae^{i\varphi}, g_n \otimes \overline{f_n} \otimes M_{\alpha k_2} T_{\alpha k_1} \phi \rangle \\ &= \sum_{k_1, k_2 \in \mathbb{Z}^d} \overline{\langle 1, M_{\alpha k_2} T_{\alpha k_1} \phi \rangle} \langle A_{k_1, k_2} f_n, g_n \rangle, \end{aligned}$$

where A_{k_1, k_2} is the integral operator with kernel

$$k_{k_1, k_2}(x, y) = \int a(x, y, \xi) e^{i\varphi(x, y, \xi)} \overline{M_{\alpha k_2} T_{\alpha k_1} \phi(\xi)} d\xi.$$

In the case $p = 1$ we have

$$\begin{aligned} \sum_{n \in \mathbb{N}} |\langle Af_n, g_n \rangle| &= \sum_{n \in \mathbb{N}} \left| \sum_{k_1, k_2 \in \mathbb{Z}^d} \overline{\langle 1, M_{\alpha k_2} T_{\alpha k_1} \phi \rangle} \langle A_{k_1, k_2} f_n, g_n \rangle \right| \\ &\leq \sum_{k_1, k_2 \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} \left| \overline{\langle 1, M_{\alpha k_2} T_{\alpha k_1} \phi \rangle} \langle A_{k_1, k_2} f_n, g_n \rangle \right| \\ &= \sum_{k_1, k_2 \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} \left| \hat{\phi}(\alpha k_2) \right| |\langle A_{k_1, k_2} f_n, g_n \rangle| \\ &= \sum_{k_1, k_2 \in \mathbb{Z}^d} \left| \hat{\phi}(\alpha k_2) \right| \sum_{n \in \mathbb{N}} |\langle A_{k_1, k_2} f_n, g_n \rangle| \end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{k_2 \in \mathbb{Z}^d} |\hat{\phi}(\alpha k_2)| \right) \left(\sup_{k_2 \in \mathbb{Z}^d} \sum_{k_1 \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} |\langle A_{k_1, k_2} f_n, g_n \rangle| \right) \\
&\leq C \left\| \hat{\phi} \right\|_{W(L^1(\mathbb{R}^d))} \sup_{k_2 \in \mathbb{Z}^d} \sum_{k_1 \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} |\langle A_{k_1, k_2} f_n, g_n \rangle| \\
&\leq C \left\| \hat{\phi} \right\|_{W(L^1(\mathbb{R}^d))} \sup_{k_2 \in \mathbb{Z}^d} \sum_{k_1 \in \mathbb{Z}^d} \|A_{k_1, k_2}\|_{\mathcal{I}_1}
\end{aligned}$$

By the proof of Theorem 4.5.1, we have

$$\begin{aligned}
\|A_{k_1, k_2}\|_{\mathcal{I}_1} &\leq \sum_{n_1, n_2 \in \mathbb{Z}^d} \left(\sum_{m_1, m_2 \in \mathbb{Z}^d} |V_{\phi \otimes \phi} k_{k_1, k_2}(\alpha n_1, \alpha m_1, \alpha n_2, \alpha m_2)|^2 \right)^{\frac{1}{2}} \\
&= \sum_{n_1, n_2 \in \mathbb{Z}^d} \left(\sum_{m_1, m_2 \in \mathbb{Z}^d} |V_{\Phi}(ae^{i\varphi})(\alpha n_1, \alpha m_1, \alpha k_1, \alpha n_2, \alpha m_2, \alpha k_2)|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

where $\Phi = \phi \otimes \phi \otimes \phi$. Thus if

$$\sup_{k_2 \in \mathbb{Z}^d} \sum_{k_1 \in \mathbb{Z}^d} \sum_{n_1, n_2 \in \mathbb{Z}^d} \left(\sum_{m_1, m_2 \in \mathbb{Z}^d} |V_{\Phi}(ae^{i\varphi})(\alpha n_1, \alpha m_1, \alpha k_1, \alpha n_2, \alpha m_2, \alpha k_2)|^2 \right)^{\frac{1}{2}} < \infty,$$

then $A \in \mathcal{I}_1(L^2(\mathbb{R}^d))$. Notice that this quantity is finite if and only if $ae^{i\varphi} \in M(c)^{p_1, p_2, \dots, p_{6d}}$.

For the case $p = 2$ we have

$$\begin{aligned}
\left(\sum_{n \in \mathbb{N}} |\langle A f_n, g_n \rangle|^2 \right)^{\frac{1}{2}} &= \left(\sum_{n \in \mathbb{N}} \left| \sum_{k_1, k_2 \in \mathbb{Z}^d} \overline{\langle 1, M_{\alpha k_2} T_{\alpha k_1} \phi \rangle} \langle A_{k_1, k_2} f_n, g_n \rangle \right|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{k_1, k_2 \in \mathbb{Z}^d} \left(\sum_{n \in \mathbb{N}} |\hat{\phi}(\alpha k_2)|^2 |\langle A_{k_1, k_2} f_n, g_n \rangle|^2 \right)^{\frac{1}{2}} \quad (27) \\
&\leq \left(\sum_{k_2 \in \mathbb{Z}^d} |\hat{\phi}(\alpha k_2)| \right) \left(\sup_{k_2 \in \mathbb{Z}^d} \sum_{k_1 \in \mathbb{Z}^d} \left(\sum_{n \in \mathbb{N}} |\langle A_{k_1, k_2} f_n, g_n \rangle|^2 \right)^{\frac{1}{2}} \right) \\
&\leq C \left\| \hat{\phi} \right\|_{W(L^1(\mathbb{R}^d))} \sup_{k_2 \in \mathbb{Z}^d} \sum_{k_1 \in \mathbb{Z}^d} \|A_{k_1, k_2}\|_{\mathcal{I}_2},
\end{aligned}$$

where (27) holds by Minkowski's integral inequality. Again, by the proof of Theorem

4.5.1, we have

$$\begin{aligned} \|A_{k_1, k_2}\|_{\mathcal{I}_2} &\leq \left(\sum_{n_1, n_2 \in \mathbb{Z}^d} \sum_{m_1, m_2 \in \mathbb{Z}^d} |V_{\phi \otimes \phi} k_{k_1, k_2}(\alpha n_1, \alpha m_1, \alpha n_2, \alpha m_2)|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{n_1, n_2 \in \mathbb{Z}^d} \sum_{m_1, m_2 \in \mathbb{Z}^d} |V_{\Phi}(ae^{i\varphi})(\alpha n_1, \alpha m_1, \alpha k_1, \alpha n_2, \alpha m_2, \alpha k_2)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus if

$$\sup_{k_2 \in \mathbb{Z}^d} \sum_{k_1 \in \mathbb{Z}^d} \left(\sum_{n_1, n_2 \in \mathbb{Z}^d} \sum_{m_1, m_2 \in \mathbb{Z}^d} |V_{\Phi}(ae^{i\varphi})(\alpha n_1, \alpha m_1, \alpha k_1, \alpha n_2, \alpha m_2, \alpha k_2)|^2 \right)^{\frac{1}{2}} < \infty,$$

then $A \in \mathcal{I}_2(L^2(\mathbb{R}^d))$. This quantity is finite if and only if $ae^{i\varphi} \in M(c)^{p_1, p_2, \dots, p_{6d}}$.

Taking the supremum of $\sum_{n \in \mathbb{N}} |\langle Af_n, g_n \rangle|$ and $(\sum_{n \in \mathbb{N}} |\langle Af_n, g_n \rangle|^2)^{\frac{1}{2}}$ over all orthonormal sequences gives the result for $p = 1$ and $p = 2$. For $1 < p < 2$, the result follows by interpolation. \square

5.3 Pointwise Multiplication in the Mixed Modulation Spaces

In this section, we find conditions on the symbol and phase function of a Fourier integral operator so that their product lies in given mixed modulation spaces. We begin by stating a special case of Proposition 1.2 in [17], which describes multiplication properties of modulation spaces.

Lemma 5.3.1. *Suppose $p, q, p_1, p_2, q_1, q_2 \in [1, \infty]$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q}$. Then there exists a finite C such that*

$$\|fg\|_{M^{p,q}(\mathbb{R}^d)} \leq C \|f\|_{M^{p_1, q_1}(\mathbb{R}^d)} \|g\|_{M^{p_2, q_2}(\mathbb{R}^d)} \quad \forall f \in M^{p_1, q_1}(\mathbb{R}^d), g \in M^{p_2, q_2}(\mathbb{R}^d).$$

In particular, there is a finite C such that

$$\|fg\|_{M^{p,q}(\mathbb{R}^d)} \leq C \|f\|_{M^{p,q}(\mathbb{R}^d)} \|g\|_{M^{\infty,1}(\mathbb{R}^d)} \quad \forall f \in M^{p,q}(\mathbb{R}^d), g \in M^{\infty,1}(\mathbb{R}^d).$$

Corollary 5.3.2. *There is a finite C such that*

$$\|f^n\|_{M^{\infty,1}(\mathbb{R}^d)} \leq C^n \|f\|_{M^{\infty,1}(\mathbb{R}^d)}^n \quad \forall f \in M^{\infty,1}(\mathbb{R}^d).$$

Now we will generalize Lemma 5.3.1 to mixed modulation spaces.

Theorem 5.3.3. *Let $p \in [1, \infty]$. Suppose c is a FIO slice permutation and $p_1 = p_2 = \dots = p_{2d} = 2$, $p_{2d+1} = p_{2d+2} = \dots = p_{4d} = p$, $p_{4d+1} = p_{4d+2} = \dots = p_{5d} = 1$ and $p_{5d+1} = p_{5d+2} = \dots = p_{6d} = \infty$. Then for some finite C we have for all $a_1 \in M(c)^{p_1, p_2, \dots, p_{6d}}$, $a_2 \in M^{\infty, 1}(\mathbb{R}^{3d})$ that*

$$\|a_1 a_2\|_{M(c)^{p_1, p_2, \dots, p_{6d}}} \leq C \|a_1\|_{M(c)^{p_1, p_2, \dots, p_{6d}}} \|a_2\|_{M^{\infty, 1}(\mathbb{R}^{3d})}.$$

Proof. We prove the result in the case c is a first FIO slice permutation and $p \in [1, \infty)$.

The proof is similar if c is a second FIO slice permutation or $p = \infty$.

Choose $\phi_1, \phi_2 \in M^{1, 1}(\mathbb{R}^{3d})$. Then by Proposition 1.2 of [18] we have $\phi_1 \phi_2 \in M^{1, 1}(\mathbb{R}^{3d})$ and

$$\begin{aligned} & V_{\phi_1 \phi_2} a_1 a_2(x_1, x_2, x_3, y_1, y_2, y_3) \\ &= \mathcal{F} \left((a_1 \cdot T_x \overline{\phi_1}) \cdot (a_2 \cdot T_x \overline{\phi_2}) \right) (y_1, y_2, y_3) \\ &= (\mathcal{F} (a_1 \cdot T_x \overline{\phi_1}) * \mathcal{F} (a_2 \cdot T_x \overline{\phi_2})) (y_1, y_2, y_3) \\ &= \iiint \mathcal{F} (a_1 \cdot T_x \overline{\phi_1}) (y_1 - t_1, y_2 - t_2, y_3 - t_3) \mathcal{F} (a_2 \cdot T_x \overline{\phi_2}) (t_1, t_2, t_3) dt_1 dt_2 dt_3 \\ &= \iiint V_{\phi_1} a_1(x_1, x_2, x_3, y_1 - t_1, y_2 - t_2, y_3 - t_3) V_{\phi_2} a_2(x_1, x_2, x_3, t_1, t_2, t_3) dt_1 dt_2 dt_3. \end{aligned}$$

Since c is a first FIO slice permutation we have

$$\begin{aligned} & \|a_1 a_2\|_{M(c)^{p_1, p_2, \dots, p_{6d}}} \\ & \equiv \sup_{y_3} \int \left(\iint \left(\iint |V_{\phi_1 \phi_2} a_1 a_2(x_1, x_2, x_3, y_1, y_2, y_3)|^2 dx_1 dy_1 \right)^{\frac{p}{2}} dx_2 dy_2 \right)^{\frac{1}{p}} dx_3. \end{aligned}$$

Thus letting

$$F_{1, x_1, x_2, x_3, y_2, y_3}(y_1) = V_{\phi_1} a_1(x_1, x_2, x_3, y_1, y_2, y_3)$$

and

$$F_{2, x_1, x_2, x_3, y_2, y_3}(y_1) = V_{\phi_2} a_2(x_1, x_2, x_3, y_1, y_2, y_3)$$

we have

$$\begin{aligned}
& \left(\int |V_{\phi_1 \phi_2} a_1 a_2(x_1, x_2, x_3, y_1, y_2, y_3)|^2 dy_1 \right)^{\frac{1}{2}} \\
&= \left(\int \left| \iint (F_{1,x_1,x_2,x_3,y_2-t_2,y_3-t_3} * F_{2,x_1,x_2,x_3,t_2,t_3})(y_1) dt_2 dt_3 \right|^2 dy_1 \right)^{\frac{1}{2}} \\
&\leq \iint \left(\int |(F_{1,x_1,x_2,x_3,y_2-t_2,y_3-t_3} * F_{2,x_1,x_2,x_3,t_2,t_3})(y_1)|^2 dy_1 \right)^{\frac{1}{2}} dt_2 dt_3 \quad (28) \\
&= \iint \|F_{1,x_1,x_2,x_3,y_2-t_2,y_3-t_3} * F_{2,x_1,x_2,x_3,t_2,t_3}\|_{L^2} dt_2 dt_3 \\
&\leq \iint \|F_{1,x_1,x_2,x_3,y_2-t_2,y_3-t_3}\|_{L^2} \|F_{2,x_1,x_2,x_3,t_2,t_3}\|_{L^1} dt_2 dt_3, \quad (29)
\end{aligned}$$

where (28) holds by Minkowski's integral inequality and (29) holds by Young's convolution inequality.

Hence

$$\begin{aligned}
& \int |V_{\phi_1 \phi_2} a_1 a_2(x_1, x_2, x_3, y_1, y_2, y_3)|^2 dy_1 \\
&\leq \left(\iint \|F_{1,x_1,x_2,x_3,y_2-t_2,y_3-t_3}\|_{L^2} \|F_{2,x_1,x_2,x_3,t_2,t_3}\|_{L^1} dt_2 dt_3 \right)^2,
\end{aligned}$$

which implies

$$\begin{aligned}
& \left(\iint |V_{\phi_1 \phi_2} a_1 a_2(x_1, x_2, x_3, y_1, y_2, y_3)|^2 dx_1 dy_1 \right)^{\frac{p}{2}} \\
&\leq \left(\int \left(\iint \|F_{1,x_1,x_2,x_3,y_2-t_2,y_3-t_3}\|_{L^2} \|F_{2,x_1,x_2,x_3,t_2,t_3}\|_{L^1} dt_2 dt_3 \right)^2 dx_1 \right)^{\frac{p}{2}} \\
&\leq \left(\iint \left(\int \|F_{1,x_1,x_2,x_3,y_2-t_2,y_3-t_3}\|_{L^2}^2 \|F_{2,x_1,x_2,x_3,t_2,t_3}\|_{L^1}^2 dx_1 \right)^{\frac{1}{2}} dt_2 dt_3 \right)^p \quad (30) \\
&\leq \left(\iint \left(\int \|F_{1,x_1,x_2,x_3,y_2-t_2,y_3-t_3}\|_{L^2}^2 dx_1 \right)^{\frac{1}{2}} \left(\sup_{x_1} \|F_{2,x_1,x_2,x_3,t_2,t_3}\|_{L^1} \right) dt_2 dt_3 \right)^p \\
&= \left(\iint G_{1,x_2,x_3,y_3-t_3}(y_2 - t_2) G_{2,x_2,x_3,t_3}(t_2) dt_2 dt_3 \right)^p \\
&= \left(\int (G_{1,x_2,x_3,y_3-t_3} * G_{2,x_2,x_3,t_3})(y_2) dt_3 \right)^p,
\end{aligned}$$

where

$$G_{1,x_2,x_3,y_3}(y_2) = \left(\int \|F_{1,x_1,x_2,x_3,y_2,y_3}\|_{L^2}^2 dx_1 \right)^{\frac{1}{2}}$$

and

$$G_{2,x_2,x_3,y_3}(y_2) = \left(\sup_{x_1} \|F_{2,x_1,x_2,x_3,y_2,y_3}\|_{L^1} \right).$$

Note that (30) holds by Minkowski's integral inequality.

Consequently, we have

$$\begin{aligned} & \left(\iint \left(\iint |V_{\phi_1\phi_2} a_1 a_2(x_1, x_2, x_3, y_1, y_2, y_3)|^2 dx_1 dy_1 \right)^{\frac{p}{2}} dx_2 dy_2 \right)^{\frac{1}{p}} \\ & \leq \left(\iint \left(\int (G_{1,x_2,x_3,y_3-t_3} * G_{2,x_2,x_3,t_3})(y_2) dt_3 \right)^p dx_2 dy_2 \right)^{\frac{1}{p}} \\ & \leq \left(\int \left(\int \left(\int |(G_{1,x_2,x_3,y_3-t_3} * G_{2,x_2,x_3,t_3})(y_2)|^p dy_2 \right)^{\frac{1}{p}} dt_3 \right)^p dx_2 \right)^{\frac{1}{p}} \quad (31) \end{aligned}$$

$$\begin{aligned} & = \left(\int \left(\int \|G_{1,x_2,x_3,y_3-t_3} * G_{2,x_2,x_3,t_3}\|_{L^p} dt_3 \right)^p dx_2 \right)^{\frac{1}{p}} \\ & \leq \left(\int \left(\int \|G_{1,x_2,x_3,y_3-t_3}\|_{L^p} \|G_{2,x_2,x_3,t_3}\|_{L^1} dt_3 \right)^p dx_2 \right)^{\frac{1}{p}} \\ & \leq \int \left(\int \|G_{1,x_2,x_3,y_3-t_3}\|_{L^p}^p \|G_{2,x_2,x_3,t_3}\|_{L^1}^p dx_2 \right)^{\frac{1}{p}} dt_3 \quad (32) \end{aligned}$$

$$\begin{aligned} & \leq \int \left(\int \|G_{1,x_2,x_3,y_3-t_3}\|_{L^p}^p dx_2 \right)^{\frac{1}{p}} \left(\sup_{x_2} \|G_{2,x_2,x_3,t_3}\|_{L^1} \right) dt_3 \\ & = \int H_{1,y_3-t_3}(x_3) H_{2,t_3}(x_3) dt_3 \quad (33) \end{aligned}$$

where $H_{1,y_3}(x_3) = \left(\int \|G_{1,x_2,x_3,y_3}\|_{L^p}^p dx_2 \right)^{\frac{1}{p}}$ and $H_{2,y_3}(x_3) = \left(\sup_{x_2} \|G_{2,x_2,x_3,y_3}\|_{L^1} \right)$.

Note that (31) and (32) both follow from Minkowski's integral inequality.

From (33) we see

$$\begin{aligned} & \sup_{y_3} \int \left(\iint \left(\iint |V_{\phi_1\phi_2} a_1 a_2(x_1, x_2, x_3, y_1, y_2, y_3)|^2 dx_1 dy_1 \right)^{\frac{p}{2}} dx_2 dy_2 \right)^{\frac{1}{p}} dx_3 \\ & \leq \sup_{y_3} \iint H_{1,y_3-t_3}(x_3) H_{2,t_3}(x_3) dt_3 dx_3 \\ & \leq \sup_{y_3} \int \|H_{1,y_3-t_3}\|_{L^1} \|H_{2,t_3}\|_{L^\infty} dt_3 \\ & = \sup_{y_3} (K_1 * K_2)(y_3) \\ & = \|K_1 * K_2\|_{L^\infty} \end{aligned}$$

$$\leq \|K_1\|_{L^\infty} \|K_2\|_{L^1},$$

where $K_1(y_3) = \|H_{1,y_3}\|_{L^1}$ and $K_2(y_3) = \|H_{2,y_3}\|_{L^\infty}$.

Notice that

$$\begin{aligned} \|K_1\|_{L^\infty} &= \sup_{y_3} \|H_{1,y_3}\|_{L^1} \\ &= \sup_{y_3} \int |H_{1,y_3}(x_3)| \, dx_3 \\ &= \sup_{y_3} \int \left(\int \|G_{1,x_2,x_3,y_3}\|_{L^p}^p \, dx_2 \right)^{\frac{1}{p}} \, dx_3 \\ &= \sup_{y_3} \int \left(\iint |G_{1,x_2,x_3,y_3}(y_2)|^p \, dy_2 \, dx_2 \right)^{\frac{1}{p}} \, dx_3 \\ &= \sup_{y_3} \int \left(\iint \left(\iint |F_{1,x_1,x_2,x_3,y_2,y_3}(y_1)|^2 \, dy_1 \, dx_1 \right)^{\frac{p}{2}} \, dy_2 \, dx_2 \right)^{\frac{1}{p}} \, dx_3 \\ &= \sup_{y_3} \int \left(\iint \left(\iint |V_{\phi_1} a_1(x_1, x_2, x_3, y_1, y_2, y_3)|^2 \, dy_1 \, dx_1 \right)^{\frac{p}{2}} \, dy_2 \, dx_2 \right)^{\frac{1}{p}} \, dx_3 \\ &\equiv \|a_1\|_{M(c)^{p_1, \dots, p_{6d}}} \end{aligned}$$

and

$$\begin{aligned} \|K_2\|_{L^1} &= \int \|H_{2,y_3}\|_{L^\infty} \, dy_3 \\ &= \int \left(\sup_{x_3} |H_{2,y_3}(x_3)| \right) \, dy_3 \\ &= \int \left(\sup_{x_3} \left(\sup_{x_2} \|G_{2,x_2,x_3,y_3}\|_{L^1} \right) \right) \, dy_3 \\ &= \int \left(\sup_{x_3} \left(\sup_{x_2} \int \sup_{x_1} \|F_{2,x_1,x_2,x_3,y_2,y_3}\|_{L^1} \, dy_2 \right) \right) \, dy_3 \\ &= \int \left(\sup_{x_3} \left(\sup_{x_2} \int \sup_{x_1} \left(\int |V_{\phi_2} a_2(x_1, x_2, x_3, y_1, y_2, y_3)| \, dy_1 \right) \, dy_2 \right) \right) \, dy_3 \\ &\leq \iiint \sup_{x_3} \sup_{x_2} \sup_{x_1} |V_{\phi_2} a_2(x_1, x_2, x_3, y_1, y_2, y_3)| \, dy_1 \, dy_2 \, dy_3 \\ &\equiv \|a_2\|_{M^{\infty,1}(\mathbb{R}^{3d})}. \end{aligned} \quad \square$$

The following lemma comes from Proposition 3.2 in [10].

Lemma 5.3.4. *There exists a C such that*

$$\|\tau(t\cdot)\|_{M^{\infty,1}(\mathbb{R}^{3d})} \leq C \|\tau\|_{M^{\infty,1}(\mathbb{R}^{3d})} \quad \forall \tau \in M^{\infty,1}(\mathbb{R}^{3d}), t \in [0, 1].$$

Theorem 5.3.5. *Suppose $p \in [1, \infty]$. Suppose c is a FIO slice permutation and $p_1 = p_2 = \dots = p_{2d} = 2$, $p_{2d+1} = p_{2d+2} = \dots = p_{4d} = p$, $p_{4d+1} = p_{4d+2} = \dots = p_{5d} = 1$ and $p_{5d+1} = p_{5d+2} = \dots = p_{6d} = \infty$. If $a \in M(c)^{p_1, p_2, \dots, p_{6d}}$ has compact support and $\varphi \in C^2(\mathbb{R}^{3d})$ is real valued and satisfies $D^\alpha \varphi \in M^{\infty,1}(\mathbb{R}^{3d})$ for all multi-indices α with $|\alpha| = 2$, then $ae^{i\varphi} \in M(c)^{p_1, p_2, \dots, p_{6d}}$.*

Proof. By our suppositions on φ we have the following Taylor expansion.

$$\varphi(w) = \varphi(0, 0, 0) + \sum_{|\alpha|=1} (D^\alpha \varphi)(0, 0, 0) w^\alpha + \sum_{|\alpha|=2} \frac{2}{\alpha!} \left(\int_0^1 (1-t) (D^\alpha \varphi)(tw) dt \right) w^\alpha.$$

Let

$$\psi_1(w) = \varphi(0, 0, 0) + \sum_{|\alpha|=1} (D^\alpha \varphi)(0, 0, 0) w^\alpha$$

and

$$\psi_2(w) = \sum_{|\alpha|=2} \frac{2}{\alpha!} \left(\int_0^1 (1-t) (D^\alpha \varphi)(tw) dt \right) w^\alpha.$$

Choose χ such that $\chi(w) = 1$ for all w in the support of a and

$$\chi(w) w^\alpha \in M^{\infty,1}(\mathbb{R}^{3d}) \quad \forall \alpha, |\alpha| = 2.$$

Then

$$\begin{aligned} \|ae^{i\varphi}\|_{M(c)^{p_1, \dots, p_{6d}}} &= \|ae^{i\psi_1} e^{i\psi_2}\|_{M(c)^{p_1, \dots, p_{6d}}} \\ &= \|ae^{i\psi_1} e^{i\chi\psi_2}\|_{M(c)^{p_1, \dots, p_{6d}}} \\ &\leq \|ae^{i\psi_1}\|_{M(c)^{p_1, \dots, p_{6d}}} \|e^{i\chi\psi_2}\|_{M^{\infty,1}}, \end{aligned} \quad (34)$$

where (34) holds by Proposition 5.3.3.

Choose finite C such that

$$\|fg\|_{M^{\infty,1}(\mathbb{R}^{3d})} \leq C \|f\|_{M^{\infty,1}(\mathbb{R}^{3d})} \|g\|_{M^{\infty,1}(\mathbb{R}^{3d})} \quad \forall f \in M^{\infty,1}(\mathbb{R}^{3d}), g \in M^{\infty,1}(\mathbb{R}^{3d}).$$

Since

$$e^{i\chi(w)\psi_2(w)} = \sum_{n \geq 0} \frac{(i\chi(w)\psi_2(w))^n}{n!},$$

we have

$$\begin{aligned} \|e^{i\chi\psi_2}\|_{M^{\infty,1}} &= \left\| \sum_{n \geq 0} \frac{(i\chi\psi_2)^n}{n!} \right\|_{M^{\infty,1}} \\ &\leq \sum_{n \geq 0} \frac{\|(i\chi\psi_2)^n\|_{M^{\infty,1}}}{n!} \\ &\leq \sum_{n \geq 0} \frac{C^n \|\chi\psi_2\|_{M^{\infty,1}}^n}{n!} \\ &= e^C \|\chi\psi_2\|_{M^{\infty,1}} \end{aligned}$$

By Lemma 5.3.4, we can choose C' so that

$$\|\tau(t \cdot)\|_{M^{\infty,1}(\mathbb{R}^{3d})} \leq C' \|\tau\|_{M^{\infty,1}(\mathbb{R}^{3d})} \quad \forall \tau \in M^{\infty,1}(\mathbb{R}^{3d}), t \in [0, 1].$$

Hence

$$\begin{aligned} &\|\chi\psi_2\|_{M^{\infty,1}} \\ &= \left\| \sum_{|\alpha|=2} \chi(w)w^\alpha \frac{2}{\alpha!} \left(\int_0^1 (1-t) (D^\alpha \varphi)(tw) dt \right) \right\|_{M^{\infty,1}} \\ &\leq \sum_{|\alpha|=2} \frac{2}{\alpha!} \left\| \chi(w)w^\alpha \left(\int_0^1 (1-t) (D^\alpha \varphi)(tw) dt \right) \right\|_{M^{\infty,1}} \\ &\leq \sum_{|\alpha|=2} \frac{2C}{\alpha!} \|\chi(w)w^\alpha\|_{M^{\infty,1}} \left\| \int_0^1 (1-t) (D^\alpha \varphi)(tw) dt \right\|_{M^{\infty,1}} \\ &= \sum_{|\alpha|=2} \frac{2C}{\alpha!} \|\chi(w)w^\alpha\|_{M^{\infty,1}} \int \sup_{x \in \mathbb{R}^{3d}} \left| \int \int_0^1 (1-t) (D^\alpha \varphi)(tw) \overline{M_\xi T_x \phi(w)} dt dw \right| d\xi \\ &\leq \sum_{|\alpha|=2} \frac{2C}{\alpha!} \|\chi(w)w^\alpha\|_{M^{\infty,1}} \int \sup_{x \in \mathbb{R}^{3d}} \int_0^1 (1-t) \left| \int (D^\alpha \varphi)(tw) \overline{M_\xi T_x \phi(w)} dw \right| dt d\xi \\ &\leq \sum_{|\alpha|=2} \frac{2C}{\alpha!} \|\chi(w)w^\alpha\|_{M^{\infty,1}} \int_0^1 (1-t) \int \sup_{x \in \mathbb{R}^{3d}} \left| \int (D^\alpha \varphi)(tw) \overline{M_\xi T_x \phi(w)} dw \right| d\xi dt \\ &= \sum_{|\alpha|=2} \frac{2C}{\alpha!} \|\chi(w)w^\alpha\|_{M^{\infty,1}} \int_0^1 (1-t) \|(D^\alpha \varphi)(tw)\|_{M^{\infty,1}} dt \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{|\alpha|=2} \frac{2C}{\alpha!} \|\chi(w)w^\alpha\|_{M^\infty,1} \int_0^1 (1-t) C' \|(D^\alpha\varphi)(w)\|_{M^\infty,1} dt \\
&\leq \sum_{|\alpha|=2} \frac{2CC'}{\alpha!} \|\chi(w)w^\alpha\|_{M^\infty,1} \|(D^\alpha\varphi)(w)\|_{M^\infty,1} \\
&< \infty.
\end{aligned}$$

Notice that

$$ae^{i\psi_1} = M_b(e^{i\varphi(0,0,0)}a),$$

where the components of $b \in \mathbb{R}^{3d}$ are $(D^\alpha\varphi)(0,0,0)$ for multi-indices α with $|\alpha| = 1$. Thus since $a \in M(c)^{p_1,p_2,\dots,p_{6d}}$, we have $\|ae^{i\psi_1}\|_{M(c)^{p_1,\dots,p_{6d}}} = \|a\|_{M(c)^{p_1,\dots,p_{6d}}} < \infty$ as well. Hence $\|ae^{i\varphi}\|_{M(c)^{p_1,\dots,p_{6d}}} < \infty$. \square

Note that Theorem 5.3.5 is similar in spirit to Lemma 2.2 in [17].

In the remainder of this section, we develop alternate conditions on the symbol and phase function of a Fourier integral operator so that their product lies in mixed modulation spaces relevant to Schatten class integral operators. To this end, the following definition will be useful.

Definition 5.3.6. *A first FIO symbol permutation c is a permutation of $\{1, 2, \dots, 6d\}$ such that*

- (a) c maps $\{5d+1, 5d+2, \dots, 6d\}$ to $\{1, 2, \dots, d\}$,
- (b) c maps $\{1, 2, \dots, d, 3d+1, 3d+2, \dots, 4d\}$ to $\{d+1, d+2, \dots, 3d\}$,
- (c) c maps $\{d+1, \dots, 2d, 4d+1, 4d+2, \dots, 5d\}$ to $\{3d+1, 3d+2, \dots, 5d\}$, and
- (d) c maps $\{2d+1, 2d+2, \dots, 3d\}$ to $\{5d+1, 5d+2, \dots, 6d\}$.

A second FIO symbol permutation c is a permutation of $\{1, 2, \dots, 6d\}$ such that

- (a) c maps $\{5d+1, 5d+2, \dots, 6d\}$ to $\{1, 2, \dots, d\}$,
- (b) c maps $\{d+1, d+2, \dots, 2d, 4d+1, 4d+2, \dots, 5d\}$ to $\{d+1, d+2, \dots, 3d\}$,

(c) c maps $\{1, 2, \dots, d, 3d + 1, 3d + 2, \dots, 4d\}$ to $\{3d + 1, 3d + 2, \dots, 5d\}$, and

(d) c maps $\{2d + 1, 2d + 2, \dots, 3d\}$ to $\{5d + 1, 5d + 2, \dots, 6d\}$.

Under certain smoothness assumptions on φ , we can show that the mixed modulation space norm of $ae^{i\varphi}$ appearing in Theorem 5.2.2, which is determined by a FIO slice permutation, is dominated by a mixed modulation space norm on a determined by a FIO symbol permutation. First, a technical lemma is needed.

Lemma 5.3.7. *Suppose $\Phi \in M^{1,1}(\mathbb{R}^{3d})$ and M is a $3d$ -by- $3d$ self-adjoint matrix. Define a operator S_M by*

$$S_M f(w) = e^{\pi i w \cdot M w} f(w), \quad \forall f \in M^{\infty, \infty}(\mathbb{R}^{3d}).$$

Then

$$|V_\Phi S_M f(x, \xi)| = |V_{S_{-M}\Phi} f(x, \xi - Mx)|, \quad \forall x, \xi \in \mathbb{R}^{3d}.$$

Proof.

$$\begin{aligned} & V_{S_{-M}\Phi} f(x, \xi - Mx) \\ &= \int f(w) \overline{\Phi(w-x)} e^{\pi i (w-x) \cdot M(w-x)} e^{-2\pi i (\xi - Mx) \cdot w} dw \\ &= e^{\pi i x \cdot Mx} \int e^{\pi i w \cdot M w} f(w) \overline{\Phi(w-x)} e^{-\pi i x \cdot M w} e^{-\pi i w \cdot Mx} e^{-2\pi i (\xi - Mx) \cdot w} dw \\ &= e^{\pi i x \cdot Mx} \int e^{\pi i w \cdot M w} f(w) \overline{\Phi(w-x)} e^{-2\pi i \xi \cdot w} dw \\ &= e^{\pi i x \cdot Mx} V_\Phi S_M f(x, \xi) \quad \square \end{aligned}$$

Theorem 5.3.8. *Let $p \in [1, \infty]$. Suppose c is a first FIO slice permutation and $p_1 = p_2 = \dots = p_{2d} = 2$, $p_{2d+1} = p_{2d+2} = \dots = p_{4d} = p$, $p_{4d+1} = p_{4d+2} = \dots = p_{5d} = 1$ and $p_{5d+1} = p_{5d+2} = \dots = p_{6d} = \infty$. Suppose the following conditions hold.*

(a) c' is a first FIO symbol permutation.

(b) $q_1 = \dots = q_d = \infty$, $q_{d+1} = q_{d+2} = \dots = q_{3d} = 2$, $q_{3d+1} = q_{3d+2} = \dots = q_{5d} = p$ and $q_{5d+1} = q_{5d+2} = \dots = q_{6d} = 1$.

(c) $a \in M(c')^{q_1, q_2, \dots, q_{6d}}$.

(d) All the second order partial derivatives of φ are constant and $\varphi_{x_i y_j} = 0$ for all $i, j \in \{1, 2, \dots, d\}$.

Then $ae^{i\varphi} \in M(c)^{p_1, p_2, \dots, p_{6d}}$.

Proof. Again, we have $\varphi = \psi_1 + \psi_2$, where

$$\psi_1(w) = \varphi(0, 0, 0) + \sum_{|\alpha|=1} (D^\alpha \varphi)(0, 0, 0) w^\alpha$$

and

$$\psi_2(w) = \sum_{|\alpha|=2} \frac{2}{\alpha!} \left(\int_0^1 (1-t) (D^\alpha \varphi)(tw) dt \right) w^\alpha.$$

Notice that $e^{i\psi_2(w)} = e^{i\pi w \cdot M w}$ where M is the block matrix

$$M = \begin{bmatrix} M_1 & M_2 & M_3 \\ M_2^* & M_4 & M_5 \\ M_3^* & M_5^* & M_6 \end{bmatrix},$$

with

$$(M_1)_{i,j} = \frac{\varphi_{x_i x_j}(0, 0, 0)}{2} \quad \forall i, j \in \{1, 2, \dots, d\},$$

$$(M_2)_{i,j} = \frac{\varphi_{x_i y_j}(0, 0, 0)}{2} = 0 \quad \forall i, j \in \{1, 2, \dots, d\},$$

$$(M_3)_{i,j} = \frac{\varphi_{x_i \xi_j}(0, 0, 0)}{2} \quad \forall i, j \in \{1, 2, \dots, d\},$$

$$(M_4)_{i,j} = \frac{\varphi_{y_i y_j}(0, 0, 0)}{2} \quad \forall i, j \in \{1, 2, \dots, d\},$$

$$(M_5)_{i,j} = \frac{\varphi_{y_i \xi_j}(0, 0, 0)}{2} \quad \forall i, j \in \{1, 2, \dots, d\},$$

and

$$(M_6)_{i,j} = \frac{\varphi_{\xi_i \xi_j}(0, 0, 0)}{2} \quad \forall i, j \in \{1, 2, \dots, d\}.$$

Thus

$$|V_\Phi(ae^{i\varphi})(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)|$$

$$\begin{aligned}
&= |V_{\Phi}(ae^{i\varphi})(x, \xi)| \\
&= |V_{\Phi}(ae^{i\psi_1}e^{i\psi_2})(x, \xi)| \\
&= |V_{\Phi}S_M(ae^{i\psi_1})(x, \xi)| \\
&= |V_{S_{-M}\Phi}(ae^{i\psi_1})(x, \xi - Mx)| \\
&= |V_{S_{-M}\Phi}(ae^{i\psi_1})(x_1, x_2, x_3, \xi_1 - M_1x_1 - M_2x_2 - M_3x_3, \\
&\quad \xi_2 - M_2^*x_1 - M_4x_2 - M_5x_3, \xi_3 - M_3^*x_1 - M_5^*x_2 - M_6x_3)| \\
&= |V_{S_{-M}\Phi}(ae^{i\psi_1})(x_1, x_2, x_3, \xi_1 - M_1x_1 - M_3x_3, \\
&\quad \xi_2 - M_4x_2 - M_5x_3, \xi_3 - M_3^*x_1 - M_5^*x_2 - M_6x_3)|
\end{aligned} \tag{35}$$

where (35) follows from Lemma 5.3.7.

Since c is a first FIO slice permutation we have

$$\begin{aligned}
&\|ae^{i\varphi}\|_{M(c)^{p_1, \dots, p_{6d}}} \\
&= \sup_{\xi_3} \int \left(\iint \left(\iint |V_{\Phi}(ae^{i\varphi})(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)|^2 d\xi_1 dx_1 \right)^{\frac{p}{2}} d\xi_2 dx_2 \right)^{\frac{1}{p}} dx_3 \\
&= \sup_{\xi_3} \int \left(\iint \left(\iint |V_{S_{-M}\Phi}(ae^{i\psi_1})(x_1, x_2, x_3, \xi_1 - M_1x_1 - M_3x_3, \right. \right. \\
&\quad \left. \left. \xi_2 - M_4x_2 - M_5x_3, \xi_3 - M_3^*x_1 - M_5^*x_2 - M_6x_3)|^2 d\xi_1 dx_1 \right)^{\frac{p}{2}} d\xi_2 dx_2 \right)^{\frac{1}{p}} dx_3 \\
&\leq \int \left(\iint \left(\iint \sup_{\xi_3} |V_{S_{-M}\Phi}(ae^{i\psi_1})(x_1, x_2, x_3, \xi_1 - M_1x_1 - M_3x_3, \right. \right. \\
&\quad \left. \left. \xi_2 - M_4x_2 - M_5x_3, \xi_3 - M_3^*x_1 - M_5^*x_2 - M_6x_3)|^2 d\xi_1 dx_1 \right)^{\frac{p}{2}} d\xi_2 dx_2 \right)^{\frac{1}{p}} dx_3 \\
&= \int \left(\iint \left(\iint \sup_{\xi_3} |V_{\Phi}(ae^{i\varphi})(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)|^2 d\xi_1 dx_1 \right)^{\frac{p}{2}} d\xi_2 dx_2 \right)^{\frac{1}{p}} dx_3 \\
&\equiv \|ae^{i\psi_1}\|_{M(c')^{q_1, \dots, q_{6d}}}.
\end{aligned}$$

As in the proof of Theorem 5.3.5, we have

$$ae^{i\psi_1} = M_b(e^{i\varphi(0,0,0)}a),$$

where the components of $b \in \mathbb{R}^{3d}$ are $(D^\alpha \varphi)(0, 0, 0)$ for multi-indices α with $|\alpha| = 1$.

Therefore

$$\|ae^{i\psi_1}\|_{M(c')^{q_1, \dots, q_{6d}}} = \|a\|_{M(c')^{q_1, \dots, q_{6d}}} < \infty,$$

which implies $ae^{i\varphi} \in M(c)^{p_1, p_2, \dots, p_{6d}}$. □

Theorem 5.3.9. *Let $p \in [1, \infty]$. Suppose c is a second FIO slice permutation and $p_1 = p_2 = \dots = p_{2d} = 2$, $p_{2d+1} = p_{2d+2} = \dots = p_{4d} = p$, $p_{4d+1} = p_{4d+2} = \dots = p_{5d} = 1$ and $p_{5d+1} = p_{5d+2} = \dots = p_{6d} = \infty$. Suppose the following conditions hold.*

(a) c' is a second FIO symbol permutation.

(b) $q_1 = \dots = q_d = \infty$, $q_{d+1} = q_{d+2} = \dots = q_{3d} = 2$, $q_{3d+1} = q_{3d+2} = \dots = q_{5d} = p$
and $q_{5d+1} = q_{5d+2} = \dots = q_{6d} = 1$.

(c) $a \in M(c')^{q_1, q_2, \dots, q_{6d}}$

(d) All the second order partial derivatives of φ are constant and $\varphi_{x_i y_j} = 0$ for all $i, j \in \{1, 2, \dots, d\}$.

Then $ae^{i\varphi} \in M(c)^{p_1, p_2, \dots, p_{6d}}$.

Proof. The proof is similar to that of Theorem 5.3.8. □

5.4 Sharp Time-Frequency Conditions on the Symbol of a Fourier Integral Operator

In this section, we combine results from the previous two sections to give smoothness and time-frequency conditions on the phase function and symbol, respectively, of a Fourier integral operator that ensure the operator is Schatten class and prove the sharpness of these conditions.

Theorem 5.4.1. *Let c be a FIO slice permutation and $p_1 = p_2 = \dots = p_{2d} = 2$, $p_{2d+1} = p_{2d+2} = \dots = p_{4d} = p$ for some $p \in [1, 2]$, $p_{4d+1} = p_{4d+2} = \dots = p_{5d} = 1$ and $p_{5d+1} = p_{5d+2} = \dots = p_{6d} = \infty$. Suppose A is a Fourier integral operator with*

symbol a and phase function φ satisfying $\varphi \in C^2(\mathbb{R}^{3d})$ and $D^\alpha \varphi \in M^{\infty,1}(\mathbb{R}^{3d})$ for all multi-indices α with $|\alpha| = 2$. If $a \in M(c)^{p_1, p_2, \dots, p_{6d}}$ has compact support then $A \in \mathcal{I}_p(L^2(\mathbb{R}^d))$. Furthermore, this result is sharp in the sense that if one of the following conditions holds, then there are Fourier integral operators that are not in $\mathcal{I}_p(L^2(\mathbb{R}^d))$ with symbols in $M(c)^{q_1, q_2, \dots, q_{5d}, p_{5d+1}, p_{5d+2}, \dots, p_{6d}}$ and phase functions φ satisfying $\varphi \in C^2(\mathbb{R}^{3d})$ and $D^\alpha \varphi \in M^{\infty,1}(\mathbb{R}^{3d})$ for all multi-indices α with $|\alpha| = 2$.

(a) At least one of q_1, q_2, \dots, q_{2d} is larger than 2.

(b) At least one of $q_{2d+1}, q_{2d+2}, \dots, q_{4d}$ is larger than p .

(c) At least one of $q_{4d+1}, q_{4d+2}, \dots, q_{5d}$ is larger than 1.

Proof. The sufficiency of $a \in M(c)^{p_1, \dots, p_{6d}}$ follows from Theorems 5.2.2 and 5.3.5. Hence all that remains to be shown is that this result is sharp.

Notice that if we fix $\varphi = 0$ and $a(x, y, \xi) = a_1(x, y)a_2(\xi)$ and let A be the Fourier integral operator with phase function φ and symbol a , then A is the integral operator with kernel equal to $Ca_1(x, y)$, where $C = (\int a_2(\xi) d\xi)$.

Let c_1 be the permutation of $\{1, 2, \dots, 4d\}$ such that

$$c_1(1) = c(1), c_1(2) = c(2), \dots, c_1(d) = c(d)$$

$$c_1(d+1) = c(d+1), c_1(d+2) = c(d+2), \dots, c_1(2d) = c(2d)$$

$$c_1(2d+1) = c(3d+1), c_1(2d+2) = c(3d+2), \dots, c_1(3d) = c(4d)$$

and

$$c_1(3d+1) = c(4d+1), c_1(3d+2) = c(4d+2), \dots, c_1(4d) = c(5d)$$

and let c_2 be the permutation of $\{1, 2, \dots, 2d\}$ such that

$$c_2(1) = c(2d+1) - 4d, c_2(2) = c(2d+2) - 4d, \dots, c_2(d) = c(3d) - 4d$$

and

$$c_2(d+1) = c(5d+1) - 4d, c_2(d+2) = c(5d+2) - 4d, \dots, c_2(2d) = c(6d) - 4d.$$

Then c_1 is a slice permutation and

$$\begin{aligned} \|a\|_{M(c)^{q_1, q_2, \dots, q_{5d}, p_{5d+1}, p_{5d+2}, \dots, p_{6d}}} &= \|a_1 \otimes a_2\|_{M(c)^{q_1, q_2, \dots, q_{5d}, p_{5d+1}, p_{5d+2}, \dots, p_{6d}}} \\ &= \|a_1\|_{M(c_1)^{q_1, q_2, \dots, q_{4d}}} \|a_2\|_{M(c_2)^{q_{4d+1}, q_{4d+2}, \dots, q_{5d}, p_{5d+1}, p_{5d+2}, \dots, p_{6d}}} \end{aligned}$$

Suppose (a) or (b) holds. By Theorem 4.5.2, we can choose $a_1 \in M(c_1)^{q_1, q_2, \dots, q_{4d}}$ so that the integral operator with kernel a_1 is not in $\mathcal{I}_p(L^2(\mathbb{R}^d))$. Hence the Fourier integral operator with symbol $a(x, y, \xi) = a_1(x, y)a_2(\xi)$ and phase function $\varphi = 0$ is not in $\mathcal{I}_p(L^2(\mathbb{R}^d))$ either (for any choice of a_2).

Now suppose (c) holds. Choose $\lambda \in \ell^{q_{4d+1}, q_{4d+2}, \dots, q_{5d}}(\mathbb{Z}^d) \setminus \ell^{1, \dots, 1}(\mathbb{Z}^d)$ and set

$$a_2 = \sum_{j, k \in \mathbb{Z}^d} |\lambda_j| \langle 1, \psi_{j, k} \rangle \psi_{j, k},$$

where $\{\psi_{j, k}\}_{j, k \in \mathbb{Z}^d} = \{M_{\alpha k} T_{\alpha j} \psi\}_{j, k \in \mathbb{Z}^d}$ is a Parseval frame for $L^2(\mathbb{R}^d)$ with $\psi \in M^{1,1}(\mathbb{R}^d)$. Then by Theorem 4.3.19, $a_2 \in M(c_2)^{q_{4d+1}, q_{4d+2}, \dots, q_{5d}, p_{5d+1}, \dots, p_{6d}}$. But

$$\int a_2(\xi) d\xi = \langle a_2, 1 \rangle = \sum_{j, k \in \mathbb{Z}^d} |\lambda_j| \langle 1, \psi_{j, k} \rangle \langle \psi_{j, k}, 1 \rangle = \sum_{j, k \in \mathbb{Z}^d} |\lambda_j| \left| \widehat{\psi}(\alpha k) \right| = \infty,$$

so that A is not a well defined operator, and hence, not in $\mathcal{I}_p(L^2(\mathbb{R}^d))$. \square

Theorem 5.4.2. *Let $p \in [1, 2]$ and A be a Fourier integral operator with symbol a and phase function φ . Suppose the following conditions hold.*

(a) c is a FIO symbol permutation.

(b) $p_1 = \dots = p_d = \infty$, $p_{d+1} = p_{d+2} = \dots = p_{3d} = 2$, $p_{3d+1} = p_{3d+2} = \dots = p_{5d} = p$
and $p_{5d+1} = p_{5d+2} = \dots = p_{6d} = 1$.

(c) $a \in M(c)^{p_1, p_2, \dots, p_{6d}}$.

(d) All the second order partial derivatives of φ are constant and $\varphi_{x_i y_j} = 0$ for all $i, j \in \{1, 2, \dots, d\}$.

Then $A \in \mathcal{I}_p(L^2(\mathbb{R}^d))$. Furthermore, this result is sharp in the sense that if one of the following conditions hold, then there exist Fourier integral operators with phase functions satisfying (d) and symbols in $M(c)^{p_1, p_2, \dots, p_d, q_{d+1}, q_{d+2}, \dots, q_{6d}}$ that are not in $\mathcal{I}_p(L^2(\mathbb{R}^d))$.

(e) At least one of $q_{d+1}, q_{d+2}, \dots, q_{3d}$ is larger than 2.

(f) At least one of $q_{3d+1}, q_{3d+2}, \dots, q_{5d}$ is larger than p .

(g) At least one of $q_{5d+1}, q_{5d+2}, \dots, q_{6d}$ is larger than 1.

Proof. Sufficiency of conditions (a), (b), (c) and (d) follows from Theorems 5.2.2, 5.3.8 and 5.3.9.

If we fix $\varphi = 0$ and $a(x, y, \xi) = a_3(x, y)a_4(\xi)$ and let A be the Fourier integral operator with phase function φ and symbol a , then A is the integral operator with kernel equal to $Ca_3(x, y)$, where $C = (\int a_4(\xi) d\xi)$.

Let c_3 be the permutation of $\{1, 2, \dots, 4d\}$ such that

$$c_3(1) = c(1) - d, c_3(2) = c(2) - d, \dots, c_3(d) = c(d) - d$$

$$c_3(d+1) = c(d+1) - d, c_3(d+2) = c(d+2) - d, \dots, c_3(2d) = c(2d) - d$$

$$c_3(2d+1) = c(3d+1) - d, c_3(2d+2) = c(3d+2) - d, \dots, c_3(3d) = c(4d) - d$$

and

$$c_3(3d+1) = c(4d+1) - d, c_3(3d+2) = c(4d+2) - d, \dots, c_3(4d) = c(5d) - d$$

and let c_4 be the permutation of $\{1, 2, \dots, 2d\}$ such that

$$c_4(1) = c(2d+1) - 4d, c_4(2) = c(2d+2) - 4d, \dots, c_4(d) = c(3d) - 4d$$

and

$$c_4(d+1) = c(5d+1), c_4(d+2) = c(5d+2), \dots, c_4(2d) = c(6d).$$

Then c_3 is a slice permutation and

$$\|a\|_{M(c)^{p_1, p_2, \dots, p_d, q_{d+1}, q_{d+2}, \dots, q_{6d}}} = \|a_3\|_{M(c_3)^{q_{d+1}, q_{d+2}, \dots, q_{5d}}} \|a_4\|_{M(c_4)^{p_1, p_2, \dots, p_d, q_{5d+1}, \dots, q_{6d}}}.$$

If (e) or (f) hold, then by Theorem 4.5.2 we can choose $a_3 \in M(c_3)^{q_{d+1}, q_{d+2}, \dots, q_{5d}}$ so that the integral operator with kernel a_3 is not in $\mathcal{I}_p(L^2(\mathbb{R}^d))$. Hence the Fourier integral operator with symbol $a(x, y, \xi) = a_3(x, y)a_4(\xi)$ and phase function $\varphi = 0$ is not in $\mathcal{I}_p(L^2(\mathbb{R}^d))$ either.

Suppose (g) holds. Choose $\lambda \in \ell^{q_{5d+1}, q_{5d+2}, \dots, q_{6d}}(\mathbb{Z}^d) \setminus \ell^{1, \dots, 1}(\mathbb{Z}^d)$ and set

$$a_4 = \sum_{j, k \in \mathbb{Z}^d} |\lambda_j| \langle 1, \psi_{j, k} \rangle \psi_{j, k},$$

where $\{\psi_{j, k}\}_{j, k \in \mathbb{Z}^d} = \{M_{\alpha k} T_{\alpha j} \psi\}_{j, k \in \mathbb{Z}^d}$ is a Parseval frame for $L^2(\mathbb{R}^d)$ with $\psi \in M^{1,1}(\mathbb{R}^d)$. Then $a_4 \in M(c_4)^{p_1, \dots, p_d, q_{5d+1}, \dots, q_{6d}}$ but

$$\int a_4(\xi) d\xi = \langle a_4, 1 \rangle = \sum_{j, k \in \mathbb{Z}^d} |\lambda_j| \langle 1, \psi_{j, k} \rangle \langle \psi_{j, k}, 1 \rangle = \sum_{j, k \in \mathbb{Z}^d} |\lambda_j| \left| \widehat{\psi}(\alpha k) \right| = \infty,$$

so that A is not a well defined operator. □

The previous theorem has implications for a common type of Fourier integral operator, namely the type with phase function $\varphi(x, y, \xi) = 2\pi x \cdot \xi - 2\pi y \cdot \xi$.

Corollary 5.4.3. *Suppose $p \in [1, 2]$ and $p_1 = \dots = p_d = \infty$, $p_{d+1} = p_{d+2} = \dots = p_{3d} = 2$, $p_{3d+1} = p_{3d+2} = \dots = p_{5d} = p$ and $p_{5d+1} = p_{5d+2} = \dots = p_{6d} = 1$. Let c be a FIO symbol permutation. If A is a Fourier integral operator with phase function $\varphi(x, y, \xi) = 2\pi x \cdot \xi - 2\pi y \cdot \xi$ and symbol $a \in M(c)^{p_1, p_2, \dots, p_{6d}}$, then $A \in \mathcal{I}_p(L^2(\mathbb{R}^d))$.*

APPENDIX A

FOURIER FRAME LEMMA

In this section, we give a detailed proof of Lemma 2.2.12. Our proof relies on the following theorem, a special case of Theorem 7 in [2].

Theorem A.0.4. *Suppose $\mathcal{F} = \{f_i\}_{i \in I}$, $\mathcal{E} = \{e_j\}_{j \in \mathbb{Z}}$ are frames for Hilbert space H with frame bounds A, B and E, F respectively. Also suppose there is a map $a : I \rightarrow \mathbb{Z}$ so that the following properties hold.*

(a) $\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}$

$$\sum_{\{i \in I : |a(i) - j| > \frac{N_\epsilon}{2}\}} |\langle f_i, e_j \rangle|^2 < \epsilon \quad \forall j \in \mathbb{Z}.$$

(b) $\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}$

$$\sum_{\{j \in \mathbb{Z} : |a(i) - j| > \frac{N_\epsilon}{2}\}} |\langle f_i, e_j \rangle|^2 < \epsilon \quad \forall i \in I.$$

Then

$$\frac{A \liminf \|e_j\|^2}{F \limsup \|f_i\|^2} \leq D^-(I, a) \leq D^+(I, a) \leq \frac{B \limsup \|e_j\|^2}{E \liminf \|f_i\|^2},$$

where

$$D^+(I, a) = \limsup_{K \rightarrow \infty} \sup_{j \in \mathbb{Z}} \frac{|\{i \in I : |a(i) - j| \leq \frac{K}{2}\}|}{|\{n \in \mathbb{Z} : |n - j| \leq \frac{K}{2}\}|}$$

and

$$D^-(I, a) = \liminf_{K \rightarrow \infty} \inf_{j \in \mathbb{Z}} \frac{|\{i \in I : |a(i) - j| \leq \frac{K}{2}\}|}{|\{n \in \mathbb{Z} : |n - j| \leq \frac{K}{2}\}|}.$$

Proof of Lemma 2.2.12. Suppose $V \subset \mathbb{R}$ and $\mathcal{F} = \{e^{2\pi i v w}\}_{v \in V} = \{f_v\}_{v \in V}$ is a frame for $L^2[-r, r]$ with bounds A, B . By Lemma 2 in [49], V is relatively separated in \mathbb{R} . Notice that $\mathcal{E} = \left\{ (2r)^{-\frac{1}{2}} e^{2\pi i \frac{n}{2r} w} \right\}_{n \in \mathbb{Z}} = \{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2[-r, r]$

Define $a : V \rightarrow \mathbb{Z}$ so that $|2rv - a(v)| \leq \frac{1}{2}$ for all $v \in V$. We will show that conditions (a) and (b) in Theorem A.0.4 are satisfied.

First, we observe that

$$|\langle f_v, e_j \rangle|^2 = \left| (2r)^{-\frac{1}{2}} \int_{[-r, r]} e^{2\pi i(v - \frac{j}{2r})w} dw \right|^2 \leq \frac{1}{2r\pi^2 |v - \frac{j}{2r}|^2} \quad \forall v \in V, j \in \mathbb{Z}.$$

Fix $\epsilon > 0$. Since V is relatively separated, so is $4rV$. Hence by Lemma 2.2.8, we can choose $C \in (0, \infty)$ so that

$$|4rV \cap x + [-M - N, M + N] \setminus [-M, M]| \leq 2CN \quad \forall M, N \in \mathbb{N}, x \in \mathbb{R}.$$

Choose $N_\epsilon \in \mathbb{N}$ so that

$$\frac{48rC}{\pi^2} \sum_{n \geq N_\epsilon} \frac{1}{(n-1)^2} < \epsilon$$

and

$$\frac{16r}{\pi^2} \sum_{n \geq N_\epsilon} \frac{1}{(n-1)^2} < \epsilon.$$

Then for any $j \in \mathbb{Z}$, we have

$$\sum_{\{v \in V : |a(v) - j| > \frac{N_\epsilon}{2}\}} |\langle f_v, e_j \rangle|^2 = \sum_{n \geq N_\epsilon} \sum_{\{v \in V : \frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}\}} |\langle f_v, e_j \rangle|^2.$$

But if $\frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}$ then $\frac{n+2}{4r} \geq |v - \frac{j}{2r}| > \frac{n-1}{4r}$. Thus

$$\begin{aligned} & \sum_{\{v \in V : |a(v) - j| > \frac{N_\epsilon}{2}\}} |\langle f_v, e_j \rangle|^2 \\ &= \sum_{n \geq N_\epsilon} \sum_{\{v \in V : \frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}\}} |\langle f_v, e_j \rangle|^2 \\ &\leq \sum_{n \geq N_\epsilon} \sum_{\{v \in V : \frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}\}} \frac{1}{2r\pi^2 |v - \frac{j}{2r}|^2} \\ &\leq \sum_{n \geq N_\epsilon} \sum_{\{v \in V : \frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}\}} \frac{16r^2}{2r\pi^2 (n-1)^2} \\ &= \frac{8r}{\pi^2} \sum_{n \geq N_\epsilon} \frac{|\{v \in V : \frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}\}|}{(n-1)^2} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{8r}{\pi^2} \sum_{n \geq N_\epsilon} \frac{|\{v \in V : \frac{n+2}{4r} \geq |v - \frac{j}{2r}| > \frac{n-1}{4r}\}|}{(n-1)^2} \\
&= \frac{8r}{\pi^2} \sum_{n \geq N_\epsilon} \frac{|\{v \in V : n+2 \geq |4rv - 2j| > n-1\}|}{(n-1)^2} \\
&= \frac{8r}{\pi^2} \sum_{n \geq N_\epsilon} \frac{|4rV \cap 2j + [-n-2, n+2] \setminus [-n+1, n-1]|}{(n-1)^2} \\
&\leq \frac{48rC}{\pi^2} \sum_{n \geq N_\epsilon} \frac{1}{(n-1)^2} \\
&< \epsilon.
\end{aligned}$$

Hence (a) is satisfied.

To prove that (b) is satisfied, we note that for any $v \in V$ and any $n \in \mathbb{N}$, there can be at most two $j \in \mathbb{Z}$ satisfying $\frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}$. Thus for any $v \in V$ we have

$$\begin{aligned}
&\sum_{\{j \in \mathbb{Z} : |a(v) - j| > \frac{N_\epsilon}{2}\}} |\langle f_v, e_j \rangle|^2 \\
&= \sum_{n \geq N_\epsilon} \sum_{\{j \in \mathbb{Z} : \frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}\}} |\langle f_v, e_j \rangle|^2 \\
&\leq \sum_{n \geq N_\epsilon} \sum_{\{j \in \mathbb{Z} : \frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}\}} \frac{1}{2r\pi^2 |v - \frac{j}{2r}|^2} \\
&\leq \sum_{n \geq N_\epsilon} \sum_{\{j \in \mathbb{Z} : \frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}\}} \frac{16r^2}{2r\pi^2 (n-1)^2} \\
&\leq \frac{8r}{\pi^2} \sum_{n \geq N_\epsilon} \frac{|\{j \in \mathbb{Z} : \frac{n+1}{2} \geq |a(v) - j| > \frac{n}{2}\}|}{(n-1)^2} \\
&\leq \frac{16r}{\pi^2} \sum_{n \geq N_\epsilon} \frac{1}{(n-1)^2} \\
&< \epsilon.
\end{aligned}$$

Thus (b) is also satisfied.

Since $\liminf_{v \in V} \|f_v\|_{L^2[-r,r]}^2 = \limsup_{v \in V} \|f_v\|_{L^2[-r,r]}^2 = 2r$, Theorem A.0.4 implies

$$\frac{A}{2r} \leq D^-(I, a) \leq D^+(I, a) \leq \frac{B}{2r}.$$

Fix $N \in \mathbb{N}$. Choose $K_N \in \mathbb{N}$ such that $\frac{K_N-1}{2} < 2rN + 1 \leq \frac{K_N}{2}$. Notice that if $|v - x| \leq N$ then $|a(v) - a(x)| = |a(v) - 2rv + 2rv - 2rx + 2rx - a(x)| \leq 2rN + 1$.

Thus

$$\begin{aligned}
& \sup_{x \in \mathbb{R}} \frac{|V \cap x + [-N, N]|}{2N} \\
& \leq \sup_{x \in \mathbb{R}} \frac{|\{v \in V : |a(v) - a(x)| \leq 2rN + 1\}|}{2N} \\
& \leq \sup_{j \in \mathbb{Z}} \frac{|\{v \in V : |a(v) - j| \leq 2rN + 1\}|}{2N} \\
& \leq 2r \cdot \sup_{j \in \mathbb{Z}} \frac{|\{v \in V : |a(v) - j| \leq \frac{K_N}{2}\}|}{K_N - 3} \\
& = 2r \cdot \sup_{j \in \mathbb{Z}} \frac{|\{n \in \mathbb{Z} : |n - j| \leq \frac{K_N}{2}\}| |\{v \in V : |a(v) - j| \leq \frac{K_N}{2}\}|}{K_N - 3} \\
& \leq 2r \frac{K_N + 1}{K_N - 3} \cdot \sup_{j \in \mathbb{Z}} \frac{|\{v \in V : |a(v) - j| \leq \frac{K_N}{2}\}|}{|\{n \in \mathbb{Z} : |n - j| \leq \frac{K_N}{2}\}|}
\end{aligned}$$

which implies

$$\begin{aligned}
& D_{\mathbb{R}}^+(V) \\
& = \limsup_{N \rightarrow \infty} \sup_{x \in \mathbb{R}} \frac{|V \cap x + [-N, N]|}{2N} \\
& \leq \limsup_{N \rightarrow \infty} \left(2r \frac{K_N + 1}{K_N - 3} \cdot \sup_{j \in \mathbb{Z}} \frac{|\{v \in V : |a(v) - j| \leq \frac{K_N}{2}\}|}{|\{n \in \mathbb{Z} : |n - j| \leq \frac{K_N}{2}\}|} \right) \\
& \leq 2r \left(\limsup_{K \rightarrow \infty} \frac{K + 1}{K - 3} \right) \left(\limsup_{K \rightarrow \infty} \sup_{j \in \mathbb{Z}} \frac{|\{v \in V : |a(v) - j| \leq \frac{K}{2}\}|}{|\{n \in \mathbb{Z} : |n - j| \leq \frac{K}{2}\}|} \right) \\
& = 2rD^+(I, a) \\
& \leq B.
\end{aligned}$$

Similar arguments show $A \leq 2rD^-(I, a) \leq D_{\mathbb{R}}^-(V)$. □

REFERENCES

- [1] BALAN, R., “Stability theorems for Fourier frames and wavelet Riesz bases,” *J. Fourier Anal. Appl.*, vol. 5, pp. 499–504, 1997.
- [2] BALAN, R., CASAZZA, P., HEIL, C., and LANDAU, Z., “Density, overcompleteness, and localization of frames, I. Theory,” *J. Fourier Anal. Appl.*, vol. 12, pp. 105–143, 2006.
- [3] BALAN, R., CASAZZA, P., HEIL, C., and LANDAU, Z., “Density, overcompleteness, and localization of frames, II. Gabor systems,” *J. Fourier Anal. Appl.*, vol. 12, pp. 309–344, 2006.
- [4] BALAN, R., POOR, H., RICKARD, S., and VERDU, S., “Canonical time-frequency, time-scale, and frequency-scale representations of time-varying channels,” *Comm. Info. Sys.*, vol. 5, pp. 197–226, 2005.
- [5] BALAN, R., POOR, V., RICKARD, S., and VERDU, S., “Canonical time-frequency, time-scale, and frequency-scale representations of time-varying channels,” *Commun. Inf. Syst.*, vol. 5, pp. 197–226, 2005.
- [6] BASDEVANT, C. and PERRIER, V., “Besov norms in terms of the continuous wavelet transform,” *Math. Models and Meth.*, vol. 5, pp. 649–664, 1996.
- [7] BENEDEK, A. and PANZONE, R., “The spaces L^p , with mixed norm,” *Duke Math. J.*, vol. 28, pp. 301–324, 1961.
- [8] BERGH, J., *Interpolation spaces: an introduction*. Berlin: Springer-Verlag, 1976.
- [9] BISHOP, S., “Mixed modulation spaces and their application to pseudodifferential operators,” *J. Math. Anal. Appl.*, vol. 363, pp. 255–264, 2010.
- [10] BOULKHEMAIR, A., “Remarks on a Wiener type pseudodifferential algebra and Fourier integral operators,” *Math. Res. Lett.*, vol. 4, pp. 53–67, 1997.
- [11] CANDÈS, E., “Ridgelets: Theory and applications,” tech. rep., Stanford University, Stanford, California, 1998.
- [12] CANDÈS, E. and DEMANET, L., “Curvelets and Fourier integral operators,” *C. R. Math. Acad. Sci. Paris*, vol. 336, pp. 395–398, 2003.
- [13] CANDÈS, E. and DONOHO, D., “Ridgelets: A key to higher-dimensional intermittency?,” *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, vol. 357, pp. 2495–2509, 1999.

- [14] CHRISTENSEN, O., *An Introduction to Frames and Riesz Bases*. Boston: Birkhäuser, 2002.
- [15] CHUI, C. and SHI, X., “Inequalities of Littlewood-Paley type for frames and wavelets,” *SIAM J. Math. Anal.*, vol. 24, pp. 263–277, 1993.
- [16] CONCETTI, F., GARELLO, G., and TOFT, J., “Trace ideals for Fourier integral operators with nonsmooth symbols, II,” *preprint*.
- [17] CONCETTI, F. and TOFT, J., “Trace ideals for Fourier integral operators with nonsmooth symbols, I,” *Fields Ins. Comm.*, vol. 52, pp. 255–264, 2007.
- [18] CONCETTI, F. and TOFT, J., “Schatten-von Neumann properties for Fourier integral operators with non-smooth symbols, I,” *Ark. Math.*, vol. 47, pp. 295–312, 2009.
- [19] CORDERO, E. and NICOLA, F., “Boundedness of Schrödinger type propagators on modulation spaces,” *preprint*.
- [20] CORDERO, E., NICOLA, F., and RODINO, L., “Time-frequency analysis of Fourier integral operators,” *preprint*.
- [21] CZAJA, W., “Boundedness of pseudodifferential operators on modulation spaces,” *J. Math. Appl. Anal.*, vol. 284, pp. 389–396, 2003.
- [22] CZAJA, W. and RZESZOTNIK, Z., “Pseudodifferential operators and Gabor frames: spectral asymptotics,” *Math. Nachr.*, vol. 233/234, pp. 77–88, 2002.
- [23] DAUBECHIES, I., “On the distributions corresponding to bounded operators in the Weyl quantization,” *Comm. Math. Phys.*, vol. 75, pp. 229–238, 1980.
- [24] DAUBECHIES, I., “The wavelet transform, time-frequency localization and signal analysis,” *IEEE Trans. Inform. Theory*, vol. 36, pp. 961–1005, 1990.
- [25] DAUBECHIES, I., *Ten Lectures on Wavelets*. Philadelphia: SIAM, 1992.
- [26] DUOANDIKOETXEA, J., *Fourier Analysis*. Providence, RI: American Mathematical Society, 2001.
- [27] FEICHTINGER, H., “Modulation spaces: looking back and ahead,” *Sampl. Theory Signal Image Process.*, vol. 5, pp. 109–140, 2006.
- [28] FEICHTINGER, H., SUN, W., and ZHOU, X., “Two Banach spaces of atoms for stable wavelet frame expansions,” *J. Approx. Theory*, vol. 146, pp. 28–70, 2007.
- [29] FERNANDEZ-RUBIO, J. and REBOLLO-NEIRA, L., “On wideband deconvolution using wavelet frames,” *IEEE Signal Proc. Letters*, vol. 4, pp. 207–209, 1997.
- [30] FOLLAND, G., *Harmonic Analysis in Phase Space*. Princeton, NJ: Princeton Univ. Press, 1989.

- [31] FOLLAND, G., *A Course in Abstract Harmonic Analysis*. New York: CRC Press, 1995.
- [32] GRAFAKOS, L., *Classical Fourier Analysis*. New York: Springer, 2008.
- [33] GRÖCHENIG, K., *Foundations of Time-Frequency Analysis*. Boston: Birkhäuser, 2001.
- [34] GRÖCHENIG, K., “The Homogeneous Approximation Property and the Comparison Theorem for coherent frames,” *Sampl. Theory Signal Image Process.*, vol. 7, no. 3, pp. 271–279, 2008.
- [35] GRÖCHENIG, K. and HEIL, C., “Modulation spaces and pseudodifferential operators,” *Integral Equations Operator Theory*, vol. 34, pp. 439–457, 1999.
- [36] GRÖCHENIG, K. and HEIL, C., “Counterexamples for boundedness of pseudodifferential operators,” *Osaka J. Math.*, vol. 41, pp. 681–691, 2004.
- [37] GRÖCHENIG, K., KUTYNIOK, G., and SEIP, K., “Landau’s necessary density conditions for LCA groups,” *J. Funct. Anal.*, vol. 255, no. 7, pp. 1831–1850, 2008.
- [38] GRÖCHENIG, K. and STROHMER, T., “Pseudodifferential operators on locally compact groups and Sjöstrand’s symbol class,” *J. Reine Angew. Math.*, vol. 613, pp. 237–249, 2007.
- [39] GROSSMANN, A., MORLET, J., and PAUL, T., “Transforms associated to square integrable group representations. I. General results,” *J. Math. Phys.*, vol. 26, pp. 2473–2479, 1985.
- [40] GUO, K. and LABATE, D., “Representation of Fourier integral operators using shearlets,” *J. Fourier Anal. Appl.*, vol. 14, pp. 327–371, 2008.
- [41] HEIL, C., “Modulation spaces and pseudodifferential operators,” in *Advances in Gabor Analysis*, Boston: Birkhäuser, 2003.
- [42] HEIL, C., “History and evolution of the Density Theorem for Gabor frames,” *J. Fourier Anal. Appl.*, vol. 13, pp. 113–166, 2007.
- [43] HEIL, C. and KUTYNIOK, G., “The Homogeneous Approximation Property for wavelet frames,” *J. Approx. Theory*, vol. 147, pp. 28–46, 2007.
- [44] HEIL, C. and KUTYNIOK, G., “Density of frames and Schauder bases of windowed exponentials,” *Houston J. Math.*, vol. 34, pp. 565–600, 2008.
- [45] HEIL, C., RAMANATHAN, J., and TOPIWALA, P., “Singular values of compact pseudodifferential operators,” *J. Funct. Anal.*, vol. 150, pp. 426–452, 1997.
- [46] HOLSCHNEIDER, M., *Wavelets: An Analysis Tool*. Oxford: Clarendon Press, 1995.

- [47] HÖRMANDER, L., “The Weyl calculus of pseudodifferential operators,” *Comm. Pure Appl. Math.*, vol. 32, pp. 360–444, 1979.
- [48] HÖRMANDER, L., *The Analysis of Partial Differential Equations*. Berlin: Springer-Verlag, 1990.
- [49] JAFFARD, S., “A density criterion for frames of complex exponentials,” *Michigan Math. J.*, vol. 38, pp. 339–348, 1991.
- [50] KUTYNIOK, G., “Affine density, frame bounds and the admissibility condition for wavelet frames,” *Constr. Approx.*, vol. 25, pp. 239–253, 2007.
- [51] KUTYNIOK, K., *Computation of the density of weighted wavelet systems*, vol. 5207. 2003.
- [52] LABATE, D., “Pseudodifferential operators on modulation spaces,” *J. Math. Appl. Anal.*, vol. 262, pp. 242–255, 2001.
- [53] LANDAU, H., “Necessary density conditions for sampling and interpolation of certain entire functions,” *Acta. Math.*, vol. 117, pp. 37–52, 1967.
- [54] MARKOE, A., *Analytic Tomography*. Cambridge: Cambridge University Press, 2006.
- [55] NYQUIST, H., “Certain topics in telegraph transmission theory,” *AIEE Trans.*, vol. 47, pp. 617–644, 1928.
- [56] OEHRING, C. and COCHRAN, J., “Integral operators and an analogue of the Hausdorff-Young theorem,” *J. London Math. Soc.*, vol. 15, pp. 511–520, 1997.
- [57] QUINTO, E., “The Radon transform, inverse problems, and tomography,” *Proc. Sympos. Appl. Math.*, vol. 63, pp. 1–23, 2006.
- [58] RAMANATHAN, J. and STEGER, T., “Incompleteness of sparse coherent states,” *Appl. Comput. Harmon. Anal.*, vol. 2, pp. 128–153, 1995.
- [59] ROCHBERG, R. and TACHIZAWA, K., *Pseudodifferential operators, Gabor frames and local trigonometric bases*, vol. Gabor Analysis and Algorithms. Boston: Birkhäuser, 1998.
- [60] RUSSO, B., “On the Hausdorff-Young theorem for integral operators,” *Pacific J. Math.*, vol. 68, pp. 241–253, 1977.
- [61] SEEGER, A., SOGGE, C., and STEIN, E., “Regularity properties of Fourier integral operators,” *Ann. of Math.*, vol. 134, pp. 231–251, 1991.
- [62] SHANNON, C., “Communication in the presence of noise,” *Proc. IRE.*, vol. 37, pp. 10–21, 1949.

- [63] SIMON, B., *Trace ideals and their applications*. London: Cambridge University Press, 1979.
- [64] SJÖSTRAND, J., “An algebra of pseudodifferential operators,” *Math. Res. Lett.*, vol. 1, pp. 185–192, 1994.
- [65] SOLMON, D., “Asymptotic formulas for the dual Radon transform and applications,” *Math. Z.*, vol. 195, pp. 321–343, 1987.
- [66] STEFANOV, A. and TORRES, T., “Calderón-Zygmund operators on mixed Lebesgue spaces and applications to null forms,” *J. London Math. Soc.*, vol. 70, pp. 447–462, 2004.
- [67] STEIN, E., *Harmonic Analysis*. Princeton, NJ: Princeton Univ. Press, 1993.
- [68] STROHMER, T., “Pseudodifferential operators and Banach algebras in mobile communications,” *Appl. Comput. Harmon. Anal.*, vol. 20, pp. 237–249, 2006.
- [69] SUN, W. and ZHOU, X., “Density and stability of wavelet frames,” *Appl. Comput. Harmon. Anal.*, vol. 15, pp. 117–133, 2003.
- [70] SUN, W. and ZHOU, X., “Density of irregular wavelet frames,” *Proc. Amer. Math. Soc.*, vol. 132, pp. 2377–2387, 2004.
- [71] TACHIZAWA, K., “The boundedness of pseudodifferential operators on modulation spaces,” *Math. Nachr.*, vol. 168, pp. 263–277, 1994.
- [72] TACHIZAWA, K., “The pseudodifferential operators and Wilson bases,” *J. Math. Pures Appl.*, vol. 9, pp. 509–529, 1996.
- [73] TOFT, J., “Continuity and Schatten properties for pseudo-differential operators on modulation spaces,” *Oper. Theory Adv. Appl.*, vol. 172, pp. 173–206, 2007.
- [74] TREVES, F., *Introduction to Pseudodifferential and Fourier Integral Operators*, vol. 2. New York: Plenum Press, 1980.
- [75] WEISS, L., YOUNG, R., and SIBUL, L., “Wideband processing of acoustic signals using wavelet transforms. Part I. Theory,” *J. Acoust. Soc. Am*, vol. 96, pp. 850–856, 1994.
- [76] WONG, M., *Trace class Weyl transforms*, vol. Pseudo-differential operators and related topics. Basel: Birkhäuser, 2005.
- [77] YANG, X. and ZHOU, X., “An extension of the Chui-Shi frame condition to nonuniform affine operations,” *Appl. Comput. Harmon. Anal.*, vol. 16, pp. 148–157, 2004.
- [78] YOUNG, R., *An Introduction to Nonharmonic Fourier Series, revised first edition*. San Diego: Academic Press, 2001.