

SPECTRAL NEVANLINNA-PICK INTERPOLATION THEORY AND ROBUST STABILIZATION

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Abstract

In this note we will discuss a new kind of interpolation theory in which one bounds the spectral radius of the matrix-valued interpolating functions instead of the norm as is the case with ordinary Nevanlinna-Pick interpolation. We show how this is related to certain kinds of multivariate stability margin problems of the kind considered by Doyle [2] and Safonov [8].

1. Introduction

In the past few years there has appeared a large number of papers concerned with the application of Nevanlinna-Pick interpolation theory to various problems in control, most notably the area of H^∞ -optimization theory. We refer the reader to the excellent new monograph of Francis [4] for a survey of these results.

Now the connection of interpolation theory to problems involving LTI, finite dimensional plants is very simple. Namely one may easily show that the question of internal stabilization amounts to a Lagrange-type interpolation problem for such plants. For SISO systems this was probably first observed in the paper of Youla, Bongiorno, and Lu [12], and in the MIMO case by a number of people (see e.g. [1], [6], [7]). The way then that sensitivity H^∞ - optimization reduces to a Nevanlinna-Pick interpolation problem is that one imposes a bound on the norm of the corresponding interpolating functions.

For SISO systems gain margin optimization works in the same way in that via a conformal equivalence, we can transform the question of finding an interpolating function whose range is a simply-connected subset of the complex plane to one whose range is the unit disc and hence once more derive an interpolation problem of the Nevanlinna-Pick kind. For MIMO systems, we will show below that if one wants to play the same game with multivariate generalizations of gain margin ([2], [8], [11]) one derives interpolation not with a norm constraint but with a spectral radius constraint. (Helton [5] in his discussion of Doyle's work exactly characterizes μ as a generalized spectral radius.)

The purpose of this preliminary note is simply to reduce precisely a very special case of multivariate gain margin optimization to this new and highly difficult interpolation problem. A mathematical discussion of its solution will appear in some joint work of the author with the operator theorists Hari Bercovici and Ciprian Foias. The reader will see even in this case how difficult such problems are, and why in the SISO setting sensitivity minimization is equivalent to gain margin optimization. The author would like to thank Professor Tryphon Georgiou for valuable discussion of the material here and especially for the neat formulation of (3.1).

2. Internal Stability as Interpolation

In this section we would like to briefly show how the problem of internal stabilization for LTI, finite dimensional plants reduces to one of interpolation. The facts here are rather well-known. See [12], [1], [6], [7], [9].

More precisely, let $P(s)$ denote a $p \times m$ LTI finite dimensional plant and $C(s)$ an $m \times p$ internally stabilizing compensator. In the usual way we define the sensitivity function to be $S(s) := (I + P(s)C(s))^{-1}$. Then invoking the standard coprime factorizations we get that

$$S(s) = L - UZ$$

where L and U are completely determined by $P(s)$ and the "free parameter" $Z \in RH_{m \times p}^\infty$ (= the space of $m \times p$ matrices with entries which are real rational functions bounded in the right half plane H). For simplicity we assume that the transmission zeros of U are simple. If we denote these zeros by z_1, \dots, z_n then there exist nonzero vectors v_1, \dots, v_n such that

$$v_j^* U(z_j) = 0$$

for $j = 1, \dots, n$. If we now set $w_j^* := v_j^* L(z_j)$, we get that the requirement of internal stability translates into the following interpolation conditions on S :

$$v_j^* S(z_j) = w_j^* \tag{1}$$

for $j = 1, \dots, n$. In other words finding an internally stabilizing compensator for the plant $P(s)$ is equivalent to constructing a matrix-valued function $S(s)$ with entries rational functions which are analytic and bounded in the right half plane and which

obeys the interpolating conditions (1). We should finally note that sensitivity minimization (in the H^∞ sense) from this point of view simply amounts to interpolating by matrix-valued functions satisfying the interpolating conditions (1). The mathematical solution for this problem (including the parametrization of the optimal solutions) appears in several places. This particular version of the Nevanlinna-Pick problem is called *tangential Nevanlinna-Pick* for obvious reasons. See e.g. [3] and [7] and the references therein.

3. Spectral Formulation of Nevanlinna-Pick Theory

In this section we formulate a control problem which reduces to a new kind of interpolation problem in which one bounds the spectral radius of the interpolant instead of the norm. In order to do this we will first need some remarks on the effect of linear fractional transformations on interpolation problems.

Let $F(s)$ and $G(s) \in RH_{p \times m}^\infty$. Moreover let $F = ND^{-1} = D_l^{-1}N_l$, and $G = AB^{-1} = B_l^{-1}A_l$ be the corresponding coprime factorizations. Suppose

$$F = (\alpha + \beta G)(\gamma + \delta G)^{-1}$$

$$G = (\gamma_l + G\gamma_l)^{-1}(\alpha_l + G\beta_l)$$

where

$$\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \neq 0$$

and

$$\begin{bmatrix} \beta_l & -\delta_l \\ \alpha_l & \gamma_l \end{bmatrix} \begin{bmatrix} \gamma & \delta \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

With this notation we can now state the following simple lemma whose proof is obvious:

LEMMA (3.1): Let $z_j \in H$ (the right half plane) for $j = 1, \dots, n$. Then

(i) $v^* F(z_j) = w^*$ if and only if $y^* G(z_j) = x^*$ where

$$[-x^* \ y^*] := [-w^* \ v^*] \begin{bmatrix} \gamma & \delta \\ \alpha & \beta \end{bmatrix};$$

(ii) $F(z_j)v = w$ if and only if $G(z_j)y = x$ with

$$\begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} \gamma_l & \alpha_l \\ \delta_l & \beta_l \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}.$$

We are now ready to formulate our control problem. Let $P(s)$ be a $p \times m$ MIMO plant as above. Consider the following family of plants (see also [9-11]):

$$P_k(s) := \{kP(s) : k \in K\}$$

$$K := \{k \in \mathbf{C} : k = (1 + s), |s| \leq r\}.$$

Then using the same arguments as in [9-11], we may show that $C(s)$ internally stabilizes the closed loop for the family $P_k(s)$, $k \in K$ if and only if there exists a rational matrix-valued function $S(s)$ which is analytic and bounded in the right half plane such that $S : \bar{H} \rightarrow G$ and which moreover satisfies the interpolation conditions (1), where \bar{H} denotes the closed right half plane $\cup \infty$, and

$$G := \{M \ m \times m \text{ matrix} : \det(I + kM) \neq 0, k \in K\}.$$

But now it is easy to construct a conformal equivalence (which is a linear fractional transformation) $\phi : G \rightarrow \Omega$ where Ω denotes the space of $m \times m$ matrices with spectral radius less than one.

From (3.1), the interpolation constraints (1) on S may be translated into equivalent constraints on $\phi \circ S : \bar{H} \rightarrow \Omega$. In other words, we have a tangential Nevanlinna-Pick type problem in which instead of bounding the norm of the interpolant we bound its spectral radius.

Notice that for $m = p = 1$ (i.e. the SISO case) we have that $\Omega = D$ (the unit disc), and hence the kind of robust stabilization problem we have been considering becomes equivalent to ordinary scalar Nevanlinna-Pick interpolation.

We should also add that a similar trick works for real parameter variations ($k \in [a, b]$ a real interval with $0 < a < 1 < b$), and we can even take parameter variation sets of the form

$$K := \{\text{diag}(k_1, \dots, k_p)\}$$

where the k_j 's may be either real or complex (μ in the 1×1 block case). See [11] for details about the conformal mapping theory associated to this set.

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