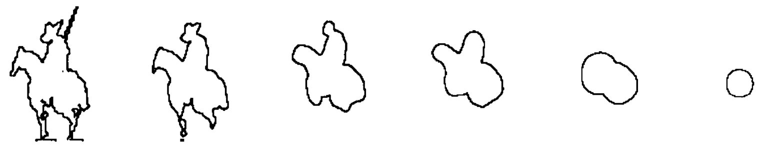

Conservation-Minded Evolution of Shape

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Abstract

Most natural and artificial systems rely heavily on vision to recognize, manipulate, and navigate within a world of objects. Although shape is a key element in this process, its representation and analysis has proven to be a difficult, multi-faceted problem. We propose a framework based on conservation laws which gives rise to computational elements for shape: *parts*, *protrusions*, and *bends*. The computation takes place in the context of a *reaction-diffusion* space and is highly robust. This scheme is ideally suited to object recognition, and has applications in areas ranging from robotics to the psychology and physiology of form. [1] [2]

Introduction

How should the shape of objects be described to enable recognition? This is one of the key problems in perception, and two views have emerged. One view holds that composite objects are formed when distinct components interpenetrate each other [9], as when two lumps of clay are put together. We refer to this as the *composition (parts)* view because it suggests that shapes are broken into "parts" at the junctions between lumps. The other, *deformation (protrusions and bends)* view, holds that existing parts should be deformed, as when clay is drawn out (or pushed in) from a lump [17]. While each of these views has some intuitive appeal, taken in the pure form neither seems completely right nor completely wrong. For example, a key missing ingredient is that of "necks", or the nature of the join between parts. Rather, this composition vs. deformation distinction has emerged as one of the frustrating dilemmas around shape; others are discussed in figure 1.

We propose an approach to representing shape, based on a reaction-diffusion equation, which resolves these dilemmas. Observe that, for two-dimensional curves, slightly deformed shapes are visually similar. We therefore study the evolution of shapes under general deformations, and show that they decompose into two types, a deformation that is constant (along the normal) and corresponds to a non-linear, hyperbolic (wave) type of process; and a deformation which varies with the curvature and corresponds to a quasi-linear diffusive one. The two types of processes interact, analagously to the way forces in physics interact at interfaces, and related questions involving conservation laws and entropy arise. Together the two processes give rise to shocks, the singularities of shape, which then provide a hierarchical decomposition of a shape into our proposed shape elements, parts, protrusions, and bends. Intuitively, necks then emerge as intersecting protrusions connecting coupled parts. Examples show that our proposed scheme is reliably computable. Moreover, the requirements of the algorithm are compatible with a physiologically-plausible model of curve detection [21] and with psychophysical evidence [1].

Shape from an Evolutionary Sequence

Since slightly deformed shapes are visually similar, we begin by studying the evolution of a shape under various deformations. Our immediate goal is to demonstrate that deformations which depend on the local geometry of the objects can be regarded as the linear sum of two basis deformations: constant motion and motion proportional to curvature. This will then lead us naturally into the study of a PDE and finally to its application to shape.

Consider the most general deformation of a curve $\mathcal{C}$, namely a deformation of some arbitrary amount along the tangent $\vec{T}$ and some other arbitrary amount along the normal $\vec{N}$ (Fig. 2), $\frac{\partial \mathcal{C}}{\partial t} = a(u,t)\vec{T} + b(u,t)\vec{N}$, where

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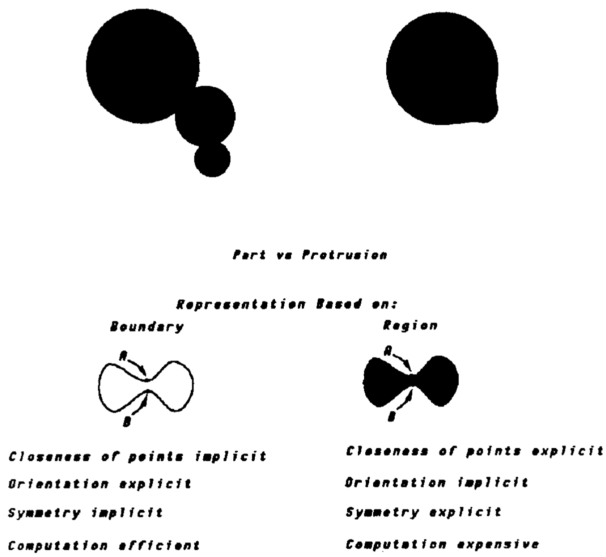

Figure 1: a) The part vs. protrusion dilemma: Some objects are naturally described as the result of composition of parts [9], e.g. the overlapping discs (left), while others are more naturally described as deformations [17], or protrusions, on a basic component (right). These two views are taken as competitive, but, intuitively, each has a certain appeal. Our theory provides a framework in which they both participate, eliminating the need to arbitrarily trade one off against the other. b) The boundary versus region dilemma: There are two complementary ways to approach a figure, either as a collection of boundary points or as a collection of interior points, and representations of shape have been based on each of these approaches; for example, boundary representations have been based on the chain code [5] and Fourier descriptors [20], while interior representations have been based on skeletons and medial-axis transformations [2]. Although the two representations are equivalent, in that one may be derived from the other, they each make different information explicit. This leads to trade-offs in stability and efficiency of computations. For example, while the structure of a "neck" at points A and B is explicit in a region-based representation, it is implicit in a boundary-based representation. The computation of a neck is local in a region-based representation, but global (thereby unstable in presence of occlusion and noise) in a boundary-based one. Our scheme makes both kinds of information explicit simultaneously, thereby enjoying much greater stability properties.

u denotes position along the curve and t is the evolutionary step (time). Without loss of generality, this deformation can be written as a deformation along the normal by some other magnitude [6]. In addition, for a theory of shape the deformation should be restricted to a local function of the geometry of the curve, and should be time invariant. Now, since the local geometry of the curve is completely determined by its curvature function [4], a time-invariant, local deformation is equivalent to a deformation along the normal as a function of curvature

$$\frac{\partial \mathcal{C}}{\partial t} = \beta(\kappa)\vec{N}. \qquad (2)$$

Qualitatively, the behaviour of this deformation is governed by the first two terms in the Taylor expansion of $\beta(\kappa) \approx \beta_0 + \beta_1\kappa$. The first term describes constant motion outwards or inwards along the normal (fig 2*ii*), and the second term describes a motion along the normal that is proportional to the curvature (fig 2*iii*). Observe that, for the curvature term, highly curved segments will move faster than slightly curved ones [12].

Conservation Laws

We now show that a deformation composed of constant motion and curvature motion satisfies a *viscous conservation law*. In particular, constant motion along the normal satisfies a hyperbolic conservation law for the slope of the boundary $u_t + \beta_0[H(u)]_x = 0$ where u is the slope in an extrinsic cartesian coordinate system (with horizontal axis x), $H(u) = -\sqrt{1+u^2}$ is the slope-flux [10], and β_0 is the extent of constant motion. When curvature motion is introduced, "viscosity" is added to the system $u_t + \beta_0[H(u)]_x = \beta_1\left[\frac{u_x}{1+u^2}\right]_x$, where β_1 is the extent of curvature motion.

This viscous conservation law is a parabolic equation ($\beta_1 \neq 0$), and contains two terms [19]. The β_0 term, which is hyperbolic and corresponds to the constant motion, is the *wave* part. The β_1 term, which is parabolic and corresponds to the curvature motion, is the *diffusion* part. The diffusion term is quasi-linear, and tends to "dampen" and smooth u, while the wave term is *non-linear* and tends to produce large solutions, steep gradients, and discontinuities. Alternatively, curvature satisfies the intrinsic evolution equation $\kappa_t + \beta_1\kappa_{ss} + \beta_0\kappa^2 + \beta_1\kappa^3 = 0$, where s is the arclength parameter along the curve and κ is curvature. This equation is a *reaction-diffusion* equation, a common model of chemical and biological phenomena. Observe that for $\beta_1 = 0$, the only effect is that of *reaction*. However, when $\beta_1 \neq 0$ *diffusion* is introduced to the system.

A second conservation law involves the product of the metric and the curvature of the evolv-

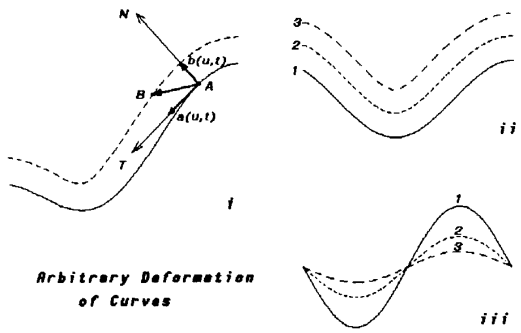

Figure 2: An arbitrary deformation of a curve is captured by two basis deformations: a constant motion (reaction) and a motion proportional to curvature (diffusion). Figures (*ii*) and (*iii*) illustrate the constant and curvature motions, respectively. Note that for constant motion all points move with the same speed, so that concave segments become more curved while convex ones become less so. However, under the curvature term, highly curved segments will move faster than slightly curved ones, so that all curved segments become smoother.

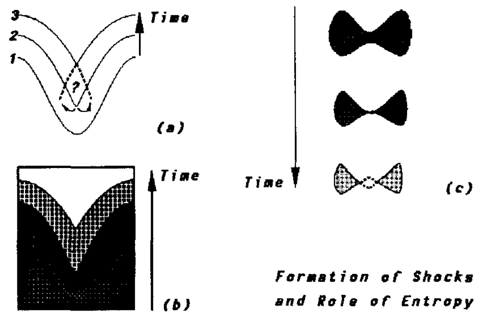

Figure 3: The formation of shocks and the role of entropy. Nonlinear processes can transform initially smooth functions to functions with singularities. (a) shows a curve with a negative curvature extremum which, when evolved by constant motion along the normal, leads to a singularity. This evolution can be based entirely on boundary information until the singularity arises. However, at this point the entropy condition is required to further control evolution, so that the curve does not cross over itself and the swallowtail configuration can be properly handled (b). The entropy condition is region-based, and controls how interior information interacts with the boundary. It plays another key role in controlling topological evolution, by globally managing the splitting of a single boundary into two closed boundaries (c).

ing curve. In [10], based on the evolution equations of the metric and the curvature [12] for the case $\beta_1 = 0$, we show that the product of curvature and metric is a conserved quantity.

Entropy and Shocks

In order to solve these equtions one must address the question of the space of solutions to these equations. In order to deal with formed singularities the space of measurable and bounded functions (*generalized functions*) is employed [16]. To restrict solutions to physically significant ones and further to constrain them to satisfy conservation across singularities, notion of *entropy* and *jump conditions* were developed [18, 14, 15, ?]. Singularities which satisfy both conditions are called *shocks*. Generalized functions whose only discontinuities are shocks enjoy existence and uniqueness properties as solutions to conservation laws [16, 19, 18, 3]. To relate the concepts of entropy, jump condition, and shocks to the problem of shape representation see [13, 10]. For example, the role of entropy, is one of handling discontinuities by explicitly introducing region-based information into the boundary-based approach whereas shocks support the decomposition of shapes into parts, protrusions, and bends, figure 3.

The Reaction-Diffusion Space

Thus far, we have modelled the deformation of a curve as a viscous conservation law and a reaction-diffusion equation. We now view reaction and diffusion as two complementary forces acting on shape, where the relative strength of the hyperbolic wave process to the parabolic diffusive one determines the nature of deformation. Together with the time of evolution, they give rise to a two-dimensional space, the *reaction-diffusion space*, spanned by the ratio β_1/β_0 and time t.

A pure *diffusion* process (no wave $\beta_0 = 0$) is a *quasi-linear* heat process. It is formally equivalent to the coordinates of the parametrized equation of the curve satisfying the heat equation [7]. Thus evolution in time under pure diffusion is tantamount to filtering the coordinates by a Gaussian kernel. Its role therefore is one of *smoothing* the boundary of the shape. In fact, the boundary converges to a circle [8]. Since the heat equation spreads information globally with infinite speed, diffusion is a *global* process. Finally, since diffusion operates soley on the boundary curvature information it is a *boundary process.*

On the other hand, a pure *reaction* process (no diffusion $\beta_1 = 0$) is a *non-linear* hyperbolic process. It can be shown to create *singularities* from the negative minima in curvature [10]. It is a *local* process in that only a limited portion of the shape affects the evolution of any single point. Finally, it requires no boundary information and is a *region process.*

The pure reaction and pure diffusion processes are extremes along one axis of the reaction-diffusion space; intermediate combinations of reaction and diffusion are compromises on their various features. Traversing along the other axis in the reaction-diffusion space, namely time, the process has the effect of simplifying the shape: diffusion spreads information instantaneously and globally along the boundary, while the reaction process removes information nonlinearly and locally through the region. This provides the basic structure for a scale-space for shape, as we show in the next section.

In summary, then, reaction and diffusion contrast and complement each other on issues of linearity vs. nonlinearity, smoothness vs. singularity, global vs. local, and boundary vs. region.

Parts, Protrusions, and Bends

The reaction-diffusion space's significance is in the segmentation of a shape into pieces. More precisely, a shape is hierarchically analysed into a composition of parts, protrusions, and bends, our proposed computational elements of shape. Examples show how certain parts protrude into one another, thus naturally giving rise to necks, the neglected aspect of shape.

But these notions are formal ones within our framework, and differ somewhat from standard usage. To clarify, recall that some traditional approaches to shape representation argue for a decomposition into "parts" or components by segmentation at the negative minima of curvature of the boundary [9, 1]. This appears reasonable because, when two distinct objects interpenetrate, the intersections are almost always transversal, projecting to negative minima in curvature. However, figure 4 shows that boundary curvature is not sufficient to determine "parts"; region information must also be taken into account. Moreover, although the statement "parts are bounded by negative curvature minima" is true, the converse does not necessarily hold. In fact, deformations of objects can give rise to negative curvature extrema, as is illustrated in figure 4. Observe that, when strictly applied the decomposition at negative curvature minima leads to counterintuitive results.

Therefore, in addition to the negative curvature minima criterion, a further condition is needed to recover parts. The intuition must be captured that parts are bounded by *pairs* of negative curvature minima that are close in the distance through the region (and not necessarily along the boundary) forming a "neck". Such a partitioning of objects along necks makes sense because it is easiest, physically, to break objects at their narrowest regions, namely the necks. Furthermore, for objects with moving parts, the joints are often narrower than the components, and joints map onto necks. For further support of this argument see [10].

How does one then interpret the unpaired curvature

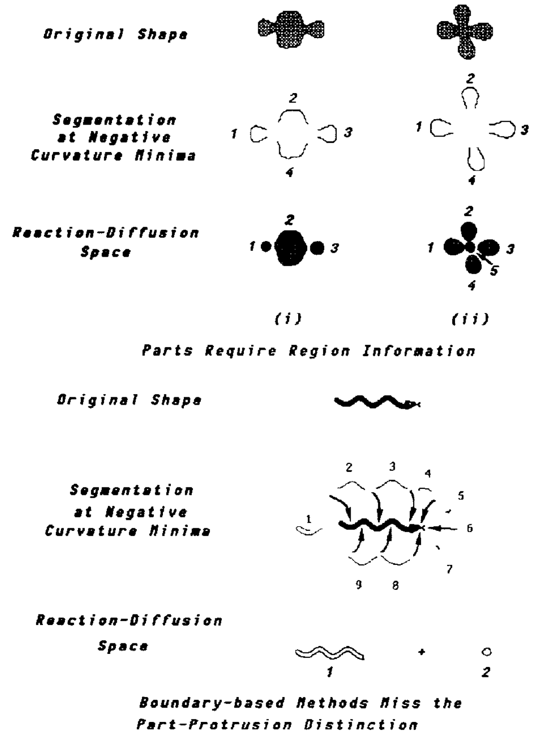

Figure 4: a) Indentification of parts requires region information. Although the objects in this figure clearly have different parts, each is segmented into four pieces based only on the boundary information and segmented at negative curvature minima. The reaction-diffusion space captures the natural difference between them, and also illustrates the non-linear nature of shape descriptions by the transition from "two petals stuck onto a blob" to "four petals composed around a common center". Such families of shapes can also serve as stimuli for psychological and physiological experimentation, and our theory makes both quantitative and qualitative predictions about when transitions like that between *i*) and *ii* will occur. b) Pure boundary-based methods miss the composition-deformation distinction. The segmentation of the snake at negative curvature minimma leads to inappropriate parts. The reaction-diffusion space, however, correctly distinguishes the body of of the snake as one deformed part.

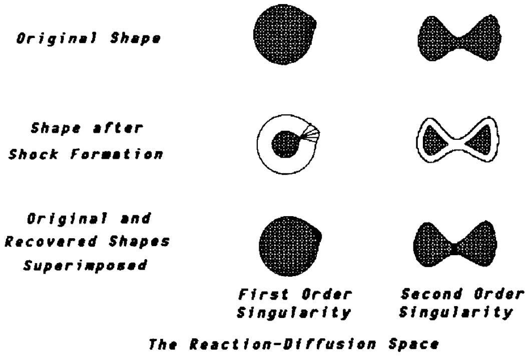

Figure 5: The Reaction-Diffusion space. The deformed disc forms a first order singularity, while the peanut-shaped object splits into two parts by forming a second order singularity. Thus, the order of the singularity differentiates between parts and protrusions. Observe the neck that develops between the two parts; this is what distinguishes the peanut shape from overlapping circles of figure 1.

extrema? Consider a circular ring of flexible material which is deformed as if someone had attempted to push a finger through it, figure 5 (left). This deformation creates a single curvature extremum. It is plausible, then, to associate the unpaired curvature extrema with deformations.

The reaction-diffusion space provides the framework for making these arguments precise. The key is in the formation of singularities (shocks) under the simulataneous influence of reaction and diffusion. Note that, for the deformed circle (Fig. 5), a single shock of the first order develops. For the peanut, figure 5 (right), however, a topological split occurs, and second order shocks are formed. For the snake example, a collection of singularities form simulataneously from the body of the snake (the symmetry axis). *Therefore, the distinction between parts, protrusions, and bends is captured by the singularities in the reaction-diffusion space: first-order singularities signal protrusions while isolated second-order singularities signal parts. Bends are signaled by a dense collection of sigularities.* To completely recover the history of the shape as a coalition of parts, protrusions, and bends, the evolved shape is run backwards through the reaction-diffusion space so it may be compared with the original shape, Fig 6. For a rigorous treatment see [10].

Discussion

In summary, we have presented a framework for a theory of shape based on the geometry of curves and their interiors. This framework resolves some of the classical dilemmas of shape perception, and results from viewing shape as a tension between reac-

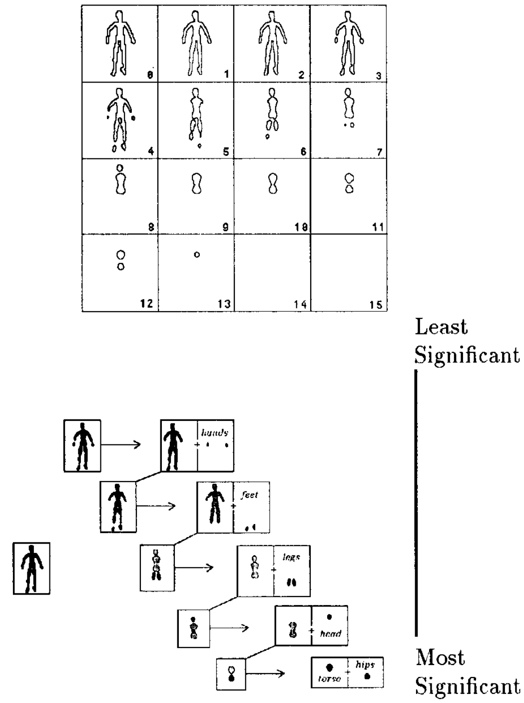

Figure 6: a) The evolution of shocks leads to parts and protrusions. This figure shows the development of an image of a doll (National Research Council of Canada Laser Range Image Library CNRC9077 Cat No 422; $128X128$). The contour shown in box N corresponds to some time step. Observe that the "feet" partition from the "legs"(via second-order shocks) between frames 3 and 4, and the "hands" from the "arms" between frames 2 and 3. Following these second-order shocks, first-order shocks develop as the "arms" are "absorbed" into the chest. Running this process in the other direction would illustrate how the arms "protrude" from the chest. b) The Hierarchical decomposition of a doll into parts. Selected frames were organized into a hierarchy acoording to the principle that significance of part is directly proportional to its survival duration.

tion and diffusion in the context of a conservation law. It defines a hierarchy of parts, protrusions, and bends based on singularities (shocks) in a reaction-diffusion space, and elucidates a mechanism for decomposing shapes into them reliably and consistently. Furthermore, a notion of scale naturally arises within this mechanism [11]. Finally, a whole family of qualitative predictions are opened up by the reaction-diffusion approach to shape, e.g. regarding the similarity between shapes expressed as a metric over the reaction-diffusion space [10].

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