Risk Measures At Risk- Are we missing the point? Discussions around sub-additivity and distortion

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Risk Measures At Risk- Are we missing the point?

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Abstract

This paper[3] discusses the regulatory requirements (Basel Committee, ECB-SSM and EBA) to measure the major risks of financial institutions, for instance Market, Credit and Operational, regarding the choice of the risk measures, the choice of the distributions used to model them and the level of confidence. We highlight and illustrate the paradoxes and issues observed when implementing one approach over another, the inconsistencies between the methodologies suggested and the goals required to achieve them. We focus on the notion of sub-additivity and alternative risk measures, providing the supervisor with some recommendations and risk managers with some tools to assess and manage the risks in a financial institution[4].

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[4]This paper has been written over a very specific period of time as most regulatory papers written in the past 20 years are currently being questioned by both practitioners and regulators themselves. Some disarray has been observed among risk managers as most models required by the regulations have not been consistent with their own objective of risk management. The enlightenment brought by this paper is based on an academic analysis of the issues engendered by some pieces of regulation and its purpose is not to create any sort of controversy.

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Key words: Risk measures - Sub-additivity - Level of confidence - Distributions - Financial regulation - Distortion - Spectral measure

1 Introduction

The ECB-SSM[5], the EBA [6] and the Basel Committee are currently reviewing the methodological framework of risk modelling. In this paper, we analyse some of the issues observed when measuring the prescribed risks that would be worth addressing in future regulatory documents.

1.1 Problematic

During the current crisis, the failure of models and the lack of capture of extreme exposures have led regulators to change the way risks are measured, either by requiring financial institutions to use particular families of distributions (Gaussian (BCBS (2005)), sub-exponential (EBA (2014*b*))), or by changing the way dependencies are captured (EBA (2014*b*)) or by suggesting a change from the Value-at-Risk (VaR)[7] to sub-additive risk measures like the Expected Shortfall (ES)[8] (BCBS (2013)). Indeed, risk modelling had played a major role during the crisis which began in 2008 either as a catalyst or trigger. The latest changes proposed by the authorities have been motivated by the will to come closer to the reality of financial markets.

In a recent paper we have discussed the importance of the choice of the distributions in measuring the risks (Guégan and Hassani (2016)). In this paper we focus on the role of the notion of sub-additivity which interested many researchers around 2000 (Artzner et al. (1999), Jorion

[5]European Central Bank - Single Supervisor Mechanism

[6]European Banking Authority

[7]Given a confidence level $p \in [0,1]$, the VaR associated to a random variable X is given by the smallest number x such that the probability that X exceeds x is not larger than $(1-p)$

$$VaR_p = \inf(x \in \mathbb{R} : P(X > x) \leq (1-p)). \tag{1.1}$$

[8]For a given p in $[0,1]$, η the VaR_p, and X a random variable which represents losses during a pre-specified period (such as a day, a week, or some other chosen time period) then,

$$ES_p = E(X|X > \eta). \tag{1.2}$$

(2006)) and brought about a change in the requirements from regulators since 2010 (BCBS (2011*a*), BCBS (2011*b*), BCBS (2013), EBA (2014*a*)). The question is to understand if this problematic is really interesting from a practical point of view, and to help address the objective set by the regulation. We will also discuss in more detail the interest of other risk measures like the spectral risk measure and a new way to take into account extreme events using distortion risk measures.

Thus, the purpose of this paper is to discuss some methodological aspects of the regulatory framework related to risk modelling and its evolution since 1995, focusing on supervisors' strong incentive to use: (i) specific distributions to characterise the risks, (ii) specific risk measures, (iii) specific associated confidence level, and to apply these strategies independently from each other. We argue that the approaches proposed by the regulator engender a bias (positive or negative) in the assessment of the risks, and consequently a distortion in both the corresponding capital requirements and the management decision taken since the problem of the measurement is not dealt with in its entirety.

Some of the following points are addressed in this paper: (i) Is the choice of a particular risk measure ensuring conservativeness? (ii) When moving from a VaR_p to sub-additive risk measures such as the ES_p, for which distributions is the sub-additivity[9] property fulfilled given that we consider several risk factors? (iii) Given that each risk type is modelled based on different distributions and using different p-s, how can the sub-additivity criterion be fulfilled? Is that really important in practice? These different points are linked to the choice of a particular distribution and to the choice of the confidence level p.

The regulatory documents state with respect to market risk - since 1995 (BCBS for instance) -

[9] A coherent risk measure is a function $\rho : \mathcal{L}^\infty \to \mathbb{R}$:

- Monotonicity:If $X_1, X_2 \in \mathcal{L}$ and $X_1 \leq X_2$ then $\rho(X_1) \leq \rho(X_2)$
- Sub-additivity: If $X_1, X_2 \in \mathcal{L}$ then $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$
- Positive homogeneity: If $\lambda \geq 0$ and $X \in \mathcal{L}$ then $\rho(\lambda X) = \lambda\rho(X)$
- Translation invariance: $\forall k \in \mathbb{R}$, $\rho(X + k) = \rho(X) - k$

that "the VaR risk measure is inadequate for measuring the risks because it does not take into account the extreme events" and also "one of the problems of recognising banks' value-at-risk measures as an appropriate capital charge is that the assessments are based on historical data and that, even under a 99% confidence interval, extreme market conditions are excluded". To confirm this fact, in the Consultative Document concerning the Fundamental review of the trading book (BCBS (2013)), the Basel Committee proposes "to move from Value-at-Risk (VaR) to Expected Shortfall (ES) as a number of weaknesses have been identified using VaR for determining regulatory capital requirements, including its inability to capture tail risk". The Committee has agreed "to use a 97.5th ES for the internal models-based approach and to use it to calibrate capital requirements under the revised market risk standardised approach".

In these documents the regulator says that the choice of the VaR as a risk measure does not take into account extreme values. This statement is not correct as the choice of the VaR is not the issue; it is the choice of the underlying distribution with which the associated quantile is evaluated that determines if the extreme events are captured or not. This question actually implies a second question about what an extreme event is and answering this question would suppose a complete information set. Then in 2013, it seems that the regulator thought that the use of the ES instead of the VaR is more effective at capturing the most relevant information to measure the risks. This is not necessarily true as once again, it depends on the choice of the distributions used for the computation of this ES. Nevertheless, we know that this last measure is more interesting than the VaR when considering the same distribution because it provides better information concerning the amplitude of the risk, but if the fitted distribution is inappropriate the problem of capturing extreme events remains the same. Besides, the choice of the level of confidence, for instance 97.5 is also arbitrary (this point will be illustrated in the next section). Indeed, why did the regulator move from 99% (in 1995) to 97.5 % (in 2013)? - Why did they not suggest 95% or another value p?

Another point is considered by the regulators for operational risk modelling, see EBA (2014*b*)[10]. In these documents they consider that a "risk measure means a single statistic extracted from the aggregated loss distribution at the desired confidence level, such as Value-at-Risk (VaR), or

[10] The discussed philosophy is also implied in the final version of the document.

shortfall measures (e.g. Expected Shortfall, Median Shortfall)". This definition is particularly limiting and dangerous. How can the risk measures computed for different factors with different levels be aggregated? If we use the ES measure, it loses its sub-additivity property in that latter case. Thus, other approaches could be more robust and realistic, for instance the use of spectral measure.

It appears that some documents are too prescriptive, preventing banks from going beyond the proposals and focusing more on the capital calculations than on the risk management itself. Regarding the calculation of the capital requirement from the knowledge of the risk factors, the main points concern the choice of the distribution, the choice of the risk measure and the choice of p: these choices are not studied in a uniform way and the approach proposed by the regulators does not constitute a robust approach for measuring the risk of a bank.

In Section Two, we investigate the notion of sub-additivity for a risk measure showing that this property also depends on the choice of the distribution and not only on the choice of the risk measure. We illustrate the point that the restriction imposed by regulators prevents a reliable approach to measure the risk. We illustrate our statements with examples. In Section Three we show that it is the choice of the distributions which is definitively the key point in risk modelling. Then in Section Four, we propose a new way to measure the risk, working in two directions; using the spectral measure and/or using a measure based on the distortion of the distribution in order to have multi-modal distributions to model the risk factors. Section Five concludes.

2 Sub-additivity property: a real added-value for risk management?

The concept of sub-additivity which has been largely studied in the 2000's appears interesting if we consider that the measure of the risk of a portfolio[11] is obtained when we calculate the risks of each factor of this portfolio. This is a very restrictive approach for measuring these exposures. An appropriate solution would be to use a multivariate quantile approach based on copula or vines (Guégan and Maugis (2010*a*), Guégan and Hassani (2013), etc). Nevertheless,

[11] The term portfolio is here used *stricto sensus* and not necessarily as a combination of assets.

if we maintain this method of calculating the risks, the idea of the sub additive risk measure is that the measure of the sum has to be smaller than the sum of the risk, and following the works of Artzner et al. (1999), it seems that this property is only verified by the ES risk measure. In fact, this property is also verified by the VaR measure: it depends on the distribution used (Degen and Embrechts (2008)). Indeed, VaR is known to be sub-additive (i) for stable distribution, (ii) for all log-concave distribution, (iii) for the infinite variance stable distributions with finite mean, (iv) and for distribution with Generalised Pareto Distribution type tails when the variance is finite. The non-sub-additivity of VaR can occur (i) when assets in portfolios have greatly skewed loss distributions; (ii) when the loss distributions of assets are smooth and symmetric, (iii) when the dependency between assets is highly asymmetric, and (iv) when underlying risk factors are independent but very heavy-tailed. To illustrate our purpose, we have selected a data set provided by a Tier European bank representing "Execution, Delivery and Process Management" risks from 2009 to 2014. "Execution, Delivery and Process Management" risk is a sub-category of operational risk[12]. This data set is characterised by a distribution right skewed (positive skewness) and leptokurtic.

In order to follow the regulatory guidelines, we choose to fit on this data set some of the distributions prescribed and also others which seem more appropriate regarding the properties of the data set. We retain seven distributions. They are estimated (i) on the whole sample: the empirical distribution, the lognormal distribution (asymmetric and medium tailed), the Weibull distribution (asymmetric and thin tailed), a Generalised Hyperbolic (GH) distribution (symmetric or asymmetric, fat tailed on an infinite support), an Alpha-Stable distribution (symmetric, fat tailed on an infinite support), a Generalised Extreme value (GEV) distribution (asymmetric and fat tail), (ii) on an adequate subset: the Generalised Pareto (GPD) distribution (asymmetric, fat tailed) calibrated on a set built over a threshold, a Generalised Extreme Value (GEVbm) distribution (asymmetric and fat tailed) fitted using maxima coming from the original set. The whole data set contains 98082 data points, the sub-sample used to fit the GPD contains 2943 data points and the sub-sample used to fit the GEV using the block maxima approach contains 3924 data points. The objective of these choices is to evaluate the impact of the selected dis-

[12]In our demonstration, the data set which has been sanitised here is not of particular importance since the same data set has been used for each and every distribution tested.

tributions on the risk representation, i.e. how the initial empirical exposures are captured and transformed by the model. It is interesting to note that using empirical distributions instead of fitted analytical distributions could be of interest as the former one captures multi-modality by construction. Unfortunately, this solution was initially rejected by regulators as this non-parametric approach is not considered able to capture tails properly which, as shown in the table, might be a false statement. However, recently the American supervisor seems to be re-introducing empirical strategies in practice for CCAR[13] purposes.

Table 1 exhibits parameter estimates for each distribution selected[14]. The parameters are estimated by maximum likelihood, except for the GPD which implied a POT (Guégan et al. (2011)) approach and the GEV fitted on the maxima of the data set (maxima obtained using a block maxima method (Gnedenko (1943))). The quality of the adjustment is measured using both the Kolmogorov-Smirnov and the Anderson-Darling tests. The results presented in Table 1 show that none of the distributions is adequate. This is usually the case when fitting uni-modal distributions to a multi-modal data set. Indeed, multi-modality of the distributions is a frequent issue when modelling operational risks as the risk categories combine multiple kinds of incidents; for instance a category combining external frauds will contain the fraud card on the body, commercial paper fraud in the middle, cyber attack and Ponzi scheme in the tail, but we have also observed a similar pattern using market data. It could be more appropriate to consider empirical distributions than fitted analytical distributions as it may help to capture multi-modality. We will come back to this last point in Section 4.

In the introduction we indicated that the regulator recommends the use of the ES instead of the VaR because the former is sub-additive, property unverified by the VaR. In the following section, we question these assertions showing that, even if it is true that the ES is sub-additive, (i) the VaR also has this property for a lot of distributions as we have mentioned before; (ii) the sub-additivity property can be verified for some values of p, and not verified for others; (iii) the sub additivity of the VaR is very often verified for fat-tailed distributions; (iv) the sub-additivity is not verified anymore for the ES when we aggregate them. We illustrate these different facts

[13] Comprehensive Capital Analysis and Review

[14] In order not to overload the table the standard deviation of the parameters are not exhibited but are available upon request.

making some simulations computing $\text{VaR}_p(X+Y)$ and $\text{VaR}_p(X) + \text{VaR}_p(Y)$ for X and Y two risk factors. We proceed in the following way:

- As $\text{VaR}_p(X)$ is a quantile, $p \in [0,1]$, the entire spectrum of the VaR has been built, considering the inverse of the cumulative distribution function. Summing $\text{VaR}_p(X)$ and $\text{VaR}_p(Y)$ for each value of p provides us with $\text{VaR}_p(X) + \text{VaR}_p(Y)$.

- To obtain $\text{VaR}_p(X+Y)$, another approach is adopted. In a first step we randomly generate X and Y using specific distributions. Then X and Y are aggregated. The resulting cumulative distribution function is built and its inverse provides the spectrum of $\text{VaR}_p(X+Y)$[15].

In Table 2 we provide the values obtained for both the VaR and the ES for fully correlated random variables. It is interesting to note that the risk measures obtained on fully correlated random variables and the sum of the risk measures obtained univariately are really similar. This means that as soon as we sum the VaR obtained on two variables we mechanically assume an upper tailed correlation for the random variables. Therefore, as well as being conservative, the sum of univariate VaRs taken at the same level prevents the capture of any diversification benefit. Fully correlated random variables do not embed any diversification benefit by definition. Consequently, the analysis regarding the sub-additivity of the risk measures has to be performed in another way.

Then we work with the data sets we have previously introduced. We randomly generated values from the distribution fitted before and combined them two by two. By carrying out this process we generated some random correlations and incidentally some diversification. Then, we compared the risk measures obtained from the combination of random variables and the sum of the risk measures computed on the random variables taken independently. From Table 3, for fixed p, we observe that the VaR is never sub-additive if the lognormal distribution is associated with a GPD; while if the lognormal distribution is associated with any of the others, the VaR is usually sub-additive in the tails but not at the end of the body part. Note that if the lognormal is associated with an identical lognormal, the differences we have observed are only due to numerical errors related to sampling. We expect the two values to be absolutely identical. However, it is interesting to note that the random generation of numbers can be at the origin of

[15] We acknowledge the numerical error that this process may engender, though this one appears negligible here.

non sub-additive results. An identical analysis can be done on other combinations. From Table 5 it appears that when the GPD has a positive location parameter, this prevents any combination from being sub-additive, because by construction the 0th percentile of the GPD is equal to the location parameter which should, according to Pickand's theorem (Pickands (1975)), be sufficiently high. At the 95th percentile, the VaR is always sub-additive whenever a lognormal distribution is involved, except if it is combined with a GPD. For the other distributions, it is not always true. For example, the VaR obtained after combining a Weibull and a GEV fitted on the whole sample is not sub-additive. Table 6 shows that the use of an Alpha-Stable combined with any other distribution, except for the GPD, provides sub-additive risk measures at the 99% level. Other examples are provided in Table 6 with the Weibull distribution.

Building the ES always leads to sub-additive values (see Tables 7 - 10), contrary to the VaR for which this property is not always verified and depends on the underlying distribution as discussed previously: the results for the ES can be compared to those obtained with the VaR looking at Tables 3 and 7, Tables 4 and 8, then Tables 5 and 9 and finally Tables 6 and 10. It is interesting to note that if we combine two ES measures taken at two different levels of confidence p, the ES may not be sub-additive anymore. This is a point that the regulators do not discuss when they imply that the risk measures have to be aggregated. This issue is particularly important for risk managers, since the level of confidence prescribed in the regulation guidelines is different from one risk factor to another and appears totally arbitrary.

In parallel, Figures 5 to 9 allow a more discriminating analysis of the behaviour of the component $VaR_p(X+Y)$ versus $VaR_p(X)+VaR_p(Y)$. In Figure 5, we show that the sub-additivity property is only verified for high percentiles when we combine a Weibull and a GH distribution, i.e. for $p > 90\%$. In addition, the gap tends to widen as the percentiles increase. Figure 6 exhibits a non sub-additive VaR from the 95th percentile, when we use the combination of an Alpha-Stable distribution and a GEV fitted with the block maxima method, but the differences are not as great as in Figure 5. Figure 7 shows that combining two identical distributions does not always produce sub-additive risk measures though it should always be the case: this can be due to numerical errors caused by the random generation of data points and the discretisation of the distribution. In Figures 8 and 9 we observe that the VaRs obtained from the combination of an Alpha-Stable distribution and a GH distribution or an Alpha-stable distribution and a GEV distribution calibrated on maxima are never sub-additive below 70%. For comparison purposes, Figures 10 and 11 illustrate the fact that the combination of two elliptical distributions (respectively the Gaussian and the Student-t distributions) always leads to sub-additive VaRs.

3 Role of the distributions in the computation of VaR and ES measures

In this section we illustrate the influence of the distributions on the risk measure evaluations. Table 11 provides the values obtained for the VaR_p and the ES_p computed from the eight distributions fitted on the data set or some sub-samples. We also illustrate the fact that an a priori on the choice of a distribution provides arbitrary results which can be disconnected from reality.

From Table 11 we note that the values provided by VaR_p can be bigger than the values derived for an ES_p and conversely. We observe that the results obtained from the GPD and the alpha-stable distributions are of the same order. Second, the differences between the GPD and the GEV fitted on the block maxima are huge, illustrating the fact that, despite being two extreme value distributions, the information captured is quite different. The ES calculations are also linked to the distribution used to model the underlying risks. Looking at Table 11, at 95%, we observe that the ES goes from 1979 for the Weibull to 224 872 for the GPD. Therefore, depending on the distribution used to model the same risk, at the same p level, the ES obtained is completely different. The corollary of that issue is that the ES obtained for a given distribution at a lower percentile will be higher than the ES computed with another distribution at a higher percentile. For example, Table 11 shows that the 90% ES obtained from an Alpha-Stable distribution is much higher than the 99.9% ES computed on a lognormal distribution.

Thus one question arises. What should the regulator ask to use: the VaR or the Expected Shortfall? To answer this question, we can consider several points: (i) Conservativeness: Regarding that point, the choice of the risk measure is only relevant for a given distribution, i.e. for any given distribution the VaR_p will always be inferior to the ES_p (assuming only positive values) for a given p. But, if we consider two distributions to characterise a risk it may happen that for a given level p, the VaR_p obtained from a distribution is superior to the ES_p for another distribution. For example, Table 11 shows that the 99.9% VaR obtained using the GH distribution is superior to the ES obtained for the Weibull or the lognormal distributions at the same level p; (ii) Distribution and p impacts: Table 11 shows that potentially a 90% level ES obtained from a given distribution is larger than a 99.9% VaR obtained with another distribution, e.g. the ES obtained from a GH distribution at 90% is higher than the VaR obtained from a lognormal distribution at 97.5%. Thus is it always pertinent to use a high value for p? (iii) Parameterisation and estimation: the impact of the calibration of the estimates of the parameters is not negligible (Guégan et al. (2011)), for instance when we fit a GPD. Indeed, in that latter case, due to the instability of the estimates of the threshold, the practitioners can largely overfit the risks.

4 Alternative approaches: Spectral measure and Distortion

4.1 Spectral Risk Measure vs Spectrum

In this subsection, we briefly introduce the concept of spectral risk measure as the work presented in this paper can easily be extended to this particular tool. Besides, in order to avoid any mis-understanding, we point out the difference between a spectral risk measure and the spectrum of a risk measure as used in this paper.

A spectral risk measure is obtained considering a weighted average of outcomes. Contrary to the approach discussed above, a spectral risk measure is always a coherent risk measure. Spectral measures found their usefulness in the fact that they can be related to risk aversion through the weights chosen for the possible risk exposures.

Acerbi (2002) introduces the formal notion of spectral risk measure: a spectral risk measure $\rho : \mathcal{L} \to \mathbb{R}$ is defined by

$$\rho(X) = -\int_0^1 \phi(p) F^{-1}(x)(p) dp \tag{4.1}$$

where ϕ is positive or null, non-increasing, right-continuous, integrable function defined on $[0,1]$ such that $\int_0^1 \phi(p)dp = 1$ and $F(x)$ is the cumulative distribution function for x. Any spectral risk measure satisfies the following condition making them useful in practice (Adam et al. (2007)):

- Positive Homogeneity: for a risk X and positive value $\psi > 0$, $\rho(\psi X) = \psi \rho(X)$;

- Translation-Invariance: for a risk X and $\alpha \in \mathbb{R}$, $\rho(X+a) = \rho(X) - a$;
- Monotonicity: for any combination of risks X and Y such that $X \geq Y$, $\rho(X) \geq \rho(Y)$;
- Sub-additivity: for any combination of risks, $\rho(X+Y) \leq \rho(X) + \rho(Y)$;
- Law-Invariance: for any combination of risks X and Y with respective cumulative distribution functions $F(x)$ and $F(y)$, if $F(x) = F(y)$ then $\rho(X) = \rho(Y)$;
- Comonotonic Additivity: for every comonotonic random variables (for instance these random variables are representing risks) X and Y, $\rho(X+Y) = \rho(X) + \rho(Y)$.

Note that the Expected Shortfall discussed in this paper is a spectral risk measure for which $\phi(p) = 1$, $\forall p$. However, the VaR is not a spectral risk measure but as discussed below may have a spectral representation. We refer here to the spectrum of the risk measure, i.e. the value obtained, considering various levels of p_i, and justify the use of the spectrum in practice.

Indeed, while the use of several levels $p_i, i = 1, \cdots, k$ allows a spectral representation of the risk measures (VaR or ES) and could be interesting for risk management, the approach proposed by regulators which combines distribution and confidence level is questionable. Indeed, the 70% ES of some combinations may lead to a much higher value than the 99.9th (Table 8, WE-GPD vs WE-GH); on the contrary we provide in Tables 12, 13 and 14, the differences $VaR(X) + VaR(Y) - VaR(X+Y)$ for several distributions. In Table 12 we use Weibull and a lognormal distributions, in Table 13 Weibull and α-stable distributions and in Table 14 two GEV distributions. We do this exercise for $90\% < p < 99.9\%$ in Tables 12 and 13, and for $1\% < p < 99.9\%$ with a step of 1% in Table 14. In that last table when the values are positive, the VaR is sub-additive, when the values are negative it is not sub-additive. The turning points are highlighted in bold. This provides an interesting picture of the property of these distributions. This spectral representation of the VaR given in these tables provides good information in terms of risk management; indeed, it shows that relying directly on risk measures to evaluate a capital requirement may not be representative of the risk profile of the target entity. In fact, it can even be misleading, as from one p_i to another because the risk measures may have dramatically different orders. The spectrum of the VaR approach shows a risk measure obtained at a particular level cannot be representative of the whole risk profile, and assuming the contrary could lead to dreadful failure and mismanagement. Thus we can encourage the risk managers to compute the spectrum to have a better understanding of these risks.

4.2 Distorted distributions

In terms of risks, the first point to consider, as we have seen previously, is to fit a "correct" distribution on the data. Indeed, looking at Figure 1, we observe that the "natural" distribution fitted on the underlying market data[16] set is multi-modal. In practice, we often observe this kind of pattern for financial or economic data sets. We observe from this graph that we can separate large losses from the other ones and then obtain a better understanding of the probability of these outcomes. In this section we propose an alternative to paragraph 3 for the fit of the distributions characterising the risk factors, and by doing so we introduce a new risk measure approach. First we need to find a way to build multi modal distributions and second we need to associate a way to measure the risks with this class of distributions and provide interesting interpretations in terms of management. Given a risk factor X characterised by a distribution function F_X, we are going to transform this distribution into another one using specific functions g.

Indeed, to build multi modal distributions is not a new problem. It has been investigated by many statisticians considering mainly multimodal distributions inside the exponential family (Fisher (1922)) and more recently by economists within the dual theory of choice (Yaari (1987)). Both these approaches extend the notion of multimodality appearing as a mixture of normal or possibly other unimodal densities and suggest transforming the original distribution into a new one using a distortion function $g(.)$ with appropriate properties. A function $g : [0,1] \to [0;1]$ is a distortion function if (i) $g(0) = 0$ and $g(1) = 1$, (ii) g is a continuous increasing function. Different distortion functions have been proposed in the literature. A wide range of parametric families of distortion functions is mentioned in Wang (2000), and Hardy and Wirch (2001). Cobb et al. (1987) also proposes an interesting approach which is more general and whose applicability is based on robust statistical techniques.

[16] Dow Jones Index

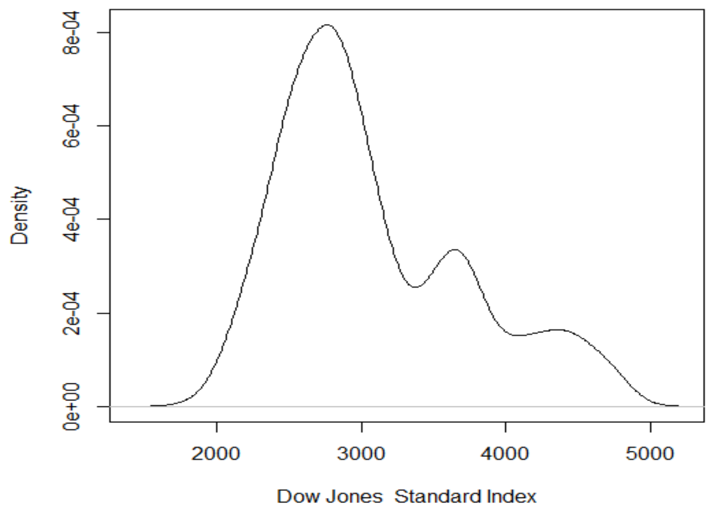

Figure 1: This figure presents the density of the Dow Jones Index. We observe that this one cannot be characterised by a Gaussian distribution, or any distribution that does not capture humps for that matter.

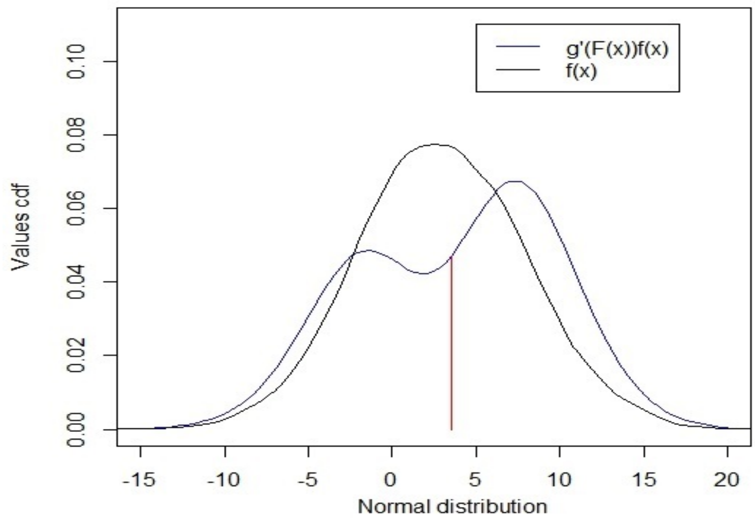

Figure 2: This figure presents a distorted Gaussian distribution. We can observe that the weight taken in the body is transferred on the tails.

We begin to introduce some functions g resulting in a bimodal distribution[17]. To do so we need to use a function g which creates saddle points. The saddle point generates a second mode in the new distribution which allows us to take into account different patterns located in the tails. The distortion function g fulfilling this objective can be an inverse S-shaped polynomial function of degree 3 - for for instance given by the following equation and characterised by two parameters δ and β :

$$g_\delta(x) = a\left[\frac{x^3}{6} - \frac{\delta}{2}x^2 + \left(\frac{\delta^2}{2} + \beta\right)x\right]. \tag{4.2}$$

We note that $g_\delta(0) = 0$, and to get $g_\delta(1) = 1$ this implies that the coefficient of normalisation is equal to $a = (\frac{1}{6} - \frac{\delta}{2} + \frac{\delta^2}{2} + \beta)^{-1}$. The function g_δ will increase if $g'_\delta > 0$ requiring $0 < \delta < 1$. The parameter $\delta \in [0,1]$ allows us to locate the saddle point. The curve exhibits a concave part and a convex part. The parameter $\beta \in \mathbb{R}$ controls the information under each mode in the distorted distribution. Illustration of the role of δ on the location of the saddle points and of β for the shape of this bimodal distribution can be found in Guégan and Hassani (2015*a*). We provide two graphs (3 and 4) below which show the creation of a bi-modal distribution using the transformation $g(F_X)$.

To create a multi-modal distribution with more than two modes, we can use a polynomial g of higher degree to have more saddle points in the interval $[0,1]$. This is important if we seek to model distributions with multiple modes to represent multiple behaviours. For example, we can consider a polynomial of degree 5 and 2 saddle points in the interval $[0,1]$:

$$\begin{aligned} g(x) = {} & a_0\Big(a_1^2a_3^2\frac{x^5}{5} + a_1^2a_4\frac{x^3}{3} + a_2^2a_3\frac{x^3}{3} + a_2^2a_4^2x - 2a_1^2a_3a_4\frac{x^4}{4} - 2a_1a_2a_3^2\frac{x^4}{4} \\ & + 4a_1a_2a_3a_4\frac{x^3}{3} - 2a_1a_2a_4^2\frac{x^2}{2} - 2a_2^2a_3a_4\frac{x^2}{2}\Big) \end{aligned}$$

[17]Here our approach is mainly descriptive, in another paper we provide robust estimation from original data sets using maximum likelihood and the weighted moment method.

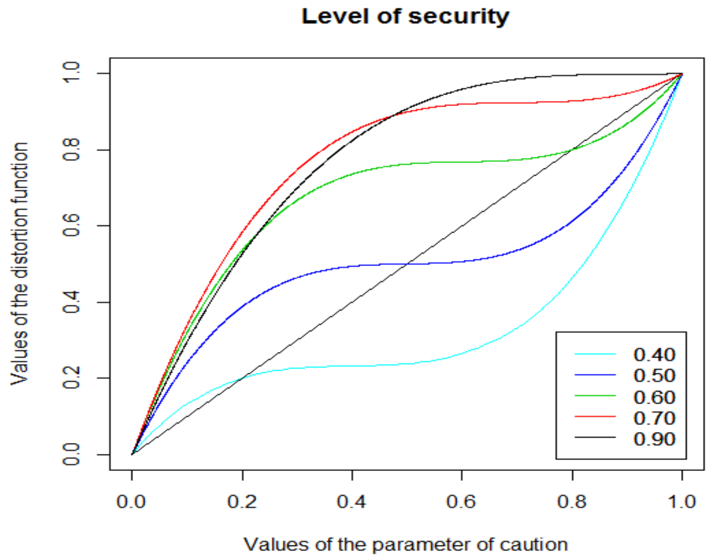

Figure 3: Curves of the distortion function g_δ introduced in equation (5) for several values of δ and fixed values of $\beta = 0.001$.

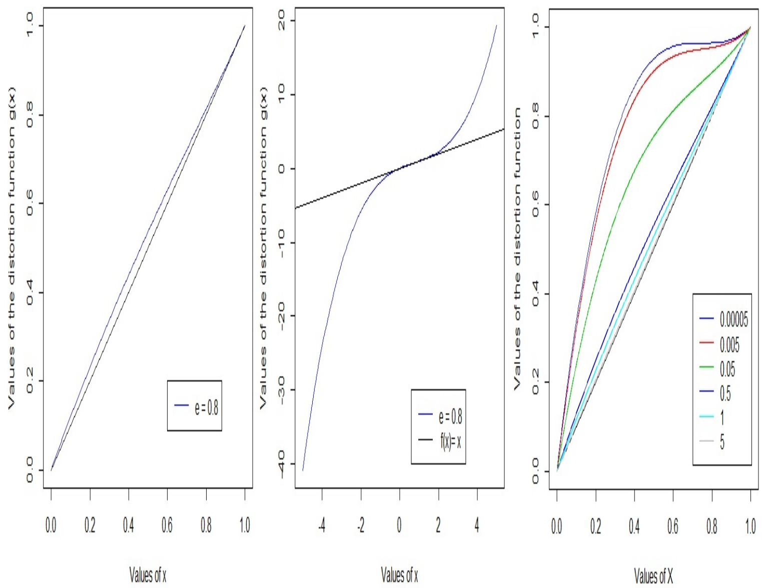

Figure 4: The effect of β on the distortion function for a level of security $\delta = 0.75$ showing that if β tends to 1 the distortion function tends to the identity function.

with first and second derivatives :

$$
\begin{aligned}
g'(x) =& a_0(a_1x - a_2)^2(a_3x - a_4)^2 = a_0(a_1a_3x^2 - a_1a_4x - a_2a_3x + a_2a_4)^2 \\
=& a_0(a_1^2a_3^2x^4 + a_1^2a_4x^2 + a_2^2a_3x^2 + a_2^2a_4^2 - 2a_1^2a_3a_4x^3 - 2a_1a_2a_3^2x^3 \\
& + 4a_1a_2a_3a_4x^2 - 2a_1a_2a_4^2x - 2a_2^2a_3a_4x), \\
g''(x) =& 2a_0a_1(a_1x - a_2)(a_3x - a_4)^2 + 2a_0a_3(a_1x - a_2)^2(a_3x - a_4).
\end{aligned}
$$

This function satisfies all the properties of a distortion function and can be used to generate a trimodal distribution on the condition that:

1. $a_i > 0$ for all $i \in \{1, 2, 3, 4\}$,
2. $\delta_1 = \dfrac{a_2}{a_1}$ and $\delta_2 = \dfrac{a_4}{a_3}$.

As we can see, the number of parameters increases as the number of saddle points increases.

The "distorted risk measure" $\rho_g(X)$ associated with a risk factor X admitting a cumulative distribution $S_X(x) = \mathbb{P}(X > x)$, transformed using a distortion function g , is defined below provided that at least one of the two integrals is finite:

$$\rho_g(X) = \int_{-\infty}^{0} [g(S_X(x)) - 1]dx + \int_{0}^{+\infty} g(S_X(x))dx. \tag{4.3}$$

Such a risk measure computed from a distorted distribution corresponds to the expectation of a new variable whose probabilities have been re-weighted. Finding appropriate "distorted risk measures" reduces the choice of an appropriate distortion function g. Properties for the choice of a distortion function include continuity, concavity, and differentiability. Assuming g is differentiable on $[0, 1]$ and $F_X(x)$ is continuous, then a distortion risk measure can be re-written as:

$$\rho_g(X) = \mathbb{E}[xg'(S_X(x))] = \int_{0}^{1} F_X^{-1}(1-p)dg(p) = \mathbb{E}_g[F_X^{-1}]. \tag{4.4}$$

Distortion functions arose from empirical observations that people do not evaluate risk as a linear function of the actual probabilities for different outcomes but rather as a non-linear distortion function. It is used to transform the probabilities of the loss distribution to another probability distribution by re-weighting the original distribution. This transformation increases the weight given to desirable events and deflates others. Different distortions g have been proposed in the literature. A wide range of parametric families of distortion functions is mentioned in Fisher (1922), and Wang (2000). We can also use several distortion functions if we want to increase the influence of asymmetry in the transformation of the original distribution and work for instance as follows (Sereda et al. (2010)):

$$\rho_{g_i}(X) = \int_{-\infty}^{0} [g_1(S_X(x)) - 1]dx + \int_{0}^{+\infty} g_2(S_X(x))dx. \tag{4.5}$$

with $g_i(u) = u + k_i(u - u^2)$ for $k \in]0, 1]$ et $\forall i \in \{1, 2\}$. With this approach one models loss and gains differently, relative to the values of the parameters $k_i, i = 1, 2$. Thus upside and downside risks are modelled in different ways. Nevertheless the calibration of the parameters $k_i, i = 1, 2$ remains an open problem. Estimation procedures are provided in a companion paper.

Thus, if we want to use coherent risk measures using the previous distributions, we can consider the following relationship:

$$\rho(X) = \mathbb{E}_g[F_X^{-1}(x)|F_X^{-1}(x) > F_X^{-1}(\delta)]. \tag{4.6}$$

It is a well defined measure, similar to the expected shortfall but computed under the distribution $g \otimes F_X$. Moreover, it verifies the coherence axiom. With this new measure we solve our problem in defining a risk measure that takes into account the information in the tails.

5 Conclusion and Recommendations

In the introduction, which analysed several guidelines issued by the EBA and the Basel Committee, we pointed out the fact that the regulators impose specific distributions, risk measures and confidence levels to analyse the risk factors in order to evaluate capital requirements of financial institutions. It appears that their approach is non holistic and their analysis of the risks relies on a disconnection between the components outlined in the previous sentence, i.e. the tools necessary to assess the risks.

In this paper we show that risk measurement in financial institutions depends intrinsically on how the tools are chosen, i.e. the distribution, the combinations of these distributions, the type of risk measure and the level of confidence. Therefore, the existence of a risk measure as discussed in the regulation is questionable, as for example modifying the level of confidence by a few percent would result in completely different interpretations. The regulators fail to propose an appropriate approach to measure these risks in financial institutions as soon as they do not take into account the problem of risk modelling in its globality.

Regulators are far too prescriptive and their choices questionable:

- Imposing distributions does not really make sense whatever the risks to be modelled as these may change quite quickly. We may wonder where these *a priori* are coming from.

- The regulation reflects some misunderstanding regarding distribution properties (probabilistic approach) and of the particular properties surrounding their fittings (statistical approach).

- The levels of confidence p seem rather arbitrary. They neither take into account the flexibility of risk measures nor the impact of the underlying distribution, misleading risk managers.

While these fundamental problems are not addressed, others are completely ignored such as the concept of spectral analysis, or of distortion risk measures (Sereda et al. (2010), Guégan and Hassani (2015a)). Despite the cosmetic changes included in Basel II and III, the propositions do not enable a better risk management, and the response of banks to regulatory points is not appropriate as they do not correspond to the reality. It is therefore not surprising that capital calculations and stress testing are still unclear, and that these are not able to capture asymmetric shocks corresponding to extreme incidents.

Some other questions should also be addressed:

- Is it more efficient in terms of risk management to measure the risk and then build a capital buffer or to adjust the risk taken, considering the capital we have? In other words, maybe banks should start optimising their income generation with respect to the capital they already have.

- The previous points are all based on uni-modal parametric distributions to characterise each risk factor. What is the impact of using multi-modal distributions in terms of risk measurement and management? We believe that an empirical evaluation of the risks provides bank with a reliable benchmark and a starting point in terms of what would be an acceptable capital charge or risk assessment.

- One of the biggest issues lies in the fact that we do not know how to combine or aggregate $VaR(X)_{p_1}$, $VaR(Y)_{p_2}$ and $VaR(Z)_{p_3}$ evaluated on three different kinds of risks at three different confidence levels p_1, p_2, p_3. This mechanically prevents banks from building a holistic approach from a capital point of view. How should we proceed to solve the problem, should we use $p = max(p_1, p_2, p_3)$, or the minimum or the average?

- Although in this paper we have focused on each factor taken independently, the question of dependence is quite important too. Maybe not as important as the impact of the distribution selected for the risk factor (Guégan and Hassani (2013)) but not addressing this issue properly could lead to a mis-interpretation of the results. The choice of the copula has a direct impact on the dependence structure we would like to apply and the capture of shocks. For instance, a Gaussian or Student t-copula is symmetric, despite the fact that a t-copula with a low number of degrees of freedom could capture tail dependencies; these would not capture asymmetric shocks. Archimedean or extrema value copulas associated with a vine strategy would be more appropriate (Guégan and Maugis (2010*b*)).

- In a situation such as one depicted by the stress-testing process with a forward looking perspective, if the risks are not correctly measured then the foundations will be very fragile and the outcome of the exercise not reliable. Indeed, stressing a situation requires an appropriate initial assessment of the real exposure, otherwise the stress would merely model what should have been captured originally and therefore be useless (Bensoussan et al. (2015), Guégan and Hassani (2015*b*), Hassani (2015)).

We came up to the conclusion that the debate related to the selection of a risk measure over another is not really relevant, and considering issues raised in the previous sections our main recommendation would be to leave as much flexibility as possible to the modellers to build the most appropriate models for risk management purposes initially and then extend with conservative buffers for capital purposes. The objective would be to suggest that good risk management would mechanically limit the exposures and the losses and therefore ultimately reduce the regulatory capital burden. Models should only be a reflection of the underlying risk framework and not a tool to justify a reduced capital charge. We would like to see the supervisory face of the authorities more and their regulatory face less; in other words we would like them to stop focusing so much on a bank's risk measurement comparability and more on financial institutions risk understanding. It would probably be wise if both regulators and risk managers worked together (e.g., academic formation open to both corpus, regular workshops, etc., (Guégan (2009))) rather than as opponents, in order to reach their objective of stability of the financial system first and profitability second.

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Distribution / Parameters	LogNormal	Weibull	GPD	GH	α-Stable	GEV	GEVbm
μ	4.412992	-	1541.558	-5.5846599	-	587.855749	50.272385
σ	1.653022	-	-	-	-	2137.940297	64.720971
α	-	0.5895611	-	0.1906536	0.86700	-	-
β	-	182.9008432	1185.8083087	0.1906304	0.95000	-	-
ξ	-	-	0.9039617	-	675 923	3.634536	1.030459
δ	-	-	-	22.5118547	58.18019	-	-
λ	-	-	-	-0.871847		-	-
γ	-	-	-	-	54.21489	-	-
KS	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16
AD	6.117e-09	6.117e-09	6.117e-09	NA	6.117e-09 (d)	6.117e-09	6.117e-09

Table 1: This table provides the estimated parameters for the seven parametric distributions fitted on the operartional risk data set. If for the α stable distribution $\alpha < 1$, for the GEV, GEVbm and GPD $\xi > 1$ we are in the presence of an infinite mean model. The p-values of both Kolmogorov-Smirnov and Anderson-Darling tests are also provided for the fit of each distribution.

X_1 \ X_2		LogNormal		Weibull		GPD		GH		Alpha-Stable		GEV		GEVbm	
		VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}
LogNormal	p_1	2 493	6 103	2 328	4 820	75 546	217 916	2 618	7 753	2 552	44 407	2.834332e+07	1.721696e+21	2 658	19 225
	p_2	4 087	9 057	3 528	6 805	93 442	353 270	4 662	12 064	4 731	85 397	3.665483e+08	3.443392e+21	4 872	34 914
	p_3	7 252	14 693	5 735	10 419	142 038	716 560	9 291	20 539	10 725	203 193	1.072611e+10	8.608479e+21	10 723	76 829
	p_4	24 250	43 132	16 710	27 468	746 560	4 871 970	36 925	59 361	111 991	1 789 756	5.390204e+13	8.608479e+22	83 960	551 875
Weibull	p_1	-	-	2 230	3 637	75 270	219 140	2 424	6 507	2 345	80 360	2.874992e+07	1.648438e+21	2 459	32 126
	p_2	-	-	3 110	4 663	92 818	356 117	4 019	9 939	4 029	157 714	3.746901e+08	3.296876e+21	4 202	61 102
	p_3	-	-	4 425	6 176	140 307	725 572	7 651	16 774	9 003	385 667	1.062522e+10	8.242191e+21	8 856	143 962
	p_4	-	-	8 542	10 777	744 266	4 982 719	29 929	48 745	102 987	3 644 010	4.051415e+13	8.242191e+22	81 979	1 249 637
GPD	p_1	-	-	-	-	150 276	481 590	75 882	198 596	76 003	253 293	2.900050e+07	8.175899e+21	2.852749e+07	8.991075e+20
	p_2	-	-	-	-	185 210	799 161	93 953	314 173	94 929	423 142	3.804500e+08	1.635180e+22	3.719388e+08	1.798215e+21
	p_3	-	-	-	-	281 725	1 667 960	143 016	618 297	149 135	886523	1.055955e+10	4.087950e+22	1.038451e+10	4.495538e+21
	p_4	-	-	-	-	1 482 342	12 136 572	735 168	3 911 547	869 960	6 369 873	5.299107e+13	4.087950e+23	4.773943e+13	4.495538e+22
GH	p_1	-	-	-	-	-	-	2 784	9 385	2 705	73 357	2.880134e+07	3.149731e+20	2.875817e+07	6.426626e+19
	p_2	-	-	-	-	-	-	5 338	14 981	5 470	142 939	3.771517e+08	6.299462e+20	3.674012e+08	1.285325e+20
	p_3	-	-	-	-	-	-	11 469	25 984	13 280	344 980	1.092553e+10	1.574866e+21	1.088133e+10	3.213313e+20
	p_4	-	-	-	-	-	-	47 282	75 037	120 486	3 167 054	5.156340e+13	1.574866e+22	4.991674e+13	3.213313e+21
Alpha-Stable	p_1	-	-	-	-	-	-	-	-	2 649	146 535	2.868921e+07	2.877576e+22	2 822	31 932
	p_2	-	-	-	-	-	-	-	-	5 648	289 288	3.667449e+08	5.755152e+22	5 644	59 930
	p_3	-	-	-	-	-	-	-	-	15 890	709 490	1.007457e+10	1.438788e+23	13 339	137 177
	p_4	-	-	-	-	-	-	-	-	225 543	6 659 012	3.907140e+13	1.438788e+24	96 356	1 107 042
GEV	p_1	-	-	-	-	-	-	-	-	-	-	1.822300e+08	1.096423e+28	2.875582e+07	1.635820e+19
	p_2	-	-	-	-	-	-	-	-	-	-	2.615063e+09	2.192846e+28	3.640431e+08	3.271639e+19
	p_3	-	-	-	-	-	-	-	-	-	-	8.028848e+10	5.482116e+28	1.075443e+10	8.179098e+19
	p_4	-	-	-	-	-	-	-	-	-	-	4.437624e+14	5.482116e+29	4.820065e+13	8.179098e+20
GEVbm	p_1	-	-	-	-	-	-	-	-	-	-	-	-	2 894	59 150
	p_2	-	-	-	-	-	-	-	-	-	-	-	-	5 985	114 221
	p_3	-	-	-	-	-	-	-	-	-	-	-	-	15 597	271 596
	p_4	-	-	-	-	-	-	-	-	-	-	-	-	170 339	2 336 019

Table 2: Correlated Risk Measures - this table presents the VaRs and the ESs obtained on fully correlated random variables simulated with seven distributions.

LN-LN 1	393	663	$1,373$	$2,503$	$7,721$	$11,661$	$27,292$
LN-LN 2	395	667	$1,376$	$2,503$	$7,721$	$11,677$	$27,517$
LN-WE 1	447	742	$1,439$	$2,427$	$6,299$	$8,924$	$18,498$
LN-WE 2	564	826	$1,374$	$2,068$	$4,654$	$6,406$	$14,066$
LN-GPD 1	$4,321$	$6,181$	$11,432$	$21,158$	$88,382$	$163,788$	$689,569$
LN-GPD 2	$58,968$	$60,766$	$65,759$	$74,945$	$138,510$	$209,859$	$726,643$
LN-GH 1	364	611	$1,313$	$2,569$	$9,882$	$16,037$	$41,329$
LN-GH 2	480	742	$1,418$	$2,528$	$8,205$	$12,765$	$30,592$
LN-AS 1	377	614	$1,269$	$2,461$	$10,965$	$21,402$	$111,987$
LN-AS 2	476	725	$1,374$	$2,472$	$9,657$	$18,319$	$101,929$
LN-GEV 1	$25,132$	$137,464$	$2,097,977$	$28,700,959$	$10.73e9$	$134.51e9$	$47,029e9$
LN-GEV 2	$25,313$	$138,221$	$2,095,098$	$29,156,891$	$10.47e9$	$135.38e9$	$45,501e9$
LN-GEVbm 1	366	614	$1,312$	$2,579$	$11,037$	$20,542$	$91,109$
LN-GEVbm 2	481	742	$1,423$	$2,571$	$9,670$	$17,603$	$80,694$

Table 3: The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

WE-WE 1	501	820	1,505	2,352	4,878	6,187	9,703
WE-WE 2	501	821	1,510	2,352	4,879	6,185	9,807
WE-GPD 1	4,376	6,259	11,498	21,082	86,961	161,051	680,774
WE-GPD 2	58,916	60,639	65,520	74,662	138,368	209,701	726,035
WE-GH 1	418	690	1,379	2,494	8,460	13,300	32,534
WE-GH 2	533	795	1,379	2,208	6,472	10,534	27,998
WE-AS 1	431	692	1,335	2,386	9,544	18,665	103,193
WE-AS 2	528	779	1,341	2,148	7,556	16,025	101,095
WE-GEV 1	25,186	137,542	2,098,044	28,700,884	$10.73e9$	$134.51e9$	$47,029e9$
WE-GEV 2	25,197	138,107	2,094,946	29,156,852	$10.47e9$	$135.38e9$	$45,501e9$
WE-GEVbm 1	420	692	1,379	2,504	9,616	17,805	82,315
WE-GEVbm 2	534	796	1,381	2,237	7,710	15,281	79,250

Table 4: The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

GPD-GPD 1	$8,250$	$11,699$	$21,490$	$39,812$	$169,044$	$315,915$	$1,351,846$
GPD-GPD 2	$117,080$	$120,546$	$130,394$	$148,749$	$276,271$	$418,831$	$1,452,006$
GPD-GH 1	$4,292$	$6,129$	$11,372$	$21,224$	$90,543$	$168,164$	$703,606$
GPD-GH 2	$59,005$	$60,888$	$66,096$	$75,538$	$139,002$	$209,869$	$726,103$
GPD-AS 1	$4,305$	$6,131$	$11,328$	$21,116$	$91,627$	$173,528$	$774,264$
GPD-AS 2	$58,987$	$60,890$	$66,273$	$76,314$	$147,644$	$229,984$	$834,971$
GPD-GEV 1	$29,061$	$142,981$	$2,108,036$	$28,719,614$	$10.73e9$	$134.51e9$	$47,029e9$
GPD-GEV 2	$92,215$	$210,767$	$2,181,852$	$29,254,626$	$10.47e9$	$135.38e9$	$45,501e9$
GPD-GEVbm 1	$4,292$	$6,129$	$11,372$	$21,224$	$90,543$	$168,164$	$703,606$
GPD-GEVbm 2	$59,005$	$60,888$	$66,096$	$75,538$	$139,002$	$209,869$	$726,103$
GH-GH 1	335	559	$1,253$	$2,635$	$12,043$	$20,413$	$55,366$
GH-GH 2	335	559	$1,253$	$2,635$	$12,043$	$20,413$	$55,366$
GH-AS 1	348	562	$1,209$	$2,527$	$13,126$	$25,778$	$126,024$
GH-AS 2	442	683	$1,393$	$2,778$	$12,596$	$23,446$	$104,497$
GH-GEV 1	$25,103$	$137,412$	$2,097,918$	$28,701,025$	$10.73e9$	$134.51e9$	$47,029e9$
GH-GEV 2	$25,635$	$138,429$	$2,095,206$	$29,157,735$	$10.47e9$	$135.38e9$	$45,501e9$
GH-GEVbm 1	336	562	$1,252$	$2,645$	$13,198$	$24,917$	$105,146$
GH-GEVbm 2	446	703	$1,451$	$2,895$	$12,502$	$22,224$	$84,680$

Table 5: The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

AS-AS 1	361	564	1,165	2,419	14,210	31,142	196,682
AS-AS 2	360	562	1,159	2,428	14,153	31,459	201,447
AS-GEV 1	25,116	137,414	2,097,873	28,700,918	10.73e9	134.51e9	47,029e9
AS-GEV 2	26,139	140,091	2,099,977	29,175,188	10.47e9	135.38e9	45,501e9
AS-GEVbm 1	349	564	1,208	2,537	14,282	30,282	175,804
AS-GEVbm 2	443	683	1,399	2,849	15,645	33,285	189,589
GEV-GEV 1	49,871	274,264	4,194,582	57,399,416	21.46e9	269e9	94,058e9
GEV-GEV 2	49,844	275,821	4,189,583	58,313,419	20.94e9	271e9	91,002e9
GEV-GEVbm 1	25,105	137,414	2,097,917	28,701,036	10.73e9	134.51e9	47,029e9
GEV-GEVbm 2	26,105	139,855	2,099,195	29,174,309	10.47e9	135.38e9	45,501e9
GEVbm-GEVbm 1	338	564	1,252	2,656	14,353	29,422	154,927
GEVbm-GEVbm 2	340	565	1,251	2,663	14,609	29,967	158,273

Table 6: The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

LN-LN 1	$1,895$	$2,587$	$4,226$	$6,616$	$16,572$	$23,652$	$50,725$
LN-LN 2	$1,895$	$2,587$	$4,226$	$6,616$	$16,572$	$23,652$	$50,725$
LN-WE 1	$1,727$	$2,302$	$3,574$	$5,293$	$11,739$	$16,021$	$31,500$
LN-WE 2	$1,541$	$1,970$	$2,882$	$4,092$	$8,841$	$12,269$	$25,675$
LN-GPD 1	$87,496$	$101,478$	$140,329$	$211,059$	$682,080$	$1,191,608$	$4,513,150$
LN-GPD 2	$87,065$	$100,726$	$138,767$	$208,277$	$674,213$	$1,180,157$	$4,488,157$
LN-GH 1	$2,146$	$2,984$	$5,081$	$8,347$	$23,114$	$33,777$	$71,406$
LN-GH 2	$1,996$	$2,698$	$4,383$	$6,898$	$17,681$	$25,214$	$50,397$
LN-AS 1	$16,694$	$24,801$	$48,732$	$95,726$	$459,981$	$905,044$	$4,350,967$
LN-AS 2	$16,545$	$24,525$	$48,067$	$94,322$	$454,147$	$895,398$	$4,326,497$
LN-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
LN-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
LN-GEVbm 1	$5,608$	$8,174$	$15,460$	$29,105$	$126,073$	$237,314$	$1,032,332$
LN-GEVbm 2	$5,457$	$7,888$	$14,762$	$27,640$	$120,229$	$227,765$	$1,008,148$

Table 7: The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

WE-WE 1	1,559	2,016	2,921	3,970	6,905	8,390	12,276
WE-WE 2	1,559	2,016	2,921	3,970	6,905	8,390	12,276
WE-GPD 1	87,328	101,193	139,676	209,736	677,247	1,183,977	4,493,926
WE-GPD 2	86,887	100,505	138,515	208,044	674,087	1,180,072	4,488,101
WE-GH 1	1,978	2,698	4,428	7,024	18,280	26,146	52,182
WE-GH 2	1,810	2,389	3,739	5,758	15,192	22,257	46,312
WE-AS 1	16,526	24,516	48,079	94,403	455,148	897,413	4,331,742
WE-AS 2	16,359	24,217	47,423	93,172	452,023	893,523	4,325,897
WE-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2447e18$
WE-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2447e18$
WE-GEVbm 1	5,440	7,889	14,807	27,782	121,240	229,683	1,013,108
WE-GEVbm 2	5,270	7,579	14,119	26,506	118,106	225,770	1,007,256

Table 8: The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

GPD-GPD 1	173,097	200,369	276,431	415,503	1,347,588	2,359,564	8,975,575
GPD-GPD 2	173,097	200,369	276,431	415,503	1,347,588	2,359,564	8,975,575
GPD-GH 1	87,747	101,874	141,183	212,791	688,622	1,201,732	4,533,832
GPD-GH 2	87,330	101,092	139,298	208,887	674,421	1,180,208	4,488,112
GPD-AS 1	102,295	123,692	184,834	300,169	1,125,489	2,073,000	8,813,392
GPD-AS 2	101,891	122,938	182,933	295,782	1,098,582	2,016,042	8,499,442
GPD-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GPD-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GPD-GEVbm 1	91,209	107,065	151,562	233,548	791,581	1,405,270	5,494,758
GPD-GEVbm 2	90,787	106,267	149,558	229,042	766,781	1,355,085	5,243,081
GH-GH 1	2,397	3,380	5,935	10,078	29,655	43,901	92,088
GH-GH 2	2,397	3,380	5,935	10,078	29,655	43,901	92,088
GH-AS 1	16,945	25,197	49,586	97,457	466,523	915,168	4,371,648
GH-AS 2	16,809	24,941	48,924	95,926	458,199	899,741	4,327,096
GH-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GH-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GH-GEVbm 1	5,858	8,571	16,314	30,836	132,615	247,439	1,053,014
GH-GEVbm 2	5,722	8,305	15,616	29,227	124,294	232,340	1,009,422

Table 9: The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

AS-AS 1	$31,493$	$47,015$	$93,237$	$184,836$	$903,390$	$1,786,436$	$8,651,209$
AS-AS 2	$31,493$	$47,015$	$93,237$	$184,836$	$903,390$	$1,786,436$	$8,651,209$
AS-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
AS-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
AS-GEVbm 1	$20,406$	$30,388$	$59,965$	$118,215$	$569,482$	$1,118,706$	$5,332,574$
AS-GEVbm 2	$20,270$	$30,130$	$59,302$	$116,655$	$559,704$	$1,097,691$	$5,212,237$
GEV-GEV 1	$16e18$	$24e18$	$48e18$	$97e18$	$489e18$	$979e18$	$4,895e18$
GEV-GEV 2	$16e18$	$24e18$	$48e18$	$97e18$	$489e18$	$979e18$	$4,895e18$
GEV-GEVbm 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GEV-GEVbm 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GEVbm-GEVbm 1	$9,320$	$13,761$	$26,693$	$51,593$	$235,574$	$450,977$	$2,013,940$
GEVbm-GEVbm 2	$9,320$	$13,761$	$26,693$	$51,593$	$235,574$	$450,977$	$2,013,940$

Table 10: The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

%tile \ Distribution	Empirical		LogNormal		Weibull		GPD		GH		Alpha-Stable		GEV		GEVbm	
	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES
90%	575	2 920	686	2 090	753	1 455	10 745	146 808	627	2 950	582	29 847 $+\infty$	2.097291e+06	$+\infty$	626	12 755
95%	1 068	5 051	1 251	3 264	1 176	1 979	19 906	224 872	1 317	5 006	1 209	58 872 $+\infty$	2.869971e+07	$+\infty$	1 328	24 616
97.5%	1 817	8 775	2 106	4 925	1 674	2 572	37 048	368 703	2 608	8 187	2 563	116 016 $+\infty$	3.735524e+08	$+\infty$	2 762	47 356
99%	3 662	18 250	3 860	8 148	2 439	3 468	84 522	758 667	5 917	14 721	7 105	283 855 $+\infty$	1.073230e+10	$+\infty$	7 177	111 937
99.9%	31 560	104 423	13 646	24 784	4 852	6 191	675 923	5 328 594	28 064	46 342	98 341	2 649 344 $+\infty$	4.702884e+13	$+\infty$	77 463	945 720

Table 11: Univariate Risk Measures - This table exhibits the VaRs and ESs for the height types of distributions considered - for instance empirical, lognormal, Weibull, GPD, GH, α-stable, GEV and GEV fitted on a series of maxima - for five confidence level (for instance, 90%, 95%, 97.5%, 99% and 99.9%) evaluated on the period 2009-2014. Note that the parameters obtained for the α-stable and the GEV fitted on the entire data set lead to infinite mean model and therefore, the ES are hardly applicable.

80.00%	81.00%	82.00%	83.00%	84.00%	85.00%	86.00%	87.00%
-81.843	-74.943	-66.539	-57.410	-47.461	-35.212	-20.496	-3.984
88.00%	89.00%	90.00%	91.00%	92.00%	93.00%	94.00%	95.00%
16.247	40.129	67.997	102.443	144.756	196.882	266.676	360.135
96.00%	97.00%	98.00%	99.00%	99.50%	99.90%	99.95%	99.99%
489.356	677.618	1,011.196	1,696.400	2,581.672	4,858.396	5,761.766	10,964.930

Table 12: This table shows the differences between the sum VaR(X) and the VaR(Y) and the VaR(X + Y). The random variable X has been generated using a Weibull and Y has been obtained from a lognormal distribution. When the values are positive, the VaR is sub-additive, when the values are negative the VaR is not. The turning points are highlighted in bold.

80.00%	81.00%	82.00%	83.00%	84.00%	85.00%	86.00%	87.00%
-86.104	-82.891	-80.004	-75.764	-69.887	-63.385	-55.082	-45.380
88.00%	89.00%	90.00%	**91.00%**	92.00%	93.00%	94.00%	95.00%
-34.810	-21.030	-2.510	**23.340**	54.970	99.660	159.200	249.830
96.00%	97.00%	98.00%	99.00%	99.50%	99.90%	99.95%	**99.99%**
393.730	632.630	1,098.500	2,170.800	3,052.900	4,784.190	17,905.440	**−633,422.500**

Table 13: This table shows the differences between the sum VaR(X) and the VaR(Y) and the VaR(X + Y). The random variable X has been generated using a Weibull and Y has been obtained from an Alpha-stable distribution. When the values are positive, the VaR is sub-additive, when the values are negative the VaR is not. The turning points are highlighted in bold.

-0.012	0.022	-0.013	-0.018	-0.015	-0.031	-0.020	-0.026
-0.038	**0.011**	0.028	0.023	0.022	0.024	0.044	0.073
0.074	0.080	0.139	0.144	0.194	0.171	0.167	0.163
0.142	0.141	0.134	0.150	0.179	0.175	0.105	0.107
0.016	**-0.001**	-0.002	-0.003	0.013	-0.021	-0.048	-0.011
-0.016	**0.045**	0.074	0.032	0.074	0.166	0.124	0.104
0.098	0.019	**-0.037**	-0.079	-0.100	-0.120	-0.144	-0.047
-0.070	-0.086	-0.136	-0.234	-0.291	-0.352	-0.272	-0.197
-0.098	0.038	0.121	-0.313	-0.299	-0.483	-0.621	-0.422
-0.457	**0.099**	0.272	0.381	0.430	0.656	0.754	0.533
0.693	1.035	0.715	1.087	0.778	$\mathbf{-0.167}$	-0.479	-0.522
-0.759	-3.391	-2.265	-4.190	-3.137	-6.484	-1.975	**9.502**
6.873	16.636	69.495	50.091	$7,118.689$	$8,798.144$	$\mathbf{-148,979.500}$	NA

Table 14: This table shows the differences between the sum VaR(X) and the VaR(Y) and the VaR(X + Y). The random variable X and Y have been obtained on 2 identical GEV distributions. When the values are positive, the VaR is sub-additive, when the values are negative the VaR is not. The turning points are highlighted in bold. The percentiles represented are sequentially going from 1% to 99% by 1%, and to capture the tail, the 99.95th, 99.9th, 99.95th and 99.99th percentiles are added.

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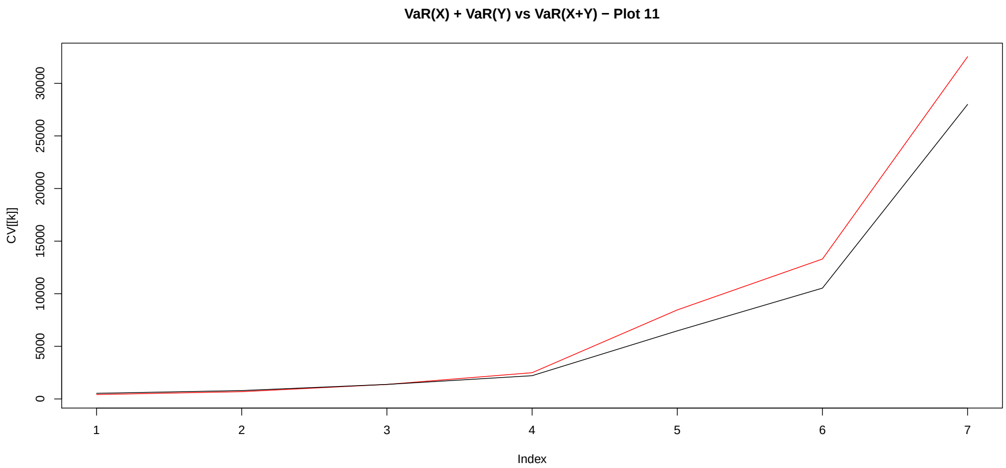

Figure 5: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using a Weibull distribution and Y has been obtained from a Generalised Hyberbolic distribution. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. For high percentiles, the VaR seems to be sub-additive.

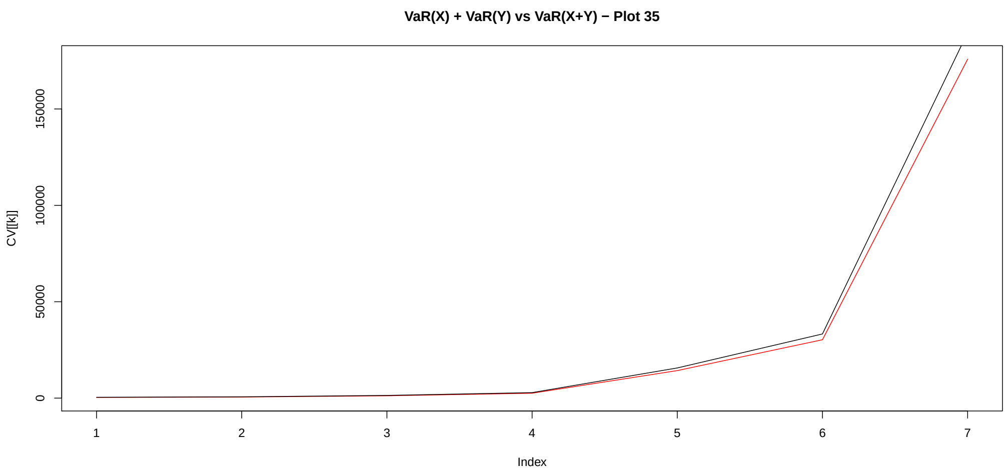

Figure 6: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using an Alpha-stable distribution and Y has been obtained from a GEV distribution calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. For high percentiles, the VaR is not sub-additive.

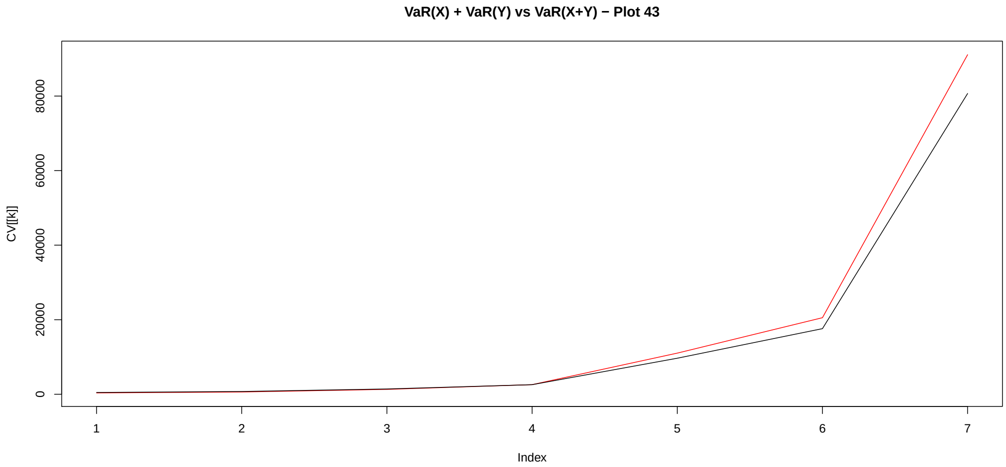

Figure 7: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variables X and Y have been obtained from two identical GEV distributions. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. For high percentiles, the VaR is sub-additive.

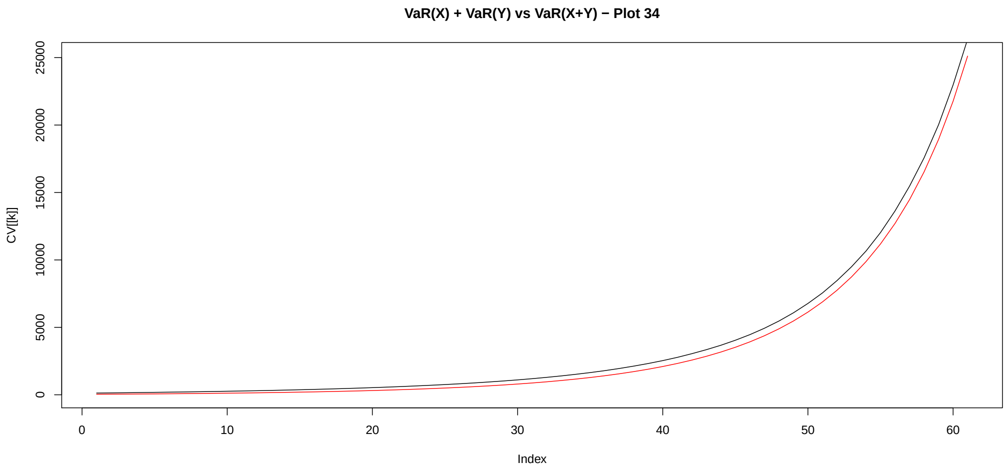

Figure 8: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using an Alpha-stable distribution and Y has been obtained from a Generalised Hyperbolic distribution. The percentiles represented are sequentially going from the 10th to the 70th with a step of 1% between two points. The VaR represented are never sub-additive.

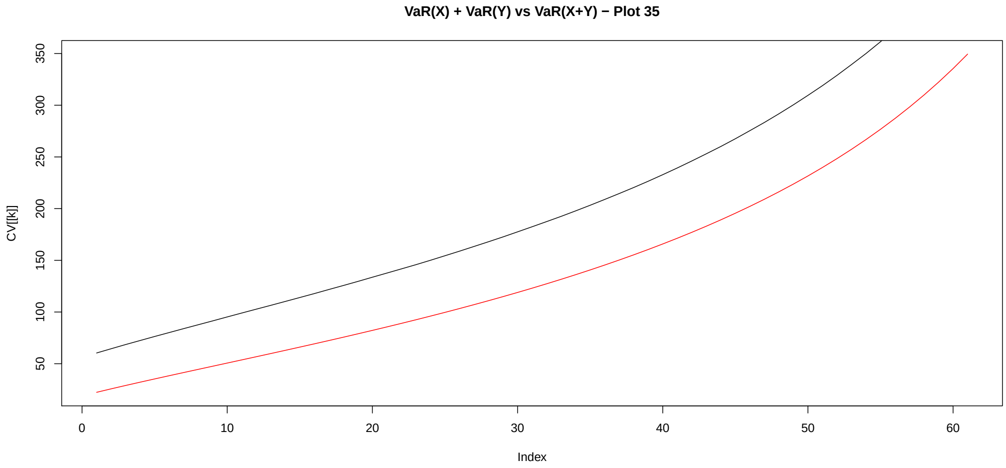

Figure 9: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using a Alpha-stable distribution and Y has been obtained from a GEV distribution calibrated on maxima. The percentiles represented are sequentially going from the 10th to the 70th with a step of 1% between two points. The VaR represented are never sub-additive.

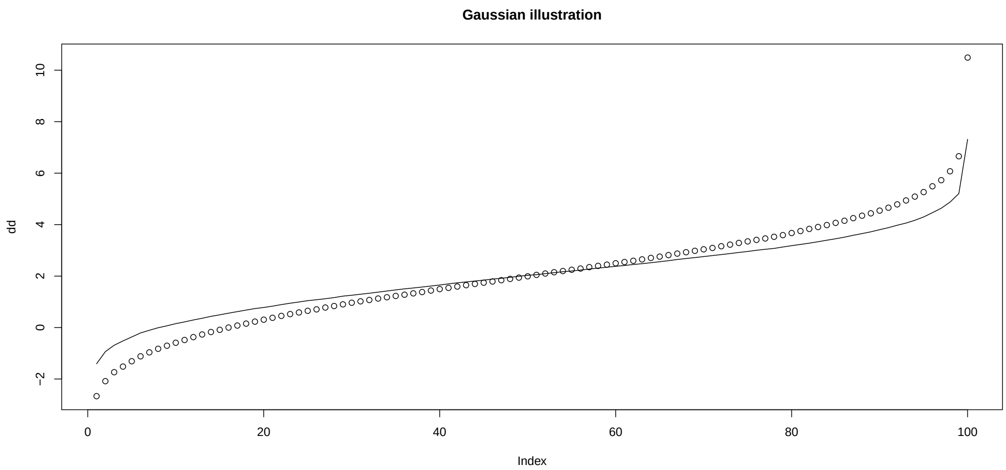

Figure 10: This plot represents the sum of VaR(X) and VaR(Y) (doted line) versus VaR(X + Y) (solid line). The random variable X has been generated using a Gaussian distribution (0, 1) and Y has been obtained from a Gaussian distribution (2, 1). The VaRs represented are always sub-additive.

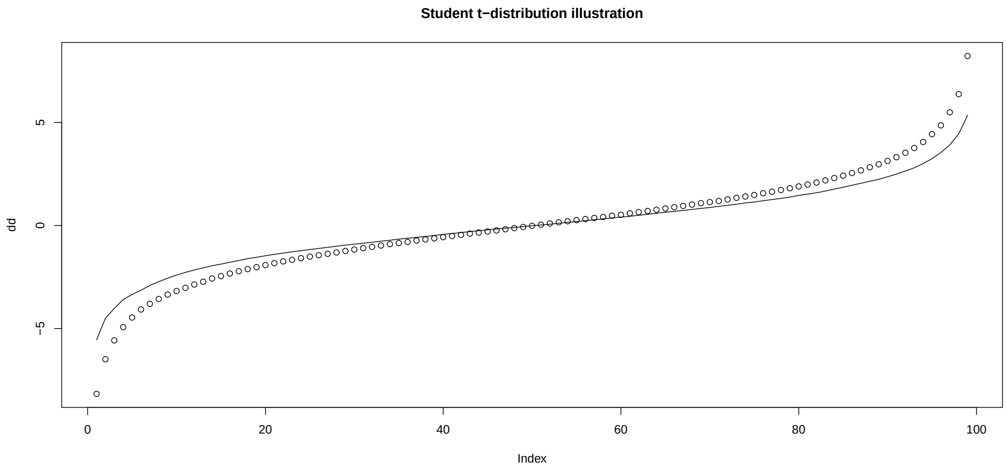

Figure 11: This plot represents the sum of VaR(X) and VaR(Y) (doted line) versus VaR(X + Y) (solid line). The random variable X has been generated using a Student-t distribution (3 df) and Y has been obtained from a Student-t distribution (4 df). The VaRs represented are always sub-additive.