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Equality statements as rules for transforming
arithmetic notation

Ian Jones

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Abstract

This thesis explores children's conceptions of the equals sign from the vantage point of notating task design. The existing literature reports that young children tend to view the equals sign as meaning "write the result here". Previous studies have demonstrated that teaching an "is the same as" meaning leads to more flexible thinking about mathematical notation. However, these studies are limited because they do not acknowledge or teach children that the equals sign also means "can be exchanged for".

The thesis explores the "sameness" and "exchanging" meanings for the equals sign by addressing four research questions. The first two questions establish the distinction, in terms of task design, between the two meanings.

Does the "can be exchanged for" meaning for the equals sign promote attention to statement form?

Are the "can be exchanged for" and "is the same as" meanings for the equals sign pedagogically distinct?

The final two research questions seek to establish how children might coordinate the two meanings, and connect them with their existing implicit knowledge of arithmetic principles.

Can children coordinate "can be exchanged for" and "is the same as" meanings for the equals sign?

Can children connect their implicit arithmetical knowledge with explicit transformations of notation?

The instrument used is a specially designed notational computer-microworld called *Sum Puzzles*. Qualitative data are generated from trials with pairs of Year 5 (9 and 10 years), and in one case Year 8 (12 and 13 years), pupils working collaboratively with the microworld toward specified task goals.

It is discovered that the "sameness" meaning is useful for distinguishing equality statements by truthfulness, whereas the "exchanging" meaning is useful for distinguishing statements by form. Moreover, a duality of both meanings can help children connect their own mental calculation strategies with transformations of properly formed notation.

Chapter 1

Introduction

The modern equals sign was invented by Robert Recorde in 1557 in *The Whetstone of Witte*. He set out his reasoning for the symbol = as follows:

to avoide the tedious repetition of these wordes : is equalle to :
I will sette as I doe often in woorke use, a paire of paraleles, or
Gemowe [twin] lines of one length, thus: =, bicause noe .2. thynges,
can be moare equalle. (cited in Cajori, 1923, p.176).

Nowadays the equals sign can indicate a variety of things, such as a computational result, as in $2 + 2 = 4$, an identity, as in $(x + y)(x - y) = x^2 - y^2$, a definition, as in $i = \sqrt{-1}$, a function, as in $f(x) = \cos x$, a substitution, as in $x = \frac{1}{2}$, and so on. Digital technology supports further uses of the equals sign. On traditional calculators = appears on a button pressed to get the result to a programmed sequence of numerals and operator signs. Within computer programming languages = can assign values or compare two inputs and return a Boolean result.

The focus in this thesis is on young children's conceptions of the symbol = in arithmetic contexts with integers > 0 . Primary pupils' expectations of a numerical result on the right hand side of the equals symbol is something I first became aware of as a school teacher. My research interests in the issue began when I was involved in trialling the *VisualFractions* microworld (Kalas & Blaho, 2003) prior to its commercial release. During these trials I noticed that pupils could quite easily connect various mathematical symbols together

in meaningful ways, except for the equals sign which caused them significant difficulties (see Section 5.2.1). Intrigued by this I developed and trialled a variant on the traditional school calculator, called the *Equivalence Calculator* (see Section 5.2.2), and this in turn led to the work reported in this thesis.

The first three chapters review the literature pertaining to the studies reported in the thesis. Chapter 2 reviews and critiques the published literature on young children's conceptions of the equals sign in arithmetic contexts. Chapter 3 reviews the psychology of mathematical equivalence relations more generally, in both concrete and notational contexts, in order to identify limitations with the literature on children's conceptions of the equals sign. Chapter 4 reviews three large-grain theories of learning (constructivism, social constructivism and situated cognition) and their implications for equivalence relations in notational contexts.

Chapter 5 recounts preliminary work with *VisualFractions* and the *Equivalence Calculator* (see above), as well as the initial rationale and early design work for this thesis. Chapter 6 sets out and justifies the "diagrammatic" approach used and describes the software-based instrument (a microworld called *Sum Puzzles*) used in the studies. Chapters 7-10 report the methods used and studies conducted, and presents qualitative data from trials of the *Sum Puzzles* microworld with pairs of Year 5 pupils. The final two chapters analyse the findings from the three studies and draw conclusions regarding the theory and pedagogy of children's conceptions of the equals sign.

Chapter 2

The equals sign and arithmetical equivalence

2.1 Introduction

There is a substantial body of literature on pupils' conceptions of the equals sign in arithmetic contexts. In this chapter I review this literature and highlight three issues. First, there is a broad consensus that young children accept only *expression = numeral* statement forms. This leads many authors to claim children view the equals sign as a “do something signal”, although terms such as “place-indicator” are also used and, in my view, more appropriately emphasise the notational role of the symbol =. Second, children develop a “basic” relational view when they come to accept a wider variety of statement forms. However, the place-indicator and basic relational views both involve computing and comparing results and so are cognitively similar. Some authors have explored how children can develop a “full” relational view, which involves reflecting on the structural properties of statements rather than results. Third, obstacles to developing relational views of the equals sign arise from how arithmetical notation is presented and talked about in the classrooms. As such, many authors recommend that an “is the same as” meaning for the symbol = be promoted, and a wide variety of statement forms be presented, in classrooms. I argue this recommendation is valid but limited because it arises from research designs that do not acknowledge the interchangeability aspect of arithmetical

equivalence.

2.2 A place-indicator view of the equals sign

Three decades ago Behr, Erlwanger, and Nichols (1976) claimed young children consider the symbol $=$ as a “do something signal” (p.10). Their evidence came from semi-structured clinical interviews in which children aged 6 to 12 years were presented with a variety of equality statements of the forms *expression* = *numeral*, *numeral* = *expression*, *numeral* = *numeral* and *expression* = *expression* (where *expression* is any closed string of operator signs and numerals). They found most of the children expected a numeral to follow the equals sign, as in $2 + 3 = 5$ and $2 + 3 = 7$ (p.2). However, for sentences such as $8 = 3 + 5$ and $6 = 4 + 1$, children read them aloud as “5 plus 3 equals 8”, or deemed them “backwards” and modified them to $6 = 4 + 10$, and read them aloud as “6 and 4 makes 10” (p.3). This was also the case for missing number sentences. For example, one child commented on $\square = 3 + 4$ by saying “you asked the question backwards ... you can’t go 7 equals 3 plus 4” (p.2).

There were similar results when children were presented with expressions on both sides of the equals sign. One child split $2 + 3 = 3 + 2$ into $2 + 3 = 5$ and $3 + 2 = 5$; another modified and extended it to become $3 + 2 + 2 + 3 = 10$; another extended $1 + 5 = 1 + 5$ to become $1 + 5 = 1 + 5 =$ (although no result was written). One child wrote the result to each side of $1 + 2 = 2 + 1$ beneath each term and read it aloud as “1 plus 2 equals 3”, touching each number on the left-hand side as each was read, and then repeated the process for the right-hand side. She then said “you forgot to put the 5” and rewrote the whole thing as the two statements $3 + 2 = 5$ and $2 + 3 = 5$ (p.9).

As well as expecting a numeral on the right of the symbol $=$, the children also expected an expression containing operator signs on the left. The authors found that the children considered “non-action” (operator free) equality statements to be incomplete. For example, when presented with $3 = 3$ one child wrote $0+$ in front of it to give $0 + 3 = 3$; another modified it to become $3 - 3 = 0$; another included the answer as a subscript, as in $3 =_6 3$ and then inserted an operator sign, giving $3+ =_6 3$. Similarly, when presented with the false non-action statement $3 = 5$, responses were to modify it to $2 + 3 = 5$ or $3 - 5 = 0$ or $3+ =_8 5$.

The authors argued these data demonstrated that young children tend to view the equals sign as a signal to “do something” to the left-hand side and put the result on the right-hand side. The phrase “do something signal” was subsequently adopted in a widely cited review (Kieran, 1981) that described pupils’ tendency to accept only *expression = numeral* statement forms. Pupils who accept a wider variety of statement forms have moved beyond this view:

They justified [*expression = expression* statements] in terms of both sides being equal because they had the same value. The comparison that subjects were eventually able to make between left and right sides of the equal sign suggest that the equality symbol was being seen at this stage more as a relational symbol than as a “do something signal”. The right side, by this time, did not have to contain the answer, but rather could be some expression that had the same value as the left side (p.321).

The phrase “do something signal” to describe pupils’ conceptions of the equals sign that correlate with exclusive acceptance of *expression = numeral* statement forms has remained in currency since Kieran adopted it (e.g. Cobb, 1987; Falkner, Levi, & Carpenter, 1999; Hiebert, 1984; Jones & Pratt, 2006; Knuth, Stephens, McNeil, & Alibali, 2006; Li, Ding, Capraro, & Capraro, 2008; Lima & Tall, 2006). Synonymous phrases have also been used, including “operator view” (Baroody & Ginsburg, 1983; Denmark, Barco, & Voran, 1976), “operational concept” (Knuth et al., 2006) and “a command to find the result” (Sáenz-Ludlow & Walgamuth, 1998). Some authors have drawn an analogy, or claimed a causal relationship, between pupils’ views of the equals sign and the function of the “=” button on a calculator (Ginsburg, 1989; Jones & Pratt, 2005; Li et al., 2008).

However, it is unclear on closer examination what is meant by claiming pupils view the equals sign as a “do something signal”. Should we expect pupils to compute $2 + 4 =$ but not compute the same expression lacking a “do something signal” (i.e. $2 + 4$ with no equals sign)? No we should not, according to Behr, Erlwanger, and Nichols (1980): “Even in the absence of the = symbol . . . $2+4$ serves as a stimulus *to do something*” (p.13 — emphasis in original). If expressions are a stimulus to do something anyway then it seems the equals sign is a partially irrelevant “do something signal” at best. A related observation has been made by Frieman and Lee (2004):

Rather than interpreting the equal sign as a “do something” signal, it is possible that children perceive $+$ $=$ as a single operator symbol where the $+$ indicates the type of operation to be performed (p.417).

Behr et al. perhaps offer a more helpful description of their findings with the following:

There is a strong tendency among all of the children to view the $=$ symbol as being acceptable in a sentence only when one (or more) operation signs ($+$, $-$, etc) precede it. Some children, in fact, tell us that the answer must come after the $=$. (p.10)

Here they seem to be describing the equals sign as a “place-indicator” for a notated result, rather than as a signal to compute that result. In fact many authors similarly use phrases that suggest place-indicator conceptions of the equals sign. For example, Renwick (1932) reported that children aged 10 to 12 years “use the ‘ $=$ ’ sign simply to separate an expression from its answer” (p.182). Pirie and Martin (1997) described children as viewing the equals sign as “indicating the result of an operation” (p.159). Kilpatrick, Swalford, and Findell (2001) argued that for pupils a “sum such as $8+5$ is a signal to compute” and the equals sign is “a signal to write the result of performing the operations indicated to the left of the sign” (p.261). Falkner et al. (1999) hinted that children possess both “do something” and place-indicator conceptions: “children in the elementary grades generally think that the equals sign means that they should carry out the calculation that precedes it and that *the number after the equals sign is the answer to the calculation*” (p.233 — emphasis added). Knuth et al. (2006) reported pupils’ written definitions for the written symbol $=$ which they classified as “operational concepts” of the equal sign. However, some of these definitions appear to reflect place-indicator concepts:

“What the sum of the two numbers are.”

“A sign connecting the answer to the problem.”

“How much the numbers added together equal.” (p.302, 303)

Indeed, Knuth et al. described what they call operational conceptions as “viewing the symbol as an announcement of the result of an arithmetic operation” (p.298).

Further evidence that pupils' more commonly view the equals sign as a "place-indicator" than a "do something signal" comes from their use of notation as computational aids. Sáenz-Ludlow and Walgamuth (1998) reported a pupil expressing an arithmetical word problem as the "running statement" $15 + 20 = 35 + 5 = 40 + 1 = 41$. They commented that:

the equal symbol did not have for the child the meaning of quantitative sameness (same quantity on both sides of the equal sign); it was only used as an *index* to represent a final point of each binary state in his over-all additive process (p.166 — emphasis added).

Likewise, pupils who completed the missing number statement $246 + 14 = \square + 246$ as $246 + 14 = 260 + 246 = 506$ viewed the equals sign "as a separator of their sequence of operations" (p.177). The equals sign here was used as a place-indicator ("index") for writing down the result once it had been computed. Similarly, Renwick (1932) argued running statements such as

$$\mathcal{L}2 \times 7 = \mathcal{L}14 - 3\frac{1}{2}\text{d.} = \mathcal{L}13 \text{ 19s. } 8\frac{1}{2}\text{d.}$$

demonstrate that

children regard the sign of equality a symbol of *distinction* rather than as a connecting link bringing into relation two expressions which are numerically or quantitatively equivalent: its function, to the child mind, would appear to be to separate rather than to bridge. (p.173 — emphasis in original)

Likewise, Hewitt (2003) likened the equals sign in running statements to a full stop: "the word *equals* breaks up the flow of left to right by creating a new beginning after the word *equals*" (p.65 — emphasis in original). In the context of computational aids, then, the symbol $=$ is not so much a signal to *do* something, as a signal to *stop doing it* and write down the result.

There are, however, some examples in the literature that more explicitly suggest pupils view the symbol $=$ as an operator. For example, N. McNeil and Alibali (2005) identified some pupils' verbalised definitions of the equals sign as operational, as in "add up all those together", "the total of the problem" and "what the problem is" (p.888) and identified other pupils' definitions as notational, as in "it's like where you end the problem". Similarly, Sáenz-Ludlow

and Walgamuth (1998) reported on a classroom interaction in which a pupil, Sh, articulated a meaning for the = in $6 + 6 = 6 + 6$:

- T: Do you think six plus six equals six plus six? Do you think that is right?
Sh: I disagree.
T: Tell me why.
Sh: Because equals doesn't mean you put six plus six again. You're supposed to add the numbers up. That's what equal means and you put the answer down. (p.172)

Sh can be seen to possess both an operator conception (“You’re supposed to add the numbers up. That’s what equal means”) and a notation-indicator (“equals doesn’t mean you put six plus six again . . . and you put the answer down”).

In sum, the evidence provided in the literature suggests young children tend to view the equals sign as a “place-indicator” for the result of an operation. There is secondary evidence that, at times, some children view the equals sign as a “do something signal”. Overall, the consistent theme throughout all the studies cited in this section is that pupils expect arithmetical equality statements to adhere to the form *expression = numeral*.

In the next section I review studies that report the responses of children who do in fact accept a wider variety of statement forms.

2.3 A relational view of the equals sign

The original motivation for Behr et al.’s (1976) study was to establish “whether children consider equality to be an operation or a relation” (p.15). The difference is that “as an operator symbol, = would be a ‘do something signal’. As a relational symbol, = suggests a comparison of two members of an equality sentence” (p.10). They did not clarify the term “members” but some authors took it to mean a comparison of both sides of an equality statement. For example, Theis (2004) described young pupils who accept only $a + b = c$ statements as having an operator view and those who accept the numerical equivalence of $c = a + b$ and $a + b = b + a$ statements as having a relational view. Carpenter and Levi (2000) and Kieran (1981— see quote on page 5) have used the same classification. Knuth et al. (2006) provided four examples of pupils’ definitions

of the = symbol that they deemed to be relational. These examples suggest those pupils viewed the equals sign in terms of value-sameness:

“It means that what is to the left and right of the sign mean the same thing.”

“The same as, same value.”

“The left side of the equals sign and the right side of the equals sign are the same value.”

“The expression on the left side is equal to the expression on the right side.” (p.303)

However, Baroody and Ginsburg (1983) suggested that pupils accepting a wide range of statement forms as true or false is evidence for only a “basic” relational view of the equals sign:

the “equals” sign cannot be divorced from an operator view. Unlike equations such as $8 = 8$, which embody the identity principle, or $4 + 3 = 3 + 4$, which embody the commutativity principle, equations such as $7 + 6 = 4 + 9$, $7 + 6 = 6 + 6 + 1$, or $2 + 4 = 3 \times 2$ require computing the expressions on each side of the “equals” sign in order to establish the equality. In other words, children regularly had to *do something* (employ arithmetic operations) in order to judge whether the equality made sense (were the “same number”). (p.210 – emphasis added)

To reiterate, accepting the truth of, say, $7 + 6 = 4 + 9$ is not cognitively very different from accepting the pair $7 + 6 = 13$ and $4 + 9 = 13$ and noticing they have the same answer; either way equivalence is established by computing $7 + 6$ and $4 + 9$. Baroody and Ginsburg argued that a “full relational view would require understanding the three identifying properties of an equivalence relationship” (p.208), namely symmetry (if $13 = 7 + 6$ then $7 + 6 = 13$), reflexivity (if $8 = 8$ then 8 is 8) and transitivity (if $7 + 6 = 13$ and $13 = XIII$ then $7 + 6 = XIII$). Behr et al. (1976) enigmatically hinted that a truly relational view might include seeing “ $2 + 4$ as being a name for 6” (p.2), seeing “ $\frac{6}{2}$ and 3 as the same number, rather than as different *numbers* having the same *value*” (p.8 — emphasis in original), and the ability to “reflect, make judgments, and infer meanings” (p.10) about equality statements.

Nevertheless, the studies cited above demonstrate that pupils can be satisfactorily grouped according to the “place-indicator” and “basic relational” views of the equals sign because “full relational” views are almost non-existent amongst young children. This dichotomy is therefore appropriate to their research questions and is in the most part set out clearly by authors. However, for classroom intervention researchers seeking ways to help pupils move beyond the “basic relational” view the indicator-relation dichotomy is insufficient. As such, there have been efforts to clarify the distinction between “basic” and “full” relational views. Carpenter, Franke, and Levi (2003) described four “benchmarks” in the development of conceptions. The first corresponds to the place-indicator view outlined in the previous section. The second is when pupils accept as true some statement forms other than *expression = numeral*. The third is when pupils understand the equals sign as a relation between two numbers and systematically compute both sides of statements to assess their truth. These second and third benchmarks are closely related and correspond to Baroody and Ginsburg’s “basic relational” view. The fourth benchmark is when pupils can establish the truth of a statement without calculation by drawing on its structural properties. This fourth benchmark, called “relational thinking”, is more complicated than the first three and has been elaborated by some researchers.

In a study that investigated the impact of presenting statements that appeal to structural readings (e.g. $7 + 7 + 9 = 14 + 9$ and $10 + 4 = 4 + 10$), Molina, Castro, and Mason (2008) grouped pupils in terms of “simple” and “sameness” relational views. A “simple” relational view involves drawing on arithmetical facts, such as knowing that $a - a$ is 0 and so immediately seeing that $24 - 24 = 0$ is true without having to work anything out. However this is an inflexible strategy for assessing the truthfulness of statements because it draws on isolated facts and pupils frequently have to resort to computing both sides of the equals sign. A “sameness” relational view involves drawing not on isolated arithmetical facts but identifying the relationships between parts of the two expressions in an equality statement. For example, one pupil saw that $7 + 7 + 9 = 14 + 9$ is true on the grounds that $7 + 7$ is 14 which is the same as the right hand side; and saw that $13 + 11 = 12 + 12$ is true on the grounds that if you take 1 from the 13 and add it to the 11 both sides will be the same. In each case this pupil was satisfied the statements were true without working out the computational value of $14 + 9$ or $12 + 12$. Other authors have similarly alluded to or elaborated “full relational” conceptions of the equals sign (e.g. Carraher, Schliemann, Brizuela,

& Earnest, 2006; Jones & Pratt, 2006; Linchevski & Livneh, 1999; Warren, 2003).

2.4 An “is the same as” view of the equals sign

In this section I consider the recommendations for pedagogy made by researchers of pupils’ conceptions of the equals sign. These can be grouped as recommendations for moving pupils from the place-indicator to a basic relational view, and recommendations for moving pupils from a basic to full relational view.

There is a broad consensus amongst researchers that place-indicator views of the equals sign arise from the way arithmetic is presented and talked about in classrooms (Baroody & Ginsburg, 1983; Cobb, 1987; Knuth et al., 2006; Li et al., 2008; Lima & Tall, 2006; N. McNeil & Alibali, 2004; Pirie & Martin, 1997; Renwick, 1932; Thomas & Tall, 2001; Warren & Cooper, 2005). Some of these authors have argued pupils should be taught the equals sign means “is the same as” and be presented with a wide variety of statement forms throughout their schooling. This recommendation is backed by three convincing pieces of evidence. The first is Baroody and Ginsburg’s (1983) opportunistic study of a primary school they chanced upon that just happened to deliver a mathematics curriculum based on these recommendations. They discovered that not only did the pupils accept the truth of a wide variety of statement forms, they were flexible enough in their thinking to accept forms they had not encountered before. The second is a study showing that pupils in China, where textbooks present $=$ in a wide variety of contexts and statement forms, solve missing-number *expression=expression* statements with a 98.6% success rate, compared to 28.6% for comparable U.S. pupils (Li et al., 2008). The third is a series of studies demonstrating that pupils who have an “is the same as” conception perform better on arithmetic and algebraic equations (e.g. Knuth et al., 2006; N. McNeil & Alibali, 2004).

Researchers seeking ways to help pupils move to a full relational view of the equals sign have also recommended presenting a wide variety of statement forms to pupils (Carraher et al., 2006; Carpenter & Levi, 2000; De Corte & Verschaffel, 1981; Denmark et al., 1976; Molina et al., 2008). However, the focus here is on the careful selection and variation of statement examples such that they appeal to the flexible use of a variety of relationships and operations when

assessing numerical balance (Molina et al.’s “sameness relational” view set out in the previous section). These studies provide detailed examples of the kinds of interventions that can help, but the long term impact remains an open empirical question (Carraher et al., 2006; Dörfler, 2008; Kilpatrick et al., 2001; Linchevski & Livneh, 1999; Tall, 2001).

There are also theoretical grounds to suspect that, despite the promising outcomes of such interventions, they are limited. These limitations are the central motivation for this thesis, and are threefold.

(i) The first limitation was discussed in the previous section. In sum, the contrived statements presented to pupils are rare in non-pedagogical mathematics, and designers are restricted to a pool of example statement types that embody arithmetical principles, as in $4 + 3 = 3 + 4$ and $13 + 11 = 11 + 13$. Given the broad consensus that conceptual difficulties arise from what was encountered previously, learning is likely to be situated within such contrived examples.

(ii) When pupils do come to work competently with various statement forms they are likely to be seeing notation in qualitatively different ways to one another (Tall, 2001). This limitation is elaborated in Section 3.3.

(iii) Tasks in which pupils assess the numerical balance of a variety of carefully selected equality statements do not contain all the necessary elements for a “full” relational view. This limitation is elaborated in the next chapter, which is concerned with the psychological basis of numerical equivalence relations in concrete and symbolic contexts.

2.5 Summary

Pupils’ conceptions of the equals sign fall into three categories. The first is the place-indicator view which correlates with accepting only statements of the form *expression = numeral*. This view is common amongst young children because of the way equality statements are presented and talked about in school. The second is the basic relational view, which correlates with accepting statements of a variety of forms. The basic relational view can be fostered by presenting the equals sign as meaning “is the same as” in different statement forms. The third is the full relational view, which is more complicated and somewhat vague, but involves exploiting structural shortcuts and arithmetical facts in subtle and flexible ways.

Researchers have demonstrated how presenting pupils with contrived statements intended to appeal to structural readings can help towards a “full” relational view of the equals sign. However, there are theoretical limitations to this approach, as elaborated in the following chapter.

Chapter 3

The psychology of arithmetical equivalence

3.1 Introduction

In this chapter I describe the psychological nature of equivalence relation conceptions in both concrete and symbolic contexts, and reflect on what a full relational view of the equals sign might be like.

Piaget (1952) found that in concrete contexts children's conceptions of relational thinking were intimately tied up with the development of number itself. In symbolic contexts mathematical development involves the "encapsulation" of processes, such as counting, into concepts, such as number, with both process and concept represented by a single symbol (Gray & Tall, 1994).

Drawing on this, I illustrate how tasks in which the equals sign implicitly or explicitly means "is the same as" do not contain all the elements of an operationalised conception of numerical equivalence relations. A complementary, "can be exchanged for" meaning is proposed, and I argue a "full relational" view of the equals sign involves a duality that embraces both sameness and exchanging meanings.

3.2 Piagetian equivalence relations

Piaget (1952) sought to describe the general stages of number development from infancy to adulthood. His ideas, and their application to the domain of formal mathematical notation, are discussed in this section¹.

According to Piaget arithmetical equivalence and non-equivalence relationships are not a property or consequence of number but are integral to its very development. Rather than numbers correlating to concepts and relationships correlating to connections between them, numbers are conceptualised through their relationships to one another. Consider the case of the cardinality of continuous quantities. Piaget described clinical experiments in which young children were shown two identical containers holding an identical amount of liquid. One container was then poured into a third (differently shaped) container and children were asked whether the quantity of liquids was the same in each. Surprisingly, they typically thought the quantities were different. Some said there was less in the third container, and some said there was more, but none said there was the same amount. The principle of conservation of quantity seems so intuitive to adults that it is hard to imagine how children cannot see it too. We might ask, where did the children think the extra liquid has come from or the missing liquid gone?

Piaget argued that children's beliefs are strongly influenced by immediate perceptual factors. Each child said there was more or less depending whether she attended to the height or width of the differently shaped container. For example, if the container was shorter and wider than the original container then children attending to the height said the quantity was less and children attending to the width said more. They did not coordinate the two dimensions of height and width into a *compensatory* relation. In this context, the conservation of quantity requires seeing that more height compensates less width, and *vice versa*. Piaget refers to this compensatory coordination as a *multiplicative* relationship.

Piaget also considered the case of the cardinality of discontinuous quantities. The experiment involving liquids and containers was repeated using beads arranged in two uneven lines and asking children which line of beads would make the longer necklace. In this case they needed to coordinate the length and density (c.f. height and width of continuous quantities) of each line. Some children carried out a one-to-one correspondence, arranging the lines of beads

¹See Section 4.2 for a discussion of Piaget's more general contribution to learning theory

adjacently in order to compare the two sets. However, when the experimenter returned the beads to their original distribution many children claimed they were no longer equivalent (i.e. the necklaces would not be identical). The prioritising of perceptual factors over multiplicative (compensatory) relations was again apparent.

Conceptions of the conservation of quantity, then, are dependent upon the ability to see multiplicative relationships between attributes of two continuous objects or sets of discontinuous elements. Piaget referred to this ability as *operationalised thinking*. Operationalised thinking is further characterised by the *reversibility* of the multiplicative relations. We know two differently shaped containers contain the same quantity of liquid when we pour it from one to the other because we can imagine pouring the liquid back again and it coming up to the same level. Likewise, we know two sets of beads arranged in one-to-one correspondence maintain equivalence when we spread one line out because we can imagine restoring the one-to-one correspondence. Piaget contended that “the triumph of operation over perceptual intuition is the outcome of the progressive reversibility of thought” (p.56).

The concept of number depends on not only conservation of quantity but also the relation between ordinance and cardinality. Young children do not innately make a correspondence between the n th element of a set of n elements and the total number of elements in the set. The ordinance of elements is only apparent by focusing on what Piaget calls *asymmetrical* relations: that is, the difference or non-equivalence between one element and another by virtue of unique attributes (such as position). Indeed, it is this non-equivalence (i.e. the distinctiveness of each element in a set) that allows enumeration thereby giving rise to number. Conversely, the cardinality of a set is apparent by focusing on *symmetrical* relations: that is, the equivalence of each element in its contribution to the total number of elements. It is this equivalence of elements (i.e. their identical properties) that allows groupings to be made giving rise to number. Therefore number arises from both the ordinance (asymmetrical, non-equivalence) and the cardinality (symmetrical, equivalence) of the elements in the set:

Number is indeed possible to the extent that the elements ... are viewed no longer as being either equivalent or non-equivalent, but as being *at the same time* equivalent and non-equivalent ... Finite numbers are therefore necessarily at the same time cardinal and

ordinal, since it is of the nature of number to be both a system of classes and of asymmetrical relations blended into one operational whole (p.156, 157 – emphasis in original).

Piaget’s ideas present a view of arithmetical relations as intimately tied up with the concept of number. Relational thinking is not dichotomous with operations (as implicit in the discourse in the literature on pupils’ conceptions of the equals sign — see Section 2.3) but complementary to operations. It is the reversibility of multiplicative attributes of sets and the ability to perform transformative operations in the mind that supports relations and indeed number itself.

3.2.1 Applying Piaget’s ideas to the equals sign

Some researchers have applied Piagetian explanations of cognitive development to pupils’ difficulties with the conventions of formal arithmetical notation. For example, Kieran (1981) identified “two intuitive meanings of equality” (p.318) in young children’s number development. One corresponds to a comparison of two cardinal sets, the other to combining two sets and counting the resulting set. Kieran deemed the former to be the relational notion of equality and the latter to be “the operator notion of equality, which is appealed to when the equal sign is introduced in school” (p.318). Her proposed correspondence of object manipulation with operator notions and (physically) passive object observation with relational notions seems an attractive proposition. However, on closer examination, it is problematic because it rests on assumed synonymies of physical manipulation with combining sets, and non-manipulative observation with comparing sets. This is unwarranted given that Piaget also reported children manipulating objects (sets of beads) to perform a one-one correspondence in order to make a comparison, as well as children non-manipulatively counting all the elements in two subsets (beads of different colours) in order to get a total.

A more systematic approach was undertaken by Collis (1975) in a series of studies designed to investigate how the “empirical observation of children working with elementary mathematical material” (p.9) might be interpreted within a Piagetian framework. For example, in one study 330 pupils aged between 7 and 17 years were given tests that included items such as those shown in Table 3.1 and asked to assess their truthfulness. The youngest children (below 10 years)

Item	Test 1	Test 2	Test 3
1	$a \times b = b \times a$	$4 \times 6 = 6 \times 4$	$4728 \times 8976 = 8976 \times 4728$
2	$a \times (b \times c)$ $= (a \times b) \times c$	$5 \times (4 \times 6)$ $= (5 \times 4) \times 6$	$982 \times (475 \times 638)$ $= (982 \times 475) \times 638$
3	$a \times x = a$	$4 \times 5 = 4$	$4932 \times 8742 = 4932$

Table 3.1: Example test items in Collis’s study

performed well on Test 2 items because, Collis argued, they could achieve “closure”, defined as the ability to “regard the outcome of an operation as unique and ‘real’ (In the sense that it is related to the child’s reality)” (p.35). More specifically, young children more readily achieved closure on Test 2 than Test 3 (and not at all for Test 1) because the numbers were small enough that they could replace each term with a computed result and so satisfy themselves of a unique outcome. This corresponds to Piaget’s children lining up beads in one-to-one correspondence and discovering their equivalence but lacking conservation of cardinal quantity when one row of beads is then spread out. Collis explained the greater success of older children (10 to 14 years) on Test 3, which contained burdensome four-digit numerals, in terms of their ability to accept closure without needing to make an actual replacement. This is because, according to the Piagetian view, the older children were operationalised thinkers and so had conceptions of reversible relations. This corresponds to seeing the conservation of quantity of two lines of beads not in one-to-one correspondence by compensating length and density. The oldest children (14 to 17 years) were able to succeed on Test 1 items, which contain letters rather than numerals, because they were able to see each term as an entity, without needing to check (or being able to imagine) that a unique computational result exists, and so the lack of closure experienced by children below 14 years was not an obstacle.

Collis’s work has a certain elegance but fails to account for the notation-specific difficulties encountered that arise from how notation is presented to pupils in the classroom (see Section 2.2). Moreover, Tall (2004) highlights a problem with transferring a theory grounded in children manipulating concrete materials to the domain of symbolic mathematics: the physical actions that can be performed with objects are quite different to the mental actions that can be performed with

symbols. For example, a physical balance can support relationships analogous to some equality statements (Warren & Cooper, 2005) but children use different strategies when working with such devices and notation (Filloy & Rojano, 1989). A description of arithmetical equivalence grounded in children’s symbolising activity is required and is the focus of the next section.

3.3 Proceptual-Symbolic equivalence relations


Tall (2004) described the development of mathematical thinking from infant to adult in terms that accounted for the differences experienced from individual to individual. The focus is on three cumulative and interrelated modes of thought, or “worlds of mathematics”, rather than hierarchical stages. The first is the “conceptual-embodied world” and is concerned with sensory perceptions and associated mental reflections. The second is the “proceptual-symbolic world”, concerned with the computation and manipulation of conventional mathematical symbols. The third is the “formal-axiomatic world” which relates to the use of formal definitions within mathematics. It is the second, proceptual-symbolic, world that is of interest here.

The starting point for understanding symbolic mathematics is the notion of “encapsulation” by which a mathematical process (such as counting to 5) becomes a mathematical concept (such as the number 5). A concept, or encapsulated process, can then become the input to further, higher-level processes, which in turn become encapsulated as concepts, and so on. Encapsulation, sometimes known as “entification” or “reification”, is well evidenced throughout the mathematics education literature (e.g. Dubinsky, 1991; Sfard, 1991). However, whereas other theorists emphasise a hierarchy of processes preceding mental objects (concepts), Gray and Tall (1994) emphasised the importance of flexibly switching between process and object when thinking mathematically (Gilmore & Inglis, 2008). This flexibility of thought correlates to employing the “simple device of using the same notation to represent both a process and the product of that process” (Gray and Tall, p.119). Gray and Tall give several examples of this, including:

- The symbol $5 + 4$ represents both the process of adding by counting all or counting on and the concept of sum ($5 + 4$ is 9).

- The symbol $+4$ stands for both the process of ‘add four’, or shift four units along the number line, and the concept of the positive number $+4$.
- The algebraic symbol $3x + 2$ stands both for the process ‘add three times x and two’ and for the product of that process, the expression ‘ $3x + 2$ ’ (p.120).

Gray and Tall referred to such a duality of process and concept represented by a single symbol as an *elementary procept* (a concatenation of the words *process* and *concept*). In the minds of flexible thinkers any given symbol, such as “6”, is one of many names for a single mathematical object, along with $3+3$, $8-2$, 2×3 and so on. A *procept*, in contrast to an *elementary procept*, includes a single mathematical object, a collection of symbolic representations for that object, and the processes for generating that object. The term *proceptual thinking* is used to characterise viewing a symbol as a duality of process and concept, as well as making interchanges of different symbolic names for the same mental object.

Gray and Tall argued that proceptual thinking distinguishes those who enjoy success at mathematics from the unfortunate majority who do not. The latter become bogged down in the procedures of mathematics, developing problem solving *efficiency* without developing problem solving *flexibility*. Consider the case of $2 + 3$. A pupil who invariably uses a counting procedure may move from “counting all” (1, 2, 3, 4, 5) to “counting on from first” (2, 3, 4, 5) to “counting on from largest” (3, 4, 5), thereby becoming more efficient. In the latter case “3” is being seen as a procept: the process of counting three *and* the outcome, “3”. For some pupils, however, the entire symbol $2 + 3$ becomes both a process of counting five *and* the outcome, “5”; that is, $2 + 3$ is seen as another name for 5. It becomes a known fact, not memorised by rote, but meaningful as part of a concept image (Tall & Vinner, 1981) that includes the symbol “5”, word “five”, the image “”, the symbol “ $2 + 3$ ”, and so on. Now consider the case of $12 + 3$. Some pupils would use a counting strategy, with varying degrees of efficiency depending on the strategy used. Others would make use of the (meaningful) fact that $2 + 3$ is 5, and then add 10. Such qualitatively different ways of seeing symbols makes a marked difference when unfamiliar types of problems are encountered. In new contexts, old ways of working, which are highly situated, can become obstacles to further development. Tall (2004) calls an old, situated

way of working a “met-before” (a pun on *metaphor*) and provides the example from arithmetic that “every sum has an answer, for instance, $2 + 3$ is 5” (p.286). This can be a problem when algebraic expressions, which contain letters and so lack an “answer”, are encountered in later schooling because there is no answer to be processed. For the proceptual thinker, however, $2 + 3$ is both a process and the encapsulated object of that process, and so subsequent encounters with irreducible expressions, such as $2 + 3x$, are less daunting.

3.3.1 A proceptual view of the equals sign

The place-indicator view of the equals sign (see Section 2.2) might be described as the “every equals sign is followed by an answer” met-before, which arises from the exclusive presentation of *expression = numeral* forms in typical classrooms (Section 2.4). As such, the statement $8 = 4 + 4$, may seem strange:

[like saying] ‘cake the I ate’ instead of ‘I ate the cake’, the order is ‘not natural’, at least, not for the child who is thinking in terms of procedures rather than in terms of flexible procepts (Tall, 2008a).

For the proceptual child, however, the symbol $8 = 4 + 4$ can be accepted as a statement of identity: the “every equals sign is followed by an answer” met-before has not caused an obstacle, because “answers” and “sums” are different symbols for the same object anyway.

From this vantage point, presenting varied statement forms throughout primary schooling avoids the “every equals sign is followed by an answer” met-before in favour of an “is the same as” meaning (or “basic relational” view — see Section 2.3). However, researchers “must be careful not to conclude simply that making sure children are exposed to the various forms of equality sentences will remove [misconceptions of the equals sign]” (Behr et al., 1976, p.10). Just as there are qualitatively different ways of thinking about $2 + 3$ and $12 + 3$ (Gray, 1991), so contrived equality statements designed to appeal to structural readings are viewed in different ways by different pupils. For example, Gilmore and Bryant (2006) presented problems of the type $15 + 12 - 12 = \square$, $\square + 7 - 7 = 13$ and $\square + 14 - 9 = 18$ to primary pupils and discovered use of the inversion principle (i.e. a and $-a$ cancel in the statements presented) was widespread, even amongst young and low-achieving pupils, but there was significant variation in how appropriately they applied it to different statement forms. Similarly, for

the case of long division, Anghileri (2006) found that primary pupils used a variety of strategies despite an emphasis in the National Numeracy Strategy on place-value partition.

Even when pupils do employ a given strategy they do not necessarily grasp the underlying arithmetical principle (Tall, 2001). Baroody and Gannon (1984) reported on a child who was asked whether presented pairs of terms, such as $6 + 3$ and $3 + 6$, produce the same result. The child performed a calculation for each term but, interestingly, used a “count-all starting with the larger term” strategy (p.81). On the one hand his use of a “start with the larger” strategy suggested a conception of commutativity (he viewed the order of addends to be irrelevant to outcome); on the other hand his need to compute both suggested not (he needed to compute results to satisfy himself of numerical sameness). In a follow-up interview, in which he was asked whether pairs of terms (such as $7 + 2$ and $2 + 7$) *would* produce the same result, he answered no, and so did not have a conception of commutation. However, when then asked to compute the same pairs of terms he again used a count-all strategy starting with the larger numeral. Baroody and Ginsburg argued this demonstrates *protocommutation* in which the order of addends is sometimes ignored simply to save computational effort and with no regard for consistency of outcome.

To summarise, some pupils make use of computational shortcuts without seeing symbols, such as $2 + 4$, as manipulable entities in their own right. They think in terms of processes rather than in terms of the encapsulation of processes. The “is the same as” meaning for the equals sign (see Section 2.4) does not explicitly promote working with encapsulated symbols, and so researchers present pupils with statement examples that appeal to non-computational readings. However, these appeals allow proceptual thinking but do not necessitate it. One problem is that the task of assessing or establishing numerical sameness promotes seeing equivalent symbols as different processes that generate the same object; but encapsulating processes and using those encapsulations as inputs to further thinking are optional. For some pupils this is a step too far and despite making increased use of notational shortcuts they continue to see symbols as processes. As such, tasks in which pupils establish the numerical sameness of contrived statements provide limited insights because they do not promote all the elements of a proceptual-symbolic equivalence conception. This issue is discussed further in the next section.

3.4 A “can be exchanged for” view of the equals sign

In this section I revisit the limitations of presenting pupils with statement examples in which the equals sign has a numerical sameness meaning (see Section 2.4) and suggest how they might be addressed.

The limitations are as follows:

- (i) The contrived statements presented to pupils are rare in non-pedagogical mathematics and so designers are restricted to a pool of example statement types that embody arithmetical principles. This limitation was discussed in Section 2.3.
- (ii) When pupils do come to work competently with various statement forms they are likely to be seeing notation in qualitatively different ways to one another. This limitation was discussed in the previous section.
- (iii) Tasks in which pupils assess the numerical balance of a variety of carefully selected equality statements do not contain all the necessary elements for a “full” relational view. This limitation has been alluded to in this chapter and is set out explicitly in the remainder of this section.

Interchangeability and reversibility are central to the definition of arithmetical equivalence (Freudenthal, 1983; Gray & Tall, 1994; Kieran, 1981; Linchevski & Livneh, 1999; Piaget, 1952; Skemp, 1986). Collis (1975) referred to interchangeability as “replacement”:

The notion of equivalence goes hand in hand with the important mathematical idea of replacement. If two expressions are equivalent then one may be used to replace the other at any time. [Given $3 \times (2 \times 4) = (3 \times 2) \times 4$ then] $3 \times (2 \times 4)$ can replace $(3 \times 2) \times 4$ or, if one finds the counting number equivalent (in this case 24), it can be used to replace the expressions. (Collis, 1975, p.17).

The “is the same as” meaning for the equals sign suggests numerical sameness but does not suggest making replacements of equivalent symbols. Making replacements instead suggests a “can be exchanged for” meaning. Consider $23 + 18 = 29 + 12$ (which makes no specific appeals to computational shortcuts). The “is the same as” meaning asks whether $23 + 18$ and $29 + 12$ generate the

same mathematical object. The “can be exchanged for” meaning tells us that the symbol $23 + 18$ can be exchanged for $29 + 12$ (for example, the statement $23 + 18 = 141$ can be transformed into $29 + 12 = 141$), but the equivalence of generated results is not an explicit requirement.

This can be put in more formal terms by drawing on the definition of an equivalence relation:

Let R stand for any relation and let a, b, c , be any members of the set of objects for which this relation is defined. Then R is an equivalence relation iff, whichever objects we choose,

- i. $a R b$ and $b R c$ implies $a R c$
- ii. $a R a$
- iii. $a R b$ implies $b R a$

In words, an equivalence relation is (i) transitive, (ii) reflexive, (iii) symmetric. (Skemp, 1986, p.172)

A “can be exchanged for” meaning emphasises the transitive property of equivalence: given $a = b$ and $b = c$ we can substitute b for c to give $a = c$. It also contains the symmetrical property: $a = b$ is identical to $b = a$ for the purpose of making a substitution, and so transformations are reversible. The reflexive property is present, but its tautology renders it uninspiring from an educationalist stance: $a = a$ can be used to substitute a for a .

An “is the same as” meaning does not fit the definition so well, because it assumes a, b and c to be mathematical entities in their own right. They can be “whichever objects we choose” and any process required to generate them is of no concern. The sameness meaning of the equals sign, however, is concerned with the internal structure of a and b in order to establish that $a = b$ rather than seeing $a = b$ as a given definition, and then using a and b as encapsulated inputs for further thinking.

This is not to say the “is the same as” meaning is not important both arithmetically and educationally — it has a central role in moving pupils from the place-indicator to basic relational view (see Section 2.3). The point instead is that the “can be exchanged for” meaning is *distinct* from the sameness meaning because, when making replacements, it makes no difference whether a statement is numerically balanced or not. For example, if we are given the imbalanced

statement $2 + 4 = 50$, we can view it as “ $2 + 4$ can be exchanged for 50 ” and so transform $2 + 4 + 10 = 16$ into $50 = 16$. Mathematically this is nonsense, and in practice the sameness and exchanging meanings for the symbol $=$ need to be coordinated. Even given a numerically balanced statement there are situations where it cannot be used to make an exchange. For example, given $2 + 4 = 4 + 2$, we can exchange $2 + 4$ for $4 + 2$ in $2 + 4 + 10 = 16$, but not in $2 + 40 + 1 = 43$ or $2 + 4 \times 10 = 42$ because the transformed statements would be imbalanced.

It seems therefore that when moving pupils from a basic relational to a full relational view, a *duality of both* “is the same as” and “can be exchanged for” meanings for the equals sign might be fruitful. This thesis explores this possibility.

3.5 Summary

Piaget (1952) demonstrated that equivalence relations are integral to the very development of number itself, and require seeing compensatory relations between the properties of two quantities or sets of objects. These compensatory relations are reversible in the mind and enable more immediate, perceptual factors to be overcome. Some authors have applied Piaget’s ideas to children’s conceptions of equality statements but such efforts fail to account for the widely reported notation-specific difficulties that arise from how statements are presented in classrooms.

Gray and Tall (1994) offer a theory of mathematical development grounded in symbolic rather than concrete contexts. This centres on the encapsulation of processes, such as counting, into concepts, such as number, and the dual role of a symbol as representing both a process and the outcome of that process. Some pupils switch appropriately between process and encapsulated views of a given symbol, and make mental reversible interchanges of equivalent encapsulations; others get bogged down in the processes themselves.

The “is the same as” meaning for the equals sign, in which $a = b$ is presented as a question of truthfulness, emphasises symbolised processes, rather than symbolised encapsulations that can be used to make interchanges. It is valuable for promoting a basic relational view of the equals sign but is limited in scope for promoting a full relational view. A “can be exchanged for” meaning, in which $a = b$ is presented as a given truth for making substitutions, might better

emphasise the transitivity and symmetry of arithmetical equivalence relations.

Chapter 4

Theories of learning

4.1 Introduction

In this chapter I review general theories of learning that have impacted on education *per se*, and influenced the assumptions and thinking underlying this thesis. I consider in turn constructivism, social constructivism and situated cognition, and draw out the themes of particular relevance to this thesis. Recent research into learning formal notation draws on situated cognition but, I argue, contains assumptions that are implicitly Platonist rather than situative. Drawing on the ideas of Hewitt (1996) and Dörfler (2006), I argue that the learning the conventions of notation should and can be situated within symbols themselves rather than external referents.

4.2 Constructivism

The work of Piaget (2003) has had a significant influence on psychologists and educationalists. His key contribution was to view the child as an active constructor of knowledge, a stance referred to as “constructivism”, rather than a passive recipient of information. The basis of knowledge construction according to Piaget is “genetic epistemology”, a form of development so central to his theories that he named his Geneva-based research institute the *Center for Genetic Epistemology*. There are four components to the development of the child,

namely *maturation*, *experience*, *social transmission* and *equilibration*. Each will be considered in turn.

Maturation can be considered the biological development of the child, a “continuation of the embryogenesis” (p.S10). It is the main driver of Piaget’s four stages of development from infancy to adulthood (sensory-motor, pre-operational representation, concrete operations, and formal operations). Maturation, and the four stages of development, are not of central interest to this thesis and will not be elaborated further. It should be noted however that within cognitive science general stages have receded in favour of modular theories of development (Fodor, 1983; Karmiloff-Smith, 1999) that acknowledge innate specialisms such as numerosity (Butterworth, 2005) and approximation (Gilmore, McCarthy, & Spelke, 2007).

Experience comes in two psychologically distinct forms. The first is empirical and is grounded in the child’s interactions with physical objects and abstracted discoveries about their relative properties, such as weight. For example, a child may discover that one object is heavier than another and that this difference is intrinsic to the objects themselves. The second form of experience is logical and involves abstracted discoveries about relative actions on objects (rather than about the objects themselves). An example might be counting a set of objects, rearranging them, and discovering they give the same number when counted again. The kinds of logical experiences involved in developing a conception of number were discussed in detail in Section 3.2.

Social transmission (which Piaget sometimes called *linguistic* or *educational* transmission) is, in simple terms, being told something. Piaget did not deny the role or importance of the transmission of knowledge and indeed stated that it was fundamental to child development. However, he did emphasise the importance of actively “assimilating” transmitted knowledge which is possible only when the child is sufficiently mature and experienced to understand it. As such, “you cannot teach higher mathematics to a five-year-old” (p.S12). In fact Piaget (1932) argued that transmission can be a hindrance to the child’s development due to the power imbalance between learner and teacher, which forces the unready learner to dismiss or accept on authority the information. Instead, Piaget valued the role of interaction between equal peers for helping to see and assimilate alternative points of view.

iv. *Equilibration*, which Piaget regarded as the most fundamental of the four components of genetic epistemology, is an active process of self-regulation by

the child. It is the preceding three factors – maturation, experience and social transmission – that are equilibrated by acts of reversible compensation. The development of a conception of the conservation of quantity provides an example in which the child at first concentrates only on the height or the width of differently shaped containers but eventually comes to operationalise the two within a coordinated (equilibrated) conceptual whole (see Section 3.2).

In sum, Piaget portrayed the child as actively constructing knowledge. The process is driven by developmental factors grounded in physical growth, experience and reflection. *Learning*, in the sense of deliberated interventions by more mature, experienced and knowledgeable others (parents, teachers, and so on), is a secondary, subordinate aspect of knowledge construction compared to *development*.

Piaget’s ideas remain in vogue in education. This can be seen in the statutory English National Curriculum for mathematics at Key Stage 3 which emphasises pupils “combining understanding, experiences, imagination and reasoning to *construct* new knowledge” (QCA, 2007, p.140 – emphasis added); and the role of peer interaction:

talking about mathematics, evaluating their own and others’ work and responding constructively, problem-solving in pairs or small groups and presenting ideas to a wider group. (p.147).

Piagetian constructivism is a descriptive theory of learning that cannot directly inform pedagogic design (diSessa, 2004). In the next section I discuss an offshoot known as constructionism, which is not a theory of learning but of pedagogy (Noss, 2008).

4.2.1 Constructionism

One impact of constructivism on educational research and practice has been a focus on “discovery learning”, “problem-based learning” and “inquiry learning” (Kirschner, Sweller, & Clark, 2006). Such approaches promote the role of the pupil, who actively constructs knowledge, and demote the role of the teacher to providing environments that best enrich pupils’ self-development. Within mathematics education a notable movement in the constructivist tradition is “constructionism”, which was pioneered in the 1960s by Seymour Papert, a student of Piaget.

Constructionism began, according to Papert (1980), as a vision of mathematics classes as analogous to art classes, in which pupils take pride in and ownership of the paintings and sculptures they produce. A key tenet is to render mathematics intrinsically motivating and engaging to pupils so that they might construct rich, conceptual mathematical knowledge. Papert argued that the computer, and computer programming in particular, offers a powerful resource for immersing learners in mathematics. Papert, Watt, diSessa, and Weir (1979) reported a multiple-case study of pupils engaging with the *LOGO* programming language. They argued programming had a “holding power” (p.17) that kept learners engaged over the duration of the year-long study. This holding power arose from the flexibility of the resources to support learners following their own intrinsic task goals and to suit individual variation.

From the constructionist vantage point computer tasks can “concretize (and personalize) the formal” (Papert, 1980, p.21). As Turkle, Papert, and Harel (1991) put it:

The computer stands betwixt and between the world of formal systems and physical things; it has the ability to make the abstract concrete. In the simplest case, an object moving on a computer screen might be defined by the most formal of rules and so be like a construct in pure mathematics; but at the same time it is visible, almost tangible, and allows a sense of direct manipulation that only the encultured mathematician can feel in traditional formal systems (p.162).

The term “microworld” (Edwards, 1998) has come to be used to describe software that supports exploratory mathematics in this manner. Microworlds allow mathematical rules to be “discovered” (explicitly or implicitly) through trial-and-error experimentation, mental reflection and discussion. This flexibility means low-achieving pupils can engage with ideas that may otherwise be inaccessible to them and high-achieving pupils can be challenged in ways not always supported by typical classroom activities (Papert et al., 1979).

Constructionism has a long association with the role of formal notation in mathematics education. LOGO was originally devised as a teaching language, allowing children to construct precise and complex mathematical structures and test and revise them. Modern variants of LOGO support point-and-click interfaces, and learners “write” programs through interactions with icons (e.g. *ToonTalk*

– Kahn, 2008). Generally, however, all-purpose constructionist software such as *LOGO* has declined in popularity since the 1980s (Agalianos, Noss, & Whitty, 2001) and the focus has shifted to content-specific microworlds. Certain learning domains, such as geometry and Newtonian mechanics lend themselves naturally to microworld design. Arithmetic notation has enjoyed far less success and popularity as a domain for exploratory software, although exceptions exist such as *VisualFractions* (see Section 5.2.1) and *Grid Algebra* (Hewitt, 2008). One outcome of this thesis is an arithmetic microworld called *Sum Puzzles* (see Section 6.3).

4.3 Social constructivism

The term “constructivism” has diversified into various approaches including radical constructivism (Glaserfeld, 2002), constructionism (see previous section) and social constructivism. A major influence on these contemporary variants of constructivism is the work of the Soviet psychologist Vygotsky (1962). Like Piaget, Vygotsky viewed the child as an active constructor of knowledge. However, unlike Piaget, he sought to explain the transmission of culture from one generation to the next rather than the development of the individual from infancy to adulthood. As such, Vygotsky’s ideas can be referred to as “social constructivism” to distinguish them from Piaget’s (individual) constructivism (Swan, 2006).

Three key Vygotskian ideas will be considered here: semiotic mediation, the genetic law of cultural development, and the zone of proximal development (ZPD).

Semiotic mediation underpinned Vygotsky’s intention to understand the mind in terms of wider societal and historical factors and trends. Unlike the minds of animals, which sense the external world directly, the human mind can only interact with the physical environment by way of cultural tools, or “signs”. This focus on semiotic tools reflects Marx’s contention that it is the creation, adoption and adaptation of tools that distinguishes human beings from other animals – a contention that was arguably mandatory given the tumultuous and ideological times of Vygotsky (van der Veer, 2007). Of relevance to this thesis, Vygotsky explicitly included “algebraic symbol systems” (cited in Wertsch, 2007, p.178) as an example of a cultural tool. Semiotic mediation has enjoyed recent attention

in mathematics education (e.g. Otte, 2006; Radford, 2008) stemming not just from the ideas of Vygotsky but also American pragmatists such as Peirce (1991). *The genetic law of cultural development* describes the relationship between (external) speech and (internal) thought. Vygotsky argued thought and language are inseparable and come together in *word meaning*, his favoured unit of analysis.

The meaning of a word represents such a close amalgam of thought and language that it is hard to tell whether it is a phenomenon of speech or a phenomenon of thought . . . Word meaning is a phenomenon of thought only in so far as thought is embodied in speech, and of speech only in so far as speech is connected with thought and illumined by it. It is a phenomenon of verbal thought, or meaningful speech – a union of word and thought (p.120).

Vygotsky argued, however, that thought and language are not unified in *origin*. They have “different genetic roots [and] develop along different lines and independently of each other” (p.41). Pure thought is vague and not self-conscious, as in apes. In early infancy pure speech develops through imitation and is devoid of intellectual content, as in parrots. However, for humans, and humans alone something profound occurs around the second year when thought and word coincide in word meaning, at which point “thought becomes verbal and speech rational” (p.44).

Whereas Piaget viewed communicative speech and “egocentric speech” (such as a child talking to herself while playing) as separable, with the latter dying out in later stages of development, Vygotsky considered them to have the same, social, origin. From about the age of two years when thought and word coincide, egocentric and communicative speech begin to split off. The former is “when the child faces difficulties which demand consciousness and reflection” (p133) and after some time becomes “internalised” into word meaning. Recent studies have demonstrated that egocentric speech results in delayed learning gains for certain problem-solving tasks, consistent with the “internalisation” hypothesis (John-Steiner, 2007).

The amalgam of thought and language has the philosophical implication that one is not possible without the other, and this is of relevance to mathematics with its strong association with Platonic ideals. Semioticians argue that mathematical objects do not exist independently of the signs (visual symbols and

spoken words) used to represent them (Peirce, 1991). Consequently, the historical development of mathematical thought is synonymous with the historical development of notational systems (Otte, 2006). For example, the Babylonians' sexagesimal system only applied to discrete objects and more sophisticated abstraction was literally unthinkable until the introduction of place-value notation around 2000 B.C. (Damerow, 2007). Consequently, mathematics education should seek to harness notation conventions if learners are to access deep mathematical thought (Dörfler, 2006).

The Zone of Proximal Development (ZPD) relates to the role of more knowledgeable others. There are limitations with (Piagetian) individual experimentation with the physical world and unstructured interactions with peers. The ZPD, in simple terms, is the difference between an unaided child's abilities (determined by some measure such as performance on a given task) and her potential under more expert guidance. This guidance is not transmission or help but involves "encouragement, focusing, demonstrations, reminders and suggestions" (Mercer & Littleton, 2007, p.15) as part of a "sensitive, supportive intervention of a more expert other" (p.18). In particular, the teacher needs to explicitly support, guide and model patterns of language that promote externalised collaborative reasoning. Mercer and Littleton described a study in which Key Stage 1 to 3 teachers explicitly addressed the components of collaborative "exploratory talk". Teachers encouraged pupils to ask their peers "why" questions, use the words "if", "because" and "so" when talking, and to make sure everyone has had a chance to talk before coming to a consensus. Teachers also modeled exploratory talk to promote imitation by pupils, which at first was somewhat mechanical and superficial, but with sustained guidance over a period of time came to transform the quality of pupil talk. This externalised engagement with alternative vantage points should, according to Vygotsky, become internalised and improve the quality of ("verbal") thought. Mercer and Littleton tested this using non-verbal reasoning tests (Raven's matrices) in both collaborative and individual conditions. They concluded that promoting exploratory talk impacts on pupils' collaborative and individual problem-solving skills. The key insight is that to be worthwhile collaborative learning requires the internalisation of certain patterns of language that do not arise spontaneously but that can be explicitly guided and modeled by teachers. In semi-structured research settings, a participant researcher can expect to need to prompt pupils to explain their thinking, and listen and respond to the explanations of others (see Section 7.2.1).

4.4 Situated cognition

Since the late 20th century many scholars have focused not on “word meaning” as their unit of analysis but “activity” (Engeström, 1987) or “practice” (Hildreth & Kimble, 2004) within a social system or community. These approaches are broadly constructivist in nature but highlight the role of intrinsic motivation and power relations within human endeavour and interaction. In this section I consider the contribution of “situated cognition” (Lave, 1988), which has come into currency across education research generally and mathematics education research in particular.

Lave (1988) studied the distinction between “formal” and “informal” mathematics. “Formal” refers to the mathematics of experts, which is taught in school and explored in psychology laboratories. “Informal” refers to the mathematics used by non-experts in everyday settings, such as American dieters, Liberian tradesmen and Brazilian street children (Nunes, Schliemann, & Carraher, 1993). Lave argued that non-experts engage in sophisticated mathematical activity with great success and accuracy in their everyday lives. However, these same non-experts performed poorly when equivalent problems were presented as formal paper-and-pencil tests. School, she argues, mistakenly values “abstract” mathematics over and above the “situated” mathematics that learners use and understand in their out-of-school lives, and attempts to replace the latter with the former.

The situated stance views knowledge as entwined with the context in which it was constructed. As such, knowledge is not abstract and cannot readily transfer from one context to another. For example, formal economics constitutes a practice situated within a particular culture of symbolic and discursive conventions that include the principle of a universal money exchange. However, in households most people compartmentalise their budget for specific purposes such as paying rent, buying food, personal treats, and so on. This compartmentalisation results in “pots” of cash that are not seen as exchangeable or equivalent in terms of how they can be spent or what they are worth (\$400 is “a lot” when spent on treats, but not when spent on rent).

Situated cognition remains a popular paradigm in education research, particularly among mathematics educators (Kirshner & Whitson, 1997) and the learning sciences movement (Sawyer, 2006). However, there is acknowledgment that no systematic evidence base yet exists comparable to that of cognitive psychol-

ogy (Greeno, 1997; Kirshner & Whitson, 1998). In particular, the problem of how such situated knowledge transfers to other settings remains one of debate within the literature (Carraher & Schliemann, 2002; Chi & Slotta, 1993; diSessa, 2002; Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Kirshner & Whitson, 1997; Mestre, 2005; Noss & Hoyles, 1996; Pratt & Noss, 2002).

Lave's work has also come in for criticism. Greiffenhagen and Sharrock (2008) argued that the mathematics in the pencil-and-paper tests was much more sophisticated than anything evidenced in the supermarket and other settings. Greiffenhagen and Sharrock also argued that Lave's suggestion that people do not understand the universality of money on the grounds that they compartmentalise their budgets is a contentious claim. Moreover, her portrayal of formal education as seeking to replace informal with formal mathematics is unwarranted. Stronger criticism has also been leveled at the situative position. Anderson, Reder, and Simon (1996) argued that the denial of abstract knowledge, and indeed of the dualist conception of mind, is a throwback to behaviourism. They pointed to a long history of transfer experiments demonstrating the successful and conceptual application of taught, abstract knowledge to a variety of problem-solving activities.

The debate between cognitive psychologists and situated theorists is not only theoretical. Each tradition makes radically different recommendations for mathematics education (Kirshner & Whitson, 1997). Cognitive approaches focus on learning content analysis and mental processes (working memory, long-term memory, cognitive load and so on), and recommend precise and measurable interventions such as *Cognitive Tutor* (Anderson, Koedinger, & Corbett, 2007). Situated approaches focus on context analysis and social processes, and seek to embed mathematical principles in problem solving contexts that are intrinsically meaningful to learners.

The situated approach is endorsed in the statutory English National Curriculum for mathematics at Key Stage 3, which states that pupils should use

mathematics as a tool for making financial decisions in personal life and for solving problems in fields such as building, plumbing, engineering and geography. Current applications of mathematics in everyday life include internet security, weather forecasting, modeling changes in society and the environment, and managing risk (eg insurance, investments and pensions). (QCA, 2007, p.141).

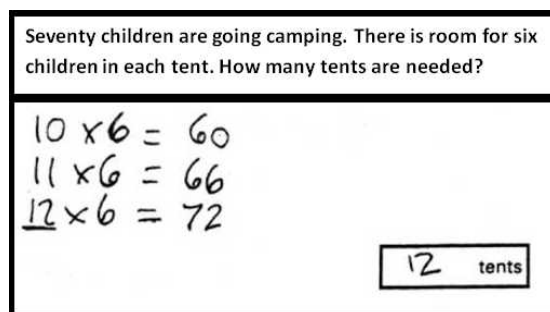


Figure 4.1: A worded problem expressed as equality statements (Reproduced from QCA, 2001b)

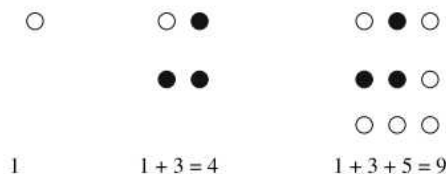
Few would disagree these are laudable *outcomes* for mathematics education. A key contribution of the situated cognition movement, however, is to view the use and application of mathematics as central to the learning of mathematics itself. In the next section I consider the implications of situated cognition for learning the conventions of formal notation.

4.4.1 Situated cognition and formal notation

Formal notation in mathematics education is associated with rote learning facts and procedures, and working through practice examples. This tradition is still prevalent in UK classrooms (Anghileri, 2006), and alienates many pupils (Ofsted, 2008). Moreover, without careful content analysis and task design this approach can create obstacles to further learning, as we saw for the case of the equals sign (Section 2.4). Since the 1960s, numerous movements (e.g. “New Math”, functional mathematics) have challenged the prominence of formal notation in mathematics classrooms, and sought to demote its role.

Nevertheless, situated approaches to learning the conventions of mathematical notation have been developed. Some of these approaches seek to “import meaning into the symbol system from external domains of reference” (Kirshner, 2001, p.84). These references might be physical models, worded problems, or pictorial patterns. For example, some researchers have explored using pan-balances and hanging-mobiles to model simple equality statements (Cooper & Warren, 2008; Aczel, 1998; Filloy & Rojano, 1989; Linchevski & Herscovics, 1996; Trigueros & García Tello, 2005; Warren & Cooper, 2005). Another approach is to provide worded problems for learners to express as equality statements (Figure 4.1

The figure shows the addition of odd numbers.



(a) Study the table and complete it.

Last Number	Total
1	$1 \times 1 = 1$
3	$2 \times 2 = 4$
5	$3 \times 3 = 9$
7	
13	

(b) Find the value of $1 + 3 + 5 + 7 + 9 + \dots + 99$.

(c) Find the value of $51 + 53 + 55 + 57 + \dots + 99$.

Figure 4.2: A generalising pattern task (Reproduced from Yeap & Kaur, 2008)

shows an example taken from QCA, 2001b). However, the prominent approach is to provide repeating patterns for learners to represent using formal notation (Amit & Neria, 2008; Britt & Irwin, 2008; Carraher et al., 2006; Cooper & Warren, 2008; Earnest & Balti, 2008; Rivera & Becker, 2008; Steele, 2008; Yeap & Kaur, 2008). (Figure 4.2 shows an example taken from Yeap & Kaur, 2008).

Such referential approaches offer a context for formal notation that is meaningful to pupils (Kirshner, 2001; Dörfler, 2008). However, they have limitations. Physical balances model only a limited pool of simple equivalence relations, and even then pupils view physical balance in qualitatively different ways to corresponding statements (Dörfler, 2008; Filloy & Rojano, 1989). Similarly, worded problems aid pupils' mental calculation for a limited pool of simple problems, but symbolic forms better aid calculation for more complex problems (Koedinger, Alibali, & Nathan, 2008).

Generalising visual patterns offers more promise and is being vigorously researched, but nevertheless suffers a limitation general to referential approaches. In short, they presuppose a Platonic view of mathematics as “something not (directly) communicable [and] that ... cannot be shown to and experienced by students” (Dörfler, 2006, p.101). This presupposition is somewhat ironic given

that referential approaches are grounded in the situated cognition movement, which denies context-free Platonic objects. Moreover, most published research uses closed tasks that strongly guide predetermined pattern continuations, and restrict other types of generalisation (Dörfler, 2008). This results in pupils producing similar or identical outputs for a given task, which masks underlying individual differences in how pupils view notation (Tall, 2001).

An alternative to referential approaches are structural approaches that “build meaning internally from the connections generated within a syntactically constructed system” (Kirshner, 2001, p.84). In Section 2.3 I described and critiqued structural approaches for the case of equality statements in which pupils exploit arithmetic principles when establishing numerical balance.

Structural approaches are situative but interpret the pedagogic implications of situated cognition quite differently to referential approaches. Rather than connecting formal notation to external referents, they enculturate learners into a discourse of symbolic mathematics (e.g. Carpenter & Levi, 2000; Cobb, 1987; Pirie & Martin, 1997; Sáenz-Ludlow & Walgamuth, 1998). Nevertheless, the criticism Dörfler (2006) makes of referential approaches applies equally here. Symbols and their manipulations simply become referents to Platonic counterparts (number, reversible equivalence relations, and so on) rather than to concrete instantiations. For example, an equality statement being assessed for numerical balance is a reference to two equal values, and, if it is a pedagogically contrived statement, to arithmetic principles such as commutativity (see Section 2.3). In this way, pedagogic mathematical representations “just have a transient character. They can be disposed of and even forgotten as soon as the corresponding abstract concept (object) has been developed by the student” (Dörfler, 2006, p.99). Dörfler contends this alienates many learners:

The common discourse in mathematics and mathematics education posits from the beginning the mathematical objects (numbers of all kind, sets, functions, geometric figures) as abstract ones which one has to learn and understand via all kinds of representations. I suggest that this (often implicit and unreflected) discourse is frightening for the students and learners. They often fail to get close to those genuine objects which mathematics purportedly is all about, they believe they lack the necessary abilities to think “abstractly”, and they are convinced that they do not understand what they are expected

to understand. They want to reach through the representations to the abstract objects but without success. (p.100)

This implicit disconnect in many approaches to mathematics education is evidenced in the Primary Framework for Literacy and Mathematics:

It is important that children’s mental methods of calculation are practised and secured *alongside* their learning and use of an efficient written method for addition (DCSF, 2007, p.6 – emphasis added).

This thesis makes the assumption that formal notation can and should be learned for its own sake (Dörfler, 2008; Hewitt, 1996; Hiebert, 1984). This is not to deny the immense importance of physical and Platonic referents of mathematics symbols. However, a critical role of symbolic conventions is to make possible certain forms of mathematical thought (Damerow, 2007). There must be moments when conventional symbols and their rule-governed manipulations become disconnected from any allocated referents, or there is no point using them (Dörfler, 2006; Kilpatrick et al., 2001). This presents a genuinely situative view of formal notation in which the mathematics to be learned is inextricably tied up with the available structuring resources.

In Chapter 5 I report preliminary work that draws on pilot studies and Hewitt’s (2001) contention that notation and mathematical reasoning are “separate pedagogic challenge[s]” (p.208). In Chapter 6 I set out the approach used in this thesis, which draws on Dörfler’s (2006) “diagrammatic” framework for presenting mathematical symbols as the literal objects of learners’ mathematical activities.

4.5 Summary

Piaget argued that children actively construct knowledge rather than passively receive information. There are four aspects to development, rooted in maturation, physical and logical experience, social transmission, and equilibration. His ideas, commonly known as “constructivism”, have had a major impact on education and continue to do so to this day. “Constructionism” is an off-shoot of constructivism which highlights the role of computers for engaging learners with rich mathematics. Constructionist principles are evident in contemporary

software packages but the conventions of formal notation are rarely the intended domain of learning.

Vygotsky sought to explain cultural transmission from one generation to the next. Learners interact with the external world by means of cultural tools, most notably spoken language. Language and thought have separate origins but come together in “word meaning”. Expert others, such as teachers, are crucial to ensuring learners develop to their full potential. As such, teachers should make learning aims explicit and model the patterns of language and behaviour that are most beneficial for learning.

Situated cognition views knowledge as entwined with the context within which it was constructed, and as such transfer to other contexts does not happen readily. Situative approaches to learning formal notation can be referential, in which meaning is imported from external referents, or structural, in which meaning emerges from relations within a symbol system. However, both approaches presuppose a Platonic view in which symbols are perfunctory to the underlying mathematics. This alienates many pupils for whom the underlying mathematics remains a mysterious, inaccessible realm. This thesis assumes that the conventions of notation are themselves an important domain of learning, and contends that physical symbols should sometimes be the very objects of learners’ activities.

Chapter 5

Preliminary work and emergent aims

5.1 Introduction

The research designs, findings and theoretical interpretations offered in this thesis developed over a period of about four years. For the most part they are presented in their final state and provide a snapshot of my own thinking at the completion of the doctoral study. The purpose of this chapter is to describe the preliminary and early work that informed the thesis, and to set out the aims.

5.2 Preliminary studies

My interest in pupils' conceptions of the equals sign stems from my brief involvement with trials of an arithmetical microworld, called *VisualFractions* (Kalas & Blaho, 2003), which illustrated the potential role of technology in researching pupils' difficulties with formal notation (Section 5.2.1). I was motivated to explore this further and developed and trialled a variant on the simple electronic calculator, called *Equivalence Calculator* (Section 5.2.2). Building on findings from these studies, and Hewitt's (2001) pedagogic techniques for connecting notation with what learners already know, I devised the task-based research instrument used in the thesis (Section 5.3).

5.2.1 The equals sign as a focus for study

In this section I describe my research involvement with the *VisualFractions* microworld. Trials were conducted with the software to test its general appropriateness, usability and robustness prior to its commercial release by *Logotron*¹, rather than explicitly to investigate conceptions of the equals sign. I first describe the microworld and then summarise a trial with a pair of secondary pupils in which the symbol = unexpectedly arose as an issue of interest. The findings correlated with the literature on pupils' conceptions of the equals sign (Chapter 2) and, I argue, demonstrate the value of technology as a window onto pupils' thinking about formal notation. (For a full report of the study see Jones & Pratt, 2006).

VisualFractions presents learners with numbers (mostly fractions), operator signs (+, −, ×, ÷) and relation signs (=, <, >) as manipulable, on-screen objects. Numbers (fractions) are available in a variety of forms including Arabic numerals, iconic graphics and number lines. The on-screen objects can be created, destroyed, altered and connected on the screen. For example, operator- and relation-objects can be connected to numbers and to other operators and relations. When connected correctly (according to the software's functionality) operators return numerical results and relations return Boolean results.

The equals sign, then, carries an “is the same as” meaning by design in the *VisualFractions* microworld. A statement made by two pupils during a trial is shown in Figure 5.1. The operator sign is connected to the addends $1\frac{3}{6}$ and $\frac{8}{12}$, and returns a numerical result (the small fraction to the top-right of + reads $\frac{26}{12}$). The equals sign is connected to two inputs, the result given by the operator sign and the statement's “answer” ($\frac{14}{6}$), and returns a Boolean result (“false” is written to the top-right of =). The “face” connected to the equals sign is a flag that conveys the Boolean result (thumbs-down means false).

The trial summarised here involved two girls aged 13 years, both of whom were confident and capable with mathematics and computing. The session lasted 80 minutes and was recorded using a microphone and screen-capture software. I was present throughout to introduce the microworld and task, and to offer support and prompt for verbal elaborations (e.g. “Why do you think that?”). The pupils were first introduced to some basic objects and allowed to experiment with making connections. They were then challenged to design an on-screen

¹<http://www.logo.com/cat/view/logotron-visual-fractions.html>

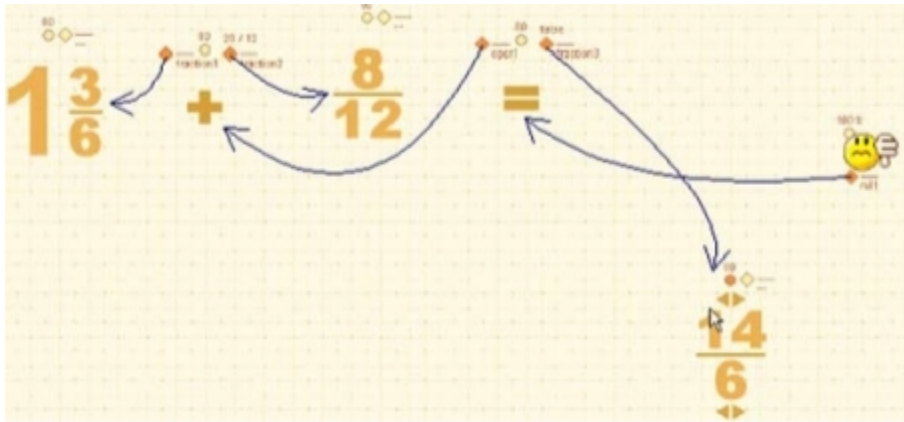


Figure 5.1: An arithmetical statement in *VisualFractions*

activity sheet that might help primary children learn about fraction arithmetic.² Audiovisual movie files were analysed using the qualitative analysis software package *Qualrus*³. A trace of the pupil’s activity was produced and used to develop a descriptive case study of key events. (A similar approach was used for the studies reported in this thesis and is fleshed out in Chapter 6.)

The two main findings were: (i) the children experienced much more difficulty when working with the equals sign than any other on-screen object; (ii) the pupils’ meanings for the equals sign were made apparent to myself as researcher by the functionality of the technology. These findings are elaborated below.

The pupils readily and effortlessly got the hang of making connections between numbers and operators. When a disallowed connection is attempted the microworld plays a flat note, which the girls correctly interpreted as an error noise. On the few occasions it sounded while working with number and operator objects they simply stopped and tried an alternative connection. However, when it came to the = object they repeatedly attempted and returned to disallowed connections. For example, they tried several times to connect the equals sign to the addend immediately preceding it rather than to the + that outputs the result to be compared. They also repeatedly tried to connect the “face” (Figure 5.1) to the “answer”, rather than to the equals sign (which outputs the

²This task design draws directly on Harel and Papert’s (1991) seminal study into the pedagogic potential of the Logo programming language.

³*Qualrus* is a software package that supports qualitative analysis of audiovisual files. <http://www.qualrus.com/>

truth of the statement). These repeated attempts suggest the pupils attended to the left-to-right directionality of equality statements as would be expected from the literature (see Section 2.2). The pupils finally succeeded at connecting the equals sign when they switched their attention from a left-to-right sequence of symbols in favour of what might be called a “data-flow” conception, as supported by the microworld. This data-flow conception was evidenced by one of the pupils, Layla, referring to the statement as “a circuit”:

LAYLA: I think I connected the equals sign straight to the smiley face instead of connecting it to the answer.

CAROL: You’ve got the equals answer to the plus and then you’ve got it to the fraction, then you’ve got a smiley face to the equals.

LAYLA: So there’s like a *circuit* there ...

CAROL: ... between them three so if you change it fraction there it will go wrong.

The pupils only constructed *expression=numeral* statement forms (Figure 5.2), although this is not evidence that they possessed only a place-indicator view of the equals sign. Indeed, given the task goal was to create a classroom resource, they were faithful to how curriculum designers present the equals sign. However, the study evidences a sequential reading of statements from left to right, and an assumption that the on-screen objects should be connected accordingly. The technology enabled them to discover this is not so, and emphasised the need for a “data-flow” conception. The technology also enabled pupils’ “relational thinking” to be explored in a context that did not involve contrived appeals to structure – the “data-flow” conception was independent of statements’ particular numbers and operations. The insight that technology can challenge, and so expose, pupils’ conceptions of notation in novel ways led to a small follow-up study, which is reported in the following section.

5.2.2 A variant on the school calculator

In this section I describe how I designed and trialled a JavaScript-based variant on the simple electronic calculator, called *Equivalence Calculator*. I first describe the software and task, and summarise trials with pairs of primary school pupils. I argue the findings suggest a “protorelational” conception of the equals sign that



Figure 5.2: An activity sheet made by the pupils in *VisualFractions*

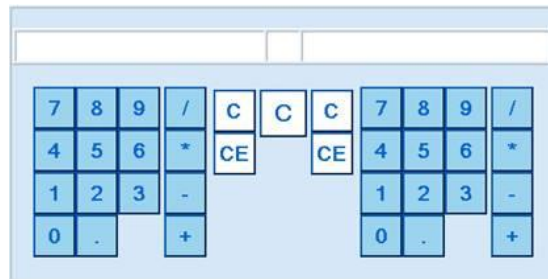


Figure 5.3: The *Equivalence Calculator*

arose from a disconnect between arithmetical processes and notating conventions within the context of the task. For a full report of the study see Jones (2006a).

The *Equivalence Calculator* was designed to resemble a traditional calculator (Figure 5.3) where a sequence of numerals and operators is programmed and the = button acts a command to generate a result. On the *Equivalence Calculator*, two expressions are inputted and the = button performs a comparison and returns a Boolean result. The middle screen is in fact a button and, when clicked, = appears if the expressions are balanced, as in

8×3	=	$12 + 12$
--------------	---	-----------

otherwise it stays blank, as in

8×3		$12 + 11$
--------------	--	-----------

This design presents pupils with a culturally familiar artifact (i.e. the school calculator) in which the equals sign suggests “get the result”, but functionally has an “is the same as” meaning. It was hoped this false expectation would challenge pupils’ views of the equals sign in an observable manner.

Inputted operator signs also act as buttons and return a numeral in place of numeral-operator-numeral, if allowed by the ordering of operations in the expression. For example, clicking the \times in

8×3	$=$	$12 + 12$
--------------	-----	-----------

would give

24	$=$	$12 + 12$
----	-----	-----------

Three trials were conducted, each involving a pair of Year 4 pupils (ages 8 and 9 years) of average or above average mathematical achievement as indicated by their class teacher. Each trial lasted about an hour and data were captured and analysed as described in the previous section. The pupils were first allowed to experiment with the device and were then challenged to create a worksheet of “number sentences” (a term most British school children are familiar with) for their peers to mark. A printed blank worksheet and pencil were provided and each time the pupils wrote a “number sentence” on the worksheet I asked them to demonstrate how other children might use the *Equivalence Calculator* to check it was correct.

The three main findings were that the pupils: (i) tended to establish balance by clicking operators to reduce expressions to numerals, rather than by clicking the middle screen to see if the equals sign appeared; (ii) repeatedly constructed non-canonical statements (i.e. forms other than *expression = numeral*) on the screen; (iii) almost invariably wrote statements down in canonical form (i.e. *expression = numeral*) or as “running” statements (i.e. *expression1 = expression2 = numeral* where $expression1 \neq expression2$ and so on) unless prompted otherwise by me. These findings are summarised in the remainder of this section.

All pupils got to grips readily with the functionality of the software including inputting expressions, clicking the middle screen to see if the equals sign appears,

and clicking operators to produce results. However, as they progressed with the task, they tended to rely more on generating results than checking the equals sign appeared in the middle screen. For example, two pupils demonstrated that 899×1000 is the same as 899000 not by entering each expression and clicking the middle screen, as in

899×1000	=	899000
-------------------	---	--------

but by entering the first term, as in

899×1000		
-------------------	--	--

and clicking the operator to see if the expected result appeared:

899000		
--------	--	--

Here they were effectively using the device in the manner of a traditional calculator where = is clicked to produce a result. This was also the case when, towards the end of each trial, I challenged pupils to make “number sentences” with operators on both sides. They found this difficult and rather than construct $a+b = c+d$ on the screen they tended to ignore the = button and click operators instead. They wrote down their results as statement “pairs” (i.e. $a + b = e$ and $c + d = f$), or in “running” form (e.g. $16 \times 12 = 192 - 56 = 136 \div 12 = 11.333 \dots$) or, in one case, “closed” form (i.e. $1200 - 34 = 1162 + 4 = 1166$).

Only one pair of pupils used the = button throughout the task without prompting. For example, they entered

19×12		
----------------	--	--

and clicked \times to give

228		
-----	--	--

and wrote $19 \times 12 = 228$ on the worksheet. When asked how someone might demonstrate it is correct, they inputted 19×12 , to give

228		19×12
-----	--	----------------

and then clicked the equals button, to give

228	=	19×12
-----	---	----------------

This demonstrates the second finding: when pupils did in fact use the equals sign, with or without prompting from me, they often constructed non-canonical statements on the computer screen. The above final two steps show how this happened. The result 228 was already on the left and so the easiest thing to do was enter 19×12 in the other side and click the = button, rather than clear everything and start again. As such, the non-canonical form of the resulting statement ($228 = 19 \times 12$) reflected how the pupils happened to use the technology and did not necessary imply a “basic” relational view of the equals sign. On occasion non-canonical forms were constructed from scratch rather than to demonstrate the truth of an existing statement. For example, one pair just happened to input a new statement starting with the right-hand side and ended up with $8 = 6 + 2$, as in

		$6 + 2$
↓		
8		$6 + 2$
↓		
8	=	$6 + 2$

The third finding is a reversal of the second: pupils tended to write statements down in canonical form (e.g. Figure 5.4). This is not surprising *per se* but is interesting in contrast to the second finding that they implicitly accepted non-canonical forms on the screen. This is perhaps analogous to Baroody and Ginsburg’s (1986) “protocommutation” where children ignore the order of addends to reduce computational burden but lack a conception that order is irrelevant to outcome (see Section 3.3). Here the pupils constructed non-canonical forms to avoid inputting statements from scratch rather than because they considered them to be properly formed. This was a “protorelational” view of the equals sign in which pupils were *indifferent* to varied form rather than necessarily *accepting* of it: on the one hand non-canonical forms were accepted, suggesting a “basic” relational view; on the other they mostly wrote statements down in canonical form (except when prompted otherwise by myself), suggesting a place-indicator view. For the most part pupils did not comment or hesitate when moving between $a = b$ and $b = a$ forms. One pair that did comment (“*you can do it backwards. We learned that in Year Three*”) nevertheless continued

The image shows a vertical list of five handwritten mathematical expressions on a worksheet. Each expression is written on a separate line with a horizontal line underneath it. The expressions are:

1. $6781 + 2976 = 9757$

2. $6719 / 2999 = 2.2404134711570522$

3. $6781 + 2976 = 9757 + 39 = 9796$

4. $5.5 + 5.5 = 11$

5. $11 + 5.6 = 16.6$

Figure 5.4: Excerpt from a “number sentence” worksheet

to write down their “number sentences” in *expression=numeral* form. It seems that despite knowing “backwards” forms are permitted in school these pupils had an intuitive preference for *expression=numeral* forms, which better reflect the *operations* \rightarrow *results* nature of arithmetic calculation. A similar finding was made by Warren and Cooper (2005) within the context of using a pan-balance to model equations:

Fifteen students reversed the order of the equation, suggesting that even though they recognised $[6 = 4 + 2]$ as true, they still tended to privilege an interpretation where the equal sign means that they should carry out the calculation that proceeds it [as in $4 + 2 = 6$] (p.68).

These findings suggest a *disconnect* between the conventions of arithmetical notation and the pupils’ thinking. At times they used the device as a traditional calculator. This involved clicking operator buttons to generate results, ignoring the = button, and writing findings down sequentially from left to right. As such, their written statements recorded their results but played no part in generating those results. When they did use the = button it was generally to demonstrate the truth of a notated statement, but they did not concern themselves with constructing an identical copy of what they had written on the worksheet. Their focus was on results, not structure, and this partly arose from the limitations of tasks in which pupils establish numerical sameness (see Section 3.4), especially given no contrived statements designed to appeal to ‘relational thinking’ were presented. These issues are elaborated in the following section.

5.3 Towards a task-based research instrument

This section describes the ideas that led to the current thesis and along with the literature review (Chapters 2-4) provides a snapshot of the intentions and thinking that informed my approach and research questions (see Chapter 6). First, I summarise Hewitt's (2001) view that notation can provide a medium for mathematical learning, a view that had a significant impact on my thinking during the early stages of the thesis. I then describe the design processes and methods involved in the development an arithmetic notational microworld.

5.3.1 Notation as a medium for mathematical learning

Arithmetic is about getting results (Hewitt, 1998). This is the case whether it is non-school mathematics, such as a back of an envelope calculation, or school mathematics, such as calculating the answers to a series of written computations. Indeed the whole point of doing arithmetic is usually to produce a numerical result of some kind and this is reflected in the design of arithmetical tools. Electronic calculators allow a programmed sequence of numerals and operator signs to be inputted and processed. The conventions of notation tend to point to a final numeral, whether it is horizontal equality statements that end “= *result*”, or vertical forms such as long division and multiplication that place the result at the top or bottom.

Pedagogically, notation is often seen as something that must be taught for its own sake (Hiebert, 1984; Hewitt, 1996). Typically, pupils generate the result on their own terms, and then are expected to represent their work using a notational convention. This disconnect arises because arithmetical thought rests on inference and conclusions are an inescapable necessity, whereas arithmetical *notation* is cultural and therefore a “mere” convention. As such, teaching notation “offers a separate pedagogic challenge” (Hewitt, 2001, p.308) to developing learners' mathematical reasoning. In practice, the way notation is taught fails to connect pupils' existing understandings of mathematical symbols with the cultural conventions of using those symbols (Hiebert, 1984, p.501). This can cause difficulties as we saw for the case of the equals sign in Chapter 2.

Hewitt (2001) described a pedagogic technique he developed in the classroom for connecting pupils' existing knowledge with the conventions of arithmetical notation. This begins as a verbal exercise in which the teacher “thinks of a

number” and challenges the class to work out what it is: “*I am thinking of a number, add 3, times by 2 and get 14. How can you get my number from the numbers I have given you?*” (p.309). This stresses inverse operations: getting the number requires starting with 14 and working backwards through the spoken operations $+3$ and $\times 2$. As the exercise continues, the teacher provides longer sequences of operations, for example “*I am thinking of a number, add 3, times by 2, subtract 4, divide by 5, add 17, times by 3, subtract 7 and get 50*”. In order to reverse these operations a written record is required that carries not just the numbers and operations, but the ordering of the operations too. Conventional notation is ideal for this purpose and can be built up on the blackboard by the teacher as the problem is spoken:

Teacher says	Teacher writes
<i>I am thinking of a number ...</i>	x
<i>add 3 ...</i>	$x+3$
<i>times by 2 ...</i>	$2(x+3)$
<i>subtract 4 ...</i>	$2(x+3) - 4$

and so on until the final equation is written:

$$3 \left(\frac{2(x+3) - 4}{5} + 17 \right) - 7 = 50$$

Note that the equation conveys ordering not in a left-to-right manner but through the very structural conventions that are the object of learning. When pupils reverse the operations starting with the final number, the teacher again builds up notation on the blackboard:

Pupil says	Teacher writes
<i>Start with 50 ...</i>	50
<i>add 7 ...</i>	$50+7$

and so on until the inverted equation is written:

$$\frac{5 \left(\frac{50+7}{3} - 17 \right) + 4}{2} - 3 = x$$

Hewitt reports that such techniques enable even young pupils to read, write and explain complex notation that they have not encountered before. This is because notation is no longer an end point to mathematical activity but “a medium through which . . . further work is carried out” (p.311). Moreover, notation is built up from mathematics that the pupils already know and understand, rather than being an unrelated extra.

Commenting approvingly on Hewitt’s techniques, Pimm (1995) made three observations. First, the task is “self-contained: no ‘real-world’ context or ‘motivation’ from the external world was offered or required” (p.92). Second, activities that involve practice with symbol-manipulation are often dismissed by mathematics educators as meaningless and rote-based, yet Hewitt’s pupils were engaged and working communally. Third, the stated task goal (“find x ”) seems results based. However, the focus is on processes as embodied by equations; the “result” ($x = 7$ in the above example) is neither generated nor of interest. Pimm observed:

Because [Hewitt] is taking care of x , [the pupils’] attention is free to focus elsewhere. In the lessons I observed, he only once evaluated the actual value for x . . . In the one instance where Hewitt decides to push through calculations to derive an actual number (using calculators), and not end with a complex arithmetic expression which *would* give the answer, the number proved to be a very messy decimal. Almost as a throwaway comment, he says, “Oh, was that the number I was thinking of? I guess it must have been” (pp.97,98 — emphasis in original).

The original (rough) design for a notational microworld was inspired by Hewitt’s ideas. Its development is described in the following two sections.

5.3.2 Initial design ideas

Initially, I envisioned an activity in which pupils use equality statements to express their existing computational strategies. For example, QCA (2001a) provides an example of a primary pupil’s explanation for how she worked out $23 + 17$: “*I added 17 and 3 to get 20, and 20 more is 40*”. There is implicit arithmetical knowledge in this spoken strategy that lends itself to being expressed as non-canonical equality statements including partition ($23 = 20 + 3$),

$$30 + 41 = 71$$

$$30 + 40 = 70 \quad 70 + 1 = 71$$

Figure 5.5: A pupil’s written strategy for $30 + 41$

associativity ($23 + 17 = 20 + 20$), and commutativity ($20 + 3 + 17 = 17 + 3 + 20$). Indeed, authors often use non-canonical statements as a short-hand to describe learners’ computational strategies. For example, Hewitt (1998) expressed the different ways 19×16 might be mentally calculated as:

$$19 \times 16 = 2 \times 16 \times 10 - 16$$

$$19 \times 16 = 20 \times 16 - 20 + 4$$

$$19 \times 16 = (19 \times 10) + (19 \times 5) + 19$$

$$19 \times 16 = 19 \times 2 \times 2 \times 2 \times 2 \quad (\text{p.23})$$

Similarly, Carpenter, Levi, Franke, and Zeringue (2005) expressed a tens-and-units mental strategy for $40 + 50$ as

$$40 + 50 = 4 \times 10 + 5 \times 10 = (4 + 5) \times 10 = 9 \times 10$$

and expressed a “relational” mental strategy for $8 + 4 = \square + 5$ as

$$8 + 4 = (7 + 1) + 4 = 7 + (1 + 4)$$

which “*implicitly* use[s] the associative property of addition” (p.54 — emphasis added). However, when asked to write their working-out for a calculation children tend to exclude this knowledge. Figure 5.5 shows a primary pupil’s written strategy for $30 + 41$ (taken from QCA, 2001c). It includes the compositions $30 + 40 = 70$ and $70 + 1 = 71$, but *assumes* the partition $41 = 40 + 1$. My original idea for the thesis was the development of a task in which pupils express their strategies as equality statements on a computer screen and then use these statements to make exchanges of notation in order to demonstrate that their strategies work. For example, given $30 + 41$, we can use $41 = 40 + 1$ to transform it into $30 + 40 + 1$; then use $30 + 40 = 70$ to give $70 + 1$, and finally use $70 + 1 = 71$ to produce the result. I envisioned activities in which pupils

made “statement puzzles” of their strategies for peers to work through in order to “guess my strategy”. Such a “puzzle” would include a starting expression, such as $(20 + 53)$, and a sequence of statements for making exchanges:

Statement	Expression
	$20 + 53$
$53 = 3 + 50$	$20 + 3 + 50$
$3 + 50 = 50 + 3$	$20 + 50 + 3$
$20 + 50 = 70$	$70 + 3$
$70 + 3 = 73$	73

This initial rough design idea needed fleshing out and testing for feasibility. This was attempted by sketching interfaces for inputting and transforming arithmetical notation. Vaguely, notation would be inputted at the keypad and appear on screen, and some buttons for processing (partitioning, commuting, composing and so on) would be provided. Every time a line of notation was processed it would scroll up for the next line. However, it was difficult to imagine a coherent design that could conceivably be implemented as a working prototype. It seemed the quirks and assumptions of pupils’ typical natural language explanations would be too removed from the precision and explicitness required by the computer to process them. Attempts to bridge this gulf by elaborating the software’s functionality led to interface sketches that were evermore unwieldy and restrictive (e.g. Figure 5.6). In light of these difficulties I implemented a storyboarding method for developing the design and capturing my ideas, which is described in the following section.

5.3.3 The storyboarding method

Storyboarding is a common method for interface design amongst the human-computer interaction community (Madsen & Aiken, 1993). However, it is novel to the design of mathematical microworlds and marks a contribution to the literature arising from this thesis⁴.

The storyboarding process lasted around six months during which time a total of ten storyboards were produced. A storyboard was abandoned and a new one started every time the design appeared to run into problems. The ten storyboards varied in length from 15 to 170 slides each and comprised just over

⁴The method described here has been published in detail as Jones (2008c)

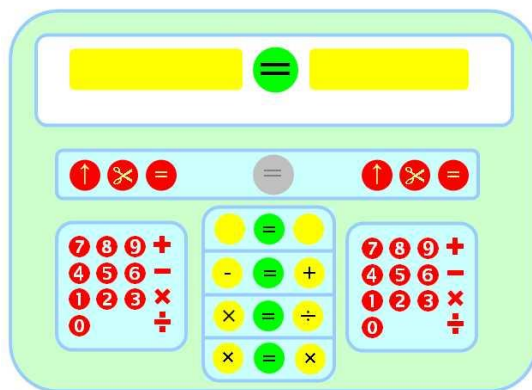


Figure 5.6: An early schematic of an interface

600 slides in total. Each storyboard contained one or more “scenes” separated by distinctive clapper-board slides to indicate a “cut”. Each scene correlated, more or less, with an imagined epistemic child, “C”, working on a particular problem such as programming a calculation strategy for 18×8 . Successive scenes in a given storyboard would typically follow alternate hypothesised strategies for the same problem, or a similar strategy applied to different problems.

Storyboarding was motivated by an impasse with using interface schematics to try and design a microworld (see previous section). To this end the interface was abstracted to a bare minimum and C (the epistemic learner) was envisioned working directly with raw notation. Figure 5.7a, the first slide of the first storyboard, shows an interface that has been abstracted to a single arithmetical statement. The following slides followed stepwise construction and transformation of notation towards the learning goal of programming a calculation strategy (Figure 5.7b-d).

The interface became more fleshed out and increasing consideration was given to manipulation tools, such as the keypad shown in Figure 5.7f. When the second storyboard was created the theme was continued but the interfaces became decreasingly abstract and began to look quite literal and concrete (Figure 5.7g). This learner-centred concretisation of the interface design continued through to the fifth storyboard (Figure 5.7h). This was no smooth journey. Many starts, stops and varied designs are apparent throughout the first five storyboards.

However, the design ran into problems. These problems emerged when the storyboards were shared with peers and presented at seminars. The interface

18×8

C is presented with the above statement and works out the answer in her head, discussing how she does so with another child.

This sum is 'difficult' because it reduces to challenging number bonds such as $18 + 18 \dots$ or 8×8 .

(a)

18×9

The next sum is presented and C notices the similarity to the previous one.

This sum is also 'difficult' but can be expected to cue the algebraic thinking used to work out the previous one.

(b)

18×10

The next sum is similar but C also notices that it is one she can do easily.

C now has a choice of building on the 'general strategy' used for the last two sums, or a 'specialised multiply by 10' strategy.

(c)

18×18

The next sum is presented. Again C notices the similarity.

C can now draw on a strategies used so far or on memory of 18×8 and prior knowledge of 18×10 .

(d)

$57 + 78$

$57 = \# + 1$
 $\# = \# + 10$
 $\# = \#0 + \#$

C drags her selection to a box.

(e)

$38 + 78 + 47$

$38 + 78 + 47 =$

She sets about working out the answer, discussing it with another child.

(f)

$206 + 19 + 100 + 100 + 100$

$\# + \# = \# + \#$
 $\# = \# + 1$
 $\# = \# + 10$
 $\# = \# + 100$

$319 = 219 + 100$
 $219 = 119 + 100$
 $119 = 19 + 100$

C clicks the = button and the sequence is executed.

(g)

$167 + 288$

$167 = 100 + 60 + 7$
 $288 = 200 + 80 + 8$

C creates the above equality statements.

(h)

Figure 5.7: Sample slides from the first five storyboards

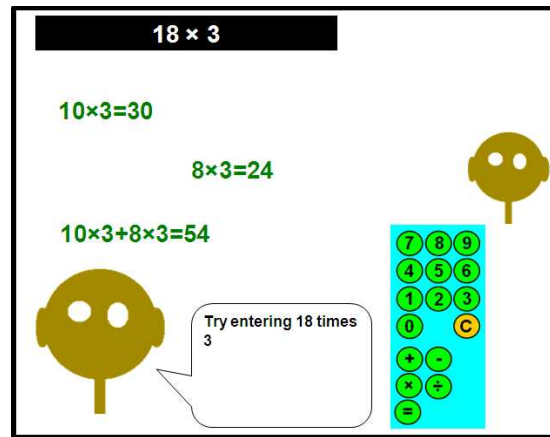


Figure 5.8: Icons of C and R with callouts

design embodied the anticipated learning scenarios in a rather rigid manner due to the technological attempts to assist learners with the difficulties of translating ambiguous and assumptive natural language arithmetic into the explicit formal notation required by the task and goals. I returned to a stripped-down (abstracted) representation of the interface for the next storyboard. Another decision made in the light of peer feedback was the inclusion of an icon for C, and call outs (speech and thought bubbles). This helped keep the focus on C's mathematical activity without a drift back to interface concretisation. In the subsequent storyboard an icon of the participant researcher, R, was also included (Figure 5.8). The icons provided a natural way to capture R's possible interventions and interactions with C as imagined in the thought experiments

I also decided at this stage to incorporate real-world arithmetical strategies from children in classrooms. The QCA website National Curriculum in Action hosts samples of pupils' work with reference to levels of attainment as specified by the National Curriculum. This data enabled the importation of real pupils' arithmetical strategies for storyboard scenes. For example, Figure 5.9 shows a slide illustrating "Tony's" written explanation for calculating $30 + 41$. Slides were created by imagining pupils inputting their own written strategies to explore the problems that might arise. These turned out to be the same problems that had emerged in the interface concretisations in early storyboards, namely, that when arithmetical strategies are spoken in natural language the partitioning and re-organising of notation is implicit. For example, "Tony" expresses com-

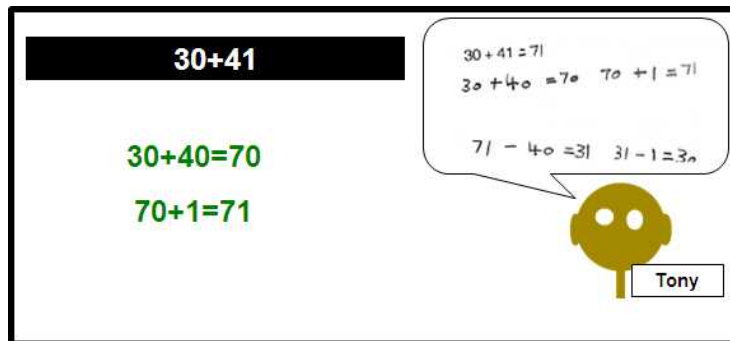


Figure 5.9: “Tony’s” strategy for solving $30 + 41$

position explicitly using equalities such as $30 + 40 = 70$, but draws implicitly on arithmetical facts such as $41 = 40 + 1$. There appeared to be a need to give learners access to the arithmetical assumptions underlying their strategies in order to express them using the computer.

Despite the remaining problems an implementable design had now been reached. Implementation became possible for two reasons. First, it had become apparent that the functionalities of the processing tools could be ascribed directly to the notation itself: notation could simply be clicked to make selections and transformations. Two keypads were then needed for entering notation. The *Equivalence Calculator* (Section 5.2.2) proved ideal without the need for much redesign (Figure 5.3, p.45). Second, the problem of making implicit arithmetic explicit, that had plagued all ten storyboards, could simply be ignored in terms of technical implementation. The introduction of the C and R icons and call outs made it clear that R could guide this process. This was somewhat heavy-handed, involving exposition, but, by drawing on guidance by R, the task could at least be implemented and piloted.

The software was subsequently programmed in *Imagine Logo* (Kalas & Blaho, 2003) which lends itself to the fast prototyping of microworlds. I made a final significant change to the task design during software testing. The impasse of inputting the partitions and commutations implicit in written strategies remained apparent and it was difficult to imagine offering learners an intuitive way to do this. However, upon programming some strategies, it was discovered the outcome made for an intriguing puzzle in which the goal is to use provided equalities to transform an arithmetical sum into its answer. Consequently, I de-

cided to reverse the task: rather than pupils expressing their existing strategies as sequences of equality statements, they would be presented with (randomly sequenced) statements and challenged to transform a given expression into a numeral. A paper-based equivalent might involve putting the following statements in the correct order for transforming $20 + 53$ into 73.

$$\begin{aligned} &20 + 53 \\ 20 + 50 &= 70 \\ 3 + 50 &= 50 + 3 \\ 70 + 3 &= 73 \\ 53 &= 3 + 50 \end{aligned}$$

It should be noted this is strictly speaking a results-based task goal but, as with Hewitt’s “guess my number”, the focus is on the processes required to achieve the goal — in this case seeing equality statements as rules for notational interchanges. Unlike Hewitt’s task this involves generating a result from an expression, but this need not be case the case. The reversibility of transformations means a “puzzle” could as easily be presented with the challenge to transform 73 into $20 + 53$, as *vice versa*. Either way, “solving” it involves transforming one term into another rather than mental calculation. Once pupils become familiar with “solving puzzles” made of statements they can be challenged to make “puzzles” that draw on the implicit knowledge in their mental strategies. This, more or less, is the nature of the computer-based task used as a research instrument in this thesis⁵.

5.4 Emergent aims of the study

The overarching aim of the study is to investigate the educational potential of presenting pupils with statements as rules for making exchanges of arithmetical notation. This aim is made possible by the software and task design described in the previous section which supports an implicit “can be exchanged for” meaning for the equals sign. There are four specific research questions that will be addressed.

1. Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?

⁵The software is described in detail in Section 6.3

2. Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct?
3. Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?
4. Can children connect their implicit arithmetical knowledge with explicit transformations of notation?

In the remainder of this section I elaborate these four research questions.

1. Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?

The statement $41 = 40 + 1$ can be used to partition the 41 in $30 + 41$. The visual transformation $30 + 41 \rightarrow 30 + 40 + 1$, when shown on a computer screen, can be seen as “splitting” the 41 into two numerals. This is not dependent on appeals to place-value or any other specific strategy — it is a general property of $c = a + b$ when observing its transformational effect. Similarly, the commutative transformational effect of $a + b = b + a$ can be seen as “swapping” numerals around. We should expect pupils to use terms such as “splitting” and “swapping” when talking about computer-generated transformational effects of $c = b + a$ and $a + b = b + a$ statements respectively. This conjecture is the focus of Study 1 (Chapter 8), and the findings are confirmed in Studies 2 and 3.

2. Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct?

If the “is the same as” meaning is ignored then false equalities can be presented that carry an *exclusive* “can be exchanged for” meaning, as in $2 + 4 = 7$. This may not be attractive pedagogically but offers an interesting research angle. Given that the two meanings are distinct by definition, pupils might accept imbalance when viewing statements as given truths for making exchanges. Such acceptance would show the two meanings to be pedagogically distinct and is the focus of Study 2 (Chapter 9).

3. Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?

In Study 3 (Chapter 10), which is the focus of this and the next question, the task goal is extended so that it requires co-ordinating the two meanings. Pupils attempt the puzzle-solving task and are then challenged to make their

own version puzzles. They are constrained by the requirement that statements must be both numerically balanced and notationally transformable. It might be expected that children’s qualitatively different ways of viewing notation (see Section 3.3) renders such co-ordination trivial to “proceptual” thinkers and inaccessible to others.

4. *Can children connect their implicit arithmetical knowledge with explicit transformations of notation?*

This is an extension of the previous question. Pupils familiar with puzzle-solving and making are challenged to use equality statements to present and then test their mental strategies for $37 + 48$ and $37 + 58$. This requires identifying implicit mental transformations, such as $37 = 30 + 7$, and expressing them as statements, and reflects the initial intent of the task design set out in Section 5.3.2. This and the previous question are the focus of Study 3 (Chapter 10).

5.5 Summary

My research interest in pupils’ conceptions of the equals sign stems from trials of the *VisualFractions* microworld in which pupils experienced difficulties connecting the = object but not number and operator objects. Their expectation of a left-to-right flow of processing was challenged and made visible by the use of technology as a research instrument. Following this, I designed the *Equivalence Calculator*, in which the expectation of a “get the answer” meaning for the = button is challenged by a technologically supported “is the same as” meaning. During trials pupils displayed a “protorelational” view of the equals sign because they constructed and accepted non-canonical statements on the screen, but wrote statements in canonical form on the worksheet. This suggests they focused exclusively on results and were indifferent to form.

Hewitt (2001) described a pedagogic technique in which formal notation is a *medium* for mathematical thinking and communication. The teacher builds notation up from spoken arithmetic, following structural conventions to record the order of operations. Drawing on this, I envisioned a task-based research instrument in which pupils identify the arithmetical knowledge implicit in their mental strategies and express it as equality statements. These statements could then be used to make substitutions and so demonstrate that the strategies work. This initial design idea was developed using a storyboarding method that revealed

difficulties with this approach and resulted in a design that presents “puzzles” to “solve” by making exchanges of notation. Once pupils are familiar with solving “puzzles” they make their own, drawing on their mental computation strategies to do so.

The overarching aim of the thesis is to investigate the educational potential of presenting pupils with statements as rules for making exchanges of arithmetical notation. There are four specific research questions:

1. Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?
2. Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct?
3. Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?
4. Can children connect their implicit arithmetical knowledge with explicit transformations of notation?

Chapter 6

Approaches to notating task design

6.1 Introduction

In Section 4.4.1 I described referential and structural approaches to presenting equality statements to learners. I also presented Dörfler’s (2006) critique that these approaches presuppose external representations to be auxiliary to accessing abstract mathematical objects, and this alienates many learners. In this chapter I describe Dörfler “diagrammatic” framework for presenting representations as the “very objects” of learning activities. I apply this framework to the case of equality statements and set out the task design used in the main studies reported in this thesis.

6.2 Diagrammatic reasoning

Recently there has been interest in the application of Charles Sanders Peirce’s “diagrammatic reasoning” to mathematics education (e.g. Bakker & Hoffmann, 2005; Ernest, 2006; Ongstad, 2006; Otte, 2006; Radford, 2000). This thesis builds in particular on Dörfler’s (2006) diagrammatic approach to task design which seeks to overcome the inherent limitations of contemporary referential and structural approaches (see Section 4.4.1). In fact the diagrammatic approach is

similar to structural approaches in which meaning emerges from relationships within a symbol system (see Section 4.4.1). However, unlike structural approaches, the diagrammatic approach explicitly disconnects symbols from their Platonic referents. The aim is for the literal, physical symbols and their rule-governed transformations to become the explicit objects of learners' activities. The term "diagrammatic reasoning" was coined by Peirce (1991) in his attempts to resolve the paradox between the deductive and discoverable aspects of mathematics. He proposed a "visual" solution:

all deductive reasoning . . . involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts (p.270).

Peirce defined diagrammatic reasoning in the following often-cited passage:

By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, performs experiments upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses this in general terms. (p.47-48).

In the Peircean sense of the term, "diagram" means arithmetical statements, algebraic equations, graphs, geometrical constructions and so on. In one sense "diagram" is being used more loosely than everyday associations with "drawings" rather than "writings" would suggest. In another sense, however, it is more restrictive, referring only to those "inscriptions" that form precise mathematical structures with grounded rules for making transformations. Consequently, reading and writing mathematical inscriptions are fundamental components of diagrammatic reasoning. These are essentially perceptive processes involving observing, comparing and recognising patterns. The "reasoning" aspect is based in seeing potential actions that can be executed and this reflects the pragmatic grounding of diagrammatic reasoning. The focus on "what is written" rather than "what is referenced" is emphasised by use of "inscription" (Dörfler, 2006)

which refers to any artificial marking on paper or a computer screen, and does not carry the theoretical baggage associated with “symbol” (referent, signal, encapsulation and so on). This is not to throw theory away – referents, signals and encapsulations are important constructs within the discipline of mathematics education – but to envisage an alternative approach to task design in which the focus is on the literal observation and transformation of notation.

Pedagogically, the essence of diagrammatic notating tasks is learners manipulating conventional representations (“inscriptions”) in an open, exploratory manner. This renders mathematical notating an empirical, creative and discovery-based activity. As such, the “diagrammatic learner” manipulates inscriptions analogously to how a Piagetian learner manipulates physical objects. However, the diagrammatic learner is no “lone scientist” (Bruner & Haste, 1987) because inscriptions are cultural tools (see Section 4.3). Diagrammatic reasoning is “public and therefore social and sharable” (p.107) and consequently diagrammatic tasks are an attempt to enculturate learners into mathematical practice (see Section 4.4). Formal notating becomes a shared, craft-like activity and to be of pedagogic value should not take place in isolation in the manner of the lone Piagetian scientist. It is a situated practice (see Section 4.4) that requires a social dimension for discussing and utilising the available structuring resources (diagrams). As Dörfler puts it:

Diagrammatic activities will always be embedded into a discursive context which offers a rich language to speak about the diagrams and their transformations. This language is a natural language plus a supply of technical terms (like: numeral, digit, place value, fraction, enumerator, graph, differential quotient, integral, arithmetic mean). The novices will have to learn this language simultaneously with their development of the diagrammatic practice (which per se is not a language and not a linguistic activity). As when learning a craft an appropriate language can give guidance, support and focus of attention. But the respective activity cannot be completely described by language, it must be done (and for that observed) and practised extensively. (p.108)

6.2.1 Diagrammatic equality statements

The “can be exchanged for” meaning for the equals sign (see Section 3.4) can support a diagrammatic approach to notating task design because it promotes the transformation of arithmetic inscriptions. In this thesis “inscription” is used to refer specifically to the case of numerals, arithmetic signs, expressions and equality statements (23, +, =, $2 + 5$, $2 + 5 = 7$ are all inscriptions)¹. Whereas the “is the same as” meaning leads to viewing equality statements as isolated questions, the “can be exchanged for” meaning allows statements to be viewed in parallel as rules for making substitutions. Consequently, sets of inter-related statements with matched inscriptions can be presented as “diagrams”, akin to simultaneous equations in algebra, as in

$$\begin{array}{c} 20 + 53 \\ 20 + 50 = 70 \\ 3 + 50 = 50 + 3 \\ 70 + 3 = 73 \\ 53 = 3 + 50 \end{array}$$

along with the *transformation rule* “ $a = b$ can be used to substitute a for b and b for a ” (this rule is in fact the “can be exchanged for” meaning of the equals sign and can be derived from the transitive and symmetrical properties of arithmetical equivalence — see Section 3.4). If the rule is known, or discovered through (technology-supported) experimentation, then diagrams can be visually searched for matches of a with a , b with b , and so on in order to determine where exchanges of notation can be made. Such visual exploration can be referred to as *iconic matching* (where “iconic” is used to emphasise visually identical inscriptions, rather than computationally identical results). When learners look for iconically matched pairs of inscriptions, and make substitutions, their engagement can be described as *diagrammatic activity* (or *diagrammatic reasoning*).

The distinction between diagrammatic activity and establishing numerical balance is not black-and-white. Arithmetical diagrams do in fact imply the rule that quantity must be conserved across transformations. As such, $2 + 4 = 4 + 2$ cannot be used to make a substitution in $2 + 40$ or $2 + 4 \times 3$, and a “full” diagrammatic view requires *co-ordinating* notational substitutions with numerical

¹See Table 6.1 for a list of all diagrammatic terms defined for the purposes of this thesis.

<i>Inscription</i>	Arabic numerals and arithmetical signs (+, =), and combinations of them, on a computer screen.
<i>Diagram</i>	An expression and collection of iconically matched equality statements, and a transformation rule.
<i>Transformation rule</i>	$a = b$ can be used to substitute a for b and b for a . This is the “can be exchanged for” meaning for the equals sign.
<i>Iconic matching</i>	Searching a diagram for multiple occurrences of a given inscription in order to determine where notational exchanges can be made.
<i>Partition</i>	The distinct transformation $c \rightarrow a + b$ as specified by $c = a + b$ (or $a + b = c$).
<i>Commutation</i>	The distinct transformation $a + b \rightarrow b + a$ as specified by $a + b = b + a$ (or $b + a = a + b$).
<i>Diagrammatic activity</i>	Observations and transformations of a diagram, particular those involving iconic matching, articulating partitional and commutative transformations, and strategising several transformational steps ahead.
<i>Structure</i>	The intra-statement relationships that can be used to assess its numerical balance. Structure is a property of the specific numerals and operators used, particularly in carefully designed examples (e.g. $a + c - a = c$, $123 = 100 + 20 + 3$).
<i>Form</i>	The inter-statement relationships within a given diagram that can be used to make distinctive transformations. Form is a general property of transformational effects ($c \rightarrow a + b$, $a + b \rightarrow b + a$ and so on) rather than specific statement examples.

Table 6.1: Summary of diagrammatic terms as defined here for the purpose of the thesis

sameness. Conversely, within the “is the same as” approach, in which the task goal is to establish numerical sameness, pupils do engage in activities that are to some extent diagrammatic. Interventionist studies have demonstrated pupils articulating statement structure in reflective and flexible ways when establishing numerical balance (Section 2.3). However, once a statement has been analysed to a learner’s satisfaction its potential for further activity comes to an end and a new statement must be presented. This curtails diagrammatic activity, which is based in discovering potential actions within a relational system.

6.2.2 Form and structure

The potential actions we wish learners to discover can be designed into diagrams. For the case of the transformation rule “ $a = b$ can be used to substitute a for b and b for a ” these discoveries can include the distinct transformational effects of different statements. For example, $16 + 5 = 5 + 16$ can be seen to have a commutative effect when substituting $16 + 5$. Similarly, the statement $21 = 16 + 5$, when used to substitute 21, can be seen to have a partitioning effect. Learners might be expected to notice, distinguish and make use of the transformational effects of the *forms*² $a + b = b + a$ and $c = a + b$ when working with an arithmetic diagram. It is the visual replacements $a + b \rightarrow b + a$ and $c \rightarrow a + b$ that can be seen as commuting and partitioning, rather than the inscriptions $a + b = b + a$ and $c = a + b$ *per se*. In fact the symmetry of equivalence means $c = a + b$ and $a + b = c$ make identical transformations and are both partitioning forms *when viewed in terms of substituting c* ; they are compositional when viewed in terms of substituting $a + b$. As such, form is viewed not in isolation but in terms of potential actions with statements within the context of a given diagram. Appeals to internal *structure*, such as associativity ($a + b = (a + 1) + (b - 1)$), place-value partition ($23 = 20 + 3$) and so on are a design option rather than a necessity.

Not that if a learner uses $a + b = b + a$ (or $b + a = a + b$) to transform $a + b \rightarrow b + a$, and notices it “swaps”, this does not necessarily imply a concept of commutation (in which the ordering of addends is irrelevant to outcome). Similarly, if they do not distinguish between $a + b \rightarrow b + a$ and other transformations this does not mean they lack the concept. The term *commutation*, as used in this thesis, might be considered shorthand for “diagrammatic-protocommutation” because

²*Form* refers here to a statement’s transformational potential irrespective of its numerals.

it applies only to the notational domain and there is no consideration of consistency of outcome. Similarly, the term *partition* is used to refer to noticing $c = a + b$ (and $a + b = c$) transforms $c \rightarrow a + b$, and can be considered shorthand for “diagrammatic-protopartition”.

6.2.3 Strategic thinking

Diagrams can also be used to support a second level of discovery related to the strategic use of transformational effects in order to achieve a task goal. This involves learners thinking two or three transformational steps ahead in order to modify a diagram into its goal state. For example, consider the diagram

$$\begin{array}{c}
 20 + 53 \\
 20 + 50 = 70 \\
 3 + 50 = 50 + 3 \\
 70 + 3 = 73 \\
 53 = 3 + 50
 \end{array}$$

and the task goal of transforming $20 + 53$ into 73 . We might start with the term $20 + 53$ and search for the inscriptions 20 , 53 and $20 + 53$ within the statements, and find we can use $53 = 3 + 50$ to make the transformation $20 + 53 \rightarrow 20 + 3 + 50$. We might then notice that in order to use the composition $70 + 3 = 73$ we need to commute 3 and 50 , and then compose 20 and 50 , giving

$$20 + 3 + 50 \rightarrow 20 + 50 + 3 \rightarrow 70 + 3 \rightarrow 73$$

Practice with several similarly structured diagrams might lead to the general strategy of starting by partitioning numerals (using $c = a + b$ forms), then commuting numerals (using $a + b = b + a$), and finally composing numerals (using $a + b = c$ forms). Alternatively, we might start with the statement that contains the target inscription $70 + 3 = 73$, search for other instances of 70 , 3 and $70 + 3$, and use $20 + 50 = 70$ to make the transformation $70 + 3 = 73 \rightarrow 20 + 50 + 3 = 73$, and so on until $20 + 53 = 73$ is produced, which can be used directly to transform $20 + 53 \rightarrow 73$. This might lead to the general strategy of starting with $a + b = c$ forms to partition, then using $a + b = b + a$ forms to commute, and finally using $c = a + b$ forms to compose.

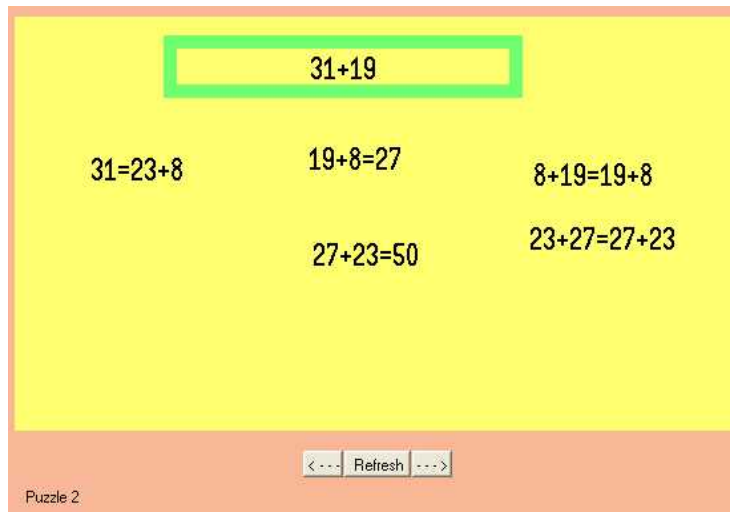


Figure 6.1: Screen-shot of the *Sum Puzzles* software

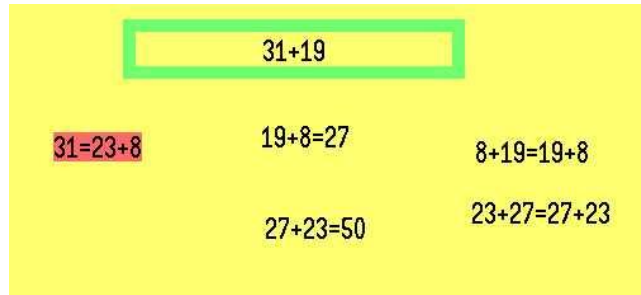
6.3 The *Sum Puzzles* microworld

This section describes a microworld I designed, called *Sum Puzzles*³. Its development took around eight months and required numerous design decisions to be taken (see Section 5.3).

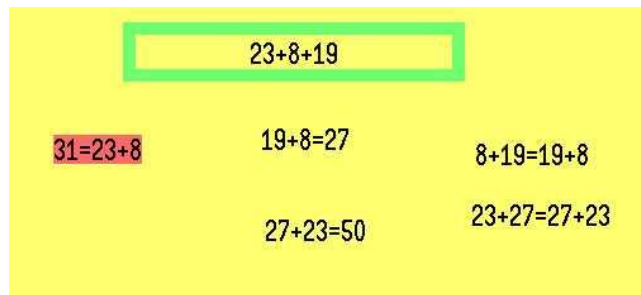
The arithmetic diagrams described in the previous section could conceivably be presented on static media. However, digital technology provides feedback, and so supports the discovery of mathematical rules (or “potential actions”) through trial-and-error, whereas static media require the rules to be understood before activity can begin. Feedback can include animated transformations, in this case the literal replacement of one inscription by another. Technology also lends itself to the construction and testing of mathematical artifacts, providing a broader scope for research than pupils’ “merely” manipulating presented artifacts (Section 4.2.1).

A screen-shot from *Sum Puzzles* is shown in Figure 6.1. An expression (the “sum”) is given prominence in a box at the top of the screen and statements are “scattered” underneath, grouped by form (partition, composition, commutation) but in no particular order. The buttons at the bottom of the screen (< - - -, Refresh, - - - >) are for browsing and refreshing the diagrams.

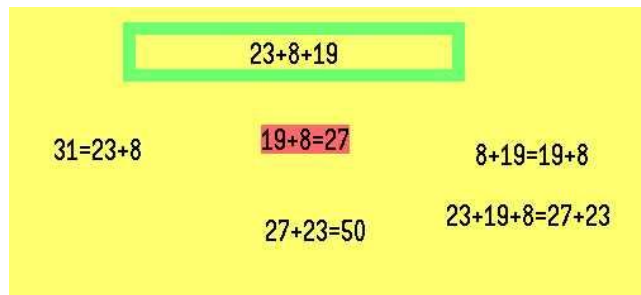
³The software is available on-line at go.warwick.ac.uk/ep-edrfae/software.



(a) The statement $31 = 23 + 8$ is selected.



(b) The contents of the box is transformed ($31 + 19 \rightarrow 23 + 8 + 19$).



(c) $19 + 8 = 27$ is used to make the transformation $23 + 27 = 27 + 23 \rightarrow 23 + 19 + 8 = 27 + 23$.

Figure 6.2: Transformations in *Sum Puzzles*

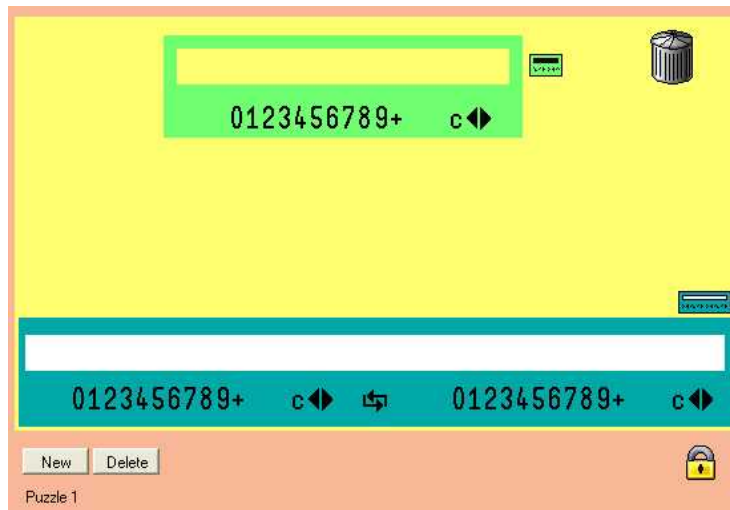


Figure 6.3: The puzzle making tools

Transforming a diagram into its goal state, or “puzzle solving”, is supported by two basic functionalities. The first is clicking an equals sign which causes a statement to become highlighted ($31 = 23 + 8$ has been highlighted in Figure 6.2a). In this sense the equals sign can be thought of as a handle for taking hold of a statement. The second functionality is clicking numerals and operator signs once a statement has been highlighted. This causes a substitution to take place, or not, as determined by the currently highlighted statement. For example, given the highlighted statement $31 = 23 + 8$, if the inscription 31 in the box is clicked the outcome will be that shown in Figure 6.2b. As such, the “can be exchanged for” meaning is supported in a way not possible using static media, namely by playing out transformations $a \rightarrow b$ as “animations” on the screen. If the highlighted statement does not determine a substitution for a clicked inscription then nothing happens. For example, clicking the inscription $+19$ in the box while the statement $31 = 23 + 8$ is highlighted (Figure 6.2a) will have no effect. Substitutions are reversible. If the inscription $23 + 8$ in the box in Figure 6.2b is clicked while the statement $31 = 23 + 8$ is still highlighted the situation returns to that shown in Figure 6.2a. Statements can also be used to manipulate other statements. For example, if the statement $19 + 8 = 27$ is highlighted and the left 27 in $23 + 27 = 27 + 23$ is clicked, it will become $23 + 19 + 8 = 27 + 23$ (Figure 6.2c).

Keypad tools are also available for learners to make diagrams (Figure 6.3). The smaller, top keypad allows the expression to be inputted; the lower keypad allows statements to be inputted and placed on the screen, and works in a similar manner to the *Equivalence Calculator* (Section 5.2.2). The lower keypad only permits true statements to be inputted (= does not appear if the two sides are imbalanced) and so supports an “is the same as” meaning for the equals sign. An important distinction between diagram-solving and diagram-making, then, is that solving emphasises only a “can be exchanged for” meaning whereas making emphasises both an “is the same as” meaning (when inputting statements) and a “can be exchanged for” meaning (when testing those statements in a diagram). This distinction enables the exchanging meaning to be explored both in isolation and in coordination with the sameness meaning.

6.4 Overview of the three studies

In Section 5.4 I set out four research questions grounded in the psychology of arithmetical equivalence. In this section I revisit them from the perspective of task testing. The questions are as follows:

1. Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?
2. Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct?
3. Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?
4. Can children connect their implicit arithmetical knowledge with explicit transformations of notation?

They translate to three studies as elaborated below.

Study 1: Articulating form

Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?

At the time of undertaking Study 1 it was at heart a viability test for the task. Three issues were of concern: the robustness and usability of the software; pupils' degree of engagement with the resources; pupils' degree of engagement with statement form. Prior opportunistic pilot trials with two individual children had given grounds for confidence that the software and interface were sound. However, the degree of engagement with the task *per se*, and statement form in particular, was somewhat unpredictable due to the novelty of the design. For simplicity, the measure of pupils' engagement with statement form was restricted to whether they articulated the distinctive transformational effects of $a + b = b + a$ ("swap", "switch around" and so on) and $c = a + b$ ("split", "separate" and so on) statements. There was no explicit concern for pupils' strategic thinking, the coordination of transformational and balance views of equality statements, or the stability and transferability of learning.

Study 2: Attention to balance

Are the "can be exchanged for" and "is the same as" meanings for the equals sign pedagogically distinct?

Study 2 builds on Study 1 in a cumulative manner. It replicates the articulation of form and strategic thinking present in Study 1. It also tests an emergent hypothesis that the truth of presented equality statements has little bearing on pupils' diagram-solving activity. This involves presenting diagrams that contained some false statements to find out whether pupils discern their presence.

Study 3: Coordinating form and balance

Study 3 more closely resembles the original vision for the task (Section 5.3.2) and sought to address two questions.

Can children coordinate "can be exchanged for" and "is the same as" meanings for the equals sign?

Whereas Studies 1 and 2 supported an exclusive utility for transformational views of equality statements in order to investigate its distinctiveness from sameness views, Study 3 supports utility for both views. This is achieved by providing pupils with diagram-solving practice (thereby replicating results for Studies 1 and 2), then challenging them to make their own diagrams. To do so they must input numerically balanced statements that integrate into a relational system.

Success at constructing arithmetical diagrams would indicate pupils switching appropriately between the two views.

Can children connect their implicit arithmetical knowledge with explicit transformations of notation?

Study 3 also contains a second component that requires pupils to identify the transformations implicit in their natural-language calculations and input them as balanced statements. Success at constructing a working diagram that reflects their articulated calculations would provide evidence for the coordination of transformational and sameness views with existing (implicit) arithmetical knowledge. This fourth question captures the original design idea that motivated the thesis (see Section 5.3.2).

6.5 Summary

The diagrammatic approach to notating tasks is similar to the structural approach but explicitly disconnects symbols from their abstract referents. The focus is on potential actions with visible “inscriptions” and the intention is to provide learners with creative, exploratory learning experiences that are shared and collaborative in nature.

The “can be exchanged for” meaning of the equals sign can be expressed as the transformational rule “ $a = b$ can be used to substitute a for b and b for a ” and forms the basis for “diagrammatic” equality statements. This rule enables statements to be presented in parallel, forming a transformational diagram akin to algebraic simultaneous equations. Such arithmetic “diagrams” support *iconic matching* activity, that is looking for occurrences of identical inscriptions in order to determine where exchanges can be made. When substitutions are made learners can be expected to distinguish statements in terms of *form*, that is a statement’s distinctive transformational potential. For example, the statement $a + b = b + a$ can be viewed as (potentially) swapping the numerals a and b ; likewise the statement $c = a + b$ can be viewed as (potentially) splitting the numeral c . Arithmetic diagrams can also support *strategic thinking* which involves working two or more transformational steps ahead when working toward a specified task goal.

The *Sum Puzzles* computer microworld has been specially designed for thesis in order to support arithmetic diagrammatic activity. The use of digital technology

enables pupils to freely explore prepared diagrams and so discover the underlying transformational rule without it being explicitly taught. Keypad tools are also provided so that pupils who are familiar with solving presented diagrams can try making their own.

The three studies reported in this thesis are cumulative and replicative. The first investigates the viability of the task in terms of engagement with statements form. The second tests an emergent hypothesis that this engagement is independent of statement truth. The third investigates whether pupils can switch appropriately between transformational and sameness views, and whether they can coordinate them with existing arithmetical knowledge.

Chapter 7

Methods

7.1 Introduction

This chapter details the methods used to trial the task, and collect and analyse data (Sections 7.2 and 7.3). Issues of validity, reliability and generality are detailed in Section 7.4.

7.2 Trial design

The trial design draws on two methods: paired triallings in a naturalistic setting, and the use of computational microworlds as research instruments. These methods are elaborated below.

7.2.1 Paired trialling

Fixed designs (Robson, 2002) that use quantitative methods such as tests, surveys and structured interviews, are not appropriate for trialling open-ended tasks. Instead, semi-structured settings are required in which pupils collaboratively investigate resources, and the researcher can ascertain the degree and quality of engagement with the intended domain of learning. Given the novelty of the design (Section 6.3), whole class and large group settings, which are complex systems and require multi-researcher iterative approaches to coordinate the many variables (Brown, 1992; Burkhardt & Schoenfeld, 2003; Somekh, 2006),

would not have been appropriate. As such paired trialling (Evens & Houssart, 2007) emerged as the default trial design. This is akin to paired, semi-structured interviews that aim to simulate discussion among participants.

Data is richest when pupils listen, explain and come to agreement when working on tasks (Mercer, Littleton, & Wegerif, 2004). However, the quality of pupils' talk is highly dependent on classroom culture and their previous educational experiences (Mercer & Littleton, 2007; Yackel & Cobb, 1996), and its development is beyond the scope of short trials. As such, a key role for myself as researcher was to prompt the pupils to explain their reasoning to one another (“*Why do you think that?*”, “*Can you explain what you did there?*” and so on). My other roles were to introduce the task resources, in this case the software and its functionality (Section 6.3), set the task goals, and offer reassurance and encouragement. A schedule for each trial was produced and an example can be seen in Appendix A.

A potential pitfall of paired trialling is that a shared computer results in one pupil operating the software, and the other advising, at any given moment. This can result in a emotionally different experience for each (Xolocotzin, 2008), and I ensured frequent turn-taking between the pupils to reduce this effect. Paired trialling can also prove more difficult to disentangle if the focus is on individual learners' cognitive processes. The analytic focus here, however, is on the content of learners' shared discussion (see Section 7.3).

7.2.2 Microworlds as research instruments

Computers, and microworlds in particular, can help make pupils' mathematical thinking visible, particularly when it makes implicit knowledge explicit (Noss & Hoyles, 1996). An example of this was provided by the trial of the *Visual Fractions* microworld (see Section 5.2.1) in which the pupils' implicit left-to-right reading of an equality statement came to the fore. Moreover, computers can also help perturb pupils' thinking in ways that causes them to shift their point of view, and makes deep thinking visible to the researcher. In the *Visual Fractions* trial, the pupils' implicit left-to-right reading was challenged by the emergence of an alternative “data-flow” view. Noss and Hoyles have referred to the computer as a “window on mathematical meanings” when used to observe and perturb pupils' thinking in this way.

In the three studies designed for this thesis (Section 6.4), pupils will encounter

something that is familiar to them from the everyday classroom: written arithmetic (albeit on a screen rather than on a page). However, the computer supports a functionality for these equality statements, namely making substitutions (Section 6.3), that can be expected to be novel for most British primary children. This is intended to perturb their thinking about equality statements in observable ways, namely articulating distinct transformational effects (“swap”, “separate” and so on — Section 6.2.2), and strategising within a relational system of notation (Section 6.2.3).

7.2.3 Data capture

Trials took place in primary schools with pairs of Year 5 (9 and 10 year old) pupils¹. Year 5 pupils were chosen for two reasons. First, there are no mandatory Standard Attainment Tests in England during Year 5, and consequently gaining access to Year 5 pupils during school hours is easier than for other years. Second, typical Year 5 pupils can be expected to use implicit arithmetic principles in the mental addition strategies and have usually been introduced to place-value partitioning statements (DCSF, 2007). Moreover, the Primary Framework for Literacy and Mathematics requires that Year 5 pupils

Represent a puzzle . . . by identifying and recording the information or calculations needed to solve it; find possible solutions and confirm them in the context of the problem (DCSF, 2007, p.5)

which fits neatly with the task design used in the studies (see Section 6.3).

For each trial I asked the class teacher to suggest a pair of pupils. I requested that both pupils be either strong or average in their level of mathematics ability as judged by the teacher. Lower ability pupils were not requested as they might struggle with components of the task such as reading notation and performing two-digit mental arithmetic (DCSF, 2007). I further requested they be friends, or at least comfortable with one another, as well as confident and talkative. This was to help ensure discussion during the trials. Prior to each trial, pupils returned a consent letter signed by their parents or carers. All

¹The exception to this is Study 2, Trial 2 (Section 9.4) in which Year 8 pupils (aged 12 and 13 years) were used in order to find out whether more mathematically advanced pupils would notice the presence of false equalities in the diagrams.

data was anonymized immediately following each trial, including providing a pseudonym for each participant.

The trials took place during pupils' normal mathematics lessons. For each trial a small room away from their normal classroom was required, and schools were selected mainly on their ability to provide a suitable space (many primary schools have limited physical space). Most of the trials took place in school libraries, music rooms and learning support rooms which were booked in advance by classroom teachers. These proved adequate for my purposes, although trials were not entirely without interruptions, as recorded in the transcripts.

For each trial a laptop computer was set up with the *Sum Puzzles* software (see Section 6.3). An external mouse was provided for pupils as the software is somewhat fiddly using the mouse-pad provided with the laptop. An external microphone was also used as the in-built laptop microphone produced inferior recordings. Pupils' on-screen interactions and discussion was recorded as screen-captured audiovisual movies using the recording software *Camtasia Studio*². Some pupils were interested in the microphone, and some a little self-conscious of being audio-recorded, but these issues were occasional and usually evaporated within the first five minutes.

7.3 Data analysis

The data produced during trials was audiovisual movies of pupils' talk and on-screen interactions (see previous section). There were four stages to analysing the data. First was *transcription*, which is itself an analytic act (Woods & Fassnacht, 2007a). Second was *description*, which involved constructing a chronological "trace" (Pratt, 1998) of events through each trial. Third was *coding*, in which articulations of statement form and transformational effects (see Section 6.2.2) were marked up. Fourth was *presentation*, in reports, journals, conferences and seminars, and which forms an important part of the analysis and review process (Yin, 2003). Each stage is elaborated below.

²*Camtasia Studio* is published by TechSmith and available from www.techsmith.com

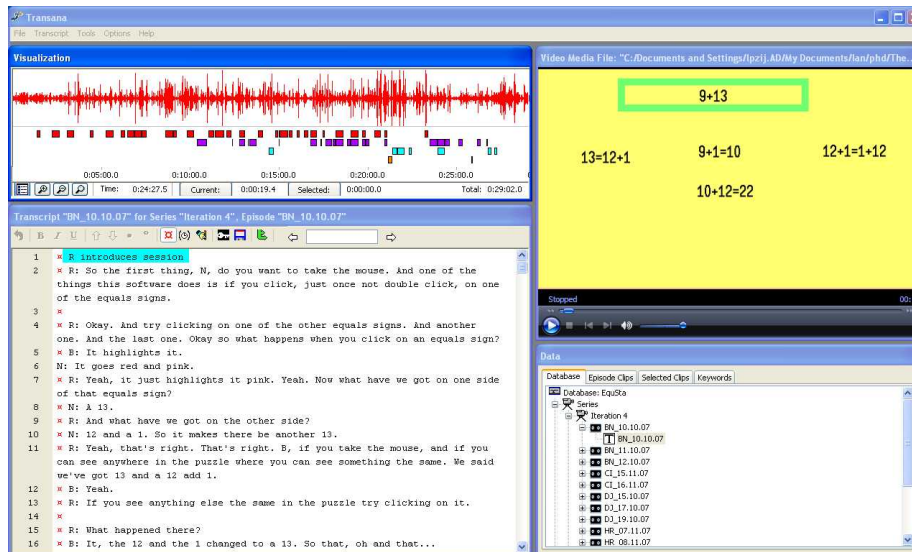


Figure 7.1: Time-coded transcription in *Transana*

7.3.1 Transcription

Transcription of the pupils' talk was undertaken using the qualitative software package *Transana* (Woods & Fassnacht, 2007b). *Transana*, which stands for *transcription-analysis*, is designed to incorporate the transcription process itself as a critical part of qualitative analysis. Transcribing familiarises the researcher with the whole data at a fine-grained level, and provides an initial sense of significant events and underlying themes. *Transana* supports time-coded transcription that can be displayed in a manner not unlike a karaoke machine (Figure 7.1). The text and audiovisual movie are dynamically linked (locating an event in the transcript loads the corresponding event in the movie, and *vice versa*) enabling the analyst to stay close to both the audiovisual and textual representations of the data. This also makes the data more accessible than transcripts alone, which are less rich than multimedia data, or movies alone, which can have sub-optimal sound quality and must be viewed in real-time. This accessibility aided the peer feedback and review that were invaluable to analysis and interpretation (see Section 7.3.4). An example transcript is provided in Appendix B.

7.3.2 Description

The second stage of analysis involved describing of the data to produce a “trace” of each trial. A trace is a text-based chronological description of key events that avoids judgments as far as possible. Traces are an evolutionary step from transcripts and comprise plain prose interspersed with transcript excerpts, and screen-shots where helpful. *Transana*’s time-linked transcripts and movies assist in this process by allowing the analyst to stay close to the raw data when constructing the trace. Traces attempt to be comprehensive, accounting for the trial from start to end, typically resulting in a document that is too long and raw for publication. The process is an important component of qualitative trial analysis, requiring the analyst to move between overviews and fine-grained views of the data. Despite care being taken to make traces as objective and fair as possible, they are in essence an act of abstraction and excerpt selection. The outcome can be shared with peers (my supervisor Dave Pratt in my case) and provide a structure for public presentation of the data (Section 7.3.4). An example trace is provided in Appendix C.

Over the duration of the three studies improved versions of *Transana* were released. For example, an improvement in *Transana*’s reporting facilities enabled the trace to be constructed more systematically and to be tied closer to the original data for Study 3. This involved annotating the transcript with text notes during the coding process (see next Section), and these text notes could be exported as a text document thereby providing a detailed foundation for the trace (see the example in Appendix C).

7.3.3 Coding

The third stage of analysis was to mark-up the data. In *Transana*, the time-codes linking transcripts and movies can be coded using specified keywords. This produces marked-up movie “clips” that can be used to navigate the data, enabling the analyst to stay close to the raw data.

Coding helps ensure the findings are tied to the hypotheses, thereby improving the validity of the study (see Section 7.4.1). The keywords relate to pupils’ articulation of commutation, partition, iconic matching (see Table 6.1), and mental calculations. Note that by coding for articulation, the focus is on the pupils’ talk, not their on-screen interactions. However, on-screen interactions were es-

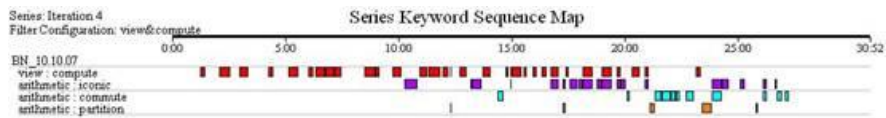


Figure 7.2: Time-sequenced coding graphic in *Transana*

essential to interpreting the talk, particularly when determining which statement the pupils’ were discussing, and *Transana*’s time-links between transcripts and movies aided this process greatly. Articulations of observed or suggested commutative and partitional transformations correlated to specific words (“swap”, “split”, and so on) and were straightforward to code. Similarly, articulated mental calculations were readily identifiable (“What’s 13 plus 18? ... 20 ... 31.” and so on). Iconic matching was slightly less clear to identify in the pupils’ talk, and was restricted to observations or suggestions to look for a second occurrence of a given numeral or term (“31 ... look for a 31 somewhere”, “there isn’t any 53 anywhere is there, so we need to change it” and so on).

Clips can also be displayed as time-sequenced graphics, annotated reports including transcript excerpts, and edited movie sequences. The time-sequenced graphics provide a visual overview of the amount, density and sequencing of keywords over the duration of a trial, allowing pattern-matching by eye (an example from the first 30 minutes of a trial reported in Study 3 is provided in Figure 7.2). Each block shows an occurrence of the pupils computing results, looking for matches of numerals across statements and the term in the box, or articulating the commuting (“swapping”) or partitioning (“splitting”) transformational effects of $a + b = b + a$ and $c = a + b$ (or $a + b = c$) forms respectively. The length of each block is somewhat arbitrary. For example, one block of (say) “commute” might reflect pupils working in a trial-and-error manner with one of them suggesting they “swap” numerals, but offering no reason. Another block of similar length might reflect pupils discussing which numerals to commute, and how and why, as part of a shared strategy. As such, Figure 2 provides a useful visual aid for summarising the trials but does not convey the quality, or the precise quantity, of the pupils’ articulations and strategising. White spaces indicate times when pupils’ discussion was ambiguous and in the main reflect periods of trial-and-error (“Click that one”, “Let’s try this one, no, that one” and so on).

The coding process seeks to be as objective as possible, but subjectivity does

creep in. First, the time-codes between transcript and movies are established manually during the transcription process. This requires decisions to be made about where to place them. For discursive data, Woods and Fassnacht (2007a) suggest inserting a time-code every time a participant speaks (as opposed to placing time-codes at regular intervals of, say, five seconds). As such, movie segments between consecutive time-codes can vary between less than a second and several minutes. These movie segments are the basic unit that can be coded, and the variable lengths of “clips” (coded movie segments) arise from the decisions made during transcription rather than reflecting the “amount” the pupils talked about the keyword. Moreover, I often coded a keyword across several time-codes in order to maintain the context of a given articulation, resulting in clips that were much longer than the specific event coded. Ideally, two or more analysts would code the data independently, but the financial constraints of a doctoral study forbade this. Instead, I had to settle to subjecting the data to peer feedback and review (Section 7.4.5).

7.3.4 Presentation

The final stage of the analysis process was presenting the data in public forums such as publications³ and conference and seminar presentations⁴. Presentation requires the researcher to rethink carefully about the data and findings, and generates informal feedback and formal review from peers (Yin, 2003).

Written reports introduce the data with a qualitative and quantitative overview, derived from the trace and coding. The bulk of a presentation comprises a discussion of key events of relevance to the research questions, exemplified by transcript excerpts and, where helpful, screen-shots. The discussion seeks to describe the data as objectively as possible, but also provides interpretation by way of explaining why and how the event is of significance, and whether it is representative or counter-representative. Conference presentations and seminars follow a similar format, but are usually shorter. The “discussion” involves presenting “subtitled” movie clips, as afforded by *Transana*’s time-codes, with verbal explanations. Feedback comes in the form of a question and answer session immediately following the presentation, and any subsequent informal,

³Publications included peer-reviewed journals (Jones & Pratt, 2006; Jones, 2008a, 2008c), conference proceedings (Jones, in press, 2007a, 2007b; Jones & Pratt, 2005), a teacher journal (Jones, 2006b, 2008d), a student journal (Jones, 2006a, 2007c, 2008b)

⁴A list is available at go.warwick.ac.uk/ep-edrfae/conferences.

opportunistic chat with peers. In some cases presentations are streamed or recorded⁵.

7.4 Scope and limitations

An important component of empirical theory generation and testing is to specify the context and domain of applicability (Burkhardt, 2006). This requires stating the validity, reliability and generality of the findings, and addressing anticipated criticisms of the methods and interpretations (Robson, 2002). Yin's (2003) acclaimed book on case study research described analytical constructs that apply to qualitative studies more generally, and these will be drawn upon here.

7.4.1 Construct validity

Construct validity is concerned with specifying the evidence that informs the research questions (Table 7.1).

Construct validity can be increased by establishing “a chain of evidence”, so that the final conclusions can be derived stepwise from the initial research questions. Study 3 closely reflects the original intent of the research (Section 5.3.2), in which pupils were envisaged expressing the implicit transformations in their calculation strategies as a sequence of equality statements. Studies 1 and 2 provide cumulative steps toward these final findings by testing components of the task (the presence of exchanging and absence of sameness views of the equals sign).

Construct validity can be further enhanced by triangulation across multiple sources, or types, of evidence. This thesis reports from only one source, paired trials, and does not include, for example, observations of teachers using the task in classrooms. This positions the thesis at the “enactment phase” (Bannan-Ritland, 2003, p.23) or “learning level” (Burkhardt, 2006, p.141) of task development. Post-doctoral work will involve the development and trialling of classroom-based materials for use by teachers (see Section 12.3.1).

⁵A video of a short presentation at the University of Nottingham (February 2008) is available at <http://www.lsri.nottingham.ac.uk/phdtalks/>

Research question	Evidence sought
1 Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?	Pupils using unprompted natural language terms (“swap”, “switch round”) when talking about the exchanging effects of $a + b = b + a$ statements; and (“split”, “separate”) when talking the effects of $c = a + b$ (or $a + b = c$) statements.
2 Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct?	Pupils’ responses (or lack of) to the presence of false equalities when solving presented diagrams, and the impact (if any) of false equalities on how they talk about and work with diagrams.
3 Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?	Pupils constructing functional, complex diagrams composed of statements that are numerically balanced and iconically matched.
4 Can children connect their implicit arithmetical knowledge with explicit transformations of notation?	Pupils constructing diagrams that reflect their spoken mental calculation strategies.

Table 7.1: Research questions and evidence sought

7.4.2 Internal validity

Internal validity is concerned with analysing and explaining the data. It requires *pattern matching*: identifying causal links between independent and dependent variables. Across the three studies evidence will be sought for pupils articulating the distinct transformational effects of presented statement forms, and for using these distinctions when thinking strategically about diagram-solving. This level of pattern matching supports internal validity if there is evidence of replication across the trials. However, it is limited as it is not concerned with explanation (such as claiming a pupil possesses a conception of commutation), or with causal sequences of events. As such, a process of *explanation building* that draws on sequential, cumulative evidence across trials is required. Within a trial, explanation building involves describing the “micro-evolution” (Pratt & Noss, 2002) of pupils’ thinking. This might involve charting articulations of (say) partitioning effects, and how such articulations become part of diagram-solving and diagram-making strategies (or not). Across trials, patterns and differences can be identified in micro-evolutions. In this thesis, I compare pupils who successfully make their own diagrams with those who fail, and look at the role articulating partition played across the two trials (see Section 11.4.2).

7.4.3 Generality

Generality, or external validity, relates to the applicability of the findings beyond the confines of the thesis itself. In quantitative studies generalisation is to a statistical population, but in qualitative studies generalisation is to a broader theory. Here, the broader theory concerns predictions about the ways typical school children view and talk about arithmetical notation (Chapter 2). An alternative approach has been set out (Chapter 6) and theoretical predictions made that this will result in children viewing and talking about notation in different ways to those predicted by the existing literature. Such evidence would support the *analytic generality* of the thesis’s findings.

Analytic generality regarding the task design principles set in Chapter 6 can be considered along two dimensions. The first is the generality of the diagrammatic approach to areas of the mathematics curriculum other than simple arithmetic, such as fractions and simple algebra (see Section 12.3.3). The second is the generality of the task design to other media (flashcards, worksheets) and to real

educational settings (typical classrooms) (see Section 12.3.4). These are in fact theoretical hypotheses for further work (see Section 12.3).

7.4.4 Reliability

Reliability relates to the replicability of the studies. Replicability can be seen across the reported trials if pupils talk about and interact with the task resources in similar ways. This suggests the same findings would emerge if another researcher were to follow the same procedures. Furthermore, by providing comprehensive details of the trial designs, and on-line access to the *Sum Puzzles* software⁶, it is perfectly possible for a researcher to repeat the studies.

7.4.5 Rival explanations

Rival explanations for the data produced by a study must be considered during analysis and reporting. There are three key sources for rival explanations: the research literature; formal peer review; and informal peer feedback. This thesis has made use of all three sources, and in particular has received a large amount of feedback and review from peers. One source for feedback has been presentations at conferences and seminars. Presentations enable formal feedback in the form of question and answer sessions, and informal feedback in the form of later conversations. Publication in peer-reviewed journals and conference proceedings provided formal, written reviews, and in some case necessitated substantial reflection and revision on my part. Both peer-reviewed and non-reviewed publications involved informal review and feedback from peers.

Another important source of feedback were consultations with relevant experts, mainly and regularly with my supervisor Dave Pratt. I also arranged consultations with experts (these consultations are summarised in Table 7.2). These consultations typically lasted an hour and involved discussing my ideas in reference to the software and audiovisual data. This feedback brought a range of interpretations of the approach and emerging findings to my attention. Some of these experts read and responded to (usually early drafts of) my publications, improving the comprehensiveness of the literature review (Chapter 2), highlighting the limitations of the preliminary work and design ideas (Chap-

⁶The *Sum Puzzles* software is available at go.warwick.ac.uk/ep-edrfae/software.

ter 5), refining the approach taken (Chapter 6), and improving the validity and reliability of the studies and findings reported in the remainder of the thesis.

7.5 Summary

The studies seek to trial the task design in semi-structured settings intended to stimulate discussion among participants. Paired trials are used and the researcher remains present to prompt pupils to explain their reasoning. Microworlds provide insights into conceptual thinking and provide rich audiovisual data in the form of screen-capture movies.

The analysis seeks evidence for pupils talking about notation in ways not predicted by the literature. There are four stages to the analysis. The first is transcription, which familiarises the researcher with the data at a fine-grained level. *Transana* is ideal for this purpose as its time-codes enables switching between text and audiovisual representations. The second is description, in which a detailed overview, or trace, of each trial is produced. This requires moving between fine- and large-grained views of the data. The third is coding, in which research questions are matched to specific events in the data. This allows quantitative overviews and, due to *Transana's* graphical functionality, pattern-matching by eye.

The validity of the studies rests on specifying the evidence sought in terms of the research questions, and seeking chains of evidence. Explanations of events within and across trials can then be built, resulting in causal claims for the data. The generality of the findings relates not to a wider population of pupils but to published research on how pupils view equality statements. Theoretical claims for how the approach applies to mathematical domains other than arithmetical statements, and media and contexts other than paired trialings of microworlds, can also be drawn, and provide hypotheses for further research. Replicability is demonstrated across trials if pupils talk and interact with the resources in similar ways, and sufficient detail and transparency must be provided to ensure other researchers can replicate the data and analysis. Consideration of rival explanations enhances the validity and robustness of a study. Substantial and continual peer review and feedback has ensured a variety of expert viewpoints has informed every stage of the thesis.

Daniel Pead, August 2008. Pead provided feedback on technical improvements (usability, aesthetics, portability, data logs) for redesigning the *Sum Puzzles* software for use in typical classrooms. His suggestions will inform the design phase of a post-doctoral project proposal.

Hugh Burkhardt and Malcolm Swan, February 2008. This consultation was stimulated by excerpts from the audiovisual data. Swan expressed enthusiasm at the quality of discussion during trials. Burkhardt stated the need to explore how teachers might use it in the classroom, and this has informed the post-doctoral project proposal.

Richard Noss and Ivan Kalas, August 2007. This took place at *EuroLogo2007*, in Bratislava. Noss expressed some concern at the closed nature of the task. Noss and Kalas, both proficient Logo programmers, provided feedback on interface issues.

Dave Hewitt, March 2007. Discussed possible learning tasks based around the *Sum Puzzles* microworld. This followed on from Hewitt writing a “reaction” (Hewitt, 2006) to a paper I published in a student journal (Jones, 2006a) that reported the *Equivalence Calculator* trials (see Section 5.2.2). Hewitt’s feedback also informed Section 5.3 of the thesis.

Willi Dörfler, February 2007. This took place at the Fifth Congress of the European Society for Research in Mathematics Education (CERME-5) in Cyprus, at which Dörfler presented a paper (Dörfler, 2007a) that introduced me to “diagrammatic reasoning” as a framework for notating task design. Dörfler agreed my approach provided an example of the diagrammatic approach for the case of arithmetic equality statements. Consequently, Dörfler wrote a “reaction” (Dörfler, 2007b) to a student journal paper (Jones, 2007c) in which I first set out a diagrammatic approach to presenting statements to pupils, which formed the basis of Chapter 6 of the thesis.

Marta Molina, February 2007. This also took place at CERME-5. Molina provided advice on the classes of statement structures that can be used within *Sum Puzzles* to appeal to structural readings.

Janet Ainley, February 2007. Ainley expressed concern that the intended role of partition and commutation when working through *Sum Puzzles* tasks might not harness but “undo” pupils’ implicit knowledge of arithmetic principles.

David Tall, February 2007. Tall expressed concern that only “proceptual” thinkers would learn from the *Sum Puzzles* tasks, and other pupils might have only a superficial understanding. Tall expressed these concerns further in a “reaction” (Tall, 2008b) to a critique of the literature on pupils’ conceptions of the equals sign that I published in a student journal (Jones, 2008b). Tall’s feedback informed Section 3.3 of the thesis.

Table 7.2: Overview of face-to-face expert consultations.

Chapter 8

Study 1: Attention to statement form

8.1 Introduction

This study investigated the first research question:

Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?

The evidence sought was pupils using unprompted terms (“swap”, “switch round”) when talking about the exchanging effects of $a + b = b + a$ statements; and (“split”, “separate”) when talking about the effects of $c = a + b$ (or $a + b = c$) statements. As will be seen, the study also evidences strategic thinking, and suggests an exclusivity of “can be exchanged for” and “is the same as” views of the equals sign.

Prior to conducting the study my concern was for the overall viability of the *Sum Puzzles* software and task design. To this end, two opportunistic pilot trials with individual children were conducted and are reported in the next section¹. The remainder of the chapter reports on two formal trials with pairs of children.

¹for a full report of the pilot trials see Jones (2007c)

8.2 Pilot trials

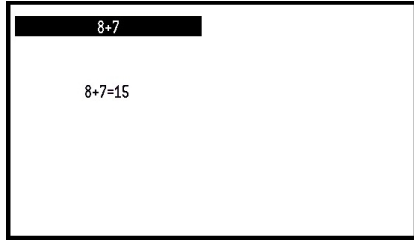
The trials reported here are atypical of the thesis because they involved only single children. These pilots were a very first trial of the software after its development and my mind was as much on technical concerns (crashes, bugs, usability and so on) as on educational issues. However, these trials provided a crucial feasibility study prior to trialling proper with pairs of children (see next section). The children involved in the trials were both girls, one aged nine years the other aged twelve years. The difference in ages reflects the opportunistic nature of the trials (both girls are known to me and the trials took place during the school breaks) and, while not ideal, was considered insignificant in light of the anticipated functional and usability difficulties. Happily, no such difficulties arose and some unexpectedly interesting data emerged instead. Although these data are limited in size and scope they support the feasibility of the design and are summarised here.

In its early version the *Sum Puzzles* software was set up to present a sequence of 13 diagrams (Figure 8.1). These diagrams contain between one and nine statements, including commutative, partitional and compositional forms, some of which appeal to tens-and-units readings, and one compensatory statement ($45 + 9 = 50 + 4$ in Figure 8.11). These diagrams were not designed with trialling in mind but were a result of testing the software and demonstrating it to peers, and reflect the opportunistic nature of the pilot trials.

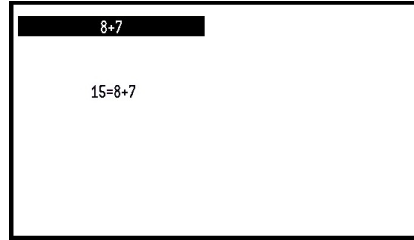
The trial with the nine year old girl, Penny, involved only five of the diagrams (Figures 8.1d-h) and lasted just ten minutes. I recorded the trial as field notes of her activity as she worked. The trial with the twelve year old girl, Lauren, involved all 13 diagrams and lasted 40 minutes. It was recorded as an audiovisual movie using screen-capture software and a microphone, which provided an opportunity to test the data capture methods used in the main studies. In each case I demonstrated the functionality of the software, asked them to “solve the puzzles” and remained present to offer guidance and encouragement.

Penny (nine years old) initially adopted a trial and error approach, selecting arbitrary statements and clicking numerals in the black box to see if anything changed. This strategy worked well for the first three diagrams attempted (Figures 8.1d-f). However her trial and error approach became guided by iconic matching² during the fourth diagram (Figure 8.1g). Penny initially selected

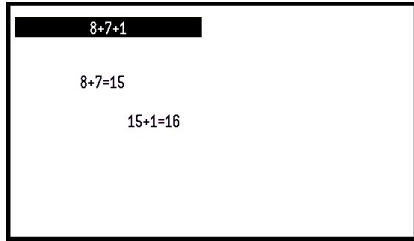
²Iconic matching is searching a diagram for multiple occurrences of a given inscription (see



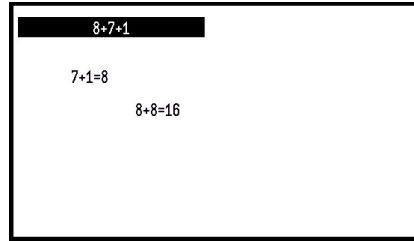
(a) Pilot trial: Diagram 1



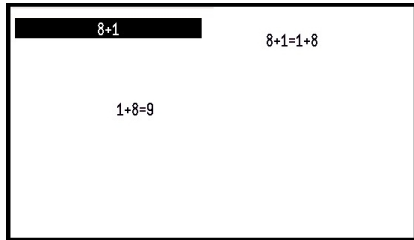
(b) Pilot trial: Diagram 2



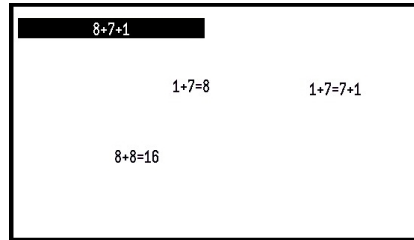
(c) Pilot trial: Diagram 3



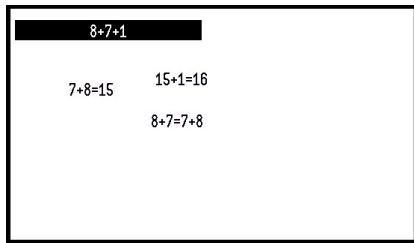
(d) Pilot trial: Diagram 4



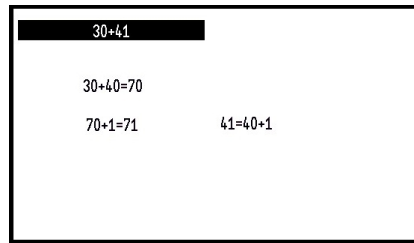
(e) Pilot trial: Diagram 5



(f) Pilot trial: Diagram 6

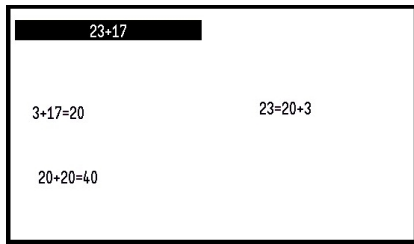


(g) Pilot trial: Diagram 7

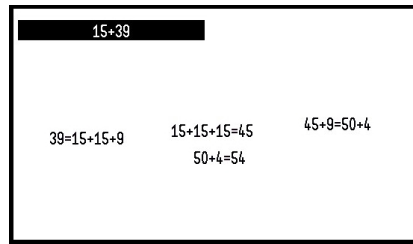


(h) Pilot trial: Diagram 8

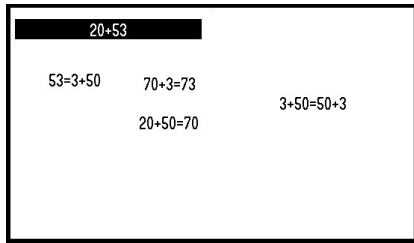
Figure 8.1: Diagrams presented in the pilot trials



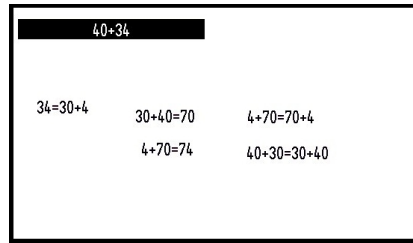
(i) Pilot trial: Diagram 9



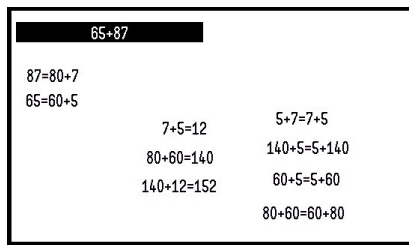
(j) Pilot trial: Diagram 10



(k) Pilot trial: Diagram 11



(l) Pilot trial: Diagram 12



(m) Pilot trial: Diagram 13

Figure 8.1: Diagrams presented in the pilot trials

the statement $7 + 8 = 15$ and attempted to substitute the term $8 + 7$ in the box. She tried clicking the term several times apparently expecting that a substitution should be possible. When I asked why she thought it was not working she answered that the 7 and the 8 are the wrong way round. She turned her attention to the statement $8 + 7 = 7 + 8$, suggesting she understood its commutative transformational potential, and then completed the diagram with no further difficulty. A second incident of interest occurred during the fifth and final diagram that Penny attempted (Figure 8.1h). After a few experimental substitutions the box had become $30 + 1 + 40$ and Penny selected the statement $31 = 1 + 30$. She clicked several times on the term $30 + 1$ and, when no substitution occurred, paused. I asked why she thought it was not working and expected her to notice the commutation of $30 + 1$ and $1 + 30$ as she had with $7 + 8$ and $8 + 7$ in the previous diagram. Instead, however, her explanation was that $31 = 1 + 30$ did not allow a substitution on $30 + 1$ because “it means 31 becomes 1 add 30, not 1 add 30 becomes 31”. This suggests Penny viewed the statement as a decomposition into constituent parts, as hypothesised by the diagrammatic approach (Section 6.2.2), rather than as a “backwards sum” as would be expected from the literature (Chapter 2). After Penny had completed the diagram shown in Figure 8.1j she asked of the equality statements “Is it like a sequence?”, suggesting an emergence of strategic thinking about the diagrams, as hypothesised by the diagrammatic approach (Section 6.2.3).

Lauren (twelve years old) also began with a trial and error approach that proved increasingly inefficient by the third diagram (Figure 8.1c) when iconic matching began to guide her experimentation. By the final few diagrams, which are complicated and contain up to nine statements (Figures 8.1j-m), iconic matching dominated Lauren’s approach. The most notable feature of Lauren’s activities was her use of a strategy from diagram six onwards (Figure 8.1f). This involved starting with the answer, wherever it appeared on-screen in an equality statement, and transforming that statement such that the left-hand side matched the inscriptions in box. To illustrate this strategy, consider the diagram shown in Figure 8.1j. Lauren initially performed a substitution on the compensation relation $45 + 9 = 50 + 4$ using $50 + 4 = 54$, transforming the diagram into that shown in Figure 8.2.

Lauren then looked for iconic matches with the left-hand side of $45 + 9 = 54$ and made substitutions until she had transformed it into $15 + 39 = 54$. She then

Table 6.1).

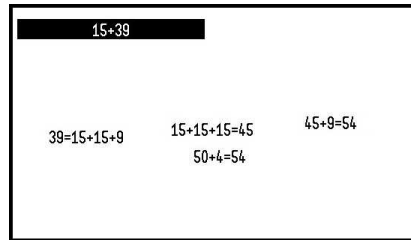


Figure 8.2: Lauren uses to $50 + 4 = 54$ transform $45 + 9 = 50 + 4 \rightarrow 50 + 4 = 54$

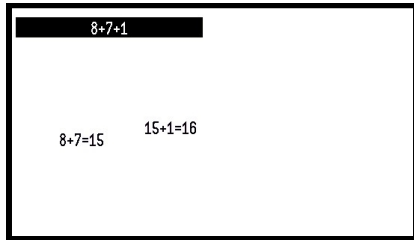
clicked on the inscription in the box and the numeral 54 appeared as required. This same strategy was used for the remaining three diagrams (Figures 8.1j-m). In sum, the pilot trials demonstrated that the software was robust and usable. The task made sense to the girls and they got to grips with it readily. The trial data suggested that the interactive diagrams might support attention to form and the use of strategic thinking.

8.3 Main study

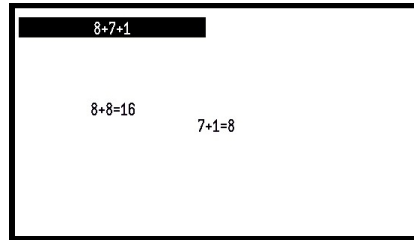
The main study reports two trials with pairs of pupils aged 9 and 10 years. The task set up was similar to that in the pilot trials, with modified diagrams and two additional task goals. I describe the task and provide an overview of each trial as well as illustrative transcript excerpts that inform the research question “Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?” The evidence sought is pupils using unprompted natural language terms (“swap”, “switch round”) when talking about the exchanging effects of $a + b = b + a$ statements, and (“split”, “separate”) when talking about the effects of $c = a + b$ (or $a + b = c$) statements (see Table 7.1).

8.3.1 Task set-up

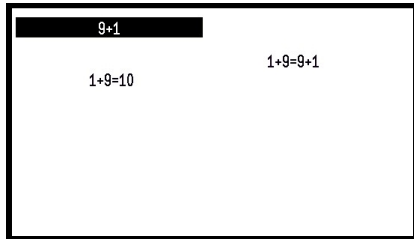
Each trial involved a pair of children working through a sequence of 11 prepared diagrams (Figure 8.3) and lasted about 40 minutes. The early version of *Sum Puzzles* (see Section 6.3) contained a feature that enables the original contents of the black box to be viewed by hovering over an equals sign to the right of the black box. This feature was dropped in the later version used in Studies 2 and



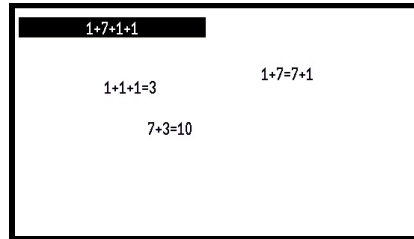
(a) Diagram 1



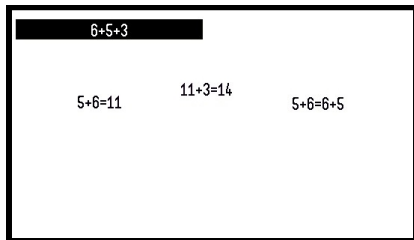
(b) Diagram 2



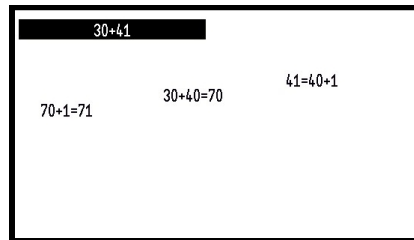
(c) Diagram 3



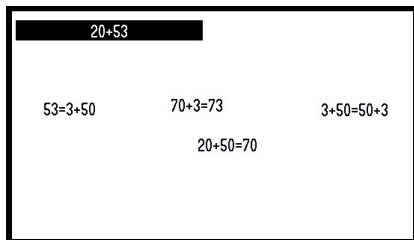
(d) Diagram 4



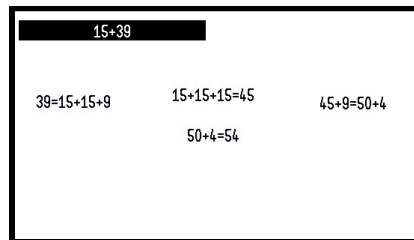
(e) Diagram 5



(f) Diagram 6

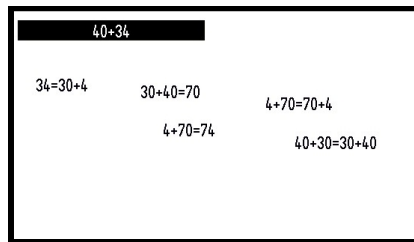


(g) Diagram 7

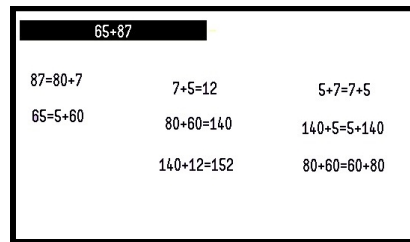


(h) Diagram 8

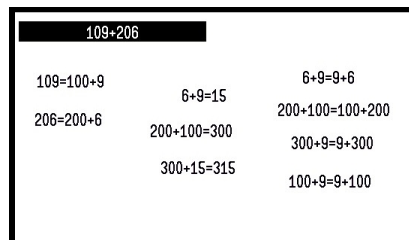
Figure 8.3: Diagrams presented in the trials



(i) Diagram 9



(j) Diagram 10



(k) Diagram 11

Figure 8.3: Diagrams presented in the trials

3 as it caused slight confusion when the children inadvertently rolled the mouse over it.

I first showed them how to operate the software, in particular how to select a statement by clicking on the equals sign and how to make substitutions by clicking on numerals, and allowed them to experiment with it for several minutes. For each diagram I challenged the children to make the “answer” to the “sum” appear in the black box (Challenge 1). Twice in each trial I then set another challenge to get the original sum back again (Challenge 2). Challenge 2 was intended to emphasise reversibility but I dropped it soon into each trial as it proved ineffectual for generating eventful data. On other occasions when the data became uneventful I set a more difficult challenge: make it so that the term in the black box can be changed into its answer with a single click (Challenge 3). The distinction between Challenge 1 and Challenge 3 can be illustrated for the case of the tenth diagram (Figure 8.3j): Challenge 1 would be to transform $65 + 87$ in the black box into 152 ; Challenge 3 would be to transform the statement $140 + 12 = 152$ into $65 + 87 = 152$, and then use it to transform the term in the black box into 152 in a single click. (Challenge 3 is in effect to use the strategy Lauren used in the pilot trials – see Section 8.2) Neither pair grasped Challenge 3 on their first attempt. I did not set the pupils in Trial

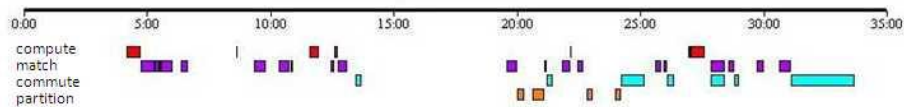


Figure 8.4: Time-sequenced coding for the Terry and Arthur trial showing articulations of computation, iconic matching, commutation and partition

2 Challenge 3 again as Challenge 1 proved sufficient to generate relevant data for the duration of the trial. The pupils in Trial 1, who worked through the diagrams more quickly, were set Challenge 3 a second time and were able to complete it.

8.3.2 Trial 1: Terry and Arthur

Terry (10 years 4 months) and Arthur (9 years 9 months) were deemed by their class teacher to be above average in mathematical ability. Both were articulate and Terry was the most talkative. They got to grips with the functionality of the software and the task goal quickly and unproblematically, completing all 11 diagrams (Figure 8.3) during the trial. Their level of engagement was high throughout the trial, and characterised by cycles of (amused) frustration and triumph. The pupils reflected on this at the end of the trial:

ARTHUR: It's quite challenging ...

TERRY: Yeah ...

ARTHUR: It's kind of addictive in a way. You don't want to stop.

TERRY: Yeah ...

ARTHUR: Kind of like, really determined.

TERRY: Bit fiddly. And it can sometimes get really annoying but it's still fun.

Figure 8.4 shows a time-sequenced map (in minutes) of codings across the trial and was produced using *Transana*³. The first thing to note is how little the pupils computed results across the trials (Terry and Arthur in fact discussed computational results much less than the pupils in the trials reported in the

³Data capture and analysis was described in Chapter 7, and time-sequenced maps were explained in Section 7.3.3

Time	Code	Pupil	Transcript
13:27	Com	Terry	that's the one that you need to use to <i>change it round</i>
20:01	Part	Terry	That's the one that you do first! It has to be ... Because it's <i>splitting up</i> the 40 and the 1
20:39	Part	Terry	Wanna <i>split</i> the 50 ... the 30 ... 3 and the 50 up
21:13	Com	Both	it <i>changed around</i> / Just <i>changed it round</i>
22:50	Part	Both	Which one's most <i>split up</i> ? That one / Forty ... 34 add ... <i>splits it up</i> there
24:00	Part	Terry	65 add 87, any <i>spli...</i> 8 ... Yeah. That one. That <i>splits it up</i>
24:15	Com	Terry	Oh you need to move 'em, you need to <i>swap 'em round</i> ... No. Those need to be <i>swapped round</i> as well ... <i>Swap round!</i>
26:06	Com	Terry	That <i>swapped it round</i> ... There
27:50	Com	Terry	I'm thinking. 140 add 5, if you <i>swap those over</i> somehow
28:47	Com	Terry	That should <i>swap these round</i>
31:13	Com	Terry	Ah but we want it <i>swapped round</i> so use that one. I think. [at computer] <i>Swap 'em round! Swap 'em round!</i>
31:37	Com	Terry	if those two could get <i>swapped round</i> somehow we could put them together
32:03	Com	Terry	That needs to be <i>swapped round</i> . Maybe. Can I <i>swap that?</i> ... 87, [at computer] <i>swap it round!</i>
32:37	Com	Terry	I wonder if we can <i>swap it round</i> now? Will these two <i>swap</i> ? No!
33:07	Com	Terry	Oh they need to be <i>swapped round</i> . And ... Yes

Table 8.1: Occurrences of “commutation” and “partition” in the Terry and Arthur trial

remainder of this thesis – see Figure 11.2, page 170 for a time-sequenced overview of all trials reported in the thesis). Conversely, they did engage in looking for matching numerals, as well as articulating the commuting properties of $a + b = b + a$ statements and, to a lesser extent, the partitioning properties of $c = a + b$ statements.

Figure 8.4 also provides a visual overview of the sequencing of the pupils' articulations. Iconic matching dominated the first half of the trial, and remained present throughout, reflecting their discussion about which substitutions could be made at any given moment. The second half of the trial also contained substantial articulation of observed or predicted commuting effects of notational transformations. The verbs “swap” (and derivatives) was used 12 times by Terry and not at all by Arthur to express commuting effects. Other terms used were “change round” (Terry twice, Arthur once) and “switch round” (Terry once). The term “split” (and derivatives) was exclusively used to express partitioning effects (in total, four times by Terry and once by Arthur). A list of all occurrences of the codes “commute” and “partition”, in the order they occurred throughout the duration of the trial, are listed in Table 8.1.

The pupils articulations of commuting and partitioning effects were increasingly strategic over the duration of the trial. This involved identifying the need to commute numerals in order to make further substitutions toward the task goal (e.g. the extract at 24:15 in Table 8.1), and articulating partitioning transformations when starting a new diagram (e.g. at 20:01 in Table 8.1). After a little practice, they would generally begin a new diagram by identifying partitioning statements, then using commuting statements to shunt the numerals in order to compose them.

Terry and Arthur: Illustrative transcript excerpts

In this section I present illustrative transcript excerpts from the trial with Terry and Arthur in order to describe how iconic matching, partitioning and commuting emerged and were used strategically to meet the task goals over the duration of the trial.

Initially, Terry and Arthur focused on computational readings of presented equality statements. For example, about four minutes into the trial during the second diagram, when discussing which statement to select, they articulated

computing results (lines 3, 5, 7)⁴.

TERRY: I think it might be this one. Yeah ... 1
ARTHUR: Yeah ... 2
TERRY: 'cause 7 add 1 equals 8, 8 add 8 is 16. There ... then if you click 3
on ... click on the equals ... 4
ARTHUR: 7 add 1 is ... 5
TERRY: Yeah. Then, like ... 6
ARTHUR: Yeah, 'cause 7 plus 7 is 14 ... isn't it? (04:10) 7

Immediately following this, they began to talk in terms of the visual appearance of numerals. Arthur commented on the transformation $8 + 7 + 1 \rightarrow 8 + 8$ in terms of its visual size (line 8), and deemed this as a step toward the solution (line 10). Terry followed this by talking in terms of the numerals on the screen rather than their computational results (lines 12-14). This is evidence of a shift to talking about numerals (rather than “numbers”), which characterised much of the trial.

ARTHUR: Yeah we're getting smaller. 8
TERRY: 8 add 7 add 1. 8 add 8. . 9
ARTHUR: No we're getting closer go to 8 add 8. We're getting closer ... We 10
just need to get the answer up on that. Give it [*the mouse*] here. 11
TERRY: What did it start as again? 8 add 7 add 1. Click on the numbers, 12
Arthur. I mean ... there. See what that [$7+1=8$] will do. Yeah! 13
Oh ... 7 add 1 add 8. (04:50) 14

Although Terry was the most talkative there is evidence of Arthur looking for iconic matching too. During the next diagram (Figure 8.3c), Terry attempted to transform $9 + 1 \rightarrow 10$ using $1 + 9 = 10$ and I asked why he thought it did not work. Arthur answered in terms of the iconic mismatch of $1 + 9$ with $1 + 9$ (lines 19-20)⁵. This enabled them to complete the diagram quickly (lines 22-24). (“R” is for researcher).

⁴Transcript excerpts include the approximate time into the trial they began in minutes and seconds.

⁵Note that while the resultant transformation was $1 + 9 \rightarrow 9 + 1$ (line 21) the pupils did not articulate this explicitly and so it was not coded as “commute”.

R: Why do you think that wasn't working? 15

TERRY: [*selects* $1 + 9 = 9 + 1$] Maybe because ... 1 and 9 is... 16

ARTHUR: Oh, because it hasn't got that sum $[9 + 1]$ in it. 17

R: What do you mean? 18

ARTHUR: Well 'cause that $[1 + 9 = 10]$ has got 1 add 9 but then the end of 19
that $[1 + 9 = 9 + 1]$ has got 9 add 1. 20

TERRY: [*transforms* $1 + 9 \rightarrow 9 + 1$] There. 1 add 9. 21

ARTHUR: Now try that one Terry, and ... 22

TERRY: Yeah. 23

ARTHUR: Yes! (09:20) 24

Throughout the trial they used a mixture of trial-and-error and iconic matching to identify and select statements. This is exemplified in the following excerpt:

ARTHUR: Oh it's changed it to that $[8 + 8 = 16 \rightarrow 7 + 1 + 8 = 16]$... No go 25
back to the, as it ... 26

TERRY: 7 add 1 add ... Okay! Erm. 27

ARTHUR: Wait a sec, keep it on that [*i.e. select* $7 + 1 + 7 + 1 = 16$]. 28

TERRY: Er ... Let me ... Let's try it with this one $[7 + 1 = 8]$ now. Select 29
that $[7 + 1 = 8]$ one. Ah, it still won't give us a sign ... No. Grrrr! 30

ARTHUR: I think we've got to try and ... 31

TERRY: Oh look, I think can ... 'cause if you click on the 7 ... 7 add 1 32
and if you click on that again. Click on the 1 ... oh no it didn't 33
work. (05:30) 34

The sixth diagram (Figure 8.3f) is the first to contain a partitional statement. Terry began by reading aloud the contents of the box ($30 + 41$) and one of the statements ($30 + 40 = 70$) (line 35), and then looked for allowable substitutions, and identified the statement that comes last in the sequence (lines 38-41). Following this he identified the first statement in the sequence ($41 = 40 + 1$) on the basis of its potential partitioning effect (line 50), and then quickly completed the diagram (line 53). This provides the first case of "partition" in the trial (see Figure 8.4 at about 20 minutes), and, interestingly, Terry *inferred* the

statement's partitional transformational effect before actually observing it on the screen (line 50):

TERRY: Er ... 30 add 41. 30 add 40 equals 70 ... Nah, maybe it's that 35
one $[30 + 40 = 70]$. 36

ARTHUR: Nah, it's that one $[70 + 1 = 71]$. 37

TERRY: I think it's, maybe ... that one $[30 + 40 = 70]$ looks like the closest 38
to it. That one $[30 + 40 = 70]$ goes with that one $[41 = 40 + 1]$, 39
because that one's $[70 + 1 = 71]$ the sum you do at the end isn't 40
it? 41

R: Which one's the sum you do at the end? 42

ARTHUR: That one $[70 + 1 = 71]$. 43

TERRY: Because 70 plus 1 because you get 70 ... that must be the same. 44

R: How do you know that's the one you do at the end? 45

ARTHUR: Well it's already got half the answer ... Because 30 add 40 46

R: Oh I see, yeah. 47

TERRY: Oh! That's $[41 = 40 + 1]$ the one that you do first! It has to be. 48

R: Why? 49

TERRY: Because it's splitting up the 40 and the 1. Try that. Yeah. 50

ARTHUR: Yes. 51

TERRY: Okay, er, yeah. I'd say that one now $[30 + 40 = 70]$. Yep. And we 52
got it. (19:18) 53

The pupils quickly worked through diagrams 7, 8 and 9 in this way. Diagram 10 contains eight statements (Figure 8.3j) and is significantly more complicated than the previous diagrams. Terry confidently started with partitional statements (lines 54-56), and completed the diagram quickly. Note he also spoke in terms of transformations he wanted to make a step ahead, identifying the need to commute numerals in order to make a subsequent substitutions. This suggests the emergence of a more strategic, less experimental approach to diagram-solving.

TERRY: Okay! 65 add 87, any spli ... 8 ... Yeah. That one. That splits 54
it up. [*highlights* $87 = 80 + 7$ and *clicks* 87 in $65 + 87$; *highlights* $65 = 5 + 60$ 55
and *clicks* 65 in $65 + 80 + 7$] Erm ... 50 ... 5 add 60 ... 80. 56

ARTHUR: Yeah. 57

TERRY: [*highlights* $80 + 60 = 140$ and *clicks* centre of $5 + 60 + 80 + 7$] No. Oh you 58
need to move 'em, you need to swap 'em round. 59

ARTHUR: Hm. 60

TERRY: [*highlights* $7 + 5 = 12$ and *clicks* 5 then 7 in $5 + 60 + 80 + 7$] No. Those need 61
to be swapped round as well. ... There. [*highlights* $5 + 7 = 7 + 5$ 62
and *clicks* 5 then 7 in $5 + 60 + 80 + 7$] Swap round! 63

R: Why wouldn't it swap them round? 64

TERRY: Because there's something in between? 65

R: Okay. 66

TERRY: There that one. [*highlights* $80 + 60 = 60 + 80$ and *clicks* centre of $5 + 60 +$ 67
 $80 + 7$] There. Now you can do that one. [*highlights* $80 + 60 = 140$ 68
and *clicks* centre of $5 + 80 + 60 + 7$] 140. Hopefully that will do it 69
now. [*highlights* $5 + 7 = 7 + 5$ and *clicks* 5 then 7 in $5 + 140 + 7$] No. 70
[*pause*] Ah! [*highlights* $140 + 5 = 5 + 140$ and *clicks* 140 in $5 + 140 + 7$] 71
That swapped it round. ... There. [*transforms* $140 + 5 + 7 \rightarrow 152$ 72
using $5 + 7 = 7 + 5$ then $7 + 5 = 12$ then $140 + 12 = 152$] And ... Yes! 73
(24:00) 74

It is interesting to note that, despite Terry's confidence and growing competence at working with the diagrams, he frequently attempted mismatched statements. For example, in lines 56-58 he attempted to transform $5 + 60 + 80 + 7$ using $7 + 5 = 12$ and $5 + 7 = 7 + 5$ respectively. However, when prompted he was able to give an iconic explanation for why it did not work (lines 64-65). The following excerpt also suggests that the pupils were far more confident about iconically matched than mis-matched substitution attempts (lines 77-78):

ARTHUR: Try 12 equals 7 plus 5 ... Try all the numbers. 75

TERRY: [*laughs*] Yeah, that's what I'm doing. 76

ARTHUR: They're not going to work though. 77

TERRY: Yeah, don't think it will. (30:02) 78

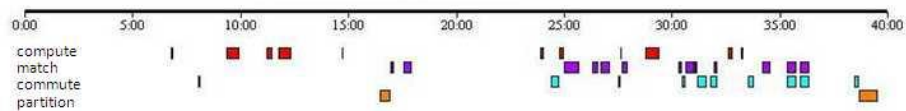


Figure 8.5: Time-sequenced coding for the Kitty and Zelda trial showing articulations of computation, iconic matching, commutation and partition

8.3.3 Trial 2: Kitty and Zelda

Kitty (10 years 6 months) and Zelda (9 years 11 months) were deemed by their class teacher to be average in mathematical ability. They were somewhat reserved at the beginning of the trial but became talkative after 15 minutes. They quickly got to grips with the software’s functionality and completed all the diagrams for Challenge 1. They solved some with apparent effortlessness and others caused them significant problems. This varied performance reflects a greater dependence on trial-and-error approaches, and less strategising, than Terry and Arthur. Challenge 2 did not generate interesting data and I abandoned it early in the trial. The girls appeared not to understand Challenge 3 when I set it, reflected by the coding gap around 20 minutes (Figure 8.5), and I abandoned it.

Kitty and Zelda were hesitantly accepting of $c = a + b$ syntaxes and unfamiliar with $a + b = b + a$ syntaxes as illustrated at the end of the trial when I asked them whether they had encountered such statements before. Only Kitty claimed to remember encountering $c = a + b$ syntaxes previously (“I remember doing one of them in Year 4”), and neither girl remembered encountering $a + b = b + a$ syntaxes.

Figure 8.5 shows all four codes (“compute”, “iconic”, “commute” and “partition”) present to a greater or lesser degree. “Compute” appears more than in the Terry and Arthur trial (see Figure 8.4), and this is fairly representative of the trials reported in the remainder of this thesis (see Figure 11.2, page 170 for a time-sequenced overview of all trials reported in the thesis). The codes “iconic”, “commute” and “partition” are also present to greater or lesser degrees.

A sequential view of Figure 8.5 reveals that the diagrammatic codes (“iconic”, “commute”, “partition”) dominated the latter half of the trial. A list of all occurrences of the codes “commute” and “partition” throughout the duration of the trial is given in Table 8.2. The girls exclusively used the verb “swap”

Time	Code	Pupil	Transcript
08:05	Com	Kitty	That one like <i>swaps them</i>
16:50	Part	Kitty	the 53 is <i>separated into</i> the 50 and the 3
24:20	Com	Both	<i>swap it</i> / That's just <i>swapped it</i> / We could try it <i>swapped</i>
27:30	Com	Kitty	I've just <i>swapped it round</i>
30:30	Com	Kitty	The 80 and the 60 are <i>swapping round</i>
31:25	Com	Both	it's just <i>swapped them</i> / Shall we try <i>swapping</i> and then we can try ...
31:50	Com	Both	<i>Swap</i> the 5 and the 7 / That's just <i>swapping them</i> / But it only <i>swapped</i>
33:45	Com	Both	that's just <i>swapping them</i> / we could do it if it was <i>swapped</i> or something
35:20	Com	Zelda	that one <i>swapped those</i> two, and there was two of those which means they've <i>swapped</i>
36:00	Com	Kitty	when we clicked on that <i>it swapped</i> / that one would <i>swap</i> and then that would <i>swap</i> as well
38:30	Com	Zelda	I think they mean that <i>they swap</i>
38:40	Part	Kitty	like <i>separate that</i> would be 100 add 9 / <i>splitting up</i> the 40 and the 1

Table 8.2: Occurrences of “commutation” and “partition” in the Kitty and Zelda trial

(and derivatives) to describe commuting effects (in total, 12 times by Kitty and 8 times by Zelda). There was only one occurrence of “partition” during diagram-solving (“separated”, Kitty), and at the end of the trial when I asked the girls to comment on $c = a + b$ statements (“splitting up”, Kitty).

Initially, they observed commuting transformations in isolation (e.g. the excerpt at 08:05 in Table 8.2), and did not immediately see a use for them in terms of the task goal (e.g. at 31:50 in in Table 8.2). Unlike Terry and Arthur, they did not make strategic use of partitioning effects, and so did not access the strategy of starting each diagram by identifying partitioning statements, then using commuting statements to shunt the numerals in order to compose them. Accordingly, they relied on trial-and-error approaches much more than Terry and Arthur, and this hampered their efficiency during the final three diagrams which are complicated (Figures 8.3i-k).

Kitty and Zelda: Illustrative transcript excerpts

In this section I present illustrative transcript excerpts from the trial with Kitty and Zelda in order to describe how iconic matching and commutating emerged and, to some extent, were used strategically toward the task goals. I also highlight how, on occasion, they noticed partitioning effects but did not exploit their strategic potential.

Kitty and Zelda were somewhat reticent during the first three diagrams and appeared to engage in arbitrary statement selecting and term clicking. The first explicit evidence for a computational reading occurred during diagram 4 (Figure 11):

KITTY: 8 add 2 are 10. 79
ZELDA: Yeah. (07:05) 80

Kitty and Zelda's grasp of iconic matching emerged gradually, and was preceded by attendance to non-iconic perceptual properties of equality statements. For example, during the fourth diagram (Figure 8.3d), when I asked why $1+7 = 7+1$ had made a substitution in $1+7+1+1$, but $7+3 = 10$ had not, Kitty answered (08:50) "is it because that's only got two numbers to add to 10 but that one's got four", and Zelda answered (09:15) "Is it because that one's got the equals sign in the middle? And that one hasn't." An emerging iconic reading first occurred during the seventh diagram (Figure 8.3g). The statement $53 = 50 + 3$ had been selected and I asked why it made a substitution in $20+50+3$ when the inscription $50 + 3$ was clicked but not when $20+$ was clicked. Zelda answered (17:00) "Because you have to click on the 50 and there's more 50s there." From then on the girls became iconically guided in their decisions of which statement to try next. However, throughout their trial they also attended to iconic near-matches, such as attempting to make a substitution in $20 + 3 + 50$ using $20 + 50 = 70$ (at 17:20), much more frequently than did Terry and Arthur.

Kitty and Zelda's articulation of commutation occurred gradually compared to Terry and Arthur's. During the fifth diagram (Figure 8.3e), the girls offered a computational equivalence (rather than a commutative) explanation for $5 + 6 = 11$ failing to transform $6 + 5 + 3$:

KITTY: Is it because ... um, that one won't work because it doesn't 81
equal the same as the top. 82

R: What do you mean? 83
 ZELDA: That one's got 3 more. 84
 KITTY: That's a different answer to that one. (11:50) 85

Just over halfway through the trial the girls began to express commutation using the verb “swap”. This first occurred when Kitty commented “that’s *just* swapped it” after transforming $40+30+4$ into $30+40+4$ using $40+30 = 30+40$ (Figure 8.3i). Note Kitty’s somewhat dismissive use of “just” when describing swapping, suggesting she saw no use for it within the context of the task goal. However, Zelda pointed out that commuting numerals is at least doing *something* (lines 88 and 91):

ZELDA: Try that on the other one. 86
 KITTY: No, it's just swapped them. 87
 ZELDA: Shall we try swapping and then we can try ... [*does not finish*] 88
 (31:15) 89
 KITTY: No, that's just swapping them. 90
 ZELDA: Yeah but, we could do it if it was swapped or something. 91
 (33:45) 92

Only toward the end of the trial did Zelda come to notice the role of commutation in a sequence of transformations (Figure 8.3k):

ZELDA: I get this now 'cause like that one swapped those two, and there 93
 was two of those which means they've swapped, and like when 94
 you clicked on that one I thought they would have added to make 95
 15 so it would be 300 and, add 15 and then click on that one 300 96
 add 15 equals 315. That make sense? (35:20) 97

Following this insight Kitty utilised commutation strategically, as a means not an end:

KITTY: So if we try and click on that one again ... 98
 ZELDA: Can we try on that one? 99

KITTY: ... then that one would swap and then that would swap as well. 100
 (36:20) 101

Only Kitty articulated partition (“separate”, “splitting up” – lines 106 and 115-115), when using $53 = 3 + 50$ to transform $20 + 53 \rightarrow 20 + 3 + 50$, and this was only in response to prompting by myself:

R: What happened there? Why do you think it didn’t work for two 102
 clicks and then it worked on the third click? [*R refreshes the screen* 103
 and repeats Kitty’s transformation $20 + 53 \rightarrow 20 + 3 + 50$] 104

ZELDA: Oh is it because ... 105

KITTY: The 50 ... the 53 is separated into the 50 and the 3? (16:27) 106

The only other articulation of partition, again by Kitty, was at the end of the trial (lines 115-116). The girls had solved the final diagram (Figure 8.3k) and I asked them about the different statement forms.

R: What do you think these different sorts of sums mean? Because 107
 we’ve got these sort [$6 + 9 = 15$], haven’t we, and we’ve got these 108
 sort [$6 + 9 = 9 + 6$] and those sort. [$109 = 100 + 9$] 109

ZELDA: Erm, I think they [*a + b = b + a statements*] mean that they swap 110
 and those [*c = a + b statements*] ... 111

KITTY: Oh can I answer? 112

ZELDA: ... don’t know. But those [*a + b = c statements*] mean, like, those 113
 two will equal that when you click on them. 114

KITTY: That ... the answer is 109 ... but like separate that would be 115
 100 add 9 so it’s just the ... sum ... is like swapped round. 116

ZELDA: Ah, okay. 117

KITTY: So like ... 118

ZELDA: So normally it would be like that first [*i.e. $100 + 9 = 109$ rather than* 119
 $109 = 100 + 9$]. 120

KITTY: ... 109 would normally be where the 100 and ... the 100 plus 121
 the 9 would normally be where the 109 is so it’s just that they’re 122
 swapped. 123

ZELDA: So it's like if those two were there ... so that would be what it was like. 124
125

R: You mean if this one $[6 + 9 = 15]$ said 15 equals 6 plus 9. Okay. 126
And why do you think they're different in this software, why do 127
you think these sums are written backwards? *[pause]* 128

KITTY: I don't know. *[giggles]* (38:43) 129

The only other occurrence of note with regards to partition was Kitty's observation of the physical effect on the boxed term of the statement $34 = 30 + 4$ (Figure 8.3i):

KITTY: Try that one. Make it bigger. A bit bigger, then the answer. 130
(27:10) 131

However the data provide no evidence for Kitty and Zelda employing partition strategically during the trial, resulting in an approach characterised by trial-and-error when selecting statements:

KITTY: Now do that one. 132

ZELDA: Or it could be that one, because there's 140 and there's 5. 133

KITTY: Yeah, but it could be that one. 134

ZELDA: Try that on the other one. 135

KITTY: No, it's just swapped them. 136

ZELDA: Shall we try swapping and then we can try ... 137

KITTY: What shall we try? 138

ZELDA: That one. 139

R: Why that one Zelda? 140

ZELDA: I don't know. (31:04) 141

8.4 Discussion

The study demonstrates that the task enabled the pupils to view and talk about notation in ways that stand in contrast to those reported in the literature (see Section 2.2). Left-to-right readings of individual statements were replaced by looking for matches of numerals across statements and terms. This arose as the pupils discussed why the software sometimes allowed a substitution to take place and other times does not, and when pupils determined which substitutions could be made within a given diagram. As such, the presentation of equality statements as transformational rules enabled the pupils in both trials to explore and talk about arithmetic notation in non-computational ways.

The pupils found it *useful* (Ainley, Pratt, & Hansen, 2006) to discern statements by form when making transformations of notation. This is because statements were presented as systems of rules for activity toward a task goal, rather than isolated questions of numerical balance. All four pupils distinguished the commuting transformational effects of $a + b = b + a$ forms and, to varying extents, used this distinction strategically to discuss possible transformations one or two steps ahead. Only two pupils, Terry and Kitty from each trial respectively, distinguished the partitioning transformational effects of $c = a + b$ forms, and only Terry explicitly used this distinction as part of a strategy that proved advantageous for later, more complicated diagrams (Figures 8.3g-k).

When the pupils articulated commuting and partitioning effects this does not mean they had a conception of the underlying arithmetic principles. Baroody and Gannon (1984) found that young children can appear to exploit commutation to reduce computational burden, but are often merely indifferent to consistency of outcome (see Section 3.3). The data from the two trials provide no evidence either way as to whether pupils concerned themselves with the numerical balance of a given statement when using it to make an exchange. All that can be inferred is that pupils did not explicitly articulate computational strategies very often during the trials (Figures 8.4 and 8.5), and rarely did so at the same time as identifying iconic matches or discussing transformational effects. This suggests the task goals *exclusively* promote a “can be exchanged for” *over* an “is the same as” meaning for the equals sign. In fact, there is no intrinsic *utility* to considering numerical balance when working through the diagrams, and it is more efficient not to do so.

The study reported in the following chapter seeks to address this by presenting

pupils with diagrams that contain some false equalities, as in $77 = 11 + 33$. If pupils do not notice and comment on the presence of such statements this suggests they are not concerned with numerical balance, and are attending exclusively to transformations of notation as visualised on the screen or in the head. False equalities lead to changes in the computational total of the boxed term across transformations, such as when using $77 = 11 + 33$ to transform $143 + 77 \rightarrow 143 + 11 + 33$. If pupils do not notice or comment on this it would suggest indifference to the total of the boxed term when transforming diagrams. Furthermore, if articulations and strategies arise that resemble those reported in this study, then the presence of false equalities has little impact on how pupils view and talk about the diagrams when working toward the task goals.

8.5 Summary

Two opportunistic pilot studies with single children suggested the task resources and goals were viable both functionally and as a research instrument. The main study, in which two pairs of primary pupils worked through a sequence of diagrams toward the task goals, sought to inform the research question:

Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?

A visual overview from each trial show a notable lack of computational readings, and the prevalence of iconic matching and distinguishing statements by their distinctive transformational properties. The pupils’ attention was drawn to numerals, and matches of numerals, when looking for allowable substitutions. All the pupils spoke of the transformational effects of $a + b = b + a$ statements in terms of commuting numerals, and one pupil from each trial spoke of the effects of $c = a + b$ statements in terms of partitioning numerals. Illustrative excerpts show that all pupils used commuting effects in a strategic manner to enable substitutions one or two steps ahead. Only one pupil systematically articulated using partitioning effects for a more sophisticated strategy, and this method proved notably more efficient in the final few diagrams, which are complicated.

It seems, then, that the “can be exchanged for” meaning for the equals sign, as instantiated by the computer and task goals, does promote attention to statement form. The requirement to identify allowable substitutions necessitates looking for matches of numerals and terms across statements, and to articulate

strategies it is useful to distinguish $a + b = b + a$ and $c = a + b$ statements in terms of their distinctive transformational effects.

This view of notation stands in contrast to that required when viewing statements as isolated questions of truthfulness. It is unclear as to whether the pupils were concerned with the numerical equivalence of terms on both sides of equals signs, or with the consistent computational value of the boxed term across transformations. In the Study 2 this will be investigated by placing false equalities into some of the diagrams to find out if pupils comment on their presence, and whether they have any visible impact on how they talk about the diagrams.

Chapter 9

Study 2: Attention to numerical balance

9.1 Introduction

This study investigates the second research question:

Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct?

It seeks to replicate the findings of Study 1, in which pupils were evidenced engaging in iconic matching, articulating statement form in terms of distinctive transformational effects, and using these distinctions to work strategically toward the task goals. However, unlike in Study 1, the diagrams presented to pupils contain some false equalities. Evidence will be sought of pupils’ responses (or lack of) to the presence of false equalities, and any notable impact their presence has on how pupils view, manipulate and talk about the diagrams.

The study comprises two trials with pairs of pupils. All pupils were deemed mathematically able by their class teachers, and were selected as being likely to discover and articulate sophisticated strategies when working with the diagrams in order to test whether they engaged with numerical balance views at the same time. One pair of pupils were aged 9 and 10 years, to provide comparison and contrast with the pupils in Study 1, and the other pair were aged 12 and 13 years, to find out if more mathematically mature pupils are more likely to pay

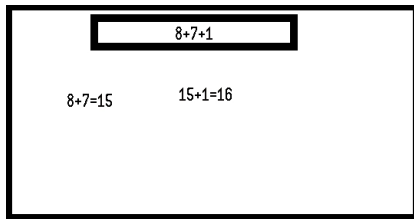
attention to the truthfulness of statements.

9.2 Task set-up

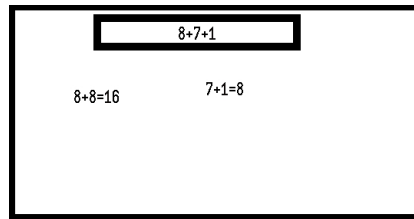
As with Study 1, the software was set up to present a sequence of eleven diagrams containing up to nine arithmetical statements (Figure 9.1). The first two diagrams contain only two compositional statements each, and are intended to familiarise learners with operating the software (Figure 9.1a-b). Diagram 3 introduces a commutative statement (Figure 9.1c) and Diagram 6 introduces a partitioning statement (Figure 9.1f). Diagrams 7 to 11 combine compositional, commutative and partitioning statements, and are increasingly complicated (Figures 9.1g-k). Diagrams 1 to 7 contain only true statements. Diagram 8 contains the first false statement ($15 + 28 = 44$), which is “subtly” false in that it is only imbalanced by 1 (Figure 9.1h). Diagram 9 contains two false equalities (Figure 9.1i). Diagram 10 contains three numerical false equalities (Figure 9.1j), two of which are less subtle than those in diagrams 8 and 9. Diagram 11 contains blatant false equalities and a large computational disparity between boxed initial term ($143 + 77$) and boxed final numeral (23).

These diagrams are modifications of those used in Study 1, and alternative designs would have been possible. First, all statements could have been false equalities from the start, or introduced much earlier in the trials. However, such a design would likely not allow the pupils sufficient time to become familiarised with the functionality of the software and nature of the task. Secondly, the imbalance of false equalities could have been introduced less gradually. For example, the first false equality met, $15 + 28 = 44$ (Figure 9.1h), could have been, say, $15 + 28 = 1$. However, initial opportunities for pupils to notice subtle imbalances was preferred as this would be more informative about the degree of their sensitivities to numerical equivalence during the trials. Thirdly, place-value partitions and compositions were used less than in previous trials because their patterns of repeating digits would make them obvious when imbalanced and would not have allowed the gradual introduction of false equalities (e.g. $41 = 40 + 2$ compared to $41 = 27 + 15$).

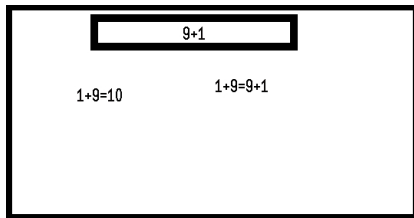
I first showed the pupils how to operate the software, in particular how to select a statement by clicking on the equals sign and how to make substitutions by clicking on numerals, and allowed them to experiment with it for several



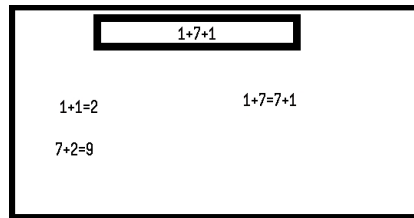
(a) Diagram 1



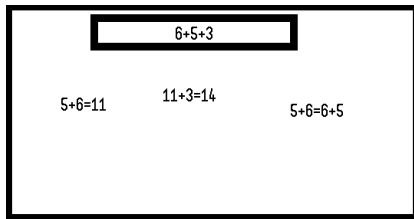
(b) Diagram 2



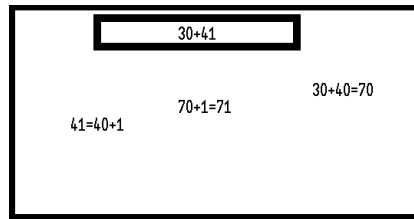
(c) Diagram 3



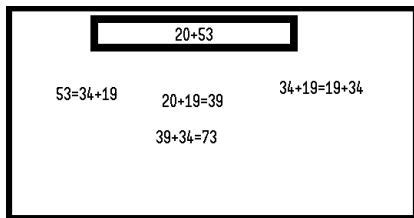
(d) Diagram 4



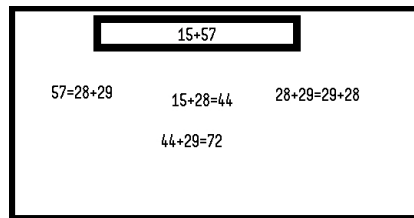
(e) Diagram 5



(f) Diagram 6



(g) Diagram 7



(h) Diagram 8

Figure 9.1: Diagrams presented in the trials

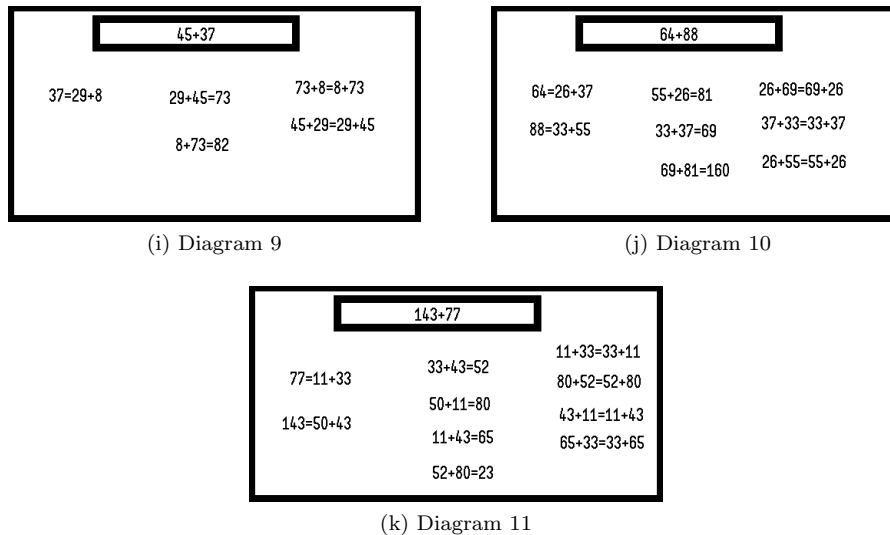


Figure 9.1: Diagrams presented in the trials

minutes. For each diagram I challenged the children to transform the boxed term into a single numeral.

9.3 Trial 1: Laura and Yaseen

Laura (female, 9 years 11 months) and Yaseen (male, 10 years 5 months) were deemed mathematically able by their class teacher. They had limited classroom exposure to partitioning ($c = a + b$) forms several months prior to the trial and no previous classroom exposure to commutative ($a + b = b + a$) forms. Both were confident and articulate, and Yaseen was the most talkative. They got to grips quickly with the functionality of the software during the first two diagrams, and went on to solve all eleven diagrams in a trial lasting about 35 minutes.

Figure 9.2 shows a time-sequenced coding mapping¹. The data replicates that found in Study 1, and mostly closely resembles the trial with Terry and Arthur (Section 8.3.2). The pupils began by computing notation, but overall the code “compute” is less present than would be expected for a task in which pupils are presented with statements as isolated questions of numerical balance. Instead the pupils quickly came to looking for iconically matched numerals and terms

¹Data capture and analysis was described in Chapter 7, and time-sequenced maps were explained in Section 7.3.3

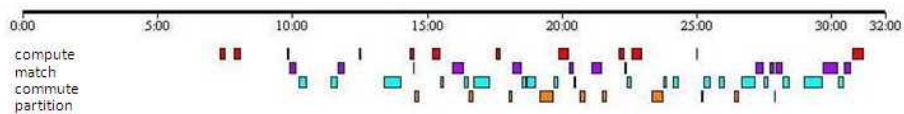


Figure 9.2: Time-sequenced coding for the Laura and Yaseen trial showing articulations of computation, iconic matching, commutation and partition

across statements and the boxed terms. Both pupils articulated the distinct transformational effects of $a + b = b + a$ and $c = a + b$ (or $a + b = c$) statements, and used these distinctions strategically. As with Terry in Study 1 (lines 50, p.104), Yaseen inferred the partitioning effect of the first $c = a + b$ statement he encountered (Figure 9.1h) and its strategic use toward the task goal. Following this the pupils adopted the strategy of beginning each diagram by using $c = a + b$ forms to partition numerals, then using $a + b = b + a$ and $a + b = c$ forms to commute and compose numerals.

They encountered the first diagram to contain a false equality (Figure 9.1h) halfway through the trial. They did not comment on its presence, or the presence of false statements in the subsequent diagrams until at the very end of the trial after they had solved Diagram 11 (Figure 9.1k). Neither did the presence of false equalities have any apparent impact on their iconic matching activity, articulations of distinctive transformational effects, or strategising. Numerical imbalance was only commented on after they had been working on Diagram 11 for several minutes and Yaseen questioned the disparity between the initial boxed term, $143 + 77$ and the final boxed numeral, 23.

Laura and Yaseen: Illustrative transcript excerpts

Early in the trial (Figure 9.1b), Laura discovered that $15 + 1 = 16$ transforms the inscription $(15 + 1)$ into the numeral (16) and I asked why it had worked. Yaseen gave a computational explanation (line 145) but Laura then offered an iconic matching explanation ($15 + 1$ is in both the statement and the box – lines 149-151):

- | | | |
|---------|--|-----|
| R: | Do you know why that one worked? | 142 |
| YASEEN: | 'cause that's ... | 143 |
| LAURA: | Oh, 'cause that's 15 add 1 [<i>unclear</i>]. | 144 |

YASEEN: That equals the 16. 145

LAURA: Yeah. 146

R: Okay, and what did you say Laura, sorry? [*Yaseen had a louder* 147
voice than Laura]. 148

LAURA: Um, because like that one's 15 add 1. 149

YASEEN: And that's all the sum. 150

LAURA: And that's what it says in the box. (07:22) 151

The next evidence for iconic matching coincided with the first evidence for commutative awareness (lines 160-164) during the third diagram (Figure 9.1c).

YASEEN: 9 plus 1, eh? 152

LAURA: Equals 10. 153

YASEEN: Click the equals and all that first. No, I think you have to get 154
it, that one [$1+9=9+1$], that's the opposite. 155

R: What's the opposite? 156

YASEEN: Of this one. 'cause that says 9 plus 1. 157

R: Oh I see what you're saying. 158

YASEEN: And 1 plus 9. 159

LAURA: It's because like that one you have to swap 'em around. 160

R: Sorry, what did you say Laura? 161

LAURA: Um, because like that was 9 add 1 and that one's 1 add 9, um, 162
that one had to be switched round for it, for that to say 1 add 163
9. (09:49) 164

From then on commutative readings occurred regularly throughout the trial. The pupils used the word “swap” a total of 29 times and “switch” 16 times when referring to $a + b = b + a$ forms or expressing a wish to transform occurrences of $a + b$ into $b + a$. Evidence for a partitioning reading first occurred during the sixth diagram (Figure 9.1f) which contains the first $c = a + b$ form. Yaseen inferred a partitioning meaning for the statement $41 = 40 + 1$ before actually using it (lines 168-169).

LAURA: Um, that box equals 71. 165

YASEEN: That's harder. 166

LAURA: You need 30 add 41. 167

YASEEN: 70, let me try that. Separate them two [40 and 1]. Let me try 168
separating these two. And then ... 169

LAURA: Yeah, then that [30+40=70]. And then that one [70+1=71]. 170

YASEEN: Easy! That's easy. It's getting easier and easier. (14:25) 171

Throughout the trial the children used the word “separate” a total of three times and “split” ten times in reference to the transformational effects of partitioning statements. There was also one occurrence of Yaseen using the phrase “split them two back together” when expressing a wish to transform an occurrence of $a+b$ back into its original form c . They came to use articulations of commutation and partition to discuss substitutions two or three steps ahead. As with Terry and Arthur in Trial 1 (Section 8.3.2), they consistently adopted the strategy of beginning by partitioning numerals, then commuting and composing them into the final boxed numeral (as in lines 165-171).

Laura and Yaseen did not comment on the false equalities in diagrams 8 to 11 (Figures 9.1h-k). This is not positive evidence that they were oblivious to these numerical imbalances, they may have spotted them and simply not mentioned them. However, two excerpts from the transcript suggest this was not the case. The first is when they were momentarily stuck on the eighth diagram, and Laura gave the incorrect result 45 in response to Yaseen. Her justification for this result suggests she had not noticed the falsity of $15 + 28 = 44$. Yaseen's acceptance of her result suggests he had not noticed it either.

YASEEN: What's 15 add, what's 15 add, er, 29? 172

LAURA: Um, it's 45. 173

YASEEN: 45? So we're one off. 174

R: How did you work that out so quickly? 175

LAURA: Um, because like that's, um, 15 add 28 [in $15 + 28 = 44$]. 176

R: Yeah. 177

LAURA: So just 15 add 29, 28, plus one more is just 45. (22:35) 178

The second observable lack of attention to numerical equivalence occurred during the final diagram (Figure 9.1k) in which many statements are very imbalanced. After working at the diagram for two and a half minutes, and then getting momentarily stuck, Yaseen began attending to the computational value of the boxed term, which had so far been transformed from $143 + 77$ to $50 + 52 + 11$. It seems Yaseen readily accepted that this value can vary without finding this variation peculiar or attributing it to the falsity of the on-screen statements (lines 184-185).

YASEEN: 50, you can't, you can't swap 'em round for some reason. Can't, 179
that's the, what's the ans ... 180

LAURA: The answer. So ... 181

YASEEN: What's ... 182

LAURA: 113 ... [*i.e.* $50 + 52 + 11$] 183

YASEEN: 143 add 77 is a hundred and ... 230. So we need to get to that 184
answer. So far we've got, er, 113. So we're nearly ... 185
(29:32) 186

The final diagram is complicated and despite the children's determination and successful emerging strategy they worked on it for a further two minutes before solving it. It was only then, when the boxed term ($143 + 77$) had been transformed into a single numeral (23), that they finally noticed something amiss with one of the statements (lines 193-197).

YASEEN: 80 add 52. Yeah make fift ... ah there! Go on, do that. Then 187
there's, is there eigh ... 188

LAURA: 80 add 52. 189

YASEEN: So you've got to swap them around there. 190

LAURA: Yeah. 191

YASEEN: Then we're done [*23 appears in the box*]. And that's it. Easy! 192

LAURA: Nope. Yeah? 193

YASEEN: What?! The answer's not 23 is it? Is that right? 194

R: Well, what do you think? 195

YASEEN: Yeah, 'cause we, that's the wrong answer isn't it? Uh?! 52 add 196
80 is not 23! (30:32) 197

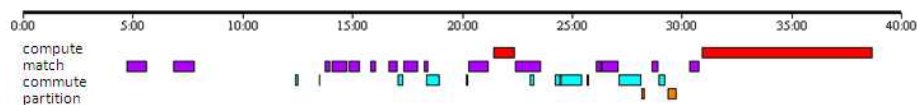


Figure 9.3: Time-sequenced coding for the Ajay and Nikisha trial showing articulations of computation, iconic matching, commutation and partition

9.4 Trial 2: Ajay and Nikisha

Ajay (male, 12 years 8 months) and Nikisha (female, 12 years 9 months) were in the top of eight ability sets for mathematics. They had 16 months experience of secondary algebra prior to the trial (excluding school holidays). Ajay and Nikisha were confident and got to grips immediately with the functionality of the software during the first diagram. They worked through all eleven diagrams in a trial lasting almost 40 minutes.

The time-sequenced coding map of the trial in Figure 9.3 shows how the pupils did not discuss computing results at all during the first half of the trial. They completed the first ten diagrams (Figures 9.1a-j) in just 12 minutes without articulating commuting or partitioning transformational effects. However, they discussed finding matched numerals and terms, and followed an unspoken strategy of beginning by partitioning, then commuting and composing numerals (the strategy used by Laura and Yaseen, as well as Terry and Arthur in Study 1). They took 18 minutes to complete the final diagram (Figure 9.1k) during which they refreshed the diagram and started again a total of six times. It was only when working on the final diagram that they explicitly articulated commuting effects and used them to discuss substitutions one or more steps ahead. Only Nikisha articulated partitioning effects, and on both occasions this was in reference to using $a + b = c$ forms to undo a composition.

Neither pupil commented on the presence of false equalities in diagrams 8 to 10 (Figures 9.1h-j). The first occurrence of “compute” just over halfway through the trial (Figure 9.3) reflects Nikisha commenting on the false equality $50 + 11 = 80$ after they had been working on diagram 11 for ten minutes. The second occurrence is after the pupils had completed Diagram 11 and I asked about the presence of false equalities. As will be seen from illustrative transcript excerpts, they had not noticed imbalances until just before Nikisha commented on them, and did not consider their presence relevant to the task goal.

Ajay and Nikisha: Illustrative transcript excerpts

After the pupils had completed the third diagram (Figure 9.1c), with confident guidance from Nikisha, I asked how she had known what to do.

- R: How did you see that Nikisha, because you seemed quite sure. 198
- NIKISHA: 'cause, erm, the first one, it said 1 plus 9 and on there it said 199
9 plus 1 and over here it says that they're the same. So if you 200
change that to m . . . to tell the, it's the same thing. You can 201
change that to make the answer. 202
- R: Okay. Did you follow that Ajay? 203
- AJAY: Yes, I was following it. 1 plus 9, if you change it to 9 plus 1. And 204
9 plus 1 equals 10. (05:11) 205

Most of their discussion in the first thirteen minutes of the trial, during which they completed the first ten diagrams, related to the functioning of the software. For example, the following excerpt is from the 25 seconds it took them to complete the fifth diagram (Figure 9.1e):

- NIKISHA: You click the equals sign 206
- AJAY: Yeah. 207
- NIKISHA: And then do that one. Then you can make that 11. Click on 208
that equals sign. And then do the 5 plus 6. And then 11 plus 3. 209
Equals sign. (08:04) 210

For diagrams 7 to 11, which include compositional, partitioning and commutative statements and are increasingly complex (Figures 9.1g-k), Ajay and Nikisha adopted the same strategy as Laura and Yaseen from Trial 1. This involved starting with $c = a + b$ statements in order to break sums up into constituent elements, and then using $a + b = b + a$ and $a + b = c$ statements to reorder and compose the elements into a final numeral. Unlike Trial 1, however, there was little explicit discussion of this strategy and no specific vocabulary to distinguish composing, partitioning and commutating; the pupils instead used the generic term “change” when referring to substitution. Also, unlike the younger children in Trial 1, their activity was far more based in searching for iconic matches than trying substitutions to see if they worked. Their approach, while speedy,

was characterised by observation not manipulation, and they rarely highlighted statements or clicked numerals and operators in an exploratory manner.

Despite this, Diagram 11 (Figure 9.1k) caused them significant difficulties and took a total of eighteen minutes to solve (interestingly, it took the Year 5 children in Trial 1 only four and a half minutes). During this struggle they came up against what they perceived to be dead ends and refreshed the diagram and started again a total of six times. They also used the term “swap” a total of 24 times during diagram 11 when referring to $a + b = b + a$ statements or expressing a wish to transform occurrences of $a + b$ into $b + a$, despite not having used it at all for the first ten diagrams. Only Nikisha used the term “split” (twice) to express a wish to transform occurrences of c into $a + b$. On both occasions this was in reference to using the statement $43 + 11 = 65$ to transform 65 back into its original form of $43 + 11$ rather than in reference to $c = a + b$ statements.

Ajay and Nikisha’s discussion displayed their increasingly sophisticated thinking toward the later stages of the trial. The following two excerpts illustrate how they imagined the effects of sequences of substitutions several steps ahead (Figure 9.1k).

- NIKISHA: Might have to make 65 and then [*pause*] then split it back. 211
- AJAY: If you can get 80 first. And then 80, and if we get 52 and then 52 212
that will change it to 52 add 80. 52 add 80 equals 23. (28:08) 213
- R: Something you said a minute ago about if you make the 65 and 214
then split it back. What did you mean by that? 215
- NIKISHA: Because you couldn’t get the positioning right so if you have the 216
two numbers to make 65 which is 11 and 43 and then if you 217
can move the 65 somewhere out and then split it again then you 218
might get it. (29:21) 219

The pupils did not comment on the presence of false equalities in diagrams 8 to 10. Notice for example Ajay’s apparent acceptance of $52 + 80 = 23$ (Figure 9.1k) in line 213. Neither did they not notice their presence in the final diagram until they had been working at it for ten minutes and thirty seconds, and Nikisha said:

- NIKISHA: Why does it say 50 add 11 equals 80? 220

R: What do you mean? 221

NIKISHA: 50 add 11 equals 80. [pause] It doesn't. It equals 61 [unclear]. 222

R: Had you spotted that before or just now? 223

NIKISHA: Before. I don't know why I didn't say it. 224

R: Um? 225

NIKISHA: I didn't say it before. 226

R: And is that the only one that's strange? 227

NIKISHA: [pause] No there's that one as well. Equals 54. Some of them are 228
right and some aren't. 229

R: And what about on the other puzzles? Are any of, were they all 230
right or were they, were some of those wrong? 231

NIKISHA: I don't know. I didn't really look at the sums, to the answers, 232
then. (21:26) 233

When I asked Nikisha when she had first noticed that $50 + 11 = 80$ is wrong she answered “about one or two minutes [before explicitly saying so]”. However, neither Nikisha nor Ajay seemed concerned by the presence of numerical non-equivalences, and immediately continued working on the diagram. During a brief semi-structured interview immediately after the trial I asked what the children made of the presence of such statements.

R: What about these sums being wrong. Do you know why that's 234
like that in the puzzle? 235

AJAY: Because it's, um, because the puzzle's different to like adding 236
and stuff like that, yeah? And, um, the puzzle, the aim of it, 237
yeah, is to like get that number and, um. I can't explain it but 238
I know why though. 239

NIKISHA: It's about your method. And not about the actually like ... 240

AJAY: Yeah, adding. (36:35) 241

9.5 Discussion

The diagrammatic activities of iconic matching (searching for visually identical inscriptions) and substituting (describing or predicting transformations) stand out as a common strand across both trials. In Trial 1, these activities manifested themselves most notably as commuting and partitioning (“separate”, “split”) descriptions of the distinct substitutive effects of $a + b = b + a$ and $c = a + b$ (or $a + b = c$) statements. In Trial 2, the pupils made fewer distinctions of form during diagrams 1 to 10, and referred to all transformations simply as “change”. However, commutative readings became explicit when they struggled for a sustained period during the final diagram and, to a lesser extent, so too partitioning readings of a $a + b = c$ statement.

The key finding is that the pupils distinguished the different transforming effects of statements, but did not articulate their truth or falsity until they had worked on the final diagram for some time. This suggests that substitutive equivalence meanings were independent of numerical equivalence meanings during the trial. The data presented here offer no direct evidence that the children lacked an “is the same as” meaning of the equals sign *per se*, as this was not tested, but rather that such a meaning was explicitly cued only infrequently during the task. It is simply unhelpful to consider the numerical balance of statements of the conservation of quantity across transformations of notation when working on the task. Therefore the best account of the findings is in terms of task design rather than any “cognitive limitations” of the pupils.

A useful construct for analysing and refining the design of pedagogic tasks is “utility” (Ainley et al., 2006). Ainley et al. argue that the usefulness of a mathematical concept is integral to its nature in a well designed task:

the learning of mathematics encompasses not just the ability to carry out procedures, but the construction of meaning for the ways in which those mathematical ideas are useful. (p.30)

In typical tasks reported in the literature, in which the equals sign means “is the same as”, it is useful to discern statements in terms of *truthfulness*. In the diagram-solving tasks reported here it is useful to discern statements by form; that is, the distinctive transformational effects of $a + b = c$, $a + b = b + a$ and $c = a + b$ statements. However, “sameness” and “exchanging” meanings for the equals sign are mutually exclusively in these two kinds of tasks. For the reason

the pupils in this study happily worked with “nonsense arithmetic” and in fact did not even notice they were doing so.

What is required then is a task that renders useful both the “is the same as” and the “can be exchanged for” meanings of the equals sign. This is the aim of Study 3, reported in the next chapter.

9.6 Summary

The study presented pupils, who were selected as likely to engage fully in diagrammatic activities, with diagrams containing some false equalities. This was done to inform the second research question:

Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct?

In both trials the data replicates that from Study 1, and the false equalities had no notable impact on how pupils viewed and talked about the diagrams. All pupils engaged in iconic matching, and articulated the commuting effects of $a+b = b+a$ statements. Both pupils in Trial 1 articulated the partitioning effects of $c = a + b$ statements, and one pupil in trial two articulated the partitioning effects of a $a + b = c$ statement when using it to undo a composition. All pupils used these distinctive transformations in order to identify and suggest substitutions one or two steps ahead. In both trials the pupils came to use the strategy of beginning with partitioning statements, then using commuting and composing statements, in order to complete each diagram. In Trial 2 this strategy was implicit and unspoken, but used consistently nevertheless.

None of the pupils noticed the presence of false equalities until they had been working on the final diagram for several minutes. In Trial 1, Yaseen noticed something was amiss when the boxed term $143+77$ had finally been transformed into 23, and then noticed the imbalance of several of the statements on screen. In Trial 2, Nikisha noticed the imbalance of $50+11 = 80$. The pupils in Trial 2 were not concerned by their presence, and attempted to explain their indifference at the end of the trial (lines 236-241).

As such, the pupils did not switch between “numerical” and “diagrammatic” views of the statements, because the task goals in Studies 1 and 2 do not support utility for viewing statements in terms of numerical equivalence. Indeed, it is more efficient when working through the diagrams to ignore the truth status

of statements and the conservation of value across transformations. In Study 3 I present trials in which the task goals are extended to require coordinating “numerical” and diagrammatic views of notation.

Chapter 10

Study 3: Coordinating form and balance

10.1 Introduction

This study investigates the third and fourth research questions:

Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?

Can children connect their implicit arithmetical knowledge with explicit transformations of notation?

It seeks to replicate the findings in Studies 1 and 2 by providing a sequence of diagrams for pupils to work through. However, in this study pupils go on to construct their own diagrams. Creating statements requires ensuring numerical balance and so supports an “is the same as” meaning for the equals sign. Placing statements in a diagram and using them to make substitutions supports the “can be exchanged for” meaning evidenced in Studies 1 and 2. In order to create functioning diagrams, then, the pupils must switch appropriately between balancing and exchanging views. Two forms of evidence for this are sought: (i) the complexity and functionality of the diagrams made by the pupils; (ii) their discussion when working together on diagram-making. In the previous studies (Chapters 8 and 9) substitution was articulated by comments such as “we can use that $[a + b = b + a]$ to swap them $[a$ and $b]$ round”, and so on.

In this study, assuming the pupils succeed at making coherent diagrams, we should expect such comments to be constructively combined with discussion about the numerical balance of statements and, when the ordering of operations is a factor, discussion about the consistency of the boxed term's value across transformations.

Once the pupils had practised freely at constructing diagrams on the screen, they were challenged to make a diagram that reflects their own mental addition strategies for a term given by the researcher. This requires identifying the implicit partitions and commutations present in typical mental calculation, and inputting them as explicit statements within a relational diagram. For example, a typical verbal strategy for $28+30$ might be “twenty plus thirty is fifty, plus the eight is 58”, which contains the explicit compositions $20+30=50$ and $50+8=58$, as well as the implicit partition $28=20+8$ which must be entered as a statement if a corresponding diagram is to be functional. If the pupils achieved this it would demonstrate they were capable of connecting the notational transformations observed on screen with their existing knowledge of the underlying arithmetical principles.

The main study comprises three trials with pairs of pupils aged 9 and 10 years. The trials lasted around 90 minutes each and took place over two or three sessions over a period of up to five days. Prior to this, however, a pilot trial was conducted with two of the pupils from Study 2 (Yaseen and Laura, see Section 9.3) in order to test the functionality and usability of the newly implemented keypad tools (Figure 10.1). Happily, the keypads performed satisfactorily and insightful data was produced, as reported in the following section.

10.2 Pilot trial

The purpose of the pilot trial was to test the robustness and usability of the keypad tools (Figure 10.1) and, assuming it worked as required, to gain initial insights into how children might coordinate balance and substitutive views of equality statements when making diagrams. The trial in fact produced interesting data which I report in this section.

The participants were Yaseen and Laura from Study 2 (Section 9.3). The pilot trial was somewhat opportunistic and took place about one month after the trials reported in Study 2. As Yaseen and Laura were already familiar with

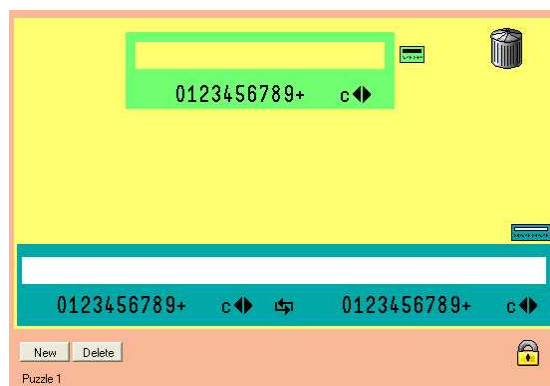
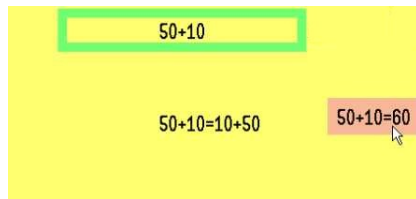


Figure 10.1: Keypads for making diagrams

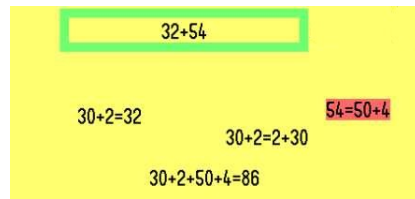
solving diagrams, and had engaged enthusiastically and successfully with the task, I presented them directly with the keypad tools and challenged them to make their own diagrams. Initially I provided keypads with only + operators and later introduced \times and then $-$ operators. The total trial lasted 43 minutes.

The *Sum Puzzles* software presents two keypads for making tools (Figure 10.1). The top keypad is for entering a term into the box at the top of each diagram, and in addition to numerals and operators also contains buttons for clearing the screen (“c”) and deleting the left-most and right-most characters in a term (\blacktriangleleft and \blacktriangleright respectively). The bottom keypad is for entering statements and functions similarly to the *Equivalence Calculator* described in Section 5.2.2. The numeral and operator buttons can be used to enter a term on either side. If the terms are numerically balanced the equals sign appears, as in $2+3 = 3+2$; if imbalanced the “not equals” sign appears, as in $2 + 3 \neq 3 + 1$; if either side contains an incomplete term nothing appears in the middle, as in $2 + 3 \quad 3+$.

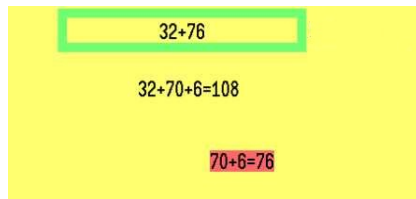
After being shown how to use the keypad gadgets, Yaseen and Laura made eight solvable diagrams during the trial (Figure 10.2). The first diagram (a) contains two statements both of which are substitutive with regard to the boxed term, but only one of which is required to solve it (i.e. produce a single numeral in the box). The second diagram (b) contains four statements, one of which is redundant ($30 + 2 = 2 + 30$). The next three diagrams (c-e) contain between two and five statements, all of which are required to solve the diagrams. The final three diagrams (f-h) all have more than one type of operator in the boxed term and all contain one or three redundant statements that play no part in



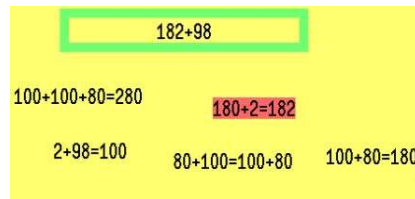
(a) Diagram 1



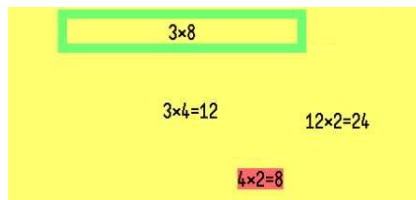
(b) Diagram 2



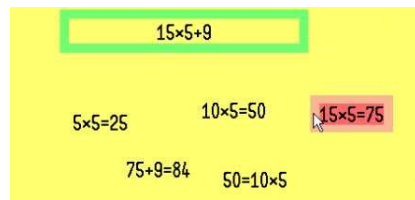
(c) Diagram 3



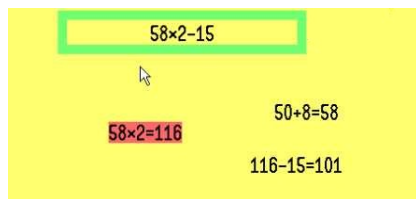
(d) Diagram 4



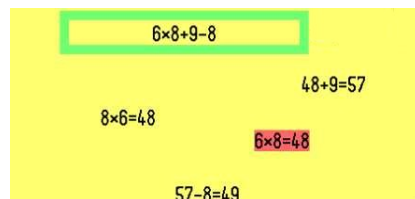
(e) Diagram 5



(f) Diagram 6



(g) Diagram 7



(h) Diagram 8

Figure 10.2: Diagrams made by the pupils in the pilot trial

solving them. (The redundant statements are $5 \times 5 = 25$, $10 \times 5 = 50$ and $50 = 10 \times 5$ in diagram f; $50 + 8 = 58$ in diagram g; and $8 \times 6 = 48$ in diagram h).

The children therefore successfully made complex diagrams comprising multiple statements that were both balanced and substitutive with regard to the boxed term. The presence of redundant statements in all the diagrams where the boxed term includes more than one type of operator (f-h) also suggests the children experienced some challenges ensuring the conservation of quantity across transformations where the ordering of operations was a factor.

The remainder of this section presents three transcript excerpts to illustrate how the children's discussion provides evidence that they considered and coordinated numerical balance and substitutive meanings for equality statements.

The first excerpt is from the start of the second diagram (Figure 10.2b) and illustrates the children attempting to make a statement that is both balanced and substitutive. Yaseen had put $32 + 54$ in the box and wanted to partition the numerals (line 242). Laura suggested $30 + 2$ for one side of the statement (line 243) and Yaseen inputted $30 + 2 = 2 + 30$ without giving a clear reason (lines 246-248). He had entered a numerically balanced statement but, as Laura pointed out (lines 249-251), it was not substitutive with regard to the boxed term. Following this Yaseen attempted to make the statement $32 = 23$ (lines 253-257), which is substitutive but imbalanced. Laura then suggested a statement that is both balanced and substitutive (i.e. $30 + 2 = 32$, lines 259-261). Note that Laura intended this statement to partition the numeral 32 suggesting a right to left substitutive reading.

YASEEN:	I want to split them two [<i>the numerals in the boxed term</i>] apart.	242
LAURA:	Let's do 30 add 2.	243
YASEEN:	You can't do that bit because that's 2, that's 30 plus 2.	244
LAURA:	Oh yeah.	245
YASEEN:	[<i>unclear</i>] We can do two more sums, innit? [<i>pause</i>] 2 plus 30, so	246
	we want, and then we just place that there. [<i>enters</i> $30 + 2 = 2 + 30$]	247
	Then...	248
LAURA:	I don't think that one works.	249
R:	Why not?	250

LAURA: Because it doesn't give 30 add 2 in the box. 251

R: Do you see what she means, Yaseen? 252

YASEEN: Mm-hm. Wait, 32 then put [*unclear*] over here, what's it called? 253
 23, and then ... [*inputs 32 and 23 at the keypad, giving $32 \neq 23$ on* 254
screen]. I don't care if the equals sign's not there but I know how 255
 to do it. Does the equals sign have to be there? [*attempts to select* 256
 $32 \neq 23$ to make a substitution but finds he can't] 257

R: Laura, have you any ideas how to help? 258

LAURA: Um, maybe you can make that sum and then put the answer 259
 on the other side, and then, make something that equals 32 and 260
 make the other one. 261

R: Okay. Yeah, I think I understand but show us what you mean. 262

YASEEN: 30 plus 2 you're saying, and what else? Okay, you get that 263
 but ... [*inputs $30 + 2 = 32$ and transforms the boxed term $32 + 54 \rightarrow$ 264
 $30 + 2 + 54$] Yeah because we can split that [32] into that [30 + 2]. 265*

The second excerpt is from the start of the sixth diagram (Figure 10.2f) and illustrates the children's awareness of the need for conservation of quantity across transformations. They had entered $15 \times 5 + 9$ in the box and Yaseen suggested entering $5 + 9 = 14$ to make a substitution but then decided against it (lines 266-270). When prompted both children indicated that $15 \times 5 + 9$ and 15×14 would give different results (lines 271-274).

YASEEN: 5 plus 9, no you can't do that. 5 plus 9, because that would make 266
 14. 15 times 14. No. [*laughs*] 267

R: Sorry, what do you mean "no"? 268

YASEEN: Swap it round [*i.e. replace $5 + 9$ with 14 in the boxed term*] makes 15 269
 plus 14. [*he presumably meant 15 times 14*] So that won't work. 270

R: Why not? 271

YASEEN: It's going to be different. 272

LAURA: [*talking at same time as Yaseen*] Because then you get another answer. 273
 274

The third excerpt is from the start of the seventh diagram (Figure 10.2g) and illustrates the children’s attempts to substitute the term $2 - 15$. Yaseen had entered $58 \times 2 - 15$ in the box and Laura suggested the statement $15 - 2 = 13$ (lines 275-279). Yaseen pointed out that it could not make a substitution in the boxed term because the 15 and 2 are commuted and attempted to input $2 - 15 = 15 - 2$. He seemed doubtful of its validity and was unsurprised when it appeared on screen as $2 - 15 \neq 15 - 2$ (lines 280-284). He then suggested the \times operator in the boxed term was the problem (probably alluding to conservation of quantity across transformations – lines 287-288) but Laura pointed out that the statement was imbalanced (lines 286-290).

LAURA:	I think 15 takeaway 2.	275
YASEEN:	15 takeaway 2?	276
LAURA:	Yeah, and then ...	277
YASEEN:	Are you sure?	278
LAURA:	... do 13.	279
YASEEN:	No, first we got to switch that [i.e. $15 - 2$ in Laura’s suggested state- ment] around. [pause] Let’s do 2 takeaway 15. [inputs $2 - 15 = 15 - 2$]	280
	Let’s see if that equals, er, recognises it, first. [the statement ap- pears on screen as $2 - 15 \neq 15 - 2$]	281
	No, the equals sign isn’t there so	282
	we can’t do it.	283
		284
R:	Why isn’t the equals sign there?	285
LAURA:	Because 2 takeaway 15 is minus 13.	286
YASEEN:	[talking over Laura ...] Because the top [boxed term] has a times. So	287
	you can’t do, because that one is ...	288
R:	Laura, say that again I don’t think Yaseen heard it.	289
LAURA:	Um, because 2 takeaway 15 equals minus 13.	290
YASEEN:	Yeah, that’s true.	291

In sum, the pilot trial demonstrated that the pupils made predictions about the effect of potential substitutions in an exploratory and reflective manner that took account of the substitutivity of statements with regard to the boxed term (lines 249-251 and 280-284), the numerical balance of statements (lines 286-290), and the conservation of quantity across transformations (lines 266-274

and 287-288). This enabled the children to confront and discuss issues such as the ordering of operations in $15 \times 5 + 9$ (lines 266-274 and the non-commutativity of $2 - 15$ (lines 275-290) in a way that was purposeful and meaningful to them. In this manner the children engaged with a duality of “is the same as” and “can be substituted for” meanings of the equals sign. This stands in contrast to the previous studies (Chapters 8 and 9) in which pupils, including Yaseen and Laura, engaged exclusively with the substitutive meaning.

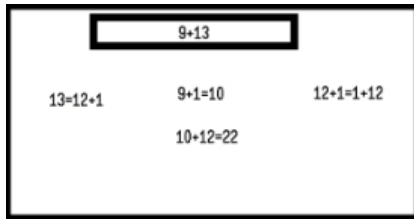
10.3 Main Study

The task for the main study comprised three key components: working through prepared diagrams as in Studies 1 and 2; constructing diagrams using provided keypads; and using spoken computational strategies to construct diagrams.

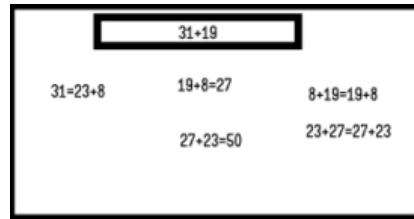
Initially, pupils were presented with a sequence of seven diagrams (Figure 10.3) in order to familiarise them with the software and task goals. The first diagrams contain only commuting and composing statements, and from the third diagram also contain partitioning statements. Diagrams contain increasing numbers of statements, with nine statements in the final diagram. The composing and partitioning statements all avoid appealing to tens-and-units decomposition. This was to ensure that “visual saliency” effects (Kirshner, 2004), such as the repetition of digits in $45 = 40 + 5$, were reduced when pupils went on to make diagrams based on their existing mental addition strategies (which are likely to exploit tens-and-units decomposition).

Following this, pupils were presented with the keypads (Figure 10.1) and challenged to make their own diagrams. They were first given a chance to familiarise themselves with inputting statements and placing them centre screen to form a diagram, and encouraged to make diagrams that contain several statements and can be completed by a hypothetical user. This provided data to inform the third research question: *Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?*

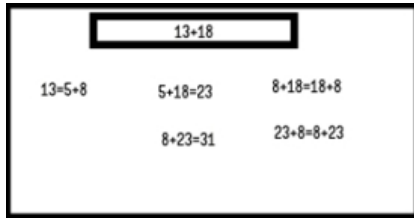
Following this pupils were asked to explain how they worked out a term presented in the box ($37 + 58$). They were then challenged to make a diagram of their spoken strategies. This provided further evidence to inform the third research question, which focuses on the pupils’ abilities to switch appropriately between numerical and diagrammatic views, as well as the fourth research ques-



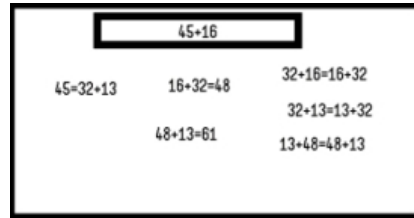
(a) Diagram 1



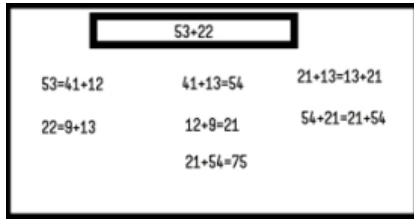
(b) Diagram 2



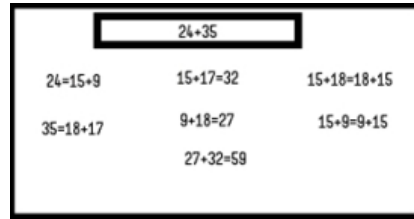
(c) Diagram 3



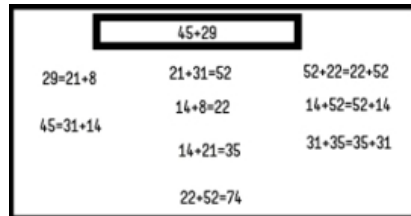
(d) Diagram 4



(e) Diagram 5



(f) Diagram 6



(g) Diagram 7

Figure 10.3: Diagrams presented in the trials

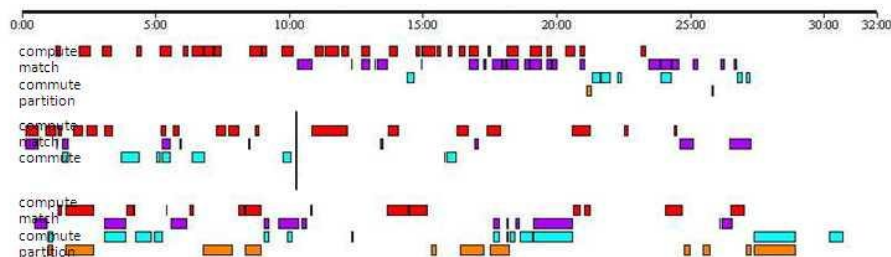


Figure 10.4: Time-sequenced coding for the Bridie and Nadine trial showing articulations of computation, iconic matching, commutation and partition

tion, which focuses on pupils’ ability to identify notational transformations with arithmetic principles: *Can children connect their implicit arithmetical knowledge with explicit transformations of notation?*. Note that in contrast to the pilot trial (Section 10.2) the pupils were only provided access to + operators throughout the whole trial. This was in order to keep their diagrams more simple and so make this final challenge more achievable.

10.4 Trial 1: Bridie and Nadine

Bridie (9 years, 3 months) and Nadine (9 years, 6 months) were deemed mathematically able by their class teacher. They were confident and talkative and got to grips quickly with the functionality of the software and task goals. Three trial sessions of about 30 minutes each took place on consecutive days. They worked through the seven prepared diagrams (Figure 10.3) during the first session and for the first ten minutes of the second session, when I introduced them to the keypads (Figure 10.1). They experimented with the keypads and made their own diagrams for the remainder of the second session and for the first half of the third session. I then challenged them to make a diagram from their mental addition strategies for $37 + 58$, which they managed to do.

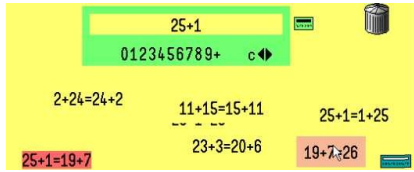
Figure 10.4 shows a time-sequenced coding of the trial¹. I will first describe how Bridie and Nadine worked through the prepared diagrams during the first session and the first ten minutes of the second session. The code “compute” is much more prevalent than in Studies 1 and 2, and this stands in contrast to all

¹Data capture and analysis was described in Chapter 7, and time-sequenced maps were explained in Section 7.3.3

other trials reported in this thesis. Bridie and Nadine were notably enthusiastic mathematicians and seemed keen to impress me with their computational prowess. Nevertheless, around halfway through the first session they began to articulate the need to find matches of numerals and terms. During this portion of the trial, both pupils frequently articulated commutation (Bridie eleven times, Nadine twelve times). Only Nadine explicitly articulated partition but, as will be seen from the transcript excerpts, there were occurrences of implicitly articulated partition. They also came to use the strategy of beginning a diagram by partitioning and then composing and commuting numerals, replicating the findings from three of the trials reported in Studies 1 and 2.

Following this, Bridie and Nadine were introduced to the keypads and made five discernible diagrams from around halfway through the second session to halfway through the third and final session (Figures 10.5a-e). The code “compute” is present throughout this portion of the trial (Figure 10.4), and in the main this reflects the pupils inputting numerically balanced statements using the keypad buttons. The codes “match”, “commute” and “partition” are also present, reflecting the pupils testing statements placed within the diagram and deciding what further statements were needed. Over the remainder of the trial, commutation was explicitly articulated thirteen times by Bridie and eight times by Nadine; partition was explicitly articulated eight times by Bridie and four times by Nadine. The pupils inputted commuting statements readily in their early diagrams, but articulated no reason for doing so. Their use of partitioning statements emerged more gradually as they identified the need to break numerals into parts in order to make more interesting diagrams.

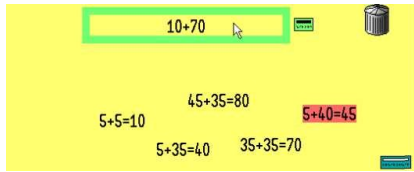
Around halfway through the third session, I asked them to work out the answer to $37+58$ in their heads, and then share their strategies aloud. I then challenged Nadine, who offered the strategy with the least steps, to make a diagram. With prompting, she entered compositional statements ($30+50=80$ and so on), and immediately identified the need for partitioning statements in order to make the diagram functional (Figure 10.5f). Similarly, Bridie quickly identified the need for partitioning statements when challenged to make a diagram using her spoken strategy for $37+58$ (Figure 10.5g). This identification of implicit partitioning allowed both pupils to produce functioning diagrams that accurately represented their spoken strategies.



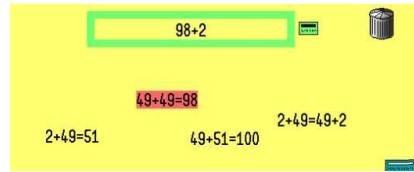
(a) Diagram 1



(b) Diagram 2



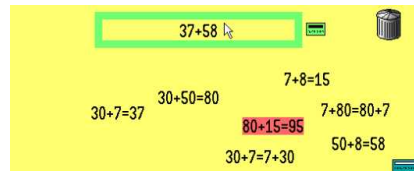
(c) Diagram 3



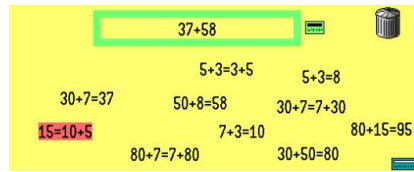
(d) Diagram 4



(e) Diagram 5



(f) Diagram 6 (based on Nadine's verbal strategy for $37 + 58$)



(g) Diagram 7 (based on Bridie's verbal strategy for $37 + 58$)

Figure 10.5: The diagrams made by Bridie and Nadine during the trial

10.4.1 Bridie and Nadine: Illustrative transcript excerpts

Prepared diagrams

Initially both pupils engaged in computing terms, and appeared to enjoy doing so. For the first half of the first session they frequently attempted substitutions of numerically equivalent rather than iconically matched terms. Iconic matching arose as they realised the need to match numerals to decide which substitutions were allowable.

BRIDIE: 31 plus 19. 292
NADINE: 19. What's that? 293
BRIDIE: 31 . . . look for a 31 somewhere. 294
NADINE: Well I found a 19 and another 19. 295
BRIDIE: But we need something that will equal 19. [pause] Aha, I found 296
a 31. (Session 1, 10:18) 297

The first explicit articulation of commutation occurred when I asked them why $27 + 23 = 50$ failed to make a substitution in $23 + 27$ (Figure 10.3b).

NADINE: Oh, we need to switch them round [numerals in the box] or switch 298
them round [left-hand side of $27+23=50$]. But we can't. 299
(Session 1, 13:52) 300

Bridie and Nadine came to articulate partition more gradually than in the trials reported in Studies 1 and 2. Precursors appear to stem from looking for matches of single numerals, such as at the start of the second diagram (lines 292-297). Partition was first explicitly articulated, as “dividing” a numeral (line 305), at the start of the fourth diagram (Figure 10.3d).

BRIDIE: Right let's see. 45 plus 16 equals fifty-, 61. [pause] Is it? 301
NADINE: Yeah. 302
BRIDIE: Yeah. So we look for 61. [pause] Oh one minute! I have an idea. 303
Equals. 304
NADINE: Because that will divide. (Session 1, 20:53) 305

The pupils completed the diagram by commuting and composing numerals, albeit with a degree of trial-and-error statement selection and dead-ends. The remaining three diagrams were completed in a similar manner, although partition was not always clearly articulated and their partition-first strategy was executed less systematically than some of the pupils in Studies 1 and 2. For example, at the start of the fifth diagram (Figure 10.3e), Nadine described a partitioning transformation in terms of the visual size of the resulting term (line 307).

BRIDIE: Now change the 53 into 41 plus 12. 306

NADINE: Okay now it's a big sum. (Session 1, 23:09) 307

However, they then worked at the diagram for a further two minutes before Nadine suggested partitioning the other numeral in the boxed term.

NADINE: The ... split that. (Session 1, 25:51) 308

Constructing diagrams

At about a third of the way into the second session I introduced the keypads (Figure 10.1) and challenged the pupils to make some diagrams. They entered $25 + 1$ in the box and then considered what statements to place in the diagram. Nadine suggested a term numerically equivalent to $25 + 1$ (lines 309-310). Bridie then suggested putting its commuted form on the right hand-side, but gave no reason (line 314).

NADINE: Now let's, plus, what equals 26, plus 11, um ... [pause] 15. 11 309
add 15 is 26. 310

R: Now on the other side of the blue, er, calculator you have to put 311
what the other side will be. So just put something in to see how 312
it works. You can always delete it and start again. 313

BRIDIE: You should put 15 add 11. [on the right hand-side] 314

NADINE: Yeah. (Session 2, 10:50) 315

They proceeded to place two more commuting statements on the screen that contained terms numerically equivalent to the boxed term, but only one was

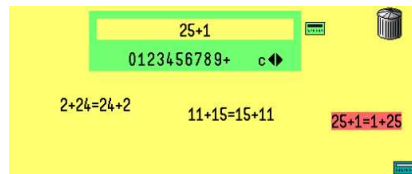


Figure 10.6: The first three statements placed in the diagram when Bridie and Nadine were introduced to the keypads

iconically matched (Figure 10.6). I asked them to test their diagram by seeing if any substitutions can be made. Nadine selected $25 + 1 = 1 + 25$ and transformed the boxed term $25 + 1 \rightarrow 1 + 25$.

- NADINE: It just switches it round. 316
- BRIDIE: At the minute it's impossible to do. 317
- NADINE: You can't do it. We need to have a sum that's only one thing. 318
- R: What do you mean? 319
- NADINE: Because all the sums are like that. [*in $a + b = b + a$ form*] We need 320
one that's just ... 321
- BRIDIE: Just so they switch round. We need one that will just ... 322
- NADINE: Is the, something. Equals thing. But not like that because ... 323
- R: Okay. Let's have a go at that because you've got the blue calcu- 324
lator [*keypad at the bottom of Figure 10.1*] so you can show me what 325
you mean. 326
- BRIDIE: So like 23 327
- NADINE: Add 3 328
- BRIDIE: 3. And then 20 plus 5. Or something like that. 329
- NADINE: 20 plus 6. 330
- BRIDIE: Yeah. That's what I meant. (Session 2, 15:31) 331

This resulted in the statement $23 + 3 = 20 + 6$ being placed in the diagram. Note it comprises terms numerically equivalent to, rather than iconically matched with, the boxed term. They did not test whether it could make a substitution, and Nadine instead suggested a statement iconically matched with the boxed term.

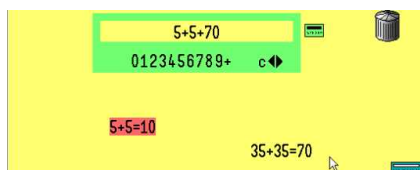


Figure 10.7: Bridie and Nadine input partitioning statements in the form $a + b = c$

NADINE: That might be an idea. Put, now let's make one of them that 332
 says 25 add 1 because then it will be easy to do. 333
 (Session 2, 16:55) 334

After some computational discussion they placed $25 + 1 = 19 + 7$ in the diagram and used it to transform the boxed term. From this point onwards they consistently inputted statements that were both numerically balanced and iconically matched with the boxed term.

During their third attempt to make a diagram, they entered $10 + 70$ as the boxed term followed by two statements of the form $a + b = c$ that were intended to partition $c \rightarrow a + b$ (Figure 10.7).

NADINE: Do 5 add 5. Equals 10. 335
 BRIDIE: And we could do [*unclear*]. 336
 NADINE: The answer. 337
 R: Just test that one. It's good to test as you go, then you won't 338
 get in so much of a muddle. 339
 BRIDIE: Yipee! 340
 NADINE: Now let's do something to make 70. Um, 35 add 35. 341
 (Session 2, 22:48) 342

Following this they went on to make the diagram shown in Figure 10.8.

Thirteen minutes into the third session I placed $37 + 58$ in the box and asked the pupils to silently work it out and then articulate their computational strategies. I then asked them to use Nadine's strategy (lines 344-348), which had less steps than Bridie's, to make a diagram.

R: Nadine, tell us how you worked it out. 343

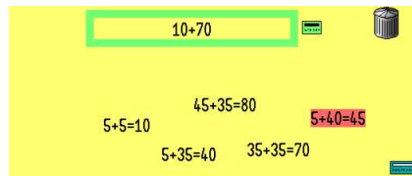


Figure 10.8: A diagram made by Bridie and Nadine

NADINE: Um, 30 add 50 is 80. 344
 R: Yeah. 345
 NADINE: And, 8 add 7 is 15. 346
 R: Yeah. 347
 NADINE: So you just have to add those together and you get 95. 348
 (Session 3, 13:46) 349

Following this Nadine entered $30 + 50 = 80$ and placed it in the diagram. Nadine spotted an impasse (line 350), and attempted to articulate the need to partition numerals (lines 352-353). Bridie then explicitly articulated the need for a partitioning statement (lines 366-367).

NADINE: Oh, now I can't do it. 350
 R: Why can't you do it? 351
 NADINE: It's impossible because you won't have them on their own unless 352
 you take them out but how do you take them out? 353
 R: Any ideas Bridie? 354
 BRIDIE: What I would do is ... 355
 NADINE: Do it then. 356
 BRIDIE: ... if ... [pause] ... you did 37 ... [enters 37 in left-hand side of 357
 bottom keypad] 358
 NADINE: ... add 58. 359
 BRIDIE: No. Actually I wouldn't. [deletes 37 from left-hand side of bottom 360
 keypad] 361
 R: Show us what you would do Bridie. 362

BRIDIE: I was going to do that and then, put that. [*places* $37 + 50 = 87$ into 363
diagram] It won't work. 364

NADINE: It won't work. 365

BRIDIE: Because first of all you need to divide 50 and the 8. Partition it. 366
 (Session 3, 15:19) 367

Bridie then deleted $37 + 50 = 87$ from the diagram. She entered $50 + 8 = 58$ and used it to transform the boxed term $37 + 58 \rightarrow 37 + 50 + 8$. Note that again the form $a + b = c$ was used for a statement intended to have a partitioning transformational effect. The pupils continued working at the diagram for another seven minutes resulting in that shown in Figure 10.5f.

Similarly, the pupils managed to make a diagram (Figure 10.5g) based on Bridie's spoken strategy (lines 368-372). Note again the use of $a + b = c$ forms for partitioning statements ($30 + 7 = 37$ and $50 + 8 = 58$) and, curiously, the use of a $c = a + b$ form for one of the composing statements ($15 = 10 + 5$). The pupils did not discuss or comment on these apparently arbitrary forms for partitioning and composing statements.

BRIDIE: I, added 30 and 50 which was 80. And then 5 plus 3 equals 8, so 368
 if you add that 3 to, um, the 87, because you have 37 and 50 ... 369

R: Mm-hm. 370

BRIDIE: So that would be 87. If you add a 3 onto the 87 you have 90 and 371
 then from that 8 you'd have 5 left, so it's 95. 372
 (Session 3, 14:00) 373

10.5 Trial 2: Derek and John

Derek (9 years, 3 months) and John (9 years, 6 months) were deemed mathematically average by their class teacher. Both pupils were confident and relaxed, and John was the more articulate and talkative. The data provide snapshots into Derek's thinking, but more substantially detail John's thinking. Three trial sessions of between 30 and 40 minutes each took place over four days. The pupils completed the seven prepared diagrams (Figure 10.3) and, with difficulty, made two distinct, functioning diagrams themselves. I then challenged them to make

a diagram from their mental addition strategies which they did not manage to do.

Figure 10.10 shows a time-sequenced coding of the trial. Consider first the prepared diagrams component of the trial, which lasted through the first session until about a third of the way into the second trial. The code “compute” appears during the first session but is absent for the bulk of the first third of the second session. This reflects a shift from computing results and looking for numerically equivalent terms across statements and the boxed term, to looking for iconic matches. During the prepared diagrams component of the trial, Derek explicitly articulated commutation four times, and John twenty times. They came to use articulations of commuting effects to discuss substitutions one or two steps ahead. However, neither pupil explicitly articulated partition, and this prevented access to the strategy of beginning a presented diagram by breaking numerals up, and they depended on trial-and-error statement selection to a high degree.

Following the prepared diagrams, the pupils were introduced to the keypads (Figure 10.1) and made two of their own diagrams (note that during the middle of the second session John left for almost ten minutes for his school photograph, and Derek worked alone). However, they struggled at diagram-making in contrast to Bridie and Nadine, and this is reflecting in the trial-and-error based evolution of the two diagrams they produced (Figures 10.9a-c and 10.9d-e). The diagram-making component ran from about a third of the way into the second session to about half way into the third session. The code “compute” is prominent throughout this component (Figure 10.10) and in the main reflects the pupils deciding and inputting numerically balanced statements at the bottom keypad. The codes “match” and “commute” are also present, reflecting the pupils placing statements in diagrams and testing they can be used to make substitutions. Over the remainder of the trial Derek explicitly articulated commuting eight times, and John ten times. Partition was articulated once only, toward the end of the trial (by John). This lack of articulation of partition appears to be the reason the pupils were less successful at making diagrams than Bridie and Nadine in Trial 1.

Finally, Derek and John attempted to make diagrams based on their own verbalised strategies for solving $37 + 48$ (Figures 10.9f and g). They were unable to make functioning diagrams that accurately represented their verbalised strategies, and failing to identify the implicit partitions in their strategies seemed key

to this.

10.5.1 Derek and John: Illustrative transcript excerpts

Prepared diagrams

Derek and John computed terms during the first session to some extent. They first commented on numerals rather than results when prompted to explain why $12 + 1 = 1 + 12$ made a transformation on the boxed term but $9 + 1 = 10$ did not (Figure 10.3a). John commented on the unfamiliarity of its $a + b = b + a$ form (line 374), and Derek expressed the need for the numeral 10 to appear in the box (line 375).

JOHN: Is it quite a different sum? Like 12 add 1 equals 1 add 12. 374

DEREK: 9 add 1, we need that 10 there. (Session 1, 07:09) 375

For the remainder of the first session they switched between computing results and looking for iconic matches, and during first third of the second almost exclusively looked for iconic matches in order to identify allowable substitutions. The following excerpt, from during the second session diagram (Figure 10.3b), illustrates this, and also contains the first occurrence of John articulating a commuting transformation (lines 382-384).

JOHN: Right. What, what is that? 23 add 19. 33. Fift-, no 42. 42. 376

DEREK: 54. Fifty-, 52. 377

JOHN: No 42. 378

DEREK: No, yeah 42. 379

JOHN: And then add 8 is 50. Right 50. We need to make it 50. Right. 380

DEREK: How are we going to do that? 381

JOHN: No try this one. Because we haven't tried that one yet. Oh there 382

we go [*transforms* $23 + 8 + 19 \rightarrow 23 + 19 + 8$]. 23 add 19 add 8. Well 383

that's the same thing but they've just mixed it round. 384

(Session 1, 15:00) 385

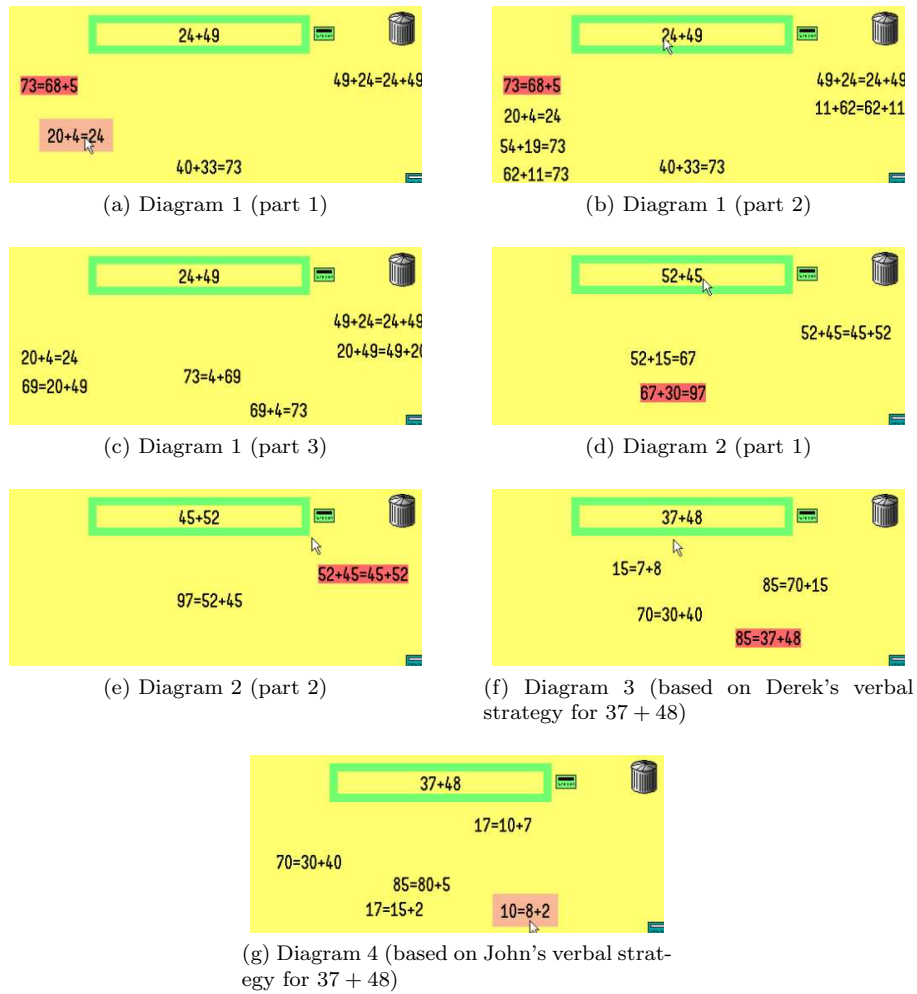


Figure 10.9: The diagrams made by Derek and John during the trial

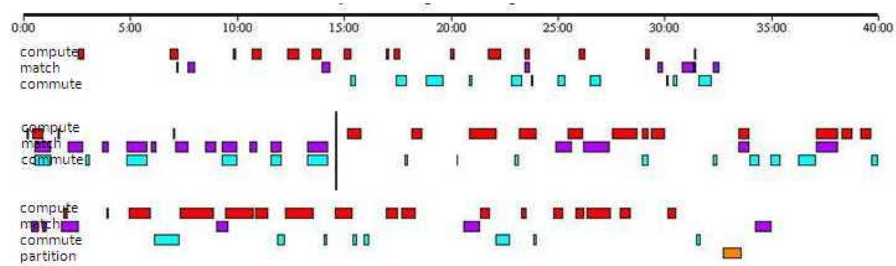


Figure 10.10: Time-sequenced coding for the Derek and John trial showing articulations of computation, iconic matching, commutation and partition

Note that John seemed somewhat sceptical of commutation's utility for achieving the task goal (lines 383-384). This theme continued during the third diagram (Figure 10.3c), when Derek used $8 + 18 = 18 + 8$ to make a transformation and John dismissed it as "just" commuting numerals. Interestingly, John apparently expected the statement $23 + 8 = 8 + 23$ to make a (non-commuting) transformation on the boxed term $5 + 8 + 18$ (line 389) despite not being iconically matched.

JOHN: That would just change it round wouldn't it? 386
 DEREK: Yeah that makes it ... 387
 JOHN: Don't, don't try that one [$18+8 = 8+18$] because it's just changing 388
 it round. Try, yeah that one, that one. [$23+8=8+23$] 389
 DEREK: No. (Session 1, 20:50) 390

However, later during the first session, John came to see a utility for commutation (lines 393-395) when prompted to explain why $16 + 32 = 48$ failed to transform $13 + 32 + 16$ (Figure 10.3d).

JOHN: Is it because it's the wrong way round? The 16 and the 32? 391
 R: Is there anything you could do about that? 392
 JOHN: Oh yes, yeah, yeah, yeah, yeah. I thought this was useless but 393
 now it's useful. These bits. Okay. Right, now we've just changed 394
 it round. Now try. There we go. Now, 13 add 48. Now that one. 395
 (Session 1, 31:14) 396

Neither pupil articulated partition explicitly or implicitly during the prepared diagrams component of the trial, and so did not gain access to the strategy of starting each diagram by partitioning numerals. This is illustrated by their trial-and-error approach when presented with the final diagram (Figure 10.3g).

DEREK: Okay. 74. 397
 JOHN: Oh no there's loads [*of statements*] now. Okay. The last one [*dia-* 398
gram], and the hardest. Wait. Try something normal like 45 and 399
 29. I'll try this one. There you go. 31 add 14. [*pause*] We need 400
 something against 35. How are we going to make it to 35? 401

DEREK: Have we tried this middle one yet? 402

JOHN: This one? 403

DEREK: No the top [*unclear*] 404

JOHN: This one? 405

DEREK: That one. Yeah. 406

JOHN: No. That wouldn't work either but, 31. No that wouldn't work. 407
 If you just try it in there that wouldn't work. Would it? No. 408
 (Session 2, 07:05) 409

Constructing diagrams

About a third of the way into the second session I introduced Derek and John to the constructing diagrams keypads (Figure 10.1). They began by entering $24+49$ in the box and John placed $24+49 = 49+24$ into the diagram but gave no reason. At this point John was called away for almost ten minutes for his school photograph, leaving Derek to work alone. He entered $40 + 33 = 73$, which is numerically equivalent to the boxed term but not iconically matched with it, and placed it at the bottom of the diagram (Figure 10.9a), saying “that’s the answer”. His positioning of the statement at the bottom of the diagram reflects the position of the final statement in each of the prepared diagrams (Figure 10.3).

When John returned they discussed another term that would be equivalent to the value of the boxed term, finally deciding upon $68 + 5$. Without comment John entered this as $73 = 68 + 5$ and placed it in the diagram. They discussed the diagram and appeared to be attempting to replicate their perception of the structure of the prepared diagrams (Figure 10.3).

JOHN: Okay. Er, can't we just do this one [$24 + 49 = 49 + 24$] as, um, [*the* 410
researcher] did. One over there [*positions* $24 + 49 = 49 + 24$ *on right of* 411
diagram]. 412

DEREK: That's a swap. 413

JOHN: That one . . . there [*positions* $73 = 68 + 5$ *on left of diagram*]. 414

DEREK: That one [$40 + 33 = 73$] needs to be at the bottom because that 415
 one's the answer though. That's the answer, that one we need 416
 to do at the bottom. 417

JOHN: Yeah, here. Okay. Can we do another one of these? [*i.e. of the* 418
form $c = a + b$] Like but a different sum, like ... 419

DEREK: 40 add 24 add, you know ... 420

JOHN: We need something else. So like, do an add sum to 73, which, so 421
basically like 40 something. (Session 2, 22:57) 422

They did not test their statements and I asked which they thought would work. This led John to articulate the need for statements that contain numerals in the box, although Derek did not appear to follow his reasoning (lines 424-427). John then identified the need for a numeral in the box, rather than the computational result of the box, to be one one side of the statement (lines 431-434).

DEREK: Er, I don't know. Like ... 423

JOHN: Oh yeah, we need to make like a 24 and 49 something. 424

R: Do you think he's right Derek? 425

DEREK: I don't really know. Kind of like ... 426

JOHN: Yes, like 24 and 49. Like we did yest... – last time. Remember? 427

R: See if you can show us what you mean. 428

JOHN: Okay. See, like ... 429

R: You need to [*unclear*] 430

JOHN: 51. Um, add something. No we need 24, um ... We don't want 431
51. 432

JOHN: 24, like kind of stuff and then, something in that sum which is 433
24. (Session 2, 24:45) 434

A moment later John suggested the statement $24 = 20 + 4$ (lines 435-440), although Derek did not understand why (line 442). John explained his reasoning not in terms of partitioning numerals, but in terms of how many $c = a + b$ statements he perceived needed to be in the diagram (lines 445-446), probably recalling the final three prepared diagrams (Figures 10.3e-g). It is also interesting to note his choice of a tens-and-units partition ($24 = 20 + 4$) despite none having appeared in any of the prepared diagrams. This was presumably to reduce computational burden.

JOHN: Ah, do some of that [*points mouse at $73 = 68 + 5$*]. 24 equals some- 435
thing like 20 add 4. 436

DEREK: Yeah. 437

JOHN: 24 add ... 438

DEREK: [*unclear*] 439

JOHN: Add 20. Oh I've done that wrong. 24 equals, um, 20 add 4. 440

R: Do you understand why he's doing that Derek? 441

DEREK: No not really. 442

JOHN: Why not?! 443

R: See if you can explain it John. 444

JOHN: Because we need more than one of them, don't we? Just can't 445
have one there. So we need another one. (Session 2, 26:14) 446

Following this, however, they placed in the diagram two $a + b = c$ statements that were numerically equivalent to the boxed term but not iconically matched with it ($54 + 19 = 73$ and $62 + 11 = 73$ in Figure 10.9b), without explaining their reasoning. During the first half of the third session they made a second diagram consisting of three statements (Figure 10.9d). One of the statements is substitutive with respect to the boxed term ($52 + 45 = 45 + 52$), and another is numerically equivalent to the boxed term but not substitutive ($67 + 30 = 97$).

Halfway through the third session I entered $37 + 48$ in the box and asked the pupils to work it out in their heads and then explain how they did it. I then asked them to use Derek's method ("I just added the 30 and 40 together, added the 8 and 7 together") to make a diagram. With guidance they inputted this spoken strategy as composing statements, which they entered without explanation or comment in the form $c = a + b$ (Figure 10.9f). At this point I asked them about the functionality of the diagram, and John suggested inputting $37 + 48 = 85$ (line 453).

R: Do any of them work at the moment? 447

JOHN: Um, well, not really no. Because look, 37 and 48. That wouldn't 448
work, definitely. 449

DEREK: Try that one, 85 and 70 ... 450

JOHN: That won't work, definitely. That won't work, definitely. 451
 R: Okay. Any ideas what else you could put in to get them working? 452
 JOHN: 37 add 48 equals 85. (Session 3, 21:04) 453

This resulted in a diagram that did not represent Derek's spoken strategy for $37 + 48$ because it contained only one functional statement ($37 + 48 = 85$). I then challenged the pupils to make a diagram using John's spoken strategy for $37 + 48$ (lines 457-460), which the pupils entered as composing statements, albeit in $c = a + b$ form (Figure 10.9g).

R: And, er, what was it we had, do you remember? 37 plus 48. So 454
 what, tell me your, before you do anything tell me your strategy 455
 again John. 456
 JOHN: Um, 30 add 40 equals 70. 457
 R: Mm-hm. 458
 JOHN: 7 add 8. I just thought the 8 equals 10, and then 7's 17. 8 to 10 459
 is 2, 17 takeaway 2 is 15. And 70 add 15 equals 85. 460
 (Session 3, 24:48) 461

Following this I prompted them to consider how to make the diagram functional. Derek suggested a commuting statement, although gave no specific example or reasoning (line 470), and John was unsure (line 472).

R: Okay. So which is the first one you put in? 462
 JOHN: 70 add 30 equals, er add. 70 equals 30 add 40. 463
 R: And will that work in the green box? 464
 JOHN: Um, no. 465
 DEREK: Won't it? 466
 JOHN: No it won't. 467
 R: Okay. Anything you can think to put in to make it work? 468
 JOHN: Um. 469
 DEREK: We need one of them swap arounds. 470
 JOHN: Um. 471

JOHN: Oh, is it [*unclear*]. No that won't work either. 472
(Session 3, 31:12) 473

I then prompted them to consider the first term they had entered, $30 + 40$.

R: Hm, but how did you know you could do that? Where did you 474
get the 30 and 40 from? Because there's not a 30 add 40 in the 475
green box. 476

JOHN: Well you just do the tens. 30 add 40 usually is a good way to 477
start for the tens. 478

R: Okay. 479

JOHN: Um, we don't know how to do it. Maybe we should do something 480
else instead of 30 add 40. (Session 3, 32:10) 481

Following this I showed them the diagram they had made during the second session (Figure 10.9c), and drew their attention to the first statement in the sequence ($20 + 4 = 24$). John made the only explicit articulation of partition that occurred during the trial (line 489), and also seemed to think the role of tens was significant (line 492).

R: Now I remember, D, you started this. And can you remember 482
which one you started with? 483

DEREK: Was it... ? 484

JOHN: No it was 20 add 4 equals 24 wasn't it? 485

DEREK: Yeah. 486

JOHN: Yeah it was that one. 487

R: Okay. So let's see how that works. 488

JOHN: That separates it. 489

R: Okay. So would something like that help us in our puzzle we're 490
doing now? 491

JOHN: Oh yes! So we do like seven-, oh no because they're both tens. 492
30 add 0. 493

DEREK: No, I know. 494

JOHN: 30 add 0. (Session 3, 33:15) 495

I brought the pupils' current diagram back on screen (Figure 10.9f) and pursued Derek's claim to know what to do (line 494). He suggested and inputted a partitioning statement (line 506), although it was numerically equivalent to the boxed term rather, than iconically matched with one of its numerals, and so was not functional.

R: So what was your idea Derek? 496

DEREK: Um. Like a, um, oh do you know on the last one we did, um. 497
What was it? Can I just have a look? 498

R: Yeah, yeah you can have a look. 499

JOHN: First one right, we did with 24. 500

DEREK: Yeah, so you do it ... 501

JOHN: Can't we just do like 30 add 0 equals 30? 502

DEREK: No. Can't you just put like ... What are we trying to make 503
again? 504

JOHN: Because, no, there's no units. 505

DEREK: 85, so couldn't you put like 80 add 5? 506

R: Okay. 507

JOHN: Let's see. 80 add 5 equals 85. 508

DEREK: Will that make anything different? 509

JOHN: No, no, no, no. I didn't think that will work. 510

(Session 3, 34:12) 511

At this point time ran out and the trial concluded.

10.6 Trial 3: Colin and Imogen

Colin (9 years, 3 months) and Imogen (9 years, 9 months) were deemed mathematically average by their class teacher. The pupils were somewhat reserved at the start of the trial, requiring more researcher prompting for verbal elaborations than the pupils in other trials, but gradually opened up over the first

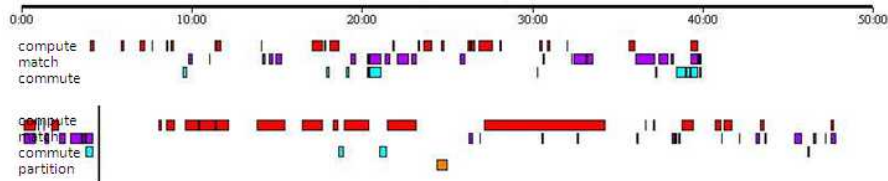


Figure 10.11: Time-sequenced coding for the Colin and Imogen trial showing articulations of computation, iconic matching, commutation and partition

session. Two trial sessions of about 40 and 50 minutes each took place on consecutive days. They worked through four of the prepared diagrams during the first session with some difficulty, and were introduced to the keypads (Figure 10.1) early in the second session. They constructed three diagrams during the second session, and struggled to make a fourth. They were not challenged to make a diagram based on their mental addition strategies for a given term.

Figure 10.11 shows a time-sequenced coding of the trial. During the first session and the first few minutes of the second session, when the pupils worked through the prepared diagrams (Figure 10.3), the codes “compute” and “match” are prominent as the pupils switched between computational and diagrammatic views. Colin articulated commuting transformations thirteen times, and Imogen nine times. This was often in response to my prompts to explain what transformations had taken place, but the pupils were somewhat dismissive of the usefulness of commutation overall. Neither pupil articulated partition during the first session and they did not access the strategy of starting each diagram by partitioning numerals.

Early into the second session, the pupils were introduced to the keypads (Figure 10.1) and made four diagrams of their own. The first three diagrams contained only one functional statement each (Figures 10.12a-c), although two of them also contain redundant statements (Figures 10.12b and d). Their fourth and final diagram contained ten statements but only some are substitutive with respect to the boxed term, and only one is required to solve the diagram (namely $149 = 55 + 94$). In contrast to Trials 1 and 2, I did not challenge Colin and Imogen to make a diagram based on their mental addition strategies for computing a given term. The constructing diagrams component of the trial was dominated by “compute” (Figure 10.11) as the pupils decided which numerically balanced statements to place in the diagrams. Imogen in particular focused on computing

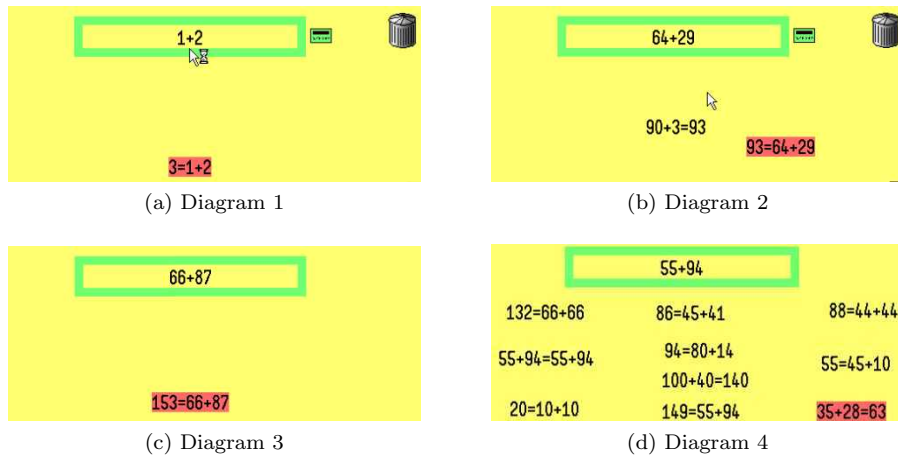


Figure 10.12: The diagrams made by Colin and Imogen during the trial

results, although Colin frequently focused on iconic matching. Imogen articulated commutation twice, and Colin not at all. Partition was articulated only once during the trial (by Colin). This lack of articulation of partition, as well as Imogen's and Colin's respective focus on computation and iconic matching with little evidence of a shared meaning, appear to be the reason they did not work strategically through the prepared diagrams and struggled to construct diagrams.

10.6.1 Colin and Imogen: Illustrative transcript excerpts

Prepared diagrams

Colin and Imogen were somewhat reticent during the first half of the first session, and I intervened substantially to try and get them to start discussing. The following excerpt illustrates this, taken from when they were presented with the second diagram (Figure 10.3b).

- COLIN: 31 add 19. [pause] 50. 512
- R: Do you agree, Imogen? 513
- IMOGEN: Er, yeah. 514
- COLIN: Because 9 add 1 is 40. 515
- IMOGEN: [unclear] 50. 516

R: If you've got any suggestions, Colin, or you have any ideas, Imo- 517
gen, you can tell each other. 518

IMOGEN: That should be red [*selected*] because that one's 50 and that one's 519
50. 520

R: What do you think about that Colin? 521

COLIN: Maybe you click some other ones first and then try and move it. 522

R: Okay. Some of them are working, and some aren't. 523

IMOGEN: Try that one. 524

COLIN: [*whispers*] 20 add . . . [*pause*] Try that one. [*pause*] 19 add 8. 525

IMOGEN: 27. [*pause*] No. 50, that should work. (Session 1, 06:59) 526

The first occurrence of articulating commutation followed shortly (line 528) in response to prompting by me.

R: And what happened when you clicked in the green box then? 527

COLIN: Turned the 27 add 23 and swapped them. 528

R: Okay. What do you think Imogen? 529

IMOGEN: I agree. (Session 1, 09:29) 530

The next handful of “commute” codes (Figure 10.11) were, similarly, descriptions of observed transformations in response to my prompting. The first occurrence of commutation being articulated toward the task goal was during the fourth diagram (Figure 10.3d).

IMOGEN: Try that one. 531

COLIN: Now we need that one that swaps them round. 532

IMOGEN: But we've got 48 so try it. No. Click on it properly. [*pause*] No, 533
that one. 534

COLIN: We've got to swap them back round again. 535

IMOGEN: Yeah now try that. Then try that. Why doesn't it work? 536

R: I think Colin knows. Why doesn't it work Colin? 537

COLIN: Because we need them, we need a 48 first and 13 after. 538

R:	What do you think Imogen?	539
IMOGEN:	I, I agree. I bet you have to swap that round [unclear] that one. And then try.	540 541
R:	Which one did you mean, Imogen, exactly?	542
IMOGEN:	Um. If you try and click on one of them two, the ones that swap them around, it might work so you can click on that one. Try that one. (Session 1, 20:18)	543 544 545

Neither pupil articulated partition and they worked through the remaining diagrams in an a trial-and-error manner, only occasionally working one or two steps substitutions ahead despite prompting from myself.

Constructing diagrams

The pupils were introduced to the keypads (Figure 10.1) early in the second session. During the first few diagrams they made they focused exclusively on computational results and inputted statements that were numerically equivalent to the boxed term (see 5 to 20 minutes of the second session in Figure 10.11). The first diagram they made consisted of $1+2$ in the box and the lone statement $3 = 1+2$ (Figure 10.12a). Note that they used a $c = a + b$ form for a composing statement with explanation or comment, as had the pupils in Trials 1 and 2. For their second diagram they put $64 + 29$ in the box and inputted the statement $93 = 90 + 3$ (Figure 10.12b – note their use of a tens and unit statement despite none having been presented in the prepared diagrams). I asked whether they expected this to make a substitution. They were uncertain, and after finding that it did not, Colin offered a results-based explanation (line 553).

R:	Before you test it. Do you think that will work? Or do you think it won't work? Just have a guess. Doesn't matter if you're wrong.	546 547 548
COLIN:	Um, works maybe.	549
IMOGEN:	A bit.	550
COLIN:	[attempting to transform the boxed term] Nope.	551
R:	Why not?	552
COLIN:	Maybe because it doesn't make 93?	553

R: What do you think Imogen? 554
 IMOGEN: I disagree because it does, 90 add 3 is 93. (Session 2, 11:24) 555

When I prompted them for a different reason, Colin suggested $90+3 = 93$ might work (as opposed to $93 = 90+3$), and inputted it into the diagram. Before Colin tested it, Imogen offered an unprompted prediction and suggested $64 + 29 = 93$ instead (lines 559-561).

IMOGEN: Try it then. I don't think it will work. 556
 R: Why did you say that? 557
 IMOGEN: Because it's just, it's the same sum but just with the numbers 558
 swapped around. So it doesn't really ... Ah, I know what will 559
 work. If you put six-, er, if you put 64 add 29 there, and you put 560
 93 there and that works. Because that's the sum. 561
 (Session 2, 12:14) 562

For the fourth diagram I asked the pupils to make it so that at least two statements were required to solve it. They entered two statements that were numerically equivalent and iconically matched with the boxed term (Figure 10.12d). I asked them to test $55 + 94 = 55 + 94$ and in the discussion that followed Colin suggested statements that equaled one of the numerals in the box (line 572).

R: What's it done? 563
 IMOGEN: Nothing. 564
 COLIN: Even though it, even though it ... 565
 IMOGEN: It works but it doesn't do anything. 566
 R: Why doesn't it do anything? 567
 COLIN: That's because we need more sums to do it. We only need that 568
 one, we have both of them, let's say, and then with 94 add 55. 569
 Then we can use that to swap it round. 570
 IMOGEN: So what shall I do? 571
 COLIN: You just need a sum that equals 55 and equals 94. 572
 IMOGEN: So what could equal 55? 44 add 10. 573
 COLIN: 45 add 10. 574

IMOGEN:	Oh yeah. 45 add 10.	575
COLIN:	Need to start a new one. Equals 55. [<i>Imogen places</i> $55 = 45 + 10$]	576
	Now you need something that equals 94.	577
IMOGEN:	[<i>transforms boxed term</i> $55 + 94 \rightarrow 45 + 10 + 94$] Oh that worked. Yeah.	578
COLIN:	Then [<i>unclear</i>] equals 94.	579
IMOGEN:	You could have 80 add 14.	580
COLIN:	Yeah. (Session 2, 21:14)	581

Following this Imogen placed $94 = 80 + 14$ into the diagram. A few moments later, when asked to show how someone might solve the diagram, Colin explicitly articulated the only occurrence of partition in the trial (line 585).

R:	So, show me how you solve that puzzle.	582
COLIN:	Um.	583
IMOGEN:	I would go to that one.	584
COLIN:	We just need to go to that one. These two first. Split them	585
	up. (Session 2, 24:24)	586

At this point they discovered they were unable to make further substitutions and began to enter more statements, including some that were deliberately redundant (Colin: “we just need some sums that don’t do anything”). During the remainder of the session, they focused mostly on computing possible statements to enter, with Imogen standing up to make use of a whiteboard for working out. This resulted in a diagram that was complicated but not particularly functional (Figure 10.12d) as the trial drew to a close.

10.7 Discussion

The prepared diagrams components of the three trials reported in this study support the findings reported in Studies 1 and 2. To varying degrees all pupils switched from computational to iconic matching views of presented diagrams, all articulated commuting transformational effects, and all used these articulations to discuss substitutions one or two steps ahead. Only Nadine from Trial 1 explicitly articulated partition when working through the prepared diagrams

(lines 305 and 308), but partition was frequently implicit in Bridie and Nadine's discussion (e.g. lines 306-307), and they alone adopted the strategy of starting new diagrams by partitioning numerals.

The pilot trial provided a first demonstration of pupils constructing their own diagrams. The focus was partly on software functionality but happily the trial demonstrated the feasibility of the task. In the main trials, pupils succeeded at constructing diagrams to differing degrees. In Trial 1, Bridie and Nadine successfully constructed their own diagrams, and successfully implemented their mental addition strategies as functioning diagrams. They were able to switch flexibly between numerical and diagrammatic views, and were able to connect the distinct transformational effects of distinct statement forms with the transformations implicit in their spoken strategies. They frequently articulated partition when making their own diagrams (e.g. lines 366-367) and this seems to be the key to their success. In Trial 2, Derek and John successfully constructed their own diagrams too, although less efficiently than Bridie and Nadine. Notably, Derek and John initially attempted to replicate from memory the structure of the presented diagrams (lines 410-422) rather focusing on placing functioning statements into the diagrams. They frequently discussed computational results and numerical sameness when constructing diagrams, but partition was articulated only once toward the very end of the trial (line 489). They did not manage to implement their mental addition strategies as functioning diagrams. In Trial 3, Colin and Imogen did not manage to construct diagrams that contained more than one or two functional statements. They frequently discussed computational results when deciding on statements to place in the diagram, and partition was articulated only once (line 585) while testing one of their own diagrams. It seems then, that pupils were able to construct functioning diagrams, to varying extents, by focusing on iconic matching, but that articulating partition was key to constructing diagrams that represent mental addition strategies.

Despite this varied success with diagram construction, there are two notable themes across all three trials. The first is indifference to statement form when inputting statements intended to have a composing or partitioning transformational effect (e.g. the intended partitioning effect of $5 + 5 = 10$ and $35 + 35 = 70$ in Figure 10.5c; and the intended composing effect of $97 = 52 + 45$ in Figure 10.9e). This indifference is in fact consistent with the diagrammatic view of equality statements in which form is a property not of a statement in isolation, but of statement in terms of the notation it acts upon. In other words, $a + b = c$

is a composition when acting on $a + b$ and a partition when acting on c (see Section 6.2.2). However, it cannot be assumed the pupils explicitly grasped this point, and it more likely arose from a quirk of the keypad for inputting statements which allows the user to start in either the left or right hand side (Figure 10.1). As was seen in the *Equivalence Calculator* study (Section 5.2.2) this lends an arbitrariness to how pupils use the keypad. Moreover, it often saves button presses to recycle the notation remaining in the keypad screens from the previous statement when entering a new one, and this effects the form of statements (see Section 5.2.2, page 44). This has implications for redesigning the software: the keypad gadget at the bottom of Figure 10.1 would perhaps be better as a simple text-box in which users enter statements at the keyboard.

The second common theme across the three diagram construction components of the three trials was pupils' use of tens-and-units decompositions. This is interesting because tens-and-units decomposition was deliberately avoided in the prepared diagrams (see Figure 10.3, page 138). In Trial 1, Bridie and Nadine identified the tens-and-units decompositions in their mental addition strategies and entered them as statements. However, in the main pupils exploited tens-and-units without explanation or comment, and this was presumably simply to reduce computational burden. The exception to tens-and-units decomposition was Bridie and Nadine's early attempts to construct diagrams in which they used halving decompositions (e.g. $5 + 5 = 10$ and $35 + 35 = 70$ in Figure 10.7). Again, this presumably was to reduce computational burden.

10.8 Summary

The study presented pupils with seven prepared diagrams to familiarise them with the software and task goals. The findings from this component of the trials reflect those from Studies 1 and 2. All pupils shifted to looking for iconic matches rather than computing results, and all articulated commutation to varying extents. Only those pupils who articulated partition, Bridie and Nadine, developed the strategy of beginning each diagram by partitioning numerals in the boxed term.

Following this, pupils were challenged to make their own diagrams. This required inputting numerically balanced statements and using them to make substitutions in a diagram. This component of the trials informs the third research

question: *Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?*

All the pupils were able to make diagrams, and so were able to coordinate sameness and exchanging meanings for the equals sign, with varying degrees of sophistication and success. Only Bridie and Nadine efficiently made diagrams that were complicated and functional. Derek and John initially tried to make diagrams that visually resembled the seven prepared diagrams, but then went on to construct functioning diagrams by focusing on iconic matches across statements and the boxed term. Colin and Imogen struggled to make diagrams that contained more than one functional statement, and tended to focus on computational results and matching terms by numerical equivalence.

The pupils in Trials 1 and 2 were then challenged to make diagrams that represented their mental addition strategies for a provided term. This informs the fourth research question: *Can children connect their implicit arithmetical knowledge with explicit transformations of notation?*

Only Bridie and Nadine identified the implicit partitioning in their spoken strategies, and they alone were able to accurately represent them as functioning diagrams on the screen. Derek and John struggled to identify the implicit partitioning in their strategies, despite prompting from me, and so failed to represent their strategies as accurate, functioning diagrams. Tentatively, this suggests that articulating partition is key to successfully identifying the transformations implicit in mental calculation and representing these transformations as explicit statements on the screen.

It was also found that all pupils were indifferent to the form of statements intended to have composing and partitioning transformational effects, and this appeared to arise from a quirk of the software interface. Pupils also tended to use tens-and-units decompositions, and in one case halving decompositions, despite these having been deliberately avoided in the prepared diagrams. Pupils appeared to do this simply to reduce computational burden.

Chapter 11

Discussion

11.1 Introduction

The thesis has presented three studies that generated qualitative data that stand in contrast to that presented in the literature reviewed in Chapter 2. In this chapter I revisit the literature and contrast it with findings presented in the thesis.

11.2 Revisiting the literature

We saw in Section 2.2 that in arithmetic contexts most pupils from Western countries view the equals sign as a place-indicator for a numerical result, as in $2+3 = 5$. The place-indicator view tends to be exclusive and stubborn, resulting in pupils rejecting or modifying presented statements that do not conform to the structure “numeral-operator-numeral-equals-numeral”. Indeed, this was the key finding of Behr et al.’s (1976) seminal study in which pupils were reported rejecting statements of the form $a + b = b + a$ and $c = a + b$, amongst others. This is also consistent with the preliminary study reported in this thesis in which pupils working with the *Equivalence Calculator* tended to write statements down using the structure “term-equals-result” without hesitation or comment, despite them often appearing on the computer screen in the form “result-equals-term” (Section 5.2.2). However, in contrast to the wider literature and the *Equivalence Calculator* study, the main studies of this thesis found that pupils readily worked

with $a + b = b + a$ and $c = a + b$ statements without rejecting or attempting to modify them.

The interventionist literature (see Section 2.3) reports that pupils accept a wider range of statement structures if they are taught the equals sign means “is the same as” and are presented with a variety of statement examples. Moreover, pupils become flexible enough in their thinking to accept and work with statement structures that they have not previously encountered. However, such interventions take careful planning and must be sustained because of the stubbornness of the place-indicator view. In other words, most pupils do not readily or effortlessly come to read equality statements in new, qualitatively different ways. This stands in contrast to the findings reported here, in which most pupils quickly came to work in substitutive ways despite being presented with atypical statement structures.

11.3 A task-based account of the findings

Up until the early 1980s, when Piagetian stage theory dominated educational research (see Section 4.2), scholars sought to explain pupils’ difficulties in terms of cognitive limitations (Section 3.2.1). However, following Baroody and Ginsburg’s (1983) seminal study it became clear that even young pupils were capable of thinking flexibly about a variety of statement syntaxes and that difficulties were a product of pupils’ previous experiences of notation in classrooms, and this remains the common view today across both educationalists and psychologists (Section 2.4). In this thesis the focus is less on cognitive limitations and classroom experience, and more on the role task design can play on how pupils talk about and work with arithmetic notation. It is the novel approach to task design, rather than cognitive limitations or classroom experience, that best explain the contrast between the findings reported here and those in the wider literature.

In Section 4.4.1 we saw that there are two main approaches to the design of notating tasks: referential and structural. Referential approaches, such as modeling, import symbol meaning from external objects. In structural approaches meaning instead arises from the relationships between symbols. In practice, however, structural approaches often imply that symbols are referents to abstractions such as numbers and arithmetic principles. When a statement is

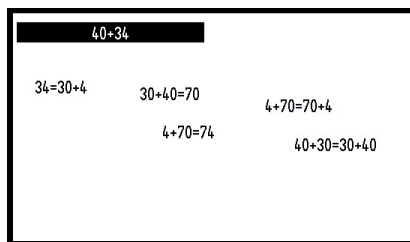


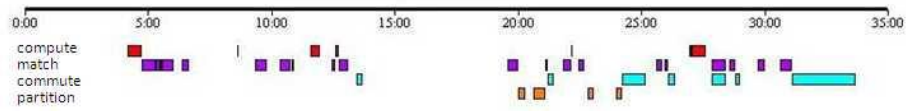
Figure 11.1: An example diagram from the studies (reproduced from page 98)

presented as an isolated question of numerical balance the notation on each side references an (abstract) number, and statements are usually carefully selected to reference a particular arithmetic principle too. For example, $2 + 3 = 3 + 2$ invites exploiting the commutativity principle in order to know the same number is on each side without knowing what the number actually is.

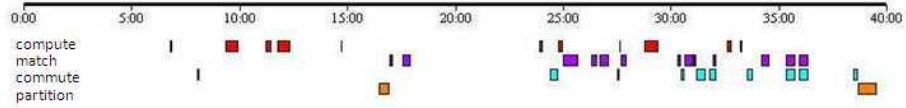
In Chapter 6 I interpreted for the case of equality statements Dörfler’s (2006) contention that learners should be presented with diagrammatic notating tasks in which symbols (or “inscriptions”) and their transformations are themselves the “very objects” of mathematical study. The substitutive meaning for the equals sign, which is integral to the very nature of equivalence relations (Section 3.4), means that equality statements can be presented as rules for making transformations of notation, and this formed the basis of the tasks used in the studies presented in this thesis.

Diagrammatic activities are based in seeing potential transformations, and so are exploratory and empirical in nature, enabling learners to make discoveries and test hypotheses. For example, if $2 + 3 = 3 + 2$ is presented as a rule for making the substitutions $2 + 3 \rightarrow 3 + 2$ and $3 + 2 \rightarrow 2 + 3$ it becomes a reusable tool for transforming notation, rather than a closed question of balance that is discarded once assessed. In the studies reported here pupils were presented with sets of inter-related arithmetic statements for making substitutions, akin to simultaneous equations in algebra. These presentations of statement sets can be considered “diagrams” in the sense that Dörfler uses the term. (A diagram from the studies is reproduced in Figure 11.1 for reference).

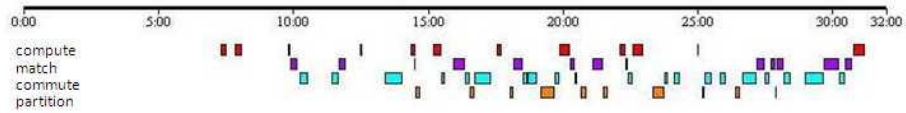
It is this diagrammatic task design that explains the disparity between the findings reported here and those in the wider literature. Most pupils in the trials initially began with computational readings of the notation on the screen, consistent with the place-indicator view of the equals sign in which the left-hand



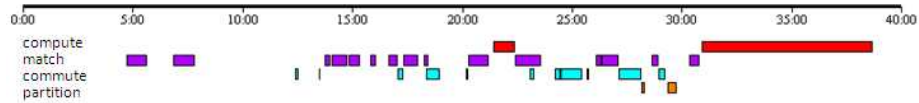
(a) Study 1, Trial 1: Terry and Arthur



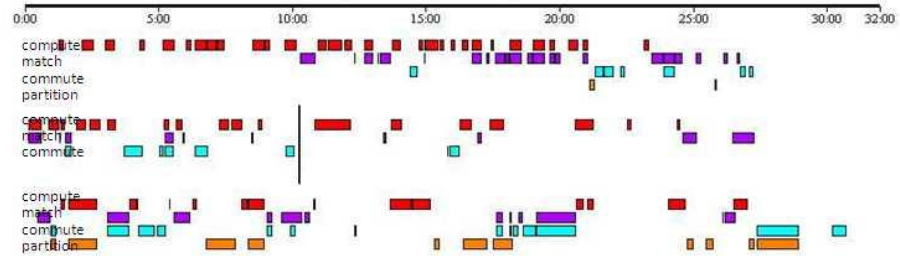
(b) Study 1, Trial 2: Kitty and Zelda



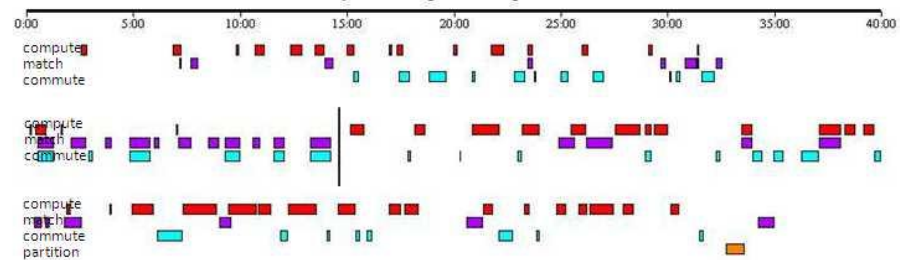
(c) Study 2, Trial 1: Laura and Yaseen



(d) Study 2, Trial 2: Ajay and Nikisha

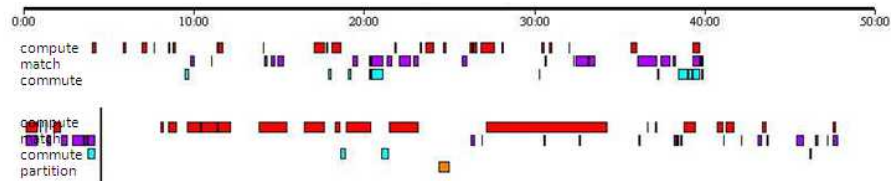


(e) Study 3, Trial 1: Bridie and Nadine



(f) Study 3, Trial 2: Derek and John

Figure 11.2: Keyword maps from across the three studies



(g) Study 3, Trial 3: Colin and Imogen

Figure 11.2: Keyword maps from across the three studies (continued)

side is computed in order to determine the result on the right-hand side. Interestingly, the pupils quickly and in most cases without apparent difficulty switched to looking for matches of notation across statements and the boxed term. This observation-based diagrammatic activity, referred to in the thesis as *iconic matching*¹, arose because it is necessary to search for identical inscriptions within a diagram in order to make transformations in the boxed term. The pupils’ shift from computing terms to iconic matching activities can be seen visually in the time-sequenced maps for each trial², which are reproduced together from the three studies in Figure 11.2. The diagram-solving components of each trial are of relevance here, and comprised the whole of Studies 1 and 2 (Figures 11.2a-d) and up to the vertical line in the second session of each trial in Study 3 (Figure 11.2e-g). (The coding after the vertical lines in the Study 3 maps represents the diagram-*making* component of the trials and is discussed later). Looking across the top rows of the time-sequenced maps, which are labeled “compute”, we can see that the pupils computed results less than they engaged in the diagrammatic activities coded “match”, “commute” and “partition”. (The exception was Bridie and Nadine who seemed keen to impress me with their arithmetic competence and computed results enthusiastically until the final ten minutes of the first session – Figure 11.2e.) More specifically, the “compute” code is prominent in the first ten minutes of each trial (bar the trial with Terry and Arthur, who did not articulate computation at the start of the trial but may well have computed results silently in their heads – Figure a), but is less present than the other codes in the final ten minutes. (The long “compute” bar in Figure 11.2d is from a post-trial semi-structured interview – see lines 234-241, page 126).

¹A table of diagrammatic terms as defined for the purpose of the thesis can be found on page 67.

²Time-sequenced maps were explained in Section 7.3.3.

In sum, then, the diagrammatic task design resulted in the pupils quickly shifting from computation to looking for iconic matches across diagrams, without the difficulties we would expect from their previous classroom experiences of equality statements in which the equals sign implicitly means “write the answer here”. In terms of the task goals, the prominence of iconic matching a few minutes into diagram-solving arose when pupils were challenged to work out why attempted substitutions sometimes failed, and to predict which substitutions would work. In this way, it is the implicit “can be exchanged for” meaning of the equals sign along with the presentation of sets of inter-related statements that can be selected and tested for substitutive effects on a computer screen that promote iconic matching activities and demote computational readings of notation. This in turn provides the basis for the four research questions that ground the studies:

1. Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?
2. Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct?
3. Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?
4. Can children connect their implicit arithmetical knowledge with explicit transformations of notation?

In the following sections I address each of these questions in turn.

11.3.1 Does the “can be exchanged for” meaning for the equals sign promote attention to statement form?

In Section 6.2.2 I distinguished between statement *structure* and statement *form*³. In much of the interventionist literature in which pupils are presented with a variety of statement types the intent is to promote “structure sense” (Linchevski & Livneh, 1999), also referred to as “relational thinking” (Carpenter et al., 2005). Statements are carefully selected to encapsulate arithmetic principles, such as commutativity and associativity, so that learners might establish

³See also the table on page 67.

numerical balance without calculating or knowing the number represented by the notation on either side of the equals sign. For example, $2 + 3 = 3 + 2$ might be presented as a referent to the principle of commutivity.

Structure, then, is an inherent property of a statement presented in isolation, and relates to the pedagogically intended arithmetic principle it encapsulates. Statement *form*, by contrast, is used in this thesis to refer to the potential transformational effect of a statement when used to make a substitution of notation. Consequently, the form of a given statement is dependent on both its particular structure and the notation it transforms. For example, both $c = a + b$ and $a + b = c$ are partitional if viewed in terms of substituting the numeral c , and both are compositional if viewed in terms of substituting the term $a + b$. The computer-based tasks used in the studies promote attending to statement form because transformations are played out on the computer screen. For example, the statement $c = a + b$ (or $a + b = c$) can be observed to partition the numeral c into $a + b$ or compose the term $a + b$ into c . This is the basis upon which the tasks were designed to engage pupils with the arithmetic principles encapsulated by statements.

The evidence sought was pupils articulating the transformational effects of $a + b = b + a$ statements as commuting (“swapping”, “switching round” and so on) the numerals a and b . As can be seen from the third row (marked “commute”) of each time-sequenced coding map in Figure 11.2, pupils across all the studies did articulate the commutative effects of $a + b = b + a$ statements to a greater or lesser extent. (Note that it is not *necessary* to articulate commutation in this way when working through the diagrams, and indeed the pair of older pupils did not do so during the first ten diagrams – see Figure 11.2d – but it is *useful*). Further evidence was sought of pupils articulating the transformational effects of $c = a + b$ (and $a + b = c$) statements as partitioning (“splitting”, “separating” and so on) the numeral c . Figure 11.2 shows that pupils did articulate partition in six of the seven trials, although in some cases this was only one of the two pupils in a given trial, and even those who did articulate partition did so to a lesser degree than they articulated commutation. These individual differences across pupils are discussed below in Section 11.4.2.

The key point here is that all pupils to some extent articulated the commutative form of statements, and about half the partitional form of statements, in ways that have not elsewhere been reported in the literature to the best of my knowledge. Moreover, these findings can be explained in terms of the diagram-

matic task design rather than their previous classroom experiences of equality statements.

11.3.2 Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct?

As shown in the previous section, the pupils tended to compute notation early in the trials and shift to more diagrammatic activities toward the end. However, it can be seen in Figure 11.2 that there is no clear switch over and that “compute” and the other codes are intertwined throughout most of the trials. Analysis of the data from the first two trials (Figures 11.2a and b) had suggested that these switches between computational and diagrammatic points of view were somewhat disconnected. More specifically, the pupils’ diagrammatic activities were often purposeful toward achieving the task goal, and they resorted to computation only when stuck and this invariably proved unhelpful and even hindered diagram-solving.

This observation is consistent with cognitive accounts of learners’ difficulties with coordinating aspects of arithmetic and algebraic understanding. For example, when a pupil exploits a computational shortcut, such as using a count-on-from-last strategy to find the result to (say) $3 + 8$, she may simply be indifferent to consistency of outcome and is not necessarily drawing on the property of commutation (see Section 3.3.1). Likewise, when the pupils in Study 1 articulated commutative and partitional transformations as “swapping” and “splitting” they were not necessarily considering the consistency of value across the transformed notation. This is because transforming the boxed term into a single numeral does not require considering conservation of quantity across transformations, or for that matter numerical balance across the statements presented in a given diagram.

This was tested in Study 2 by including false equalities in some of the diagrams presented to the pupils. (An example diagram from Study 2 is reproduced in Figure 11.3 for reference). It was found that the pupils did not comment on the presence of the false equalities, nor the lack of conservation of quantity of their transformational effects, until the very end of the trials when they had been working on the final diagram for some minutes (Section 9.5). When prompted, they were unable to recall whether diagrams prior to the final one had contained

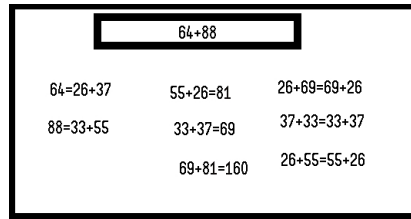


Figure 11.3: A diagram from Study 2 containing four false equalities (reproduced from page 118)

false equalities. Moreover, the presence of imbalanced statements had no effect on how the pupils worked with and talked about the task compared to the diagram-solving components of the trials in Studies 1 and 3.

This finding confirmed that the task design exclusively promoted a substitutive view of equality statements *instead of* a numerical balance view. This demonstrates that the “is the same as” and “can be exchanged for” meanings of the equals sign are pedagogically distinct: the sameness view promotes distinguishing statements by truthfulness; the exchanging view promotes distinguishing statements by form in terms of the notation they transform. A key finding arising from the thesis then is that sameness and substituting meanings for the equals sign offer task designers distinct pedagogic affordances.

This finding also helps to explain why pupils in all the studies accepted atypical statement types without difficulty, and effortlessly talked about and worked with statements in new ways without the “change resistance” (N. McNeil & Alibali, 2005) we would expect from the wider literature. When solving diagrams the pupils did not build on their existing arithmetic knowledge and did not need to coordinate a novel way of viewing statements with their previous experiences of notation. Instead, their substitutive views of the equals sign evidenced in the trials were situated to the task and so unlikely to have any impact on their understanding of equality statements in other contexts. We might say that the novel ways the pupils engaged with statement form were “easy come, easy go”. The task, then, offers learners easy access to new ways of working with equality statements because existing arithmetic conceptions simply were not “triggered” (diSessa, 2002), and consequently medium- and long-term learning and transfer to other contexts is unlikely to arise from the task.

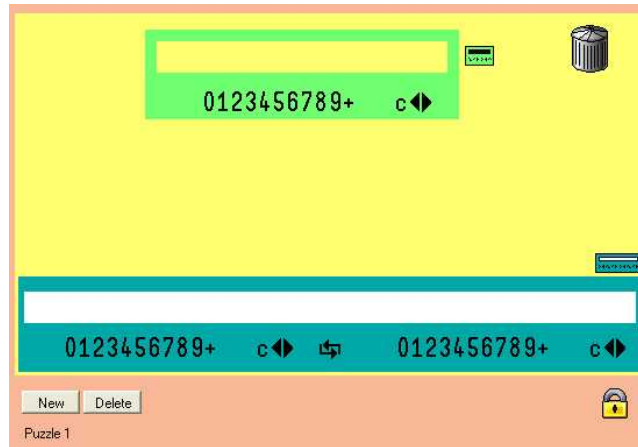


Figure 11.4: Keypads for making diagrams (reproduced from page 132)

11.3.3 Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign?

Study 3 sought to address the pupils’ disconnect between their existing views of equality statements and those supported by the diagram-solving task (see previous section). The design challenge was to modify the task such that *both* the sameness and exchanging views of statements were “useful” (Ainley et al., 2006) toward achieving the task goals. The approach used was to familiarise pupils with the diagram-solving task and then challenge them to make their own diagrams using provided keypad tools (Figure 11.4). Making diagrams necessitates ensuring the numerical balance of statements when entering them at the keypads⁴, and ensuring the statements are substitutive with respect to the boxed term in order to construct a functioning diagram.

The evidence sought for coordinating sameness and exchanging views was pupils (i) managing to construct functional (i.e. solvable) diagrams that contain several inter-related statements, and (ii) articulating the distinct substitutive effects of statements (“swapping”, “splitting”, and so on – see Section 11.3.1 above) interspersed with discussion about the numerical equivalence of terms on either side of equals signs, and the conservation of quantity across transformations of notation.

It was found that to varying extents pupils in the trials managed to construct

⁴For a description of the keypad functionalities see Section 10.2.

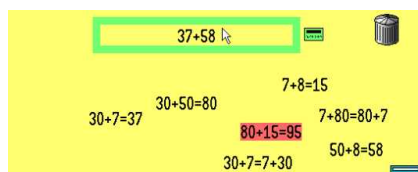
functioning diagrams, some of which contain several statements that were both substitutive with regard to the boxed term as well as necessary to solving the diagrams. (The diagrams made by the pupils in Study 3 can be seen in Figure 10.2, page 133; Figure 10.5, page 141; Figure 10.9, page 150; Figure 10.12, page 159). Moreover, to construct these diagrams the pupils needed to discuss both the numerical balance and substitutivity of the statements they inputted in a coordinated and constructive manner. For example, in the following excerpt (reproduced here from page 135) Yaseen and Laura discuss how the transformation $15 \times 5 + 9 \rightarrow 15 \times 14$ would not conserve quantity.

- YASEEN: 5 plus 9, no you can't do that. 5 plus 9, because that would make 14. 15 times 14. No. [*laughs*]
- R: Sorry, what do you mean "no"?
- YASEEN: Swap it round [*i.e. replace 5 + 9 with 14 in the boxed term*] makes 15 plus 14. [*he presumably meant 15 times 14*] So that won't work.
- R: Why not?
- YASEEN: It's going to be different.
- LAURA: [*talking at same time as Yaseen*] Because then you get another answer.

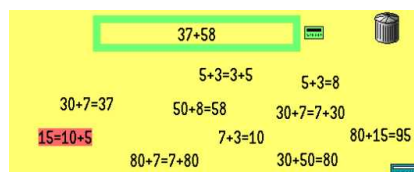
This finding, that pupils indeed coordinate sameness and substitutive views of notation when diagram-making, confirmed that it was indeed the task design rather than cognitive limitations or previous experiences that explains the disconnect when diagram-solving (see previous section).

11.3.4 Can children connect their implicit arithmetical knowledge with explicit transformations of notation?

The third research question, discussed in the previous section, asked whether pupils could coordinate sameness and exchanging views of the equals sign. The fourth research question builds on this to find out whether pupils can connect their existing knowledge of arithmetic principles and strategies to the diagrams they make on the computer screen. (This research question was in fact the original motivation for the thesis – see Section 5.3.3).



(a) Diagram based on Nadine's verbal strategy. (Reproduced from page 141).



(b) Diagram based on Bridie's verbal strategy. (Reproduced from page 141).

Figure 11.5: Bridie and Nadine's diagrams based on their verbal strategies for $37 + 58$

The pupils were presented with a computation in the box ($37 + 58$ or $37 + 48$) and asked to say aloud how they would work it out. Verbalised computation strategies typically contain implicit transformations of notation, in this case partitioning 37 and 58, and commuting the resulting numerals in order to perform place-value compositions. For example, Bridie's spoken strategy was "I, added 30 and 50 which was 80. And then 5 plus 3 equals 8, so if you add that 3 to, um, the 87, because you have 37 and 50. So that would be 87. If you add a 3 onto the 87 you have 90 and then from that 8 you'd have 5 left, so it's 95." (page 147). The pupils were then challenged to express their verbalised strategies as diagrams, which requires identifying these implicit transformations and entering them into the computer as balanced, substitutive statements ($37 = 30 + 7$) and so on. Two pairs of pupils in Study 3 were set this challenge but only one pair succeeded (Bridie and Nadine in Trial 1). Their efforts are reproduced in Figure 11.5, and the key to their success was identifying the need to partition 37 and 58 (e.g. "first of all you need to divide 50 and the 8. Partition it." – page 147). This demonstrates that the modified task design could support pupils in connecting their existing knowledge arithmetic principles and strategies with the formal notation within a diagram, and this is quite unlike arithmetic notating task data published elsewhere to the best of my knowledge.

However, the fact that only pupils in one of the three trials succeeded suggests two things. First, the task did not have the "easy come, easy go" aspect of pupils' engagement with statement form when solving presented diagrams (see Section 11.3.2), and so offers more pedagogical substance in terms of potential stable learning gains and transfer (this issue is elaborated on in the following chapter). Second, task design alone cannot explain the data from Study 3 because there are substantial individual difference across the pupils. These individual differences are the focus of the following section.

11.4 Similarities and differences across the pupils

So far in this chapter I have contrasted the findings reported in the thesis with those of the wider literature and argued that issues of task design better account for the contrast than cognitive limitations or classroom experiences. I have also demonstrated how the diagrammatic approach enabled pupils to talk about and work with notation in genuinely novel ways, and how pupils might coordinate and connect this with their existing views of equality statements and arithmetic knowledge. However, a task design explanation cannot alone account for the data because there are individual differences between how pupils engaged with the task and their successfulness at meeting the task goals. These differences were notable for more complicated diagrams of the diagram-solving components of the trials, and more marked still for the diagram-making components, and in particular for the fourth research question (“Can children connect their implicit arithmetical knowledge with explicit transformations of notation?”).

11.4.1 Similarities across all pupils

It is helpful first to review the similarities across pupils in the three studies when working toward the task goals before going on to elaborate the differences. All the pupils: (i) engaged in *iconic matching* (that is, searching for occurrences of visually identical notation – Section 11.3) when working on the tasks; (ii) articulated the distinct transformational effects of $a + b = b + a$ statements (“swapping”, “switching round”, and so on – Section 11.3.1); (iii) engaged in some form of strategising when solving diagrams; (iv) were indifferent to whether partitions and compositions implied a left-to-right or right-to-left reading (i.e. $c = a + b$ or $a + b = c$) when placing statements in diagrams; and (v) inputted partitioning statements when making diagrams that exploited tens-and-units or halving decompositions. These latter three findings have been described in Chapters 8-10 but are not explicitly addressed by the research questions and need expanding upon here.

Similarity (iii): Strategising. The pupils could be said to have worked strategically at diagram-solving when they considered transformations of notation two or three steps ahead in order to try and achieve the task goal. This happened when substitutions were made in order to enable further substitutions when solving diagrams, and, in Study 3, when deciding which statements to input

when making diagrams. The evidence for strategising is captured in the pupils' discussions about what needed to be done in order to modify a diagram in such a way that was perceived by them to be a step closer to the task goal. Examples include utterances such as “you need to swap 'em round . . . No. Those need to be swapped round as well . . . There. Now you can do that one.” (Terry, taken from lines 58-74, page 105). However, while all pupils strategised in this way to some extent, the degree to which they worked strategically, and the success of the strategies they developed, varied across individuals and this issue is returned to in the following section.

Similarity (iv): Indifference to $a + b = c$ and $c = a + b$ forms. Recall that statement *form*, as defined for the purposes of this thesis, is a property both of a statement itself and the notation it acts upon (Section 3.3.1). Accordingly, both $c = a + b$ and $a + b = c$ are partitioning statements when viewed in terms of the numeral c (Section 11.3.1). In all three studies the diagrams presented contained statements that exclusively implied a left-to-right reading in terms of the intended transformation effects (i.e. only $c = a + b$ statements were provided for partitioning c when diagram-solving; similarly, only $a + b = c$ statements were provided for composing $a + b$). However, in Study 3, when pupils constructed their own diagrams, a notable theme was an apparent indifference to left-to-right readings. Pupils showed a tendency to implement the desired transformation $c \rightarrow a + b$ as $a + b = c$ statements without comment or justification (e.g. line 335, page 145), and, in two of the trials, implemented $a + b \rightarrow c$ as $c = a + b$ (e.g. the intended composing effect of $97 = 52 + 45$ in Figure 10.9e, page 150). This arbitrariness appears to have arisen as a quirk of the design of the keypad tools (Section 10.7) but nonetheless demonstrates an indifference to left-to-right readings. With the possible exception of Penny in the early pilot trials (Section 8.2), it seems pupils viewed statement form purely in terms of transformational effects on a specific numeral or term. This stands in contrast children's rigid adherence to left-to-right readings as reported in the literature (Section 2.2).

Similarity (v): Inputting partitioning statements that use tens-and-units decomposition or halving. In Study 3 the diagrams presented to pupils for solving practice deliberately avoided tens-and-units decompositions (Figure 10.3, page 138). This was to reduce “visual saliency” effects (Kirshner, 2004) of repeating digits, as in $45 = 40 + 5$. When the pupils in Study 3 went on to make their own diagrams they almost invariably used tens-and-units decompositions for

partitioning numerals (e.g. Figure 10.2d, page 133; Figure 10.9c, page 150; Figure 10.12d, page 159) and, in one trial, halving decompositions (Figure 10.5c, page 141). This was presumably done to reduce computational burden.

11.4.2 Individual differences

In a short response to a discussion on “early algebra” at the twenty-fifth annual conference of the International Group for Psychology of Mathematics Education, Tall (2001) highlighted the importance of acknowledging children’s individual differences when evaluating interventions intended to “bring out the algebraic character of arithmetic” (p.2). The existence of measurable differences across individuals, which can reliably be clustered into groupings, has also been substantially reported in the psychology literature (e.g. Canobi, 2004; Gilmore & Bryant, 2006). The key finding of this thesis – that the task design enabled all pupils to talk about and work with equality statements in novel ways, and to do so readily – may tempt us to focus exclusively on the similarities discussed so far in this chapter. However, this temptation would ignore much previous research on the varied ways different pupils interpret mathematical situations, even if ostensibly they respond in the same way. For example, two pupils might both use a count-on-from-larger strategy for (say) $3 + 8$, yet one might be drawing on the principle of commutivity and the other merely indifferent to consistency of outcome (Section 3.3.1).

Gray and Tall (1994) more extensively explored such individual differences within the realm of formal mathematical notation. They distinguished between “procedural” and “proceptual” thinkers. The former become bogged down in the processes of arithmetic calculation and without special individual intervention might not come to “compress” such processes into thinkable concepts. Consequently the mathematics curriculum is increasingly inaccessible as procedures that worked fine for simple arithmetic problems become ever more cumbersome, inefficient and error-prone. In contrast, proceptual thinkers are able to shift appropriately between seeing the same notation in terms of processes and the product of those processes (i.e. a thinkable concept). This flexibility opens up meaningful and powerful labour-saving strategies that can be drawn upon with great success when encountering new ideas later in the mathematics curriculum. (A more detailed account of procedural and proceptual thinking was given in Section 3.3).

In light of substantial evidence base of pupils' individual differences we should expect to see them evidenced in the studies. However, it is important to distinguish differences in the data that arise from pupils' cognitive profiles from idiosyncrasies of particular trials. In order to address this the differences highlighted in the remainder of this section are those that reflect unambiguous distinctions in the data. In each of the following identified differences the pupils fell clearly into one camp or the other. Moreover, the class teachers' deemings of each pupil as either a high or middle mathematical achiever provided a reliable predictor of which camp the pupil fell into in each case. While this clarity and predictability does not guarantee that individual pupil profiles were the cause of the differences, it provides good grounds for believing it is the case when set against the psychological literature.

Given this qualification, the remainder of this section details the differences that are apparent across the studies. The most notable differences are (i) the degree to which individuals articulated the partitioning effects of presented $c = a + b$ statements; (ii) the varied degrees of the pupils' strategic sophistication when diagram-solving; (iii) the varied complexity and functionality of diagrams made by the pupils (research question 3, discussed in Section 11.3.3); and (iv) the success of only one pair of pupils at connecting diagram-making with their existing knowledge of arithmetic principles and calculation strategies (research question 4, discussed in Section 11.3.4).

Individual difference (i): The role of partition.

All pupils readily articulated commuting transformations but pupils in only four of the seven trials explicitly articulated partitioning transformations ("split", "separate", and so on) when solving diagrams. This in itself may not be of great significance; for example, the two older pupils from a high-achieving mathematics class (Ajay and Nikisha from Study 2 – see Section 9.4) did not articulate the distinctive partitioning effects of presented $c = a + b$ statements until the final diagram, despite working through the task strategically and efficiently (see Figure 11.2d). Moreover, across all trials there are candidates for *near articulations* of partition, such as commenting on the larger visual size of the resulting term in the transformation $c \rightarrow a + b$ (e.g. lines 306-307, page 143). In this way even some pupils who did not explicitly distinguish partitional transformations were sometimes able to exploit $c \rightarrow a + b$ transformations by focusing on statements that contained the single numeral c on one side or the other.

Nevertheless, those pupils deemed mathematical high-achievers by their teachers, and therefore the more likely candidates for being proceptual thinkers, tended to some extent to be those who more readily articulated partition (i.e. the pupils in Figures 11.2a, c, d and e). Moreover it was these same pupils who developed more sophisticated and consistent diagram-solving strategies, as discussed next. (It should be recalled that properly speaking the evidence only shows that pupils articulated “notational protopartition” and were not necessarily concerning themselves with conservation of quantity – see Section 6.2.2.)

Individual difference (ii): Diagram-solving strategies.

The tasks used in the thesis supported strategic thinking; that is, seeing potential transformations several steps ahead in order to achieve a task goal. Across the trials two distinct degrees of sophistication were apparent in the pupils’ strategising when solving diagrams. Some pupils, notably those deemed high-achievers by their class teachers, developed the strategy of starting by identifying $c = a + b$ statements in order to partition the notation in the boxed term into constituent parts, then used $a + b = b + a$ and $a + b = c$ statements to commute and compose the constituents into a single numeral (e.g. lines 54-74, page 105)⁵. These pupils found this strategy worked efficiently and consistently over several diagrams and became increasingly adept at articulating and exploiting it. Moreover, the pupils who developed this strategy were those who described the distinctive transformational effects of $c = a + b$ statements as “splitting” or “separating” the numeral c . This provided them with a useful vocabulary for distinguishing and identifying the statements with which to start solving each diagram (as in “That’s the one that you do first! It has to be . . . Because it’s splitting up the 40 and the 1” – row 20:01, Table 8.1, page 100), as well as describing what they wanted to do to a given numeral c in the boxed term and then searching for a statement that contained only c on one side of the equals sign (as in “Wanna split the . . . 3 and the 50 up” – row 20:39, Table 8.1, page 100).

The pupils who did not adopt this start-with-partitioning strategy were less efficient and less consistent with diagram-solving, most notably for the later,

⁵Note that another, more complicated, strategy is available for solving the diagrams involving making transformations of statements in order to produce a statement of the form “boxed-term=single-numeral”. In practice only Lauren from the first pilot trials used this strategy (Section 8.2) and it is of no particular concern here – see Section 8.3.1 for further explanation.

more complicated diagrams. These pupils all strategised in the sense of working two or three potential transformations ahead, but it was more grounded in trial-and-error and less certain. This is exemplified by the following excerpt from Study 1 (reproduced from page 111).

KITTY: Now do that one.

ZELDA: Or it could be that one, because there's 140 and there's 5.

KITTY: Yeah, but it could be that one.

ZELDA: Try that on the other one.

KITTY: No, it's just swapped them.

ZELDA: Shall we try swapping and then we can try . . .

KITTY: What shall we try?

ZELDA: That one.

R: Why that one Zelda?

ZELDA: I don't know.

There is, then, a divide amongst the pupils in terms of strategic thinking, and this hinges on the extent and consistency with which they articulated and exploited the distinctive partitioning effects of $c = a + b$ and $a + b = c$ statements. This may reflect some pupils' greater ease and comfort at working in non-computational ways, which came to the fore when pupils attempted to make their own diagrams.

Individual difference (iii): The complexity and functionality of diagrams made by the pupils.

Diagram-making necessitates considering numerical balance and substitutive potential when placing statements into a diagram. The numerical balance of inputted statements is imposed by the functionality of the *Sum Puzzles* software. An imbalanced statement can in fact be entered but will be displayed with a \neq sign (Figure 11.6) and cannot be selected to make substitutions. The substitutivity of statements is not imposed by the software itself, but by the task goal of making a functional diagram. Statements that are not iconic with notation in the diagram can be inputted and selected (provided they are numerically balanced), but cannot be used to make any substitutions (Figure 11.7). Moreover,

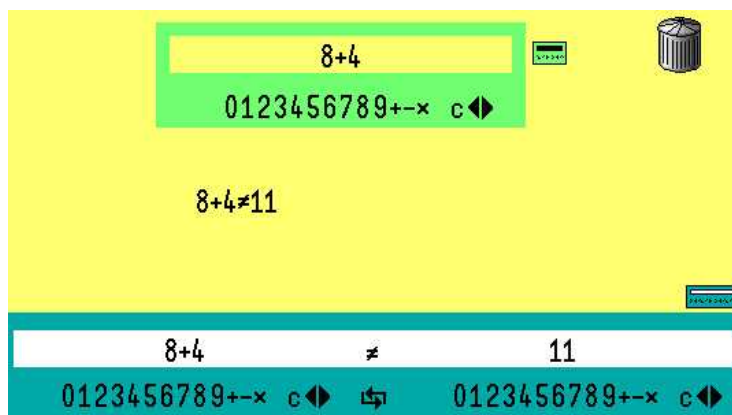


Figure 11.6: A numerically imbalanced statement in the *Sum Puzzles* software

the sameness and exchanging meanings of the equals sign must be coordinated to ensure the conservation of quantity across a transformation: $2 + 4 = 42 + 2$ cannot be used to make an exchange in $2 + 40$. It is necessary, then, to ensure that statements are both balanced and iconically matched in order to construct a working diagram.

The pupils had little difficulty entering numerically balanced statements once they had got to grips with using the keypads. (The one exception to this was in the pilot trial where the pupils had access to various operator signs and so met obstacles such as trying to use $a - b = b - a$ to make commutations – see Section 10.2). However, there was notable variation across the diagram-making components of the trials in Study 3 to the extent to which the pupils ensured statements were capable of making substitutions in the boxed term. Sophisticated, functioning diagrams were made by the pupils in the pilot trial (Yaseen and Laura – Figure 10.2, page 133) and Trial 1 (Bridie and Nadine – Figure 10.5, page 141) and, to a lesser extent, the pupils in Trial 2 (Derek and John – Figure 10.9, page 150). The pupils in Trial 3, however, struggled to construct diagrams that contained more than one functioning (iconically matched) statements (Colin and Imogen – Figure 10.12, page 159).

As with the variations in the sophistication of strategies when diagram-solving, the variation of complexity and functionality of the pupils' diagrams hinges around their articulation and exploitation of partitional transformations. Three of Bridie and Nadine's five diagrams contain statements that partition numerals in the boxed term. Nadine identified the need for partitional statements during

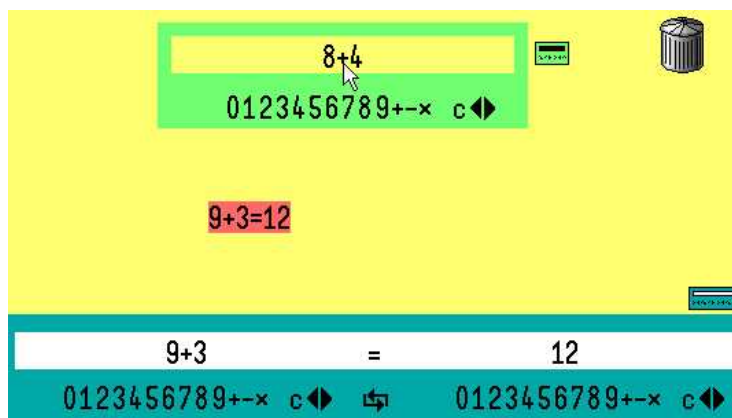


Figure 11.7: A statement that is not iconically matched cannot be used to make a substitution

the first diagram they constructed (“You can’t do it. We need to have a sum that’s only one thing” – line 318, page 144), and explicitly suggested partitioning statements for the third diagram (“Do 5 add 5. Equals 10 ... Now let’s do something to make 70. Um, 35 add 35.” – lines 335-341, page 145). The girls continued this strategy for the next two diagrams they made, and when challenged to make diagrams based on their verbalised strategies for $37 + 58$ (see below). John and Derek similarly used partitional statements in the first of the two diagrams they made (Figure 10.9c, page 150). However, this was not explicitly articulated. John simply entered $73 = 68 + 5$ without comment (Figure 10.9a), then suggested “24 equals something like 20 add 4”, although Derek could not understand why (lines 435-446, page 154). In contrast with Bridie and Nadine, neither John nor Derek again suggested partitional statements and this did not become a consistent strategy for their diagram-making activities. Colin and Imogen entered one partitional statement ($55 = 45 + 10$ in Figure 10.12d, page 159) at Colin’s suggestion (“You just need a sum that equals 55 and equals 94” – line 572, page 162). However, there were no more occurrences of Colin (or Imogen) suggesting or entering partitional statements.

The key difference, then, is not whether pupils attempted to exploit partitional transformations, but the strategic consistency with which they did so. An explanation for this difference might be found in the keyword maps for the diagram-making component of the trials, shown after the vertical lines in Figures 11.2e, f and g, pages 170-171). Notably, the trials in which the pupils did not consis-

tently exploit partitional transformations when making diagrams are dominated by the “compute” code (Figures 11.2f and g). This is in contrast to the trial in which they did exploit partitioning, where the diagrammatic codes (“iconic”, “commute”, “partition”) are dominant (Figure 11.2e). According to Gray and Tall (1994), we might expect that some of the pupils became bogged down in the procedural aspects of ensuring numerical balance when entering statements into diagrams. An examination of the transcripts excerpts would seem to support this. For example, the following excerpt (reproduced from page 572) shows that when Colin suggested partitioning the numerals 55 and 94 in the boxed term, Imogen focused on possible computations and steered attention away from transformational effects.

COLIN: You just need a sum that equals 55 and equals 94.

IMOGEN: So what could equal 55? 44 add 10.

COLIN: 45 add 10.

IMOGEN: Oh yeah. 45 add 10.

COLIN: Need to start a new one. Equals 55. [*Imogen places $55 = 45 + 10$*]
Now you need something that equals 94.

IMOGEN: [*transforms boxed term $55 + 94 \rightarrow 45 + 10 + 94$*] Oh that worked. Yeah.

COLIN: Then [*unclear*] equals 94.

IMOGEN: You could have 80 add 14.

COLIN: Yeah.

Similarly, despite suggesting partitional statements early on in Study 3 Trial 2, John’s subsequent discussion was characterised by computation when deciding on statements to input: “73. Add 11. No 73 equals ... Um, um, um, equals elev...-, no don’t do 11 yet ... Equals, um, 54 add, um, 19. Equals, um, 54 add, hold on, um, 19 ... Put a 49 in add 4 and something. That’s it. Add 4. Equals. Now go on this one. Er, 49 add 4 equals 53. Now just click on there. Would that work? Just try it.”

This is not to say that success at diagram-making required ignoring computation, rather that computation needed to be coordinated with the substitutive properties of statements with regards to the boxed term. In other words, the

pupils needed to switch appropriately and flexibly between considering the substitutive and numerical equivalence of statements. For example, in the following excerpt Bridie and Nadine had identified the substitutive property required of a statement (i.e. to transform $25 + 1$: “let’s make one of them that says 25 add 1 because then it will be easy to do” – lines 332-333, page 145) and worked out a numerically equivalent term ($19 + 7$). This resulted in them placing $25 + 1 = 19 + 7$ in the diagram, which was both numerically balanced and iconically matched with the boxed term.

BRIDIE: 25 add 1.

NADINE: And then like 19 add 7 or something like that.

BRIDIE: You can’t do minuses!

R: No, there’s no minuses.

BRIDIE: Because you could get 32 minus 8.

NADINE: 19 ... add 7. And you press ...

BRIDIE: Where you want it.

NADINE: There. And then equals that, that.

BRIDIE: Yeah we need to do something ... with 19 and 7.

Individual difference (iv): Connecting diagram-making to existing arithmetic knowledge.

This difference across individuals relates to the fourth research question and was discussed in Section 11.3.4. In sum, the pupils in Trials 1 and 2 of Study 3 were challenged to make diagrams that represent their spoken strategies for $37 + 48$ and $37 + 58$ and only the pupils in Trial 1 succeeded. The reason for this difference again seems to hinge on the extent and consistency with which the pupils articulated partition during the trials.

The spoken strategies were similar across all four pupils in the two trials:

Nadine (Trial 1): “30 add 50 is 80 ... And, 8 add 7 is 15 ... So you just have to add those together and you get 95.”

Bridie (Trial 1): “I added 30 and 50 which was 80. And then 5 plus 3 equals 8, so if you add that 3 to, um, the 87, because you have 37 and 50 ... So that

would be 87. If you add a 3 onto the 87 you have 90 and then from that 8 you'd have 5 left, so it's 95."

Derek (Trial 2): "I just added the 30 and 40 together, added the 8 and 7 together."

John (Trial 2): "30 add 40 equals 70 ... 7 add 8. I just thought the 8 equals 10, and then 7's 17. 8 to 10 is 2, 17 takeaway 2 is 15. And 70 add 15 equals 85."

Bridie and Nadine had little difficulty identifying the need to input statements corresponding to the partitions implicit in their spoken strategies. For example, Nadine began by entering $30 + 50 = 80$ and then realised that some numerals needed to be "taken out" (i.e. partitioned):

NADINE: Oh, now I can't do it.

R: Why can't you do it?

NADINE: It's impossible because you won't have them on their own unless you take them out but how do you take them out?

(Reproduced from page 146)

A few moments later Barbara stated more explicitly what needed to be done: "It won't work ... because first of all you need to divide 50 and the 8. Partition it" (reproduced from page 147).

In contrast, Derek and John failed to enter statements corresponding to the partitions implicit in their spoken strategies, despite prompting by myself (e.g. "Where did you get the 30 and 40 from? Because there's not a 30 add 40 in the green box" – see page 156). After I showed them a diagram they had constructed earlier which did include a partition ($20 + 4 = 24$ in Figure 10.9a), Derek identified the need for a partitioning statement, but incorrectly identified which numeral was in need of partition ("85, so couldn't you put like 80 add 5?" – see page 157).

11.5 Summary

The findings reported in this thesis stand in contrast to those of the broader literature. This disparity is due to the diagrammatic task designs used in this thesis, as supported by the "can be exchanged for" meaning of the equals sign.

When statements are presented as reusable rules for making transformations of notation pupils are able to see and talk about potential actions toward a specified task goal. In particular, inter-related statements can be presented in parallel and the learner must look for identical occurrences of notation in order to see where substitutions can be made. This *iconic matching* activity is quite different to computational or numerical balance readings of statements, and is the basis of the for the four research questions addressed in this thesis.

Does the “can be exchanged for” meaning for the equals sign promote attention to statement form? Pupils articulated statements in terms of their distinctive substituting effects on notation (“swap” for $a + b \rightarrow b + a$, “split” for $c \rightarrow a + b$, and so on). This was supported by transformations being played out on the computer screen, which were observed and described by pupils, as well as making predictions about transformational effects when working toward the task goals. This engagement with statement form is a unique contribution of the thesis.

Are the “can be exchanged for” and “is the same as” meanings for the equals sign pedagogically distinct? Analysis of the trials reported in Study 1 suggested pupils were not considering the numerical balance of statements or the conservation of quantity when engaging in *iconic matching* and *substitution making* activities. This was tested by including false equalities in some diagrams. It was found their presence was not noticed by pupils and had no effect on how they worked with and talked about the diagrams. This demonstrated that the pupils did not coordinate “sameness” and “exchanging” meanings for the equals sign, and did not make connections between the task and their existing knowledge, when solving diagrams.

Can children coordinate “can be exchanged for” and “is the same as” meanings for the equals sign? The disconnect between “sameness” and “exchanging” views arose from the task design (rather than cognitive limitations or classroom culture). Solving diagrams promotes the “exchanging” meaning of the equals sign at the expense of the “sameness” meaning. To address this pupils were challenged to make their own diagrams, which requires inputting statements that are numerically balanced as well as iconically matched with one another. It was found that pupils were able to construct functioning diagrams and that they switched appropriately between “sameness” and “exchanging” views when doing so. (Additionally, a pilot trial also demonstrated how providing pupils with more operators than + alone promotes coordinating the two views of the equals sign because the ordering of operations becomes an issue when making

substitutions).

Can children connect their implicit arithmetical knowledge with explicit transformations of notation? The final question investigated whether pupils could connect the explicit transformations on the computer screen with the implicit transformations in their existing mental strategies. Only one pair of pupils managed this, and their success depended on identifying the partitions implicit in their spoken strategies.

The above four questions focus on similarities across pupils in the trials. Three further similarities not explicitly addressed by the questions are the use of strategies in which pupils consider substitutions two or more steps ahead, an indifference to $a+b = c$ and $c = a+b$ statements in terms of composing and partitioning effects, and the inputting of partitioning statements that reduce computational burden by using tens-and-units and halving decompositions. There were also important individual differences across pupils that cannot be accounted for in terms of the task design, but reflect instead pupils' varied flexibility of thought. The most interesting difference was the extent to which pupils articulated partitioning effects ("split", "separate"), which provides a reliable predictor of a pupil's subsequent success with diagram-solving and making. Articulating partitioning was key to identifying and making consistent use of the diagram-solving strategy of starting by breaking up the boxed numerals into constituent parts, then using the remaining statements to commute and compose them into a single, final numeral. Articulating partitioning was also key to producing complex and functional diagrams which, as with solving diagrams, requires starting with providing statements that break up numerals in the box. Finally, identifying the implicit partitions in mental arithmetic was key to successfully making diagrams based on pupils' own spoken computation strategies.

Chapter 12

Conclusion

12.1 Introduction

In the concluding chapter I set out the key contributions made by the thesis to our theoretical understanding of children's conceptions of the equals sign. I highlight the limitations of the approach taken and suggest further work that might help address these limitations. Finally I discuss the implications of the findings for pedagogy.

12.2 Contribution to theory

The contributions of the thesis to existing theory are threefold. The first is a critique of the literature on children's conceptions of the equals sign. The second is the application of Dörfler's (2006) diagrammatic approach to pedagogic tasks to the case of arithmetic equality statements. The third is the presentation of task-based data that stands in contrast to that reported elsewhere.

12.2.1 A critique of the literature

The impetus for the studies carried out for the thesis was a critique of the literature on children's conceptions of the equals sign. This literature reports that children, from Western countries at least, tend to view the equals sign as a place-indicator for a result (Section 2.2) and that this arises from how

notation is presented and talked about in classrooms (Section 2.3). This is not contested here. Interventionist studies have demonstrated that presenting a variety of equality statement types, and teaching that the equals sign means “is the same as”, can help pupils take a more flexible view of notation (Section 2.4). The validity and reliability of these studies is not questioned here, nor are the implications for pedagogy that arise from them, but the point is made that they are limited. This is due to such studies excluding the substituting meaning for the equals sign, which is central to the very nature of equivalence relations (Section 3.4). This is, to the best of my knowledge, an original critique of the interventionist literature on children’s conceptions of the equals sign.

12.2.2 A diagrammatic approach to task design

The second contribution made is the application of Dörfler’s (2006) “diagrammatic reasoning” to the case of equality statements. There are two aspects to this.

First, the thesis applies Dörfler’s (2006) critique that much mathematics education is implicitly Platonic to the case of presenting pupils with equality statements (Section 6.2). When a statement such as $19 + 8 = 8 + 19$ is presented the stated intention is typically to engage pupils with “structure sense” (Linchevski & Livneh, 1999) or “relational thinking” (Carpenter & Levi, 2000). However, such a statement is an implicit but inescapable reference to the balance of two (abstract) numbers. Pedagogically, $19 + 8 = 8 + 19$ is also a reference to an arithmetical principle (commutation) which pupils might exploit and so establish balance without working out what the number on each side actually is. Once a statement has been assessed or completed by a pupil it must be discarded and another presented. The focus is on patterns of consistency of the abstract principles, as implicitly referenced by statement structure, across a sequence of examples.

Second, Dörfler’s (2006) suggested approach, in which inscriptions rather than abstract entities are presented as the “very objects” of study, was applied to the case of equality statements in the tasks used in the trials. The “can be exchanged for” meaning for the equals sign provides the basis for this (Section 6.2). Rather than statements being presented as isolated questions of numerical balance, they can instead be presented in parallel as reusable rules for making transformations of notation. The focus of learning is then on patterns of actions (literal or mental

transformations) and potential actions on the visible inscriptions.

12.2.3 Unique data that justifies the approach

The third contribution is the production of trial data that are novel, to the best of my knowledge, and which justify the diagrammatic approach used (Chapters 8-11). The key attributes of this were that children did not take a computational or numerical balance view of statements. Rather, they looked for matches of notation (“iconic matching”) across sets of inter-related statements in order to determine where substitutions could be made. They distinguished statements by form, that is in terms of their observed or imagined substitutive effects on given notation ($a + b \rightarrow b + a$, $c \rightarrow a + b$ and so on), rather than by truthfulness. During the diagram-solving component of the task the pupils viewed statements exclusively in terms of potential substitutions, which demonstrated that the “is the same as” and “can be exchanged for” meanings of the equals sign are pedagogically distinct (Study 2). During diagram-making tasks however the pupils were able to coordinate the sameness and exchanging meanings to varying extents in order to construct their own functional diagrams (Study 3). Two of the pupils were also able to connect the explicit transformations of notation on the screen with the implicit transformations in their own mental addition strategies (Section 10.4).

12.3 Limitations and further work

The thesis achieves its goal of addressing the research questions within the context of the trials. However, the findings lack generalisability to a statistical population. Also, the degree or transferability of any learning that occurred was not measured. The generality of the task design to non-digital media, classroom settings and other mathematical domains is also unknown. The thesis does however generate novel and promising hypotheses for further research.

12.3.1 Generality to a population

The thesis presents qualitative analyses of seven trials involving fourteen pupils and generalises to findings in the literature. Post-doctoral work is currently being planned to establish the generality of the findings to a population of

pupils¹. The study will involve pre-tests, a classroom intervention based around the *Sum Puzzles* software, and post- and delayed post-tests of Year 7 pupils (11 and 12 years). Two of the instruments used (Knuth et al., 2006; N. M. McNeil et al., 2006) will allow us to cluster the pupils according to differences in their exiting understandings of the equals sign and evaluate the impact of the intervention on these clusters. Finally, we will compare the influence of the intervention on early algebra understanding via a straightforward between-groups comparison on scores on the final task in the delayed post-test, which concentrates on algebraic skills. Two classes will form an intervention group, and two a comparison group. Note that the Hawthorne effect will not interfere with the study as we are measuring the impact on different clusters of children within the intervention group, and using the control group as a baseline.

12.3.2 Learning and transfer

The thesis describes how diagrammatic task design can enable pupils to view, talk about and work with arithmetical notation in novel ways compared to that reported in the literature. This is an important finding in light of the widespread consensus that pupils' difficulties arise from how arithmetic notation is presented and discussed in classrooms (Section 2.4). Learning and its transfer to other tasks and mathematical domains are not explicitly addressed, but are also important questions. The post-doctoral study mentioned in the previous section will seek to measure the intervention's impact on pupils' understanding of the equals sign, and on performance with establishing the numerical balance of structured statements.

A further avenue of interest is the potential impact diagrammatic arithmetic tasks might have when pupils first encounter algebraic notation at the start of secondary schooling. There is currently significant interest in the possibility of teaching elementary arithmetic in more "algebraic" ways, notably in the US (Carpenter et al., 2003; Carraher, Martinez, & Schliemann, 2008), in order to help ease the difficult transition from arithmetic to algebra. The tasks and findings reported in this thesis offer a complementary approach to this growing body of interventionist research. The post-doctoral study will be conducted immediately prior to pupils meeting algebraic notation in the classroom and delayed post-tests will measure any impact.

¹This study is being designed in collaboration with Matthew Inglis at Loughborough University and Camilla Gilmore at the University of Nottingham

12.3.3 Applicability to other domains

The thesis focuses almost exclusively on the case of arithmetic statements containing only positive integers and addition operations. There is also scope to support a wider variety of statements that include negative numbers, decimal and fraction representations, as well as other arithmetic operations, brackets and so on. The ordering of operations is a domain of particular interest for further research because it supports an alternative approach to coordinating “sameness” and “exchanging” meanings ($2 + 4 = 4 + 2$ can be used to make a substitution in $2 + 4 + 3 = 9$ but not $2 + 4 \times 3 = 14$). This hypothesis was explored and supported to a limited extent in the pilot study reported in Study 3 in which the pupils had access to $+$, $-$ and \times operators (Section 10.2).

Going further, the *Sum Puzzles* software could be reprogrammed to allow letters as well as numerals and so support algebraic equations. In fact, in principle the diagrammatic approach developed for the thesis has scope for any domain of notated mathematics that involves reversal equivalence relations represented by the symbol $=$.

12.3.4 Applicability to non-digital media

The “can be exchanged for” meaning for the equals sign can also be supported by non-digital media such as card-sorting activities and worksheets. A card-sorting activity, and corresponding interactive whiteboard version, of the *Sum Puzzles* tasks has in fact been developed (Figure 12.1) and used in the classroom by myself. This came about when one of the schools that assisted with the trials reported in the thesis requested help with their “mathematics week”. I designed and delivered a whole day lesson to a Year 5 class (9 and 10 years). Due to the opportunistic nature of this no data was captured but, anecdotally, the tasks and resources proved viable in a classroom setting. Towards the end of the day there was a particularly stimulating whole class discussion in which several pupils appeared to suddenly realise that $c = a + b$ and $a + b = c$ are identical for the purposes of making exchanges.

$75 + 30$	$75 = 70 + 5$
$70 + 5 + 30$	$5 + 30 = 30 + 5$
$70 + 30 + 5$	$70 + 30 = 100$
$100 + 5$	$100 + 5 = 105$
105	

Figure 12.1: Interactive whiteboard version of a card-matching activity

12.4 Implications for pedagogy

The key implication for pedagogy is that teachers should consider complementing the “is the same as” meaning for the equals sign with a “can be exchanged for” meaning. However, whereas the former meaning can be readily implemented by presenting carefully selected statements in sequence, the latter is not so simple. For the purposes of this thesis digital technology was exploited in order to play out substitutions of notation on the screen and to support interactive tasks. The software is freely available to be used on-line ², and is also available from the Association of Teachers of Mathematics (ATM) ³ However, teachers and pupils cannot design and save their own diagrams using the on-line version at present.

The previous section suggests how the diagrammatic approach might be adapted to non-digital media that are more readily available to teachers. However, substantial work is required to develop, test and refine these traditional media to a usable state for typical classroom teachers. The development of robust and impactful materials is no simple or short-term task (Burkhardt, 2006). In order to develop the tasks into usable classroom resources it will be necessary to produce a set of pupil and teacher resources that acknowledge the realities of the curriculum, the assessment regime, varied teaching styles, typical classroom resources, and so on. A robust methodology for achieving this is mixed-methods iterative design involving close collaboration between researchers, teachers and the learners themselves (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble,

²See go.warwick.ac.uk/ep-edrfae/software.

³See www.atm.org.uk/mt/archive/mt207.html (membership log-in required).

2003; The Design-Based Research, 2003). The post-doctoral work outlined in this chapter is seen as a step toward the iterative mixed-methods development of a diagrammatic approach to mathematical notating tasks.

12.5 Summary

The thesis makes three contributions to theory. The first is a critique of interventionist studies of children’s conceptions of the equals sign as inadequate due to an exclusive focus on the “is the same as” meaning of the equals sign. The second is the application of Dörfler’s (2006) diagrammatic approach to the analysis and design of pedagogic tasks based around relational meanings for equality statements. The third is the presentation of novel data that demonstrate pupils viewing statements in terms of their potential substitutive effects on notation.

The thesis does not inform statistical generality, learning or transfer, and these issues are to be explored in post-doctoral work. The diagrammatic design approach developed for the thesis has theoretical applicability to non-digital media and other mathematical domains of learning. As with all research-based tasks, its development into a robust and usable set of resources for mathematics classrooms will require hypothesis-driven iterative design and trialling in collaboration with teachers and pupils.

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Appendix A

Example trial schedule

Trial schedule for Study 3.

Overview

- The focus is on children solving and constructing their own puzzles. The theoretical reasons for this are not set out here.
- The participants will be four pairs of Year 5 children: one middle and one high ability pair from a suburban school; one middle and one high ability pair from an inner-city school.
- The main structure is a session on puzzle solving followed by two sessions on puzzle construction. Sessions will last around 30-40 mins.
- Trial data will be captured as audio visual movies of each pair working with the software at laptops.

Session 1

R will introduce the software and task to the children ("Pupil 1" and "Pupil 2"). The following dialogue will be used by R:

We are going to try solving some puzzles. First I'll show you how to use the software and then I'll tell you how to solve puzzles.

Pupil 1, try clicking on an equals sign. What has happened? ... What have we got on each side of the equals sign? Can you see anywhere else in the puzzle

where you can see the same thing? ... Pupil 2, try clicking on where you can see the same thing. What has happened? ... So that's everything the software does - you click on equals signs and then you click on numbers and pluses. To solve a puzzle you have to make the sum in the green box into its answer. You can do this by clicking on equals signs and numbers and pluses.

The pupils then work collaboratively on a sequence of 7 puzzles. [These were reproduced here in the original schedule and can be seen in the thesis in Figure 10.3, page 138]. These puzzles all make use of partitional, commutative and compositional forms and get progressively more complicated (i.e. the number of statements increases). Appeals to typical calculation strategies, such as dividing into tens or units, have been deliberately avoided. This is because if the use of typical strategies emerges during sessions 2 and 3, as predicted, I can then categorically state this emergence was not prompted by puzzles presented to them in session 1.

During this time R will encourage, praise and prompt for verbal elaborations (*Why do you think that didn't work? What did you say Pupil 1? etc*)

Session 2

R introduces the puzzle construction tools and allows the children to explore them. These tools proved intuitive and usable in the pilot trial.

Children work collaboratively at the laptop to produce puzzles (+ operators only).

The role of R will be to praise and encourage collaboration through comments such as "*Pupil 2, did you have an idea?*" and "*Does anyone disagree with that?*" and so on. If pupils attempt overly simplistic puzzles (2 or less statements) or overly complicated puzzles (10 or more statements) some structuring will be used such as "*Can you make a puzzle that has three number sentences?*"

Session 3

The "constructing puzzles" pilot trial suggested one session may not be enough. Session 3 is a continuation of session 2. If however session 2 proves sufficient, then it could be used for exploratory work instead. This would involve introducing the $-$ operator with a particular focus on children confronting the common misconception that $a + b = b + a$ generalizes to $a - b = b - a$.

Appendix B

Example transcript

Transcript from Study 3, Trial 1, Session 1 (Bridie and Nadine).

(00:00:00) R introduces session

(00:00:40) R: So the first thing, N, do you want to take the mouse. And one of the things this software does is if you click, just once not double click, on one of the equals signs.

(00:00:57) R: Okay. And try clicking on one of the other equals signs. And another one. And the last one. Okay so what happens when you click on an equals sign?

(00:01:13) B: It highlights it.

N: It goes red and pink.

(00:01:16) R: Yeah, it just highlights it pink. Yeah. Now what have we got on one side of that equals sign?

(00:01:23) N: A 13.

(00:01:24) R: And what have we got on the other side?

(00:01:25) N: 12 and a 1. So it makes there be another 13.

(00:01:28) R: Yeah, that's right. That's right. B, if you take the mouse, and if you can see anywhere in the puzzle where you can see something the same. We said we've got 13 and a 12 add 1.

(00:01:40) B: Yeah.

(00:01:40) R: If you see anything else the same in the puzzle try clicking on it.

(00:01:47) R: What happened there?

(00:01:48) B: It, the 12 and the 1 changed to a 13. So that, oh and that...

(00:02:02) R: What have you got highlighted there?

(00:02:05) B: A 9 and a 1 and a 10

(00:02:06) R: Okay, so is there anywhere else you can see a 9 add a 1 or a 10?

(00:02:11) B: A 9 add a 1 or a 10.

(00:02:17) R: A 9 add a 1, or a 10.

(00:02:20) B: There's a 10.

(00:02:23) R: Try clicking on the 10.

(00:02:24) R: N, what do you think's happening there?

(00:02:28) N: Erm, it changes to 9 add 1 because 9 add 1 is the same as 10.

(00:02:35) R: Okay. And what do you think B do you agree or anything else?

(00:02:37) B: Yeah.

(00:02:39) R: Okay. So that was just showing you how it works. There's only two things you can do and it's click on equals and then click on, er, things that match. If you, er, click refresh B. On the bottom there. Now the puzzle is all about the thing in the green box. What have we got in the green box?

(00:02:59) N: 9 add 13.

B: 9 plus 13.

(00:03:01) R: And what do you think you have to do to the puzzle to solve it?

(00:03:05) B: Ah! Click... if you um, if you do, ah. That equals 22.

(00:03:16) R: Yeah.

(00:03:17) B: So, there 22, so if you click on that it should equal 22.

(00:03:23) R: Okay. So what, what do you think the puzzle will look like when you've solved it? You're right.

(00:03:29) B: They'll all be 9 add 13s?

(00:03:31) R: Uh-uh. Something else. R: Have you any guesses N? It's a bit simpler than that actually.

(00:03:37) B: Is it? How can anything possibly be simpler than that!

(00:03:42) R: Well all you have to do to solve a puzzle is make it so that the 9 add the 13 in the green box becomes 22.

(00:03:52) N: Yeah.

(00:03:52) B: Oh!

(00:03:54) R: And you've seen what you can do. You can click equals and you can click other things.

(00:03:58) N: So you click 22 and then you click that.

(00:04:03) B: No!

(00:04:03) N: Oh!

(00:04:05) B: Ooh! Aha! That! Nope. One minute.

(00:04:12) (00:04:16) B: We need something that equals.

(00:04:19) N: Like that?

(00:04:20) B: 13 equals 13, no good. Got 26.

(00:04:24) N: That one's 10 add 10.

(00:04:26) B: That's 20. That equals 22 obviously.

(00:04:33) B: So what can we do now?

(00:04:36) R: Well, when you've clicked on an equals and highlighted...

(00:04:40) B: Yeah...

(00:04:40) R: You then have to click on something that's not highlighted to change something.

(00:04:45) N: But that's not highlighted but it's in the green box.

(00:04:49) B: So if we click on something that's not highlighted [unclear]

(00:04:54) N: That changes because they're not the same.

(00:04:56) B: Then can we mess about with this?

(00:04:58) R: See if it will work.

(00:04:59) N: Press the equals.

(00:05:02) B: But it won't highlight it anymore.

(00:05:04) (00:05:05) B: And then, that won't change.

(00:05:10) R: That's right but something changed earlier. Why do you think things changed earlier?

(00:05:18) B: 'cause they were the same.

(00:05:19) R: That's right. What do you, so can you see anything that's the same there? You've got a 10 add a 12, you've got a 22.

(00:05:28) B: There's a 10 add a 10. And if that equals the 13.

(00:05:32) N: Those are three 10s. One, two, three.

(00:05:40) B: And your point is?

(00:05:42) [N laughs]

(00:05:43) R: Okay, I'll say something to start you off. If you highlight the one that's 13 equals 12 plus 1.

(00:05:51) B: Yeah.

(00:05:53) R: And then in the green box, see if anything will change.

B: [unclear] 13.

(00:05:59) B: Oh!

(00:06:01) R: So what do you think happened there? Why did that change?

(00:06:04) N: Because it was the same.

(00:06:05) B: Aha! So we need something that equals...

(00:06:09) N: 9 now.

(00:06:09) B: What equals 9?

(00:06:12) N: Equals 9? Nothing equals 9.

(00:06:15) B: But here... ah.

(00:06:19) N: The lowest...

(00:06:22) B: There's no one with subtractions!

(00:06:26) N: No.

(00:06:27) B: You need a subtraction.

(00:06:29) N: But we haven't got a subtraction.

(00:06:31) R: Why do you need a subtraction?

(00:06:33) B: Because, 'cause 12 and 1 is the same as 13 we need something that's the same as 9.

(00:06:40) N: But there isn't anything the same as 9.

(00:06:43) R: Why do you need something that's the same as 9?

(00:06:46) B: So it can [unclear]

(00:06:48) R: Ah, I see. So is there anything else in the green box that you could do? You can't use a 9

(00:06:54) B: 12 and the 1.

(00:06:57) N: Ah, but you can't do the 9.

(00:07:00) B: Aha, I've got an idea. Yeah try like that.

(00:07:05) N: Wait...

B: That changed.

N: Wait, wait, wait, wait, wait, that is the same as 22, still.

(00:07:11) B: Ah! I've got an idea. If we change, something like we could change that, then we might get something that will be able to equal 9. Oh that might go horribly wrong.

(00:07:25) N: Okay you're off it.

(00:07:29) B: Are those 2s the same as they were before? They look slightly different or is it just me?

(00:07:36) N: Just me.

(00:07:37) (00:07:48) B: Yeah we're stuck.

(00:07:49) R: Okay, click the refresh button. Now I started you off, can you remember the first thing you did in the box?

(00:07:58) N: Oh, we put the 12 and the 13. 12 add the 1 up there.

(00:08:07) R: So where should that be clicked, N?

(00:08:10) N: In the green box.

(00:08:11) R: The green box?

(00:08:13) B: In the add.

(00:08:14) N: Or maybe that?

(00:08:15) B: Highlight it again. Shall I press the add?

(00:08:21) R: Um-um. N, you said what you did before. What did you click before? In the green box, but where in the green box?

(00:08:27) N: There.

(00:08:30) B: Ah!

(00:08:31) R: What do you get?

(00:08:32) B: That! If you highlight that and then click on something that equals it. What are you doing? So if you click on that...

(00:08:46) N: Click on this! And then, and then, and then you click on [unclear]

(00:08:48) B: If, the one you highlighted, and there's something the same, um, click where the add sign is...

(00:08:56) R: Okay.

(00:08:57) B: It will magically appear.

(00:08:58) N: Ooh, that's annoying now because now we've changed it we have to press on the 22 and then on the add but it won't work.

(00:09:06) R: Can you see why it won't work, B?

(00:09:07) B: Yeah. I think.

(00:09:10) N: So add them [unclear]

(00:09:15) R: Try again in the green box.

N: What's 9 add 12?

(00:09:17) R: Try somewhere else on the green box. So why did that change? It didn't when you clicked on the 9 plus, but it did when you clicked on the 12 plus 1.

(00:09:30) B: Because, it was exactly... we've got it!

(00:09:36) R: Have you?

(00:09:36) N: No.

(00:09:37) B: Yeah!

R: No?

(00:09:39) R: Brilliant.

(00:09:39) B: How did that happen?

R: Well done.

(00:09:44) R: N, how do you think it works now, are you starting to get it?

(00:09:47) N: Er, yeah, sort of. Because you have to do that and then you have to change it lots of times until there's a sum that's the same as that.

(00:09:54) R: Okay. And what do you think, B?

(00:09:56) B: I think it's very complicated.

(00:10:01) N: No, because you have to have, because there's, there has to be a sum at the bottom [B: Oh yeah!] that's the same and then you have to make that sum in the green box.

(00:10:09) R: Okay. You can see if you're right by having a go at the next puzzle. You see the button with the arrow? Yep.

(00:10:16) PUZZLE 2

(00:10:18) B: 31 plus 19.

(00:10:21) N: 19. What's that?

(00:10:22) B: 31... look for a 31 somewhere.

(00:10:27) (00:10:33) N: Well I found a 19 and another 19.

(00:10:37) B: But we need something that will equal 19.

(00:10:40) B: Aha, I found a 31.

(00:10:45) N: 28 add 8, no I mean 23 add 8. Click on the 31.

(00:10:55) B: Oh!

(00:10:55) N: Well done, B.

(00:10:57) B: 28 plus 8 equals 19, add 19. 28, 8. 23, 8 and 19.

(00:11:05) N: Ah [unclear] That's not the same, is it?

(00:11:10) B: What's 19 add 8, twenty, twenty, twenty...

(00:11:15) N: 29, 27.

B: 27.

(00:11:21) N: 27. Click on 27 and then click on that. Click on 19.

(00:11:27) B: Why on earth would that work?

(00:11:30) N: Oh yeah.

(00:11:31) R: I can see why it wouldn't work.

(00:11:35) N: Because 27 isn't the same as 19.

(00:11:39) B: I was being sarcastic actually N.

(00:11:42) R: No it's not that reason.

(00:11:45) B: Because she clicked on the 27... oh.

(00:11:51) (00:11:57) N: Wait, what's 31 add 19? We still haven't worked that out.

(00:12:02) B: 31 plus 19?

(00:12:03) N: Yeah. Because that's what we started with.

(00:12:07) B: 50.

(00:12:10) N: Oh yes.

(00:12:14) (00:12:18) N: What equals 19? Er, let's see if something equals 19.

(00:12:21) B asks about the recording

(00:12:42) N: What makes 19?

(00:12:45) B: I don't know but this equals 50. Ha ha! Oops.Ooh, wait, wait, wait, wait, wait.

(00:12:54) N: Click 19 add 8. What's 19 add 8? 27. So, click on 27.

(00:13:06) B: This is fun. It's just random.

(00:13:12) N: And then we haven't got a 27. Is there anything we can do? 19.

(00:13:16) N: We could change that into 27.

(00:13:21) R: Did you have a little mis-click there, did you mean to do that?

(00:13:26) (00:13:32) B: Aha! If we now highlighted that, clicked on that, put that there.

(00:13:45) B: And then highlighted that, clicked on that, I'll just put that there, maybe clicked on that after

(00:13:52) N: It will put, click on 50 and put it on one of the things.

(00:13:58) B: Or, click on 23, put it there.

(00:14:01) N: But that wouldn't work because it wouldn't equal the same thing.

(00:14:04) (00:14:11) B: Now I know what we're meant to be doing but it won't work!

(00:14:14) R: I can see why it's not working, I'm not surprised. You're almost there. [B: Why isn't it working?] And I can see why you thought it would work, I can see why you thought that.

(00:14:23) N: Oh, we need to switch them round or switch them round. But we can't.

(00:14:27) R: What do you need to switch round?
(00:14:28) N: Those.
B: But, what's the point when you've got that there?
(00:14:31) [N laughs]
(00:14:35) (00:14:38) N: Yay!
(00:14:40) R: Well done. Yeah, the next one.
(00:14:42) PUZZLE 3
(00:14:43) B: What's 13 plus 18?
(00:14:45) N: 20... 31.
(00:14:48) B: 31. Right, so 31. So change that...
(00:14:57) N: We need to make it 8 add 23.
(00:14:59) B: Yeah.
(00:14:59) N: So 5 add 8, no... There aren't any 8s here. What's 8 add 23?
(00:15:06) B: Definitely not that.
(00:15:08) N: I know but what's 8 add 23.
(00:15:10) B: Er, 41.
(00:15:12) N: 41. Okay that won't work.
(00:15:15) B: No, yeah, no. What's 18 plus 8? [unclear]
(00:15:24) B: 18 plus 8 is...
(00:15:27) N: 26.
(00:15:28) (00:15:31) B: Um, if we did something with this. Then clicked on that.
(00:15:42) N: Yes that would completely work.
(00:15:45) B: Oh! Wait! Ah! I think I have an idea. No, my idea just [unclear] up.
(00:15:57) N: What's 23 add 8?
(00:16:00) B: 23 plus 8 is 41.
(00:16:02) N: But there isn't any 41 here.
(00:16:05) B: [unclear] no, 31.

(00:16:06) (00:16:20) N: We need to make 26 some how.

(00:16:24) B: We need 26. I think. 5 add 18 is...

(00:16:32) N: 23.

(00:16:35) (00:16:44) B: Can we have some help?

(00:16:45) R: I would start with highlighting the 13 equals 5 plus 8.

(00:16:53) B: Oh, okay then.

(00:16:54) N: Oh yeah, yeah, yeah because that's, that's 5 add 8 isn't it and then we have an 8, and 18 add 5 is 23.

(00:17:05) B: You've changed that back!

(00:17:07) N: I know! Try and unhighlight it.

(00:17:12) B: Change back!

(00:17:13) N: Try and click on 13.

(00:17:16) B: Right now I've got an idea. What equals 18? Something here's got to be equal to 18.

(00:17:23) N: No. [unclear]

(00:17:25) B: Wait a minute. What did you say that was?

(00:17:27) N: 23. It says.

(00:17:31) B: Oh yeah.

(00:17:32) [laughter]

(00:17:35) N: Now we need to do that. Wait, wait, wait, wait. Wait, wait, wait, wait, wait, wait, wait, wait. [unclear] that like that. Oh!

(00:17:49) R: Well it looks like it should work, N, but why do you think it didn't?

(00:17:52) N: Because the 8's in the middle.

(00:17:53) R: Okay. So what can you do about that?

(00:17:56) N: Um.

(00:17:57) B: Change it back.

(00:17:58) N: Wait. We need, hm. We need to get 23 but we can't... But if we change it back then there's no point is there because that's 13 plus 18.

(00:18:08) B: What's 18 plus 8?

(00:18:09) (00:18:13) B: 18 plus 8 is 26.

(00:18:19) N: There isn't a 26 there, at all.

(00:18:21) B: I have an idea. If... change that into a 26.

(00:18:31) N: Yeah but there isn't a 26 B.

(00:18:33) B: 23 plus 8 equals 31.

(00:18:39) B: Can we have some more help?

(00:18:41) R: Yeah go back to where you were before, because you'd used the 13 equals 5 plus 8 hadn't you?

(00:18:46) B: Yeah.

(00:18:47) R: So click in the green box, like that. And then N, you said you couldn't use the 5 add 18 equals 23, why did you [interrupted]

(00:18:54) N: Because the 8

B: Is in the middle.

N: Is in the middle.

(00:18:56) R: And I said see what you can do about that.

(00:19:00) B: 8 plus 23 equals 31!

(00:19:03) N: Yes but we don't have 23.

(00:19:05) R: What do you mean you don't have 23 N?

(00:19:07) B: 18 plus 8 is 23, oh no it isn't.

(00:19:10) N: Yeah it is. No it isn't. So 18 [B: I have an idea] plus 5 is 23.

(00:19:17) B: Right so if we change that, ah! This is really starting to bug me.

(00:19:26) (00:19:36) B: That, I'd still click it in the middle.

(00:19:38) N: Yeah I know. We need to make 18 23.

(00:19:45) B: 18 plus what equals 23?

(00:19:47) N: 5.

(00:19:48) B: Oh yeah.

(00:19:50) R: And why couldn't you do that?

(00:19:51) B: Becau-, we could.

(00:19:57) N: Yes but we can't do that because the 5's over there and the 8's in the middle so you can't do that.

(00:20:06) R: So can you see anything that has an 8 in it that might help you?

(00:20:09) N: That one.

(00:20:10) B: That one.

(00:20:11) N: Press equals.

(00:20:12) R: What do you think that one's going to do?

(00:20:14) B: Not much.

(00:20:18) R: Click again more carefully.

(00:20:20) N: Look, look, look, B, B, because look. It's 26. You see, 18 add 8, and then we've got 18 add 8 here.

(00:20:29) B: We're doing well!

(00:20:31) N: So now, we can do this. Ta-daa!

(00:20:35) B: 23 plus 8 is 41. 31.

(00:20:39) N: Yes!

(00:20:41) R: Well done. Let's have a go at the next one.

(00:20:46) PUZZLE 4

(00:20:46) B: Are these meant to get harder?

(00:20:50) N: It's kind of a hint that the last one was harder than the one before.

(00:20:53) B: Right let's see. 45 plus 16 equals [both together] fifty-, 61.

(00:21:01) B: Is it?

(00:21:02) N: Yeah.

(00:21:02) B: Yeah. So we look for 61.

(00:21:06) B: Oh one minute! I have an idea. Equals.

(00:21:13) N: Because that will divide.

(00:21:16) [unclear]

(00:21:20) N: No, no.

(00:21:24) B: Yes!

(00:21:26) N: Oh yeah.

(00:21:26) B: If we change them round.

(00:21:28) N: So where, where's...

(00:21:29) B: Change those two round. So then it'd be...

(00:21:31) N: No you just, no I know a simpler way of doing it, ta-daa. Why won't it change?

(00:21:38) R: What were you saying B, that sounded a better, er...

(00:21:42) B: If you wanted those two to change round you could, um, press on that.

(00:21:50) R: Yeah. Go on then.

(00:21:52) B: And click on that. Then press that.

(00:21:56) N: Yeah but they're still the same aren't they?

(00:21:58) B: Those two just change round.

N: Ah!

(00:22:02)

(00:22:02) B: And now what you can do is this.

(00:22:06) N: Yeah, yeah, yeah, yeah, yeah. I have an idea! Equals. No that doesn't work.

(00:22:14) B: It does.

(00:22:16) (00:22:16) N: Oops.

(00:22:20) B: Equals.

(00:22:33) B: 32 plus 16.

(00:22:35) N: Is 48.

(00:22:37) B: Yeah.

(00:22:37) N: So we could change that but then...

(00:22:42) B: 13 plus, we've got 13 plus 48.

(00:22:45) N: 13 plus 48. That was, oh! We need to change that, into 61.

(00:22:54) B: Okay. Then, I can do...

(00:22:56) N: 61's there. Equals 61. And there. Yay!

(00:23:03) R: Brilliant.

(00:23:04) B: Done it!

(00:23:05) R: They're getting, it's getting harder but you're also getting better at it.

(00:23:09) B: Yeah.

PUZZLE 5

(00:23:09) N: 53 add 22 is 75. 75's there.

(00:23:17) N: 21 add 54.

(00:23:21) (00:23:26) N: 22, 22.

(00:23:28) B: Well you want...

(00:23:29) N: So change that to 9 add 13. Change that to 9 add 13.

(00:23:34) B: What with...

(00:23:35) N: Change 22 to 9 add 13.

(00:23:43) B: Now change the 53 into 41 plus 12.

(00:23:47) N: Okay now it's a big sum.

(00:23:53) B: 41 plus 13 equals 54.

(00:23:57) N: We do have a 13 but it's right at the other end so it won't work.

(00:24:01) B: Oh yeah. So we need to find a way to get those two close together.

(00:24:06) N: Swap, but we can't swap the 12.

(00:24:09) B: If, 54 plus 21, 54 plus 21 and then the other way.

N: Swap, no [unclear]

(00:24:17) B: 21. [unclear] If we change this back.

(00:24:23) N: Yeah if we change this to 53.

(00:24:27) B: If we work on this for just a while.

(00:24:31) B: 61 plus 13.

N: But there isn't any 53 anywhere is there, so we need to change it.

(00:24:36) B: Wait a minute. 22...

(00:24:44) (00:24:46) N: Yeah.

(00:24:49) B: Now...

(00:24:52) (00:25:05) N: Oh, that's 54. We need to take, we need to get 75. We need to get 21 add 54.

(00:25:14) N: How can we do that?

(00:25:19) B: Can we have some more help?

(00:25:22) R: You did really well at the start and then when you said it's gone really big, because it had four numbers in the green box, see if you can get back there again.

(00:25:30) N: Okay.

(00:25:30) B: Okay. I can do that.

(00:25:32) (00:25:36) R: If you grab the edge it will just drag back. There you go.

(00:25:40) B: 9 add 13, wait. Now I think, equals, yeah.

(00:25:46) N: No, let's add one.

(00:25:47) B: Equals.

(00:25:49) N: That.

(00:25:50) B: Those two.

(00:25:51) N: Yes. The split that.

(00:25:52) R: I'll just tell you something, B, you know when you clicked equals and then you clicked the 9 plus 13. You only need to click the equals. You only need to highlight them. Okay that's, that's looking good.

(00:26:07) N: 41 add 13.

(00:26:10) B: 21 plus 13.

(00:26:12) R: How do you know?

(00:26:14) B: Because it says it down there.

(00:26:16) R: Okay, yes.

(00:26:20) N: [unclear]

(00:26:23) B: Now four, look.

(00:26:27) [brief interruption]

(00:26:39) B: Then, when we've done that we can press that. 54 plus 21.

(00:26:45) N: So we need to switch them round.

(00:26:47) B: No we don't.

(00:26:47) N: Yeah we do.

(00:26:48) B: No we don't.

(00:26:49) N: Ah yeah, no we don't.

(00:26:52) B: 21 plus 54.

(00:26:53) N: 21 plus 54.

(00:26:57) B: Unhighlight that one.

(00:26:59) N: Click 75. Click 75!

(00:27:06) B: They're the wrong way round again. They need to be that way round. Then if we do this...

(00:27:13) N: Click the 75...

B: 75.

(00:27:15) R: Brilliant.

(00:27:18) R stops session and tells B and N we'll do puzzle 6 next time.

Appendix C

Example trace

Trace of Study 3, Trial 1, Session 1 (Bridie and Nadine).

The following trace of the data was produced using *Transana*'s "clip notes" reporting facility. Each clip note is presented here as three items: (i) "Clip", which the label attached to each "movie clip" by *Transana*; (ii) "Time", which shows the start and end time of the clip in hours:minutes:seconds; (iii) the text of the note attached to the clip.

Clip: BN10:10:07 Quick Clip 1

Time: 0:01:16.2 - 0:01:28.9 (Length: 0:00:12.7)

N says the right-hand side of $12+1$ makes "another 13".

Clip: BN10:10:07 Quick Clip 2

Time: 0:02:06.7 - 0:02:35.0 (Length: 0:00:28.3)

B and N observe their first substitution. On cue from R, N explains it in terms of $9+1$ being computationally "the same as 10".

Clip: BN10:10:07 Quick Clip 3

Time: 0:02:59.5 - 0:03:23.4 (Length: 0:00:23.9)

R asks the girls to predict what the puzzle might look like when solved. B responds that the statement $10+12=22$ should make it “equal 22”.

Clip: BN10:10:07 Quick Clip 4

Time: 0:04:16.7 - 0:04:30.0 (Length: 0:00:13.3)

B reads $13=12+1$ as 26 and $9+1=10$ as 20 but says $10+22=22$ “equals 22 obviously”.

Clip: BN10:10:07 Quick Clip 5

Time: 0:05:10.5 - 0:05:35.9 (Length: 0:00:25.4)

B reads $9+1=10$ as “10 add a 10”. N comments that $9+1=10$ and $10+12=22$ contain “three 10s”.

Clip: BN10:10:07 Quick Clip 6

Time: 0:06:01.5 - 0:06:15.4 (Length: 0:00:13.9)

R asks why $13=12+1$ made a substitution on $9+13$. N responds that it is the same. B then declares they need something that equals 9 to make another substitution.

Clip: BN10:10:07 Quick Clip 7

Time: 0:06:22.0 - 0:06:48.1 (Length: 0:00:26.1)

B declares they need a subtraction and R asks why. B answers “because 12 and 1 is the same as 13 we need something that’s the same as 9”

Clip: BN10:10:07 Quick Clip 8

Time: 0:06:48.1 - 0:07:11.0 (Length: 0:00:22.9)

N computes the total of the green box and concludes it still makes 22.

Clip: BN10:10:07 Quick Clip 9

Time: 0:07:11.0 - 0:07:29.1 (Length: 0:00:18.0)

B expresses an idea, but then withdraws it. It's unclear what her idea is exactly, but it involves making something numerically equivalent to 9.

Clip: BN10:10:07 Quick Clip 10

Time: 0:08:31.0 - 0:08:58.9 (Length: 0:00:27.9)

B has an “aha!” moment and R asks her to elaborate. She says she has realised that once a statement is highlighted you have to click on an inscription that is numerically equivalent to the statement (she's half right). In her next statement it is unclear whether she means numerically equivalent or iconically matched when she says “and there's something the same”.

Clip: BN10:10:07 Quick Clip 11

Time: 0:08:58.9 - 0:09:09.9 (Length: 0:00:11.0)

N has a model of the puzzle's functionality involving highlighting a statement then clicking on its 'answer' before attempting a substitution. She is surprised, or at least bemused, that this does not work in the case of $10+12=22$ acting on $9+12=1$.

Clip: BN10:10:07 Quick Clip 12

Time: 0:09:17.7 - 0:09:44.2 (Length: 0:00:26.5)

B suddenly sees the rest of the way to solving the puzzle and does so quickly and confidently. Although neither girl articulates it, it seems that iconic matching is the reason for this.

Clip: BN10:10:07 Quick Clip 13

Time: 0:09:44.2 - 0:10:09.8 (Length: 0:00:25.6)

R asks the pupils how they think the puzzle works. Only N offers a detailed answer. She elaborates the strategy of working back from the result of the green box (computational view). However, she also indicates that it is necessary for the green box to iconically match the sum of the 'final statement'.

Clip: BN10:10:07 Quick Clip 14

Time: 0:10:18.4 - 0:10:52.3 (Length: 0:00:33.9)

Puzzle 2 appears but the pupils do not compute the contents of the box. Instead B suggests look for a 31. N finds 19s instead but B points out that they would need a statement that equals 19. They use $31=23+8$ to make a substitution. N predicts the outcome but the pupils do not explicitly articulate the distinct partitional properties of the transformation.

Clip: BN10:10:07 Quick Clip 15

Time: 0:10:57.1 - 0:11:18.6 (Length: 0:00:21.5)

The pupils compute two numerals in the green box.

Clip: BN10:10:07 Quick Clip 16

Time: 0:11:21.1 - 0:11:52.0 (Length: 0:00:30.9)

The pupils try to use $19+8=27$ on $23+8+19$. This decision is based on the numerical equivalence of $8+19$ and 27 rather than the iconic similarity of $19+8$ and $8+19$. B adopts N mental model that a highlighted statement's 'answer' must be clicked before attempting a substitution.

Clip: BN10:10:07 Quick Clip 17

Time: 0:11:57.9 - 0:12:14.8 (Length: 0:00:17.0)

N realises they never computed the contents of the box, and they pupils do so.

Clip: BN10:10:07 Quick Clip 18

Time: 0:12:18.7 - 0:12:22.0 (Length: 0:00:03.3)

N looks for numerical equivalence matches with the numeral 19. Note the computational reading has merit, and could be interpreted as the emergence of partitioning.

Clip: BN10:10:07 Quick Clip 19

Time: 0:12:42.7 - 0:13:02.6 (Length: 0:00:19.9)

The pupils look for numerical equivalence matches with 19 and 50.

Clip: BN10:10:07 Quick Clip 20

Time: 0:13:12.2 - 0:13:15.5 (Length: 0:00:03.2)

N realises that $19+8=27$ is no use because the green box does not contain the numeral 27.

Clip: BN10:10:07 Quick Clip 21

Time: 0:13:16.2 - 0:13:41.2 (Length: 0:00:25.0)

N predicts correctly that $8+19=27$ can change $23+8+19$ into $23+27$

Clip: BN10:10:07 Quick Clip 22

Time: 0:13:45.3 - 0:14:04.2 (Length: 0:00:18.9)

Pupils attend to numerical equivalence of the box and a statement. N seems to think that the 50 should be in some sense “dragged” to the box. When this doesn’t work, B suggests dragging the 23 from the statement onto the 27 in the box, perhaps expecting this to add the 23 and 27.

Clip: BN10:10:07 Quick Clip 23

Time: 0:14:23.0 - 0:14:40.7 (Length: 0:00:17.6)

N sees the need for commuting the numerals to make the statement work.

Clip: BN10:10:07 Quick Clip 24

Time: 0:14:43.9 - 0:14:54.0 (Length: 0:00:10.1)

The pupils begin puzzle 3 by computing the contents of the box.

Clip: BN10:10:07 Quick Clip 25

Time: 0:14:57.3 - 0:14:59.8 (Length: 0:00:02.5)

N suggests the strategy of making the box iconically matched with the LHS of the final statement.

Clip: BN10:10:07 Quick Clip 26

Time: 0:14:59.8 - 0:15:28.9 (Length: 0:00:29.1)

The pupils compute terms on the screen.

Clip: BN10:10:07 Quick Clip 27

Time: 0:15:31.8 - 0:15:42.2 (Length: 0:00:10.4)

B attempts to make a substitution using the final statement. Notably, she clicks on the result of the statement then clicks on the + in the box.

Clip: BN10:10:07 Quick Clip 28

Time: 0:15:57.5 - 0:16:06.2 (Length: 0:00:08.7)

N reads $23+8=8+23$ compositionally.

Clip: BN10:10:07 Quick Clip 29

Time: 0:16:20.7 - 0:16:35.2 (Length: 0:00:14.5)

N focuses on the computational result 26, although where she gets it from is unclear.

Clip: BN10:10:07 Quick Clip 30

Time: 0:16:44.1 - 0:17:05.4 (Length: 0:00:21.2)

N attends to the numerals that appear in the box after a substitution, and then computes numerals in the box.

Clip: BN10:10:07 Quick Clip 31

Time: 0:17:16.8 - 0:17:23.5 (Length: 0:00:06.7)

B searches for a statement that matches the numeral 18. She wants it to have a partitional function, but talks in compositional structure.

Clip: BN10:10:07 Quick Clip 32

Time: 0:17:25.3 - 0:17:32.3 (Length: 0:00:07.0)

B attends to a computational result, but not a compositional reading (no apparent expectation of result to follow equals sign)

Clip: BN10:10:07 Quick Clip 33

Time: 0:17:35.4 - 0:17:57.0 (Length: 0:00:21.6)

N attends to matching numerals when trying to make a compositional transformation

Clip: BN10:10:07 Quick Clip 34

Time: 0:17:58.8 - 0:18:08.3 (Length: 0:00:09.6)

N attends to the need for the numeral 23 to appear in the box

Clip: BN10:10:07 Quick Clip 35

Time: 0:18:08.3 - 0:18:36.3 (Length: 0:00:28.0)

B computes terms, though not with an explicitly compositional reading of statements. N argues that B's result is a numeral not present on screen.

Clip: BN10:10:07 Quick Clip 36

Time: 0:18:47.7 - 0:19:00.2 (Length: 0:00:12.5)

On prompting, N again attends to matching numerals when trying to make a compositional transformation (c.f. BN10:10:07 Quick Clip 33)

Clip: BN10:10:07 Quick Clip 37

Time: 0:19:00.2 - 0:19:26.9 (Length: 0:00:26.7)

B reads a composition statement, and makes computations. N argues that B's result (23) is not a numeral in the box.

Clip: BN10:10:07 Quick Clip 38

Time: 0:19:38.6 - 0:19:50.0 (Length: 0:00:11.5)

The pupils consider how 18 might be transformed into 23.

Clip: BN10:10:07 Quick Clip 39

Time: 0:19:50.0 - 0:20:02.6 (Length: 0:00:12.6)

N explains why $5+18=23$ won't substitute in $5+8+18$

Clip: BN10:10:07 Quick Clip 40

Time: 0:20:06.0 - 0:20:15.6 (Length: 0:00:09.6)

R draws attention to the numeral 8. B comments that it won't do much.

Clip: BN10:10:07 Quick Clip 41

Time: 0:20:20.2 - 0:20:41.2 (Length: 0:00:21.0)

N explains a commutational substitution in terms numerical equivalence. She then attends to composition to complete the puzzle.

Clip: BN10:10:07 Quick Clip 42

Time: 0:20:53.4 - 0:21:05.1 (Length: 0:00:11.7)

B starts puzzle 4 by computing the contents of the box, and then suggests looking for the resulting numeral (61)

Clip: BN10:10:07 Quick Clip 43

Time: 0:21:06.1 - 0:21:20.2 (Length: 0:00:14.0)

B suggests the statement $45=32+13$ and N explains that it "will divide"

Clip: BN10:10:07 Quick Clip 44

Time: 0:21:20.2 - 0:21:38.4 (Length: 0:00:18.3)

N browses the commutative statements. B uses the phrase "change round"

Clip: BN10:10:07 Quick Clip 45

Time: 0:21:38.4 - 0:22:02.0 (Length: 0:00:23.6)

B predicts and demonstrates the transformational effect of a commutative statement

Clip: BN10:10:07 Quick Clip 46

Time: 0:22:02.9 - 0:22:16.8 (Length: 0:00:13.9)

B tries the iconic mismatch composition $16+32=48$ on $13+32+16$ N sees the need to use $32+16=16+32$ first.

Clip: BN10:10:07 Quick Clip 47

Time: 0:22:17.0 - 0:22:27.2 (Length: 0:00:10.2)

A commutation attempt goes wrong due to a mis-click

Clip: BN10:10:07 Quick Clip 48

Time: 0:22:33.2 - 0:22:42.7 (Length: 0:00:09.6)

Compositional reading of $32+16=48$ and $48=16+32$

Clip: BN10:10:07 Quick Clip 49

Time: 0:22:42.7 - 0:23:04.0 (Length: 0:00:21.2)

The pupils complete the puzzle by commuting then composing.

Clip: BN10:10:07 Quick Clip 50

Time: 0:23:09.9 - 0:23:21.9 (Length: 0:00:12.0)

The pupils begin puzzle 5 by computing the contents of the box and looking for the resulting numeral.

Clip: BN10:10:07 Quick Clip 51

Time: 0:23:26.4 - 0:23:53.1 (Length: 0:00:26.7)

N looks for matches with a numeral in the box, and suggests using $22=9+13$. B then suggests changing another numeral using $53=41+12$. N comments on the visual change: “now it’s a big sum”

Clip: BN10:10:07 Quick Clip 52

Time: 0:23:53.1 - 0:24:17.6 (Length: 0:00:24.4)

B focusses on a composition, possibly because it’s answer is a numeral in the final statement. The pupils then discuss how they can bring two numerals together in the box (in order to use the composition) by using commutative statements

Clip: BN10:10:07 Quick Clip 53

Time: 0:24:17.6 - 0:24:36.8 (Length: 0:00:19.2)

B recomposes two numerals using $53=41+12$, apparently to make the box 'easier' to work with. N points out that there is no other numeral 53 in the puzzle.

Clip: BN10:10:07 Quick Clip 54

Time: 0:24:36.8 - 0:24:52.1 (Length: 0:00:15.3)

The pupils re-compose using $22=9+13$ then re-partition using $53=41+12$. No reasons are articulated.

Clip: BN10:10:07 Quick Clip 55

Time: 0:25:05.8 - 0:25:19.5 (Length: 0:00:13.8)

N computes the contents of the box, and articulates the strategy of matching the box to the LHS of the final statement.

Clip: BN10:10:07 Quick Clip 56

Time: 0:25:47.8 - 0:25:52.7 (Length: 0:00:04.9)

When B asks for help, R tells them to go back to having four numerals in the box. When doing so N using the phrase "split that"

Clip: BN10:10:07 Quick Clip 57

Time: 0:26:07.2 - 0:26:19.1 (Length: 0:00:11.8)

B gives an iconic matching reason for choosing a commutative statement

Clip: BN10:10:07 Quick Clip 58

Time: 0:26:39.0 - 0:26:45.7 (Length: 0:00:06.6)

B uses a compositional statement.

Clip: BN10:10:07 Quick Clip 59

Time: 0:26:45.7 - 0:26:57.0 (Length: 0:00:11.3)

N sees the need for a commutation.

Clip: BN10:10:07 Quick Clip 60

Time: 0:26:59.7 - 0:27:06.3 (Length: 0:00:06.6)

N does not notice the need for a further commutation (due to a mis-click) and instead thinks the result in a compositional statement needs to be clicked.