Understanding irrational numbers by means of their representation as non-repeating decimals

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Research study on students' conceptions of irrational numbers upon entering university is of importance towards the transition to university. In this paper, we analyze students' conceptions of irrational numbers using their representation as non-repeating infinite decimals. The majority of students in the study identify the set of all decimals (finite and infinite) with the set of rational numbers. In spite of the fact that around 80% of the students claimed that they had learned about irrational numbers, only a small percentage of students (19%) showed awareness of the existence of non-repeating infinite decimals.

Keywords: extension of rational numbers to irrational numbers; irrational numbers, intuition, non-repeating infinite decimal, transition to university.

INTRODUCTION

The paper deals with students' conceptions of irrational numbers. The importance of real numbers towards the learning of analysis is well known. The understanding of irrational numbers is essential for the extension and reconstruction of the concept of number from the system of rational numbers to the system of real numbers. Therefore, research study on students' conceptions of irrational numbers upon entering university is of importance towards the transition to university. Artigue (2001) wrote about the necessarily reconstructions which deal with mathematical objects already familiar to students before the official teaching of calculus:

> Real numbers are a typical example…Many pieces of research show that, even upon entering university, students' conceptions remain fuzzy, incoherent, and poorly adapted to the needs of the calculus world…the constructions of the real number field introduced at the university level have little effect if students are not faced with the incoherence of their conceptions and the resulting cognitive conflicts (Artigue, 2001, p.212).

This study is a part of a broader study which aims to investigate students' conceptions of rational and irrational numbers upon entering university. Using epistemological considerations, three different representations of the irrational numbers were considered in the broader study. The first one relates to the decimal representation of an irrational number. The second representation relates to the fitting of the irrational numbers on the real number line. The third representation considers the relationship between incommensurability and the irrational numbers. In this paper, we consider the first representation and analyze students' understanding of irrational numbers by means of their representation as non-repeating infinite decimals.

THEORETICAL BACKGROUND

Monaghan (1986) observed that students' mental images of both repeating and non-repeating decimals often represent "*improper numbers which go on for ever"*. Because of their infinite decimal expansions, these numbers are often considered as *infinite* numbers. Tall (2013) relates to students' difficulties with irrational numbers:

> The shift from rational numbers to real numbers proves to be a major watershed for many students. In school, students meet irrational numbers such as $\sqrt{2}$, π and e, and begin to realize that the number line has numbers on it that are not rational, though it is not clear precisely what these irrational numbers are (Tall, 2013, p. 265).

Kidron and Vinner (1983) observed that the infinite decimal is conceived as one of its finite approximation or as a dynamic creature which is in an unending process- a potentially infinite process: in each next stage we improve the precision with one more digit after the decimal point. Vinner and Kidron (1985) analyzed the concept of repeating and non-repeating decimals at the senior high level. The present study is a broader study of the part that relates to non-repeating decimals. Fischbein, Jehiam and Cohen (1995) observed that the participants in their study were not able to define correctly the concepts of rational, irrational, and real numbers. Zaskis and Sirotic (2004) analyzed how different representations of irrational numbers influence participants' responses with respect to irrationality. Sirotic and Zaskis (2007) observed inconsistencies between participants' intuitions and their formal and algorithmic knowledge. The authors claim that constructing consistent connections among algorithms, intuitions and concepts is essential for understanding irrationality.

From the epistemological approach, the difficulties that are inherent to the nature of the specific domain should be taken into account (Barbé et al., 2005). Some of the cognitive difficulties in relation to the concept of irrational numbers might be a consequence of the way we conceive the concept of infinity. Fischbein, Tirosh, and Hess (1979) observed that the natural concept of infinity is the concept of potential infinity. Therefore, students' intuition of infinity might become an obstacle in the understanding of irrational numbers as non-repeating infinite decimals. Fischbein's theory which offers a rich insight in the mechanisms of intuition will serve as theoretical framework for the present study. Fischbein considers the intuitive structures as essential components of productive thinking. Fischbein (1987, pp.129-130) distinguishes different types of analogies which may intervene, tacitly or explicitly, in mathematical reasoning. He also refers to some kind of analogies that manipulate the reasoning process from "behind the scenes". Fischbein (2001) analyzes several examples of tacit influences exerted by mental models on the interpretation of mathematical concepts in the domain of actual infinity. He describes the concept of mental models as mental representations which replace, in the reasoning process, the original entities.

METHODOLOGY

The task

A questionnaire (which served as a research tool) was compiled and administered and the results concerning one of its questions is brought and analysed here.

Question: *A teacher asked his students to give him an example of an infinite decimal.*

Dan: I will look for two whole numbers such that when I divide them I would not get a finite decimal; for instance: 1 and 3

Ron: I will write down in a sequence digits that occur to me arbitrarily, for instance: 1.236418..

Dan: Such a number does not exist because what you write down is not a result of a division of 2 whole numbers

Ron: Who told you that what you write down must be the result of a division of two whole numbers?

Who is right? Please explain!

The question aimed to examine whether the students are mathematically matured for the idea of irrational numbers as infinite non-repeating decimals.

Participants and data collection

The question was posed to 91 10^{th} graders and 97 11^{th} and 12^{th} graders learning at the same academic high school in Jerusalem, which is academically selective. The 10^{th} graders learned mathematics at the same level. One group of the 11^{th} graders learned mathematics in an advanced level class (5 units). The other 11^{th} and 12^{th} graders learned in an average level class (4 units).

We asked the students in the sample whether they studied the concept of irrational numbers in the past. 77% of the 10^{th} graders (78% of the 11^{th} and 12^{th} graders) claimed that they studied it; 7% of the 10^{th} graders (12% of the 11^{th} and 12^{th} graders) claimed that they do not remember if they studied it or not and 16% of the 10^{th} graders claimed that they did not study it (10% of the 11^{th} and 12^{th} graders). The part of the questionnaire that related to irrational numbers requested more concentration from the students in comparison to the part that related to rational numbers. The 11^{th} and 12^{th} graders were focused in their work and wrote detailed answers. Even after answering the questionnaire they remained in the class and discussed their answers. The situation was different for the 10^{th} graders. It was difficult for them to concentrate on the questions on irrational numbers. In contrary to the first part of the questionnaire which dealt with rational numbers in which all of the 10^{th} graders participated, around 20% of the 10th graders did not participate in the second part which dealt with irrational numbers.

RESULTS AND ANALYSIS

The distribution of answers to question 1 is given in Table 1

Categories	Percentages of 10th graders N=91	Percentages of 11th and 12th graders N=97
A. Any decimal must be a result of a division of 2 whole numbers	56%	54%
B. An infinite decimal can be obtained not only as a result of a division of 2 whole numbers	23%	43%
C. No answer	21%	3%

Table 1: Distribution of answers

Category A : Any decimal must be a result of a division of 2 whole numbers

For 55% of all students in the sample "any decimal must be a result of a division of 2 whole numbers". Analysing students' detailed answers, we observe four sub-categories of answers.

The answers of 14 % of all students belong to the first sub-category:

Every number is a result of a division of two whole numbers

8% of all students refer only to rational numbers. For example, the following answer:

> Dan is right because we asserted that an infinite number, namely, an infinite decimal is a certain kind of a rational number and in order to obtain a rational number we should divide two whole numbers.

Some students proposed to check if the number given by Ron is a result of a division of 2 whole numbers. The students wrote:

- If a number is not the result of a division of 2 whole numbers then it is impossible to define it or to express it.
- Ron is wrong. He proposed a number which is not defined and we do not know what will be the next digits. We do not know if the number will be finite or infinite.

The students are reluctant to deal with an irrational number because there is no enough information about this number.

The answers of 17 % of all students belong to the second following sub-category:

We do not create numbers. All numbers are formed by means of division of whole numbers

> In a decimal number the digits after the decimal points should be linked to some division which give them.

Students are not ready to accept the irrational number:

- I think that Dan is right contrary to Ron who creates something out of nothing, a meaningless number.
- Ron can add as many digits as he wants it will not be a decimal number since a decimal number is a result of a division of two numbers. It will be something else.

Some students have difficulties to imagine an infinite procedure of writing digits.

> Dan is right since his decimal number has infinite digits in contrary of Ron's decimal. When Ron will stop adding digits it will result in a finite decimal.

The belief that we do not create numbers and every number is obtained by means of dividing two whole numbers was expressed in two main groups of answers. In the first group (7% of all students) the reason for this belief is that this is the only way to control the infinite number of digits in the decimal representation.

> Dan is right. Ron invented a number and he is not able to know if it is finite or not since we do not have here two whole numbers that he can divide.

2/3 of students' answers in this group are 10^{th} graders' answers.

For the second group of answers (5% of all students), every infinite decimal is "at the end" a repeating decimal. This conception might be a consequence of the fact that it is not easy for the students to give an example of an infinite non repeating decimal with a rule which guaranties the infinite digits with no repetition. A similar percentage of answers of 10^{th} graders and 11^{th} and 12^{th} graders belong to this group. The following answer was given by a student who expressed in other questions his awareness of the existence of irrational numbers. This answer shows the student's erroneous conception regarding randomness.

> At the moment you just write digits after the decimal point there will always be a repeating pattern since you only have 10 digits and an infinite number of places. Therefore there is a probability that a periodicity will appear.

The answers of 12 % of all students belong to the third following sub-category:

An "infinite decimal fraction" is identified by mistake with "fraction"

> Dan is right. A decimal fraction is another name or another way of writing a fraction a/b.

There is no fraction which might be obtained not by means of dividing two numbers. It might be a consequence of the fact that the questionnaire was in Hebrew and in Hebrew the infinite decimal is called "infinite decimal fraction".

The answers of 12 % of all students belong to the last following sub-category:

The student thinks only in terms of rational numbers

Ron is right since even when we divide 1/3:1/2 = 2/3, 2/3 is also an infinite decimal and it is a result of the division of two rational numbers which are not whole numbers.

Category B: An infinite decimal can be obtained not as a result of a division of 2 whole numbers

For 34% of all students in the sample “an infinite decimal can be obtained not only as a result of a division of 2 whole numbers”. We observe two sub-categories of answers.

The answers of 15 % of all students belong to the first sub-category: **There might be such a number with no relation to the question “what is this kind of number?”**

- Finally, such a number as the one given by Ron must exist and not every infinite decimal must be the result of a division of two whole numbers.

- The number exists although the thought process by Dan is safer

- One can obtain a number merely by writing down its digits.

In some answers, we note some reservation:

An infinite decimal has an infinite number of digits after the decimal point and we can write down its digits as we want since theoretically it exists.

The answers of 24% of 10th graders who complete the questionnaire belong to this first sub-category vs 10% of 11th and 12th graders.

The answers of 19 % of all students belong to the second sub-category: **An infinite non repeating decimal is not a result of a division of 2 whole numbers**

The situation is now different: 4% of 10th graders wrote answers that belong to this category vs 33% of 11th and 12th graders (50% of 11th graders that learn in the advanced level class). The following answers belong to this category:

- Ron is right. Dan claims that the number 1.236…is not a number. I do not agree with Dan because for him the word “number” only means rational numbers and he does not recognize irrational numbers as a number.
- Ron is right. There exist infinite decimals that cannot be obtained as a result of a division of 2 whole numbers. For example, π is an infinite non-repeating decimal. As a result of a division of 2 whole numbers, we always obtain a repeating decimal.
- Ron is right: (i) as a result of a division of 2 whole numbers, you always obtain a rational number. An irrational number cannot be obtained by means of a division of two whole numbers. (ii) it is always possible to define a new group of numbers (for example, if there is no solution to a quadratic equation, you can define a new group of numbers in which there is a solution).

Even if the students are aware of the existence of irrational numbers we note reservation especially in those answers that emphasize that one should not add verbal explanations to a mathematic object like a number.

- We should require to express the infinite decimal by means of conventional signs in order to assure that it is an infinite decimal.

Categories	Percentages of the entire sample N = 188	Percentages of 10^{th} graders N=91	Percentages of 11^{th} and 12^{th} graders N=97	Percentages of 11^{th} and 12^{th} graders at the average level N=64	Percentages of 11^{th} graders at the advanced level N=33
A. Any decimal must be a result of a division of 2 whole numbers	55%	56%	54%	58%	48%
I Every number is a result of a division of two whole numbers	14%	14%	14%	16%	12%
II We do not create numbers. All numbers are formed by means of division of whole numbers	17%	20%	15%	17%	12%
III An “infinite decimal fraction” is identified by mistake with “fraction”	12%	5%	16%	14%	21%
IV The student thinks only in terms of rational numbers	12%	16%	8%	11%	3%
B. An infinite decimal can be	34%	23%	43%	39%	52%

obtained not as a result of a division of 2 whole numbers					
I There might be such a number- no relation to the kind of number II An infinite non repeating decimal is not a result of a division of 2 whole numbers	15% 19%	19% 4%	10% 33%	16% 23%	0% 52%
C. No answer	11%	21%	3%	3%	

Table 2: Distribution of answers with sub-categories of perceptions

DISCUSSION OF FINDINGS

55% of the students identify the set of all decimals (finite and infinite) with the set of rational numbers. In spite of the fact that around 80% of the students claimed that they had learned about irrational numbers, only 4% of the 10^{th} graders and 33% of the 11^{th} and 12^{th} graders showed awareness of the existence of non-repeating infinite decimals. In addition, 21% of the 10^{th} graders could not even answer the question. The 11^{th} and 12^{th} graders did not receive any additional learning experience about irrational numbers. Therefore, the difference between the two groups may be explained by maturation. The mental ability to imagine an infinite procedure of writing digits, in an arbitrary way, to the right of the decimal point requests such maturation. A large number of students in both groups did not show awareness of the existence of non-repeating infinite decimals. In the next three subsections, we propose an explanation of students' reluctance to deal with irrational numbers.

The conception that numbers exist and we have no control on it

We can point on (natural) numbers or define rules of operating on two of these numbers (by means of addition, subtraction, multiplication or division). The larger set that the students can obtain by means of this conception is the set of rational numbers. The transition to real numbers is more problematic. The thinking "I will define a larger set of numbers that includes the previous one and keeps its properties" is a thinking which is opposed to the intuition that numbers exist without our intervention. In category A_{II}, we find explicit expression of this intuition. This view might be a consequence of the influence from the outside real world and the analogy with natural phenomena that exist without our involvement. We can investigate them but their existence does not depend on us. We noticed here a possible conflict between the learners' intuition in the sense of Fischbein and the formal rules of thinking.

The extension from the rational numbers to the real numbers is of a different kind than the previous extensions

The previous extensions from the natural numbers to the whole numbers and from the whole numbers to the rational numbers were done and expressed by means of the previous set. A rational number is simply defined by means of whole numbers. This kind of request to define the irrational number by means of rational numbers is well expressed in the historical development of irrational numbers.

> I demand that arithmetic shall be developed out of itself…Just as negative and fractional rational numbers are formed by a new creation, and as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone (Dedekind translated by Woodruff Beman in D.E. Smith).

The need to know the process that leads to the infinite decimal

In the first part of the questionnaire that deals with rational numbers and the way students conceive repeating infinite decimals we observed that the students are unable to differentiate between the result -the infinite repeating decimal and the process that gives this decimal. For example, the students identify the number 0.3333… with the process: 0.3; 0.33; 0.333; 0.3333; …; …; ….

The process (1:3) promises a single fixed result and this is important because of students' dynamic view of the repeating infinite decimal. The task in the present paper deals with non-repeating infinite decimals. Ron's number with the infinite arbitrary digits reinforces this conception of a number that changes all the time. We are also unable to predict how it changes. This situation reinforces students' dynamic process view of the infinite decimal and, as a consequence, the need for a process that promises a result. This dynamic process view of the infinite decimal corresponds to Fischbein's description of intuition of infinity as a potential infinite. When 55% of the students claim that a decimal (including infinite decimal) is the result of a division of two whole numbers they express their view that this process is a division. Why? It might be by analogy with the extension from whole numbers to rational numbers. This is right for a finite decimal and the student wants to suppose that it also works for any infinite decimal repeating or not-repeating. This need to identify the infinite decimal as a result of a division of two numbers was also observed among students who did express their awareness of the existence of irrational numbers. Even so, we read some answers like the following one

> Every number is always obtained by division of two numbers.. We can also obtain an infinite decimal by means of a division of irrational numbers.

We have here an expression of tacit influences exerted by mental models on the interpretation of mathematical concepts in the domain of actual infinity even for students who have constructed formal knowledge (Fischbein, 2001). The findings of this study help towards the effort of facing students entering university with the incoherence of some of their conceptions and the resulting cognitive conflicts.

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