Routines in the didactical and mathematical discourses of closed-book examination tasks

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A commognitive analysis of closed-book examination tasks and lecturers’ perspectives

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In this paper, we investigate the discourse of the closed-book examinations using a commognitive perspective. We analyse the routines of the discourse aiming to describe lecturers’ expectations about students’ engagement with mathematical discourse. Our data consists of the examination tasks of a year one course from a mathematics department in the United Kingdom and interviews with the lecturers of the course. In our analysis we identify the routines of the assessment discourse. The analysis reveals routines focusing largely on: directions given on the how of the mathematical routines; the gradual structure of tasks; students’ enculturation; and that the majority of the mathematical routines the students are expected to engage with are substantiation and recall.

Keywords: Closed-book examinations, commognition, routines, assessment discourse.

INTRODUCTION

The dominant method of assessment in mathematics departments in the United Kingdom is the closed-book [1] examinations (Iannone and Simpson, 2011). There is a wealth of frameworks analysing the tasks used in closed-book examinations (e.g. Bergqvist, 2007). The focus of these frameworks is on the range of skills, knowledge and reasoning assessed in the tasks. Taking a discursive approach when analysing the closed-book examinations provides us with a wider understanding of assessment practices: it allows us to characterise the mathematical discourse the students are expected to engage with and provides insight into the lecturers’ rationale for the way they pose examination tasks as well as into their expectations from student responses.

This paper focuses on a closed-book examination from a year one course in a mathematics department in the United Kingdom. The course consists of two parts: Sets, Numbers and Proofs taught in the autumn semester; and, Probability taught in the spring semester. The examination tasks are analysed using Sfard's (2008) theory of commognition. We focus on the routines of the discourse of the closed-book examination tasks and we complement the analysis of the tasks with data from interviews with the lecturers who designed the examinations.

In what follows, we first present the commognitive framework and review the literature on lecturers’ perspectives on examination tasks. We then introduce the context of our study and analyse two of the examination tasks. We conclude with a discussion of the discursive characteristics of these closed-book examination tasks.

COMMOGNITIVE FRAMEWORK AND RELEVANT LITERATURE

There is a wealth of studies in mathematics education using discursive approaches (Ryve, 2011), with Sfard's (2008) theory of commognition rapidly becoming a quite widely used one (Tabach & Nachlieli, 2016). Sfard (2008) defines discourse as “different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors” (p.93). Mathematics is seen as a discourse and doing mathematics is seen as engaging with mathematical discourse. The rules followed by the participants of the discourse are distinguished in *object-lev*el rules (“narratives about regularities in the behavior of objects of the discourse” (p. 201)) and *metarules* which “define patterns in the activity of the discursants trying to produce and substantiate object-level narratives” (p.201). Discourses are described in terms of four characteristics: *word use, visual mediators, endorsed narratives* and *routines*. More specifically, *word use* refers to the use of words specific to the discourse or everyday words (colloquial discourse) which may have different meaning when used in this discourse. In the mathematical discourse *word use* includes mathematical terminology (i.e. integers) and some words with special meaning in mathematics (i.e. disjoint sets). The *visual mediators* are objects and artifacts used to describe objects of the discourse. Some examples of visual mediators in the mathematical discourse are symbols and diagrams (i.e. Venn diagrams). *Endorsed narratives* are “sequence(s) of utterances, spoken or written, framed as a description of objects of relations between objects, or of activities with or by objects” (p.223). In the mathematical discourse an example of an endorsed narrative is a definition or a theorem. Finally, *routines* are a set of metarules describing patterns in the activity of the discursants. Some examples of routines in the mathematical discourse are the routines of proving and defining.

Sfard defines the *how* of a routine as “a set of metarules that determine, or just constrain, the course of the patterned discursive performance” (p.208) and the *when* of a routine as “a collection of metarules that determine, or just constrain those situations in which the discursant would deem this performance as appropriate” (p.208). She categorises routines in: *deeds* (effecting change on objects), *rituals* (“creating and sustaining a bond with other people”, p.241) and *explorations* (producing or substantiating an endorsed narrative, p.224). The exploration routines are further distinguished in: *construction* (resulting in new endorsable narratives); *substantiation* (aiming to decide whether to endorse previously constructed narratives); and *recall* routines (aiming to remember endorsed narratives).

The theory of commognition is being used increasingly in studies in mathematics education at university level (Nardi *et al*., 2014). Viirman’s (2014, 2015) work on the routines of the discourse is of particular relevance to this study. His study analysed in detail the routines of the teaching practices of university mathematics teachers when giving lectures. The participants of the study were teaching year one mathematics courses in three Swedish universities. The analysis of the discourse of mathematics teaching resulted in a categorisation of the mathematical routines (Viirman, 2014) and the didactical routines (Viirman, 2015). In our study we extend this focus on routines related to assessment, aiming to describe the lecturers' expectations about the students' engagement with mathematical discourse in the context of closed-book examination.

Lecturers' perspectives on mathematical tasks is not a widely researched area. Schoenfeld and Herrmann (1982) investigated the way that students and lecturers classified mathematical tasks and showed that the lecturers sorted the tasks according to the mathematical principles necessary for the solution of the task (e.g. solution by induction). The students however in the same study classified the tasks according to the items described in the problems (e.g. roots of polynomials). In a commognitive sense we could argue that in Schoenfeld and Herrmann's study the lecturers classified the tasks according to the rules of the discourse and the students categorised the tasks according to the objects of the discourse.

The lecturers' perspectives on mathematical tasks is also the focus of Tallman and Carlson (2012). These researchers produced a classification of Calculus examination tasks based on orientation, representation and format of the task. Furthermore, they investigated the lecturers' intended and implemented practices examining their views regarding the focus of the task on a mathematical concept or a procedure and whether the tasks ask students to provide explanation for their answers. The findings from the analysis of the tasks are in stark contrast with the findings from the analysis of the lecturers' questionnaire responses. Specifically, the lecturers claim that they usually require their students to explain their thinking and also believe that the proportion of tasks focusing on demonstrating the understanding of mathematical concepts was the same as the tasks focusing on procedures. However, the results of the task analysis did not substantiate those claims, showing a difference between intended and implemented assessment practices.

Similarly, focusing again on Calculus examinations, Bergqvist (2012) examines the lecturers' views on the reasoning expected from the students during the examinations. The results of the study show that the reasoning required in the exams is imitative and not creative. The lecturers, commenting on their implemented assessment practices, argue that otherwise the examinations would be too difficult for the students and this would lead to low passing rates. Also, reporting on the factors they take into account when designing examination tasks, the lecturers include student proficiency, prior knowledge, course content, perceived degree of difficulty and students' familiarity with the task. Our study, building on Bergqvist's (2012) study of intended assessment practices, seeks to provide insight into these practices by taking a discursive perspective and characterising the discourse of closed-book examinations.

METHODOLOGY

This paper reports part of a larger study which aims to analyse the assessment discourse in mathematical courses at university level using a commognitive perspective. Our focus in this paper is to characterise the routines of the closed-book examination discourse. As described in the previous section an example of a mathematical routine is proving, whereas an example of a routine of the assessment discourse evidenced in an examination task is whether, and if so to what extent, students are provided with hints regarding the *how* of the mathematical routine.

Data collection took place in a well-regarded mathematics department in the United Kingdom during the spring semester of the academic year 2014-2015. The data analysed for the purposes of this paper consists of: the tasks from a closed-book examination of a year one compulsory course; and, semi-structured interviews with the two lecturers, L1 and L2, each teaching one part of the course (Sets, Numbers and Proofs and Probability respectively). The duration of the two interviews was 110 minutes with L1 and 83 minutes with L2. The interview discussion was focused on the examination tasks set for the final examination of that academic year.

This year one course focuses on Sets, Numbers and Proofs in the autumn semester and Probability in the spring semester. The final examination has six tasks with the first two compulsory and the rest optional. One of the compulsory and two of the optional tasks are from the Sets, Numbers and Proofs part of the module and the rest are from the Probability part. The duration of the examination was two hours and the students had to respond to both of the compulsory tasks and choose three from the optional tasks. Non-programmable calculators were permitted and the statistical tables were provided to the students. The total grade of the examination was one hundred marks, with the pass mark set at forty marks. In the following we analyse the two compulsory tasks also with reference to the interviews with the lecturers.

ANALYSIS

The compulsory task from Numbers, Sets and Proofs (figure 1) consists of two sub-tasks. In (i) the students are expected to engage in a substantiation routine as they are asked to prove that the given equality stands for all natural numbers. They are directed regarding the *how* of

(i) Prove by induction that for all natural numbers n,

$$2^1 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2.$$

(ii) (a) Suppose a, b, d, m, n are integers. Give the definition of what is meant by saying that d is a divisor of a. Using this, prove that if d is a divisor of a and d is a divisor of b, then d is a divisor of $ma + nb$.

(b) Use the Euclidean algorithm to find the greatest common divisor d of 123 and 45. Hence (or otherwise) find integers m, n with $123m + 45n = d$.

(c) Do there exist integers s, t such that $123s + 45t = 7$? Explain your answer carefully.

Figure 1: Compulsory task on Sets, Numbers and Proofs

the substantiation routine with the phrase “Prove by induction”.

At the start of sub-task (ii) the students are expected to engage with a recall routine. They have to recall an endorsed narrative; the definition of a divisor. Then, they are asked to engage in a substantiation routine, specifically in a direct proof. The students are instructed regarding the *how* of the substantiation routine as illustrated by the phrase “using this, prove that if…then…”. They are asked to use the endorsed narrative they recalled in the previous part three times: twice, to express the relationship between *a* and *d*, and *b* and *d*; and, once, to show that *d* is a divisor of the linear combination of *a* and *b*. In part (b), the students are asked to engage in a ritual, using the Euclidean algorithm, in order to compute the common divisor of 123 and 45. They are again instructed regarding the *how* of the routine, as they are explicitly asked to use the endorsed narrative of the Euclidean algorithm. After finding the greatest common divisor they are asked to find the integers for which the equality is true. This is a substantiation routine as they are asked to identify the integers which substantiate the narrative. In this part they are instructed implicitly to use the work they did with the Euclidean algorithm or use another procedure as indicated by the “Hence, (or otherwise)”. During the interview L1, comments on how the phrase “(or otherwise)” allows him to give full marks to students that use a different *how* for this substantiation routine and thus reward also those who take an alternative approach. In doing so, he talks about the creativity in the *how* of the mathematical routines and how this assessment routine allows students to be creative. This creativity in the *how* of the mathematical routines is common practice, this is also highlighted in the way L1 talks about it in the excerpt below:

> L1: “in mathematics generally, solving some mathematical problem usually there is not a unique way to do that, and that is a good thing, that is a nice thing about mathematics. So a very bright student might be able to solve some mathematical problem in some, in some completely interesting different way that you don't expect and that sometimes happens and it is really fantastic when it happens and they should get credit for it”

In the final part of the task the students have to combine the endorsed narratives they substantiated in the previous parts in order to decide whether the narrative describing the relationship between 7 and the linear combination of 123 and 45 can be endorsed. The students are expected to engage in a substantiation routine as they are asked to consider whether a linear combination which is a multiple of 3 is equal to 7. Additionally, they are instructed by the phrase “Explain your answer carefully” to provide justification for their response. L1 justifies this choice of words as follows:

> L1: “the danger would be that the student would write yes or no and then write nothing else (...) I guess it's to remind them that I want them to explain why they are saying what they are saying”

We also observe that the structure of sub-task (ii) is gradually leading the students in answering (c). It starts with the definition of the divisor and the relationship between the linear combination of two numbers that have the same divisor. Next, an example of two specific numbers, 123 and 45, is given and the students are expected to find the greatest common divisor. Then, the students are asked to express this number in the form of a linear combination of 123 and 45. Finally, they are asked to examine whether 7 could be expressed in a linear combination of these two numbers. This structure of sub-task (ii) assists the students with engaging in routines that lead to endorsing or rejecting the final narrative. We also note that parts (b) and (c) are dependent on each other: in order to answer (c) the students need to have identified the greatest common divisor of 123 and 45.

Next, we analyse the compulsory task from the Probability part of the course (figure 2). The task starts with a small introduction "In the framework of the modern probability…" situating the students in the historical context set out in L2's lecture notes (which include an account of Probability as a subject, starting from the 16th century until the modern axiomatic definition of probability given by Kolmogorov). Then the students are expected to engage in a recall routine as they are asked to remember and state some endorsed narratives. More specifically, they have to recall the definition of disjoint events and Kolmogorov's axioms. In the following excerpt, L2 illustrates how he poses the specific task aiming to guide the students in constructing complete definitions. L2 seems aware of students' previous engagement with the mathematical discourse and, more specifically, with the routine of defining. Consequently he assists students in providing a complete definition aiming to shift their discursive practices.

(i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions:
  (a) For any event $A = \varnothing$, prove that $P(A) = 0$.

  You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.
  (b) For any events A and B such that $A \subseteq B$, prove that $P(A) \leq P(B)$.

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.
  (a) Show that $P(A \cap B) = \frac{1}{4}$.
  (b) Find $P(B)$ and the range of possible values for the parameter p.
  (c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

Figure 2: Compulsory task in Probability.

L2: "Usually students, especially in the first year they are not used to give proper definitions or if they give the definitions or they write a part of the theorems they don't specify what are the objects they are talking about. Okay? So in other words is like if you are speaking a language but you, you are speaking to somebody which is able to understand you. But what I want the students is to make an effort to try to explain something in a most complete way and that is why I sometimes try to guide them in giving the right definition or writing the right axioms. Because in the third axiom they need to speak about disjoint event, pairwise disjoint event, I've asked them to give first the definition of disjoint even just to see if they really know what is a disjoint event and they are able to explain it in the axiom."

The prompt "then use them to demonstrate" illustrates the *how* of the substantiation routines the students have to engage in while proving the two propositions. Both of these substantiation routines are direct proofs. Also, L2 gives to the students another narrative, namely Proposition 2, which can be endorsed without substantiation.

The structure of sub-task (ii) assists the students in solving it. At first, the students are asked to engage in a substantiation routine in order to substantiate that the probability of the intersection is ¼. In engaging with this substantiation routine, the students have to recall the multiplication rule or the definition of conditional probability. For sub-task (b) the students are expected to engage in a ritual based on recalling a proposition to calculate *P(B)*. Then, using Kolmogorov's axioms, they need to determine the range of the possible values for *p*. Finally, in (c) the students are asked to calculate two probabilities. In order to do that they have to engage in a ritual, recalling a proposition and the multiplication rule. First, they are asked to calculate the probability of the complement of *B* given *A*, which assists in calculating the probability of *A* intersecting the complement of *B*. The parts (a) and (b) are dependent, meaning that the value from (a) is needed to engage with the ritual in (b). However, the lecturer, knowing this and aiming to make all the tasks independent, he decided to phrase the task as a "Show that" task providing the value of the probability and thus assisting the students with achieving the desired response. This is evident in the following excerpt.

L2: "(...) they need to use this value in the second part. So, I don't want them to penalise if they are not able to get the first solution (...) while for part c they don't need this value to solve any other task, any other part of that task. So, I can ask them find. Of course it would be better to ask them to find everything but it's just to again help them in order to do to let's say separate all subsections of exercise so that they can do it. They can do them separately without the need of any other values."

In this task, we note that the structure of the sub-tasks indicates helpful ways to solve the sub-tasks. In (i) the students are asked first to recall the endorsed narratives which are needed to engage in the substantiation routines in (a) and (b). Similarly, for sub-task (ii) the relation in (a) is needed in order to engage with (b) and (c).

DISCUSSION

In this paper we deployed a commognitive perspective in order to describe the assessment discourse at university level. Sfard's theory of commognition, alongside Systemic Functional Linguistics (Halliday, 1978), has been the basis of a framework [2] introduced to examine changes in the nature of students' participation in the mathematical discourse over the years 1987-2011. The aim of this framework is to identify changes in the form of the tasks and the expectations from students' responses focusing on the public examinations (GCSE – General Certificate of Secondary Education) in the United Kingdom taken at age 16. Our analysis is in the spirit of this framework: it focuses on closed-book examination tasks at university level, as well as the interviews with the lecturers who designed the tasks, and aims to offer a characterisation of the assessment discourse and examine the choices the lecturers make when designing the tasks. The analysis highlights discursive routines some aimed at the mathematical discourse the students are asked to engage with and others aimed at assisting the students in their engagement.

Regarding the mathematical routines the students are expected to engage with, we observed engagement with rituals and explorations. The analysis of the explorations in the compulsory and optional tasks showed that the students are asked to engage mostly in substantiation and recall routines, with only two instances being construction routines. Examining further the substantiation routines from the whole examination paper (we note that here we elaborated only on two of the six tasks in the paper), the *how* of these routines can be distinguished as follows: there are sub-tasks using proof by induction, direct proof and proof by counterexample. Examining the recall routines, the students are asked mostly for definitions of mathematical objects. However, we note that in engaging in the rituals and the substantiation routines, the students have to recall endorsed narratives and thus also engage in recall routines. For example, in 1(ii)b from the Sets, Numbers and Proofs task (figure 1), the students have to recall the Euclidean algorithm in order to be able to engage with the ritual.

One assessment routine aimed at assisting students' engagement, is the gradual structure of the task. This step-by-step structure assists the students in gradually recalling or substantiating endorsed narratives needed in subsequent sub-tasks. Furthermore, the sub-tasks are independent or dependent, allowing the students to engage with further tasks or requiring them to find answers that will allow them to proceed further with the task.

Another assessment routine concerns the different degrees of guidance provided to the students in terms of the *how* of the routine or the endorsed narratives that would be needed in order to substantiate a narrative. Similarly, there are instances where students are guided to justify their response – thus assisting the students in providing arguments for the claims they make and helping them to understand this quintessential characteristic of the university mathematics discourse. These directions are aimed at helping students to shift their mathematical discourse towards what is required at university level. Also we observe that, to assist this shift, the lecturers illustrate some of the routines of the new, for the students, discourse. For example, L1 uses the directions for justification and L2 assists students to explain the term disjoint sets in the third axiom and thus provide a complete narrative of Kolmogorov's axioms. With these requests in the examination tasks the lecturers are encouraging students to demonstrate enculturation into the practices of university mathematics. Sfard (2014) comments on some of the characteristics of the university mathematical discourse as follows when she observes

> "first, this discourse's extreme objectification; secondly, its reliance on rules of endorsement that privilege analytic thinking and leave little space for empirical evidence; and thirdly, the unprecedented level of rigour that is to be attained by following a set of well-defined formal rules." (p.200).

In our analysis we focus on the lecturers' expectations regarding students' engagement with the mathematical discourse. We examine the choices the lecturers made while posing the tasks and how they justified those choices. Their choices seem to be guided by their experience of where the students might face problems. The gradual structure, the guidance in terms of the *how* of the routines and the explicit directions regarding justification aim to assist students in achieving a correct and complete response. These choices though could potentially foster a somewhat limited image of mathematics as they suggest mathematics as a predominantly step by step, highly directed activity. However, we have to consider also that the lecturers take into account the context for which these tasks are designed for. In the examination the students have limited time – and they are stressed – and lecturers calibrate their examination task design accordingly.

Our analysis resonates with the analysis of others in the field such as Schoenfeld and Herrmann (1982) and Bergqvist (2012). However, the results of our analysis slightly digress from the results of Tallman and Carlson (2012) regarding the directions for justification. They noted an inconsistency between the lecturers' claims regarding justification and the results of the task analysis. Whereas, we observed that the students are explicitly directed regarding the justification of their response. Of course we recognize that this is based on a small set of data, one examination from a year one course and interviews with the two lecturers. More examination tasks and interviews with the lecturers are needed to provide a richer characterisation of assessment routines.

Finally, we note that in this paper we sampled from our analysis of the examination tasks in terms of routines, one of several aspects of discourse that our analysis is focussing on. In addition to other aspects of said discourse, we are now analysing the students' written responses to the same tasks in order to examine their actual engagement with the mathematical discourse and whether there are differences between what the lecturers intended and what the students actually did.

ENDNOTE

1. Closed-book examination means that the students are not allowed to use textbooks or notes during the examination.

2. An ESRC funded project: "The evolution of school mathematics discourse as seen through the lens of GCSE examinations" (http://gtr.rcuk.ac.uk/project/D23BF129-B7CC-4BEA-83E2-8EB9D0EDBF17 accessed on 28/04/2016)

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