Relevant knowledge concerning the derivative concept for students of economics - A normative point of view and students' perspectives

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University of Paderborn, Germany, feudel@math.uni-paderborn.de

The concept of derivative plays a major role in economics. A proper understanding of the concept and its application in economics is therefore important for students of economics. In this paper two perspectives on the relevant knowledge concerning the derivative for students of economics are presented: a normative point of view based on literature and the students' perspectives identified in an empirical study. It can be seen that the students' perspectives differed from the normative point of view. An interesting result is that, although emphasized in the course, the students of economics seemed to consider the economic interpretation of the derivative and the corresponding mathematical background knowledge to be much less important than pure mathematical procedures like differentiation rules.

Keywords: derivative, economics, concept image, economic interpretation.

INTRODUCTION AND BACKGROUND

The concept of derivative is one of the very important mathematical concepts used in economics. Therefore, students of economics should have an adequate concept image in the sense of Tall and Vinner (1981) of the derivative to be able to deal with the concept in economics in a reflective manner. The study presented in this paper is part of a larger research project (my PhD-Thesis, supervisor: Prof. Dr. Rolf Biehler) at the Centre for Higher Mathematics Education in Germany (khdm, www.khdm.de) about the understanding of the derivative by students of economics. This research project has the following three research questions:

1. Which understanding of the concept of derivative do students of economics need to have?
2. Which understanding of the concept of derivative do students of economics have before attending any mathematical course at university?
3. Which understanding of the derivative do students of economics have after the math course, especially concerning the use of the derivative in economics in the example of marginal cost?

In this paper the focus mainly lies on the question what knowledge concerning the derivative concept is relevant for students of economics (question 1). Besides a normative point of view based on literature (which also serves as theoretical framework of the study presented in this paper) the students' of economics perspectives are the main issue in this paper. If knowledge is not considered to be relevant by them, they probably will not have that knowledge later on. So the results are also relevant for question 3.

LITERATURE REVIEW ON RELEVANT KNOWLEDGE FOR STUDENTS OF ECONOMICS CONCERNING THE CONCEPT OF DERIVATIVE

The mathematical concept of derivative

According to Zandieh (2000) the concept of derivative is connected with three other mathematical concepts that she calls layers of the derivative:

1. The concept of ratio/rate (relevant for understanding the difference quotient as the first step for getting from a function f to its derivative function f')
2. The concept of limit (relevant when taking the limiting process of difference quotients)
3. The concept of function (relevant for the transition from the single value of the derivative $f'(x_0)$ to the derivative function f')

Each of the concepts can be seen as a process-object pair. For the layer of limit for example, the process is the limiting process, and the object is the value of the limit. Furthermore, Zandieh (2000) mentions multiple representations for the derivative that students ought to know: a) graphically as the slope of the tangent line at a point, b) verbally as the instantaneous rate of change, c) physically as speed or velocity, or d) symbolically as the limit of the difference quotient. These representations should be part of an adequate concept image of the derivative after the math course.

Furthermore, the students are supposed to know the connections between the derivative and other mathematical concepts like monotonicity or convexity to be able to use the concept for finding maximal or minimal values of economic functions.

The economic interpretation of the derivative

Students of economics also need to give an interpretation of the derivative in economic contexts (mostly with discrete units in the independent variable). In economics, the derivative is often interpreted as the absolute change of the values of the function if the independent variable of the function increases or decreases by one unit. In case of a cost function C, for example, the derivative $C'(x)$ is often interpreted as the additional cost while increasing the production from x to $x+1$ units (Schierenbeck, 2003). However, that additional cost for the next unit (exactly calculated by $C(x+1)-C(x)$) actually represents a different mathematical object. The derivative $C'(x)$ as a mathematical object represents the rate of change of the cost function C at the point x while $C(x+1)-C(x)$ is the absolute change of the cost while increasing the output x by one unit. Both objects differ in its numerical value and in the corresponding unit (if the output is measured in units per quantity and the cost $C(x)$ is measured in Euro, the unit of $C'(x)$ would be Euro per unit of quantity).

Although $C'(x)$ and the additional cost are different mathematical objects, they are connected via the approximation formula $C(x+h)-C(x)\approx C'(x)\cdot h$ for h close to 0. This formula can either be derived from the symbolic representation of the derivative as limit of the difference quotient by using the approximation aspect of the limit (Çetin, 2009) or from the property of the derivative being the slope of the tangent line that is the best approximating linear function of C near the point x (Danckwerts & Vogel, 2006). Because $h=1$ can be considered as small in economics the numerical values of $C(x+1)-C(x)$ and $C'(x)$ are often close for cost functions, which justifies the interpretation of $C'(x)$ as additional cost while increasing the output x by one unit.

The knowledge concerning the economic interpretation of the derivative is not included in the framework of Zandieh (2000) directly. One could extend the framework with an extra column "economics" like it was done by Roorda, Vos, and M. (2007). But the interpretation of $C'(x)$ as the additional cost $C(x+1)-C(x)$ still does not match one of the resulting layers (average cost $\Delta C/\Delta x$, derivative $C'(x)$, derivative function C') directly. For getting from the derivative $C'(x)$ to the additional cost $C(x+1)-C(x)$ one would have to go backwards from $C'(x)$ to the average cost per unit again and then to the additional cost $C(x+1)-C(x)$ by specifying the interval as just one unit. However, the differences in the units between $\Delta C/\Delta 1$ and $C(x+1)-C(x)$ would still remain (the first term is a rate, the second is not).

The usual approach to the economic interpretation of the derivative in math courses for students of economics" that is found in many math books for students of economics, e.g. Sydsæter and Hammond (2009) or Tietze (2010), and that is also the approach in the course in which the study presented in this paper takes place, is different and avoids the layer of the average cost. Instead of starting with average cost, this approach starts with the derivative as a pure mathematical concept (with all the representations mentioned by Zandieh directly). Afterwards, the approximation formula $f(x+h)-f(x)\approx f'(x)\cdot h$ for h close to 0 is derived by the above mentioned approximation arguments, and then the argument, that $h=1$ is small in economics comes into account that finally justifying the identification of $C'(x)$ and $C(x+1)-C(x)$. For that approach Zandieh's framework should rather be extended by approximation aspects of the derivative (like for example presented in Serhan (2009)) than with another representation "economics" containing the classic layers.

After the math course, the students of economics should know the above mentioned differences between the derivative as a mathematical concept and its economic interpretation as additional cost (differences in the numerical values and the unit), but also know the connection between $C'(x)$ and $C(x+1)-C(x)$ to justify the economic interpretation of the derivative, which is normally not done in books of economics (e.g. (Wöhe & Döring, 2010)) and should therefore be aim of the math course.

A STUDY ABOUT THE SUTDENTS' OF ECONOMICS PRSPECTIVES ON RELEVANT KNOWLEDGE CONCERNING THE DERIVATIVE

Aim of the study

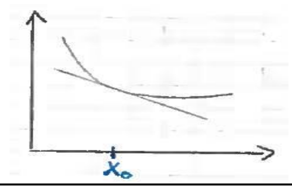
The aim of the study was to find out which of the aspects of the expected knowledge concerning the derivative, that were mentioned above, are considered to be important by students of economics. If students do not consider knowledge as relevant, they will probably soon forget it shortly after the exam (or even never absorb it).

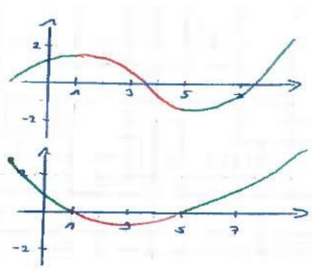
Data Collection

In January 2015, three weeks before the final exam, the students of economics at the University of Paderborn were given the homework to write a "concept summary" about the relevant knowledge concerning the derivative (which they would use when preparing for the exam) in the math course. Since the task was given as homework, the use of books and the lecture notes was allowed. The students had practiced writing such concept summaries, called concept bases in the course (Dietz, 2015), for the concept "relation" in the tutorials two weeks before and were therefore familiar with the given task. They should know that concept summaries should contain the definition of the concept, examples and counterexamples, visualizations, important statements involving the concept, and applications. So this summary should contain the definition of the derivative, the aspects of the concept image, and the economic interpretation. The task was given to them just after having dealt with the derivative in the math course. So all the relevant knowledge mentioned above was covered in the course. Since the task was voluntary, only 146 students handed a solution in (in a course with over 700 students).

Data Analysis

The summaries were analyzed with the help of quantitative content analysis. Different parts of the summaries were assigned to different categories, which were mainly deduced from the intended knowledge mentioned above: representations of the derivative (Zandieh, 2000), connections to other concepts like monotonicity, the economic interpretation of the derivative (Tietze, 2010) and relevant mathematical background knowledge especially the approximation aspect of the derivative (Çetin, 2009). During the coding process, the layers in Zandieh's framework were first coded separately, but later aggregated for this paper due to limited space. Some categories like differentiation rules, often found in the analysis, were then also included into the scheme. In the end, this led to a system of 12 categories (table 1).

Category	Description	Prototypical example or examples
Definition	The formal definition of the derivative is mentioned.	$f'(x)=\lim\limits_{h\to 0}\dfrac{f(x)-f(x_0)}{x-x_0}$
Slope of tangent line	The geometric interpretation as slope of the tangent line (or shortly as slope of the function at one point) is mentioned – either verbalized or illustrated with the help of a visualization.	1. $f'(x_0)$ is the slope of the tangent line t at the graph of f at the point $(x_0,f(x_0))$. 2. 

Rate of Change	The interpretation of the function as (local) rate of change of a function is mentioned (verbalized as rate of growth by the lecturer). A part of the summary is already sorted into that category if the interpretation of the difference quotient as average rate of change is mentioned.	1. The derivative is the limit of growth rates and represents therefore the local rate of growth of a function. 2. $\frac{\Delta f}{\Delta x} = \frac{\text{(absolute) growth of the values of the function}}{\text{(absolute) growth in the independent variable}}$
Derivatives of elementary functions	Elementary functions are written down together with their derivative.	$f(x) = ax + b \quad f'(x) = a$ $f(x) = x^p \quad f'(x) = px^{p-1}$ $f(x) = e^x \quad f'(x) = e^x$ $f(x) = \ln(x) \quad f'(x) = \frac{1}{x}$ $f(x) = \sin(x) \quad f'(x) = \cos(x)$ $f(x) = \cos(x) \quad f'(x) = -\sin(x)$
Differentiation rules	The rules of differentiation like the product rule, the quotient rule, or the chain rule are mentioned.	$(f+g)' = f' + g'$ (Sum-Rule) $(\lambda f)' = \lambda f'$ (Factor-Rule) $(fg)' = f'g + g'f$ (Product rule) $(f(g(x))' = f'(g(x))g'(x)$ (Chain rule)
Algebraic Example	A concrete sample function (differing from elementary functions) is written down with its derivative.	$f(x)=4x^3+3x^2+26$ $f'(x)=12x^2+6x$
Derivative and monotonicity	The connection between the derivative and monotonicity for differentiable functions is mentioned or is clearly visualized.	1. f is increasing <=> $f' \geq 0$ f is strictly increasing <= $f'>0$ 2. 
Derivative and convexity	The connection between the derivative and convexity for functions being two times differentiable is mentioned.	$f'' \geq 0$ <=> f is convex $f''>0$ => f is strictly convex

Derivative and extreme values	It is mentioned that the derivative is used for finding extreme values of a function.	Use: determining the maximum and the minimum value of a function
Approximation aspect	It is mentioned that the derivative can be used for approximation – either verbalized or with an approximation formula.	1. $\Delta f \approx f'(x_0)\Delta x$ 2. Approximate determination of the values of a given function
Term "marginal" means derivative	It is mentioned (either with an example of an economic function or in general) that the term "marginal" means to take its derivative.	1. Let *K* be a cost function. The derivative *K'* is called marginal cost. 2. Marginal means derivative
Economic interpretation	An economic interpretation of the derivative in an economic context is mentioned.	$K'(21) = \frac{1}{3}$ means: If one increases the output from 21 units of quantity by one unit, the cost increase round about 1/3 units of money.

Table 1: Categories of the students' summaries concerning relevant knowledge of the derivative (that has been addressed in the course "Mathematics for students of economics" at the University of Paderborn)

For each concept summary, a category occurring in the summary was coded with "1", a missing category was coded with "0". It is important to mention in addition, that it did not matter for the coding process if there were any mistakes in the summary. For example, a wrong connection between monotonicity and the derivative like "*f* is strictly increasing <=> $f'(x)>0$" was nevertheless coded with "1" in the category as "Derivative and monotonicity" because the connection between the derivative and monotonicity is still considered as important even if it is not known correctly.

The concept summaries were later coded again by a student, who successfully completed the course one and a half year ago, to check reliability. All categories have been proven to be reliable (κ>0.8 for all categories) except for the category "approximation aspect", where the student first wrongly sorted the notation $f'(x) = \frac{df}{dx}$ in that category. After recoding, this category was also reliable with κ=.97.

Results of the study

Some aspects of the derivative seem to be more important than other aspects to the students of economics. Concerning the category-scheme related to the aspects of the derivative covered in the course (table 1), the percentages of students whose summary contained a certain aspect of the derivative can be found in figure 1.

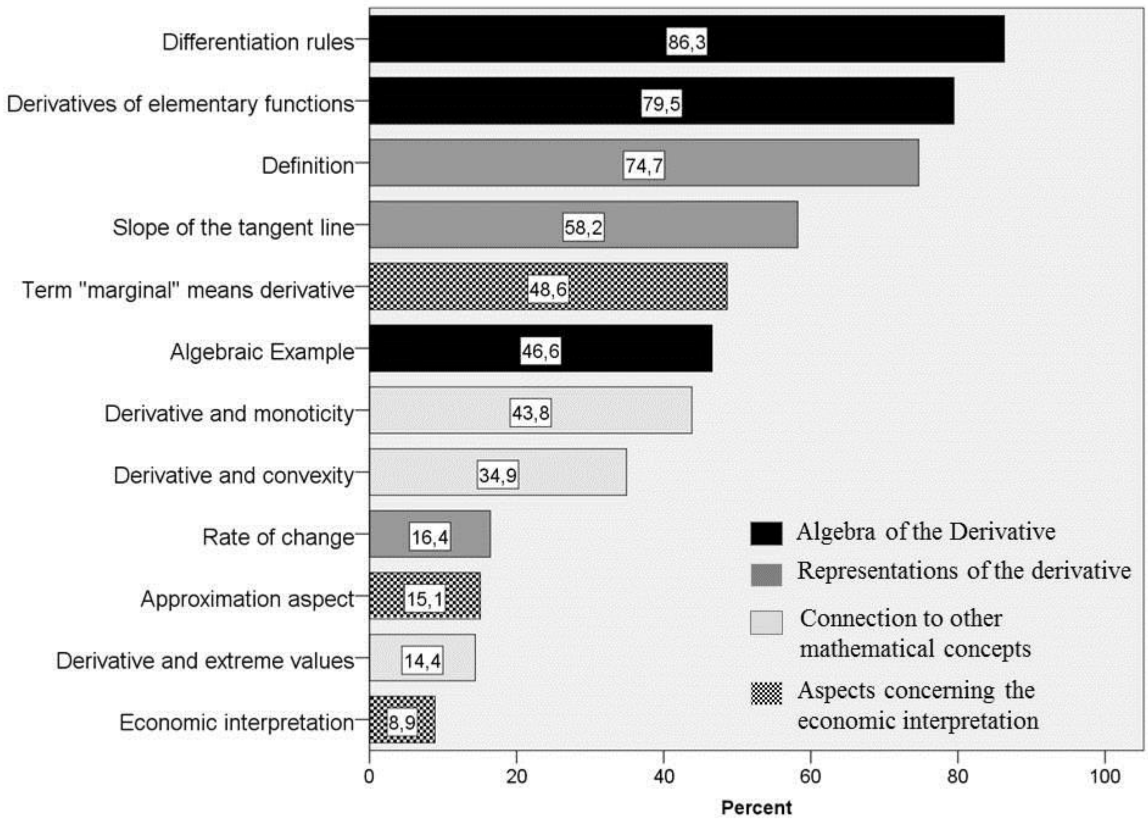


Figure 1: Percentage of students, whose summary contained the different aspects of the derivative addressed in the course "Mathematics for students of economics" (N=146)

These percentages yield several interesting direct findings:

i. The calculation of derivatives by using algebraic rules is considered as being most important.

ii. The geometric representation as slope of the tangent line is clearly preferred in comparison to the representation as rate of change

iii. Only about 75% of the students included the definition in their summary after the course "Mathematics for students of economics".

iv. The economic interpretation of the derivative and the corresponding mathematical background knowledge concerning the approximation aspect of the derivative seemed to be least important, although these aspects are particularly relevant for students of economics

Ad i: This is not a surprising result and coincides with the often mentioned result that students are able to differentiate but often do not understand the concept of the derivative (see for example Orton (1983)). A possible explanation for our students could be that the students probably often experienced at school that calculus mainly consists of calculating derivatives with the help of the differentiation rules and using those calculations to find extreme values and turning points of functions.

Ad ii: The students of economics clearly preferred the geometric interpretation as slope of the tangent line in comparison to rate of change. This result shows that although also studying mathematics for application like engineering students, students of economics do not seem to appreciate the representation as rate of change unlike engineering students (Maull & Berry, 2000). A reason might be that using graphical arguments, when dealing with functions, is very common in books of economics (Wöhe & Döring, 2010). This suggests that justifying the economic interpretation by using the "best approximation property of the tangent line" could reach more students than using rate of change arguments or symbolic arguments.

Ad iii: Interesting about that finding is that although the students were explicitly told that they ought to know the definition in the exam about 25% did not seem to consider the definition as important. A possible reason might be that definitions of mathematical concepts had rarely been part in exam tasks at High School. The students considering the definition to be unimportant will probably not be able to solve any task involving the definition of the derivative. They will especially not be able to justify the economic interpretation of the derivative by de-encapsulating the limiting process behind the derivative and using the resulting approximation formula $C(x+h)-C(x) \approx C'(x) \cdot h$ for h close to 0, as intended in the course.

Ad iv: This is the most interesting result. The only aspect concerning the use of the derivative in economics that many students considered to be important seems to be that the term "marginal" in economics means to take the derivative of an economic function (e.g. of a cost function). The detailed interpretation of the derivative and the relevant background knowledge for understanding it (approximation aspect of the derivative) were seen as very unimportant. This was surprising since these aspects were emphasized in the lecture and the tutorials very much. The students even had to work on problems involving these aspects by themselves. A possible reason could be that the economic interpretation seemed trivial to them and therefore did not have to be explicitly learned. Another reason could be that they cared more about procedures and vocabulary needed when dealing with economic functions in economic contexts rather than about understanding the mathematical background. Especially a justification of the identification of the derivative $f'(x)$ with the value $f(x+1)-f(x)$ with approximation arguments, used when interpreting the derivative in economic contexts, might not have been seen as necessary to them for using that interpretation (although it definitely is from a mathematical point of view).

Result iv also gives a possible explanation why students often perform poorly on tasks involving the use of the derivative for approximation, even at the end of a course (Bingolbali & Monaghan, 2008). Maybe they did not recognize that aspect to be relevant knowledge concerning the derivative and never even tried to learn it.

Limitations of the study

The students were asked to write a concept summary they could use when preparing for the exam. The task was formulated that way to motivate them really writing it and handing it in because that task could not be made obligatory for all students because there was not enough staff in the course to correct the summaries for all of the students. Therefore, expectation concerning exam tasks could have influenced the results. Maybe some aspects would have less and others more often occurred if the students were asked to write a summary about the derivative that they could use as a reference in their later courses of economics.

DISCUSSION AND CONCLUSION FOR FUTURE RESEARCH

From the results of the study it can be clearly concluded that the students' of economics perspectives on the relevant knowledge concerning the derivative differed from the normative point of view (based on literature). In detail two results can be seen from this study:

1. The students of economics considered procedural knowledge concerning the derivative to be more important than conceptual knowledge even although they were told that the conceptual knowledge (the definition, examples, visualizations, connections to other concepts, or applications) is required for the exam.
2. The students of economics did not consider the mathematical background concerning the use of the derivative in economics as important learning material even although that background was emphasized in the course.

From the first result the following conclusion can be drawn: If conceptual knowledge is an important goal in a math course, the students must experience the importance of the conceptual knowledge by themselves already during the semester, e.g. through an adequate proportion between tasks involving conceptual knowledge and tasks involving procedural knowledge in the exercises. This is even more important for math service courses, in which many students just plan to pass the exam with 50% of the points (at least half of the tasks should then involve conceptual knowledge).

The second result concerning the fact that the economic interpretation of the derivative (and the relevant mathematical knowledge to understand that interpretation properly) was not considered to be important by many students of economics, even although it was emphasized in the course very much, was surprising. That result is a problem because students not having that knowledge will not be able to work with the derivative in economics in a reflective manner. Several reasons for the felt unimportance are possible, e.g. (felt) triviality of the economic interpretation, expectation that the economic interpretation would not occur in the exam, interest only in the procedures and vocabulary when dealing with economic functions in economic contexts and not in the mathematical background knowledge, or even no interest in economics itself because the study subject "economics" was mainly chosen because of an expected high salary.

The reasons for the felt unimportance of mathematical background knowledge directly related to the own study subject in comparison to pure mathematical procedures could be a starting point for future research. Similar phenomena might occur in case of other mathematical concepts or procedures used in economics (e.g. elasticity, differentials, or Lagrange’s method), but can also occur in other math service courses. Only if the reasons for a felt unimportance of the mathematical background knowledge concerning the use of mathematical concepts in other sciences are discovered, adequate conclusions can be drawn so that students might feel a need to understand the mathematical background properly that enables them to use the concepts in a reflective manner.

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