CDD Poles in Dispersion Relations for Strong Interactions

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Background

In a previous analysis[1] of pion-pion scattering numerical solutions for the scattering amplitude have been found by iterating partial wave inverse amplitude dispersion relations coupled by unitarity and crossing equations. Only S and P waves are included so that there are three amplitudes to consider $A_0(v)$, $A_1(v)$ and $A_2(v)$ corresponding to isospin 0, 1, and 2 respectively. A_0 and A_2 are S wave amplitudes and A_1 is P wave; $v = (k/\mu)^2$ where k is the magnitude of the center of mass 3 momentum and μ is the pion mass. Dispersion relations with one subtraction are used and first derivative crossing conditions applied so that only the pion-pion coupling constant, λ, remains as a free parameter.

Some of the solutions included a P wave resonance and these can be characterized as follows. For the S waves the inverse amplitude right cut is only slightly (less than 10%) dependent on the left cut in the dispersion relations. Thus we have for $v \geq 0$, I = 0, 2

$$\text{Im } A_I^{-1}(v) = -\left[v/(v+1)\right]^{1/2} \qquad (1)$$

by unitarity and from the dispersion relations

$$\mathrm{Re}\, A_I^{-1}(v) = a_I^{-1} + \frac{v-v_0}{\pi}\, P\int_{L+R} \frac{\mathrm{Im}\, A_I^{-1}(v')}{(v'-v)(v'-v_0)}\, dv' \tag{2}$$

where a_I^{-1} are the subtraction constants, v_0 is the subtraction point ($v_0 = -2/3$) and the integral runs over both the left cut, $(-\infty, -1)$, and the right cut, $(0, \infty)$. If we neglect the left cut and use eqn. (1) for $\mathrm{Im}\, A_I^{-1}(v)$ for $v \geq 0$ we find

$$\mathrm{Re}\, A_I^{-1}(v) \approx a_I^{-1} + h(v) \tag{3}$$

where

$$h(v) = \frac{1}{\pi}\sqrt{\frac{v}{v+1}}\, \ln \left| \frac{1+\sqrt{\frac{v}{v+1}}}{1-\sqrt{\frac{v}{v+1}}} \right| - 0.554 \tag{4}$$

Equation (3) will be used whenever an expressed form of $\mathrm{Re}\, A_I^{-1}(v)$ is needed. For the P wave inverse amplitude one finds an effective range behavior until well above the resonance energy. Thus we may write for $v \geq 0$

$$v\, \mathrm{Re}\, A_1^{-1}(v) = C(v_R - v) \tag{5}$$

and

$$v\, \mathrm{Im}\, A_1^{-1}(v) = -\left[v^3/(v+1) \right]^{1/2} \tag{6}$$

with C and v_R dependent on λ.

The amplitudes for $v \leq -1$ may be found by first using the crossing equations for the imaginary parts and then the dispersion relations for the real parts. For present purposes it is more practical to use dispersion relations for the direct amplitude

$$\mathrm{Re}\, A_I(v) = a_I + \frac{v-v_0}{\pi}\, P\int_{L+R} dv'\, \frac{\mathrm{Im}\, A_I(v')}{(v'-v)(v'-v_0)} \tag{7}$$

where again the integral runs over both the left and right cuts. The crossing equations are, for $v \lessgtr -1$, I = 0 or 2

$$\text{Im } A_I(v) = \frac{1}{v} \int_0^{-v-1} dv' \left[\frac{2}{3} \text{ Im } A_0(v') + b_I \text{ Im} A_2(v') \right.$$

$$\left. + d_I \; 3(1+2 \frac{v+1}{v'}) \text{ Im } A_1(v') \right] \tag{8}$$

where $b_0 = \frac{10}{3}$, $b_2 = \frac{1}{3}$, $d_0 = 2$ and $d_2 = -1$ and

$$\text{Im } A_1(v) = \frac{1}{v} \int_0^{-v-1} dv' \; (1+2 \frac{v'+1}{v}) \left[\frac{2}{3} \text{ Im } A_0(v') - \frac{5}{3} \text{ Im } A_2 \; (v') \right.$$

$$\left. + 3(1+2 \frac{v+1}{v'}) \text{ Im } A_1(v') \right] \tag{9}$$

The parameters a_0, a_1 and a_2 as well as C and v_R are determined for any particular λ, by the consistency requirements of the crossing conditions. For the partial wave analyses these become

$$\frac{3}{v_0} \; a_1 = \frac{1}{2} \; \frac{\partial A_0}{\partial v} \bigg|_{v=v_0} = - \frac{\partial A_2}{\partial v} \bigg|_{v=v_0} \tag{10}$$

In particular one finds $a_0 = -5\lambda$ and $a_2 = -2\lambda$. a_1, C and v_R are not so simply determined but for $\lambda = -0.15$ one finds $a_1 = -0.005$, C = 19 and $v_R = 4.1$.

Problem

The present problem centers around a modification of the solutions discussed above. The dispersion relation analysis is arbitrary to the extent of poles in the inverse amplitudes (CDD poles) and these cannot be found by numerical iteration on a computer since they introduce infinities in the physical region. These correspond to zeros of the amplitude itself. However, in the presence of any inelastic scattering the elastic scattering has a lower bound which is non zero. The CDD poles still may exist but are moved off the real axis into the negative half v-plane. Such poles can be treated by the computer and when a program including inelastic scattering was run solutions with CDD poles appeared but with one peculiar property. The CDD pole always occurred close to the original resonance position.(2) The result of this was to split the resonance into two, one each side of the CDD pole, but still close together.(3) It appears then that in any P wave resonance type solution a CDD pole may be placed close to the resonance position without disturbing the consistency requirements of the analysis, i.e. the crossing equations. Our purpose here is to justify this statement.

Analysis

In the original discussion of CDD poles(4) their position and strength were arbitrary. On the other hand for that analysis a very simple form of crossing, (actually direct reflection) was used. Here we have partial wave coupling in the crossing equations so that more restrictions on the CDD pole may be expected. To start with we modify eqns. (5) and (6) to include a CDD pole at $v = v_p$ with Re $v_p = v_R$. Thus

$$\mathrm{Re}\, A_1^{-1}(v) = C(v_R - v)/v + K \frac{\tau(v-v_R)}{\tau^2+(v-v_R)^2} \tag{11}$$

$$\mathrm{Im}\, A_1^{-1}(v) = -\left[\frac{v}{v+1}\right]^{1/2} - \frac{K\tau^2}{\tau^2+(v-v_R)^2} \tag{12}$$

where $\tau = -\text{Im}\, v_p$ and $\kappa\tau$ is the strength of the CDD pole. The parameters κ and τ may be related to the real and imaginary parts of the P wave whose shift. For κ large

$$\text{Im}\,\delta_1(v)\Big|_{\text{Re}\,\delta_1 = \pi} = \kappa^{-1},\ \frac{d}{dv}\,\text{Re}\,\delta_1(v)\Big|_{\text{Re}\,\delta_1 = \pi} = (\kappa\tau)^{-1} \qquad (13)$$

Now using eqns. (1), (3), (4), (11) and (12) $A_I(v)$ $(I = 0, 1, 2)$ may be found by inverting and then the $A_I(v)$ for $v \lesssim -1$ are determined by using first the crossing equations, eqns. (8) and (9), and then the dispersion relations, eqns. (7), since $\text{Im}\, A_I(v)$ has then been determined for all v. The resulting curves for Re A_I, Im A_I, Re A_I^{-1} and Im A_I^{-1} are shown in Figs. 1, 2 and 3 both with and without the CDD pole. We are basically only interested in the difference between the results with and without the pole and this corresponds to making further subtractions in the dispersion relations. This point will be clarified by a careful consideration of the crossing conditions.

The salient point shown in the curves in Fig. 1 is the double resonance behavior in Re A_1 and Im A_1. The two resonances are symmetrically positioned about $v = v_R$ and are both lower and narrower than the original resonance. As shown in an earlier paper[(3)] there combined effect in any dispersion integral is similar to that of the simple resonance except close to $v = v_R$. On the other hand in the crossing equations the variable upper limit produces a change of slope of Im A(v) for $v \lesssim -1$ at the reflected resonance positions. When the CDD pole is present this occurs in two places and gives a somewhat different behavior as shown in Im $A_0(v)$ and Im $A_2(v)$ for $v \lesssim -1$. These changes on the left cut are not important in the dispersion integrals in the low energy physical region due to the subtractions: their effect, if any, will be felt in the subtraction constants. Thus the consistency of the scheme with the CDD pole depends on satisfying the crossing conditions and, in particular leaving the P wave subtraction constant unchanged.

Using the dispersion relations (7) the crossing conditions (10) may be rewritten

$$\frac{3\,a_1}{v_0} = \frac{1}{2\pi}\int_{L+R} \frac{\text{Im}\,A_0(v)}{(v-v_0)^2}\,dv = -\frac{1}{\pi}\int_{L+R}\frac{\text{Im}\,A_2(v)}{(v-v_0)^2}\,dv \tag{14}$$

The curves in Fig. 1 were obtained by fairly crude numerical calculations, mainly for purposes of illustration, and are not accurate enough for use in eqn. (14). Instead we set up a precise analytic expression and then evaluate only the part which is effected by the CDD pole. For I = 0, 2 consider then

$$\int_{L+R} \frac{\text{Im}\,A_I(v)}{(v-v_0)^2}\,dv = \int_0^{\infty} \frac{\text{Im}\,A_I(v)}{(v-v_0)^2}\,dv + \int_{-v_1-1}^{-1} \frac{\text{Im}\,A_I(v)}{(v-v_0)^2}\,dv$$

$$+ \int_{-\infty}^{-v_1-1} \frac{dv}{(v-v_0)^2}\;\frac{1}{v}\int_0^{-v-1} dv' \left[\frac{2}{3}\,\text{Im}\,A_0(v') + b_I\,\text{Im}\,A_2\,(v')\right]$$

$$+ \int_{-\infty}^{-v_1-1} \frac{dv}{(v-v_0)^2}\;\frac{1}{v}\int_0^{v_1} dv'\; d_I\; 2\left(1+2\,\frac{v+1}{v'}\right)\,\text{Im}\,A_1(v')$$

$$+ d_I\;\; R_{\pm} \tag{15}$$

where

$$R_{\pm} = \int_{-\infty}^{-v-1} \frac{dv}{(v-v_0)^2}\;\frac{1}{v}\int_{v_1}^{-v-1} dv'\; 3\left(1+2\,\frac{v+1}{v'}\right)\,\text{Im}\,A_1(v') \tag{16}$$

and by a suitable choice of v_1 ($v_1 = v_R - 4\tau$ say) this is the only term effected by the CDD pole. The $\pm$ designates the presence or absence of the CDD pole in Im $A_1(v')$. Writing P_I for the CDD pole - independent terms we have

$$\int_{L+R} \frac{\text{Im } A_I(v)}{(v-v_0)^2}\, dv = P_I + d_I\, R_{\pm} \tag{17}$$

Thus because the CDD pole is only effective in a limited range of v we may treat it as a perturbation which takes us from R_- to R_+. But in the unperturbed case we have

$$\frac{3a_1}{v_0} = \frac{1}{2\pi}\,(P_0 + d_0\, R_-) = \frac{-1}{\pi}\,(P_2 + d_2\, R_-) \tag{18}$$

and to satisfy the crossing conditions with the CDD pole we require

$$\frac{3a_1}{v_0} = \frac{1}{2\pi}\,(P_0 + d_0\, R_+) = \frac{-1}{\pi}\,(P_2 + d_2\, R_+) \tag{19}$$

i.e.,

$$0 = \frac{d_0}{2\pi}\,(R_+ - R_-) = -\frac{d_2}{\pi}\,(R_+ - R_-) \tag{20}$$

The second condition is identically satisfied, even if the first is not, since by definition

$$\frac{d_0}{2\pi} = \frac{1}{\pi} = -\frac{d_2}{\pi} \tag{21}$$

For the first condition to be satisfied we must have $R_+ - R_- = 0$ or, more in keeping with the spirit of the analysis, we require

$$\left| R_+ - R_- \right| \ll \left| \frac{3\pi a_1}{v_0} \right| \tag{22}$$

For the case $\lambda = -0.15$, $a_1 = -0.005$ and this reduces to

$$\left| R_+ - R_- \right| \ll 0.07 \tag{23}$$

We now turn to the evaluation of R_+ and R_-.

Since the CDD pole effect is concentrated about $v = v_R$ we shall only keep lowest order terms in $v-v_R$. These approximations will be reviewed later. Thus

$$\mathrm{Re}\ A_1^{-1} = \frac{C\tau}{v_R}\left[\frac{v_R - v}{\tau}\right] + \frac{K(v-v_R)/\tau}{1+\left[(v-v_R)/\tau\right]^2}$$

$$= -\phi\, x + K\, x/(1+x^2) \qquad (24)$$

and

$$\mathrm{Im}\ A_1^{-1} = -\left[\frac{v_R}{v_R+1}\right]^{1/2} - \frac{K}{1+x^2}$$

$$= -\ u - K/(1+x^2) \qquad (25)$$

where

$\phi = C\tau/v_R$, $u = \left[v_R/(v_R+1)\right]^{1/2}$ and $x = (v-v_R)/\tau$. Inverting we find

$$\mathrm{Im}\ A_1 = \frac{ux^2 + (K + u)}{\phi^2 x^4 - (2K\phi - \phi^2 + u^2)x^2 + (K + u)^2} \qquad (26)$$

Making similar approximations in R_+ we write

$$R_+ \approx R_+' = \int_{-}^{-v_1-1} \frac{dv}{(v - \frac{2}{3}v_0)^3}\ 3(1+2\frac{v+1}{v_R}) \int_{v_1}^{-v-1} \mathrm{Im}\ A_1(v')\ dv'$$

Using eqn. (26) we find

$$\int_{v_1}^{-v-1} dv'\ \mathrm{Im}\ A_1(v') = f(-v-1-v_R) - f(v+1+v_R)$$

$$- f(v_1 - v_R) + f(v_R - v_1)$$

with

$$f(v) = \frac{\tau}{2\phi}\arctan\left(\frac{p+v}{q}\right) + \frac{(\phi-u)\tau^2}{8\phi^2 p}\ln\left[1 + \left(\frac{p+v}{q}\right)^2\right]$$

where

$$p = \tau\left[4K\phi - (\phi-u)^2\right]/2\,\phi$$

and

$$q = \tau(\phi+u)/2\,\phi$$

Further define

$$\gamma = (3\,v_R + 6 + 4\,v_0)/12\,q$$

$$\omega = (v_1 + 1 + \tfrac{2}{3}\,v_0)/q$$

$$x_1 = (v_R + 1 + \tfrac{2}{3}\,v_0 - p)/q$$

$$x_2 = (v_R + 1 + \tfrac{2}{3}\,v_0 + p)/q$$

$$\alpha = 6/v_R\,(\phi+u)$$

and

$$\beta_1 = -\beta_2 = \tau(\phi-u)/2p$$

Then

$$R'_+ = \alpha \sum_{i=1}^{2} \frac{1}{1+x_i^2} \left\{ \frac{1}{2}\ \ln\left[\frac{1+(x_i+\omega)^2}{\omega^2}\right] \times \right.$$

$$\times \left[(1 - \frac{2\gamma\,x_i}{1+x_i^2}) - \beta_i\ (x_i - \gamma + \frac{2\,\gamma}{1+x_i^2}) \right] +$$

$$+ \left[\frac{\pi}{2} + \arctan\,(x_i - \omega) \right] \times$$

$$\times \left[(x_i - \gamma + \frac{2\gamma}{1+x_i^2}) - \beta_i \left(1 - \frac{2\gamma\,x_i}{1+x_i^2}\right) \right]$$

$$\left. - \gamma\,(1 - \beta_i\ x_i)/\omega \right\}$$

This is to be compared with R_- or rather

$$R_-' = \int_-^{-v_1-1} \frac{dv}{(v - \frac{2}{3} v_0)^3} \; 3/1+2 \frac{v+1}{v_R}) \int_{v_1}^{-v-1} dv' \; \frac{1/u}{1+\phi^2(v'-v_R)^2/u^2\tau^2}$$

where we have made the same approximations as in R_+'. Values of R_+' and R_-' and $\Delta R = R_+' - R_-'$ are given in table 1 for various values of κ, τ, and v_1 assuming C = 19 and $v_R = 4.1$. The calculations indeed show that ΔR is very small for a wide range of values of τ and κ and is not sensitive to the value of v_1 provided it is somewhat less than v_R. In fact we may use the freedom in v_1 to test the efficacy of the dispersion relations. By making $v_1 > v_R$ we may evaluate the infinite tail of the integral. As shown in table 1 its effect is negligible both in $R_\pm'$ and ΔR.

Finally let us consider the approximations made in the analysis: Firstly the computational approximations.

$$\left[\frac{v'}{v'+1}\right]^{1/2} \rightarrow \left[\frac{v_R}{v_R+1}\right]^{1/2} \qquad v' > v_1$$

$$C(v_R-v')/v' \rightarrow C(v_R-v')/v_R \qquad v' > v_1$$

$$(1+2 \frac{v+1}{v'}) \rightarrow (1+2 \frac{v+1}{v_R}) \qquad v' > v_1$$

$$(v-v_0)^2 \, v \rightarrow (v - \frac{2}{3} v_0)^3 \qquad v < -v_1 -1$$

The first two just constitute variations on the analytic approximations to the physical amplitudes and are not related to the CDD pole. The third is the most suspect since it distinguishes between the unperturbed case with Im A_1 large on each side of v_R. This approximation will be refined in future work by the addition of higher order terms in $(v-v_R)$. The fourth approximation is proportional to

$$\epsilon = \frac{(-v\ -1\ -\frac{2}{3}\,v_0)^3}{(-v\ -1\ -v_0)^2(-v_1\ -1)} - 1$$

$$= \frac{v_0{}^2}{(v+1+v_0)^2}$$

and since the main contribution to R_+ and R_- occurs near $v = v_R \pm \tau$ and $v = v_R$ respectively the correction to Δ R will be of the order

$$\epsilon' = R_{\pm} \left\{ \frac{1}{2} \left[\frac{v_0{}^2}{(v_R-\tau+1+v_0)^2} + \frac{v_0{}^2}{(v_R+\tau+1+v_0)^2} \right] - \frac{v_0{}^2}{(v_R+1+v_0)^2} \right\}$$

$$\approx \frac{3\ v_0{}^2\ \tau^2\ R_{\pm}}{v_R{}^4} \simeq 10^{-5}$$

which is negligible.

Secondly there are the assumptions that the S wave right cuts are not effected by the modifications to the left cuts. However, if the changes near $v = -v_R -1$ are not noticeable at $v = v_0$ in the subtracting constant calculation they should also be negligible for $v \geq 0$ in the dispersion integrals.

Thirdly, there is the possibility of changes in the P wave right cut altering the effective range behavior independent of a_1. But arguments similar to these for the S wave right cut suggest that these may also be neglected. However, a more detailed analysis will be undertaken.

One situation which has not been considered in detail here is that of a very weak CDD pole. In that case the resonance is not split (the variable P becomes imaginary) but only broadened and decreased in height.

Summary

The effect of adding a CDD pole at the resonance position in pion-pion scattering has been treated analytically and it has been shown that, provided the CDD pole is not too strong, i.e., the inelastic scattering is not too weak, the crossing conditions are still satisfied. It is interesting to note that for the ρ -meson which may be a split resonance there is a strong inelastic contribution from the ω-meson.

References

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Table 1.

Values of R_+', R_-' and ΔR for C = 19, v_R = 4.1 and various values of κ , τ and v_1.

κ	τ	v_1	R_+'	R_-'	ΔR
0.01	0.2	2.5	0.145534	0.145535	-1×10^{-6}
0.1	0.2	2.5	0.145528	0.145535	-7×10^{-6}
1	0.2	2.5	0.145460	0.145535	-8×10^{-5}
10	0.2	2.5	0.144540	0.145535	-1×10^{-3}
10	0.01	2.5	0.145520	0.145535	-1.5×10^{-5}
10	0.05	2.5	0.145429	0.145535	-1×10^{-4}
10	0.1	2.5	0.145234	0.145535	-3×10^{-4}
10	0.3	2.0	0.146730	0.147246	-5×10^{-4}
10	0.5	1.5	0.147749	0.148290	-5×10^{-4}
10	0.2	1.5	0.148420	0.148290	1.3×10^{-4}
10	0.2	2	0.147088	0.147246	-1.6×10^{-4}
10	0.2	2.5	0.144540	0.145535	-1×10^{-3}
10	0.2	6	0.003348	0.002865	5×10^{-4}
10	0.2	8	0.001058	0.001018	4×10^{-5}

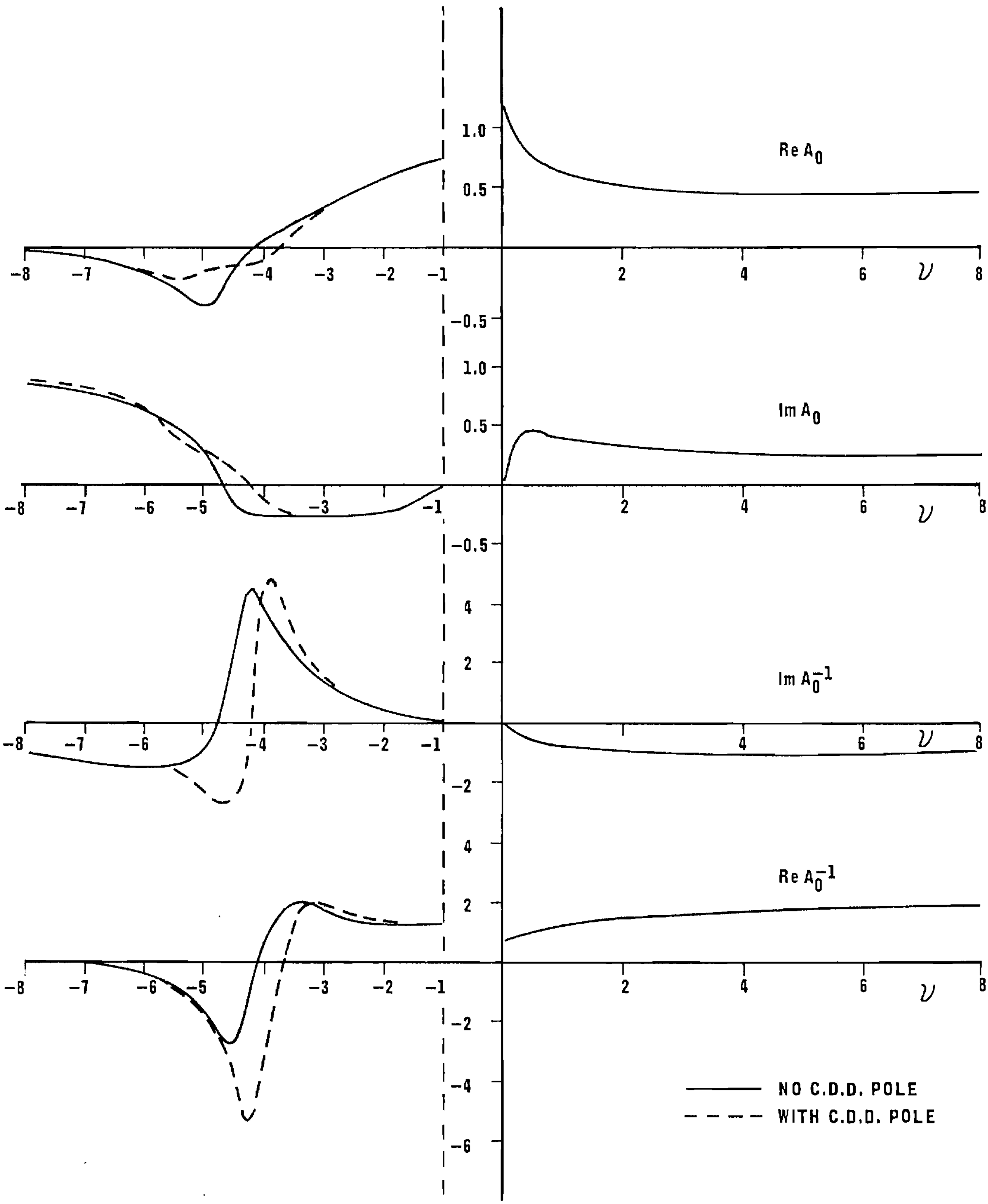

Figure 1. The S Wave, Isospin 0, Amplitude without a CDD Pole (Continuous Line) and the Modifications Introduced by the CDD Pole (Broken Line).

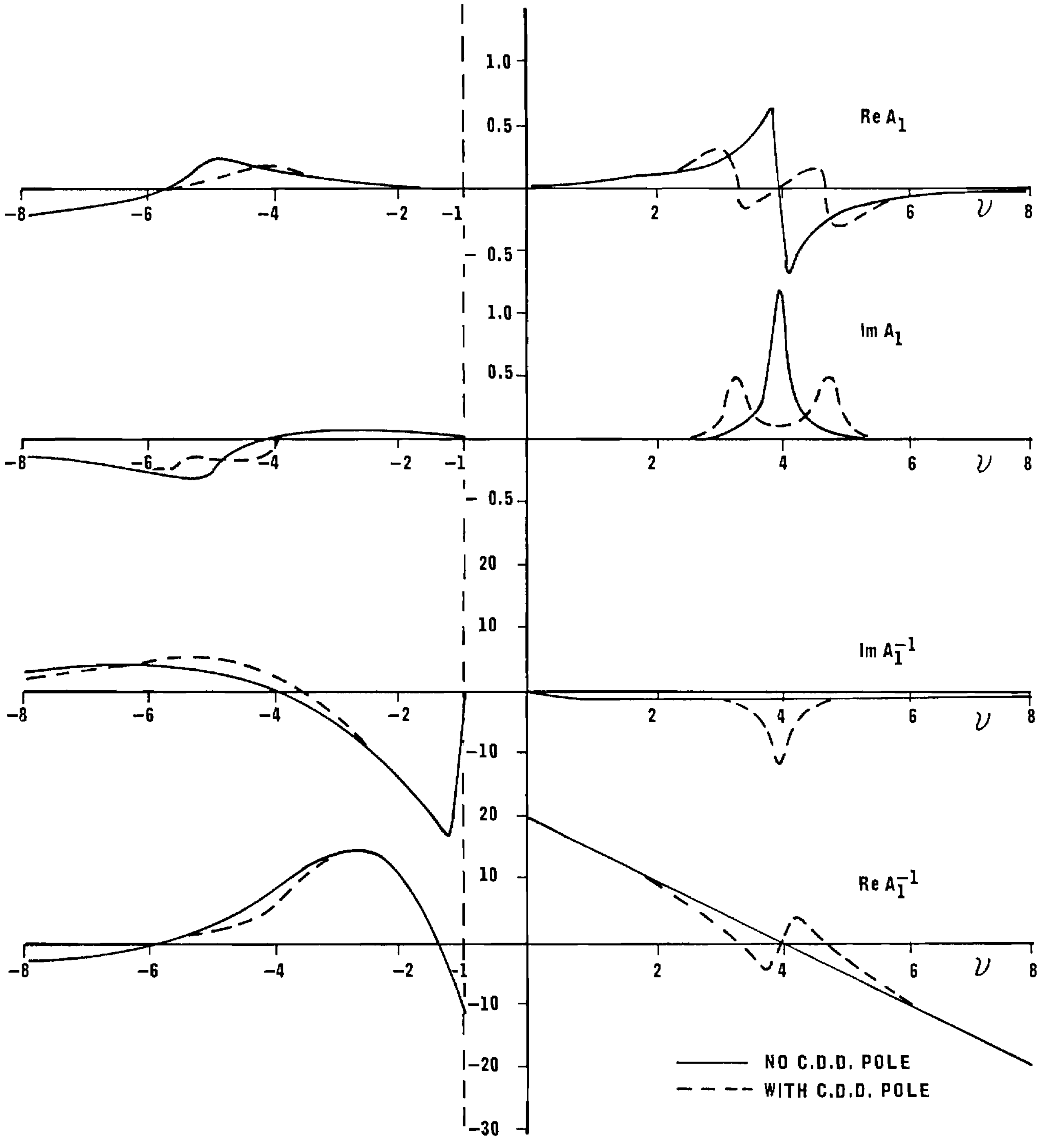

Figure 2. The P Wave Amplitude without a CDD Pole (Continuous Line) and the Modifications Introduced by the CDD Pole (Broken Line).