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Learning Financial Shocks and the Great Recession*

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Abstract

This paper develops a simple business-cycle model in which financial shocks have large macroeconomic effects when private agents are gradually learning the uncertain environment. When agents update their beliefs about the parameters that govern the unobserved process driving financial shocks to the leverage ratio, the responses of output, investment, and other aggregates under adaptive learning are significantly larger than under rational expectations. In our benchmark case calibrated using US data on leverage, debt-to-GDP and land value-to-GDP ratios for 1996Q1-2008Q4, learning amplifies leverage shocks by a factor of about three, relative to rational expectations. When fed with actual leverage innovations observed over that period, the learning model predicts that the persistence of leverage shocks is increasingly overestimated after 2002 and that a sizeable recession occurs in 2008-10, while its rational expectations counterpart predicts a counter-factual expansion. In addition, we show that procyclical leverage reinforces the amplification due to learning and, accordingly, that macro-prudential policies that enforce countercyclical leverage dampen the effects of leverage shocks.

Keywords: Collateral Constraints, Learning, Financial Shocks, Great Recession

Journal of Economic Literature Classification Numbers: E32, E44, G18

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1 Introduction

Whether or not banks and other financial institutions, policy-makers, households and firms relied on a decent approximation of the "true" probability distribution prior to the 2007-08 financial collapse is a key question to address if one is to understand the Great Recession. On the theoretical side, answering such a question requires relaxing the assumption that the data-generating process is known when agents make decisions in an economy that is subject to random disturbances (see Woodford [47] for a recent survey). New tractable approaches to tackle parameter uncertainty have recently been proposed. Hebert, Fuster and Laibson [17] show that asset price booms and busts are more satisfactorily explained when forecasters are assumed to use simple models that typically underestimate mean-reversion and overestimate the persistence of the impact of shocks. Ilut and Schneider [27] show that shocks driving the unknown mean level of productivity contribute significantly, under ambiguity aversion, to business cycles. In both contributions, the key assumption is that there is uncertainty about the "true" parameters (e.g. the mean and the autocorrelation) governing the random shocks that affect the economy.

Our distinctive contribution to this strand of literature is the introduction of statistical learning in a setting where consumption, investment and labor supply decisions depend on aggregate credit availability, which itself varies over time in a stochastic fashion. We focus on how decision-makers set and revise their beliefs about the parameters of the stochastic process governing financial shocks as new observed data arrive, following Marcet and Sargent [38] and Evans and Honkapohja [15] (see also the related discussion in Evans [14]). Although our analysis is similar in spirit to Hebert, Fuster and Laibson [17], Ilut and Schneider [27], since agents do not know these parameters, the key dimension we add is that agents learn their economic environment by estimating the unknown parameters driving aggregate credit conditions and by updating each period such estimates. In turn, those beliefs about the shock process are used to make forecasts that affect decisions and hence determine economic outcomes.

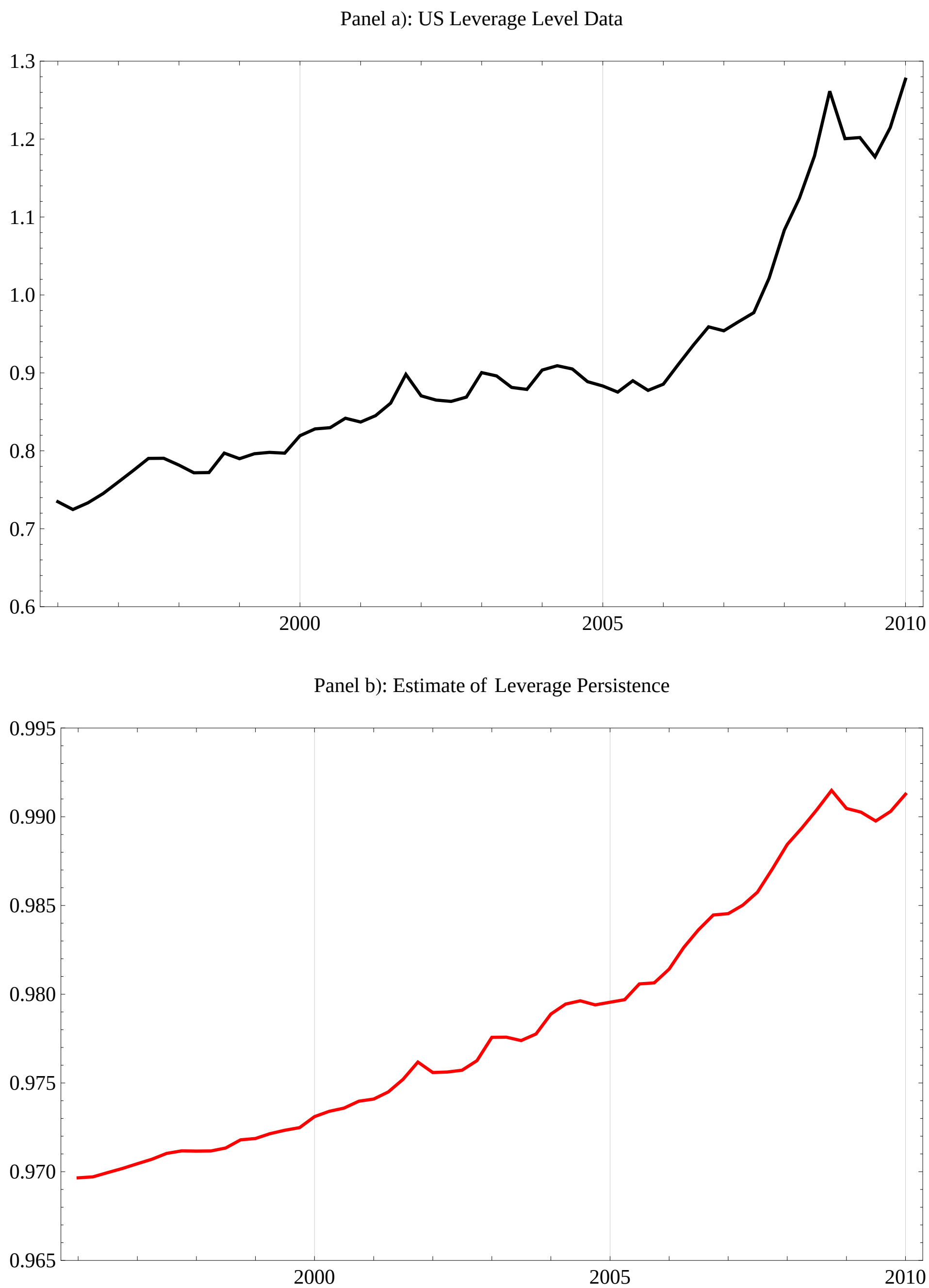
In a simple business-cycle model with collateral constraints and stochastic leverage, we show that dynamics under learning can differ significantly from the dynamics under rational expectations. More precisely, we compare two settings: (*i*) the model with full information (rational expectations), in which agents know the parameters governing the VAR(1) process governing the behavior of the economy; (*ii*) the model of incomplete information with learning, in which agents do not know the "true" parameters of the VAR(1) model and update their estimates as new data arrive. We find that the amplification of financial shocks is particularly large when agents *overestimate* either the persistence of financial shocks or the long-run level of credit conditions. When we simulate the stylized model using actual financial innovations we find that our learning model delivers a sizeable recession in 2008-2010, in contrast to the full information rational expectations that predicts a counterfactual *expansion* when subjected to the same financial shocks. The key random variable in our analysis is the leverage ratio defined by how much households can borrow out of the land market value. We show that when agents update their beliefs about the parameters that govern both the dynamics of endogenous variables as well as the unobserved process driving shocks to the leverage ratio, the responses of output and other aggregates under adaptive learning are significantly larger than under rational expectations.

Our results can be anticipated by looking at the panels in Figure 1. Panel a) of Figure 1 plots the US quarterly households' leverage data (provided by Boz and Mendoza [6]) over the period 1996Q1-2010Q1 that covers the latest boom-bust behavior in the housing market[1]. Panel b) of Figure 1 reports the autocorrelation coefficient of the exogenous leverage component that is estimated under learning. The autocorrelation coefficient graphed in panel b) is obtained through constant-gain recursive estimation in real time, as new data is collected. Panel b) shows that when confronted with the data in panel a), learning agents think of the AR(1) leverage process as moving towards unit root at the end of the period. Essentially, the level of leverage trends up and accelerates in 2007, when the land price stops expanding and starts falling while borrowing is sticky - panel a) - and this translates into an increasing estimate of autocorrelation by learning agents, which ends up being very close to one in the last two quarters of 2008. As a result, both the observed level of leverage and its estimated persistence peak at the same time, in 2008Q4. On the other hand, the rational expectations estimate, obtained by ordinary least squares over the whole sample period, is lower than its learning counterpart over the period shown in panel b) and it is around 0.976. The learning model generates the estimate shown in panel b) when fed with the actual leverage innovations and predicts that the impact of the negative shock to leverage observed in 2008Q4 is about three times bigger than under full information. When believed under learning to be close to permanent, shocks to credit conditions have a larger effect on the economy, compared to rational expectations. As a consequence, the learning model generates, under a negative leverage shock, a contraction that is similar in magnitude to the Great Recession.

We focus on financial shocks that drive up and down the leverage ratio, which according to the data in panel a) of Figure 1 are very persistent. We first perform two theoretical experiments. The first one assumes that agents know the economy's steady state and, in particular, the mean level of leverage but not its autocorrelation, which is allowed to be time-varying. We calibrate the model using data on leverage, debt-to-GDP and land value-to-GDP ratios for the period 1996Q1-2008Q4 and we subject the economy to the large negative shock to leverage that was observed in 2008Q4 (see panel a) in Figure 1) under the assumption that learning agents overestimate the autocorrelation of the leverage shock, which is believed to be close to unity according to panel b) in Figure 1. We compare the responses of the linearized economy under adaptive learning and under rational expectations.

[1] We take out of the raw data the endogenous component of leverage that has been (moderately) elastic to land prices prior to 2008 (Mian and Sufi [39]) and we estimate an AR(1) process on the residual (exogenous) component, see details in Section 4.

Figure 1: in Panel a) US Household Leverage Ratio 1996Q1-2010Q1 (Source: Boz and Mendoza [6]); in Panel b) Model Estimate of Leverage Autocorrelation 1996Q1-2010Q1



Our typical sample of results shows that learning amplifies leverage shocks by a factor of about 2.5 (see Figure 2). More precisely, our model predicts that after a negative leverage shock of about -5% observed in 2008Q4, the output falls by about 3.2%. In addition, aggregate consumption and the capital stock fall by about 3.6% and 5%, respectively. Under rational expectations, however, output drops only by about 1.3% while the responses of consumption and investment are divided by more than two at impact. Consumption and investment go down by a significantly larger margin under learning because de-leveraging is more severe: land price and debt are much more depressed after the negative leverage shock hits when its persistence is overestimated by agents who are constantly learning their environment and, because of recent past data, temporarily pessimistic.

We next show that the magnitude of the consequent recession may in part be attributed to the high *level* of leverage (and the correspondingly high level of the debt-to-GDP ratio) observed in 2008Q4. When the same negative leverage shock occurs in the model calibrated using 1996Q1 data, when leverage was much lower, the impact on output's response is reduced by about two thirds (see Figure 3). In this sense, our model points at the obvious fact that financial shocks to leverage originate larger aggregate volatility in economies that are more levered. In addition, we ask whether procyclical leverage may act as an aggravating factor and our answer is positive. The assumption that households' leverage responds to land price is motivated by the recent evidence provided by Mian and Sufi [39]. The counter-factual experiment with countercyclical leverage shows dampened effects of leverage shocks, with responses of aggregate variables under learning that are close to their rational expectations counterpart (see Figure 3). One possible interpretation of this finding is that macro-prudential policies enforcing countercyclical leverage have potential stabilizing effects on the economy in the face of financial shocks, at small cost provided that non-distortionary policies are implemented (e.g. through regulation).

Our second theoretical experiment is carried out under the assumption that learning agents do not know the steady state of the economy and, in particular, that they do not know the long-run level of leverage. This is our preferred model in the sense that it is arguably a more realistic description of the difficulties that forecasting agents/econometricians face when trying to figure out the parameters governing the data generating process. In such a setting, we again feed the model with the negative leverage shock of about -5% observed in 2008Q4 and we show that the responses of the economy are further amplified under learning when agents' belief about the mean level of leverage is overestimated (see Figure 4). Summing up the results from our two model experiments, our main conclusion is that in a world where agents overestimate the persistence of financial shocks and/or the mean level of leverage, learning amplifies the disturbances to borrowing capacity.

We next derive our set of quantitative results about the model-generated recession for 2008-10. In line with the literature (see Kiyotaki, Michaelides, Nikolov [31], Liu, Wang, Zha [36], Justiniano, Primiceri, Tombalotti [29], Kaas, Pintus, Ray [30], among others), we show that replicating the observed boom-bust pattern of land prices over the 2000s requires another source of shocks in addition to leverage shocks. We introduce a land price shock that we calibrate to ensure that the behavior of the endogenous land price matches its observed counterpart.[2] We also feed the model with the actual innovations to leverage and show that the model predicts a sizeable fall in output, of similar magnitude to that observed during the Great Recession. More precisely, we do that in a setting where agents do not know the steady state and the autocorrelation matrix in the VAR representation of the economy, that they have to estimate using constant-gain learning. When we let agents revise their estimates in reaction to the actual leverage innovations observed up to 2008, the learning model predicts a boom that is followed by a sizeable recession in 2008-10 (see Figure 8). In sharp contrast, in the 2000s the rational expectations model predicts a long recession that is followed by an expansion, which are both at odds with the data.

Related Literature: Our paper connects to several strands of the literature. The macroeconomic importance of financial shocks has recently been emphasized by Jermann and Quadrini [28], among others, and our paper contributes to this literature about credit shocks by showing how learning under parameter uncertainty matters. Closest to ours are the papers by Adam, Kuang and Marcet [1], who focus on exogenous interest rate changes, and by Boz and Mendoza [6], who show how changes in the leverage ratio have large macroeconomic effects under Bayesian learning and Markov regime switching.[3] As in Boz and Mendoza [6], we focus on leverage shocks but our setting is different. First, our model with adaptive learning is easily amenable to simulations and we solve for equilibria through usual linearization techniques. Because we assume that agents are adaptively learning through VAR estimation, it is possible to enrich the model by adding capital accumulation and endogenous production. Most importantly, our model predicts large output drops when the economy is hit by negative leverage shocks. In sharp contrast, absent TFP shocks, output remains constant after a financial regime switch in Boz and Mendoza [6]. In addition, we show that our results are robust to the introduction of heterogeneous agents and endogenous interest rate. Since in such setting the interest rate is endogenously procyclical, it could completely defeat the effect of an increase in credit supply even under learning. Our robustness analysis makes clear that this is not the case and that amplification due to learning does not rely upon the small-open economy assumption, an issue that is addressed neither in Adam, Kuang and Marcet [1] nor in Boz and Mendoza [6].[4] To sum up, this paper follows the literature by emphasizing how financial shocks affect asset

[2] Alternative settings have been proposed to explain boom-bust patterns in housing price. For example, Garriga, Manuelli, Peralta-Alva [19] assume segmented financial markets, He, Wright, Zhu [24] focus on search environments, while Cao and L'Huillier [9] introduce noisy news about future productivity.

[3] In independent research, Kuang [35] introduces learning in the original model of Kiyotaki and Moore [32] with risk neutrality and linear technologies. In contrast, utility and production functions are assumed to be concave in this paper. See also Gelain and Lansing [22] for a related analysis in a Lucas-type asset pricing model.

[4] Another notable difference with Adam, Kuang and Marcet [1] is that our setting does not rule out mean reversion in agents' beliefs.

prices, but it also differs by measuring to what extent financial shocks help explain the fall in output and investment observed over the Great Recession period.

Our paper also relates to some of the insights in Howitt [25], Hebert, Fuster and Laibson [17, 18]. Contrary to Hebert, Fuster and Laibson [17, 18] who assume that agents use a misspecified model, in our case the overestimated persistence of shocks arises endogenously under adaptive learning when agents face the sequence of financial innovations that was observed in the run-up to the crisis.[5] In addition, our paper stresses that endogenous changes in the beliefs about the long-run level of leverage may also matter for explaining why shocks get amplified under adaptive learning. This is also where our paper departs from Ilut and Schneider [27], who do not consider learning in their setting with exogenously driven ambiguity about TFP shocks. Although, in theory, endogenous persistence could arise under iid shocks when agents learn the steady-state leverage level, this effect turns out not to be quantitatively important in our setting. In contrast, persistent slumps are shown by Kozlowski, Veldkamp and Venkateswaran [34] to arise under reasonable calibrations when learning is about the tails of the real shocks distribution. Close to our macro perspective is Pancrazi and Petrunti (2015), who use survery evidence to show that financial experts overestimated long-run prices and did not forecast a mean reversion in long-run housing price dynamics, implying the overestimation of the persistence of housing prices. In a similar vein, Piazzesi and Schneider (2009) identify momentum traders, using Michigan Survey of Consumers data, and show that their size (and optimism) increased during the housing price boom.

In the literature, the idea that procyclical leverage has adverse consequences on the macroeconomy is forthfully developed in Geanakoplos [20] (see also Cao [8], and Geerolf [21] for a tractable model of endogenous leverage distribution). Although our formulation of elastic leverage is derived in an admittedly simple setup, it allows us to examine its effect in a full-fledged macroeconomic setting. Lastly, the notion that learning is important in business-cycle models when some change in the shock process occurs has been discussed by, e.g., Bullard and Duffy [7] and Williams [46]. More recently, Eusepi and Preston [13] have shown that learning matters in a standard RBC model when the economy is hit by shocks to productivity growth. Our paper adds to this literature by focusing on financial shocks under collateral constraints.

The paper is organized as follows. Section 2 presents the model and derives its rational expectations equilibria. Section 3 relaxes the assumption that agents form rational expectations in the short run and it shows how financial shocks are amplified under learning when agents update their estimates about the parameters of the stochastic process driving financial shocks. Section 4 shows that the model with learning predicts a sizeable recession in 2008-10 while its rational expectations counterpart does not. Section 5 gathers concluding remarks and all proofs are exposed in the appendices.

[5] Along this dimension, we address some concerns raised by Evans [14]. Beshears, Choi, Fuster, Laibson and Madrian [4] report results from experiments where subjects underestimate mean reversion.

2 The Leveraged Economy with Financial Shocks

2.1 Model

The model is essentially an extension of Kocherlakota's [33] to partial capital depreciation, endogenous labor and, most importantly, adaptive learning. This is arguably the simplest setting within which one can study how learning affects the business cycle in a leveraged economy. A representative agent solves:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left[C_t - \psi \frac{N_t^{1+\chi}}{1+\chi}\right]^{1-\sigma} - 1}{1-\sigma} \tag{1}$$

where $C_t \geq 0$ is consumption, $N_t \geq 0$ is hours worked, $\sigma \geq 0$ denotes relative risk aversion, $\psi \geq 0$ is a scaling parameter, $\chi \geq 0$ is the inverse of the Frisch labor supply elasticity, subject to both the budget constraint:

$$C_t + K_{t+1} - (1-\delta)K_t + T_t Q_t (L_{t+1} - L_t) + (1+R)B_t = B_{t+1} + AK_t^{\alpha} L_t^{\gamma} N_t^{1-\alpha-\gamma} \tag{2}$$

and the collateral constraint:

$$\tilde{\Theta}_t E_t[Q_{t+1}] L_{t+1} \geq (1+R)B_{t+1} \tag{3}$$

where K_{t+1}, L_{t+1} and B_{t+1} are the capital stock, the land stock and the amount of new borrowing, respectively, all chosen in period t, Q_t is the land price, R is the exogenous interest rate, and A is the total factor productivity (TFP thereafter). In the model, leverage $\tilde{\Theta}_t$ is subject to random shocks whereas both the interest rate and TFP are constant over time.[6] As we focus on financial shocks, we ignore TFP disturbances. We also introduce a land price shock T, essentially because the model with only leverage disturbances can hardly replicate the land price behavior that has been observed in the 2000s. In line with the literature (see e.g. Kiyotaki, Michaelides, Nikolov [31], Liu, Wang, Zha [36], Justiniano, Primiceri, Tombalotti [29], Kaas, Pintus, Ray [30], among others), we find that the model with both shocks does a better job along this dimension. Although the formulation we use is rather agnostic, it is easy to show that it is essentially equivalent to land preference shocks or other "political" (e.g. tax) shocks that push the demand for land and the land price up or down. We assume that the land price shock process is $T_t = T_{t-1}^{\rho_\tau} \Psi_t$ and, absent shocks, that it does not cause any distortions in the steady state. We present first the results obtained under the collateral constraint (3), which follows Kiyotaki and Moore [32]. However, quantitatively similar results hold under the margin requirement timing stressed in Aiyagari and Gertler [3] (see Section 3.2 for robustness analysis).

Denoting Λ_t and Φ_t the Lagrange multipliers of constraints (2) and (3), respectively, the

[6] In Section 3.2, we show that our main results are robust to the introduction of heterogeneous agents and endogenous interest rate.

borrower's first-order conditions with respect to consumption, labor, land stock, capital stock, and loan are given by:

$$\left[C_t - \psi \frac{N_t^{1+\chi}}{1+\chi}\right]^{-\sigma} = \Lambda_t \tag{4}$$

$$\psi N_t^{\chi+\alpha+\gamma} = (1-\alpha-\gamma) A K_t^{\alpha} L_t^{\gamma} \tag{5}$$

$$T_t Q_t \Lambda_t = \beta E_t[T_{t+1} Q_{t+1} \Lambda_{t+1}] + \beta\gamma E_t[\Lambda_{t+1} Y_{t+1}/L_{t+1}] + \Phi_t \tilde{\Theta}_t E_t[Q_{t+1}] \tag{6}$$

$$\Lambda_t = \beta E_t[\Lambda_{t+1}(\alpha Y_{t+1}/K_{t+1} + 1 - \delta)] \tag{7}$$

$$\Lambda_t = \beta(1+R) E_t[\Lambda_{t+1}] + (1+R)\Phi_t \tag{8}$$

Consistent with the "shadow price learning" approach proposed by Evans and Mc Gough [16], we keep track of the Lagrange multipliers in our Euler-equation learning procedure. Although the analysis of nonlinear decision rules for control variables is beyond the scope of this paper, we conjecture that Euler-equation learning is in our setting similar to shadow-price learning when applied to the linearized model, similarly to Section 6 of Evans and McGough [16] in which is studied a simpler Ramsey economy.

We also incorporate into the model the feature that leverage responds to changes in the land price, which accords with the US micro data evidence documented by Mian and Sufi [39]. More precisely, we posit that:

$$\tilde{\Theta}_t \equiv \Theta_t \left\{ \frac{E_t[Q_{t+1}]}{Q} \right\}^{\varepsilon} \tag{9}$$

where Q is the steady-state value of land price and the log of Θ_t follows an AR(1) process, that is, $\Theta_t = \overline{\Theta}^{1-\rho_\theta} \Theta_{t-1}^{\rho_\theta} \Xi_t$. One can think of (9) as a decomposition of the leverage into an exogenous component Θ_t and a component $\{E_t[Q_{t+1}]/Q\}^{\varepsilon}$ that is endogenous and responds to land price.[7] While our qualitative results do not depend on this assumption, we set the parameter ε to a positive value in our benchmark calibration to be consistent with the evidence reported in Mian and Sufi [39] and, then, to examine the predictions of our model under the counterfactual assumption that leverage is countercyclical.

In what follows, we assume that Θ_t and T_t are subject to the innovations Ξ_t and Ψ_t. We compare two cases regarding what agents know about the data generating process of the economy:
(i) rational expectations (with full information): agents know with certainty all the structural parameters of the model including "true" values of ρ_τ, ρ_θ and $\overline{\Theta}$,
(ii) learning (with incomplete information): the exact structure of the economy and, importantly, ρ_τ, ρ_θ and $\overline{\Theta}$ are unknown and agents have to learn and estimate unknown parameters based on available data. We consider two experiments which are reported in Sections 3.1 and 3.3. In Section 3.1, we first assume that the steady state is known but

[7] In Appendix A.1, we show how (9) can be derived in a simple setting with ex-post moral hazard and costly monitoring.

that learning agents do not know and have to estimate, among other parameters, the persistence parameter ρ_θ. We then discuss the robustness of the main result in Section 3.2. Next, in Section 3.3, we assume that agents are uncertain about the steady state, including level of leverage $\overline{\Theta}$, as well. Before turning to that, we present the benchmark case of full information rational expectations equilibria.

2.2 Rational Expectations Equilibria

A rational expectations competitive equilibrium is a sequence of positive prices $\{Q_t\}_{t=0}^{\infty}$ and positive allocations $\{C_t, K_{t+1}, L_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ such that, given the exogenous sequence $\{\Theta_t, T_t\}_{t=0}^{\infty}$ of the leverage and price shocks, and the exogenous interest rate $R \geq 0$:

(i) $\{C_t, N_t, K_{t+1}, L_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ satisfies the first-order conditions (4)-(8), the transversality conditions, $\lim_{t\to\infty} \beta^t \Lambda_t L_{t+1} = \lim_{t\to\infty} \beta^t \Lambda_t K_{t+1} = 0$, and the complementarity slackness condition $\Phi_t \left[\tilde{\Theta}_t E_t[Q_{t+1}] L_{t+1} - (1+R)B_{t+1}\right] = 0$ for all $t \geq 0$, where $\tilde{\Theta}_t \equiv \Theta_t \{E_t[Q_{t+1}]/Q\}^{\varepsilon}$, given the initial endowments $L_0 \geq 0, B_0 \geq 0, K_0 \geq 0$;

(ii) The good and land markets clear for all t, that is, $C_t + K_{t+1} - (1-\delta)K_t + (1+R)B_t = B_{t+1} + A_t K_t^{\alpha} N_t^{1-\alpha-\gamma}$ and $L_t = 1$, respectively.

The above definition assumes that the interest rate is exogenous. Therefore, a natural interpretation of the model is that it represents a small, open economy. However, in Section 3.2 we show that our main results are robust to the introduction of heterogeneous agents and endogenous interest rate in a closed-economy variant of Iacoviello's [26] model. The details of such an extension are presented in Appendix A.3. As our focus is on how borrowers adaptively learn how the economy settles after financial shocks, we abstract both from TFP shocks and from further details regarding the lender's side, and we focus on the small-open-economy setting as in Adam, Kuang and Marcet [1], Boz and Mendoza [6]. However, our contribution with respect to the latter is to show that amplification due to learning does not critically depend on the interest rate being exogenous.

There is a unique (deterministic) stationary equilibrium such that the credit constraint (3) binds, provided that the interest factor $1 + R \equiv 1/\mu$ is such that $\mu \in (\beta, 1)$, that is, if lenders are more patient than borrowers. This follows from the steady-state version of (8), $\Phi = \Lambda(\mu - \beta) > 0$. The steady state is characterized by the following ratios, that fully determine the linearized dynamics around the steady state. From (6) and (7), it follows that the land price-to-GDP and capital-to-GDP ratios are given by $Q/Y = \gamma\beta/[1 - \beta - \overline{\Theta}(\mu - \beta)]$ and $K/Y = \alpha\beta/[1 - \beta(1 - \delta)]$, respectively. In addition, (3) yields the debt-to-GDP ratio $B/Y = \mu\overline{\Theta}Q/Y$ and (2) yields the consumption-to-GDP ratio $C/Y = 1 - \delta K/Y - (1/\mu - 1)(B/Y)$. Finally, (5) gives that $\psi N^{\chi+\alpha+\gamma} = (1-\alpha-\gamma)AK^{\alpha}$ so that if one defines $X \equiv C - \psi N^{1+\chi}/(1+\chi)$, it follows that $X/Y = C/Y - (1-\alpha-\gamma)/(1+\chi)$.

Appendix A.2 provides a log-linearized version in levels of the set of equations (2)-(8) defining, together with (9) and the laws of motion $\Theta_t = \overline{\Theta}^{1-\rho_\theta} \Theta_{t-1}^{\rho_\theta} \Xi_t$ and $T_t = T_{t-1}^{\rho_\tau} \Psi_t$, intertemporal equilibria. The linearized expectational system can be written as:

$$X_t = \mathbf{A}X_{t-1} + \mathbf{B}E_{t-1}[X_t] + \mathbf{C}E_t[X_{t+1}] + \mathbf{N} + \mathbf{D}\xi_t + \mathbf{F}\psi_t \tag{10}$$

where $X_t' \equiv (c_t, q_t, \lambda_t, \phi_t, b_t, k_t, \theta_t, \tau_t)$; ξ_t, ψ_t are exogenous shocks, and all variables in lowercase letters denote variables in log (e.g. $k_t \equiv \log(K_t)$). The derivation and the expressions of the 8-by-8 matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{F}$, $\mathbf{N}$ as functions of parameters are given in Appendix A.2.

The linearized rational expectations equilibrium can be obtained as the unique E-stable Minimal-State-Variable solution (MSV thereafter) such that

$$E_{t-1}[X_t] = \mathbf{H^{re}} + \mathbf{M^{re}}X_{t-1} \tag{11}$$

where $\mathbf{M^{re}}$ and $\mathbf{H^{re}}$ solve

$$\mathbf{M} = [\mathbf{I}_8 - \mathbf{C}\mathbf{M}]^{-1}[\mathbf{A} + \mathbf{B}\mathbf{M}], \tag{12}$$

$$\mathbf{H} = [\mathbf{I}_8 - \mathbf{C}\mathbf{M^{re}}]^{-1}[\mathbf{B}\mathbf{H} + \mathbf{C}\mathbf{H} + \mathbf{N}] \tag{13}$$

and $\mathbf{I}_8$ is the 8-by-8 identity matrix.
It is important to underline that all parameters, including both the autocorrelation of the leverage shock process, that is, ρ_θ, and the leverage level, that is $\overline{\Theta}$, are known under rational expectations. In contrast, the next sections relax such an assumption and assume instead that agents have to form estimates about ρ_θ and $\overline{\Theta}$ using the available data.

3 Adaptive Learning and Financial Shocks

Following Marcet and Sargent [38] and Evans and Honkapohja [15], we now relax the assumption that agents form rational expectations in the short-run. We first assume that the steady state of the economy is known, which implies that the steady state level of leverage is a common knowledge. However, the parameters governing the dynamics of the economy are not known. In particular, ρ_θ is not known with certainty by agents. We can still use the linearized dynamic system in log levels, which is now:

$$X_t = \mathbf{A}X_{t-1} + \mathbf{B}E^*_{t-1}[X_t] + \mathbf{C}E^*_t[X_{t+1}] + \mathbf{N} + \mathbf{D}\xi_t + \mathbf{F}\psi_t \tag{14}$$

where the operator E^*_t indicates expectations that are taken using all information available at t but that are possibly nonrational. More precisely, agents behave as econometricians by embracing the following perceived law of motion (PLM thereafter):

$$X_t = \mathbf{M}X_{t-1} + \mathbf{H} + \mathbf{G}\xi_t + \mathbf{J}\psi_t \tag{15}$$

which agents use for forecasting. In particular, (15) yields $E_t[X_{t+1}] = \mathbf{M}_{t-1}X_t + \mathbf{H}_{t-1}$ and $E_{t-1}[X_t] = \mathbf{M}_{t-2}X_{t-1} + \mathbf{H}_{t-2}$. The actual law of motion (ALM thereafter) results from combining (14) and (15) which gives:

$$[\mathbf{I}_8 - \mathbf{C}\mathbf{M}_{t-1}]X_t = [\mathbf{A} + \mathbf{B}\mathbf{M}_{t-2}]X_{t-1} + [\mathbf{B}\mathbf{H}_{t-2} + \mathbf{C}\mathbf{H}_{t-1} + \mathbf{N}] + \mathbf{D}\xi_t + \mathbf{F}\psi_t \tag{16}$$

When $\mathbf{M}$ and $\mathbf{H}$ coincide with $\mathbf{M^{re}}$ and $\mathbf{H^{re}}$ (as derived in Section 2.2) then agents hold rational expectations. However, beliefs captured in $\mathbf{M}$ and $\mathbf{H}$ may differ from rational expectations. Following Evans and Honkapohja [15], we assume they are updated in real time using recursive learning algorithms which means that the belief matrices $\mathbf{M}$ and $\mathbf{H}$ are time-varying. The coefficients are updated according to

$$\begin{aligned} \mathbf{\Omega}_t &= \mathbf{\Omega}_{t-1} + \nu_t (X_t - \mathbf{\Omega}_{t-1} Z_{t-1}) Z'_{t-1} \mathbf{R}_t^{-1} \quad (17) \\ \mathbf{R}_t &= \mathbf{R}_{t-1} + \nu_t (Z_{t-1} Z'_{t-1} - \mathbf{R}_{t-1}) \quad (18) \end{aligned}$$

where $Z'_t = [1, X'_t]$ and $\mathbf{\Omega} = [\mathbf{H} \quad \mathbf{M}]$. $\mathbf{R}$ is the estimate of the variance-covariance matrix and ν_t is the gain sequence (which equals $1/t$ under ordinary least squares and ν under constant gain, respectively OLS and CG thereafter). One difference with rational expectations that is key to our results is that agents' estimates may differ from true parameter values, that is $(\mathbf{M_t}, \mathbf{H_t}) \neq (\mathbf{M^{re}}, \mathbf{H^{re}})$. This implies, for example, that agents may overestimate the autocorrelation parameter ρ_θ (or overestimate the steady state level of leverage $\bar{\Theta}$ as later explained, in Section 3.3).

More generally, our aim is to compare:
(i) dynamic equilibria under (full information) rational expectations: the sequence of endogenous variables $X'_t \equiv (c_t, q_t, \lambda_t, \phi_t, b_t, k_t, \theta_t, \tau_t)$ satisfy (15), given exogenous innovations ξ_t, ψ_t, with $(\mathbf{M}, \mathbf{H}) = (\mathbf{M^{re}}, \mathbf{H^{re}})$ and $(\mathbf{G}, \mathbf{J}) = ([\mathbf{I}_8 - \mathbf{C}\mathbf{M^{re}}]^{-1}\mathbf{D}, [\mathbf{I}_8 - \mathbf{C}\mathbf{M^{re}}]^{-1}\mathbf{F})$,
(ii) dynamic equilibria under learning: the sequence of endogenous variables X_t satisfy (16), given ξ_t, ψ_t, where $\mathbf{\Omega}_t = [\mathbf{H}_t \quad \mathbf{M}_t]$ follow the updating rules (17)-(18), given initial conditions $(\mathbf{H}_0, \mathbf{M}_0, \mathbf{R}_0)$, as described in more details in Appendix A.4.

The mapping from the PLM (15) into the ALM (16) is given by:

$$\begin{aligned} T_{\mathbf{M}}(\mathbf{M}, \mathbf{H}) &= [\mathbf{I}_8 - \mathbf{C}\mathbf{M}]^{-1} [\mathbf{A} + \mathbf{B}\mathbf{M}] \quad (19) \\ T_{\mathbf{H}}(\mathbf{M}, \mathbf{H}) &= [\mathbf{I}_8 - \mathbf{C}\mathbf{M}]^{-1} [\mathbf{B}\mathbf{H} + \mathbf{C}\mathbf{H} + \mathbf{N}] \quad (20) \end{aligned}$$

We verify numerically that under parameterizations that we consider, the MSV solution is locally E-stable, that is, all eigenvalues of both $DT_{\mathbf{M}}(\mathbf{M^{re}}, \mathbf{H^{re}})$ and $DT_{\mathbf{H}}(\mathbf{M^{re}}, \mathbf{H^{re}})$ lie within the interior of the unit circle.[8] Because E-stability conditions hold in all simulations that we report, we conjecture that the results of Evans and Honkapohja [15] about convergence in distribution to the rational-expectations equilibrium for small enough gain values apply as well in our setting.[9]

[8] The Jacobian expression is derived in Appendix A.4.2.

[9] Since our purpose is not to establish convergence results, we abstract from the analytical conditions stated in Evans and Honkapohja [15, p.165], which turn out to be quite demanding. In practice, we numerically compute the E-stable solutions by iterating the T-map (19)-(20), as described in Evans and Honkapohja [15, p.232].

3.1 Learning the Persistence of Leverage Shocks

In this section, we show that learning amplifies leverage shocks when agents' beliefs about the model parameters are allowed to differ from rational expectations. In particular, we assume that learning agents incorrectly believe that ρ_θ is closer to one than the "true" value. This is meant to capture the trend in leverage that is observed in the run-up to the 2008Q4 crisis. On the other hand, we make our theoretical experiment more transparent by first subjecting the model to a single source of shock and we shut down the land price shock, that is, $T_t = 1$ for all t.

The quarterly data on households's debt, land holdings, land price and leverage we use are borrowed from Boz and Mendoza [6]. The model is calibrated to deliver average values for leverage, debt-to-GDP and land value-to-GDP ratios observed over the housing market "bubble" period 1996Q1-2008Q4, that is $\overline{\Theta} \approx 0.88$, $B/Y \approx 0.52$ and $QL/Y \approx 0.59$, see Table 1 for all parameter values. To calibrate those ratios, we fix the quarterly interest rate to 1% (that is, $\mu = 0.99$) and set $\beta = 0.96\mu$, as in Iacoviello [26]. From Gertler et al. [23], we get the labor elasticity parameter $\chi = 1/3$ and the capital share $\alpha = 0.33$. From the data 1996Q1-2008Q4, we compute the leverage mean level $\overline{\Theta} \approx 0.88$ and we pick the land share $\gamma = 0.0093$ to target the land price-to-GDP ratio $QL/Y \approx 0.59$ and the debt-to-GDP is $B/Y \approx 0.52$ as in the data. Although leverage stationarity may appear as questionable for the 2006-09 period, it is arguably not for longer periods in the data and also in theory. The work disutility scaling factor is simply taken to be $\psi = 1 - \alpha - \gamma$. On the other hand, we set $\sigma = (C/Y - (1 - \alpha - \gamma)/(1 + \chi))/(C/Y)$ to ensure unitary relative risk aversion. In addition, we take the value $\varepsilon = 0.5$ from the estimates of Mian and Sufi [39, Table 2, column 6], who regress leverage growth on house price growth.[10] Finally, the standard deviation of the innovations to leverage, that is, σ_ξ, comes from the OLS estimate over the whole sample period of the AR(1) leverage process for the log of Θ. In all the simulations reported below, we have checked numerically that the borrowing constraint is always binding.

Table 1. Parameter Values (1996Q1-2008Q4)

μ	β	δ	α	γ	$\overline{\Theta}$	χ	ε	ν	σ_ξ
0.99	0.96μ	0.025	0.33	0.0093	0.88	1/3	0.5	0.004	0.034

In our first experiment we assume that in the period preceding the financial collapse of 2008Q4, the agents in our model economy have learned that ρ_θ was close to one, reflecting the leverage trend that starts in the early 1990s. This means that agents' beliefs encapsulated in matrix $\mathbf{M}$ of the PLM (15) reflect that ρ_θ is closer to one than under RE. Then in 2008Q4 a large negative shock to leverage of about -5% happens (see Figure 1). The (pseudo-)impulse response functions in Figure 2 report the reaction of the economy's

[10] The value chosen for ε implies, for instance, that a 10% increase in land price triggers a 5% increase in leverage, which under our calibration would raise leverage from 0.88 to about 0.92.

aggregates under the assumptions that agents wrongly believe that $\rho_\theta \approx 0.9904$ whereas the true value is 0.9756. Such a calibration is strictly disciplined by data in the sense that the above values come from the CG and OLS estimates obtained from the data, as shown in panel b) of Figure 1 (see also Figure 7 in Section 4). More precisely, panel b) in Figure 1 shows that $\rho_\theta \approx 0.9904$ in 2008Q3, which is the value that learning agents use to forecast 2008Q4. To initialize the model, we simulate a million times the RE model calibrated according to Table 1, using $\mathbf{M^{re}}$, $\mathbf{H^{re}}$, and we estimate the variance-covariance matrix $\mathbf{R}$ that is used as initial condition to generate the impulse responses under learning.[11] The blue dotted line in Figure 2 represents the RE equilibrium with $\rho_\theta = 0.9756$. The solid red curve in Figure 2 occurs when agents gradually learn using (17)-(18) under the initial belief that $\rho_\theta = 0.9904$, with the true value being 0.9756. Our chosen value of $\nu = 0.004$ for the constant-gain learning parameter implies that learning agents regress past data using a forgetting half-length of about 45 years, that is, data older than 45 years are weighted less than 50%. Such a low value falls within the range estimated by Slobodyan and Wouters [45, Table 3] in a medium-scale DSGE model with VAR beliefs.[12] While Figure 2 relies on $\nu = 0.004$, similar impulse-response patterns would occur with values that belong to $(0.001, 0.04)$ (which would imply similar effects at impact but slower or faster recovery). However, to guarantee local E-stability in the quantitative exercise reported in Section 4, ν cannot be much larger than 0.004. Another advantage of using such a low value for the constant-gain parameter is that we do not need to resort to any projection facility.

Figure 2 shows that the negative leverage shock is significantly amplified under learning. In all figures, the numbers reported on the y-axis are deviations from steady-state values expressed in percentage terms. For example, Figure 2 reports that the output fall in period two is about -3.2% under learning and about -1.3% under rational expectations. In particular, the impact on output and capital is roughly 2.5 times larger and the consumption drop is multiplied by about four compared to the rational expectations outcome. This follows from the fact that deleveraging is much more severe under learning: the fall in land price is about four times larger and the debt decrease is multiplied by around 2.5 compared to RE.[13]

In summary, because agents incorrectly believe that the negative leverage shock will be *more persistent*, they expect a much tighter future borrowing constraint leading to a much larger fall in land price than under rational expectations. When confronted with negative credit conditions, agents are pessimistic due to incorrect beliefs and this pessimism de-

[11] The learning model is stable enough that in this exercise we do not need to make use of any projection facility.

[12] Larger estimates for the gain parameter have been reported in Branch and Evans [5], Chakraborty and Evans [11], Malmendier and Nagel [37], Milani [40, 41]. Although there seems to be no empirical estimate of the gain parameter corresponding to actual forecasts of housing or land prices, the dataset exploited in Pancrazi and Pietrunti [42] could in principle be used to that purpose.

[13] In Figure 2, debt falls by much more than output. This implies that the debt-to-GDP ratio - a common definition of aggregate leverage - falls by a large amount as well.

presses consumption, investment and output much more than under rational expectations.

It is worth emphasizing that our assumption that agents overestimate the persistence of leverage agrees with forecasting evidence. Piazzesi and Schneider [43], using data from the Michigan Survey of Consumers, provide evidence that a "momentum" cluster exists, which groups households who were optimistic in the sense that they expected both credit conditions to improve and housing prices to increase as late as in the second part of the boom phase, in 2004-05. Using a different dataset provided by a professional forecasting firm, Pancrazi and Pietrunti [42] similarly document how forecasts made by financial experts in the 2000s follow a pattern that typically lacks mean-reversion dynamics, at least at a 80-quarter horizon.

3.2 Mechanism and Robustness Under Alternative Assumptions

To shed some light on the mechanism at work, we now conduct a counterfactual analysis in three steps. More specifically, we report output's response (*i*) under lower leverage, (*ii*) if variations in land price do not affect the borrowing constraint, (*iii*) if leverage is countercyclical. To measure how the leverage level matters for the response to a financial shock, we set $\overline{\Theta} \approx 0.73$, which is value of leverage observed in the first quarter of 1996 (the other values are as in Table 1), which leads to $B/Y \approx 0.34$ and $QL/Y \approx 0.48$. According to most measures, this corresponds to the starting point of the housing price "bubble". The lower level of leverage implies that both the debt-to-GDP and the land value-to-GDP are correspondingly lower than their averages over the 1996Q1-2008Q4 period. Panel a) in Figure 3 replicates the same experiment as above, when a -5% shock to leverage hits the economy and ρ_θ is believed to equal 0.9904 while its true value is 0.9756. Direct comparison of Figures 2 and 3 - panel a) - reveals that higher leverage increases the effect of the shock on aggregates under learning both in absolute and in relative terms. In this sense, the larger the level of leverage the deeper the recession that follows after a negative financial shock.[14]

It is important to stress that the economy's responses to a leverage shock are larger under learning because the land price forecast interacts with the borrowing constraint. To illustrate this fact, we also report the responses of output when the land price is assumed to be fixed in the borrowing constraint, that is, when (3) is replaced by:

$$\Theta_t Q L_{t+1} \geq (1+R) B_{t+1} \tag{21}$$

while the land price is allowed to respond according to the Euler condition (6). Panel b) in Figure 3 reports the response of output, which is about the same under learning and under rational expectations, in contrast to Figure 2. This unambiguously shows that it is the interaction of land price with the borrowing constraint that generates our results under

[14] Output's response and capital's response are proportional so we report only the former and not the latter.

Figure 2: Responses to a −5% Leverage Shock under Learning (Solid Red) and RE (Dotted Blue); % Deviations From Steady-State; Parameter Values in Table 1.

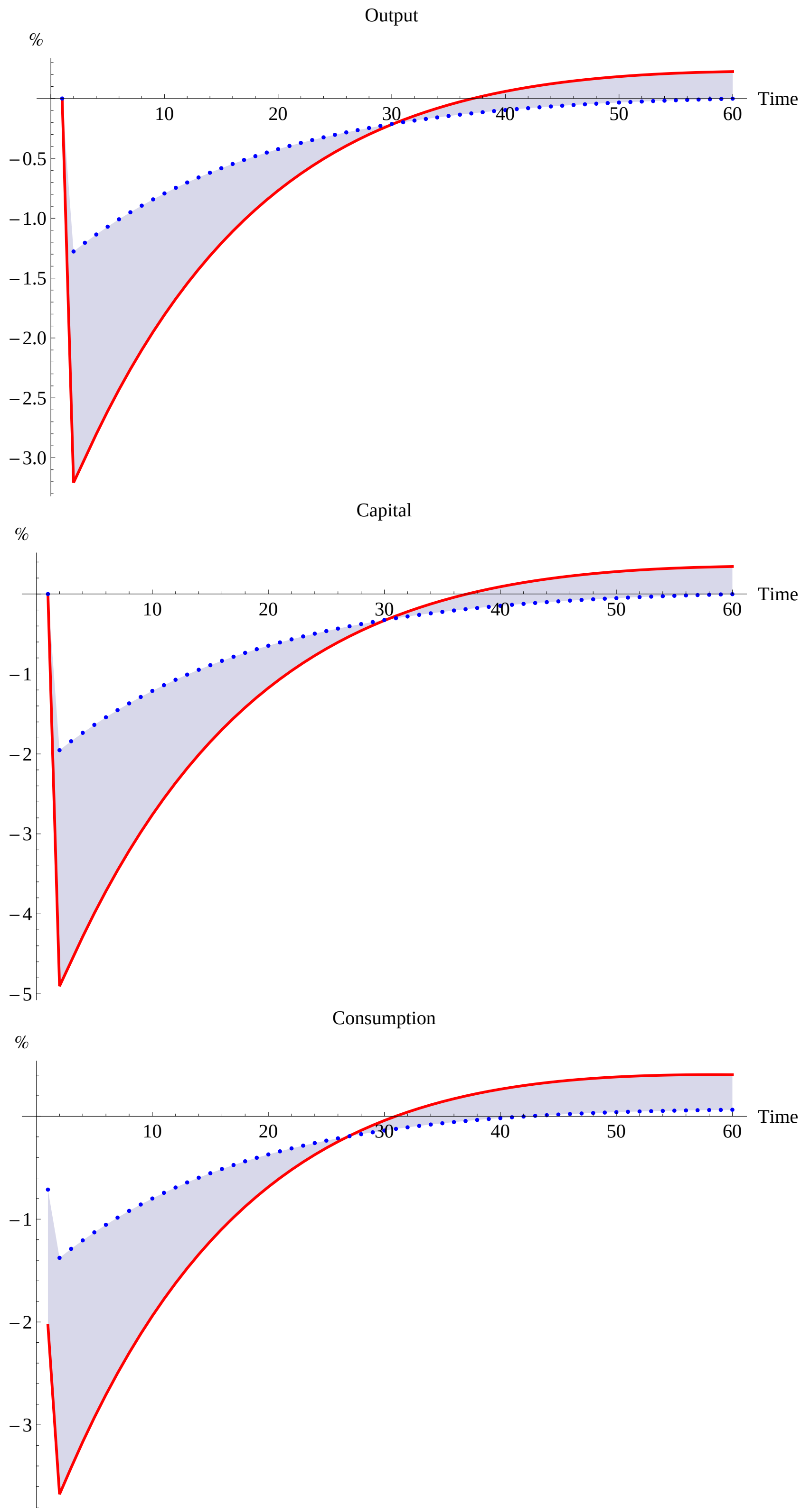
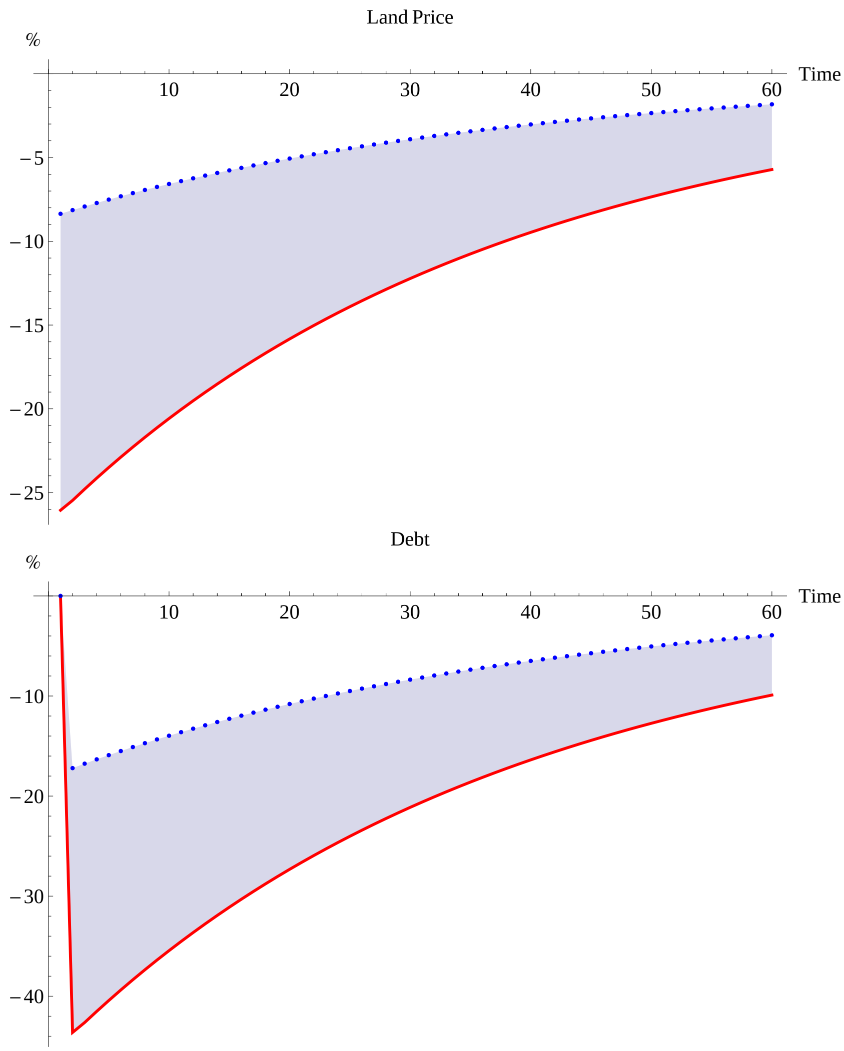


Figure 2 continued.

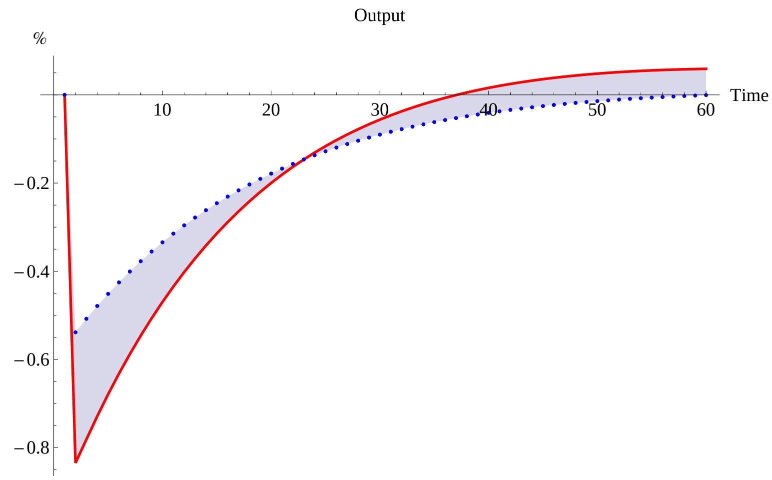


learning. That is to say, if a declining leverage would not translate into less collateral and less borrowing, our learning economy would behave like a RE economy. To the extent that borrowing constraints reflect the market value of collateralizable assets, such as land or real estate, the latter configuration may appear unrealistic. It turns out, however, that a simple macroprudential policy can come very close to eliminating the effect of land price swings on the borrowing constraint and, hence, the amplification of leverage shocks due to learning. To show this, we now ask the counter-factual question: what would be the reaction of the economy to the same shock, under the same parameter values but with the leverage being now mildly countercyclical?[15] More precisely, we assume that $\varepsilon = -0.5$ while the other parameters are kept unchanged and set as in Table 1. The economy's responses are reported in panel c) of Figure 3. The comparison of Figures 2 and 3 - panel c) - shows

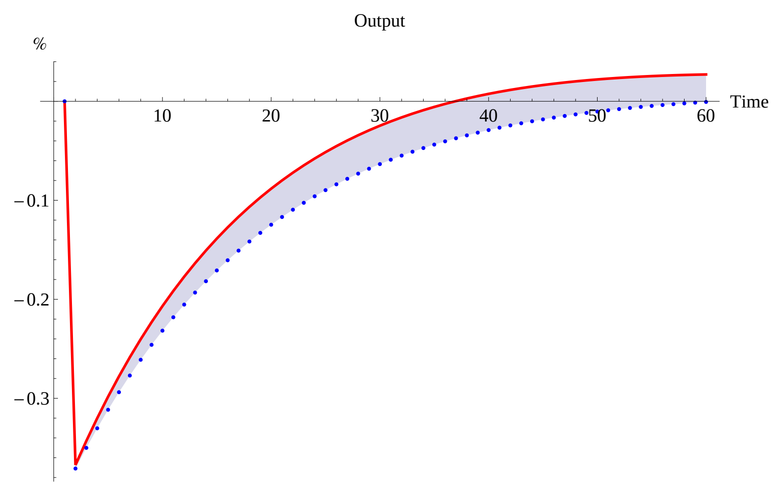
[15]This feature could possibly be enforced by appropriate regulation of credit markets. Alternatively, Appendix A.1 shows how it arises if government sets procyclical taxes on the recovery value of collateral.

Figure 3: Reduced Output Response to a -5% Leverage Shock
(% Deviations From Steady-State; Learning: Solid Red; RE: Dotted Blue)

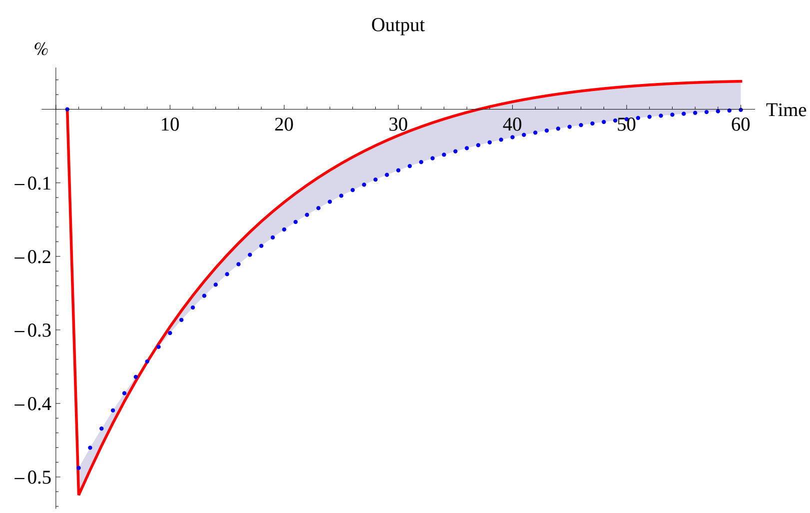
a) Lower Steady-State Leverage $\overline{\Theta} = 0.73$



b) Fixed Land Price



c) Countercyclical Leverage $\varepsilon = -0.5$



that countercyclical leverage dampens by a significant margin the responses to financial shocks and it brings learning dynamics closer to its rational expectations counterpart. As a consequence, a much smaller recession follows a negative leverage shock: though agents anticipate a too large deleveraging effect because they overestimate the persistence of the adverse leverage shock, the land price fall now triggers an *increase* in countercyclical leverage, which dampens the impact of the negative shock. In other words, the negative shock to the exogenous component of leverage is now dampened by an increase of the endogenous part, which is itself triggered by a fall in land price. As a consequence, borrowing falls only moderately and the resulting recession is much smaller and similar under learning and under RE. Our experiment thus suggests that simple rules that enforce countercyclical leverage are potentially powerful. In addition, comparing panels b) and c) of Figure 3 reveals that the dampening effect of countercyclical leverage essentially works as if land price variations were close to being neutralized in the borrowing constraint.

To assess the robustness of the findings reported in Section 3.1, we now relax several assumptions one by one.[16] First, we adopt the timing assumption that is implied by the margin requirement interpretation of the borrowing constraint (Aiyagari and Gertler [3]). That is, borrowing is limited to the current market value of collateral as opposed to tomorrow's market value. In other words, we replace both (3) by $\tilde{\Theta}_t Q_t L_{t+1} \geq (1+R)B_{t+1}$ and (9) by $\tilde{\Theta}_t \equiv \Theta_t \{Q_t/Q\}^{\varepsilon}$. Next, we relax the small-open economy assumption and introduce heterogeneous agents and endogenous interest rate. Finally, we examine the impact of assuming inelastic leverage on our results.

In Table 2, we report output amplification that obtains at impact under learning, as a fraction of that under rational expectations. For example, the impact of a -5% leverage shock on output's deviation (from its steady-state value, in percentage terms) is about -3.2% under learning and -1.3% under RE (see Figure 2) when parameters are set according to Table 1. Therefore, the first column of Table 2 reports that the ratio is about $2.51 \approx 3.2/1.3$.

Table 2. Output Amplification Factor Under Learning And Misperception

Benchmark	Margin	Heterogeneous	$\varepsilon = 0$	$\varepsilon = -0.5$	Fixed Land Price
2.51	2.55	2.31	1.28	1.08	0.99

The second column in Table 2 reports the ratio in the margin requirement model. The third column in Table 2 reports relative output amplification in a closed-economy version of the model with heterogeneous agents and endogenous interest rate (see Appendix A.3 for modeling details). Finally, the fourth column reports amplification when the procyclicality of leverage is shut down, that is, when $\varepsilon = 0$. Finally, the fifth and last columns reports output amplification levels that correspond to panels c) and b) of Figure 3 discussed above.

[16] Our results are unaltered when relative risk aversion moves away from 1 within reasonable bounds so we abtract from discussing that point.

Direct inspection of Table 2 shows that our main findings are robust both to changes in the timing assumption and to endogenizing the interest rate. Output amplification is quantitatively similar across different models and this turns out to be the case for the other variables (not reported) as well.

To further stress that initial beliefs about the persistence of the leverage process are important for our results, we now report the amplification that comes from the self-referentiality of learning alone, *without* the assumption that agents over-estimate persistence. To compare the volatility under learning relative to rational-expectations we proceed as follows. The learning model is initialized with the beliefs centered at the RE equilibrium, simulated for 400 periods to allow estimates to converge to its long-run distribution and, finally, run next for 60 quarters to assess the volatility of endogenous variables under learning. Table 3 reports those volatilities. More precisely, the numbers in Table 3 show the ratio of variances of deviations from the steady state under learning relative to full information case for alternative values of the CG gain. Table 3 makes clear that the amplification reported in Table 2 is the result of the assumption that ρ_θ is overestimated under learning. When learning agents are assumed to know the true value of leverage persistence, amplification is modest, especially if the gain parameter is not large. We also use those stochastic simulations to assess the frequency of values for ρ_θ that are larger than or equal to 0.9904, which is our calibrated value. This rare event has a non-negligible frequency of around 0.85 percent in the benchmark scenario such that $\nu = 0.004$. This means that such high values for persistence would be observed on average every 30 years.

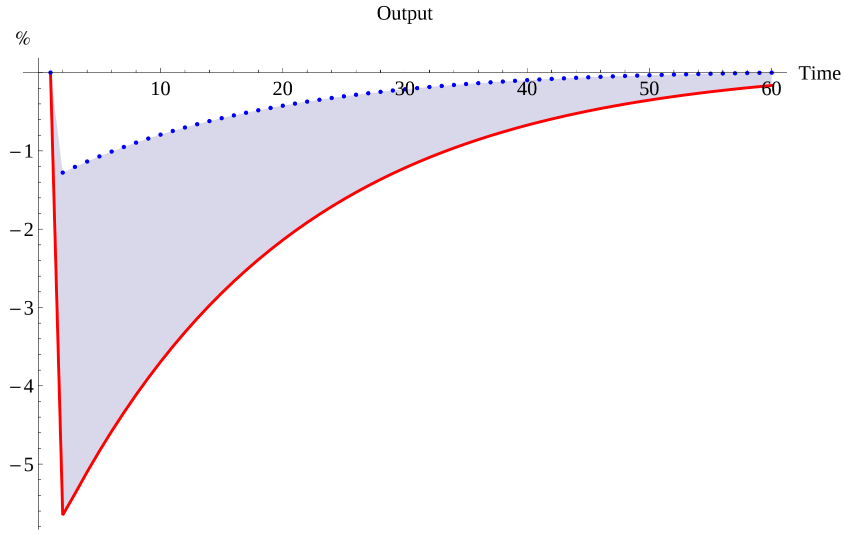
Table 3. Amplification Factor Under Learning Alone

Variable	Benchmark: $\nu = 0.004$	$\nu = 0.01$	$\nu = 0.04$
Output	1.037	1.07	1.15
Capital	1.037	1.07	1.15
Consumption	1.051	1.10	1.19
Land price	1.054	1.09	1.23
Debt	1.029	1.06	1.16

3.3 Learning the Mean Level of Leverage

The purpose of this section is to report the outcome of our second experiment. We now subject the economy to the same shock that was considered in Section 3.1 but we assume that agents overestimate both the leverage shocks' persistence and the mean leverage level. That is, agents believe that $\rho_\theta = 0.9904$ while the true value is 0.9756. We also set the RE value $\overline{\Theta} = 0.88$ just as in Table 1 and we assume learning agents believe that $\overline{\Theta} = 0.92$, which is the average of leverage in the data over 2001Q1-2008Q4, the period over which most of the land price "bubble" materialized. Starting with such wrong beliefs about both parameters governing the AR(1) process for leverage, ρ_θ and $\overline{\Theta}$, agents then update using the algorithm

Figure 4: Larger Output Response to a -5% Leverage Shock When Agents Learn About Both Persistence and Level of Leverage: Belief set to $\overline{\Theta} = 0.92$
(% Deviations From Steady-State; Learning: Solid Red; RE: Dotted Blue)



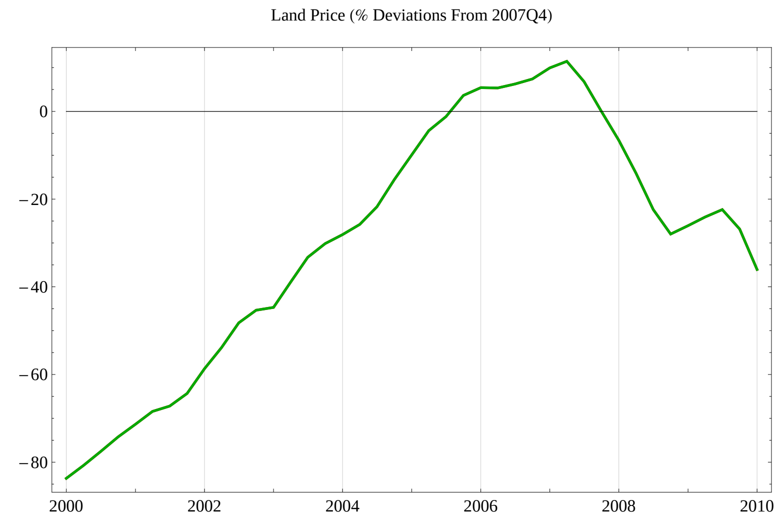
described in Section 3. The responses of output to a -5% shock to leverage is reported in Figure 4, which features substantially larger deviations under CG learning compared to the RE benchmark. The fall in output under CG learning is more than 4.4 times larger than that under RE, compared to 2.5 times in Figure 2.[17] Overestimating the mean level of leverage on top of its persistence adds a extra kick to the amplification mechanism that arises under learning. Although, in principle, errors about the long-run leverage level could result in endogenous persistence under iid shocks, this effect turns out to be unimportant in our quantitative analysis developed in Section 4. In contrast, Kozlowski, Veldkamp and Venkateswaran [34] develop an interesting setting where endogenous persistence arises under reasonable calibrations. Taken together, our two experiments suggest that learning amplifies negative shocks to leverage such as the one observed in 2008Q4. A natural question that we now ask is whether or not the learning model accords better with the actual path followed by the US output over the Great Recession than its rational expectations counterpart.

4 Does Learning Help Account For The Great Recession?

The purpose of this section is to argue that learning is a plausible mechanism that helps explaining the magnitude of the Great Recession. More precisely, we show that the learning model predicts both a boom in the early 2000s and a sizeable recession beginning

[17] Alternatively, setting $\varepsilon = 0$ implies that the relative output amplification is about 1.61 under learning, compared to about 1.28 according to Table 2 when learning agents know the long-run leverage level.

Figure 5: Effects of Land Price Shocks on Land Price



2007Q3, while the model with rational expectations generates a fall in output up to 2007Q3 followed by an expansion which are both at odds with the data. All parameter values are set according to Table 1 and we reverse-engineer the land price i.i.d. shocks that are required to replicate the observed path for land price, as shown in Figure 5.[18]

To derive leverage shocks, we use the data provided by Boz and Mendoza [6] for the period 1975Q1-2010Q1 to decompose the exogenous and endogenous components of leverage using definition (9). That is, we obtain the exogenous component Θ_t by removing the part of the leverage that is explained by land price. We then estimate AR(1) processes on the log of Θ_t both under CG and under OLS and we compute the residuals from such estimated processes that we use to feed our model with.[19] The resulting innovations, reported in Figure 6, do not significantly differ, which indicates that our results derived below do not rely on disturbances being different under learning and under rational expectations.

Figure 6 makes clear why the model is unable to explain the Great Recession when fed with only the actual leverage innovations: the fall in land price that starts in early 2007 generates positive shocks in 2007 and 2008 that produce a large expansion that is hardly reversed when the negative shock happens in 2008Q4. Consistent with the literature, we find that the model requires another source of disturbance to accord with the data and this is why we use i.i.d. land price shocks to replicate the observed land price behavior as shown in Figure 7.

Figure 7 reports the OLS and CG estimates of ρ_θ. The OLS estimate is obtained from a univariate regression using the data over sample period 1975Q1-2010Q1. This is consistent

[18] Although we set ρ_τ to zero so as to isolate the effect of learning a positive ρ_θ, similar results obtain when the land price shock process has some autocorrelation. More precisely, unreported simulations show that assuming large persistence in the process driving land price shocks helps the RE model to predict a boom prior to 2007 but still does not produce a realistically large recession in 2008-09 under RE.

[19] We have also checked that ARMA processes do not better describe our ε-adjusted data on leverage.

Figure 6: Estimated Leverage Innovations Over Time
(Constant-Gain Learning: Solid Red; RE: Dotted Blue)

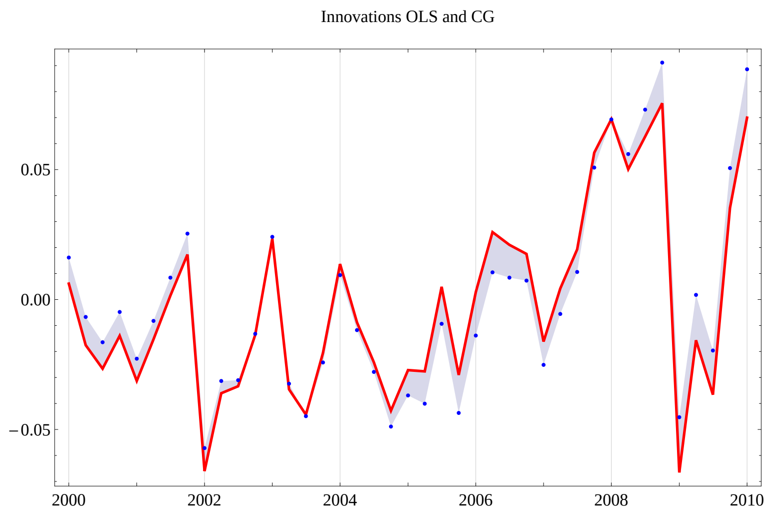
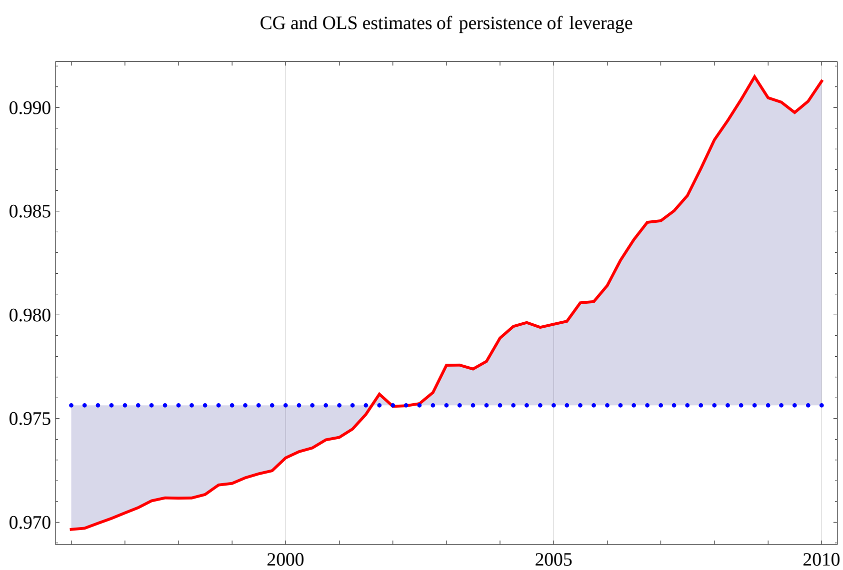


Figure 7: Estimates of Leverage Persistence ρ_θ Over Time
(Constant-Gain Learning: Solid Red; RE: Dotted Blue)



with the notion that RE agents know the process governing leverage. The CG estimate is obtained from the VAR estimation when learning agents use the full model to forecast and update their beliefs in real time. Figure 7 partly replicates panel b) of Figure 1, to which we also add the OLS estimate which is $\rho_\theta \approx 0.9756$. Although Figure 7 may seem to imply that learning does not converge to rational expectations, it does so in the whole sample period and also in theory, as the CG estimates converge in distribution to the RE estimates. Figure 7 shows that learning agents overestimate the autocorrelation parameter consistently after 2002 and, more importantly, that their estimate drifts toward unit root.[20] In particular, the model predicts that the VAR estimate of ρ_θ is around 0.9904 in 2008Q3 (which is the value agents are assumed to use in Section 3.1 to forecast about 2008Q4) and 0.9914 in 2008Q4 before falling down. This means that when the negative leverage shock of 2008Q4 occurs, learning agents think of leverage as essentially having unit root and they expect any innovations at that time to be close to permanent. As a consequence, deleveraging is much more severe than what would happen under RE and the resulting outcome is reported in Figure 8.

Figure 8 shows that the model predicts a sizeable recession during the Great Recession period, with a fall in output about 4.7% between 2007Q3 and 2010Q1 and a significant boom prior to that. The learning model explains almost all the actual output drop reported by the NBER to be about 5%, and it does much better than the RE model. The latter predicts a continuous fall in output from 2000 to 2007 followed by a significant expansion over the 2007-2010 period, both features being at odds with the data. The major reason behind such a stark contrast is that the RE model does not allow for belief revision, while the latter feature precisely explains why learning agents were overestimating the impact of the negative leverage shock in 2008Q4 and why this leads to a sizeable output fall at that time in the model. Given that the model is overly too simple to fully account for the data, our main claim here is that the learning model explains a sizeable fraction of the Great Recession, while the RE model does not.[21] To fully account for the fall in observed output during the Great Recession, the model would need labor productivity to fall by about half a percentage point, which is in the ballpark of estimated values.

In view of our theoretical results on countercyclical leverage reported in Section 3.2, it is natural to ask whether the fact that leverage is procyclical aggravates the recession, which is what intuition suggests. As a counter-factual, Figure 9 reports the output response that occurs under mildly countercyclical leverage, with $\varepsilon = -0.5$ (implying that a 10% fall in land price increases leverage by 5%). Comparing Figures 8 and 9 suggests that a simple macroprudential policy may substantially attenuate the impact of leverage shocks on aggregates under learning. In Figure 9, output would have decreased in 2010Q1 by only half a percent relative to its level in 2007Q4, a very moderate fall compared to what really happened.

5 Conclusion

A large part of business-cycle theory relies on the assumption that agents know all parameters governing the stochastic process underlying the disturbances that hit the economy. This paper has shown how relaxing such an assumption in a simple model predicts that the

[20] Before 2002 agents would underestimate the persistence of the leverage shocks.

[21] Under the assumption that $\varepsilon = 0$, the fall in output is still about -2.3%.

Figure 8: Model-Generated Great Recession
(Constant-Gain Learning: Solid Red; RE: Dotted Blue)

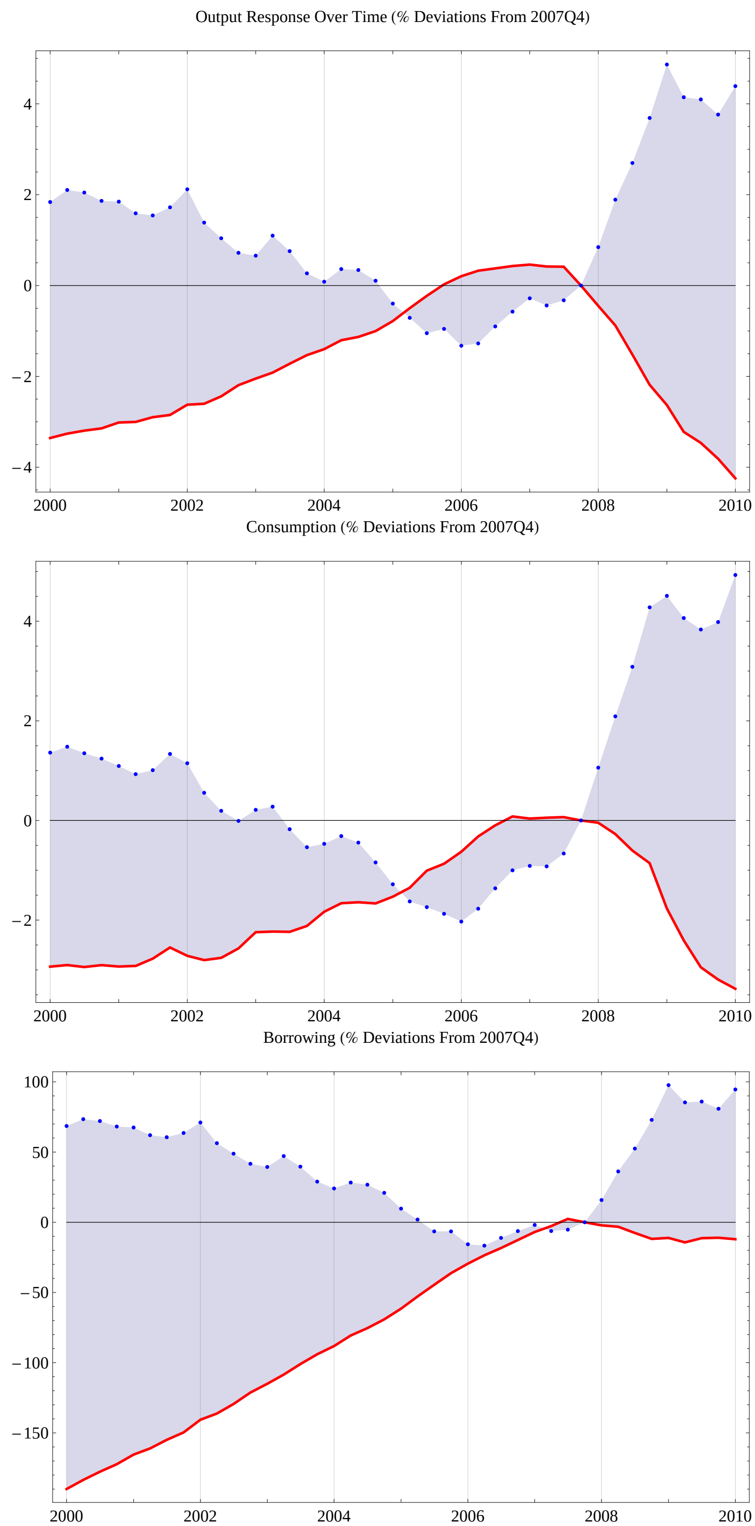
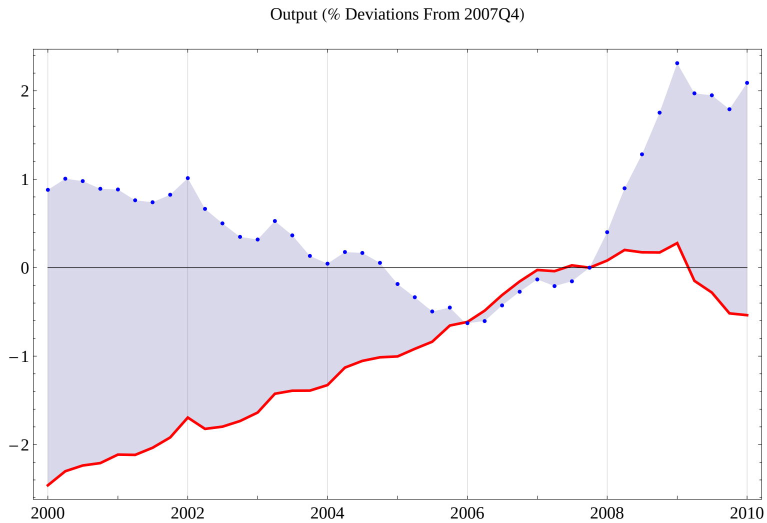


Figure 9: Counter-factual Model-Generated Recession when $\varepsilon = -0.5$
(Constant-Gain Learning: Solid Red; RE: Dotted Blue)



economy's aggregates respond very differently to financial shocks when agents are gradually learning their environment, compared to rational expectations. More specifically, our theoretical experiments with a calibrated model suggest that reasonable parameter configurations can lead to much larger amplification of the impact of shocks to leverage. This is for instance the case when learning agents overestimate either the autocorrelation parameter governing the persistence of leverage shocks or the long-run level of leverage. We have provided evidence that both cases are not inconsistent with the US data prior to the Great Recession, when borrowers probably believed that credit collateralized by real estate assets was being permanently extended by the financial sector. In addition, the more empirically oriented counterparts of our two theoretical experiments are informative about which assumption better stands against the data. Our preferred model with agents updating their estimates of the long-run level of leverage and of leverage persistence as new data arrive is not unsuccessful in that respect. In particular, it predicts a sizeable fall in output from peak to trough, as reported by the NBER, whereas the rational expectations model predicts a continued, counter-factual expansion in 2008 and 2009. Our analysis could of course be extended to incorporate other margins (e.g. capacity utilization, labor productivity) and would be useful to measure the contribution of learning in middle-scale models like that proposed by Christiano, Eichenbaum, Trabandt [12]. In addition, whether the boom-bust and cyclical patterns in Garriga, Manuelli, Peralta-Alva [19] and He, Wright, Zhu [24] are stable under learning remains an open question that should be addressed.

We believe that the main results of this paper may also be relevant for studying other settings. For example, they are suggestive about how one could try to measure to what extent unemployment variations are driven by beliefs formed by firms about either the persistence of demand shocks or the steady-state level of demand, or both. Monetary policy perhaps provides still another example in which the beliefs formed by the private sector about the persistence or about the long-run stance of monetary policy matter, in particular when the economy hits the zero lower bound, as they could change the effects of policy on the economy. These are but a few examples for which extensions of the setting used in this paper could lead to fruitful research. In the same vein, another potential avenue for future research would be to model how perceptions about the process driving uncertainty shocks affect how those shocks propagate in the real economy. This requires solving higher-order approximations of nonlinear models and we believe this calls for further inquiries.

A Appendix

A.1 Elastic Leverage: Simple Micro-Foundations

This section derives some simple micro-foundations for the assumption of elastic leverage captured in (9). The case when leverage is procyclical (that is, $\varepsilon > 0$) obtains in a setting with ex-post moral hazard and costly monitoring similar to Aghion et al. [2, p.1391]. Suppose that the borrower has wealth QL and has access to investment opportunities, which can be financed by credit in the amount B. If the borrower repays next period, his income is $I - (1+R)B$, where I is whatever income was generated by investing. If the borrower defaults next period, his income is now $I - pQL$, assuming that he loses his collateral with some probability p, which represents for example the frequency of foreclosures. Strategic default is avoided provided that $I - (1+R)B \geq I - pQL$, that is, if $pQL \geq (1+R)B$. The lender incurs a cost $C(p)L$ when collecting collateral, with $C'(p) > 0$ and $C''(p) > 0$, and he chooses the optimal monitoring policy by solving:

$$\max_p pQL - C(p)L \tag{22}$$

which gives $Q = C'(p)$. The higher the land price, the larger the incentives to increase effort to collect collateral. Assuming now that the cost function is $C(p) = \phi p^{1+1/\varepsilon}/(1+1/\varepsilon)$, with $\varepsilon > 0$, gives that $p = (Q/\phi)^{\varepsilon}$. Setting the scaling parameter $\phi = Q^{*}\Theta^{-1/\varepsilon}$, where Q^{*} is steady-state land value and Θ is leverage, gives (9). Therefore, ex-post moral hazard leads to procyclical leverage.

In contrast, countercyclical leverage obtains if government implements procyclical taxes as follows. Suppose now that the lender gets $(1-\tau)pQL - C(p)L$ when monitoring, where $1 \geq \tau \geq 0$ is the tax rate. Under the assumption that the cost function is isoelastic, the optimal p is now $p = ((1-\tau)Q/\phi)^{\varepsilon}$. If the government sets time-varying taxes such that $1 - \tau = (Q/\phi)^{-\eta/\varepsilon - 1}$, for some $\eta \geq 0$, then it follows that $p = (Q/\phi)^{-\eta}$ and that leverage is countercyclical. Note that this happens provided that the tax rate goes up when the land price goes up.

A.2 Log-Linearized Model in Levels

We now derive the log-linearized version of the set of equations (2)-(8) defining, together with the laws of motion of leverage $\Theta_t = \overline{\Theta}^{1-\rho_\theta}\Theta_{t-1}^{\rho_\theta}\Xi_t$ and of land price shock $T_t = T_{t-1}^{\rho_\tau}\Psi_t$, intertemporal equilibria near steady state. In all equations below, lowercase letters denote logs and $\tilde{x}_t$ denotes the log of X_t/X, where X is the steady-state value of X_t. For example, $\tilde{k}_t \equiv k_t - k$, with $k_t = \log(K_t)$ and $k = \log(K)$, so that lowercase variables without time subscript are steady-state levels in log. Eliminating from the other equations Φ_t by using (8) and N_t by using (5), one gets the following linearized equations corresponding to (2)-(8) and the exogenous states' transition equations, respectively:

$$\tfrac{K}{Y}\tilde{k}_t - \tfrac{B}{Y}\tilde{b}_t \quad = \quad -\tfrac{C}{Y}\tilde{c}_{t-1} - \tfrac{(1+R)B}{Y}\tilde{b}_{t-1} + \left(\alpha + \alpha\tfrac{1-\alpha-\gamma}{\chi+\alpha+\gamma} + (1-\delta)\tfrac{K}{Y}\right)\tilde{k}_{t-1} \tag{23}$$

$$\tilde{b}_t = (1+\varepsilon)E_{t-1}[\tilde{q}_t] + \tilde{\theta}_{t-1} \tag{24}$$

$$\frac{C/Y}{X/Y}\tilde{c}_t + \frac{1}{\sigma}\tilde{\lambda}_t = \frac{1-\alpha-\gamma}{X/Y}\tilde{n}_t \tag{25}$$

$$\begin{aligned} \tilde{q}_t + \tilde{\tau}_t + \tilde{\lambda}_t(1-\mu\overline{\Theta}) \quad &= \quad \beta E_t[\tilde{\tau}_{t+1}] + E_t[\tilde{q}_{t+1}]\left(\beta + \overline{\Theta}(1+\varepsilon)(\mu-\beta)\right) \\ &+ \quad E_t[\tilde{\lambda}_{t+1}]\left(\beta(1-\overline{\Theta}) + \gamma\beta\tfrac{Y}{Q}\right) \\ &+ \quad \alpha\gamma\beta\tfrac{Y}{Q}(1+\tfrac{1-\alpha-\gamma}{\chi+\alpha+\gamma})E_t[\tilde{k}_{t+1}] + \tilde{\theta}_t\overline{\Theta}(\mu-\beta) \end{aligned} \tag{26}$$

$$\tilde{\lambda}_t \quad = \quad E_t[\tilde{\lambda}_{t+1}]\left(\beta(1-\delta) + \alpha\beta\tfrac{Y}{K}\right) + \alpha\beta\tfrac{Y}{K}(\alpha - 1 + \alpha\tfrac{1-\alpha-\gamma}{\chi+\alpha+\gamma})E_t[\tilde{k}_{t+1}] \tag{27}$$

$$(\mu-\beta)\tilde{\phi}_t = \mu\tilde{\lambda}_t - \beta E_t[\tilde{\lambda}_{t+1}] \tag{28}$$

$$\tilde{\theta}_t = \rho_\theta\tilde{\theta}_{t-1} + \xi_t \tag{29}$$

$$\tilde{\tau}_t = \rho_\tau\tilde{\tau}_{t-1} + \psi_t \tag{30}$$

where $\tilde{\tau}_t = \log(T_t) = \tau_t$ as the steady state value of T_t is set to one by assumption.

Define $P_t' \equiv (b_t, k_t, \theta_t, \tau_t)$ and $S_t' = (c_t, q_t, \lambda_t, \phi_t)$ the vectors of predetermined and jump variables in log, respectively. Then equations (23)-(29) can be decomposed into two subsystems, each pertaining to P_t and S_t. The first block composed of (23), (24), (29) and (30) can be written:

$$M_0P_t = M_1S_{t-1} + M_2E_{t-1}[S_t] + M_3P_{t-1} + N_0 + V_1\xi_t + V_2\psi_t \tag{31}$$

where:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{B}{Y} & \frac{K}{Y} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{C}{Y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 & 1+\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -(1+R)\frac{B}{Y} & \alpha + \alpha\frac{1-\alpha-\gamma}{\chi+\alpha+\gamma} + (1-\delta)\frac{K}{Y} & 0 & 0 \\ 0 & 0 & \rho_\theta & 0 \\ 0 & 0 & 0 & \rho_\tau \end{pmatrix},$$

$$N_0 = \begin{pmatrix} b-(1+\varepsilon)q-\theta \\ \frac{K}{Y}k+\frac{RB}{Y}b+\frac{C}{Y}c-(\alpha+\alpha\frac{1-\alpha-\gamma}{\chi+\alpha+\gamma}+\frac{(1-\delta)K}{Y})k \\ (1-\rho_\theta)\theta \\ 0 \end{pmatrix}$$

and $V_1' = (0,0,1,0)$, $V_2' = (0,0,0,1)$. Note that (23) and (24) are the linearized, lagged versions of (2) and (3).

The second block (25)-(28) can be written:

$$M_4 S_t = M_5 E_t[S_{t+1}] + M_6 P_t + M_7 E_t[P_{t+1}] + N_1 \tag{32}$$

where:

$$M_4 = \begin{pmatrix} 0 & 1 & 1-\mu\overline{\Theta} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{C/Y}{X/Y} & 0 & 1/\sigma & 0 \\ 0 & 0 & -\mu & \mu-\beta \end{pmatrix},\ M_5 = \begin{pmatrix} 0 & \beta+\overline{\Theta}(1+\varepsilon)(\mu-\beta) & \beta(1-\overline{\Theta})+\gamma\beta\frac{Y}{Q} & 0 \\ 0 & 0 & \beta(1-\delta)+\alpha\beta\frac{Y}{K} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 \end{pmatrix},$$

$$M_6 = \begin{pmatrix} 0 & 0 & \overline{\Theta}(\mu-\beta) & -1 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha(1-\alpha-\gamma)}{(\chi+\alpha+\gamma)X/Y} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ M_7 = \begin{pmatrix} 0 & \alpha\gamma\beta\frac{Y}{Q}(1+\alpha\frac{1-\alpha-\gamma}{\chi+\alpha+\gamma}) & 0 & \beta \\ 0 & \alpha\beta\frac{Y}{K}(\alpha-1+\alpha\frac{1-\alpha-\gamma}{\chi+\alpha+\gamma}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} q(1-\beta-\overline{\Theta}(1+\varepsilon)(\mu-\beta))+\lambda(1-\overline{\Theta}\mu-\beta(1-\overline{\Theta})-\beta\gamma\frac{Y}{Q})-\alpha\gamma\beta\frac{Y}{Q}(1+\alpha\frac{1-\alpha-\gamma}{\chi+\alpha+\gamma})k-\overline{\Theta}(\mu-\beta)\theta \\ \lambda(1-\beta(1-\delta)-\alpha\beta\frac{Y}{K})-\alpha\beta\frac{Y}{K}(\alpha-1+\alpha\frac{1-\alpha-\gamma}{\chi+\alpha+\gamma})k \\ \frac{C/Y}{X/Y}c+\frac{\lambda}{\sigma}-\frac{\alpha(1-\alpha-\gamma)}{(\chi+\alpha+\gamma)X/Y}k \\ (\mu-\beta)(\phi-\lambda) \end{pmatrix}.$$

Finally, substituting the expression of P_t from (31) in (32) and piling up the resulting two blocks of equations allows one to rewrite the system as:

$$X_t = \mathbf{A}X_{t-1} + \mathbf{B}E_{t-1}[X_t] + \mathbf{C}E_t[X_{t+1}] + \mathbf{N} + \mathbf{D}\xi_t + \mathbf{F}\psi_t \tag{33}$$

where $X_t' = \text{vec}(S_t', P_t')$ and:

$$\mathbf{A} = \begin{pmatrix} M_4^{-1}M_6M_0^{-1}M_1 & M_4^{-1}M_6M_0^{-1}M_3 \\ M_0^{-1}M_1 & M_0^{-1}M_3 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} M_4^{-1}M_6M_0^{-1}M_2 & O_4 \\ M_0^{-1}M_2 & O_4 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} M_4^{-1}M_5 & M_4^{-1}M_7 \\ O_4 & O_4 \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} M_4^{-1}M_6M_0^{-1}V_1 \\ M_0^{-1}V_1 \end{pmatrix},\ \mathbf{F} = \begin{pmatrix} M_4^{-1}M_6M_0^{-1}V_2 \\ M_0^{-1}V_2 \end{pmatrix},$$

$$\mathbf{N} = \begin{pmatrix} M_4^{-1}N_1 + M_4^{-1}M_6M_0^{-1}N_0 \\ M_0^{-1}N_0 \end{pmatrix}$$

where O_4 is a 4-by-4 zeroes matrix.

A.3 Extension: Closed-Economy Model with Endogenous Interest Rate

The purpose of this appendix is to show that, similar to the open-economy model developed in Section 2, learning generates amplification in a closed-economy version with domestic borrowers and lenders and endogenous interest rate.

Let us now assume that lenders are domestic agents (instead of foreign countries as in Section 2), whose unique role is to provide loans to borrowers. Following Iacoviello [26], lenders derive utility from consumption and land holdings, and they get interest income from last period's loan payments. As discussed in Pintus and Wen [44], lenders may be interpreted as financial intermediaries. The representative lender solves:

$$\max E_0 \sum_{t=0}^{\infty} \mu^t \left\{ \frac{(C_t^l)^{1-\sigma_c} - 1}{1-\sigma_c} + \psi \frac{(L_t^l)^{1-\sigma_l} - 1}{1-\sigma_l} \right\} \tag{34}$$

with σ_c, σ_l, ψ all strictly greater than zero and $\mu \in (0,1)$, subject to the budget constraint:

$$C_t^l + Q_t(L_{t+1}^l - L_t^l) + B_{t+1} = (1+R_t)B_t \tag{35}$$

where C_t^l and L_t^l denotes the lender's consumption and land holdings, respectively, Q_t is the land price, B_{t+1} is the new loan. The interest rate R_t is now endogenous and it is determined by the equality between the demand and supply of loans.

The first-order conditions obtained from (34)-(35) with respect to consumption, land, and lending are, respectively:

$$(C_t^l)^{-\sigma_c} = \chi_t \tag{36}$$

$$\chi_t Q_t = \mu E_t[\chi_{t+1} Q_{t+1}] + \mu\psi (L_{t+1}^l)^{-\sigma_l} \tag{37}$$

$$\chi_t = \mu E_t[\chi_{t+1}(1+R_{t+1})] \tag{38}$$

where χ_t is the Lagrange multiplier of constraint (35) in period t.

Assuming that lenders' utility is linear in consumption (that is, $\sigma_c = 0$), one gets from (36) that in any rational expectations equilibrium $\chi_t = 1$ for all $t \geq 0$ so that, in view of (38), the interest factor is constant and given by $1 + R = 1/\mu$. As in the small-open economy model developed in Section 2, the interest rate is constant and the land price moves over time.

The borrower side of the model is still described by (1), (2) and (3), as in Section 2, with the addition that the total amount of land is now divided between lenders and borrowers according to:

$$L_t + L_t^l = \bar{L}.$$

where $\bar{L}$ is the fixed supply of land. How exactly is land divided depends on both the sequence of land price and the lender's preferences, as reflected in the first-order condition (37). In addition, the representative borrower's first-order conditions are given by (4)-(8). As in Section 2, if $\mu \in (\beta, 1)$, then the borrower's credit constraint (3) is binding. Therefore, the main difference is that the closed-economy model allows some reallocation of land from lenders to borrowers when a shock hits the economy. This is why collateral constraints generate boom-bust patterns even when both the land price and the interest rate are constant over time (see Pintus and Wen [44] for a complete analysis). Under our calibration (see

Figure 10: Ouput Response to a -5% Leverage Shock in Model with Endogenous Interest Rate (Learning: Solid Red; RE: Dotted Blue)

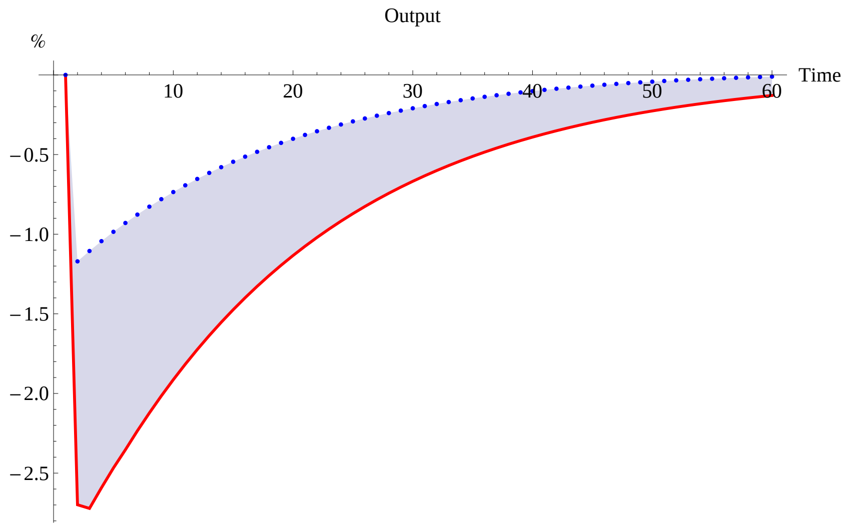


Table 1), however, the effect of land reallocation is quantitatively unimportant because the land share γ is reasonably small. To ease comparison with Figure 2, Figure 10 reports the response of output in the model when the endogenous interest rate is constant (that is, when $\sigma_c = 0$). Output amplification is more than twice larger under learning, compared to rational expectations. When the lender's utility for consumption no longer exhibits risk neutrality, output amplification remains much larger under learning provided that σ_c is not too large. For example, if we assume that the lender is less risk averse than the borrower and that $\sigma_c = 0.5$, output amplification is almost twice as big under learning. Such robustness reflects the result that in this class of models, the borrowing interest rate is not much volatile if the lender's utility function is between linear and logarithmic. It follows that amplification due to learning arises as long as lenders are not too risk averse.

A.4 Learning Procedure of VAR Model

A.4.1 VAR Estimation

Denoting $X_t' = \text{vec}(S_t', P_t')$ the system can be written as before:

$$X_t = \mathbf{A}X_{t-1} + \mathbf{B}E_{t-1}[X_t] + \mathbf{C}E_t[X_{t+1}] + \mathbf{N} + \mathbf{D}\xi_t + \mathbf{F}\psi_t \tag{39}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{F}$ and $\mathbf{N}$ are given in Appendix A.2. The rational expectations solution has a VAR form:

$$X_t = \mathbf{M}X_{t-1} + \mathbf{H} + \mathbf{G}\xi_t + \mathbf{J}\psi_t. \tag{40}$$

Given this form of equilibrium, the law of motion of endogenous variables can be represented using $E_{t-1}X_t = \mathbf{M}X_{t-1} + \mathbf{H}$ and $E_t X_{t+1} = \mathbf{M}X_t + \mathbf{H}$ as:

$$X_t = \mathbf{A}X_{t-1} + \mathbf{B}\left[\mathbf{M}X_{t-1} + \mathbf{H}\right] + \mathbf{C}\left[\mathbf{M}X_t + \mathbf{H}\right] + \mathbf{N} + \mathbf{D}\xi_t + \mathbf{F}\psi_t, \tag{41}$$

or

$$\begin{aligned} X_t &= [\mathbf{I} - \mathbf{CM}]^{-1}[\mathbf{A} + \mathbf{BM}]X_{t-1} + [\mathbf{I} - \mathbf{CM}]^{-1}[\mathbf{BH} + \mathbf{CH} + \mathbf{N}] \\ &+ [\mathbf{I} - \mathbf{CM}]^{-1}[\mathbf{D}\xi_t + \mathbf{F}\psi_t], \end{aligned}$$

Matrices **M**, **H**, **G**, and **J** are given by:

$$\mathbf{M} = [\mathbf{I} - \mathbf{CM}]^{-1}[\mathbf{A} + \mathbf{BM}] \tag{42}$$

$$\mathbf{H} = [\mathbf{I} - \mathbf{CM}]^{-1}[\mathbf{BH} + \mathbf{CH} + \mathbf{N}] \tag{43}$$

$$\mathbf{G} = [\mathbf{I} - \mathbf{CM}]^{-1}\mathbf{D} \tag{44}$$

$$\mathbf{J} = [\mathbf{I} - \mathbf{CM}]^{-1}\mathbf{F} \tag{45}$$

To estimate the VAR we represent the model as

$$X_t = \mathbf{\Omega} Z_{t-1} + \Sigma_t, \tag{46}$$

where $Z'_{t-1} = [1', X'_{t-1}]$ and $\mathbf{\Omega} = [\mathbf{H} \quad \mathbf{M}]$.

The estimator for Ω equals

$$\hat{\mathbf{\Omega}} = XZ'(ZZ')^{-1}, \tag{47}$$

and its time T estimates, $\hat{\mathbf{\Omega}}_T$, can be computed from

$$\hat{\mathbf{\Omega}}_T = \left(\frac{1}{T}\sum_{t=2}^{T} X_t Z'_{t-1}\right)\left(\frac{1}{T}\sum_{t=2}^{T} Z_{t-1}Z'_{t-1}\right)^{-1}. \tag{48}$$

The recursive OLS updating takes form of

$$\hat{\mathbf{\Omega}}_{T+1} = \hat{\mathbf{\Omega}}_T + \nu\left(X_{T+1} - \hat{\mathbf{\Omega}}_T Z_T\right) Z'_T \mathbf{R}^{-1}_{T+1} \tag{49}$$

and

$$\mathbf{R}_{T+1} = \mathbf{R}_T + \nu\left(Z_T Z'_T - \mathbf{R}_T\right) \tag{50}$$

given constant gain $\nu > 0$. Equations (49) and (50) show how the estimates of matrix $\mathbf{\Omega}$ are updated as new data become available. In the above expression, $X_{T+1} - \hat{\mathbf{\Omega}}_T Z_T$ corresponds to a forecast error made using last period estimates.

A.4.2 Learning

Assume agents re-estimate the consistency with the RE model each period and use their estimates to make forecasts. These forecasts affect the behavior of the economy through equation (39).

Agents' perceived low of motion is

$$X_t = \mathbf{M}X_{t-1} + \mathbf{H} + \Sigma_t = \mathbf{\Omega} Z_{t-1} + \Sigma_t. \tag{51}$$

The forecasts agents make use the estimates of this PLM over available data. Since X_t depends on agents' forecasts (so it is not available at time t regression) at time t agents have run the regression:

$$\begin{aligned} E_t X_{t+1} &= \mathbf{M}_{t-1}X_t + \mathbf{H}_{t-1} = \mathbf{\Omega}_{t-1}Z_t & (52) \\ E_{t-1}X_t &= \mathbf{M}_{t-2}X_{t-1} + \mathbf{H}_{t-2} = \mathbf{\Omega}_{t-2}Z_{t-1} & (53) \end{aligned}$$

where now we allow agents to depart from running simply OLS regression (least-squares learning) and use constant gain,

$$\begin{aligned} \mathbf{R}_t &= \mathbf{R}_{t-1} + \nu_t \left(Z_{t-1}Z'_{t-1} - \mathbf{R}_{t-1} \right) \\ \mathbf{\Omega}_t &= \mathbf{\Omega}_{t-1} + \nu_t \left(X_t - \mathbf{\Omega}_{t-1}Z_{t-1} \right) Z'_{t-1}\mathbf{R}_t^{-1}. \end{aligned}$$

Substituting in agents' expectations, we can write the actual law of motion as

$$X_t = \mathbf{A}X_{t-1} + \mathbf{B}\left[\mathbf{M}_{t-2}X_{t-1} + \mathbf{H}_{t-2}\right] + \mathbf{C}\left[\mathbf{M}_{t-1}X_t + \mathbf{H}_{t-1}\right] + \mathbf{N} + \mathbf{D}\xi_t + \mathbf{F}\psi_t \tag{54}$$

or

$$\begin{aligned} X_t &= \left[\mathbf{I} - \mathbf{C}\mathbf{M}_{t-1}\right]^{-1}\left[\mathbf{A} + \mathbf{B}\mathbf{M}_{t-2}\right] + \left[\mathbf{I} - \mathbf{C}\mathbf{M}_{t-1}\right]^{-1}\left[\mathbf{C}\mathbf{H}_{t-1} + \mathbf{B}\mathbf{H}_{t-2} + \mathbf{N}\right] \\ &+ \left[\mathbf{I} - \mathbf{C}\mathbf{M}_{t-1}\right]^{-1}\left[\mathbf{D}\xi_t + \mathbf{F}\psi_t\right] \end{aligned} \tag{55}$$

There is a mapping $\{\mathbf{M}, \mathbf{H}\} = T(\mathbf{M}, \mathbf{H})$ from PLM to ALM,

$$\begin{aligned} T_{\mathbf{M}}(\mathbf{M}, \mathbf{H}) &= \left[\mathbf{I} - \mathbf{C}\mathbf{M}\right]^{-1}\left[\mathbf{A} + \mathbf{B}\mathbf{M}\right] & (56) \\ T_{\mathbf{H}}(\mathbf{M}, \mathbf{H}) &= \left[\mathbf{I} - \mathbf{C}\mathbf{M}\right]^{-1}\left[\mathbf{B}\mathbf{H} + \mathbf{C}\mathbf{H} + \mathbf{N}\right]. & (57) \end{aligned}$$

Rational expectations equilibrium is a fixed-point of this mapping:

$$\mathbf{M}^{\mathbf{re}} = \left[\mathbf{I} - \mathbf{C}\mathbf{M}^{\mathbf{re}}\right]^{-1}\left[\mathbf{A} + \mathbf{B}\mathbf{M}^{\mathbf{re}}\right]. \tag{58}$$

Conditional on $\mathbf{M}^{\mathbf{re}}$ we can solve for $\mathbf{H}^{\mathbf{re}}$:

$$\mathbf{H}^{\mathbf{re}} = \left[\mathbf{I} - \left[\mathbf{I} - \mathbf{C}\mathbf{M}^{\mathbf{re}}\right]^{-1}(\mathbf{B} + \mathbf{C})\right]^{-1}\left[\mathbf{I} - \mathbf{C}\mathbf{M}^{\mathbf{re}}\right]^{-1}\mathbf{N}. \tag{59}$$

Adapting Proposition 10.3 from Evans and Honkapohja [15], we check that all eigenvalues of $DT_{\mathbf{M}}(\mathbf{M}, \mathbf{H})$ and $DT_{\mathbf{H}}(\mathbf{M}, \mathbf{H})$ have real parts less than 1 when evaluated at the fixed-point solutions of the T-map (19), that is, $\mathbf{M} = \mathbf{M}^{\mathbf{re}}$ and $\mathbf{H} = \mathbf{H}^{\mathbf{re}}$. Using the rules for vectorization, we get:

$$\begin{aligned} DT_{\mathbf{M}}(\mathbf{M}^{\mathbf{re}}, \mathbf{H}^{\mathbf{re}}) &= \left([\mathbf{I}_8 - \mathbf{C}\mathbf{M}^{\mathbf{re}}]^{-1}[\mathbf{A} + \mathbf{B}\mathbf{M}^{\mathbf{re}}]\right)' \otimes [\mathbf{I}_8 - \mathbf{C}\mathbf{M}^{\mathbf{re}}]^{-1}\mathbf{C} \\ &\quad + \mathbf{I}_8 \otimes [\mathbf{I}_8 - \mathbf{C}\mathbf{M}^{\mathbf{re}}]^{-1}\mathbf{B} \\ DT_{\mathbf{H}}(\mathbf{M}^{\mathbf{re}}, \mathbf{H}^{\mathbf{re}}) &= [\mathbf{I}_8 - \mathbf{C}\mathbf{M}^{\mathbf{re}}]^{-1}[\mathbf{B} + \mathbf{C}]. \end{aligned}$$

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