ALGORITHMS FOR STOCHASTIC APPROXIMATIONS OF CURVATURE FLOWS

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ABSTRACT

Curvature flows have been extensively considered from a deterministic point of view. They have been shown to be useful for a number of applications including crystal growth, flame propagation, and computer vision. In some previous work [1], we have described a random particle system, evolving on the discretized unit circle, whose profile converges toward the Gauss-Minkowsky transformation of solutions of curve shortening flows initiated by convex curves. The present note shows that this theory may be implemented as a new way of evolving curves and as a possible alternative to level set methods.

1. INTRODUCTION

In some previous work Ben-Arous, Tannenbaum, and Zeitouni [1], described a stochastic interpretation of curve shortening flows. This brought together the theories of curve evolution and hydrodynamical limits, and as such impacted on the growing use of joint methods from probability and pde's in image processing and computer vision. In this present note we will indicate how this theory may be implemented to forge a novel stochastic curve evolution algorithm.

Following [1], we will now set the background for our results, to which we refer the reader for all the technical details. Let $\mathcal{C}(p,t) : S^1 \times [0,T) \mapsto \mathbb{R}^2$ be a family of embedded curves where t parameterizes the family and p parameterizes each curve. We consider stochastic interpretations of certain *curvature driven flows*, i.e., starting from an initial embedded curve $\mathcal{C}_0(p)$ we consider the solution (when it exists) of an equation of the form

$$\frac{\partial \mathcal{C}(p,t)}{\partial t} = \hat{V}(\kappa(p,t))\mathcal{N}, \quad \mathcal{C}(\cdot,t) = \mathcal{C}_t(\cdot), \qquad (1)$$

where $\kappa(p,t)$ denotes the curvature and $\mathcal{N}$ denotes the inner unit normal of the curve $\mathcal{C}(\cdot,t)$ at p. Of particular interest is the case in which $\hat{V}(x) = \pm x^{\alpha}$. Note that the case $\hat{V}(x) = x$ corresponds to the *Euclidean* curve shortening flow [3] while $\hat{V}(x) = x^{1/3}$ corresponds to the *affine* curve shortening, which is of strong relevance in computer vision and image processing [5]. Since in both cases we get gradient flows and resulting heat equations, a stochastic interpretation seems quite natural.

We will be dealing with convex curves here and so we employ the standard parameterization via the Gauss map, that is fixing $p = \theta$, the angle between the exterior normal to the curve and a fixed axis. It is well known that the Gauss map can be used to map smooth convex curves $\mathcal{C}(\cdot)$ into positive functions $m(\cdot)$ on S^1 such that $\int_{S^1} e^{2\pi i\theta} m(\theta)d\theta = 0$, and that this map can be extended to the *Gauss-Minkowsky* bijection between convex curves with $\mathcal{C}(0) = 0$ and positive measures on S^1 with zero barycenter; see [2, Section 8] for details. We denote by $\mathcal{M}_+^0$ the latter set of measures. Under this parameterization, a convex curve $\mathcal{C}(\theta)$ can be reconstructed from a $\mu \in \mathcal{M}_+^0$ by the formula $\mathcal{C}(\theta) = \int_0^\theta e^{2\pi i\Theta}\mu(d\Theta)$, using linear interpolation over jumps of the function $\mathcal{C}(\theta)$. Further, whenever μ possesses a strictly positive density $m(\theta)d\theta$ then the curvature of the curve at θ is $\kappa(\theta) = 1/m(\theta)$.

Another useful property in working with measures $\mu \in \mathcal{M}_+^0$ is that the evolution of the density $m(\cdot)$ takes a particularly simple form:

$$\frac{\partial m(t,\theta)}{\partial t} = -\frac{\partial^2 V(m(t,\theta))}{\partial \theta^2} - V(m(t,\theta)),$$
$$V(x) := \hat{V}(1/x). \qquad (2)$$

Our interest is in constructing stochastic approximations to the solutions of the equations (2). Approximations corresponding to polygonal curves have been discussed in the literature under the name "crystalline motion"; see [7] for a description of recent results and references. The approach in [1] is different and can be thought of as a *stochastic* crystalline algorithm: we construct a stochastic particle system whose profile defines an *atomic* measure on S^1, such that the corresponding curve is a convex polygon. Applying standard tools from hydrodynamic limits, it is proven in [1] that the (random) evolution of this polygonal curve converges, in the limit of a large number of particles, to curve evolution under the curve shortening flow.

2. APPROXIMATIONS TO CURVATURE FLOWS

We present in this section a general result concerning the existence and uniqueness of a certain class of quasilinear parabolic equations, and show how such equations are approximations of the curve-shortening equations described above.The proofs may be found in [1].

Let $\Phi, V : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfy the following:

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Assumption C

(C–1) $\Phi \in C^3(\mathbb{R}_+)$, $V \in C^1(\mathbb{R}_+)$.

(C–2) *For every $L > 0$ there exist constants $c_L, d_L > 0$ such that*

$$\min_{x\in[0,L]} \Phi'(x) \geq c_L\,, \quad \max_{x\in[0,L]} |\Phi''(x)| \leq d_L\,.$$

(C–3) *$V(\cdot)$ is bounded and $V(0) \geq 0$.*

Define the operator $L : C^{1,2}(\mathbb{R}_+ \times S^1) \mapsto C(\mathbb{R}_+ \times S^1)$ as

$$L\rho(t,x) = -\partial_t\rho(t,x) + \frac{1}{2}\partial_{xx}\Phi(\rho)(t,x) + V(\rho(t,x))\,. \quad (3)$$

Proposition 1 *Suppose Φ, V satisfy Assumption C, and let $m(\cdot)$ be $C^{2+\beta}(S^1)$ for some $1 \geq \beta > 0$, be a strictly positive function. Then there exists a unique solution $\rho \in C^{2+\beta}(S^1)$ to the equation*

$$L\rho(t,x) = 0\,, \quad \rho(0,x) = m(x)\,. \quad (4)$$

Further, $\rho(t,x)$ is strictly positive.

Note that the curve shortening flow (2) is not covered by Proposition 1, for the functions $V(x) = \Phi(x) = -x^{-\alpha}$ do not satisfy Assumption C (and indeed, the curve shortening flow does possess a finite blow-up time, contrary to the conclusion of Proposition 1). We thus wish to approximate this flow, e.g. by using functions of the form $\Phi_{\alpha,\epsilon}(x) = (1/\epsilon) - (1/(x+\epsilon^{1/\alpha})^\alpha)$ and $V_{\alpha,\epsilon}(x) = -x/(x+\epsilon^{1/\alpha})^{\alpha+1}$. (see Section 3.3). We thus establish next a convergence result for solutions of quasilinear parabolic equations that approximate curve-shortening equations. In what follows, set $\mathbb{R}^0_+ = (0,\infty)$.

Theorem 1 *Suppose functions $\Phi \in C^2(\mathbb{R}^0_+)$, $V \in C^1(\mathbb{R}^0_+)$ and $m \in C^{2+\beta}(S^1)$ are given such that $m(\cdot)$ is strictly positive and (4) holds on $[0,T)$ with ρ strictly positive. Let $\Phi_\epsilon, V_\epsilon$ satisfy Assumption C and assume that $\Phi'_\epsilon, \Phi''_\epsilon, V_\epsilon$ converge uniformly on compact subsets of $(0,\infty)$ to Φ', Φ'', V. Let L^ϵ denote the operator L with the functions $\Phi_\epsilon, V_\epsilon$ substituted for the functions Φ, V, and let $\rho_\epsilon(t,x)$ satisfy $L^\epsilon\rho_\epsilon(t,x) = 0$, $\rho_\epsilon(0,x) = m(x)$. Then, for any $\delta > 0$,*

$$\limsup_{\epsilon\to 0} \sup_{(t,x)\in[0,T-\delta]\times S^1} \frac{\rho_\epsilon(t,x)}{\rho(t,x)} = \limsup_{\epsilon\to 0} \sup_{(t,x)\in[0,T-\delta]\times S^1} \frac{\rho(t,x)}{\rho_\epsilon(t,x)} = 1\,.$$

For the proof, we refer to [1]. Note that in Theorem 1, we did *not* assume that Φ, V satisfy Assumption C. On the other hand, the existence and uniqueness of $\rho^\epsilon(t,x)$ is assured by Proposition 1.

3. PARTICLE SYSTEMS AND APPROXIMATE FLOWS

We construct in this section the particle systems alluded to above, prove their hydrodynamical limits, and relate them to approximate curvature flows. Again all the proofs may be found in [1]..

3.1. Birth and Death Zero Range Particle Systems

Let $T_N = \mathbb{Z}/N\mathbb{Z}$ denote the discrete torus. Let $g : \mathbb{N} \to \mathbb{R}_+$ (the *jump rate*, with $g(0) = 0$), $b : \mathbb{N} \to \mathbb{R}_+$ (the *birth rate*), $d : \mathbb{N} \to \mathbb{R}_+$ (the *death rate*, with $d(0) = 0$) be given, and define the Markov generator on the particle configuration $E_N = \mathbb{N}^{T_N}$ by

$$(\mathcal{L}^N f)(\eta) = N^2(\mathcal{L}_0 f)(\eta) + (\mathcal{L}_1 f)(\eta)\,, \quad f \in C_b(E_N)\,,$$

where

$$(\mathcal{L}_0 f)(\eta) = \frac{1}{2}\sum_{i\in T_N} g(\eta(i))\left[f(\eta^{i,i+1}) + f(\eta^{i,i-1}) - 2f(\eta)\right]$$

$$(\mathcal{L}_1 f)(\eta) = \sum_{i\in T_N}\left[b(\eta(i))\left[f(\eta^{i,+}) - f(\eta)\right] + d(\eta(i))\left[f(\eta^{i,-}) - f(\eta)\right]\right]$$

and

$$\eta^{i,i\pm 1}(j) = \begin{cases} \eta(j)+1, & j = i\pm 1, \eta(i) \neq 0, \\ \eta(j)-1, & j = i, \eta(i) \neq 0, \\ \eta(j), & \text{else} \end{cases}$$

$$\eta^{i,+}(j) = \begin{cases} \eta(j)+1, j = i, \\ \eta(j), \text{else} \end{cases}$$

$$\eta^{i,-}(j) = \begin{cases} \eta(j)-1, j = i, \eta(i) > 0, \\ \eta(j), \text{else} \end{cases}.$$

We denote by $\mu_{t,N}$ the law of the process at time t, with initial law $\mu_{0,N}$, under this Markovian semigroup. In order to state the main limit result of [1], we need to introduce the appropriate equilibrium measure. Define $Z : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ by $Z(\varphi) := \sum_k \frac{\varphi^k}{g(k)!}$ where $g(k)! = g(1)\cdots g(k)$ and $g(0)! = 1$. Set $\mathcal{D}_g = \{\varphi \in \mathbb{R}_+ : Z(\varphi) < \infty\}$, and $\varphi^* = \sup\{\varphi : \varphi \in \mathcal{D}_g\}$. For any $\varphi \in \mathcal{D}_g$, we define the probability measure $\overline{p}_\varphi$ on $\mathbb{N}$ by,

$$\overline{p}_\varphi(k) = \frac{\varphi^k}{g(k)!Z(\varphi)},$$

and set $R(\varphi) := \varphi\frac{Z'(\varphi)}{Z(\varphi)}, \varphi \in \mathcal{D}_g$ (see [4, pg. 28–31] for background).

The following lemma is true under certain boundedness assumptions on $g(\cdot)$; see [1] (Assumption A) for the exact statement.

Lemma 1 *(a) $\varphi^* > 0, R(\varphi) \nearrow_{\varphi\nearrow\varphi^*} \infty$, and for each $\varphi < \varphi^*$ there exists a $\theta(\varphi) > 0$ such that $\overline{p}_\varphi$ possesses exponential moments with parameter $\theta(\varphi)$.*

(b) Set $\Phi(\alpha) = R^{-1}(\alpha)$ and $p_\alpha = \overline{p}_{\Phi(\alpha)}$. Then, $\Phi(\cdot)$ is a smooth function with strictly increasing derivative, $\Phi'(0) \in (0,\infty)$, and

$$E_{p_\alpha}(X) = \alpha,\ E_{p_\alpha}(g(X)) = \Phi(\alpha)$$

(c) *Set $\nu_\alpha = p_\alpha^{\otimes \mathbb{Z}}$ and let $\nu_{\alpha,N}$ denote the restriction of ν_α to T_N. Then $\nu_{\alpha,N}$ is reversible, and hence invariant, for the Markov generator L_0^N.*

In the sequel, for any function h defined on $\mathbb{N}$, we set $\tilde{h}(\alpha) := E_{p_\alpha}(h(X))$. In particular, by Lemma 1, $\tilde{g}(\alpha) = \Phi(\alpha)$. We need below the following assumption on the initial law of our Markov evolution:

Assumption B *There exists a $\delta > 0$ and an $m \in C^{2+\delta}(S^1)$ strictly positive such that*

$$\frac{1}{N} H\left(\mu_{0,N} \middle| \prod_{i=0}^{N-1} p_{m(\frac{i}{N})}\right) \xrightarrow[N\to\infty]{} 0$$

Set

$$V(\alpha) = V_+(\alpha) - V_-(\alpha) := \tilde{b}(\alpha) - \tilde{d}(\alpha).$$

Let $\rho(t,x) : [0,T] \times S^1 \mapsto \mathbb{R}_+$ denote a $C^{1,2+\delta}$ strictly positive solution of the PDE

$$\partial_t \rho(t,x) = \frac{1}{2}\partial_{xx}\Phi(\rho)(t,x) + V(\rho)(t,x),\ \rho(0,x) = m(x). \tag{5}$$

We are now ready to state the hydrodynamic limit result for the laws $\mu_{t,N}$:

Theorem 2 *Let Assumptions A of [1] and Assumption B hold. Then, for any function $G \in C(S^1)$, any $\delta > 0$, and any $t \in [0,T]$, the following limit goes to zero:*

$$\lim_{N\to\infty} \mu_{t,N}\left\{\eta : \left|\frac{1}{N}\sum_{i\in T_N} \eta(i) G\left(\frac{i}{N}\right) - \int_{S^1} G(x)\rho(t,x)\right| > \delta\right\} \tag{6}$$

3.2. Stochastic curve shortening convergence

Fixing next $\alpha > 0$, consider the functions $\Phi_\alpha(x) = -x^{-\alpha}$, $V_\alpha(x) = -x^{-\alpha}$, and define the operator L_α as in (3). Fix an m satisfying Assumption B, and let ρ_α denote the solution of (4) with operator L_α, with blow-up time T_α, and associated curve $\mathcal{C}_\alpha(t,\theta)$. Let $g_{\alpha,\epsilon}, b_{\alpha,\epsilon}, d_{\alpha,\epsilon}$ satisfy Assumption A of [1], set $\Phi_{\alpha,\epsilon}$ and $V_{\alpha,\epsilon}$ as in Section 3.1. The following assumption is needed in order to relate the particle system with the curve shortening flow:

Assumption D (D–1) *$\Phi_{\alpha,\epsilon}, V_{\alpha,\epsilon}$ satisfy Assumption C.*

(D–2) *$\Phi'_{\alpha,\epsilon}, \Phi''_{\alpha,\epsilon}, V_{\alpha,\epsilon}$ converge uniformly on compact subsets of $(0,\infty)$ to $\Phi'_\alpha, \Phi''_\alpha, V_\alpha$.*

The main result of [1] is the following:

Theorem 3 *Let $\mathcal{C}^N_{\alpha,\epsilon} : \mathbb{R}_+ \times S^1 \mapsto \mathbb{R}_+$ denote the curve corresponding to the particle system defined above. Fix $\delta, \delta' > 0$. Then the following limit converges to zero:*

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} P\left(\sup_{(t,\theta)\in[0,T_\alpha-\delta]\times S^1} |\mathcal{C}^N_{\alpha,\epsilon}(t,\theta) - \mathcal{C}_\alpha(t,\theta)| > \delta'\right) \tag{7}$$

If further $\mathcal{C}_\alpha(t,\theta) \to_{t\to T_\alpha} 0$, $\mathcal{C}_\alpha(t,\theta) := 0$ for $t > T_\alpha$, and there exists a $z_0 = z_0(\alpha)$ such that $\Phi'_{\alpha,\epsilon}(z) \geq 0$, $V_{\alpha,\epsilon}(z) < 0$ for all $0 < z < z_0$, then $T_\alpha - \delta$ in (7) can be replaced by any deterministic constant $T > 0$.

3.3. Approximate Euclidean curvature flows

In this section we give the stochastic approximation of the Euclidean curvature flow. For the general curve shortening case, see [1]. We now present candidates for the functions b, d, g defining the particle systems of Section 3.1. They relate to an approximate version of the Euclidean curvature flow [1]:

$$\begin{aligned} \bar{g}_{\epsilon,1}(1) &= \epsilon^{-2}, & \bar{g}_{\epsilon,1}(k) &= \epsilon^{-1}k/(k-1),\ k \geq 2, \\ \bar{b}_{\epsilon,1}(2) &= 2\epsilon^{-2}, & \bar{b}_{\epsilon,1}(k) &= 0,\ k \neq 2, \\ \bar{d}_{\epsilon,1}(1) &= \epsilon^{-2}, & \bar{d}_{\epsilon,1}(k) &= 0,\ k \neq 1. \end{aligned} \tag{8}$$

where $\epsilon > 0$ is a fixed parameter,

4. ALGORITHM AND SIMULATIONS

4.1. Algorithms

We assume that we have a system in state $\eta \in \mathbf{N}^{T_N}$ at time t_o. We are given 4 rates depending on 4 possible events that can occur at a site i i: b_i=birth, d_i=death, ${g_i}^+$=jump to right, ${g_i}^-$=jump to left. We let $E_i := \{b_i, d_i, N^2{g_i}^+, N^2{g_i}^-\}$ be the set of possible transition rates for system in state η at site i. We choose a possible event $e_i \in E_i$, and its rate is given by $\lambda(\eta_i, e_i) := e_i(\eta_i)$. Then the total rate at the given site i is found by summing rates of all possible events:

$$U(\eta, i) := \sum_{e_i \in E_i} \lambda(\eta_i, e_i).$$

Then there are two simple ways of getting the Poisson system for simulating the Markov process described above.

Algorithm 1: Per Site Transition

1. Get values for $T_i \sim$ exponential($U(\eta,i)$). (By this of course we mean that the T_i's are exponential random variables with parameter $U(\eta,i)$.)
2. Set
$$T := \min_{i\in T_N}\{T_i\} =: T_{i^*}.$$
i^* is the site where the transition occurs at time $t_o + T$.
3. To find the event in E_i, we then take $e_i \in E_i$ with probability
$$\frac{\lambda(\eta, e_i)}{U(\eta, e_i)},$$
the (conditional) transition probability.

Algorithm 2: Per Event Transition

1. Get $T(e_i) \sim$ exponential($\lambda(\eta, e_i)$) for all i, e_i.

2. Set $T := \min_{i,e_i}\{T(e_i))\}$.
3. Then the next event time is $t_o + T$ and the next event is arg $\min\{T(e_i)\}$.

Curve Reconstruction We begin by explicitly constructing random polygons from particle configurations. Each particle configuration $\eta(\cdot)$ defines a positive measure on S^1 by $\mu_\eta = \sum_{k\in T_N} \eta(k)\delta_{2\pi k/N}$. Unfortunately, this measure does not possess necessarily a zero barycenter, and thus does not correspond *a priori* to a closed convex curve. To remedy this situation, set

$$b_\eta = b_\eta^R + i b_\eta^I = \sum_{k\in T_N} e^{2\pi k/N}\eta(k),$$

and define

$$\bar{\mu}_\eta = \mu_\eta + |b_\eta^R|\delta_{\pi/2+(\pi/2)\mathrm{sign}(b_\eta^R)} + |b_\eta^I|\delta_{-(\pi/2)\mathrm{sign}(b_\eta^I)}\,.$$

Then $\bar{\mu}_\eta \in \mathcal{M}_+^0$, and it defines a curve by a linear interpolation between the jump points of the function $C_\eta(\theta) = \int_0^\theta e^{2\pi i\Theta}\bar{\mu}_\eta(d\Theta)$.

4.2. Simulations

Simulation 1: linear heat equation Given an initial system configuration, when each particle jumps to one of its neighbor sites at a rate $\frac{1}{2}g(k)$, linearly proportional to number of particles k, e.g. $g(k) = k, k \geq 0$, the generator corresponds to the heat equation. We simulated the system (in which particles evolve according to independent continuous time random walks), with an initial configuration η_0, where all but a few sites have zero particles, and the remaining have, say 50 particles. For display purposes, here we depict the curves reconstructed from η at several iterations in Fig. 1.

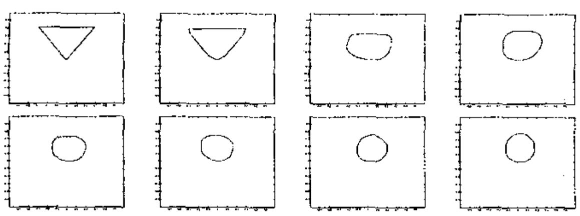

Fig. 1. System configuration η is evolved with a jump rate $g(k) = k$, and the corresponding curves are shown left-right top-bottom.

Simulation 2: geometric heat equation Given an initial system configuration with $N = 500$ sites, $\epsilon = 0.1$, the generator described in Section 3.1, and the approximate rates described in Section 3.3, we simulated the stochastic curve shortening system with an initial configuration η_0 that satisfies Assumption B. As expected, the overall number of particles in the system decreases with time to 0, as seen in Fig. 2. This corresponds graphically to the shrinkage of the initial curve to a point. For display purposes, here we depict the curves reconstructed from η at several iterations in Fig. 3.

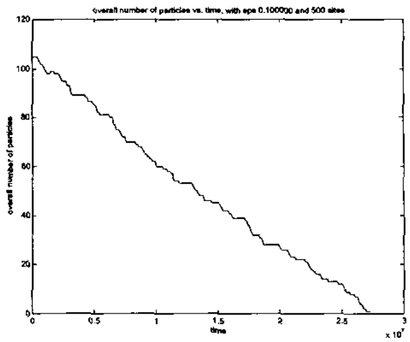

Fig. 2. Overall Number of Particles in the system versus time of evolution

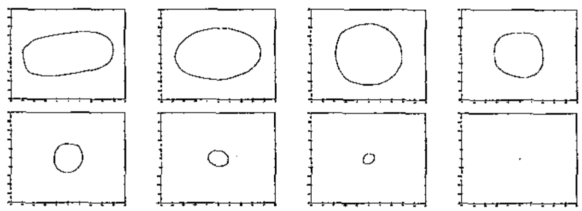

Fig. 3. System configuration η is evolved and the corresponding curves are shown left-right top-bottom.

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