Least Squares Approximate Feedback Linearization*

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Abstract. We study the least squares approximate feedback linearization problem: given a single input nonlinear system, find a linearizable nonlinear system that is close to the given system in a least squares (L_2) sense. A linearly controllable single input affine nonlinear system is feedback linearizable if and only if its characteristic distribution is involutive (hence integrable) or, equivalently, any characteristic one-form (a one-form that annihilates the characteristic distribution) is integrable. We study the problem of finding (least squares approximate) integrating factors that make a fixed characteristic one-form close to being exact in an L_2 sense. One can decompose a given one-form into exact and inexact parts using the Hodge decomposition. We derive an upper bound on the size of the inexact part of a scaled characteristic one-form and show that a least squares integrating factor provides the minimum value for this upper bound. We also consider higher order approximate integrating factors that scale a nonintegrable one-form in a way that the scaled form is closer to being integrable in L_2 together with some derivatives and derive similar bounds for the inexact part. One can use least squares approximate integrating factors in approximate feedback linearization of nonlinearizable single input affine systems. Moreover, least squares approximate integrating factors allow a unified approach to both least squares approximate and exact feedback linearization.

Keywords. Nonlinear systems, feedback linearization, differential forms, calculus of variations, Sobolev spaces, elliptic PDE's.

Introduction

Feedback linearization of nonlinear control systems has proven to be a useful tool for the design of controllers guaranteeing good performance and stability over a large region of operation. When the system is close to being feedback linearizable, one may still be able to guarantee satisfactory performance and stability. For this, one needs to develop some

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measures of distance between nonlinearizable systems and linearizable ones. So far most of the work on approximate feedback linearization has focused on applying a change of coordinates and a preliminary feedback so that the resulting system looks like linearizable part plus nonlinear terms of highest possible order around an equilibrium point [19, 17] or an equilibrium manifold [13, 22, 23]. By neglecting these nonlinear terms one obtains a linearizable system approximating the original system. If one applies to the original system a controller designed for the approximating linearizable system, performance and stability are guaranteed in a neighborhood of the equilibrium or equlibrium manifold. However, in many applications one requires a large region of operation for the nonlinearizable system. In such a case, demanding the nonlinear terms to be neglected to be of highest possible order may, in fact, be quite undesirable. One might prefer that the nonlinear terms to be neglected be small in some average or uniform sense over the region of operation. In the present paper we derive an approach that allows one to obtain upper bounds on the norm of these nonlinear terms in Sobolev spaces and thus, via Sobolev embeddings, in all L_p spaces.

Consider a single-input affine nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where f, g are smooth vector fields defined on the closure of an open, bounded, and contractible region $\mathcal{M}$ of $\mathbf{R}^n$ containing the origin and having a smooth boundary $\partial\mathcal{M}$. The classical problem of feedback linearization (cf. [15, 16]) can be stated as follows: find in a neighborhood of the origin a smooth change of coordinates $z = \Phi(x)$ (a local diffeomorphism) and a smooth feedback law $u = k(x) + l(x)u_{new}$ such that the closed loop system in the new coordinates with new control is linear:

$$\dot{z} = Az + Bu_{new}, \tag{2}$$

and controllable.

In the paper we will assume that the system (1) has the *linear controllability* property

$$\dim \operatorname{span} \{g, ad_f g, \ldots, ad_f^{n-1} g\}(x) = n, \ x \in \mathcal{M}. \tag{3}$$

We define the *characteristic distribution* for (1)

$$D := \operatorname{span} \{g, ad_f g, ..., ad_f^{n-2} g\} \tag{4}$$

(it is a smooth $(n-1)$-dimensional distribution since (1) is linearly controllable). We call any nowhere vanishing one-form annihilating D a *characteristic one-form* for (1). Frobenius' theorem implies that the system (1) is exactly feedback linearizable if and only if we can find among all characteristic one-forms one which is exact, i.e., it is the *exterior derivative* $d\alpha$ of some zero-form α. Let ω_0 be a characteristic one-form for (1). All the other characteristic one-forms are multiples of ω_0 by a smooth nowhere vanishing function (zero-form) β. Without loss of generality we can assume that β is everywhere positive. The system (1) is exactly feedback-linearizable if and only if there is a nowhere vanishing β

such that $\beta\omega_0$ is exact. Such β, if it exists, is called an (exact) *integrating factor* for ω_0. A form ω_0 that admits an integrating factor (i.e., whose multiple by some nonzero function is exact) is called *integrable*. The Poincare Lemma says that, on a contractible region, a form $\beta\omega_0$ is exact if and only if it is *closed*, i.e.,

$$d\beta\omega_0 = 0. \tag{5}$$

Thus, to find an integrating factor for an integrable ω_0, one needs to find a nontrivial solution β to first order PDE (5).

The conditions for integrability of a one-form are nontrivial for $n > 2$ so that, in general, no exact integrating factor β will exist. One implication of this is that a generic nonlinear system (1) will not be feedback linearizable. To construct an *approximate integrating factor* that is optimal in some precise sense we need a Riemannian metric—a nondegenerate pointwise inner product on the tangent space to $\mathcal{M}$. A Riemannian metric induces an inner product and norm on k-forms. In the present paper we use a special Riemannian metric constructed as follows. The linear controllability property allows one to locally define a special set of coordinates called s-coordinates (cf. [18, 16, 21]). In a neighborhood of the origin one can "reach" any point x by "traveling" along vector fields $\{ad_f^{n-1}g, \ldots, ad_f g, g\}$ with "times" $s_1, s_2, \ldots, s_n$. We define a metric on vector fields and forms on $\mathcal{M}$ by taking $\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_n}$ to be a positively oriented orthonormal basis for the tangent space to $\mathcal{M}$ at every point. This metric will be referred to as the *s-metric*. In this paper we will use a *specific* characteristic form ω_0 fixed by the requirements $|\omega_0| = 1$ and $\omega_0(ad_f^{n-1}g) > 0$, where $|\cdot|$ is the (pointwise) norm defined by the s-metric.

A global L_2 norm $\|\xi\|$ of a form ξ on $\mathcal{M}$ is obtained by integrating the pointwise one:

$$\|\xi\| := \left(\int_{\mathcal{M}} |\xi|^2 \mu\right)^{\frac{1}{2}}, \tag{6}$$

where $\mu := ds_1 \wedge \ldots \wedge ds_n$ is the standard volume element corresponding to the s-metric.

Let δ be the *codifferential*—the formal adjoint to the exterior derivative d with respect to the chosen metric. In this paper we consider finding β so that $d\beta\omega_0$ and $\delta\beta\omega_0$ are the smallest possible in a least squares sense. The motivation for making *both* $d\beta\omega_0$ *and* $\delta\beta\omega_0$ *small* is as follows.

In the exactly linearizable case we may find $\beta > 0$ such that $d\beta\omega_0 = 0$. However, if one exact integrating factor for ω_0 exists, then there are infinitely many. Namely, if β is an integrating factor for ω_0, i.e., $\beta\omega_0 = d\alpha$ for some α, then for *any* nowhere vanishing function $h(\cdot)$, $h(\alpha)\beta$ is also an integrating factor for ω_0, for $h(\alpha)\beta\omega_0(x) = d\int_0^{\alpha(x)} h(y)dy$. If we do not impose an assumption on the smoothness of $h(\cdot)$, the integrating factors $h(\alpha)\beta$ include some *discontinuous* ones (for integrable but discontinuous $h(\cdot)$). We see that the discontinuity may occur in the direction *transversal* to the integral manifolds of ω_0. The reason is that the exterior derivative in $d\beta\omega_0$ controls only behavior of β *along* the integral manifolds of ω_0. Now requiring also $\delta\beta\omega_0 = 0$ has a *smoothing effect* on β, for the codifferential δ in $\delta\beta\omega_0$ controls the behavior of β in the direction *transversal* to the integral manifolds of ω_0. Thus, if we can assure both $d\beta\omega_0 = 0$ and $\delta\beta\omega_0 = 0$ for some $\beta > 0$, we will have a *smooth* integrating factor β for ω_0. We will also show that the integrating factor satisfying *both* $d\beta\omega_0 = 0$ and $\delta\beta\omega_0 = 0$ is unique up to multiplication by a nonzero constant. It is not clear at the moment that, for integrable ω_0, such an integrating factor *exists*. We will show that it does, if the corresponding Riemannian metric is an s-metric.

In the case of nonlinearizable (1) we will look for β minimizing the following functional. For the *fixed* (by the requirements $|\omega_0| = 1$ and $\omega_0(ad_f^{n-1}g) > 0$) characteristic form ω_0, we define

$$I_1(\beta) := \frac{||d\beta\omega_0||^2 + ||\delta\beta\omega_0||^2}{||\beta\omega_0||^2}. \tag{7}$$

Note that minimizing the functional $I_1(\beta)$ forces both $d\beta\omega_0$ and $\delta\beta\omega_0$ to be *small*. Making $d\beta\omega_0$ *small* makes $\beta\omega_0$ close to being closed and hence exact (cf. [7]). Control over the norm of $\delta\beta\omega_0$ assures smoothness of β. We will show that there is a smooth positive minimizer β_0 for $I_1(\beta)$, to be called the *least squares approximate integrating factor for* ω_0, uniquely defined up to multiplication by a constant. One can obtain uniqueness by a proper normalization, for instance, $||\beta_0||^2 = 1$.

The Hodge decomposition of a characteristic one-form $\beta\omega_0$ has form

$$\beta\omega_0 = d\alpha + \epsilon, \tag{8}$$

where $d\alpha$ is a least squares approximation of $\beta\omega_0$ by an exact form and ϵ (called the *antiexact part* of $\beta\omega_0$) is a one-form whose (global) norm is a measure of how far $\beta\omega_0$ is from being exact in a least squares sense (cf. [1, 20]). We also show how to use the Hodge decomposition to obtain approximations of nonlinearizable systems (1) by linearizable ones (cf. [4, 7]).

We will show that, for the approximation of $\beta\omega_0$ by an exact form in the Sobolev space H^1 (the space of forms with coefficients in L_2 together with all first order partial derivatives), it makes good sense to choose $\beta = \beta_0$ where β_0 is an approximate integrating factor for ω_0 (of order 1). More precisely, we show that the H^1 norm of the error one-form ϵ in the Hodge decomposition of $\beta\omega_0$ can be bounded by

$$||\epsilon||_1 \leq C\sqrt{I_1(\beta)}. \tag{9}$$

when β is normalized so that $||\beta|| = 1$. Choosing β to be the (normalized) least squares approximate integrating factor provides the smallest value of this upper bound.

We will also consider *higher order approximate integrating factors for* ω_0. To be precise, any minimizer of the functional

$$I_m(\beta) = \frac{||d\beta\omega_0||^2 + ||\delta\beta\omega_0||^2 + \cdots + ||\overbrace{\cdots d\delta d}^{m}\beta\omega_0||^2 + ||\overbrace{\cdots \delta d\delta}^{m}\beta\omega_0||^2}{||\beta\omega_0||^2} \tag{10}$$

is a *least squares approximate integrating factor of order* m for ω_0. Similar to the order 1 case, we show that using an approximate integrating factor of order $2m$ provides for a guaranteed level of approximation of $\beta\omega_0$ by an exact form in the Sobolev space H^{2m}. To be precise, we show that the H^{2m} norm of the error one-form ϵ in the the Hodge decomposition of $\beta\omega_0$ can be bounded by

$$\|\epsilon\|_{2m}^{\mathcal{M}'} \leq C\sqrt{I_{2m}(\beta)}, \tag{11}$$

on any *interior* region $\mathcal{M}'$ of $\mathcal{M}$.

Finally, we show how one can use higher order approximate integrating factors in the *approximate feedback linearization* of the nonlinear system (1). Namely, it was shown in [7] that if α (obtained via the Hodge decomposition (8) of $\beta\omega_0$) and its first $n-1$ Lie derivatives along f have linearly independent differentials one can use them to define a change of coordinates taking (1) to a normal form

$$\dot{z} = Az + Bru + Bp + Eu \tag{12}$$

where E depends linearly on ϵ and its derivatives up to order $n-1$. By neglecting E one obtains a linearizable system

$$\dot{z} = Az + Bru + Bp \tag{13}$$

approximating (12) with "error" Eu. The results of this paper allow one to obtain upper bounds on the H^k norms of E of the form

$$\|E\|_k^{\mathcal{M}'} \leq C\sqrt{I_{2m_1}(\beta)}, \tag{14}$$

on any interior region $\mathcal{M}'$ of $\mathcal{M}$, where m_1 depends on n and k. Moreover, Sobolev embedding theorems allow one to obtain upper bounds on the C^k norms (uniform norm together with k derivatives) of E of the form

$$\|E\|_{k,\infty}^{\mathcal{M}'} \leq C\sqrt{I_{2m_2}(\beta)}, \tag{15}$$

on any interior region $\mathcal{M}'$ of $\mathcal{M}$, where m_2 depends on n and k. Therefore, the use of least squares integrating factors provide a means for finding sensible approximations of nonlinear single-input systems by linearizable ones. Moreover, the minimum values of the functionals $I_k(\cdot)$ provide measures of linearizability of nonlinear single-input systems in various function spaces.

The paper is organized as follows. In Section 1 we introduce notation and present some auxiliary results. In Section 2 we recall the construction of s-coordinates for a single-input affine nonlinear system. The construction is illustrated by a system in $\mathbf{R}^3$. In Section 3 we show that the minimum value of functionals $I_m(\cdot)$ is zero if and only if the corresponding system is exactly linearizable. We also show that in this case all minimizers of $I_m(\cdot)$ are constant functions. In Section 4 we show that the minimum value of $I_1(\beta)$ is attained for a smooth and positive function β. All minimizers of $I_1(\beta)$ are unique up to a multiplication by nonzero constants. We also show that the minimizers of $I_1(\beta)$ satisfy an elliptic PDE of second order with mixed boundary conditions. We provide an example of construction of a minimizer of $I_1(\beta)$ for a system in $\mathbf{R}^3$. We also propose an approximation scheme for the minimizers of $I_1(\beta)$. In Section 5 show that minimum value of functionals $I_m(\beta)$, for

$m > 1$, is obtained for a function β which is smooth in the interior of $\mathcal{M}$. Contrary to the case $m = 1$, positivity, uniqueness, and smoothness up to the boundary of $\mathcal{M}$ is not clear. However, in Section 6 we show that positivity and uniqueness up to a multiplication by nonzero constants of minimizers of $I_m(\beta)$, for $m > 1$, for systems sufficiently close to being linearizable. In Section 7 we provide an easy to calculate lower bound on the value of functionals $I_m(\beta)$. In Section 8 we use the Hodge decomposition to decompose a scaled characteristic form into exact and antiexact parts. We provide upper bounds on H^m norms of the antiexact part of $\beta\omega_0$ in terms of the value of functionals $I_m(\beta)$. In Section 9 we present a decomposition of a single-input affine nonlinear system into a linearizable part and an error term. The error term represents an obstruction to linearizability. We provide upper bounds on the H^k and C^k norms of the error term via the values of the functionals $I_m(\beta)$.

To make the paper more accessible for readers not familiar with the theory of differential forms, calculus of variations, and elliptic PDE's, we include proofs of some standard results from these theories.

1 Notation and Auxiliary Results

By $\Omega^p(\mathcal{M})$ we denote the space of smooth p-forms on $\mathcal{M}$. By $\Omega(\mathcal{M})$ we will mean the algebra of exterior differential forms on $\mathcal{M}$.

In the sequel we assume that $\mathcal{M}$ is equipped with a Riemannian metric, i.e., a positive definite (pointwise) inner product $\langle \cdot, \cdot \rangle$ on the tangent space to $\mathcal{M}$. Except for the Section 6, the metric considered in this paper is *standard* in the so-called s-coordinates s_i (see Section 2 of the present paper and [21]) which are *global* on $\mathcal{M}$ (i.e., $\mathcal{M}$ can be covered by one patch in s-coordinates). Such a metric arises by assigning $\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_n}$ to be a positively oriented orthonormal basis for the tangent space to $\mathcal{M}$ at every point. The inner product on vectors induces an inner product on p-forms [1, 9] that we will denote by the same symbol. The corresponding pointwise norm will be denoted by $|\cdot|$. We obtain a global inner product $\langle\langle \cdot, \cdot \rangle\rangle$ of p-forms on $\mathcal{M}$ by integrating the pointwise one over $\mathcal{M}$:

$$\langle\langle \xi, \zeta \rangle\rangle := \int_{\mathcal{M}} \langle \xi, \zeta \rangle \mu,$$

where $\mu := ds_1 \wedge \ldots \wedge ds_n$ is the volume element corresponding to the s-metric and some fixed orientation of $\mathcal{M}$. We define a (global) L_2 norm of forms on $\mathcal{M}$ by integrating the pointwise one:

$$||\xi|| := \left(\int_{\mathcal{M}} |\xi|^2 \mu\right)^{\frac{1}{2}}, \ \ for\ \xi \in \Omega(\mathcal{M}). \tag{16}$$

We will also need the *Sobolev space* $H^m(\mathcal{M})$ of k-forms ξ which is a Hilbert space equipped with the inner product

$$(\xi, \zeta) := \langle\langle \xi, \zeta \rangle\rangle + \int_{\mathcal{M}} \sum_I \sum_{|\alpha| \le m} \xi_{Ix^\alpha} \zeta_{Ix^\alpha} \mu, \tag{17}$$

where $\xi = \sum_I \xi_I ds_I$, $\zeta = \sum_I \zeta_I ds_I$, $\alpha = \alpha_1, \cdots, \alpha_j$ is a multiindex, $|\alpha| = j$, and

$$\xi_{Ix^\alpha} := \frac{\partial^j \xi_I}{\partial s_{\alpha_1} \cdots \partial s_{\alpha_j}}.$$

The corresponding norm will be denoted by $\|\cdot\|_m$. Note that a p-form $\xi = \sum_I \xi_I ds_I$ is in $H^m(\mathcal{M})$ if and only if all of its coefficients (in s-coordinates) ξ_I are in $H^m(\mathcal{M})$. Moreover, $\|\xi\|_m^2 = \sum_I \|\xi_I\|_m^2$.

The (Hodge) $*$ operator (see, e.g., [9, 1]) is defined as the unique operator $* : \Omega^p(\mathcal{M}) \to \Omega^{n-p}(\mathcal{M})$ such that $\xi \wedge *\eta = \eta \wedge *\xi = \langle \xi, \eta \rangle \mu$, where μ is the volume element corresponding to the Riemannian metric and some fixed orientation of $\mathcal{M}$. The $*$ operator is an isomorphism between $\Omega^p(\mathcal{M})$ and $\Omega^{n-p}(\mathcal{M})$. For example, let $\mathcal{M}$ be 3-dimensional with coordinates s_1, s_2, s_3 and the metric chosen so that the vector fields $\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3}$ form a positively oriented orthonormal basis. Then the dual one-forms ds_1, ds_2, ds_3, are also orthonormal and we may choose $\mu := ds_1 \wedge ds_2 \wedge ds_3$. We have

$$\begin{aligned}
*1 &= ds_1 \wedge ds_2 \wedge ds_3, \\
*ds_1 &= ds_2 \wedge ds_3, \\
*ds_2 &= ds_3 \wedge ds_1, \\
*ds_3 &= ds_1 \wedge ds_2, \\
*ds_1 \wedge ds_2 &= ds_3, \\
*ds_2 \wedge ds_3 &= ds_1, \\
*ds_3 \wedge ds_1 &= ds_2, \\
*ds_1 \wedge ds_2 \wedge ds_3 &= 1.
\end{aligned}$$

Let $\xi \in \Omega^p(\mathcal{M})$. We define the *codifferential* of ξ as

$$\delta\xi := (-1)^{n(p-1)+1} * d * \xi.$$

Note that the codifferential depends on the choice of Riemannian metric. The codifferential is a formal adjoint to the exterior derivative due to the following result (cf. [1]).

Proposition 1.1 *Let $\xi \in \Omega^p(\mathcal{M})$ and $\zeta \in \Omega^{p+1}(\mathcal{M})$. Then*

$$\int_{\mathcal{M}} \langle d\xi, \zeta \rangle \mu = \int_{\mathcal{M}} \langle \xi, \delta\zeta \rangle \mu + \int_{\partial\mathcal{M}} \xi \wedge *\zeta \ . \tag{18}$$

Proof: Since, for any $\gamma \in \Omega^p(\mathcal{M})$, $* * \gamma = (-1)^{p(n-p)}\gamma$ (see [1]), we have

$$d(\xi \wedge *\zeta) = d\xi \ \wedge *\zeta + (-1)^p \xi \ \wedge \ d * \zeta = d\xi \ \wedge *\zeta - \xi \wedge *\delta\zeta$$

so that

$$d\xi \wedge *\zeta = \xi \wedge *\delta\zeta + d(\xi \wedge *\zeta)$$

Integrate this expression over $\mathcal{M}$ and apply Stokes Theorem. □

Note that on manifolds without boundary the above result states that $\langle\langle d\xi, \zeta\rangle\rangle = \langle\langle \xi, \delta\zeta\rangle\rangle$ where $\langle\langle\cdot,\cdot\rangle\rangle$ is the global inner product of p-forms on $\mathcal{M}$ obtained by integrating the pointwise one over $\mathcal{M}$.

We will also need the differential operators

$$d_i := \overbrace{\cdots d\delta d}^{i} \quad \delta_i := \overbrace{\cdots \delta d\delta}^{i},$$

so that $d_1 = d$, $\delta_1 = \delta$, $d_2 = \delta d$, $\delta_2 = d\delta$, etc. With this notation we can define for any positive integer m the quadratic form

$$Q_m(\xi, \zeta) := \sum_{i=1}^{m} (\langle\langle d_i\xi, d_i\zeta\rangle\rangle + \langle\langle \delta_i\xi, \delta_i\zeta\rangle\rangle) \tag{19}$$

By $\mathcal{D}Q_m$ we denote the domain of $Q_m(\cdot,\cdot)$, i.e., the set of one-forms ξ in L_2 such that $\delta_i\xi$ and $d_i\xi$ are in L_2 for $i \leq m$.

In the sequel we will deal with a fixed one-form ω_0 that satisfies $|\omega_0| = 1$ and $ad_f^{n-1}g > 0$. For any zero forms β, η we define $q_m(\beta, \eta) := Q_m(\beta\omega_0, \eta\omega_0)$. The domain of $q_m(\cdot,\cdot)$ is denoted $\mathcal{D}q_m$.

Suppose that ω_0 is the unique characteristic form for the system (1) with $|\omega_0| = 1$ and $\omega_0(ad_f^{n-1}g) > 0$. Using notation introduced above we may rewrite the functional I_m defined by (10) as

$$I_m(\beta) = \frac{Q_m(\beta\omega_0, \beta\omega_0)}{||\beta\omega_0||^2} = \frac{q_m(\beta, \beta)}{||\beta\omega_0||^2} \ . \tag{20}$$

The *Laplacian* of a k-form ξ is defined as

$$\Delta\xi := (\delta d + d\delta)\xi \ .$$

Any form ξ such that $d\xi = 0$ and $\delta\xi = 0$ (which implies $\Delta\xi = 0$) on $\mathcal{M}$ will be called *harmonic*.

We will also need the following simple result.

Lemma 1.1 *Let a, b be any real numbers and $\epsilon > 0$. Then*

$$|a|\,|b| \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$$

If the closure of $\mathcal{M}'$ is included in $\mathcal{M}$ we write $\mathcal{M}' \subset\subset \mathcal{M}$. In the sequel we will need the following result.

Theorem 1.1 *Let $\mathcal{M} \subset \mathbf{R}^n$ be an open and bounded region with a smooth boundary $\partial\mathcal{M}$ and m be a positive integer. Let $\mathcal{M}' \subset\subset \mathcal{M}$. Then, for any one-form ξ one has*

$$||\xi||_{2m}^{\mathcal{M}'} \leq C_1(\sqrt{Q_{2m}(\xi, \xi)} + ||\xi||), \tag{21}$$

where C_1 depends on $\mathcal{M}'$, $\mathcal{M}$, and m.

Proof: From a standard interior regularity result (see, e.g., [11, Thm 16.1]) it follows that for any function γ one has

$$\|\gamma\|_{2m}^{\mathcal{M}'} \leq c_1(\|\Delta^m\gamma\| + \|\gamma\|). \tag{22}$$

Note that this result extends to one-forms. For this, express ξ in s-coordinates as $\xi = \sum \xi_i ds_i$. Then $\Delta^m\xi = \sum \Delta^m\xi_i ds_i$ and thus $\|\Delta^m\xi\|^2 = \sum \|\Delta^m\xi_i\|^2$. Obviously then

$$\|\xi\|_{2m}^{\mathcal{M}'} \leq c_2(\|\Delta^m\xi\| + \|\xi\|). \tag{23}$$

Since $dd = 0$ and $\delta\delta = 0$ we have $\Delta^m\xi = (\delta d + d\delta)^m\xi = d_{2m}\xi + \delta_{2m}\xi$. Therefore $\|\Delta^m\xi\| \leq \|d_{2m}\xi\| + \|\delta_{2m}\xi\| \leq \sqrt{2Q_{2m}(\xi,\xi)}$. This and (23) yield the result. □

We will also use the spaces C^k of k-times continuously differentiable vector fields equipped with the norm

$$\|v\|_{k,\infty} := \sum_i \sum_{|\alpha|\leq k} \sup_{x\in\mathcal{M}} |v_{ix^\alpha}|.$$

The set of smooth functions with compact support on $\mathcal{M}$ will be denoted by $C_0^\infty(\mathcal{M})$.

2 s-Coordinates

The linear controllability property allows one to locally define a special set of coordinates called s-coordinates (cf. [21]). In a neighborhood of the origin one can "reach" any point x by "traveling" along vector fields $\{ad_f^{n-1}g, \ldots, ad_f g, g\}$ with "times" $s_1, s_2, \ldots, s_n$, i.e.,

$$x \;=\; \phi_{s_n}^{g} \circ \phi_{s_{n-1}}^{ad_f g} \circ \cdots \circ \phi_{s_1}^{ad_f^{n-1} g}(0)$$

with $\phi_s^h(\cdot)$ being the flow of a vector field h. We define the *s-metric* to be the Riemannian metric in which the vector fields $\{\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}, \ldots, \frac{\partial}{\partial s_n}\}$ are orthonormal.

Note that the construction of the s-metric makes sense only on the subset of $\mathcal{M}$ on which the s-coordinates are valid. If necessary, we restrict to a subset of $\mathcal{M}$ which is an open and bounded subset of $\mathbf{R}^n$ containing the origin and having a smooth boundary on which the s-coordinates are valid.

The following example illustrates the construction of s-coordinates for a system in $\mathbf{R}^3$.

Example 2.0 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + h_1(x_3) + h_2(x_1, x_2) \\ \dot{x}_2 &= x_3 + h_3(x_3) + h_4(x_1, x_2) \\ \dot{x}_3 &= u, \end{aligned} \tag{24}$$

where $h_i(\cdot)$ are smooth functions with $h_i(0) = \frac{\partial h_i}{\partial x_j}(0) = 0$ and $\frac{\partial h_2}{\partial x_2}(x_1, 0) = \frac{\partial h_4}{\partial x_2}(x_1, 0) = 0$. We have

$$\begin{aligned} g &= \tfrac{\partial}{\partial x_3}, \\ ad_f g &= -h_1'(x_3)\tfrac{\partial}{\partial x_1} - (1 + h_3'(x_3))\tfrac{\partial}{\partial x_2}, \\ ad_f^2 g &= ((1 + h_3'(x_3))(1 + \tfrac{\partial h_2(x_1,x_2)}{\partial x_2}) + h_1'(x_3)\tfrac{\partial h_2(x_1,x_2)}{\partial x_1})\tfrac{\partial}{\partial x_1} \\ &\quad + ((1 + h_3'(x_3))\tfrac{\partial h_4(x_1,x_2)}{\partial x_2} + h_1'(x_3)\tfrac{\partial h_4(x_1,x_2)}{\partial x_1})\tfrac{\partial}{\partial x_2}. \end{aligned} \tag{25}$$

Note that $ad_f^2 g(x_1, 0, 0) = \frac{\partial}{\partial x_1}$ and $ad_f^2 g(x_1, x_2, 0) = -\frac{\partial}{\partial x_2}$. We see that the s-coordinates

$$\begin{aligned} s_1 &:= x_1 \\ s_2 &:= -x_2 \\ s_3 &:= x_3. \end{aligned} \tag{26}$$

are valid on all of $\mathbf{R}^3$. □

Remark 2.1 It is possible to consider a more general construction of s-coordinates than the one using the flow of $\{ad_f^{n-1} g, \ldots, ad_f g, g\}$. Actually, all results of this paper, except those of Section 6, remain true, if to construct the s-coordinates one uses the flow of any set of vector fields $\{g_1, g_2, g_3, \ldots, g_n\}$ with the property that g_1 is transversal to the characteristic distribution D in a neighborhood of 0 and $\{g_2, g_3, \ldots, g_n\}$ is any basis for D. In [8] we construct another set of s-coordinates (in this more general sense) for the system (24). □

3 Least Squares Exact Integrating Factors

Clearly, the value of the functional $I_1(\beta)$ given by (7) is always nonnegative. We will show that it attains zero if and only if the system (1) is exactly feedback linearizable. Thus the minimum value of the functional provides a *measure of linearizability* of (1) in a least squares sense.

Theorem 3.1 *Let $\mathcal{M} \subset \mathbf{R}^n$ be an open, bounded, and contractible region with a smooth boundary $\partial\mathcal{M}$. Then the system (1) is exactly feedback linearizable if and only if there exists a smooth positive zero-form β such that $I_1(\beta) = 0$. Moreover, in this case any minimizer has the form $\beta = c = const$ for some $c \neq 0$.*

Proof: ($\Rightarrow$) Note that in the case of exactly feedback linearizable system (1) the vector fields $\{ad_f^{n-2} g, \ldots, ad_f g, g\}$ span the tangent spaces to the $n-1$ dimensional manifolds $s_1 = const$. Thus, we have span $\{\frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3}, \ldots, \frac{\partial}{\partial s_n}\} = D$. Therefore, in s-coordinates any characteristic form ω_0 has the form

$$\omega_0 = \omega_0^1 ds_1,$$

where ω_0^1 is a smooth nonvanishing function of $s_1, \ldots, s_n$. Moreover, the normalization $|\omega_0| = 1$ and $\omega_0(ad_f^{n-1} g) > 0$ implies $\omega_0^1 = 1$.Thus

$$d\beta\omega_0 = d\beta \wedge ds_1 = \sum_{i=2}^{n} \frac{\partial \beta}{\partial s_i} ds_i \wedge ds_1,$$

$$\delta\beta\omega_0 = \frac{\partial \beta}{\partial s_1}.$$

Note that in the s-metric (or, more precisely in the metric on forms induced by s-metric on vector fields) $ds_i \wedge ds_1$ are orthonormal. Therefore

$$|d\beta\omega_0|^2 + |\delta\beta\omega_0|^2 = |d\beta|^2. \tag{27}$$

Choosing $\beta_0 = c$, where c is a nonzero constant, we obtain $I_1(\beta_0) = 0$. Changing sign of c, if necessary, we may assume that β_0 is strictly positive.

($\Leftarrow$) $I_1(\beta) = 0$ implies in particular $d\beta\omega_0 = 0$ on $\mathcal{M}$. Thus $\beta\omega_0$ is exact in a neighborhood of the origin. □

One can also show that

Theorem 3.2 *The system (1) is exactly feedback linearizable if and only if there exists a smooth positive zero-form β such that $I_m(\beta) = 0$. Moreover, in this case any minimizer has the form $\beta = c = const$ for some $c \neq 0$.*

Proof: It is similar to that of Theorem 3.1. □

Corollary 3.1 *Let $\mathcal{M} \subset \mathbf{R}^n$ be an open, bounnded, and contractible region with a smooth boundary $\partial\mathcal{M}$. A linearly controllable system (1) is exactly feedback linearizable if and only if there exists a characteristic form ω for (1) which is harmonic in the s-metric.*

A nice feature of an s-metric is that in the case of linearizable systems the value of s_1 parametrizes the integral manifolds of D. The calculation of terms $d\beta\omega_0$ and $\delta\beta\omega_0$ in the proof of Theorem 3.1 shows precisely that the exterior derivative in $d\beta\omega_0$ controls the behavior of β *along* the integral manifolds of ω_0, while the codifferential δ in $\delta\beta\omega_0$ controls the behavior of β in the direction *transversal* to the integral manifolds of ω_0.

A natural question arises of what happens if we skip the assumption that the Riemannian metric is an s-metric. As we shall see in the next section, the minimum of the integral is still obtained for some smooth and positive β. However, we can no longer guarantee that the minimum value of the functional will be zero for a linearizable system. (Actually, for a generic choice of metric this will not be the case.) Hence, we cannot claim that the minimum value of the functional is a measure of linearizability if we do not define it with an s-metric.

4 Least Squares Approximate Integrating Factors of Order One

It can be shown that the minimum of $I_1(\beta)$ exists for any positive definite Riemannian metric. To simplify calculations we will use the standard metric associated with some coordinate system x_i. In applications we would seek for the minimum of the functional in some s-metric, since only for those metrics we know that we obtain an exact integrating factor for a characteristic form ω_0 if one exists (i.e. when the corresponding system (1) is exactly linearizable).

Theorem 4.1 *Let $\mathcal{M} \subset \mathbf{R}^n$ be an open, bounded, and contractible region with a smooth boundary $\partial\mathcal{M}$ with the standard metric associated with some coordinate system x_i. Then, among all zero-forms β in $H^1(\mathcal{M})$ there exists a smooth positive zero-form β that minimizes the functional $I_1(\beta)$ defined by (7), determined uniquely up to a multiplication by a positive constant. Define*

$$\lambda_0 := \inf_{\beta \in H^1(\mathcal{M})} I_1(\beta) \ . \tag{28}$$

Any minimizing zero-form β is a solution to the boundary value problem

$$\langle \omega_0, \Delta(\beta\omega_0) \rangle - \lambda_0 \beta = 0, \ \text{on } \mathcal{M}, \tag{29}$$

with the boundary conditions

$$\omega_0 \wedge *d(\beta\omega_0) - \delta(\beta\omega_0) \wedge *\omega_0 = 0, \ \text{on } \partial\mathcal{M}. \tag{30}$$

Moreover, any $\beta \not\equiv 0$ in $H^1(\mathcal{M})$ satisfying (29) and (30) is a minimizer of $I_1(\beta)$.

Proof: Step 1. Boundedness. Note that $I_1(\beta) \geq 0, \forall \beta \in H^1(\mathcal{M})$, so that $\lambda_0 \geq 0$. Let $\beta_i \in H^1(\mathcal{M})$ be any minimizing sequence, i.e. $\lim I_1(\beta_i) = \lambda_0$. Without loss of generality we can assume that the minimizing sequence is normalized in $L_2(\mathcal{M})$, so that

$$||\beta_i \omega_0|| = ||\beta_i|| = 1. \tag{31}$$

Note also that we can assume that for some real C_0 we have

$$||d\beta_i\omega_0||^2 + ||\delta\beta_i\omega_0||^2 \leq C_0. \tag{32}$$

Let $l(\beta) := |d\beta\omega_0|^2 + |\delta\beta\omega_0|^2$. Note that $l(\beta)$ is a quadratic form in β and its first partial derivatives. We will show that there are positive constants C_1, C_2 such that

$$l(\beta) \geq C_1 |d\beta|^2 - C_2 \beta^2 \, \forall x \in \mathcal{M}. \tag{33}$$

For this, by direct calculation we verify that

$$l(\beta) = |d\beta|^2 + l(1)\beta^2 + \sum_{i=1}^{n} c_i(x) \beta \frac{\partial \beta}{\partial x_i}, \tag{34}$$

where $c_i(x)$ are smooth functions (depending on ω_0 and the partial derivatives of its components). Thus,

$$l(\beta) \geq |d\beta|^2 + l(1)\beta^2 - C_4 \sum_{i=1}^{n} |\beta| |\frac{\partial \beta}{\partial x_i}|, \tag{35}$$

where $C_4 \geq |c_i(x)| \, \forall i \, \forall x \in \mathcal{M}$ Now, using Lemma 1.1 we have

$$|\beta| |\frac{\partial \beta}{\partial x_i}| \leq \frac{\beta^2}{2\epsilon} + \frac{\epsilon}{2} (\frac{\partial \beta}{\partial x_i})^2 \ \ \forall i, \, \forall \epsilon > 0, \tag{36}$$

so that $\forall\epsilon > 0$ we have

$$l(\beta) \geq (1 - \frac{C_4\epsilon}{2})|d\beta|^2 + (l(1) - \frac{nC_4}{2\epsilon})\beta^2. \tag{37}$$

Put

$$C_1 := 1 - \frac{C_4\epsilon}{2},\ C_2 := \frac{nC_4}{2\epsilon} - l(1). \tag{38}$$

Note that the constants are positive for ϵ small enough. (Actually, we only need positivity of C_1.) The normalization (31) and the inequalities (32) and (33) imply that

$$||d\beta_i|| \leq C_3 \tag{39}$$

for some real C_3, so that the mimimizing sequence β_i is bounded in $H^1(\mathcal{M})$. In particular, we proved that

$$\mathcal{D}q_1 = H^1(\mathcal{M}). \tag{40}$$

Step 2. Compactness. By Rellich-Kondrasov Theorem (a ball in $H^1(\mathcal{M})$ is precompact in $L_2(\mathcal{M})$), β_i has a subsequence, to be also denoted by β_i, converging to some $\beta_0 \in H^1(\mathcal{M})$ weakly in $H^1(\mathcal{M})$ and strongly in $L_2(\mathcal{M})$. Note that since $||\beta_i|| = 1$ we also have $||\beta_0|| = 1$. Since the norm $||\cdot||$ is lower semicontinuous with respect to the weak convergence,

$$I_1(\beta_0) \leq \liminf I_1(\beta_i) = \lambda_0,$$

so that β_0 is a minimizer.

Step 3. Regularity. Since we know that a minimizer $\beta_0 \in H^1(\mathcal{M})$ exists, we can write the corresponding Euler-Lagrange equations in a weak form. We know that $I_1(\beta_0) = \lambda_0$ so that the associated functional

$$J(\beta) := q_1(\beta, \beta) - \lambda_0||\beta||^2 = \int_{\mathcal{M}} (|d\beta\omega_0|^2 + |\delta\beta\omega_0|^2 - \lambda_0\beta^2)\mu \tag{41}$$

has value zero for $\beta = \beta_0$ and $J(\beta_0 + t\eta) \geq 0$ for any fixed $\eta \in H^1(\mathcal{M})$ and any real t (cf. [12, Section 8.12]). Thus, for fixed $\eta \in H^1(\mathcal{M})$ the first variation $\frac{\partial J(\beta_0+t\eta)}{\partial t}$ must vanish at $t = 0$. A straightforward calculation gives the following Euler-Lagrange equation in a weak form.

$$\left.\frac{\partial J(\beta_0 + t\eta)}{\partial t}\right|_{t=0} = 2(q_1(\beta_0, \eta) - \lambda_0\langle\langle\beta_0, \eta\rangle\rangle) = 0, \tag{42}$$

for all $\eta \in H^1(\mathcal{M})$. The estimate (33) shows that the form $q_1(\cdot,\cdot) - \lambda_0\langle\langle\cdot,\cdot\rangle\rangle$ defined is coercive in $H^1(\mathcal{M})$ and hence a standard elliptic regularity argument (cf. [10]) shows that $\beta_0 \in C^\infty(\overline{\mathcal{M}})$. Therefore, integration by parts is legal. We use Proposition 1.1 to obtain the strong version of the Euler-Lagrange PDE (29) with the boundary conditions (30).

Step 4. Positivity. One can check that if β_0 is a minimizer of $I_1(\cdot)$, so is $|\beta_0|$. From Step 3 we see that $\beta_0 \in H^1(\mathcal{M})$ is a minimizer of $I_1(\cdot)$ if and only if it is an eigenfunction (the ground state) of the strongly elliptic operator $L\beta := \langle\omega_0, \Delta(\beta\omega_0)\rangle$ corresponding to the minimal eigenvalue λ_0 of L. Therefore, if β_0 is a minimizer of $I_1(\cdot)$, then $|\beta_0|$ is an eigenfunction of L. The Harnack inequality (cf. [12, Thm 8.21]) shows that $|\beta_0| > 0$ so that β_0 must have been either strictly positive or negative. One can assume that β_0 is strictly positive.

Step 5. Uniqueness. If there were two linearly independent minimizers of $I_1(\beta)$, they would be both eigenfunctions of L corresponding to the eigenvalue λ_0. But then one could choose them to be orthogonal. Yet, Step 4 shows that each of them is either strictly positive or negative, so that they cannot be orthogonal. Thus, the eigenspace of L corresponding to the minimal eigenvalue λ_0 is one-dimensional, so that the minimizer is determined uniquely up to a multiplication by a nonzero constant. (cf. [12, Thm 8.38]) □

Remark 4.1 It can be shown that in the case when ω_0 is not normalized, a zero-form β minimizing $I_1(\beta)$ is a solution to the boundary value problem

$$\langle \omega_0, \Delta(\beta\omega_0)\rangle - \lambda_0\beta|\omega_0|^2 = 0 \tag{43}$$

with the boundary conditions (30). □

Below we show an example of construction of a least squares approximate integrating factor for a system in $\mathbf{R}^3$.

Example 4.0 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + \log(\cos(ax_3)^{\frac{1}{a}}) \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u, \end{aligned} \tag{44}$$

where $a > 0$. Let $\mathcal{M}$ be any open, bounded, and contractible region on which $|x_3| < \frac{\pi}{2a}$. Note that this system is a particular case of (24), so that the s-coordinates are $s_1 := x_1$, $s_2 := -x_2$, and $s_3 := x_3$. We have

$$\begin{aligned} g &= \tfrac{\partial}{\partial x_3} &&= \tfrac{\partial}{\partial s_3}, \\ ad_f g &= \tan(ax_3)\tfrac{\partial}{\partial x_1} - \tfrac{\partial}{\partial x_2} &&= \tan(as_3)\tfrac{\partial}{\partial s_1} + \tfrac{\partial}{\partial s_2}, \\ ad_f^2 g &= \tfrac{\partial}{\partial x_1} &&= \tfrac{\partial}{\partial s_1}. \end{aligned} \tag{45}$$

The normalized characteristic one-form is

$$\omega_0 := \cos(as_3)ds_1 - \sin(as_3)ds_2.$$

One can show that β minimizing $I_1(\beta)$ depends only on s_3. After straightforward calculations we obtain

$$\begin{aligned} |d\beta\omega_0|^2 &= \beta'^2 + a^2\beta^2 \\ |\delta\beta\omega_0|^2 &= 0. \end{aligned} \tag{46}$$

Therefore,

$$I_1(\beta) = \frac{\int_{\mathcal{M}}(\beta'^2 + a^2\beta^2)\mu}{\int_{\mathcal{M}}\beta^2\mu}. \tag{47}$$

Note that for every $\beta \in H^1(\mathcal{M})$ we have $I_1(\beta) \geq a^2$. Moreover, for any nonzero constant c, for $\beta = c$, we have $I_1(\beta) = a^2$. Thus $\beta = c$ are the minimizers of $I_1(\beta)$. We also see that a^2 is a measure of linearizability of the system (44). □

One can rarely expect to be able to find least squares integrating factors directly by investigating the functional $I_1(\beta)$ or solving the corresponding eigenvalue problem. An important feature of our variational problem is that one can *approximate* least squares integrating factors by a sequence of minimizers β_m of the corresponding functionals $I_1^m(\beta)$ where $I_1^m(\beta)$ is simply $I_1(\beta)$ with β is restricted to a fixed m-dimensional subspace E_m of $H^1(\mathcal{M})$.

Theorem 4.2 *Let $\mathcal{M} \subset \mathbf{R}^n$ be an open and bounded region with a smooth boundary $\partial\mathcal{M}$. Let $\{e_j\}$ be a basis for the space $H^1(\mathcal{M})$, which is orthonormal in $L_2(\mathcal{M})$. Let β_m be the unique positive minimizer of $I_1(\beta)$ for $\beta \in E_m := \mathrm{span}\,\{e_1, \ldots, e_m\}$ with $\|\beta_m\omega_0\| = 1$. Then the sequence $\{\beta_m\}$ converges strongly in $L_2(\mathcal{M})$ to the unique positive minimizer β_0 of $I_1(\beta)$ with $\|\beta\omega_0\| = 1$.*

Proof: Let b_j be the coefficients of β_0 with respect to $\{e_j\}$. Then the sequence of partial sums $\gamma_m := \sum_{j=1}^m b_j e_j$ converges to β_0 strongly in $H^1(\mathcal{M})$. Thus $I_1(\gamma_m)$ converges to $I_1(\beta_0)$. Obviously, $I_1(\beta_m) \le I_1(\gamma_m)$. As in Step 1 of the proof of Theorem 4.1 one shows that $\|d\beta_m\|$ are bounded. It follows from the uniqueness of positive normalized minimizer β_0 of $I_1(\beta)$ that β_m converges strongly in $L_2(\mathcal{M})$ to β_0. $\square$

Note that minimizing the functional $I_1(\beta)$ over a *finite dimensional space* E_m is a standard quadratic minimization problem. The solution for the coefficients of β_m in basis $\{e_1, \ldots, e_m\}$ is an eigenvector of a nonnegative definite $m \times m$ matrix $\{q_1(e_i, e_j)\}$ corresponding to its smallest eigenvalue. By solving the problem for each m we obtain a sequence of approximate minimizers that converges in $L_2(\mathcal{M})$ to the unique positive minimizer β_0 of $I_1(\beta)$ with $\|\beta\omega_0\| = 1$.

5 Higher Order Approximate Integrating Factors

In this section we show that the minimum of $I_m(\beta)$ is attained. We begin with the following important result.

Proposition 5.1 *For any $m \ge 1$, one has $\mathcal{D}q_m \subset H^1(\mathcal{M})$ and*

$$\|\beta\|_1^2 \le C(q_m(\beta,\beta) + \|\beta\|^2), \ \forall \beta \in \mathcal{D}q_m. \tag{48}$$

Proof: In Step 1 of the proof of Theorem 4.1 we showed that $\mathcal{D}q_1 = H^1(\mathcal{M})$. It is obvious that $\mathcal{D}q_m \subset \mathcal{D}q_1$. $\square$

Theorem 5.1 *Let $\mathcal{M} \subset \mathbf{R}^n$ be an open, bounded, and contractible region with a smooth boundary $\partial\mathcal{M}$ and m be a positive integer. Then, among all zero-forms β in $\mathcal{D}q_m$, there exists a smooth zero-form β that minimizes the functional $I_m(\beta)$ defined by (20).*

Proof: Let $\lambda := \inf I_m(\beta)$. Note that $I_m(\beta) \geq 0, \forall \beta \in \mathcal{D}q_m$, so that $\lambda \geq 0$. Let $\beta_i \in \mathcal{D}q_m$ be any minimizing sequence, i.e., $\lim I_m(\beta_i) = \lambda$. Without loss of generality we can assume that the minimizing sequence is normalized in $L_2(\mathcal{M})$ by (31). Thus $q_m(\beta_i, \beta_i) = I_m(\beta_i)$ and for some real C_0 we have

$$q_m(\beta_i, \beta_i) \leq C_0. \tag{49}$$

Therefore, by Proposition 5.1, we have

$$\|\beta_i\|_1 \leq C_1, \tag{50}$$

for some C_1. Thus, by Rellich-Kondrasov Theorem (cf. [20]), there is $\beta_0 \in \mathcal{D}q_m$ such that there is a subsequence of β_i (to be also denoted by β_i) which converges to β_0 strongly in L_2 and all the terms $d_j\beta_i\omega_0$ and $\delta_j\beta_i\omega_0$ converge weakly in L_2 to $d_j\beta_0\omega_0$ and $\delta_j\beta_0\omega_0$. To explain the latter, we notice that $d_j\beta_i\omega_0$ and $\delta_j\beta_i\omega_0$ converge weakly in L_2 to something and in the distributional sense to $d_j\beta_0\omega_0$ and $\delta_j\beta_0\omega_0$, respectively, so that the limits coincide. Moreover, since the norm is lower semicontinuous with respect to weak convergence, we obtain $q_m(\beta_0, \beta_0) = I_m(\beta_0) = \lambda$.

Note that for every $\eta \in \mathcal{D}q_m$ one has

$$q_m(\beta_0, \eta) - \lambda\langle\langle \beta_0, \eta \rangle\rangle) = 0. \tag{51}$$

Since the smooth functions with compact support on $\mathcal{M}$ are in $\mathcal{D}q_m$, it follows from Proposition 1.1 that β_0 satisfies the following Euler-Lagrange equation in a weak form

$$\int_{\mathcal{M}} (\langle\langle (\Delta + \Delta^2 + \cdots + \Delta^m - \lambda)\eta\omega_0, \beta\omega_0 \rangle\rangle)\mu = 0, \tag{52}$$

for all $\eta \in C_0^\infty(\mathcal{M})$. Standard interior elliptic regularity arguments (see, e.g., [11, Part 1, 15-16]) show that β is smooth in $\mathcal{M}$. □

Note that Theorem 5.1 is a much weaker result than Theorem 4.1. First of all, contrary to the case $m = 1$, we have not proved positivity and uniqueness (up to multiplication by a constant) of minimizers of I_m for $m > 1$ in a general situation. Since we do not have positivity of minimizers β of I_m for $m > 1$, we cannot claim that $\beta\omega_0$ is a characteristic one-form for (1) on $\mathcal{M}$. However, by a perturbation argument, we will show in the next section that we have positivity and uniqueness of higher order approximate integrating factors for system sufficiently close to being linearizable. In general, positivity of approximate integrating factors seems to be very difficult to prove, for there is no general theory that would guarantee positivity of the ground state of higher order elliptic operators.

Another weakness of Theorem 5.1 is that it only guarantees smoothness of minimizers of I_m for $m > 1$ in the interior of $\mathcal{M}$ and we do not have smoothness on the boundary. The reason is that, except when $m = 1$, the form $q_m(\cdot, \cdot)$ is not coercive in $H^m(\mathcal{M})$ and thus $q_m(\beta, \beta)$ does not control the behavior of β on the boundary of $\mathcal{M}$. This, in turn, does not allow one to claim that the minimizers satisfy certain boundary conditions in the classical sense. One can treat boundary conditions by abstract trace techniques (see [3]), but it does not improve the regularity of the minimizer on the boundary. It seems to be possible to obtain smoothness up to the boundary and classical boundary conditions for the minimizer (as in the case $m = 1$), by modifying the functional I_m so that it involves some more higher order partial derivatives.

One can derive an approximation scheme for the minimizer of I_m for $m > 1$ similar to that given by Theorem 4.2.

6 Higher Order Approximate Integrating Factors for Systems Close to Being Linearizable

In this section we show that the higher order approximate integrating factors do not vanish in $\mathcal{M}$ if the system is sufficiently close to being linearizable. Since we are going to work with several systems defined by several pairs f, g, each defining its own $s^{f,g}$-coordinates, and thus a metric and a functional, to compare them we need to introduce some common fixed x-coordinates and Sobolev spaces. Denote by $\|\cdot\|^x$, $\|\cdot\|^x_m$, $\|\cdot\|^x_{k,\infty}$ respectively the L_2, H^m, and C^k norms in some fixed x-coordinates. We will denote the corresponding spaces by L^x_2, $H^{x,m}$, and $C^{x,k}$, respectively. For every pair f, g the corresponding L_2 norms, Laplacians, functionals, and quadratic forms defined using $s^{f,g}$-coordinates will be denoted by $\|\cdot\|^{f,g}$, $\Delta_{f,g}$, $I^{f,g}_m(\cdot)$, and $q^{f,g}_m(\cdot,\cdot)$, respectively. We will assume that all the $s^{f,g}$-coordinates are global on $\mathcal{M}$ and all Jacobian determinants of changes of coordinates from x to $s^{f,g}$ and from $s^{f,g}$ to $x^{f,g}$ are uniformly bounded away from zero. We also assume that all $s^{f,g}$-coordinates have common origin. (These assumptions are not essential, they are made mainly for simplicity of presentation.)

In the sequel we will need the following basic result.

Proposition 6.1 *For any k the mapping $(f, g) \mapsto s^{f,g}(\cdot)$ assigning s-coordinates to f, g is a continuous mapping from $C^{x,k+n-1} \times C^{x,k+n-1}$ to $C^{x,k}$.*

The main result of this section is as follows.

Theorem 6.1 *Let f_0, g_0 define a linearizable system on $\mathcal{M}$. Then for a sufficiently large integer k and a sufficiently small real number ρ if $\|f - f_0\|^x_{k,\infty} + \|g - g_0\|^x_{k,\infty} < \rho$ then the zero-forms β minimizing the functional $I^{f,g}_m(\beta)$ do not vanish in $\mathcal{M}$ and are unique up to multiplication by a nonzero constant.*

Proof: Let k, k' be positive integers to be determined later. It follows from straightforward calculations that, after change of coordinates from s to x, the operator $L_{f,g} := \Delta_{f,g} + \Delta^2_{f,g} + \cdots + \Delta^m_{f,g} - \lambda_{f,g}$ is a strongly elliptic differential operator of order $2m$ in coordinates x, whose coefficients are smooth functions and change continuously in $C^{x,k'}$ if f, g change continuously in $C^{x,k} \times C^{x,k}$, for sufficiently large k.

To prove the theorem it is enough to show that, for any $\mathcal{M}' \subset\subset \mathcal{M}$ and any sequence of normalized minimizers β_i of the functionals $I^{f_i,g_i}_m(\cdot)$ corresponding to f_i, g_i converging in $C^{x,k}$ to f_0, g_0, one has $\liminf_{\mathcal{M}'} |\beta_i| > 0$. Suppose to the contrary that there is $\mathcal{M}'$ and a sequence f_i, g_i, β_i violating this statement. Observe that L_{f_i,g_i} have ellipticity constant in x-coordinates uniform in i and common bounds on $C^{x,k'}$ norms of coefficients. Recall that β_i are smooth on $\mathcal{M}$ and satisfy $L_{f_i,g_i}\beta_i = 0$ on $\mathcal{M}$. Moreover, we can also assume $||\beta_i||^{f_i,g_i} = ||\beta_i\omega_{f_i,g_i}||^{f_i,g_i} = 1$ (note that we have a standing assumption $|\omega_{f_i,g_i}|^{f_i,g_i} = 1$), and hence $q_m^{f_i,g_i}(\beta_i,\beta_i) = I_m^{f_i,g_i}(\beta_i)$. One can easily show that for sufficiently large i one has

$$c_1 \leq ||\beta_i||^x \leq c_2 \tag{53}$$

for some positive c_1, c_2. Note that for fixed η_1, η_2, $q_m^{f_i,g_i}(\eta_1,\eta_2)$ change continuously if f_i, g_i change continuously in $C^{x,k} \times C^{x,k}$, for sufficiently large k. Therefore, since $\beta = c \neq 0$ (see Theorem 3.2) is a minimizer of the functional $I_m^{f_0,g_0}(\cdot)$ and $q_m^{f_i,g_i}(\beta_i,\beta_i) \leq q_m^{f_i,g_i}(c,c)$, we have $q_m^{f_i,g_i}(\beta_i,\beta_i) \to 0$. Thus, one can prove that

$$||\beta_i||_1^x \leq c_3, \tag{54}$$

for some positive c_3. Choose a subsequence of β_i, also denoted by β_i, that converges strongly in L_2^x. Expressing all terms $d_j\beta_i\omega_i$ and $\delta_j\beta_i\omega_i$ as explicit differential operators in x-coordinates and arguing as in the proof of Theorem 5.1 it can be shown (passing to a subsequence, if necessary) that $d_j\beta_i\omega_i$ and $\delta_j\beta_i\omega_i$ converge weakly in L_2^x to $d_j\beta_0\omega_0$ and $\delta_j\beta_0\omega_0$, for some $\beta_0 \in \mathcal{D}q_m^{f_0,g_0}$. Thus $q_m^{f_0,g_0}(\beta_0,\beta_0) = 0$. Therefore, β_0 is a constant that we can assume to be positive. Since the operators L_{f_i,g_i} have ellipticity constant in x-coordinates uniform in i and common bounds on $C^{x,k'}$ norms of coefficients, by standard interior regularity results (see, e.g., [11, Part 1, 15-16]), we get $||\beta_i||_{m'}^{x,\mathcal{M}'} \leq C_{m'}$, where m' can be arbitrarily large by choice of k'. From Sobolev embeddings ([2]) it follows that β_i are uniformly continuous on $\mathcal{M}'$, uniform in i. Hence, β_i converge uniformly to β_0, which is a nonzero constant. We have reached a contradiction.

Assume now that the k and ρ are such that the corresponding minimizers of $I_m(\beta)$ are positive. To prove uniqueness, observe that one can easily show (using (51)) that the space of minimizers of $I_m(\beta)$ is linear. Thus, if the space of minimizers of $I_m(\beta)$ were of dimension two or more, there would be two linearly independent minimizers β_1 and β_2 of $I_m(\beta)$. Moreover, one could choose them to be orthogonal in $L_2(\mathcal{M})$. But then at least one of them would have to change sign, contradicting the positivity. Thus, the space of minimizers of $I_m(\beta)$ has dimension one. □

Note that the optimal value of integer k can be explicitly computed.

7 A Lower Bound

In this section we establish a lower bound for the value of the functionals $I_m(\beta)$. For this we need a decomposition of the exterior derivative of a characteristic form ω_0 (cf. [7, 6]).

Proposition 7.1 *Let ω_0 be a given one-form on $\mathcal{M}$. Then there is a one-form γ and a two-form τ such that*

1. $d\omega_0 = \gamma \wedge \omega_0 + \tau$

2. $|\tau| = |d\omega_0 - \gamma \wedge \omega_0|$ *is pointwise minimal possible*

3. for any one-form ζ *the two-forms* $\zeta \wedge \omega_0$ *and* τ *are pointwise orthogonal.*

The two-form τ *is determined uniquely, while the one-form* γ *is determined uniquely up to an addition of a multiple of the one-form* ω_0 *by any zero-form. The forms* γ *and* τ *can be explicitly expressed as*

$$\begin{array}{rcl} \gamma & = & (-1)^{n+1} \frac{*(\omega_0 \wedge * d\omega_0)}{|\omega_0|^2} \\ \tau & = & \frac{*(\omega_0 \wedge *(d\omega_0 \wedge \omega_0))}{|\omega_0|^2}, \end{array} \tag{55}$$

where $*$ *is the Hodge star operator.*

Proof: To prove that γ and τ given by (55) satisfy $d\omega_0 = \gamma \wedge \omega_0 + \tau$ is a tedious, but straightforward calculation. To prove the third statement, let ζ be any one-form on $\mathcal{M}$. Note that

$$\langle \zeta \wedge \omega_0, \tau \rangle \mu = \zeta \wedge \omega_0 \wedge * \tau = \zeta \wedge \omega_0 \wedge \frac{(\omega_0 \wedge *(d\omega_0 \wedge \omega_0))}{|\omega_0|^2} = 0.$$

Thus, $\zeta \wedge \omega_0$ and τ are pointwise orthogonal. It was proved in [7] that the second statement follows from the third one. □

An alternative expression for γ and τ in terms of interior products instead of $*$-operator can be found in [7, 6]. It can be shown that $\tau = 0$ if and only if ω_0 is integrable and $\tau = 0$ and $\gamma = 0$ if and only if ω_0 is closed. With those forms we can express $d\beta\omega_0$ as

$$d\beta\omega_0 = (d\beta + \beta\gamma) \wedge \omega_0 + \beta\tau. \tag{56}$$

Now, we can obtain the following lower bound for $I_m(\beta)$

Proposition 7.2 *Let* $m \geq 1$*. For every* $\beta \in \mathcal{D}q_m$ *we have*

$$I_m(\beta) \geq (\inf |\tau|)^2, \tag{57}$$

where the infimum is taken over $\mathcal{M}$*.*

Proof: Since $|\omega_0| = 1$ and the two-forms $(d\beta + \beta\gamma) \wedge \omega_0$ and τ are pointwise orthogonal for every β, we have

$$\begin{array}{l} I_m(\beta) \geq I_1(\beta) = \frac{\int_{\mathcal{M}} (|d\beta\omega_0|^2 + |\delta\beta\omega_0|^2)\mu}{\int_{\mathcal{M}} |\beta\omega_0|^2 \mu} \geq \frac{\int_{\mathcal{M}} |d\beta\omega_0|^2 \mu}{\int_{\mathcal{M}} |\beta\omega_0|^2 \mu} = \frac{\int_{\mathcal{M}} |(d\beta + \beta\gamma) \wedge \omega_0 + \beta\tau|^2 \mu}{\int_{\mathcal{M}} |\beta\omega_0|^2 \mu} \\ \quad = \frac{\int_{\mathcal{M}} |(d\beta + \beta\gamma) \wedge \omega_0|^2 \mu}{\int_{\mathcal{M}} |\beta\omega_0|^2 \mu} + \frac{\int_{\mathcal{M}} |\beta\tau|^2 \mu}{\int_{\mathcal{M}} |\beta\omega_0|^2 \mu} \geq \frac{\int_{\mathcal{M}} |\beta|^2 |\tau|^2 \mu}{\int_{\mathcal{M}} |\beta|^2 |\omega_0|^2 \mu} \geq (\inf |\tau|)^2. \end{array}$$

□

Note that the lower bound for $I_m(\beta)$ derived above is positive if and only if τ is everywhere nonzero. (It can be shown that the latter is the case whenever $d\omega_0 \wedge \omega_0 \neq 0$ on $\mathcal{M}$.) It follows from Frobenius' Theorem that an exact integrating factor for $\omega_0 \neq 0$ exists if and only if $\tau = 0$ on $\mathcal{M}$. Thus, $|\tau|$ provides a pointwise measure of integrability of ω_0. From this point of view we may say that Proposition 7.2 gives a positive lower bound for the average measure of integrability in terms of a pointwise one. This lower bound is easy to find, for it requires only pointwise calculations. If one uses an arbitrary (i.e., not normalized) characteristic form ω_0, one should replace $(\inf|\tau|)^2$ with $\frac{\inf|\tau(x)|}{\sup|\omega_0(x)|}$ in (57). It can be shown that the ratio $\frac{\inf|\tau(x)|}{\sup|\omega_0(x)|}$ is independent of the choice of ω_0.

8 Hodge Decomposition

In this section we show how to approximate a characteristic one-form $\beta\omega_0$ by an exact form $d\alpha$ in a least-squares sense. The most important result is the *Hodge Decomposition Theorem* (see, e.g., [1, 20] and the application in [4]) which provides a least squares approximation of a one-form by an exact form.

Theorem 8.1 *Let ω_0 be a fixed smooth one-form on $\mathcal{M}$. For any β there is a unique (up to a constant) smooth zero-form α on $\mathcal{M}$ and a smooth one-form ϵ such that*

$$\beta\omega_0 = d\alpha + \epsilon, \tag{58}$$

and such that $\|\epsilon\|$ is minimized over all possible smooth zero-forms α. The one-form ϵ satisfies

$$\delta\epsilon = 0$$

in $\mathcal{M}$ and is tangent to the boundary $\partial\mathcal{M}$ of $\mathcal{M}$, i.e., $\epsilon(v) = 0$, where v is the unit outward normal vector field to $\partial\mathcal{M}$. The smooth function α is given uniquely up to an addition of a constant by the solution of Poisson's equation

$$\Delta\alpha = \delta\beta\omega_0 \tag{59}$$

with Neumann boundary conditions

$$d\alpha(v) = \beta\omega_0(v). \tag{60}$$

The one-forms $d\alpha$ and ϵ appearing in the Hodge decomposition (58) of $\beta\omega_0$ will be called, respectively, the *exact part* and *antiexact part* of $\beta\omega_0$.

In the sequel we assume that ω_0 is a characteristic one-form for (1) such that $|\omega_0| = 1$ and $\omega_0(ad_f^{n-1}g) > 0$, β is a smooth nonvanishing function normalized in L_2 so that $\|\beta\| = 1$, and ϵ is the antiexact part of $\beta\omega_0$.

We can obtain the following upper bound on the H^1 norm of ϵ.

Theorem 8.2 *There is a positive constant C (depending only on $\mathcal{M}$) such that*

$$\|\epsilon\|_1 \leq C\sqrt{I_1(\beta)}. \tag{61}$$

Proof: It follows from Theorem 7.7.9 of [20]. □

We see that a "small" value of $I_1(\beta)$ guarantees a "small" value of $\|\epsilon\|_1$. In particular, $I_1(\beta) = 0$ implies $\|\epsilon\|_1 = 0$. Since the bound (61) in monotone in $I_1(\beta)$, we obtain the smallest value of this upper bound by choosing β to be a least squares integrating factor.

We can obtain a similar upper bound on the $H^{2m}(\mathcal{M}')$ norm of ϵ, where $\mathcal{M}' \subset\subset \mathcal{M}$.

Theorem 8.3 *Let $\mathcal{M}' \subset\subset \mathcal{M}$. There is a positive constant C (depending only on $\mathcal{M}'$, $\mathcal{M}$, and m) such that*

$$\|\epsilon\|_{2m}^{\mathcal{M}'} \leq C\sqrt{I_{2m}(\beta)}. \tag{62}$$

Proof: It follows from Theorem 1.1 that $\|\epsilon\|_{2m}^{\mathcal{M}'} \leq C_1(\sqrt{Q_{2m}(\epsilon,\epsilon)} + \|\epsilon\|)$. Observe that $d\epsilon = d\beta\omega_0$ and $\delta\epsilon = 0$ imply that $Q_{2m}(\epsilon,\epsilon) \leq Q_{2m}(\beta\omega_0,\beta\omega_0)$. On the other hand we have $\|\epsilon\| \leq \|\epsilon\|_1$, and it follows from Theorem 8.2 that $\|\epsilon\|_1 \leq C_2\sqrt{I_1(\beta)} = C_2\sqrt{Q_1(\beta\omega_0,\beta\omega_0)}$. Thus $\|\epsilon\|_1 \leq C_2\sqrt{Q_{2m}(\beta\omega_0,\beta\omega_0)}$. This proves the result. □

We see that a "small" value of $I_{2m}(\beta)$ guarantees a "small" value of $\|\epsilon\|_{2m}$. In particular, $I_{2m}(\beta) = 0$ implies $\|\epsilon\|_{2m} = 0$. Again, since the bound (62) in monotone in $I_m(\beta)$, we obtain the smallest value of this upper bound by choosing β to be a higher order least squares integrating factor.

9 Application of Approximate Integrating Factors to Approximate Feedback Linearization

In this section we show how higher order approximate integrating factors can be applied to obtain feedback linearizable systems approximating a given nonlinearizable system. Namely, in the previous section we have shown that construction of least-squares integrating factors provides good least-squares approximation of characteristic one-forms by exact forms. In this section we show that this, in turn, leads to good better approximation of nonlinearizable systems (1) by linearizable systems. To see that, recall that it was shown in [7] that if α and its first $n-1$ Lie derivatives along f have linearly independent differentials one can use them to define a change of coordinates taking (1) to a normal form

$$\dot{z} = Az + Bru + Bp + Eu \tag{63}$$

where A, B are in Brunovsky form and r, p, and $E := [e_1, e_2, \cdots, e_n]^T$ are given by

$$\begin{array}{rcl} e_1 & = & -\epsilon(g), \\ e_2 & = & -(L_f\epsilon)(g), \\ & \vdots & \\ e_n & = & -(L_f^{n-1}\epsilon)(g), \\ p & = & (L_f^{n-1}\omega)(f) - (L_f^{n-1}\epsilon)(f), \\ r & = & (-1)^{n-1}\omega(ad_f^{n-1}g). \end{array} \tag{64}$$

Note that by linear controllability assumption $r \neq 0$ on $\mathcal{M}$. Observe also that E vanishes whenever ϵ does. Neglecting E one obtains a linearizable system

$$\dot{z} = Az + Bru + Bp \tag{65}$$

approximating (63) with "error" Eu. The results of this paper allow one to obtain upper bounds on the H^k and C^k norms of E providing a measure of how well (65) approximates (63) on the given region.

Note that the expression for E involves derivatives of ϵ up to order $n-1$. Therefore, it is possible to find an upper bound on the H^k norm of E depending on the H^{k+n-1} norm of ϵ. Namely, we have the following simple result.

Proposition 9.1 *Let $\mathcal{M}' \subset \mathcal{M}$. There is a positive constant C (depending only on $\mathcal{M}'$, k, f, and g) such that*

$$||E||_k^{\mathcal{M}'} \leq C||\epsilon||_{k+n-1}^{\mathcal{M}'}. \tag{66}$$

Using Sobolev embeddings it is also possible to obtain bounds on the C^k norm of E depending on the H^{m+n-1} norm of ϵ where $m > k + n/2$. The next result follows from the Sobolev embedding $H^m \hookrightarrow C^k$ for $m > k + n/2$ (see, e.g., [2]).

Proposition 9.2 *Let $\mathcal{M}' \subset \mathcal{M}$ and $m > k+n/2$. There is a positive constant C (depending only on $\mathcal{M}'$, f, and g) such that*

$$||E||_{k,\infty}^{\mathcal{M}'} \leq C||\epsilon||_{m+n-1}^{\mathcal{M}'}. \tag{67}$$

Theorem 8.3, combined with Propositions 9.1 and 9.2, allows one to obtain upper bounds on the H^m and C^k norms of E on $\mathcal{M}' \subset\subset \mathcal{M}$ depending on the value of the functional $I_{2m}(\beta)$. Namely, we have the following simple corollary.

Corollary 9.1 *Let $\mathcal{M}' \subset\subset \mathcal{M}$. Assume that ω_0 is a characteristic form for (1) such that $|\omega_0| = 1$ in the s-metric. Let β be any smooth nonzero function normalized in L_2 so that $||\beta|| = 1$ and let α be obtained from the Hodge decomposition (58) of $\beta\omega_0$. Then, if $\alpha, L_f\alpha, \cdots, L_f^{n-1}\alpha$ are valid coordinates in $\mathcal{M}'$, one can put (1) in form (63). Let k be any positive integer and let m_1 be any integer such that $k + n - 1 \leq 2m_1$. Then, there is a positive constant C_1 (depending only on $\mathcal{M}'$, $\mathcal{M}$, f, g, and k) such that*

$$||E||_k^{\mathcal{M}'} \leq C_1\sqrt{I_{2m_1}(\beta)}. \tag{68}$$

Also, let m_2 be any integer such that $k+3n/2-1<2m_2$. Then there is a positive constant C_2 (depending only on $\mathcal{M}'$, $\mathcal{M}$, f, g, and k) such that

$$\|E\|_{k,\infty}^{\mathcal{M}'} \leq C_2\sqrt{I_{2m_2}(\beta)}. \tag{69}$$

The above results allow one to calculate a minimum order of approximate integrating factor that guarantees the required bound on nonlinear perturbation term E in (63). The table below gives, for several values of the dimension n of $\mathcal{M}$, a minimum value of $2m_1$ and $2m_2$ so that $I_{2m_1}(\beta)$ and $I_{2m_2}(\beta)$ provide upper bounds on, respectively, $\|E\|^{\mathcal{M}'}$ and $\|E\|_{0,\infty}^{\mathcal{M}'}$ (i.e. on L_2 and C^0 norms of E) in (68) and (69).

Minimum order of approximate integrating factor								
dimension of $\mathcal{M}$	3	4	5	6	7	8	9	10
$2m_1$ that provides a bound on $\|E\|^{\mathcal{M}'}$ in (68)	2	4	4	6	6	8	8	10
$2m_2$ that provides a bound on $\|E\|_{0,\infty}^{\mathcal{M}'}$ in (69)	4	6	8	8	10	12	14	14

Conclusion

In this paper we have developed a unified approach to the exact and least squares approximate feedback linearization problems. This approach uses the Hodge decomposition to obtain the closest (in a least squares sense) exact form to a given scaled characteristic one-form. We have studied the problem of finding approximate integrating factors that scale a nonintegrable one-form in a way that the scaled form is close to being integrable in L_2 together with some derivatives. Our solution(s), the (order one and higher order) least squares approximate integrating factor, possesses a number of useful properties. In particular, a least squares integrating factor will be an exact integrating factor if the one-form is integrable and will provide a guaranteed level of approximation when the one-form is not integrable. Our work suggests that practical approximation schemes can be developed and applied to the approximate feedback linearization of engineering systems.

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References

[1] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications*, Addison-Wesley, 1983.

[2] R. A. Adams. *Sobolev Spaces*, Academic Press, 1975.

[3] J. P. Aubin. *Approximation of Elliptic Boundary-Value Problems*, Wiley, 1972.

[4] A. Banaszuk, J. Hauser, and D. Taylor. L^2-Approximate feedback linearization using the Hodge decomposition theorem. *Proceedings of the International Symposium on Implicit and Nonlinear Systems*, Fort Worth, TX, December 14-15, 1992, 329–333.

[5] A. Banaszuk and J. Hauser. Least squares approximate feedback linearization: a variational approach. *Proceedings of the 32nd IEEE Conference on Decision and Control*, San Antonio, TX, December 1993, 2760–2765.

[6] A. Banaszuk and J. Hauser. Approximate feedback linearization: approximate integrating factors. *Proceedings of the 1994 American Control Conference*, Baltimore, MD, June 1994, 1690–1694.

[7] A. Banaszuk and J. Hauser. Approximate feedback linearization: homotopy operator approach. Submitted for publication. Also in *Proceedings of the 1994 American Control Conference*, Baltimore, MD, June 1994, 1695–1699.

[8] A. Banaszuk, A. Święch, and J. Hauser. Approximate feedback linearization: least squares approximate integrating factors. *Proceedings of 33rd IEEE Conference on Decision and Control*, Lake Buena Vista, FL, December 1994.

[9] H. Flanders. *Differential Forms with Applications to the Physical Sciences*, Academic Press, 1963.

[10] G. B. Folland. *Introduction to Partial Differential Equations*, Princeton University Press, 1976.

[11] A. Friedman. *Partial Differential Equations*, Holt, Rinehart and Winston, Inc., 1969.

[12] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.

[13] J. Hauser. Nonlinear control via uniform system approximation. *Systems and Control Letters*, 17, 1991, 145–154.

[14] J. Hauser, S. Sastry, and P. Kokotovic. Nonlinear control via approximate input-output linearization: the ball and beam example. *IEEE Trans. Auto. Contr.*, AC-37, 1992, 392–398.

[15] L. R. Hunt, R. Su, and G. Meyer. Global transformations of nonlinear systems. *IEEE Trans. Auto. Contr.*, AC-28, 1983, 24–30.

[16] B. Jakubczyk and W. Respondek. On linearization of control systems. *Bull. Acad. Pol. Sci. Ser. Sci. Math.*, 28, 1980, 517–522.

[17] A. J. Krener, S. Karahan, M. Hubbard, and R. Frezza. Higher order linear approximations to nonlinear control systems. *Proceedings of the 26th Conference on Decision and Control*, 1987, 519–523.

[18] A. J. Krener. On the equivalence of control systems and linearization of nonlinear systems. *SIAM J. Contr.*, 11, 1973, 670–676.

[19] A. J. Krener. Approximate linearization by state feedback and coordinate change. *Systems and Control Letters*, 5, 1984, 181–185.

[20] C. Morrey. *Multiple integrals in calculus of variations*, Springer-Verlag, 1966.

[21] R. Su and L.R. Hunt. A canonical expansion for nonlinear systems. *IEEE Trans. Auto. Contr.*, AC-31, 1986, 670–673.

[22] Z. Xu and J. Hauser. Higher order approximate feedback linearization about a manifold. *Journal of Mathematical Systems, Estimation and Control*, 4, 1994, 451–465.

[23] Z. Xu and J. Hauser. Higher order approximate feedback linearization about a manifold for multi-input systems. *IEEE Trans. Auto. Contr.*, 1995.