Feedback Linearization of Transverse Dynamics for Periodic Orbits in $\mathbf{R}^3$ with Points of Transverse Controllability Loss*

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Abstract. In this paper we show how one can linearize the transverse dynamics of a nonlinear affine single-input system in $\mathbf{R}^3$ in a neighborhood of a periodic orbit in the case when the transverse linear controllability fails in finite number of points along the periodic orbit. An autonomous feedback control providing stability of the periodic orbit is designed in the transverse coordinate system.

Keywords. Nonlinear systems, feedback linearization, periodic orbits, transverse dynamics.

Introduction

Stable maneuvering of a nonlinear system is an important goal in many fields including the flight of aerospace vehicles, robotic manipulation, and the manufacture of sophisticated materials. This goal can often be accomplished by providing a stable orbit (or, more generally, a maneuver) for the system by stabilizing the dynamics *transverse* to that orbit.

Consider the smooth dynamical system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

on $\mathbf{R}^n$ and suppose that $\eta \subset \mathbf{R}^n$ is a periodic orbit of (1) with minimal period T when $u \equiv 0$. We are interested in studying the *structure* of the control system in a neighborhood of the periodic orbit η.

In [1], the following problem was studied. Under what conditions is it possible to find coordinates $(\theta, \rho_1, \ldots, \rho_{n-1})$ and control v so that, after change of coordinates and feedback

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$u = k(x) + l(x)v$, the dynamics of (1) in a neighborhood of the periodic orbit η have the form

$$\begin{array}{lcl} \dot{\theta} & = & 1 + f_1(\theta,\rho) + g_0(\theta,\rho)v \\ \dot{\rho}_1 & = & \rho_2 \\ \vdots & & \\ \dot{\rho}_{n-2} & = & \rho_{n-1} \\ \dot{\rho}_{n-1} & = & v, \end{array} \tag{2}$$

where $f_1(\cdot,\cdot)$ satisfies $f_1(\theta,0) = 0$? The variable $\theta \in S^1 = [0,T]$ (we identify 0 and T) parametrizes the periodic orbit η and the coordinates $(\rho_1,\ldots,\rho_{n-1})$ parametrize the transverse dynamics.

A system (1) which admits such a feedback transformation is called *(globally) transversely feedback linearizable along* η. System (1) is called *transversely linearly controllable* at $x \in \mathbf{R}^n$ if

$$\dim \operatorname{span} \{f(x), g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)\} = n. \tag{3}$$

We say that (1) is *transversely linearly controllable* on a subset of $\mathbf{R}^n$ if (1) is transversely linearly controllable at every point of this subset.

The idea of transverse feedback linearization is a natural extension of feedback linearization [5, 7, 6] to the setting where the nominal behavior is an orbit rather than an equilibrium. Indeed, as with (standard) feedback linearization, one uses the transverse linearization to help understand the structure of the control system as well as to design feedback controls providing stabilization of the nominal behavior.

The following result has been shown in [1].

Theorem 0.1 *Let η be a periodic orbit of the undriven system (1) with $u \equiv 0$. Then, the system (1) is transversely feedback linearizable along the periodic orbit η if and only if it is transversely linearly controllable along η and there exists a smooth function α in a neighborhood $\mathcal{N}$ of η such that*

1. *$d\alpha \neq 0$ on η.*
2. *$\alpha = 0$ on η.*
3. *$L_{ad_f^i g}\alpha = 0$ in $\mathcal{N}$ for $i = 0,\ldots,n-3$.*

As shown in [1], a suitable set of linearizing transverse coordinates are given by

$$\begin{array}{lcl} \rho_1 & := & \alpha \\ \rho_2 & := & L_f\alpha \\ & \vdots & \\ \rho_{n-1} & := & L_f^{n-2}\alpha. \end{array} \tag{4}$$

The transverse linear controllability condition implies that this change of coordinates (together with θ parametrizing η) is a local diffeomorphism in a neighborhood of η.

It can be easily seen that in $\mathbf{R}^3$ the transverse linear controllability along η is necessary and sufficient for transverse feedback linearization. Observe that for systems in $\mathbf{R}^3$ transverse linear controllability (and thus linearizability) condition is generic with respect to *points*. One might tend to think that it is also a generic condition with respect to *orbits*. This is not the case. It is true that the set of transversely controllable periodic orbits is *open*. But it is not *dense*. To see that, consider f and g such that there is a two-dimensional surface Ω in $\mathbf{R}^3$ with the property that for some volume form μ we have $\mu(f, g, ad_f g) = 0$ on Ω and $\mu(f, g, ad_f g)$ changes sign on Ω. Note that transverse linear controllability fails on Ω. A small perturbation of f and g *perturbs* Ω a little, but Ω does not disappear. If a periodic orbit of f intersects Ω *transversely*, a periodic orbit of a perturbed system (if it persists, which it does if the Floquet multipliers have absolute values different from 1) will also intersect the (perturbed) set Ω. Thus, transverse linear controllability fails (at some point on η) even when f and g are slightly perturbed.

We are motivated by a simple (reduced) model of a cart-pendulum system [3, 4] with state space $\mathbf{R}^3$. This system possesses a family of periodic orbits that transversely intersect a surface of transverse linear controllability loss.

In this paper we study what can be done in such a situation. In particular, we provide conditions such that it is possible to find new coordinates (θ, ρ_1, ρ_2) and control v so that, after change of coordinates and feedback $u = k(x) + l(x)v$, the dynamics of (1) in a neighborhood of the periodic orbit η have the form

$$\begin{array}{rcl} \dot\theta &=& 1 + f_1(\theta, \rho) + g_0(\theta, \rho)v \\ \dot\rho_1 &=& a(\theta)\rho_2 \\ \dot\rho_2 &=& v, \end{array} \tag{5}$$

where $f_1(\cdot, \cdot)$ satisfies $f_1(\theta, 0) = 0$, $a(\theta)$ is a smooth function periodic in θ with values in the interval $[-1, 1]$, $a(\theta) = 0$ only on Ω, $a(\theta) = 1$ or $a(\theta) = -1$ except in an arbitrarily small neighborhood of Ω, where $a(\theta)$ changes sign. One may consider the transverse dynamics of (5) to be in a *modified* Brunovsky normal form. In the last section we construct modified Brunovsky form for the cart-pendulum system. Using the form (5), one may easily design controllers stabilizing the transverse dynamics of (1), so that all trajectories of the closed-loop system with initial conditions close to η will asymptotically approach η.

This paper is a preliminary study of a more general problem of finding suitable standard forms for the transverse dynamics of general orbits for systems in $\mathbf{R}^n$ crossing surfaces of transverse linear controllability loss. Methods presented in this paper should allow to obtain "global" transverse coordinates along the orbit, i.e., the ones working in a neighborhood along the whole orbit, in particular "through" singularities. One would expect that for a periodic orbit in $\mathbf{R}^n$ whose transverse dynamics is feedback linearizable, except for the points of the transverse controllability loss, it is possible to find a "global" (i.e., working about the whole orbit) coordinates so that the dynamics of the system has form

$$
\begin{array}{lcl}
\dot{\theta} & = & 1 + f_1(\theta, \rho) + g_0(\theta, \rho) v \\
\dot{\rho}_1 & = & a_1(\theta)\rho_2 \\
\vdots & & \\
\dot{\rho}_{n-2} & = & a_{n-2}(\theta)\rho_{n-1} \\
\dot{\rho}_{n-1} & = & a_{n-1}(\theta) v,
\end{array}
\tag{6}
$$

where the $a_i(\theta)$ are smooth functions periodic in θ with values in the interval $[-1, 1]$, $a_i(\theta) = 0$ only on the surface of transverse linear controllability loss Ω, and $a_i(\theta) = 1$ or $a_i(\theta) = -1$ except in an arbitrarily small neighborhood of Ω. In the case when the transverse dynamics is not exactly linearizable, the form (6) would still represent the linearized dynamics along the orbit.

Such coordinates can be used to understand the dynamics of the control system close to singular points, as well as to design locally stabilizing controllers for the orbit.

1 Results

Consider now the nonlinear system (1) on $\mathbf{R}^3$ with periodic orbit η (when $u \equiv 0$) and suppose that

(A) $\dim \operatorname{span} \{f(x), g(x)\} = 2$ on η.

(B) $\dim \operatorname{span} \{f(x), g(x), ad_f g(x)\} = 3$ on $\eta \setminus \Omega$, where Ω is a two-dimensional surface Ω in $\mathbf{R}^3$ with the property that $\mu(f, g, ad_f g) = 0$ on Ω, $\mu(f, g, ad_f g)$ changes sign on Ω, and η intersects Ω transversely at an even number of points.

(C) $\dim \operatorname{span} \{f(x), g(x), ad_f^2 g(x)\} = 3$ for $x \in \eta \cap \Omega$.

(All these conditions are satisfied by a reduced cart-pendulum model.)

It can be shown using methods in [1] that the condition (A) implies the existence of a smooth function α in a neighborhood $\mathcal{N}$ of η that satisfies conditions (1), (2), and (3) of Theorem 0.1. Thus, $\rho_1 := \alpha$ and $\rho_2 := L_f\alpha$ and any θ parametrizing η are locally transversely linearizing coordinates on connected neighborhoods of $\eta \setminus \Omega$ (where the transverse linear controllability condition holds). On each of these neighborhoods one can represent (1) in form (5) with $a(\theta) = 1$ (the transverse dynamics in the Brunovsky form). Even though $\rho_1 := \alpha$ and $\rho_2 := L_f\alpha$ are defined globally around η, they don't define a coordinate system around η, for $dL_f\alpha = 0$ on $\eta \cap \Omega$. Since our goal is to find transverse coordinates that would work globally around η, we cannot expect to have the transverse dynamics in the Brunovsky form globally (for then the system would be globally transversely linearly controllable around η). This is the reason for $a(\theta)$ to be a function rather than a constant in (5). It should be intuitively clear that one must have $a(\theta) = 0$ on Ω, i.e., at the points that the transverse linear controllability fails. The reason why $a(\theta)$ changes sign when crossing Ω is that the coordinate systems (θ, α, $L_f\alpha$) have *different orientations* on the opposite sides of Ω (is is easy to show that $dL_f\alpha$ crosses the plane spanned $d\alpha$ and $d\theta$ when ad_fg crosses the plane spanned by f and g), so that one cannot "glue" them smoothly. Instead, globally working transverse coordinates ρ_1 and ρ_2 can obtained by "gluing" (θ, α, $L_f\alpha$) one side of Ω with (θ, α, $-L_f\alpha$) on the other side of Ω (these have *the same orientation*).

Since both (θ, α, $L_f\alpha$) and (θ, α, $-L_f\alpha$) become singular coordinates on Ω, one needs to construct a coordinate system on an arbitrarily thin neighborhood of Ω with the same orientation as (θ, α, $L_f\alpha$) and (θ, α, $-L_f\alpha$).

One can show that it is possible to find, in a thin neighborhood of $\eta \cap \Omega$, a function β such that $\beta = 0$ on η, $L_g\beta = 1$, and, for some choice of θ and α satisfying the conditions (1), (2), and (3), one has $L_f\alpha = h\beta$, for some smooth function h. We have the following result.

Proposition 1.1 *Assume that the conditions (A)-(C) are satisfied. Then there are smooth functions θ, α, β, and h defined in a neighborhood of η such that*

(a) α satisfies conditions (1), (2), and (3) of Theorem 0.1.

(b) $L_f\alpha = h\beta$ in a neighborhood of η.

(c) $\beta = 0$ on η.

(d) $L_g\beta = 1$ in a neighborhood of $\eta \cap \Omega$.

(e) $L_f\theta = 1$ on η.

(f) $h = h(\theta)$ in a neighborhood of $\eta \cap \Omega$, i.e. θ parametrizes level sets of h.

Proof: By assumption (B), $\eta \cap \Omega = \{x_1, \ldots, x_m\}$ for some even integer m. Let $c_i \subset \Omega$ be a smooth curve passing through x_i and transverse to f and g and let v_i be any smooth nonvanishing vector field tangent to c_i. The transversality requirement can be satisfied by demanding $\mu(f(x_i), g(x_i), v(x_i)) > 0$, where μ is a volume form on a neighborhood of η. Consider the mappings $\Phi_i : \{s_1^i, s_2^i\} \mapsto x$ given by ($\phi_s^h(\cdot)$ is the flow of a vector field h)

$$x \;=\; \phi^f_{s_2^i} \circ \phi^{v_i}_{s_1^i}(x_i) \tag{7}$$

where s_1^i and s_2^i range over an open interval I of real numbers containing 0. Let $\mathcal{S}_i$ be the range of Φ_i. (In other words, $\mathcal{S}_i$ is obtained by "flowing" c_i along f.) Note that $\mathcal{S}_i$ is a two-dimensional surface transverse to Ω. Using s_1^i and s_2^i as coordinates on $\mathcal{S}_i$ we notice that on c_i we have $v_i = \frac{\partial}{\partial s_1^i}$. We can extend the vector field v_i to a vector field on $\mathcal{S}_i$ by $v_i := \frac{\partial}{\partial s_1^i}$. Note that on $\mathcal{S}_i$ we have $f = \frac{\partial}{\partial s_2^i}$, so that $[f, v_i] = 0$ on $\mathcal{S}_i$. Furthermore, one can extend v_i to a smooth vector field on a neighborhood $\mathcal{N}_i$ of $\mathcal{S}_i$, for instance by "flowing" along g.

Shrinking $\mathcal{N}_i$ if necessary, we can assume that $\mathcal{N}_i$, $i = 1, \ldots, m$, are disjoint connected open sets, each containing $x_i \in \eta \cap \Omega$. Let $\mathcal{N}_i$, $i = m+1, \ldots, 2m$, be disjoint connected open sets such that each $\mathcal{N}_i$ contains a connected segment of $\eta \setminus \Omega$, $\mathcal{N}_i \cap \Omega = \emptyset$, $i = m+1, \ldots, 2m$, and $\mathcal{N}_i$, $i = 1, \ldots, 2m$, cover a neighborhood of η. Let v_i, $i = m+1, \ldots, 2m$, be smooth vector fields on $\mathcal{N}_i$, $i = m+1, \ldots, 2m$, with the property $\mu(f(x_i), g(x_i), v(x_i)) > 0$, $i = m+1, \ldots, 2m$. Let $\{\phi_i\}_{i=1}^{2m}$ be a smooth *partition of unity* subordinate to $\{\mathcal{N}_i\}$. That is, $\phi_i \geq 0$, the support of ϕ_i is contained in $\mathcal{N}_i$, and, for each $x \in \cup_{i=1}^{2m} \mathcal{N}_i$, $\sum_{i=1}^{2m} \phi_i(x) = 1$ (see, e.g., [2]). On $\cup_{i=1}^{2m} \mathcal{N}_i$ we define a vector field $v := \sum_{i=1}^{2m} \phi_i v_i$. Note that v is a smooth vector field. Observe also that $\mu(f(x), g(x), v(x)) = \sum_{i=1}^{2m} \phi_i \mu(f(x_i), g(x_i), v(x_i)) > 0$, for at each $x \in \cup_{i=1}^{2m} \mathcal{N}_i$ at least for one i we have $\phi_i(x) > 0$ and the corresponding $\mu(f(x_i), g(x_i), v(x_i)) > 0$. Therefore, f, g, and v are independent in a neighborhood of η. Moreover, for $i \in [1, \ldots, m]$, define $\hat{\mathcal{S}}_i := \mathcal{S}_i \setminus \cup_{j=1,\, j \neq i}^{2m} \mathcal{N}_i$. On $\hat{\mathcal{S}}_i$, $v = v_i$ and thus $[f, v] = 0$.

Fix a point x_0 on η. In a neighborhood of η one can *reach* any point x by *traveling* from x_0 along vector fields f, v, g with *times* s_0, s_1, s_2, i.e., the mapping $\Phi_0 : s \mapsto x$ given by

$$x \;=\; \phi^g_{s_2} \circ \phi^v_{s_1} \circ \phi^f_{s_0}(x_0) \tag{8}$$

is a local diffeomorphism between the cylinder $S^1 \times \mathbf{R}^3$ and a tubular neighborhood of η. It is easy to verify that $\alpha := s_1$ satisfies the conditions (1), (2), and (3) of Theorem 0.1 (cf. [1]). Moreover, since $L_f\alpha = 0$ on η and $[f, v] = 0$ on $\hat{\mathcal{S}}_i$, $i = 1, \ldots, m$, we have $L_f\alpha = 0$ on $\hat{\mathcal{S}}_i$.

We claim that one can join the $\hat{\mathcal{S}}_i$ to obtain a smooth two-dimensional orientable surface $\mathcal{S}$ containing η. Note that $dL_f\alpha \neq 0$ at the points where the system is transversely linearly controllable and in particular on $\mathcal{N}_i$, $i = m+1, \ldots, 2m$ (cf. [1]). Let $\mathcal{S}_i$, $i = m+1, \ldots, 2m$, be the connected component of $\{x \in \mathcal{N}_i | L_f\alpha(x) = 0,\, dL_f\alpha(x) \neq 0\}$ containing $\eta \cap \mathcal{N}_i$. Define $\mathcal{S}$ by $\mathcal{S} := (\cup_{i=1}^{2m} \mathcal{S}_i)$. Observe that $\mathcal{S}$ is a smooth two-dimensional surface, since on $\mathcal{N}_i \cap \mathcal{N}_j$, $i \in [1, m]$, $j \in [m+1, 2m]$, $dL_f\alpha \neq 0$ implies $\mathcal{S}_i = \mathcal{S}_j$. Note that $L_f\alpha = 0$ on $\mathcal{S}$ (see Remark 1 below). Orientability of $\mathcal{S}$ follows from the fact that g is everywhere transverse to $\mathcal{S}$.

Let us repeat the construction of flow coordinates as in (8) with v and f replaced by $\hat{v}$ and $\hat{f}$ that are tangent to $\mathcal{S}$ for $x \in \mathcal{S}$. Denote the new flow coordinates by $\hat{s}_0, \hat{s}_1, \hat{s}_2$. (We do not re-define α.) Let $\hat{\mathcal{N}}$ be a neighborhood of $\mathcal{S}$ on which $\hat{s}_0, \hat{s}_1, \hat{s}_2$ are valid coordinates. Since $\hat{v}$ and $\hat{f}$ are tangent to $\mathcal{S}$, $\mathcal{S}$ is given in $\hat{s}_0, \hat{s}_1, \hat{s}_2$-coordinates by $\hat{s}_2 = 0$. Since $L_f\alpha = 0$ on $\mathcal{S}$, and $L_f\alpha$ is a smooth function in $\hat{\mathcal{N}}$, one has $L_f\alpha(\hat{s}_0, \hat{s}_1, \hat{s}_2) = \int_0^{\hat{s}_2} \frac{\partial L_f\alpha(\hat{s}_0, \hat{s}_1, \sigma)}{\partial\sigma} d\sigma$. Using the substitution $\sigma = \tau\hat{s}_2$ we get $L_f\alpha(\hat{s}_0, \hat{s}_1, \hat{s}_2) = \hat{s}_2 \int_0^1 \left.\frac{\partial L_f\alpha(\hat{s}_0, \hat{s}_1, \sigma)}{\partial\sigma}\right|_{\sigma=\tau\hat{s}_2} d\tau$. Define $\beta := \hat{s}_2$, $h := \int_0^1 \left.\frac{\partial L_f\alpha(\hat{s}_0, \hat{s}_1, \sigma)}{\partial\sigma}\right|_{\sigma=\tau\hat{s}_2} d\tau$. Note that β and h are smooth. Moreover, $\beta = 0$ on $\mathcal{S}$, and thus on η. Observe that on $\hat{\mathcal{N}}$ one has $g = \frac{\partial}{\partial\hat{s}_2}$, so that $L_g\beta = 1$ on $\hat{\mathcal{N}}$. Note also that on $\hat{\mathcal{N}}$ $L_gL_f\alpha = h + \beta L_g h$. In particular, $L_gL_f\alpha = h$ on η.

Assumption (C) guarantees that on $\eta \cap \Omega$ we have $L_f h = L_f L_g L_f \alpha = L_{ad_f^2 g}\alpha \neq 0$. Thus, in a neighborhood of $\eta \cap \Omega$ the level sets of $h(\cdot)$ are transverse to η, so that one can parametrize them by an appropriate function θ.

□

Remark 1.1 Note that, even though $L_f\alpha = 0$ on the surface $\mathcal{S}$ constructed in the proof of Proposition 1.1, we could not define $\mathcal{S}$ by the requirement $L_f\alpha = 0$. The problem is that there are *two branches* of the surface $L_f\alpha = 0$ that intersect transversely at $x_i \in \eta \cap \Omega$. This is possible since $dL_f\alpha(x_i) = 0$. The construction of $\mathcal{S}$ in the proof of Proposition 1.1 picks only the "good" branch of the surface $L_f\alpha = 0$ that contains η and continues through the singularity at x_i. The construction neglects the "bad" branches of the surface $L_f\alpha = 0$ that are transverse to η at x_i. The function β is defined in such way that $\beta = 0$ on this "good" branch of the surface $L_f\alpha = 0$, while the "bad" branches of the surface $L_f\alpha = 0$ correspond to $h = 0$. $\square$

Remark 1.2 In the proof of Proposition 1.1, we constructed α using a *special* vector field v (i.e., one commuting with f near the points $x_i \in \eta \cap \Omega$) rather than an *arbitrary* v transverse to f and g as in [1]. The reason for this is that, in general, it is not possible to factor $L_f\alpha$ as $h\beta$ for *arbitrary* α satisfying conditions (1), (2), and (3) of Theorem 0.1. The difficulty is that there may not be a "good" branch of the surface $L_f\alpha = 0$ that contains η and continues around η. For example, suppose that near a point of transverse linear controllability loss $x_i \in \eta \cap \Omega$, we have $f = \frac{\partial}{\partial\theta} + (\theta\rho_2 + \rho_1^2)\frac{\partial}{\partial\rho_1}$, $g = \frac{\partial}{\partial\rho_2}$, and η is (locally) given by $\rho_1 = \rho_2 = 0$. We have $ad_f g = -\theta\frac{\partial}{\partial\rho_1}$, so Ω is (locally) given by $\theta = 0$. Note that $\alpha := \rho_1$ satisfies conditions (1), (2), and (3) of Theorem 0.1. Consider the set $L_f\alpha = \theta\rho_2 + \rho_1^2 = 0$. Intersections of this set with the sets $\rho_1 = c$ for constant $c \neq 0$ are two branches of the hyperbola $\theta\rho_2 = -c^2$, which lie on two sides of Ω (given by $\theta = 0$). Therefore, there is no "good" branch of the surface $L_f\alpha = 0$ that contains η and intersects Ω transversely, which would be the case when $L_f\alpha = h\beta$. On the other hand, $\alpha := \rho_1 - \theta\rho_1^2$ also satisfies conditions (1), (2), and (3) of Theorem 0.1. We have $L_f\alpha = \theta(\rho_2 + 2\theta^2\rho_1\rho_2 + 2\theta\rho_1^3)$. Defining $h := \theta$ and $\beta := \rho_2 + 2\theta^2\rho_1\rho_2 + 2\theta\rho_1^3$, we obtain the required factorization $L_f\alpha = h\beta$. $\square$

Remark 1.3 The proof of Proposition 1.1 uses a detailed procedure to *construct* the suitable functions α, β, h, and θ. In many cases, it is not necessary to construct these functions in this fashion. Indeed, for the cart-pendulum system in the sequel, an obvious α was found with the property that $L_f\alpha$ could be factored into h and β. $\square$

Note that on η we have $L_gL_f\alpha = h$. In particular $L_gL_f\alpha = 0$ if and only if $h = 0$ (which happens on $\eta \cap \Omega$).

Now we are ready to show that the three groups of local coordinate systems (θ, α, $L_f\alpha$), (θ, α, $-L_f\alpha$), and (θ, α, β) can be "glued" together using a *partition of unity* [2] to form a coordinate system that works globally around η.

Proposition 1.2 *Assume that the conditions (A)-(C) are satisfied. Let θ, α, h, and β be smooth functions defined on a neighborhood $\mathcal{N}$ of η satisfying conditions of (a)-(f) of Proposition 1.1. Fix a small positive number ϵ. Define*

$$\begin{array}{rcl} \mathcal{N}_+ & := & \{x \in \mathcal{N} | h(x) > \epsilon\}, \\ \mathcal{N}_- & := & \{x \in \mathcal{N} | h(x) < -\epsilon\}, \\ \mathcal{N}_0 & := & \{x \in \mathcal{N} | -2\epsilon < h(x) < 2\epsilon\}. \end{array} \tag{9}$$

Let $\{\phi_+, \phi_-, \phi_0\}$ be a partition of unity subordinate to $\{\mathcal{N}_+, \mathcal{N}_-, \mathcal{N}_0\}$. Define

$$\begin{aligned} \rho_1 &:= \alpha \\ \rho_2 &:= \phi_+ L_f\alpha - \phi_- L_f\alpha + \phi_0\beta. \end{aligned} \tag{10}$$

Then (θ, ρ_1, ρ_2) are valid local coordinates around η. The system (1) (in (θ, ρ) coordinates) is given by

$$\begin{aligned} \dot\theta &= 1 + f_1(\theta,\rho) + g_0(\theta,\rho)u \\ \dot\rho_1 &= a(\theta,\rho)\rho_2 \\ \dot\rho_2 &= p + ru \end{aligned} \tag{11}$$

where

$$a := \frac{h}{(\phi_+ - \phi_-)h + \phi_0}, \tag{12}$$

$$p := L_f\rho_2 = L_f(\phi_+ L_f\alpha - \phi_- L_f\alpha + \phi_0\beta),$$

and

$$r := L_g\rho_2 = L_g(\phi_+ L_f\alpha - \phi_- L_f\alpha + \phi_0\beta).$$

Moreover, $r \neq 0$ on η. We have $a = 1$ on $\mathcal{N}_+ \setminus \mathcal{N}_0$, $a = -1$ on $\mathcal{N}_- \setminus \mathcal{N}_0$, and $a = h$ on $\mathcal{N}_0 \setminus (\mathcal{N}_+ \cup \mathcal{N}_-)$. Moreover, for $\epsilon < \frac{1}{2}$, $-1 < a < 1$ on $\mathcal{N}_0$. Also, for ϵ sufficiently small, one can assume that $a(\theta,\rho) = a(\theta)$ in a neighborhood of η.

Proof: Note that the open sets $\mathcal{N}_+$, $\mathcal{N}_-$, and $\mathcal{N}_0$ cover $\mathcal{N}$. To show that (θ, ρ_1, ρ_2) are valid local coordinates in a neighborhood of η it is sufficient to verify that they have linearly independent differentials on η. Let v be any vector field such that $L_v\alpha = 1$ on η. Then f, g, and v are linearly independent on η. Observe that

$$(d\theta \wedge d\rho_2 \wedge d\rho_1)(f, g, v) = \det S$$

where

$$S = \begin{bmatrix} L_f\theta & L_g\theta & L_v\theta \\ L_f\rho_2 & L_g\rho_2 & L_v\rho_2 \\ L_f\rho_1 & L_g\rho_1 & L_v\rho_1 \end{bmatrix}.$$

Note that on η one has $L_f\rho_1 = L_f\alpha = 0$, $L_g\rho_1 = L_g\alpha = 0$, $L_v\rho_1 = L_v\alpha = 1$, $L_f\theta = 1$, and $L_f\rho_2 = L_f(\phi_+ L_f\alpha - \phi_- L_f\alpha + \phi_0\beta) = 0$. Moreover, since on η $L_g L_f\alpha = h$, we have $L_g\rho_2 = L_g(\phi_+ L_f\alpha - \phi_- L_f\alpha + \phi_0\beta) = \phi_+ h - \phi_- h + \phi_0$. Thus, on η, $\det S = \phi_+ h - \phi_- h + \phi_0$. Note that $\phi_+ h \geq 0$, $-\phi_- h \geq 0$, and $\phi_0 \geq 0$. Moreover, for each $x \in \eta$, at least one of the quantities $\phi_+ h$, $-\phi_- h$, and ϕ_0 is strictly positive. Therefore, $\det S > 0$ on η and hence in a neighborhood. We have proved that (θ, ρ_1, ρ_2) are valid local coordinates in a neighborhood of η and $r := L_g\rho_2 \neq 0$ on η.

The system (in (θ, ρ) coordinates) is given by

$$\begin{aligned} \dot\theta &= 1 + f_1(\theta,\rho) + g_0(\theta,\rho)u \\ \dot\rho_1 &= L_f\rho_1 \\ \dot\rho_2 &= L_f\rho_2 + L_g\rho_2 u = p + ru. \end{aligned} \tag{13}$$

Note that $((\phi_+ - \phi_-)h + \phi_0)L_f\rho_1 = ((\phi_+ - \phi_-)h + \phi_0)L_f\alpha = h((\phi_+ - \phi_-)L_f\alpha + \phi_0\beta) = h\rho_2$. Thus $L_f\rho_1 = a\rho_2$, where a is given by (12). The facts $a = 1$ on $\mathcal{N}_+ \setminus \mathcal{N}_0$, $a = -1$ on $\mathcal{N}_- \setminus \mathcal{N}_0$, and $a = h$ on $\mathcal{N}_0 \setminus (\mathcal{N}_+ \cup \mathcal{N}_-)$ follow immediately. A simple proof of the fact that, for $\epsilon < \frac{1}{2}$, $-1 < a < 1$ on $\mathcal{N}_0$ is left to the reader. Now, on a small neighborhood of $\eta \cap \Omega$, $h = h(\theta)$. Hence, for ϵ sufficiently small and shrinking $\mathcal{N}$ if necessary, we see that h is a function of θ in $\mathcal{N}_0$. Also ϕ_+, ϕ_-, ϕ_0 can then be chosen as functions of θ. It follows that a is a function of θ in $\mathcal{N}_0$. Outside of $\mathcal{N}_0$, a is constant so that we can say that a is a function of θ in a neighborhood of η.

□

The preliminary feedback

$$u = \frac{1}{r}(-p + v) \tag{14}$$

puts the system (11) into the modified Brunovsky form

$$\begin{aligned} \dot\theta &= 1 + f_1(\theta, \rho) + g_0(\theta, \rho)v \\ \dot\rho_1 &= a\rho_2 \\ \dot\rho_2 &= v, \end{aligned} \tag{15}$$

This form represents smooth transitions from the Brunovsky form

$$\begin{aligned} \dot\rho_1 &= \rho_2 \\ \dot\rho_2 &= v. \end{aligned} \tag{16}$$

(corresponding to $(\theta,\ \alpha,\ L_f\alpha)$) to *twisted* Brunovsky form

$$\begin{aligned} \dot\rho_1 &= -\rho_2 \\ \dot\rho_2 &= v. \end{aligned} \tag{17}$$

(corresponding to $(\theta,\ \alpha,\ -L_f\alpha)$) of the transverse dynamics as one travels around η.

The form (15) can be used to design a locally stabilizing feedback for η.

Proposition 1.3 *Consider the system (15) and suppose that, for all real c sufficiently close to* 0, *we have*

$$a(\theta, c, 0) \not\equiv 0 \ .$$

Then, the control law

$$v = -k_1 a\rho_1 - k_2\rho_2 \tag{18}$$

where k_1 and k_2 are positive constants locally stabilizes η (i.e., $\rho_1 = 0$, $\rho_2 = 0$).

Proof: Consider, as Lyapunov function,

$$V = (k_1\rho_1^2 + \rho_2^2)/2 \ .$$

Along the closed loop system (15), (18), we have

$$\dot V = -k_2\rho_2^2 \le 0 \ .$$

By LaSalle's principle, the trajectories converge to the largest invariant set such that $\dot{V} = 0$, namely, $\rho_2 = 0$, $\rho_1 = c$, where c is a constant. Furthermore, since $\dot{\rho}_2 = 0$, we have $\dot{\rho}_2 = -k_1 a(\theta, c, 0)c = 0$ on the invariant set. If $a \neq 0$ for at least one θ then c must be 0. $\square$

Note that the set $a(\theta, \rho_1, \rho_2) = 0$ intersects η only at points of transverse linear controllability loss. Furthermore, since such intersections are transverse, it is clear that, for each c close to 0, one can easily find θ such that $a(\theta, c, 0)$ is nonzero.

Remark 1.4 Note that if our goal is local stabilization of η, then rather than using ρ_2 defined by (10) as one of the (θ, ρ) coordinates, one can instead use $\rho_2 := \beta$ in a neighborhood of η. It is easily seen that the representation of the system (1) in the coordinates $(\theta, \rho) := (\theta, \alpha, \beta)$ has still form (11) with $a = h$ and $r = 1$. In particular, the stabilization scheme presented above still applies. Alternatively, one can interpret the choice $\rho_2 := \beta$ as a particular case of ρ_2 defined by (10) with $\epsilon > \sup_{x \in \mathcal{N}} |h(x)|$. $\square$

2 Example

A reduced cart-pendulum model (with the cart velocity as input rather than the acceleration—see [3], [4]) is described by the equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1) - \cos(x_1)u \\ \dot{x}_3 &= u. \end{aligned} \tag{19}$$

Note that the undriven system ($u \equiv 0$) has a family of periodic orbits

$$\eta_{c_0} = \{x \in \mathbf{R}^3 : x_2^2 - 2\cos(x_1) + c_0 = 0, x_3 = 0\}.$$

We have $ad_f g = \cos(x_1)\frac{\partial}{\partial x_1} + x_2 \sin(x_1)\frac{\partial}{\partial x_2}$ and for $\mu := dx_1 \wedge dx_2 \wedge dx_3$ we have $\mu(f, g, ad_f g) = -\sin(x_1)(x_2^2 + \cos(x_1))$, so that the system is not transversely linearly controllable on the set

$$\Omega := \{x | \mu(f, g, ad_f g)(x) = 0\} = \{x | x_1 = 0 \,\mathrm{or}\, x_2^2 + \cos(x_1) = 0\}.$$

It is easy to verify that the functions $\alpha_1(x) := x_1$ and $\alpha_2(x) := x_2 + x_3\cos(x_1)$ are constant along g. One can observe that the function

$$\alpha \;\; := (x_2 + x_3\cos(x_1))^2 - 2\cos(x_1) + c_0$$

is zero on η_{c_0}. We have

$$L_f \alpha = -2x_3(x_2^2 + \cos(x_1) + x_2 x_3 \cos(x_1))\sin(x_1).$$

Defining

$$\beta := x_3$$

and

$$h := -2(x_2^2 + \cos(x_1) + x_2 x_3 \cos(x_1)) \sin(x_1)$$

we see that

$$L_f \alpha = h\beta.$$

In a neighborhood of Ω η intersects the level sets of $-2(x_2^2 + \cos(x_1) + x_2 x_3 \cos(x_1)) \sin(x_1)$ transversely. Therefore we may assume that θ in this neighborhood has been chosen so that h is a function of θ.

Let $\mathcal{N}$ be an open neighborhood of η. Define $\mathcal{N}_+$, $\mathcal{N}_-$, and $\mathcal{N}_0$ by (9), for some $\epsilon > 0$. A smooth partition of unity subordinate to $\{\mathcal{N}_+, \mathcal{N}_-, \mathcal{N}_0\}$ can be constructed as follows. Let

$$\begin{array}{rcl} \xi(y) & := & e^{-\frac{1}{y^2}}, \text{ for } y > 0 \\ \xi(y) & := & 0, \ for\, y \leq 0. \end{array} \tag{20}$$

Let $a_1 < a_2$. Define

$$\phi_{a_1,a_2}(y) \ := \ \frac{\xi(a_2-y)}{\xi(a_2-y)+\xi(y-a_1)}. \tag{21}$$

Note that $\phi_{a_1,a_2}(.)$ is a nonincreasing function, $\phi_{a_1,a_2}(y) = 1$ for $y \leq a_1$, and $\phi_{a_1,a_2}(y) = 0$ for $y \geq a_2$. One can choose

$$\phi_-(h) := \phi_{-2\epsilon,-\epsilon}(h)$$

$$\phi_+(h) := \phi_{-2\epsilon,-\epsilon}(-h)$$

$$\phi_0(h) := 1 - \phi_+(h) - \phi_-(h).$$

The change of coordinates (10) and preliminary feedback (14) puts the system in the form (15). Taking advantage of this form, we design a stabilizing feedback

$$v = -k_1 a \rho_1 - k_2 \rho_2$$

with $k_1 = 9$ and $k_2 = 4$. By Proposition 1.3, this feedback provides local stability of η. For easier implementation, we used simple piecewise linear partition of unity (continuous but not smooth) rather than the smooth one constructed using function ξ defined by (20). Figure 1 shows the closed loop trajectory for the system starting from $x = (0.5, 0.5, 1)$ for the case where $c_0 = 2\cos 2$. Figure 2 shows the projection of the closed loop trajectory onto the $x_1 - x_2$ plane. Also shown are three components of the $h(x) = 0$ set.

Conclusion

We have presented a normal form for a nonlinear affine single-input system in $\mathbf{R}^3$ in a neighborhood of a periodic orbit in the case when the transverse linear controllability fails in finite number of points along the periodic orbit. This normal form can be used to understand the dynamics of the control system close to singular points, as well as to design locally stabilizing controllers for the orbit.

We hope that the methods applied in the present paper can be applied for finding suitable normal forms for the transverse dynamics of general orbits for systems in $\mathbf{R}^n$ crossing surfaces of transverse linear controllability loss. Methods presented in this paper should allow to obtain "global" transverse coordinates along the orbit, i.e., the ones working in a neighborhood along the whole orbit, in particular "through" singularities.

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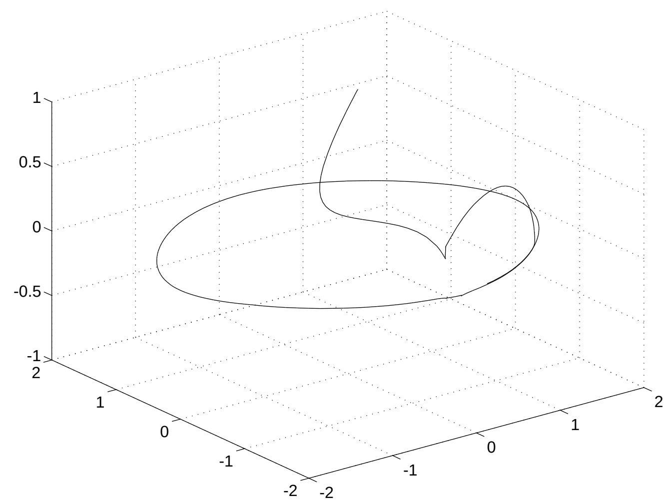

Figure 1: Closed loop trajectory from $(0.5, 0.5, 1)$ with $c_0 = 2\cos 2$.

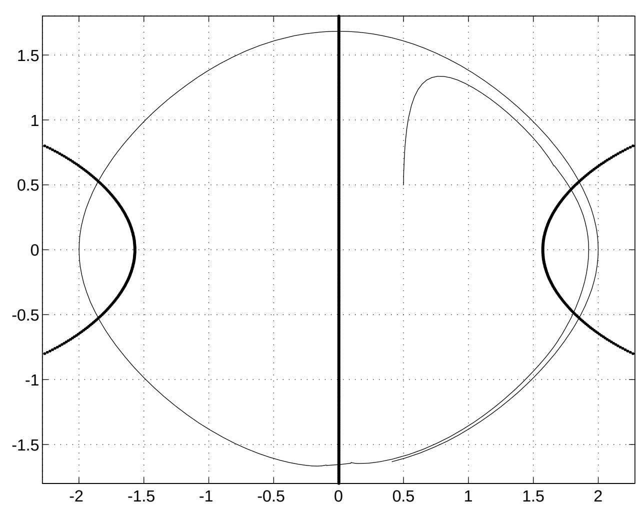

Figure 2: Projection of the closed loop trajectory onto the $x_1 - x_2$ plane. The sets $h = 0$ are also shown.