A CURIOUS BINOMIAL IDENTITY

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In this note we shall prove the following curious identity of sums of powers of the partial sum of binomial coefficients.

1. An identity

Theorem. $\sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n}{k}\right)^3 = n2^{3n-1} + 2^{3n} - \frac{3n}{4}2^n\binom{2n}{n}$.

Proof. Define $f_n = \sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n}{k}\right)^3$. It is sufficient to show that

$$f_{n+1} - 8f_n = 4\ 2^{3n} - 3\ 2^n\binom{2n}{n}$$

Write $A_l = \sum_{k=0}^{l}\binom{n}{k}$. Then $f_n = \sum_{l=0}^{n} A_l^3$.

$$\begin{aligned}
f_{n+1} &= \sum_{l=0}^{n+1}\left(\sum_{k=0}^{l}\binom{n+1}{k}\right)^3 \\
&= 2^{3n+3} + \sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n+1}{k}\right)^3 \\
&= 2^{3n+3} + \sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n}{k} + \binom{n}{k-1}\right)^3 \\
&= 2^{3n+3} + \sum_{l=0}^{n}\left(2A_l - \binom{n}{l}\right)^3 \\
f_{n+1} - 8f_n &= 2^{3n+3} + \sum_{l=0}^{n}\left(2A_l - \binom{n}{l}\right)^3 - (2A_l)^3 \\
&= 2^{3n+3} - \sum_{l=0}^{n} 12A_l^2\binom{n}{l} + \sum_{l=0}^{n} 6A_l\binom{n}{l}^2 - \sum_{l=0}^{n}\binom{n}{l}^3
\end{aligned}$$

Observation 1:

$$\sum_{l=0}^{n} A_l\binom{n}{l}^2 = \frac{1}{2}2^n\binom{2n}{n} + \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}^3$$

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Indeed;

$$\sum_{l=0}^{n} A_l \binom{n}{l}^2 = \sum_{l=0}^{n} A_{n-l} \binom{n}{n-l}^2 = \sum_{l=0}^{n} A_{n-l} \binom{n}{l}^2$$

and since

$$A_l + A_{n-l} = 2^n + \binom{n}{l}$$

we have

$$\begin{aligned}\sum_{l=0}^{n} A_l \binom{n}{l}^2 &= \frac{1}{2}\sum_{l=0}^{n} \left(2^n + \binom{n}{l}\right)\binom{n}{l}^2 \\ &= \frac{1}{2}\sum_{l=0}^{n} 2^n \binom{n}{l}^2 + \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}^3 \\ &= \frac{1}{2}2^n\binom{2n}{n} + \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}^3\end{aligned}$$

Observation 2:

$$\sum_{l=0}^{n} A_l^2 \binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2}2^n\binom{2n}{n} + \frac{1}{6}\sum_{l=0}^{n}\binom{n}{l}^3$$

Indeed,

$$\begin{aligned}2^{3n} = A_n^3 &= \sum_{l=0}^{n} A_l^3 - A_{l-1}^3 \\ &= \sum_{l=0}^{n} A_l^3 - \left(A_l - \binom{n}{l}\right)^3 \\ &= \sum_{l=0}^{n} 3A_l^2\binom{n}{l} - \sum_{l=0}^{n} 3A_l\binom{n}{l}^2 + \sum_{l=0}^{n}\binom{n}{l}^3 \\ &= \sum_{l=0}^{n} 3A_l^2\binom{n}{l} - \frac{3}{2}2^n\binom{2n}{n} - \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}^3\end{aligned}$$

Hence

$$\sum_{l=0}^{n} A_l^2 \binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2}2^n\binom{2n}{n} + \frac{1}{6}\sum_{l=0}^{n}\binom{n}{l}^3$$

Putting these together, we indeed find that

$$f_{n+1} - 8f_n = 4\ 2^{3n} - 3\ 2^n\binom{2n}{n}$$

as required. □

2. An application

In this section we shall discuss an application of this to order statistics. Observe that the expected value of the maximum of three independent Bernoulli random variables $B(n, \frac{1}{2})$ is

$$\sum_{l=0}^{n}\left(1-\left(\sum_{k=0}^{l}2^{-n}\binom{n}{k}\right)^3\right) = n - 2^{-3n}f_n$$
$$= \frac{n}{2} + \frac{3}{4}n2^{-2n}\binom{2n}{n}.$$

Hence, by the central limit theorem, the expected value m_3 of the maximum of three independent normal $N(0,1)$ random variables is

$$m_3 = \lim_{n\to\infty}\frac{\frac{3}{4}n2^{-2n}\binom{2n}{n}}{\frac{\sqrt{n}}{2}} = \frac{3}{2\sqrt{\pi}}$$

subracting off the mean, dividing by the standard deviation and applying Stirling's formula for the asymptotics of n!

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