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Abstract

We formulate the Bethe Ansatz equations for the open super spin chain based on the super Yangian of $osp(M|2n)$ and with diagonal boundary conditions. We then study the bulk and boundary scattering of the $osp(1|2n)$ open spin chain.

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1 Introduction

The notion of the reflection equation associated with solutions of the Yang–Baxter equation [1, 2], goes back to the key works of Cherednik [3] and Sklyanin [4]. The subject has recently attracted a great deal of activity as was summarised in [5] (and references therein). More specifically, starting from a quantum R-matrix $R(\lambda)$ depending on the spectral parameter λ and satisfying the (*super*) Yang–Baxter equation [1, 2, 6]

$$R_{12}(\lambda_1 - \lambda_2)\; R_{13}(\lambda_1)\; R_{23}(\lambda_2) = R_{23}(\lambda_2)\; R_{13}(\lambda_1)\; R_{12}(\lambda_1 - \lambda_2)\,, \tag{1.1}$$

one derives the reflection equation for an object $K(\lambda)$ as

$$R_{12}(\lambda_1 - \lambda_2)\; K_1(\lambda_1)\; R_{12}(\lambda_1 + \lambda_2)\; K_2(\lambda_2) = K_2(\lambda_2)\; R_{12}(\lambda_1 + \lambda_2)\; K_1(\lambda_1)\; R_{12}(\lambda_1 - \lambda_2)\,. \tag{1.2}$$

We have proposed in [5] a classification of c-number solutions $K(\lambda)$ of the reflection equation (1.2) for rational (super) Yangian R-matrices [7, 8] associated to the infinite series $so(m)$, $sp(2n)$ and $osp(m|2n)$. This classification entailed K matrices with purely diagonal, anti-diagonal and mixed (diagonal, anti-diagonal) non-zero entries. The explicit values of the K matrices were then used within the analytical Bethe Ansatz formulation [9, 10, 11, 12, 5] for the derivation of the spectrum and the bulk and boundary S-matrices for the $so(m)$, $sp(2n)$ open spin chains.
There exists a substantial body of work on $gl(m|n)$ super spin chains. Interest in these systems stemmed from the existence of physically relevant particular cases such as supersymmetric t-J and extended Hubbard models. They have been the object of many studies. Supersymmetric t-J models were considered, e.g. in [13] (thermodynamical aspects), [14] (diagonal boundary K matrices) and [15] (boundary S-matrix). Extended Hubbard models were considered in [16] (closed chain) and in [17, 18, 19] (open chains with integrable boundary conditions), whilst spin ladder systems associated to some $sl(m|n)$ superalgebras were obtained in [20, 21]. General results for continuum limit of the $gl(m|n)$ super spin chains were derived in [22].
A natural alternative to these models with $gl(m|n)$ underlying superalgebras is provided by super spin chains with underlying $osp(m|2n)$ superalgebras. A connection to intersecting loop models and hence polymer field theories was pointed out in [23], where the analytical Bethe Ansatz equations were written for the closed spin chain. Algebraic methods were used for some specific cases in [24, 25] including nested Bethe Ansatz in [25]. Field theoretical limits were also considered in the literature: the exact bulk $osp(2|2)$ S-matrix was conjectured in [26] in the framework of disordered systems. Investigation of the thermodynamics of $osp(1|2n)$ closed spin chains was undertaken in [27] using the thermodynamical Bethe Ansatz formalism. An algebraic construction using Birman–Wenzl–Murakami algebra then yielded conjectural S-matrices for field theoretical $osp(m|2n)$ models, and allowed a subsequent thermodynamical Bethe Ansatz analysis of their thermodynamical properties [28]. However a systematic thermodynamic treatment of these models with more general boundaries is still missing.
Our purpose is to make an exhaustive study of the more complicated case of open spin chains with $osp(m|2n)$ underlying superalgebra and any integrable (diagonal at a first step) boundary conditions. The strategy is to establish (insofar as the methods are available) Bethe Ansatz

equations for ground state and excited states (note that there is no obvious relation between closed spin chains Bethe Ansatz equations and open spin chain Bethe Ansatz equations, particularly when non trivial boundary conditions are involved); solve them within the non trivial string hypothesis (discussed in the closed case in [13, 22]); and use the results to obtain the S-matrix and thermodynamical quantities, with explicit evaluation of the effect of boundary conditions.

This paper is our first step in this direction: using the analytical Bethe Ansatz method, we derive the Bethe Ansatz equations for all orthosymplectic superalgebras, and all *diagonal* K matrices. Restricting ourselves then to $osp(1|2n)$, we solve these equations in the thermodynamic limit, we derive the ground state and low-lying excitations, and compute explicitly the bulk and boundary S-matrices. Further generalisations will be left for future investigations.

2 Bethe Ansatz equations for the osp(M|2n) open spin chain

2.1 Conventions and notations

The Bethe Ansatz equations will be derived here for the $osp(M|2n)$ N-site open spin chain with diagonal reflection conditions by means of the analytical Bethe Ansatz method (see e.g. [9, 10, 12]). As customary to construct the open chain transfer matrix we introduce the R-matrix which is a solution of the *super* Yang–Baxter equation. We focus on the $osp(M|2n)$ invariant R-matrix given by [8]

$$R(\lambda) = \lambda(\lambda + i\kappa)1 + i(\lambda + i\kappa)P - i\lambda Q \,, \qquad 2\kappa = \theta_0(M - 2n - 2) \tag{2.1}$$

where P is the (super)permutation operator (i.e. $X_{21} \equiv PX_{12}P$)

$$P = \sum_{i,j=1}^{M+2n} (-1)^{[j]} E_{ij} \otimes E_{ji} \tag{2.2}$$

and

$$Q = \sum_{i,j=1}^{M+2n} (-1)^{[i][j]} \theta_i \theta_j E_{\bar{\jmath}\bar{\imath}} \otimes E_{ji} \equiv P^{t_1} \,. \tag{2.3}$$

For each index i, we have introduced a conjugate index

$$\bar{\imath} = M + 2n + 1 - i \,. \tag{2.4}$$

We also introduce a sign θ_i and a $\mathbb{Z}_2$-grading $[i]$ whose definition, due to the conventions we adopt (see below), depend whether we consider the superalgebra $osp(2|2n)$ or any other $osp(M|2n)$ superalgebra:

For $osp(M|2n)$ **superalgebras,** $M \neq 2$**:**

$$\theta_i = \begin{cases} +1 & \text{for} \quad 1 \leq i \leq M + n \\ -1 & \text{for} \quad M + n + 1 \leq i \leq M + 2n \end{cases} \tag{2.5}$$

$$(-1)^{[i]} = +1 \qquad \text{for} \quad 1 \le i \le n \qquad \text{and} \quad M+1 \le i \le M+2n \tag{2.6}$$
$$(-1)^{[i]} = -1 \qquad \text{for} \quad n+1 \le i \le n+M \tag{2.7}$$

We will associate to this choice the sign $\theta_0 = -1$.

For $osp(2|2n)$ superalgebras:

$$\theta_i = \begin{cases} +1 & \text{for} \quad 1 \le i \le n+1 \qquad \text{and} \quad i = 2n+2 \\ -1 & \text{for} \quad n+2 \le i \le 2n+1 \end{cases} \tag{2.8}$$

$$(-1)^{[i]} = +1 \qquad \text{for} \quad i = 1 \qquad \text{and} \quad i = 2n+2 \tag{2.9}$$
$$(-1)^{[i]} = -1 \qquad \text{for} \quad 2 \le i \le 2n+1 \tag{2.10}$$

The sign corresponding to this choice will be $\theta_0 = +1$.

The transposition ^{t} used in (2.3) and below is defined, for $A = \sum_{ij} A^{ij} E_{ij}$, by

$$A^t = \sum_{ij} (-1)^{[i][j]+[j]} \theta_i \theta_j \, A^{ij} \, E_{\overline{j}\overline{i}} = \sum_{ij} \left(A^t\right)^{ij} E_{ij} \tag{2.11}$$

The R-matrix (2.1) satisfies crossing and unitarity, namely

$$R_{12}(\lambda) R_{12}(-\lambda) = (\lambda^2 + \kappa^2)(\lambda^2 + 1)\, 1, \quad R_{12}(\lambda) = R_{12}^{t_1}(-\lambda - i\kappa)\,. \tag{2.12}$$

We finally define the super trace operation according to the $\mathbb{Z}_2$-grading we have introduced:

$$\mathrm{Tr}\, A = \sum_{j=1}^{M+2n} (-1)^{[j]} A_{jj} \text{ for } A = \sum_{i,j=1}^{M+2n} A_{ij} \, E_{ij}\,. \tag{2.13}$$

2.2 Transfer matrix and pseudo vacuum

The open chain transfer matrix is given by [4]

$$t(\lambda) = \mathrm{Tr}_0 \, K_0^+(\lambda) \; T_0(\lambda) \; K_0^-(\lambda) \; \hat{T}_0(\lambda)\,, \tag{2.14}$$

where Tr_0 denotes here the *super* trace (2.13) over the auxiliary space,

$$T_0(\lambda) = R_{0N}(\lambda) R_{0,N-1}(\lambda) \cdots R_{02}(\lambda) R_{01}(\lambda)\,, \quad \hat{T}_0(\lambda) = R_{10}(\lambda) R_{20}(\lambda) \cdots R_{N-1,0}(\lambda) R_{N0}(\lambda)\,, \tag{2.15}$$

$K_0^-(\lambda)$ is any solution of the *super* boundary Yang–Baxter equation

$$R_{12}(\lambda_1 - \lambda_2)\, K_1(\lambda_1)\, R_{12}(\lambda_1 + \lambda_2)\, K_2(\lambda_2) = K_2(\lambda_2)\, R_{12}(\lambda_1 + \lambda_2)\, K_1(\lambda_1)\, R_{12}(\lambda_1 - \lambda_2) \tag{2.16}$$

and $K_0^+(\lambda)$ is a solution of a closely related reflection equation defined to be:

$$R_{12}(\lambda_2 - \lambda_1)\, K_1^{t_1}(\lambda_1)\, R_{12}(-\lambda_1 - \lambda_2 - 2i\kappa)\, K_2^{t_2}(\lambda_2) = K_2^{t_2}(\lambda_2)\, R_{12}(-\lambda_1 - \lambda_2 - 2i\kappa)\, K_1^{t_1}(\lambda_1)\, R_{12}(\lambda_2 - \lambda_1)\,. \tag{2.17}$$

It is clear that any solution $K^-(\lambda)$ of (2.16), e.g. given in [5], gives rise to a solution $K^+(\lambda)$ of (2.17), defined by $K^+(\lambda) = K^-(-\lambda - i\kappa)^t$.

To determine the eigenvalues of the transfer matrix and the corresponding Bethe Ansatz equations, we use the analytical Bethe Ansatz method [9, 10, 5]. We follow the same procedure as in [5], by imposing certain constraints on the eigenvalues, deduced from the crossing symmetry of the model, the symmetry of the transfer matrix, the analyticity of the eigenvalues, and the fusion procedure for open spin chains. These constraints allow to determine the eigenvalues by solving a set of coupled non-linear consistency equations or Bethe Ansatz equations.

We first describe the case with trivial boundaries, $K^-(\lambda) = K^+(\lambda) = 1$.
We recall that the fusion procedure for the open spin chain [12, 29] yields the fused transfer matrix

$$\tilde{t}(\lambda) = \zeta(2\lambda + 2i\kappa)\ t(\lambda)\ t(\lambda + i\kappa) - \zeta(\lambda + i\kappa)^{2N} q(2\lambda + i\kappa) q(-2\lambda - 3i\kappa)\,, \tag{2.18}$$

where we define

$$\zeta(\lambda) = (\lambda + i\kappa)(\lambda + i)(\lambda - i\kappa)(\lambda - i)\,, \quad q(\lambda) = \theta_0(\lambda - i\theta_0)(\lambda - i\kappa). \tag{2.19}$$

Note that the value of $q(\lambda)$ is related to the specific choice of the position of the orthogonal and the symplectic part in the R-matrix. We choose for the general case (apart from the $osp(2|2n)$ case) the symplectic part to be "outside" and the orthogonal part to be "inside". This formulation corresponds to a specific Dynkin diagram: the so-called distinguished one (see fig. 1). For the case of $osp(2|2n)$ the distinguished Dynkin diagram has a special form (see fig. 1): it corresponds to the orthogonal part being "outside" and the symplectic part being inside. These considerations justify the conventions we have adopted in (2.5)-(2.6) and (2.8)-(2.9).
From the crossing symmetry of the R-matrix (2.12) it follows that: $t(\lambda) = t(-\lambda - i\kappa)$. The transfer matrix with $K^- = K^+ = 1$ is obviously $osp(M|2n)$ invariant, since the corresponding R-matrix (2.1) is $osp(M|2n)$ invariant, namely

$$\Big[R_{12},\ \ U_1 + U_2\Big] = 0, \tag{2.20}$$

where U is any generator of the $osp(M|2n)$ algebra. Finally, from the assumption of analyticity of the eigenvalues, we require that no singularity appears in the Bethe eigenvalues. The aforementioned set of constraints uniquely fix the eigenvalues.

We now choose an appropriate pseudo-vacuum, which is an exact eigenstate of the transfer matrix; it is the state with all "spins" up, i.e.

$$|\omega_+\rangle = \bigotimes_{i=1}^{N} |+\rangle_i \ \text{ where } \ |+\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{M+2n}\,. \tag{2.21}$$

Our choice of θ_0 ensures that this state is always bosonic, whichever orthosymplectic superalgebra we consider.

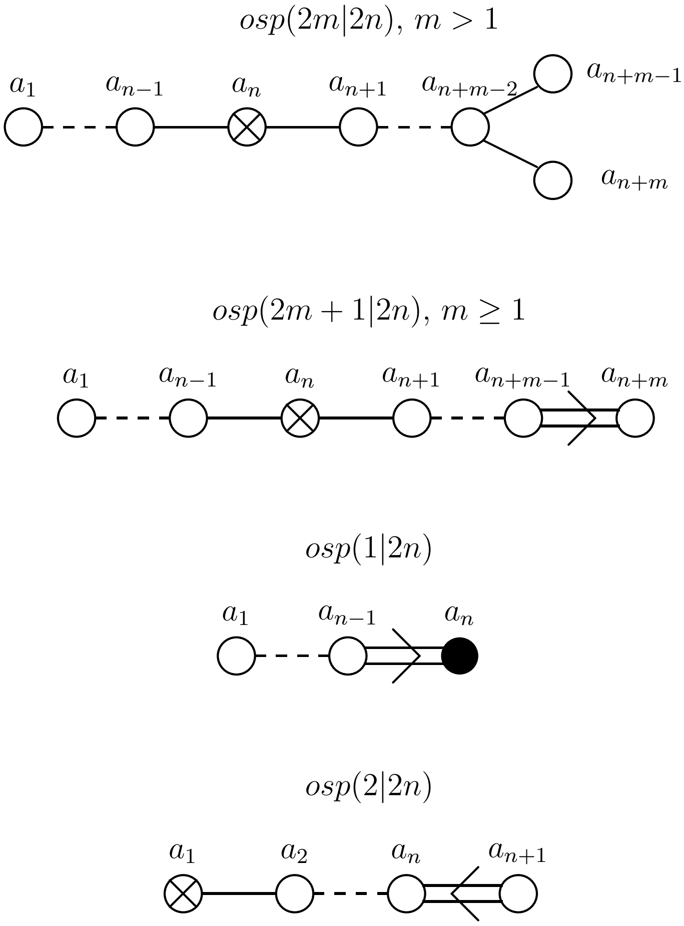

Figure 1: Distinguished Dynkin diagrams of the $osp(M|2n)$ superalgebras.

After some lengthy computation, we determine explicitly the action of the transfer matrix as $t(\lambda)|\omega_+\rangle = \Lambda^0(\lambda)|\omega_+\rangle$, where $\Lambda^0(\lambda)$ is given by the following expression

$$\Lambda^0(\lambda) = a(\lambda)^{2N} g_0(\lambda) + b(\lambda)^{2N} \sum_{l=1}^{2n+M-2} (-1)^{[l+1]} g_l(\lambda) + c(\lambda)^{2N} g_{2n+M-1}(\lambda) \tag{2.22}$$

with

$$a(\lambda) = (\lambda + i)(\lambda + i\kappa), \quad b(\lambda) = \lambda(\lambda + i\kappa), \quad c(\lambda) = \lambda(\lambda + i\kappa - i) \tag{2.23}$$

The expressions of the functions $g_l(\lambda)$ depend on the case we consider.

For the generic $osp(M|2n)$ **case,** $M \neq 2$, they are given by (with $M = 2m$ or $M = 2m+1$):

$$\begin{aligned}
g_l(\lambda) &= \frac{\lambda(\lambda+\frac{i\kappa}{2}-\frac{i}{2})(\lambda+i\kappa)}{(\lambda+\frac{i\kappa}{2})(\lambda+\frac{il}{2})(\lambda+\frac{i(l+1)}{2})}, \quad l=0,\dots,n-1, \\
g_l(\lambda) &= \frac{\lambda(\lambda+\frac{i\kappa}{2}-\frac{i}{2})(\lambda+i\kappa)}{(\lambda+\frac{i\kappa}{2})(\lambda+in-\frac{il}{2})(\lambda+in-\frac{i(l+1)}{2})}, \quad l=n,\dots,n+m-1 \\
g_{n+m}(\lambda) &= \frac{\lambda(\lambda+i\kappa)}{(\lambda+i\frac{n-m}{2})(\lambda+i\frac{n-m+1}{2})} \qquad \text{if } M=2m+1 \\
g_l(\lambda) &= g_{2n+M-l-1}(-\lambda-i\kappa), \quad l=0,1,...,M+2n
\end{aligned} \tag{2.24}$$

In this case, we have $\kappa = n+1-\frac{M}{2}$. We also set $k = n+m$.

For the case of $osp(2|2n)$, due to the different conventions, one has:

$$\begin{aligned}
g_0(\lambda) &= \frac{(\lambda+\frac{i\kappa}{2}+\frac{i}{2})(\lambda+i\kappa)}{(\lambda+\frac{i\kappa}{2})(\lambda+\frac{i}{2})}, \\
g_l(\lambda) &= \frac{\lambda(\lambda+\frac{i\kappa}{2}+\frac{i}{2})(\lambda+i\kappa)}{(\lambda+\frac{i\kappa}{2})(\lambda+i-\frac{il}{2})(\lambda+i-\frac{i(l+1)}{2})}, \quad l=1,\dots,n, \\
g_l(\lambda) &= g_{2n+1-l}(-\lambda-i\kappa), \quad l=0,\dots,2n+1
\end{aligned} \tag{2.25}$$

We remind that in this latter case, $\kappa = -n$.

2.3 Dressing functions

From the exact expression for the pseudo-vacuum eigenvalue, we introduce the following assumption for the structure of the general eigenvalues:

$$\Lambda(\lambda) = a(\lambda)^{2N} g_0(\lambda)A_0(\lambda) + b(\lambda)^{2N} \sum_{l=1}^{2n+M-2} (-1)^{[l+1]} g_l(\lambda)A_l(\lambda) + c(\lambda)^{2N} g_{2n+M-1}(\lambda)A_{2n+M-1}(\lambda) \tag{2.26}$$

where the so-called "dressing functions" $A_i(\lambda)$ need now to be determined.
We immediately get from the crossing symmetry of the transfer matrix:

$$A_l(\lambda) = A_{2n+M-l-1}(-\lambda-i\kappa) \qquad l=0,...,M+2n-1\,. \tag{2.27}$$

Moreover, we obtain from the fusion relation (2.18) the following identity, by a comparison of the forms (2.26) for the initial and fused auxiliary spaces:

$$A_0(\lambda+i\kappa)A_{2n+M-1}(\lambda) = 1\,. \tag{2.28}$$

Gathering the above two equations (2.27), (2.28) we conclude

$$A_0(\lambda)A_0(-\lambda) = 1\,. \tag{2.29}$$

Additional constraints are then imposed on the "dressing functions" from analyticity properties. Studying carefully the common poles of successive g_l's, we deduce from the form of the g_l functions (2.24) that g_l and g_{l-1} have common poles at $\lambda = -\frac{il}{2}$ or $\lambda = -in + \frac{il}{2}$, therefore from analyticity requirements

$$\begin{aligned} A_l(-\frac{il}{2}) &= A_{l-1}(-\frac{il}{2}), \quad l = 1, \ldots, n-1, \\ A_l(-in + \frac{il}{2}) &= A_{l-1}(-in + \frac{il}{2}), \quad l = n, \ldots, n+m-1 \end{aligned} \tag{2.30}$$

There is an extra constraint when $M = 2m+1$, namely

$$A_{n+m}(-in + \frac{ik}{2}) = A_{n+m-1}(-in + \frac{ik}{2}). \tag{2.31}$$

Having deduced the necessary constraints for the "dressing functions", we determine them explicitly. The "dressing functions" A_l are basically characterised by a set of parameters $\lambda_j^{(l)}$ with $j = 1, \ldots, M^{(l)}$, where the integer numbers $M^{(l)}$ are related to the diagonal generators of $osp(M|2n)$. These generators are defined as:

$$S^{(l)} = \sum_{i=1}^{N} s_i^{(l)}, \quad s^{(l)} = (e_{ll} - e_{\bar{l}\bar{l}})/2, \quad (e_{kl})_{ij} = \delta_{ik}\delta_{jl}. \tag{2.32}$$

The precise identification of $M^{(l)}$ follows from the symmetry of the transfer matrix (see also [9]):

$$S^{(l)} = M^{(l-1)} - M^{(l)}, \quad l = 1, \ldots, n-1, n+1, ..., n+m-2 \tag{2.33}$$

$$S^{(n)} = M^{(n-1)} - 2M^{(n)} \tag{2.34}$$

$$S^{(n+m-1)} = M^{(n+m-2)} - M^{(n+m-1)}, \quad S^{(n+m)} = M^{(n+m-1)} - M^{(n+m)}, \quad \text{if } M = 2m+1 \tag{2.35}$$

$$S^{(n+m-1)} = M^{(n+m-2)} - M^{(+)} - M^{(-)}, \quad S^{(n+m)} = M^{(+)} - M^{(-)}, \quad \text{if } M = 2m \tag{2.36}$$

and $M^{(0)} = \frac{N}{2}$.

A. $\mathbf{osp(2m+1|2n)}$

The dressing functions take the form:

$$\begin{aligned} A_0(\lambda) &= \prod_{j=1}^{M^{(1)}} \frac{\lambda + \lambda_j^{(1)} - \frac{i}{2}}{\lambda + \lambda_j^{(1)} + \frac{i}{2}} \frac{\lambda - \lambda_j^{(1)} - \frac{i}{2}}{\lambda - \lambda_j^{(1)} + \frac{i}{2}}, \\ A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda + \lambda_j^{(l)} + \frac{il}{2} + i}{\lambda + \lambda_j^{(l)} + \frac{il}{2}} \frac{\lambda - \lambda_j^{(l)} + \frac{il}{2} + i}{\lambda - \lambda_j^{(l)} + \frac{il}{2}} \\ &\times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda + \lambda_j^{(l+1)} + \frac{il}{2} - \frac{i}{2}}{\lambda + \lambda_j^{(l+1)} + \frac{il}{2} + \frac{i}{2}} \frac{\lambda - \lambda_j^{(l+1)} + \frac{il}{2} - \frac{i}{2}}{\lambda - \lambda_j^{(l+1)} + \frac{il}{2} + \frac{i}{2}}, \qquad l = 1, \ldots, n-1 \end{aligned}$$

$$
\begin{aligned}
A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda+\lambda_j^{(l)}+in-\frac{il}{2}-i}{\lambda+\lambda_j^{(l)}+in-\frac{il}{2}} \frac{\lambda-\lambda_j^{(l)}+in-\frac{il}{2}-i}{\lambda-\lambda_j^{(l)}+in-\frac{il}{2}} \\
&\times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda+\lambda_j^{(l+1)}+in-\frac{il}{2}+\frac{i}{2}}{\lambda+\lambda_j^{(l+1)}+in-\frac{il}{2}-\frac{i}{2}} \frac{\lambda-\lambda_j^{(l+1)}+in-\frac{il}{2}+\frac{i}{2}}{\lambda-\lambda_j^{(l+1)}+in-\frac{il}{2}-\frac{i}{2}}, \qquad l=n,\ldots,n+m-1 \\
A_{n+m}(\lambda) &= \prod_{j=1}^{M^{(k)}} \frac{\lambda+\lambda_j^{(k)}+in-\frac{ik}{2}+i}{\lambda+\lambda_j^{(k)}+in-\frac{ik}{2}} \frac{\lambda-\lambda_j^{(k)}+in-\frac{ik}{2}+i}{\lambda-\lambda_j^{(k)}+in-\frac{ik}{2}} \\
&\times \frac{\lambda+\lambda_j^{(k)}+in-\frac{ik}{2}-\frac{i}{2}}{\lambda+\lambda_j^{(k)}+in-\frac{ik}{2}+\frac{i}{2}} \frac{\lambda-\lambda_j^{(k)}+in-\frac{ik}{2}-\frac{i}{2}}{\lambda-\lambda_j^{(k)}+in-\frac{ik}{2}+\frac{i}{2}}, \qquad (2.37)
\end{aligned}
$$

and $A_l(\lambda) = A_{2n+2m-l}(-\lambda - i\kappa)$ for $l > n+m$, $\kappa = n - m + \frac{1}{2}$.

B. osp(2m|2n) with m > 1

The dressing functions are the same as in the previous case for $l = 0, \ldots, n+m-3$, but

$$
\begin{aligned}
A_{n+m-2}(\lambda) &= \prod_{j=1}^{M^{(k-2)}} \frac{\lambda+\lambda_j^{(k-2)}+in-\frac{ik}{2}}{\lambda+\lambda_j^{(k-2)}+in-\frac{ik}{2}+i} \frac{\lambda-\lambda_j^{(k-2)}+in-\frac{ik}{2}}{\lambda-\lambda_j^{(k-2)}+in-\frac{ik}{2}+i} \\
&\times \prod_{j=1}^{M^{(+)}} \frac{\lambda+\lambda_j^{(+)}+in-\frac{ik}{2}+\frac{3i}{2}}{\lambda+\lambda_j^{(+)}+in-\frac{ik}{2}+\frac{i}{2}} \frac{\lambda-\lambda_j^{(+)}+in-\frac{ik}{2}+\frac{3i}{2}}{\lambda-\lambda_j^{(+)}+in-\frac{ik}{2}+\frac{i}{2}} \\
&\times \prod_{j=1}^{M^{(-)}} \frac{\lambda+\lambda_j^{(-)}+in-\frac{ik}{2}+\frac{3i}{2}}{\lambda+\lambda_j^{(-)}+in-\frac{ik}{2}+\frac{i}{2}} \frac{\lambda-\lambda_j^{(-)}+in-\frac{ik}{2}+\frac{3i}{2}}{\lambda-\lambda_j^{(-)}+in-\frac{ik}{2}+\frac{i}{2}}, \\
A_{n+m-1}(\lambda) &= \prod_{j=1}^{M^{(+)}} \frac{\lambda+\lambda_j^{(+)}+in-\frac{ik}{2}+\frac{3i}{2}}{\lambda+\lambda_j^{(+)}+in-\frac{ik}{2}+\frac{i}{2}} \frac{\lambda-\lambda_j^{(+)}+in-\frac{ik}{2}+\frac{3i}{2}}{\lambda-\lambda_j^{(+)}+in-\frac{ik}{2}+\frac{i}{2}} \\
&\times \prod_{j=1}^{M^{(-)}} \frac{\lambda+\lambda_j^{(-)}+in-\frac{ik}{2}-\frac{i}{2}}{\lambda+\lambda_j^{(-)}+in-\frac{ik}{2}+\frac{i}{2}} \frac{\lambda-\lambda_j^{(-)}+in-\frac{ik}{2}-\frac{i}{2}}{\lambda-\lambda_j^{(-)}+in-\frac{ik}{2}+\frac{i}{2}} \qquad (2.38)
\end{aligned}
$$

and $A_l(\lambda) = A_{2n+2m-l-1}(-\lambda - i\kappa)$ for $l > n+m-1$, $\kappa = n + 1 - m$.

C. osp(2|2n)

As already mentioned, this case must be treated separately, because of the different position of the orthogonal and symplectic parts in the R-matrix. Here the orthogonal part of the R-matrix is considered to be "outside" and the symplectic part "inside" as opposed to the previous cases.

The corresponding dressing functions have the form

$$
\begin{aligned}
A_0(\lambda) &= \prod_{j=1}^{M^{(1)}} \frac{\lambda+\lambda_j^{(1)}-\frac{i}{2}}{\lambda+\lambda_j^{(1)}+\frac{i}{2}}\ \frac{\lambda-\lambda_j^{(1)}-\frac{i}{2}}{\lambda-\lambda_j^{(1)}+\frac{i}{2}}\,, \\
A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda+\lambda_j^{(l)}-\frac{il}{2}}{\lambda+\lambda_j^{(l)}+i-\frac{il}{2}}\ \frac{\lambda-\lambda_j^{(l)}-\frac{il}{2}}{\lambda-\lambda_j^{(l)}+i-\frac{il}{2}} \\
&\times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda+\lambda_j^{(l+1)}+\frac{3i}{2}-\frac{il}{2}}{\lambda+\lambda_j^{(l+1)}+\frac{i}{2}-\frac{il}{2}}\ \frac{\lambda-\lambda_j^{(l+1)}+\frac{3i}{2}-\frac{il}{2}}{\lambda-\lambda_j^{(l+1)}+\frac{i}{2}-\frac{il}{2}}\,, \qquad l=1,\dots,n-1 \\
A_n(\lambda) &= \prod_{j=1}^{M^{(n)}} \frac{\lambda+\lambda_j^{(n)}-\frac{in}{2}}{\lambda+\lambda_j^{(n)}+i-\frac{in}{2}}\ \frac{\lambda-\lambda_j^{(n)}-\frac{in}{2}}{\lambda-\lambda_j^{(n)}+i-\frac{in}{2}} \\
&\times \prod_{j=1}^{M^{(n+1)}} \frac{\lambda+\lambda_j^{(n+1)}+2i-\frac{in}{2}}{\lambda+\lambda_j^{(n+1)}-\frac{in}{2}}\ \frac{\lambda-\lambda_j^{(n+1)}+2i-\frac{in}{2}}{\lambda-\lambda_j^{(n+1)}-\frac{in}{2}}\,,
\end{aligned}
\tag{2.39}
$$

and $A_l(\lambda) = A_{2n+1-l}(-\lambda - i\kappa)$ for $l > n$, $\kappa = -n$.

2.4 Bethe Ansatz equations

We define the function

$$
e_x(\lambda) = \frac{\lambda+\frac{ix}{2}}{\lambda-\frac{ix}{2}}\,. \tag{2.40}
$$

From the analyticity requirements one obtains the Bethe Ansatz equations which read as:

A. osp(2m + 1|2n)

$$
\begin{aligned}
e_1(\lambda_i^{(1)})^{2N} &= \prod_{j=1,j\neq i}^{M^{(1)}} e_2(\lambda_i^{(1)}-\lambda_j^{(1)})\ e_2(\lambda_i^{(1)}+\lambda_j^{(1)}) \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)}-\lambda_j^{(2)})\ e_{-1}(\lambda_i^{(1)}+\lambda_j^{(2)})\,, \\
1 &= \prod_{j=1,j\neq i}^{M^{(l)}} e_2(\lambda_i^{(l)}-\lambda_j^{(l)})\ e_2(\lambda_i^{(l)}+\lambda_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l+\tau)})\ e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l+\tau)}) \\
& \quad l=2,\dots,n+m-1,\ \ l\neq n \\
1 &= \prod_{j=1}^{M^{(n+1)}} e_1(\lambda_i^{(n)}-\lambda_j^{(n+1)})\ e_1(\lambda_i^{(n)}+\lambda_j^{(n+1)}) \prod_{j=1}^{M^{(n-1)}} e_{-1}(\lambda_i^{(n)}-\lambda_j^{(n-1)})\ e_{-1}(\lambda_i^{(n)}+\lambda_j^{(n-1)}) \\
1 &= \prod_{j=1,j\neq i}^{M^{(n+m)}} e_1(\lambda_i^{(n+m)}-\lambda_j^{(n+m)})\ e_1(\lambda_i^{(n+m)}+\lambda_j^{(n+m)}) \\
&\times \prod_{j=1}^{M^{(n+m-1)}} e_{-1}(\lambda_i^{(n+m)}-\lambda_j^{(n+m-1)})\ e_{-1}(\lambda_i^{(n+m)}+\lambda_j^{(n+m-1)})
\end{aligned}
\tag{2.41}
$$

In particular for $M=1$ the Bethe Ansatz equations become

$$\begin{aligned}
e_1(\lambda_i^{(1)})^{2N} &= \prod_{j=1,j\neq i}^{M^{(1)}} e_2(\lambda_i^{(1)}-\lambda_j^{(1)})\ e_2(\lambda_i^{(1)}+\lambda_j^{(1)}) \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)}-\lambda_j^{(2)})\ e_{-1}(\lambda_i^{(1)}+\lambda_j^{(2)})\,, \\
1 &= \prod_{j=1,j\neq i}^{M^{(l)}} e_2(\lambda_i^{(l)}-\lambda_j^{(l)})\ e_2(\lambda_i^{(l)}+\lambda_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l+\tau)})\ e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l+\tau)}) \\
& l=2,\ldots,n-1, \\
1 &= \prod_{j=1,j\neq i}^{M^{(n)}} e_{-1}(\lambda_i^{(n)}-\lambda_j^{(n)})\ e_{-1}(\lambda_i^{(n)}+\lambda_j^{(n)}) e_2(\lambda_i^{(n)}-\lambda_j^{(n)})\ e_2(\lambda_i^{(n)}+\lambda_j^{(n)}) \\
&\times \prod_{j=1}^{M^{(n-1)}} e_{-1}(\lambda_i^{(n)}-\lambda_j^{(n-1)})\ e_{-1}(\lambda_i^{(n)}+\lambda_j^{(n-1)})
\end{aligned} \tag{2.42}$$

B. osp(2m|2n) with m > 1

The first $n+m-3$ equations are the same as in the previous case for $M=2m+1$, see eq. (2.41), but the last three equations are modified, and they become identical to the last three equations of the $so(2n+2m)$ open spin chain, namely,

$$\begin{aligned}
1 &= \prod_{j=1,j\neq i}^{M^{(n+m-2)}} e_2(\lambda_i^{(n+m-2)}-\lambda_j^{(n+m-2)})\ e_2(\lambda_i^{(n+m-2)}+\lambda_j^{(n+m-2)}) \\
&\times \prod_{j=1}^{M^{(n+m-3)}} e_{-1}(\lambda_i^{(n+m-2)}-\lambda_j^{(n+m-3)})\ e_{-1}(\lambda_i^{(n+m-2)}+\lambda_j^{(n+m-3)}) \\
&\times \prod_{\tau=\pm}\prod_{j=1}^{M^{(\tau)}} e_{-1}(\lambda_i^{(n+m-2)}-\lambda_j^{(\tau)})\ e_{-1}(\lambda_i^{(n+m-2)}+\lambda_j^{(\tau)}) \\
1 &= \prod_{j=1,j\neq i}^{M^{(\tau)}} e_2(\lambda_i^{(\tau)}-\lambda_j^{(\tau)})\ e_2(\lambda_i^{(\tau)}+\lambda_j^{(\tau)}) \prod_{j=1}^{M^{(n+m-2)}} e_{-1}(\lambda_i^{(\tau)}-\lambda_j^{(n+m-2)})\ e_{-1}(\lambda_i^{(\tau)}+\lambda_j^{(n+m-2)})
\end{aligned} \tag{2.43}$$

C. osp(2|2n)

$$\begin{aligned}
e_1(\lambda_i^{(1)})^{2N} &= \prod_{j=1}^{M^{(2)}} e_1(\lambda_i^{(1)}-\lambda_j^{(2)})\ e_1(\lambda_i^{(1)}+\lambda_j^{(2)})\,, \\
1 &= \prod_{j=1,j\neq i}^{M^{(l)}} e_2(\lambda_i^{(l)}-\lambda_j^{(l)})\ e_2(\lambda_i^{(l)}+\lambda_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l+\tau)})\ e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l+\tau)}) \\
& l=2,\ldots,n-1,
\end{aligned}$$

$$
\begin{aligned}
1 &= \prod_{j=1,}^{M^{(n+1)}} e_{-2}(\lambda_i^{(n)} - \lambda_j^{(n+1)})\ e_{-2}(\lambda_i^{(n)} + \lambda_j^{(n+1)}) \\
&\times \prod_{j=1}^{M^{(n-1)}} e_{-1}(\lambda_i^{(n)} - \lambda_j^{(n-1)})\ e_{-1}(\lambda_i^{(n)} + \lambda_j^{(n-1)}) \prod_{j=1,j\neq i}^{M^{(n)}} e_2(\lambda_i^{(n)} - \lambda_j^{(n)})\ e_2(\lambda_i^{(n)} + \lambda_j^{(n)}) \\
1 &= \prod_{j=1,j\neq i}^{M^{(n+1)}} e_4(\lambda_i^{(n+1)} - \lambda_j^{(n+1)})\ e_4(\lambda_i^{(n+1)} + \lambda_j^{(n+1)}) \\
& \prod_{j=1}^{M^{(n)}} e_{-2}(\lambda_i^{(n+1)} - \lambda_j^{(n)})\ e_{-2}(\lambda_i^{(n+1)} + \lambda_j^{(n)}).
\end{aligned}
\tag{2.44}
$$

In particular the equations for the $osp(2|2)$ open chain are given by

$$
\begin{aligned}
e_1(\lambda_i^{(1)})^{2N} &= \prod_{j=1}^{M^{(2)}} e_2(\lambda_i^{(1)} - \lambda_j^{(2)})\ e_2(\lambda_i^{(1)} + \lambda_j^{(2)})\,, \\
1 &= \prod_{j=1}^{M^{(1)}} e_{-2}(\lambda_i^{(2)} - \lambda_j^{(1)})\ e_{-2}(\lambda_i^{(2)} + \lambda_j^{(1)}) \prod_{j=1,j\neq i}^{M^{(2)}} e_4(\lambda_i^{(2)} - \lambda_j^{(2)})\ e_4(\lambda_i^{(2)} + \lambda_j^{(2)})
\end{aligned}
\tag{2.45}
$$

Notice that there is a one-to-one correspondence between the distinguished Dynkin diagrams (see fig. 1) and the Bethe Ansatz equations derived for each case. The Bethe Ansatz equations for $osp(1|2n)$, $osp(2|2n)$, $osp(2m|2)$, $osp(2m+1|2)$ can now be compared with the corresponding results obtained in [25] for super spin chains with periodic boundaries. Let us point out however that we derived explicitly the Bethe Ansatz equations for any $osp(M|2n)$ open spin chain, and we expect the corresponding equations for a chain with periodic boundary conditions to be "halved" (i.e. half of the factors in the products should be missing) compared to the ones we found.

2.5 Non-trivial diagonal boundary conditions

Until now we have derived the Bethe Ansatz equations for trivial boundary conditions, namely $K^- = K^+ = 1$. We shall now insert non-trivial boundary effects and then rederive the modified Bethe Ansatz equations. We choose K^- to be one of the diagonal solutions D1, D2, D3 found in Proposition 3.1 of [5]. We consider, for simplicity but without loss of generality, $K^+ = 1$. Note that the pseudo-vacuum remains an exact eigenstate after this modification. We rewrite the solutions D1, D2 and D3 of [5] in a slightly modified notation, which we are going to use from now on.

D1: The solution D1 can be written in the following form

$$
K(\lambda) = diag(\alpha, \ldots, \alpha, \beta, \ldots, \beta)\,. \tag{2.46}
$$

The number of $\alpha's$ is equal to the number of $\beta's$, so that this solution exists only for the $osp(2m|2n)$ cases as stated in Proposition 3.1 of ref. [5], and

$$
\alpha(\lambda) = -\lambda + i\xi, \quad \beta(\lambda) = \lambda + i\xi, \tag{2.47}
$$

where ξ is the free boundary parameter.

D2: Solution D2 can be written in the $osp(M|2n)$ case ($M > 2$) as

$$K(\lambda) = diag(\underbrace{1,\dots,1}_{n}, \bar{\alpha}, \underbrace{1,\dots,1}_{M-2}, \bar{\beta}, \underbrace{1,\dots,1}_{n}) \tag{2.48}$$

and for the $osp(2|2n)$ case as

$$K(\lambda) = diag(\bar{\alpha}, \underbrace{1,\dots,1}_{2n}, \bar{\beta}) \tag{2.49}$$

with

$$\bar{\alpha}(\lambda) = \frac{-\lambda + i\xi_1}{\lambda + i\xi_1}, \quad \bar{\beta}(\lambda) = \frac{-\lambda + i\xi_n}{\lambda + i\xi_n}, \tag{2.50}$$

where ξ_1 and ξ_n are the boundary parameters which satisfy the constraint

$$\xi_1 + \xi_n = \kappa - \theta_0. \tag{2.51}$$

Obviously, this solution does not exist for the $osp(1|2n)$ superalgebras.

D3: Solution D3 has the form in the $osp(M|2n)$ case ($M \neq 2$)

$$K(\lambda) = diag(\underbrace{\beta,\dots,\beta}_{n-n_1}, \underbrace{\alpha,\dots,\alpha}_{n_1+m_1}, \underbrace{\beta,\dots,\beta}_{M-2m_1}, \underbrace{\alpha,\dots,\alpha}_{n_1+m_1}, \underbrace{\beta,\dots,\beta}_{n-n_1}). \tag{2.52}$$

while for the $osp(2|2n)$ case it takes the form

$$K(\lambda) = diag(\alpha, \underbrace{\alpha,\dots,\alpha}_{n_1}, \underbrace{\beta,\dots,\beta}_{2n-2n_1}, \underbrace{\alpha,\dots,\alpha}_{n_1}, \alpha). \tag{2.53}$$

The $osp(1|2n)$ case is recovered by taking $M = 1$ and $m_1 = 0$ in (2.52).
Again, α and β are given by

$$\alpha(\lambda) = -\lambda + i\xi, \quad \beta(\lambda) = \lambda + i\xi \tag{2.54}$$

where $\xi = (\kappa + 2m_1 - 2n_1 - 1)/2$ has a fixed value, the integers m_1 and n_1 being restricted to $0 \leq n_1 \leq n$ and $0 \leq m_1 \leq m$ ($M = 2m$ or $M = 2m + 1$) respectively, n_1 and m_1 being neither both zero nor taking maximal values simultaneously.

We now come to the explicit expression of the eigenvalues when K^- is one of the above mentioned solutions. We should point out that the dressing functions are related to the bulk behaviour of the chain and thus they are form-invariant under changes of boundary conditions. Indeed the only modifications in the expression of the eigenvalues (2.26) occur in the g_l functions, which basically characterise the boundary effects. We call the new g_l functions $\tilde{g}_l$.

D1: As already mentioned, the solution D1 can only be applied to $osp(2m|2n)$ with $m \geq 1$. In this case we have

$$\begin{aligned}
\tilde{g}_l(\lambda) &= (-\lambda + i\xi)g_l(\lambda), \quad l = 0, \dots, n+m-1 \\
\tilde{g}_l(\lambda) &= (\lambda + i\xi + i\kappa)g_l(\lambda), \quad l = n+m, \dots, 2n+2m-1
\end{aligned} \tag{2.55}$$

where $g_l(\lambda)$ are given by (2.24)–(2.25). The system with such boundaries has a residual symmetry $sl(m|n)$, which immediately follows from the structure of the corresponding K matrix.

D2: We have to separate the cases $osp(M|2n)$ with $M \neq 2$ and $osp(2|2n)$. In the $osp(M|2n)$ case with $M \neq 2$, one gets

$$\begin{aligned}
\tilde{g}_l(\lambda) &= g_l(\lambda), \quad l = 0, \dots, n-1, \qquad \tilde{g}_n(\lambda) = \frac{(-\lambda + i\xi_1 - in)}{(\lambda + i\xi_1)}\, g_n(\lambda), \\
\tilde{g}_l(\lambda) &= \frac{(\lambda + i\xi_1 - i)}{(\lambda + i\xi_1)}\, g_l(\lambda), \quad l = n+1, \dots, n+M-2 \\
\tilde{g}_{n+M-1}(\lambda) &= \frac{(\lambda + i\xi_1 - i)}{(\lambda + i\xi_1)}\, \frac{(-\lambda + i\xi_n - in + iM - 3i)}{(\lambda + i\xi_n)}\, g_{n+M-1}(\lambda), \\
\tilde{g}_l(\lambda) &= \frac{(\lambda + i\xi_1 - i)}{(\lambda + i\xi_1)}\, \frac{(\lambda + i\xi_n - i)}{(\lambda + i\xi_n)}\, g_l(\lambda), \quad l = n+M, \dots, 2n+M-1
\end{aligned} \tag{2.56}$$

In the $osp(2|2n)$ case, the formulae are similar to the $sp(2n)$ case:

$$\begin{aligned}
\tilde{g}_0(\lambda) &= \frac{(-\lambda + i\xi_1)}{(\lambda + i\xi_1)}\, g_0(\lambda), \qquad \tilde{g}_{2n+1}(\lambda) = \frac{(\lambda + i\xi_1 + i)}{(\lambda + i\xi_1)}\, \frac{(\lambda + i\xi_1 + i\kappa)}{(-\lambda - i\kappa + i\xi_1 + i)}\, g_{2n+1}(\lambda), \\
\tilde{g}_l(\lambda) &= \frac{(\lambda + i\xi_1 + i)}{(\lambda + i\xi_1)}\, g_l(\lambda), \quad l = 1, \dots, 2n
\end{aligned} \tag{2.57}$$

The functions $g_l(\lambda)$ are given by (2.24)–(2.25). The system with such boundaries has a residual symmetry $osp(M-2|2n)$.

D3: For the D3 solution we find the following modified g functions in the case of $osp(M|2n)$ with $M \neq 2$:

$$\begin{aligned}
\tilde{g}_l(\lambda) &= (\lambda + i\xi)\, g_l(\lambda), \quad l = 0, \dots, n-n_1-1 \\
\tilde{g}_l(\lambda) &= (-\lambda + i\xi - in + in_1)\, g_l(\lambda), \quad l = n-n_1, \dots, n+m_1-1 \\
\tilde{g}_l(\lambda) &= (\lambda + i\frac{\kappa}{2} - \frac{i}{2})\, g_l(\lambda), \quad l = n+m_1, \dots, n+m-1 \\
\tilde{g}_{M+2n-1-l}(\lambda) &= \frac{(\lambda + i\frac{\kappa}{2} - \frac{i}{2})}{(\lambda + i\frac{\kappa}{2} + \frac{i}{2})}\, \tilde{g}_l(-\lambda - i\kappa), \quad l = 0, \dots, n+m-1
\end{aligned} \tag{2.58}$$

and in the case of $osp(2m+1|2n)$

$$\tilde{g}_{n+m}(\lambda) = (\lambda + i\frac{\kappa}{2} - \frac{i}{2})\, g_{n+m}(\lambda) \tag{2.59}$$

In the $osp(2|2n)$ case, the formulae become

$$\begin{aligned}\tilde{g}_l(\lambda) &= (-\lambda + i\xi)\; g_l(\lambda), \quad l = 0, \ldots, n_1 \\ \tilde{g}_l(\lambda) &= (\lambda + i\frac{\kappa}{2} + \frac{i}{2})\; g_l(\lambda), \quad l = n_1 + 1, \ldots, 2n - n_1 \\ \tilde{g}_l(\lambda) &= (-\lambda - i\kappa - i\xi)\; \frac{(\lambda + i\frac{\kappa}{2} + \frac{i}{2})}{(\lambda + i\frac{\kappa}{2} - \frac{i}{2})}\; g_l(\lambda), \quad l = 2n - n_1 + 1, \ldots, 2n + 1 \end{aligned} \tag{2.60}$$

The system with such boundaries has a residual symmetry $osp(M-2m_1|2n-2n_1) \oplus osp(2m_1|2n_1)$.

We now formulate the Bethe Ansatz equations for the general diagonal solutions. The only modifications induced on Bethe Ansatz equations are the following for each solution:

D1 The factor $-e^{-1}_{2\xi+\kappa}(\lambda)$ appears in the LHS of the $(n+m)^{th}$ Bethe equation.

D2 The factor $-e_{2\xi_1-n}(\lambda)$ appears in the LHS of the n^{th} Bethe equation.
The factor $-e^{-1}_{2\xi_1-n-1}(\lambda)$ appears in the LHS of the $(n+1)^{th}$ Bethe equation.

D3 The factor $-e_{2\xi-(n-n_1)}(\lambda)$ appears in the LHS of the $(n-n_1)^{th}$ Bethe equation.
The factor $-e^{-1}_{2\xi+2n_1-m_1-n}(\lambda)$ appears in the LHS of the $(n+m_1)^{th}$ Bethe equation.

3 Scattering for the $osp(1|2n)$ open spin chain

3.1 Low lying excitations

Before we derive explicitly the bulk and boundary scattering amplitudes for the $osp(1|2n)$ case we first need to determine the ground state and the low-lying excitations of the model. We recall that the energy is derived via the relation $H = \frac{d}{d\lambda} t(\lambda)|_{\lambda=0}$. It is given by

$$E = -\frac{1}{2\pi} \sum_{j=1}^{M^{(1)}} \frac{1}{\left(\lambda_j^{(1)}\right)^2 + \frac{1}{4}}\,. \tag{3.1}$$

In what follows we write the Bethe Ansatz equations for the ground state and the low-lying excitations (holes) of the models under study. Bethe Ansatz equations may in general only be solved in the thermodynamic limit $N \to \infty$. In this limit, a state is described in particular by the density functions $\sigma^l(\lambda)$ of the parameters $\lambda_i^{(l)}$.

A. osp(1|2)

Let us first consider the $osp(1|2)$ case, for which the ground state consists of one filled Dirac sea with real strings (all λ_i's real). The set of Bethe Ansatz equations for the $osp(1|2)$ case takes the form

$$e_1(\lambda_i)^{2N} e_1(\lambda_i) e_{-\frac{1}{2}}(\lambda_i) = \prod_{j=1}^{M} e_2(\lambda_i - \lambda_j)\; e_2(\lambda_i + \lambda_j)\; e_{-1}(\lambda_i - \lambda_j)\; e_{-1}(\lambda_i + \lambda_j)\,, \tag{3.2}$$

The reason why we study this case separately is basically because we wish to point out the striking similarity between the latter Bethe Ansatz equations (3.2) and the corresponding equations appearing in the study of the $SU(3)$ open spin chain with "soliton non-preserving" boundary conditions [12]. This is indeed a remarkable connection, which can presumably be extended to open spin chains with "soliton non-preserving" boundary conditions, for higher rank algebras. In particular, we expect that for any $SU(n)$ (n odd) chain with "soliton non-preserving" boundary conditions the resulting Bethe Ansatz equations will have the same form as in the $osp(1|n-1)$ open spin chain with certain diagonal boundaries. We hope to report on this in detail elsewhere.

We now study the low-lying excitations, which are holes in the filled Dirac sea. In order to convert the sums into integrals, after taking the thermodynamic limit ($N \to \infty$), we employ the following approximate relation

$$\frac{1}{N}\sum_{i=1}^{M} f(\lambda_i^{(l)}) = \int_0^{\infty} d\lambda f(\lambda)\sigma^l(\lambda) - \frac{1}{N}\sum_{i=1}^{\nu^{(l)}} f(\tilde{\lambda}_i^{(l)}) - \frac{1}{2N} f(0) \tag{3.3}$$

where the correction terms take into account the $\nu^{(l)}$ holes located at values $\tilde{\lambda}_i^{(l)}$ and the halved contribution at 0^+. For $osp(1|2)$ in particular $\nu^{(l)} \equiv \nu$ and $\tilde{\lambda}_i^{(l)} \equiv \tilde{\lambda}_i$.
We shall denote by $\hat{f}(\omega)$ the Fourier transform of any function $f(\lambda)$. Once we take the logarithm and the derivative of (3.2), we derive the densities from the equation

$$\hat{\mathcal{K}}(\omega)\hat{\sigma}(\omega) = \hat{a}_1(\omega) + \frac{1}{N}\hat{F}(\omega) \tag{3.4}$$

where $a_\ell(\lambda) = \dfrac{i}{2\pi}\dfrac{d}{d\lambda}\ln e_\ell(\lambda)$ and $\hat{a}_\ell(\omega) = e^{-\frac{\ell\omega}{2}}$. Moreover

$$\hat{\mathcal{K}}(\omega) = e^{-\frac{\omega}{2}}\frac{\cosh\frac{3\omega}{4}}{\cosh\frac{\omega}{4}}, \quad F(\lambda) = a_2(\lambda) - a_{\frac{1}{2}}(\lambda) + \sum_{i=1}^{\nu}\Big((a_2 - a_1)(\lambda - \tilde{\lambda}_i) + (a_2 - a_1)(\lambda + \tilde{\lambda}_i)\Big). \tag{3.5}$$

Finally equation (3.4) can be written as

$$\sigma(\lambda) = 2\epsilon(\lambda) + \frac{1}{N}\Phi(\lambda), \tag{3.6}$$

where

$$\hat{\epsilon}(\omega) = \hat{a}_1(\omega)\hat{\mathcal{R}}(\omega), \quad \hat{\Phi}(\omega) = \hat{\mathcal{R}}(\omega)\hat{F}(\omega), \quad \text{and} \quad \hat{\mathcal{R}}(\omega) = \hat{\mathcal{K}}^{-1}(\omega) \tag{3.7}$$

In particular the energy ϵ can be written in terms of hyperbolic functions as $\hat{\epsilon}(\omega) = \frac{\cosh\frac{\omega}{4}}{\cosh\frac{3\omega}{4}}$.

B. osp(1|2n)

We recall that we can only consider the D3 solution in this case. The ground state consists of n filled Dirac seas with real strings. With the help of relation (3.3) we derive the densities that describe the state with $\nu^{(l)}$ holes in the l sea from the equation

$$\hat{\mathcal{K}}(\omega)\hat{\sigma}(\omega) = \hat{a}(\omega) + \frac{1}{N}\hat{F}(\omega) + \frac{1}{N}\hat{G}(\omega,\ \xi) \tag{3.8}$$

where we have introduced

$$a(\lambda)=\begin{pmatrix} 2a_1(\lambda)\\ 0\\ \vdots\\ 0\end{pmatrix}, \quad \sigma(\lambda)=\begin{pmatrix} \sigma^1(\lambda)\\ \vdots\\ \sigma^l(\lambda)\\ \vdots\\ \sigma^n(\lambda)\end{pmatrix}. \tag{3.9}$$

$F(\lambda)$ is a n-vector as well with

$$\begin{aligned} F^j(\lambda) &= a_1(\lambda)\delta_{j1}-a_1(\lambda)+a_2(\lambda)+\sum_{i=1}^{\nu^{(l)}}\Big(a_2(\lambda-\tilde\lambda_i^{(l)})+a_2(\lambda+\tilde\lambda_i^{(l)})\Big)\delta_{lj}\\ &- \sum_{i=1}^{\nu^{(l)}}\Big(a_1(\lambda-\tilde\lambda_i^{(l)})+a_1(\lambda+\tilde\lambda_i^{(l)})\Big)(\delta_{j,l+1}+\delta_{j,l-1}), \qquad (j=1,\dots,n-1)\\ F^n(\lambda) &= a_1(\lambda)-a_{\frac{1}{2}}(\lambda)+a_2(\lambda)+\sum_{i=1}^{\nu^{(l)}}\Big((a_2-a_1)(\lambda-\tilde\lambda_i^{(l)})+(a_2-a_1)(\lambda+\tilde\lambda_i^{(l)})\Big)\delta_{ln}\\ &- \sum_{i=1}^{\nu^{(l)}}\Big(a_1(\lambda-\tilde\lambda_i^{(l)})+a_1(\lambda+\tilde\lambda_i^{(l)})\Big)\delta_{l,n-1} \end{aligned} \tag{3.10}$$

and $\hat{\mathcal{K}}$ is a $n\times n$ matrix with entries given by

$$\begin{aligned} &\hat{\mathcal{K}}_{ij}(\omega)=(1+\hat a_2(\omega))\delta_{ij}-\hat a_1(\omega)(\delta_{i,j+1}+\delta_{i,j-1}), \quad i,j=1,\dots n-1,\\ &\hat{\mathcal{K}}_{nn-1}(\omega)=\hat{\mathcal{K}}_{n-1n}(\omega)=\hat a_1(\omega), \quad \hat{\mathcal{K}}_{nn}(\omega)=1-\hat a_1(\omega)+\hat a_2(\omega) \end{aligned} \tag{3.11}$$

Finally, the n-vector G carries all the explicit dependence on the boundary parameter ξ of the D3 solution

$$G^j(\lambda)=a_{2\xi-\tilde n_1}(\lambda)\ \delta_{j,\tilde n_1} \tag{3.12}$$

where $\tilde n_1=n-n_1$. Solving equation (3.8) we find the densities σ^i which describe a Bethe Ansatz state. The solution of (3.8) has the following form

$$\sigma(\lambda)=2\epsilon(\lambda)+\frac{1}{N}\Phi_0(\lambda)+\frac{1}{N}\Phi_1(\lambda,\ \xi) \tag{3.13}$$

where ϵ and $\Phi_{0,1}$ are n-vectors (columns) with

$$\hat\epsilon^i(\omega)=\hat{\mathcal{R}}_{i1}(\omega)\hat a_1(\omega), \quad \hat\Phi_0^i(\omega)=\sum_{j=1}^n\hat{\mathcal{R}}_{ij}(\omega)\hat F^j(\omega), \quad \hat\Phi_1^i(\omega,\ \xi)=\sum_{j=1}^n\hat{\mathcal{R}}_{ij}(\omega)\hat G^j(\omega,\ \xi). \tag{3.14}$$

$\hat{\mathcal{R}}=\hat{\mathcal{K}}^{-1}$ and ϵ^j is the energy of a hole in the j sea; they are written in terms of hyperbolic functions as

$$\hat{\mathcal{R}}_{ij}(\omega)=e^{\frac{\omega}{2}}\ \frac{\sinh\Big(\min(i,j)\frac{\omega}{2}\Big)\ \cosh\Big(n+\frac{1}{2}-\max(i,j)\Big)\frac{\omega}{2}}{\cosh(n+\frac{1}{2})\frac{\omega}{2}\ \sinh\frac{\omega}{2}}, \quad i,j=1,\dots,n \tag{3.15}$$

$$\hat{\epsilon}^j(\omega) = \frac{\cosh(n+\frac{1}{2}-j)\frac{\omega}{2}}{\cosh(n+\frac{1}{2})\frac{\omega}{2}}, \quad j = 1,\dots,n \tag{3.16}$$

3.2 Scattering

As already mentioned the main aim here is the derivation of the exact bulk and boundary S-matrices. We follow the standard formulation developed by Korepin [30], and later by Andrei and Destri [31]. One first implement the so-called quantisation condition,

$$(e^{2iNp^l}S - 1)|\tilde{\lambda}_i^l\rangle = 0 \tag{3.17}$$

where p^l is the momentum of the particle (in our case, the hole) with rapidity $\tilde{\lambda}_1^l$. For the case of ν (even) holes in l sea we insert the integrated density (3.13) into the quantisation condition (3.17). We use the dispersion relation

$$\epsilon^l(\lambda) = \frac{1}{2\pi}\frac{d}{d\lambda}\ p^l(\lambda) \tag{3.18}$$

and the sum rule $N\int_0^{\tilde{\lambda}_i} d\lambda\sigma(\lambda) \in \mathbf{Z}_+$. We end up with the following expression for the boundary scattering amplitudes:

$$\alpha^{+l}\alpha^{-l} = \exp\Big\{2\pi N\int_0^{\tilde{\lambda}_1} d\lambda\Big(\sigma^l(\lambda) - 2\epsilon^l(\lambda)\Big)\Big\} \tag{3.19}$$

with

$$\alpha^{-l}(\lambda,\ \xi) = k_0(\lambda)\ k_1(\lambda,\ \xi), \quad \alpha^{+l}(\lambda) = k_0(\lambda) \tag{3.20}$$

where α^{+l} is realised just as the overall factor in front of the unit matrix at the left boundary (recall that $K^+ = 1$, whereas K^- is given by the solution D3). Moreover,

$$k_0(\tilde{\lambda}_1^l) = \exp\Big\{i\pi\int_0^{\tilde{\lambda}_1^l} d\lambda\Phi_0^l(\lambda)\Big\}, \quad k_1(\tilde{\lambda}_1^l,\ \xi) = \exp\Big\{2i\pi\int_0^{\tilde{\lambda}_1^l} d\lambda\Phi_1^l(\lambda,\ \xi)\Big\}, \tag{3.21}$$

with Φ^l given by (3.14), (3.10). We finally restrict ourselves to $l = 1$ in the first sea and we write the latter expression in term of the Fourier transform of Φ^1 (3.14),

$$k_0(\lambda) = \exp\Big\{-\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\hat{\Phi}_0^1(\omega)e^{-i\omega\lambda}\Big\}, \quad k_1(\lambda,\ \xi) = \exp\Big\{-\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\hat{\Phi}_1^1(\omega,\ \xi)e^{-i\omega\lambda}\Big\}. \tag{3.22}$$

Let us discuss first the form of the exact bulk S-matrix. It is easy to compute the scattering amplitude between two holes in the first sea. The bulk scattering amplitude comes from the contribution of the terms of Φ^1 given by eqs. (3.10), (3.14), (3.15), with argument $\lambda \pm \tilde{\lambda}_j$. After some algebra and using the following identity

$$\frac{1}{2}\int_0^{\infty}\frac{d\omega}{\omega}\frac{e^{-\frac{\mu\omega}{2}}}{\cosh\frac{\omega}{2}} = \ln\frac{\Gamma(\frac{\mu+1}{4})}{\Gamma(\frac{\mu+3}{4})} \tag{3.23}$$

we conclude that the hole-hole scattering amplitude is given by the expression

$$S_0(\lambda) = \frac{\tan \pi(\frac{i\lambda-1}{2n+1})}{\tan \pi(\frac{i\lambda+1}{2n+1})} \frac{\Gamma(\frac{i\lambda}{2n+1})}{\Gamma(\frac{-i\lambda}{2n+1})} \frac{\Gamma(\frac{-i\lambda}{2n+1}+\frac{1}{2})}{\Gamma(\frac{i\lambda}{2n+1}+\frac{1}{2})} \frac{\Gamma(\frac{-i\lambda+1}{2n+1})}{\Gamma(\frac{i\lambda+1}{2n+1})} \frac{\Gamma(\frac{i\lambda+1}{2n+1}+\frac{1}{2})}{\Gamma(\frac{-i\lambda+1}{2n+1}+\frac{1}{2})}. \tag{3.24}$$

As a consistency check we compute one further eigenvalue of the S-matrix. In particular if one considers the state with two holes in the first Dirac sea, and a two-string located at the midpoint of the two holes, the corresponding eigenvalue is given by (see also [32])

$$S_b(\lambda) = e_1(\lambda) S_0(\lambda). \tag{3.25}$$

The explicit bulk S-matrix, which is a solution of the *super* Yang–Baxter equation has the following structure

$$S(\lambda) = \frac{S_0(\lambda)}{(i\lambda+\kappa)(i\lambda+1)}(\lambda(\lambda+i\kappa)1 + i(\lambda+i\kappa)P - i\lambda Q). \tag{3.26}$$

We now give the expressions for the boundary S-matrix, which follow from (3.21), (3.22), and the duplication formula for the Γ function

$$2^{2x-1}\Gamma(x+\frac{1}{2})\Gamma(x) = \pi^{\frac{1}{2}}\Gamma(2x). \tag{3.27}$$

The overall factor k_0, (3.21), is given by

$$k_0(\lambda) = Y_0(\lambda) \frac{\Gamma(\frac{i\lambda}{2n+1})}{\Gamma(\frac{-i\lambda}{2n+1})} \frac{\Gamma(\frac{-i\lambda}{2n+1}+\frac{3}{4})}{\Gamma(\frac{i\lambda}{2n+1}+\frac{3}{4})} \frac{\Gamma(\frac{i\lambda}{2n+1}+\frac{1}{2(2n+1)}+\frac{3}{4})}{\Gamma(\frac{-i\lambda}{2n+1}+\frac{1}{2(2n+1)}+\frac{3}{4})} \frac{\Gamma(\frac{-i\lambda}{2n+1}+\frac{1}{2(2n+1)}+\frac{1}{2})}{\Gamma(\frac{i\lambda}{2n+1}+\frac{1}{2(2n+1)}+\frac{1}{2})} \tag{3.28}$$

where

$$Y_0(\lambda) = \frac{\sin \pi(\frac{i\lambda}{2n+1}+\frac{1}{2(2n+1)}-\frac{1}{4})}{\sin \pi(\frac{i\lambda}{2n+1}-\frac{1}{2(2n+1)}+\frac{1}{4})} \frac{\sin \pi(\frac{i\lambda}{2n+1}-\frac{1}{2(2n+1)}+\frac{1}{2})}{\sin \pi(\frac{i\lambda}{2n+1}+\frac{1}{2(2n+1)}-\frac{1}{2})} \frac{\tan \pi(\frac{i\lambda}{2n+1}+\frac{1}{4(2n+1)}+\frac{1}{4})}{\tan \pi(\frac{i\lambda}{2n+1}-\frac{1}{4(2n+1)}-\frac{1}{4})}. \tag{3.29}$$

The ξ dependent part for the D3 solution k_1, (3.21) reads

$$k_1(\lambda,\ \xi) = \frac{\Gamma(\frac{i\lambda}{2n+1}+\frac{\xi'}{2n+1}+\frac{1}{2})}{\Gamma(\frac{-i\lambda}{2n+1}+\frac{\xi'}{2n+1}+\frac{1}{2})} \frac{\Gamma(\frac{-i\lambda}{2n+1}+\frac{\xi'}{2n+1})}{\Gamma(\frac{i\lambda}{2n+1}+\frac{\xi'}{2n+1})} \frac{\Gamma(\frac{-i\lambda}{2n+1}+\frac{\xi'-\tilde{n}_1}{2n+1}+\frac{1}{2})}{\Gamma(\frac{i\lambda}{2n+1}+\frac{\xi'-\tilde{n}_1}{2n+1}+\frac{1}{2})} \frac{\Gamma(\frac{i\lambda}{2n+1}+\frac{\xi'-\tilde{n}_1}{2n+1}+1)}{\Gamma(\frac{-i\lambda}{2n+1}+\frac{\xi'-\tilde{n}_1}{2n+1})+1)} \tag{3.30}$$

where $\xi' = \xi - \frac{1}{2}$ has the fixed value found for D3, so that the boundary S-matrix satisfies the reflection equation. Note that our solutions include the necessary CDD factors both for the bulk and boundary matrices.

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