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# EVERY CONNECTED SUM OF LENS SPACES IS A REAL COMPONENT OF A UNIRULED ALGEBRAIC VARIETY

JOHANNES HUISMAN AND FRÉDÉRIC MANGOLTE

**ABSTRACT.** Let  $M$  be a connected sum of finitely many lens spaces, and let  $N$  be a connected sum of finitely many copies of  $S^1 \times S^2$ . We show that there is a uniruled algebraic variety  $X$  such that the connected sum  $M \# N$  of  $M$  and  $N$  is diffeomorphic to a connected component of the set of real points  $X(\mathbb{R})$  of  $X$ . In particular, any finite connected sum of lens spaces is diffeomorphic to a real component of a uniruled algebraic variety.

*MSC 2000:* 14P25

*Keywords:* uniruled algebraic variety, Seifert fibered manifold, lens space, connected sum, equivariant line bundle, real algebraic model

## 1. INTRODUCTION

A famous theorem of Nash states that any differentiable manifold is diffeomorphic to a real component of an algebraic variety [11]. More precisely, for any compact connected differentiable manifold  $M$ , there is a nonsingular projective and geometrically irreducible real algebraic variety  $X$ , such that  $M$  is diffeomorphic to a connected component of the set of real points  $X(\mathbb{R})$  of  $X$ . The question then naturally rises as to which differentiable manifolds occur as real components of algebraic varieties of a given class. For example, one may wonder which differentiable manifolds are diffeomorphic to a real component of an algebraic variety of Kodaira dimension  $-\infty$ . That specific question is the question we will address in the current paper, for algebraic varieties of dimension 3.

In dimension  $\leq 3$ , an algebraic variety  $X$  has Kodaira dimension  $-\infty$  if and only if it is *uniruled* [10, 9], i.e., if and only if there is a dominant rational map  $Y \times \mathbb{P}^1 \dashrightarrow X$ , where  $Y$  is a real algebraic variety of dimension  $\dim(X) - 1$ . Therefore, the question we study is the question as to which differentiable manifolds occur as a real component of a uniruled algebraic variety of dimension 3. In dimension 0 and 1, that question has a trivial answer. In dimension 2, the answer is due to Comessatti.

**Theorem** (Comessatti 1914 [1]). *Let  $X$  be a uniruled real algebraic surface. Then, a connected component of  $X(\mathbb{R})$  is either nonorientable, or diffeomorphic to the sphere  $S^2$  or the torus  $S^1 \times S^1$ . Conversely, a compact connected differentiable surface that is either nonorientable or diffeomorphic to  $S^2$  or  $S^1 \times S^1$ , is diffeomorphic to a real component of a uniruled real algebraic surface.*

Roughly speaking, a compact connected differentiable surface does not occur as a connected component of a uniruled real algebraic surface if and only if it is orientable of genus greater than 1.

We have deliberately adapted the statement of Comessatti's Theorem for the purposes of the current paper. Comessatti stated the result for real surfaces that

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are geometrically rational, i.e., whose complexification is a complex rational surface. The more general statement above easily follows from that fact.

In dimension 3, much progress has been made, due to Kollár, in classifying the differentiable manifolds that are diffeomorphic to a real component of a uniruled algebraic variety.

**Theorem** (Kollár 1998 [8, Th. 6.6]). *Let  $X$  be a uniruled real algebraic variety of dimension 3 such that  $X(\mathbb{R})$  is orientable. Let  $M$  be a connected component of  $X(\mathbb{R})$ . Then,  $M$  is diffeomorphic to one of the following manifolds:*

- (1) *a Seifert fibered manifold,*
- (2) *a connected sum of finitely many lens spaces,*
- (3) *a locally trivial torus bundle over  $S^1$ , or doubly covered by such a bundle,*
- (4) *a manifold belonging to an a priori given finite list of exceptions, or*
- (5) *a manifold obtained from one of the above by taking the connected sum with a finite number of copies of  $\mathbb{P}^3(\mathbb{R})$  and a finite number of copies of  $S^1 \times S^2$ .*

Recall that a *Seifert fibered manifold* is a manifold admitting a differentiable foliation by circles. A *lens space* is a manifold diffeomorphic to a quotient of the 3-sphere  $S^3$  by the action of a cyclic group. In case the set of real points of a uniruled algebraic variety is allowed not to be orientable, the results of Kollár are less precise due to many technical difficulties, but see [7, Theorem 8.3]. In order to complete the classification in the orientable case, Kollár proposed the following conjectures.

**Conjecture** (Kollár 1998 [8, Conj. 6.7]). (1) *Let  $M$  be an orientable Seifert fibered manifold. Then there is a uniruled algebraic variety  $X$  such that  $M$  is diffeomorphic to a connected component of  $X(\mathbb{R})$ .*

- (2) *Let  $M$  be a connected sum of lens spaces. Then there is a uniruled algebraic variety  $X$  such that  $M$  is diffeomorphic to a connected component of  $X(\mathbb{R})$ .*
- (3) *Let  $M$  be a manifold belonging to the a priori given list of exceptional manifolds or a locally trivial torus bundle over  $S^1$  which is not a Seifert fibered manifold. Then  $M$  is not diffeomorphic to a real component of a uniruled algebraic variety  $X$ .*

Let us also mention the following result of Eliashberg and Viterbo (unpublished, but see [4]).

**Theorem** (Eliashberg, Viterbo). *Let  $X$  be a uniruled real algebraic variety. Let  $M$  be a connected component of  $X(\mathbb{R})$ . Then  $M$  is not hyperbolic.*

In an earlier paper, we proved Conjecture (1) above, i.e., that any orientable Seifert fibered manifold  $M$  is diffeomorphic to a connected component of the set of real points of a uniruled real algebraic variety  $X$  [3, Th. 1.1]. Unfortunately, we do not know whether  $X(\mathbb{R})$  is orientable, in general. Indeed, the uniruled variety  $X$  we constructed may have more real components than the one that is diffeomorphic to  $M$ , and we are not able to control the orientability of such additional components.

Recently, we realized that the methods used to prove Th. 1.1 of [3] can be generalized in order to obtain a similar statement concerning connected sums of lens spaces. In fact, we prove, in the current paper the following, slightly more general, statement.

**Theorem 1.1.** *Let  $N_1$  be a connected sum of finitely many lens spaces, and let  $N_2$  be a connected sum of finitely many copies of  $S^1 \times S^2$ . Let  $M$  be the connected sum  $N_1 \# N_2$ . Then, there is a uniruled real algebraic variety  $X$  such that  $M$  is diffeomorphic to a connected component of  $X(\mathbb{R})$ .*

**Corollary 1.2.** *Let  $M$  be a connected sum of finitely many lens spaces. Then, there is a uniruled real algebraic variety  $X$  such that  $M$  is diffeomorphic to a connected component of  $X(\mathbb{R})$ .*

This proves Conjecture (2) above. Conjecture (3) remain open.

As explained in [6, Example 1.4], if a connected 3-manifold  $M$  is realizable as a connected component of a uniruled real algebraic variety  $X$ , then the connected sums  $M\#\mathbb{P}^3(\mathbb{R})$  and  $M\#(S^1 \times S^2)$  are also realizable by such a variety. Indeed,  $M\#\mathbb{P}^3(\mathbb{R})$  is diffeomorphic to the uniruled real algebraic variety obtained from  $X$  by blowing up a real point. The connected sum  $M\#(S^1 \times S^2)$  is diffeomorphic to the uniruled real algebraic variety obtained from  $X$  by blowing up along a singular real algebraic curve that has only one real point. Therefore, Theorem 1.1 is, in fact, a consequence of Corollary 1.2. Our proof of Theorem 1.1, however, does not follow those lines. Since, moreover, it turns out not to be more expensive to show directly the more general statement of Theorem 1.1, we preferred to do so. One could have shown an even more general statement involving a connected sum of, on the one hand,  $M = N_1\#N_2$  and, on the other hand, a connected sum of finitely many copies of  $\mathbb{P}^3(\mathbb{R})$ . However, this greater generality is only apparent, for  $\mathbb{P}^3(\mathbb{R})$  is a lens space.

The paper is organized as follows. In Section 2, we show that  $M$  admits a particularly nice fibration over a differentiable surface  $S$  with boundary, following an idea of Kollár in [7]. We call such a fibration a *Werther fibration*, as it reminded us of an original candy by the same name. It is a Seifert fibration over the interior of  $S$ , and a diffeomorphism over the boundary of  $S$ . Roughly speaking, the 3-manifold  $M$  is pinched over the boundary of  $S$ , much like the candy.

The Werther fibration is used, in Section 3, to show that  $M$  admits a finite Galois covering  $\tilde{M}$  with the following property. The manifold  $\tilde{M}$  admits a Werther fibration over a differentiable surface  $\tilde{S}$  whose restriction over the interior of  $\tilde{S}$  is a locally trivial fibration in circles.

Let  $\tilde{T}$  be the differentiable surface without boundary obtained from  $\tilde{S}$  by gluing closed discs along its boundary components. In Section 3, we show that there is a differentiable plane bundle  $\tilde{V}$  over  $\tilde{T}$  with the following property. The manifold  $\tilde{M}$  is diffeomorphic to a submanifold  $\tilde{N}$  of the total space  $\tilde{V}$  of the plane bundle  $\tilde{V}$ . The intersection of  $\tilde{N}$  with the fibers of  $\tilde{V}/\tilde{T}$  are real conics, nondegenerate ones over the interior of  $\tilde{S}$ , degenerate ones over the boundary of  $\tilde{S}$ . Moreover, the corresponding action of  $G$  on  $\tilde{N}$  extends to an action of the plane bundle  $\tilde{V}$  over an action of  $G$  on  $\tilde{S}$ .

At that point, we need a result of a former paper [3], where we show that such an equivariant plane bundle can be realized algebraically. We recall and use that result in Section 4. In Section 5, we then derive Theorem 1.1.

## 2. CONNECTED SUMS OF LENS SPACES

Let  $S^1 \times D^2$  be the *solid torus* where  $S^1$  is the unit circle  $\{u \in \mathbb{C} \mid |u| = 1\}$  and  $D^2$  is the closed unit disc  $\{z \in \mathbb{C}, |z| \leq 1\}$ . A *Seifert fibration* of the solid torus is a differentiable map of the form

$$f_{p,q} : S^1 \times D^2 \rightarrow D^2, (u, z) \mapsto u^q z^p,$$

where  $p$  and  $q$  are natural integers, with  $p \neq 0$  and  $\gcd(p, q) = 1$ . Let  $M$  be a 3-manifold. A *Seifert fibration* of  $M$  is a differentiable map  $f$  from  $M$  into a differentiable surface  $S$  having the following property. Every point  $P \in S$  has a closed neighborhood  $U$  such that the restriction of  $f$  to  $f^{-1}(U)$  is diffeomorphic to a Seifert fibration of the solid torus. Sometimes, nonorientable local models are also

allowed in the literature, e.g. [12]. For our purposes, we do not need to include them in the definition of a Seifert fibration, since the manifolds we study are orientable.

Let  $C^2$  be the *collar* defined by  $C^2 = \{w \in \mathbb{C} \mid 1 \leq |w| < 2\}$ , i.e.,  $C^2$  is the half-open annulus of radii 1 and 2. Let  $P$  be the differentiable 3-manifold defined by

$$P = \{(w, z) \in C^2 \times \mathbb{C} \mid |z|^2 = |w| - 1\}.$$

Let  $\omega: P \rightarrow C^2$  be the projection defined by  $\omega(w, z) = w$ . It is clear that  $\omega$  is a differentiable map, that  $\omega$  is a trivial circle bundle over the interior of  $C^2$ , and that  $\omega$  is a diffeomorphism over the boundary of  $C^2$ .

**Definition 2.1.** *Let  $f: M \rightarrow S$  be a differentiable map from a 3-manifold  $M$  without boundary into a differentiable surface  $S$  with boundary. The map  $f$  is a Werther fibration if*

- (1) *the restriction of  $f$  over the interior of  $S$  is a Seifert fibration, and*
- (2) *every point  $P$  in the boundary of  $S$  has an open neighborhood  $U$  such that the restriction of  $f$  to  $f^{-1}(U)$  is diffeomorphic to the restriction of  $\omega$  over an open neighborhood of 1 in  $C^2$ .*

*Remarks 2.2.* (1) Let  $M$  be a Seifert fibered manifold without boundary which is not a connected sum of lens spaces, then for all Werther maps  $M \rightarrow S$ , we have  $\partial S = \emptyset$ , see [7, 3.7].

- (2) Let  $M$  be a 3-manifold without boundary. A Werther map  $M \rightarrow S$  is a Seifert fibration if and only if  $\partial S = \emptyset$ .
- (3) Let  $f: M \rightarrow S$  be a Werther fibration, and let  $B$  be a connected component of the boundary of  $S$ . Then, the restriction of  $f$  over any small open neighborhood  $U$  of  $B$  is not necessarily diffeomorphic to  $\omega$ . Indeed, if the restriction of  $f$  to  $f^{-1}(U)$  is diffeomorphic to  $\omega$ , then, in particular, the restriction  $TM|_B$  to  $B$  of the tangent bundle  $TM$  of  $M$  is a trivial vector bundle of rank 3. Conversely, if  $TM|_B$  is trivial, then  $f|_{f^{-1}(U)}$  is diffeomorphic to  $\omega$ .

Since  $U$  has the homotopy type of the circle  $S^1$ , there are, up to isomorphism, exactly 2 vector bundles of rank 3 over  $U$ , the trivial one, and the direct sum of the trivial plane bundle with the Möbius line bundle over  $U$ .

For an integer  $n$ , let  $\mu_n$  be the multiplicative subgroup of  $\mathbb{C}^*$  of the  $n$ -th roots of unity.

Let  $0 < q < p$  be relatively prime integers. The *lens space*  $L_{p,q}$  is the quotient of the 3-sphere  $S^3 = \{(w, z) \in \mathbb{C}^2 \mid |w|^2 + |z|^2 = 1\}$  by the action of  $\mu_p$  defined by

$$\xi \cdot (w, z) = (\xi w, \xi^q z),$$

for all  $\xi \in \mu_p$  and  $(w, z) \in S^3$ . A *lens space* is a differentiable manifold diffeomorphic to a manifold of the form  $L_{p,q}$ . It is clear that a lens space is an orientable compact connected differentiable manifold of dimension 3.

**Lemma 2.3.** *Let  $0 < q < p$  be relatively prime integers. There is a Werther fibration  $f: L_{p,q} \rightarrow D^2$ .*

*Proof.* Let  $g: S^3 \rightarrow D^2$  be the map  $g(w, z) = w^p$  for all  $(w, z) \in S^3$ . Since  $g$  is constant on  $\mu_p$ -orbits, the map  $g$  induces a differentiable map  $f: L_{p,q} \rightarrow D^2$ . It is easy to check that  $f$  is a Werther fibration.  $\square$

**Lemma 2.4.** *Let  $A^2$  be the closed annulus  $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ . There is Werther fibration  $f: S^1 \times S^2 \rightarrow A^2$ .*

*Proof.* Let  $S^2$  be the 2-sphere in  $\mathbb{C} \times \mathbb{R}$  defined by  $|z|^2 + t^2 = 1$ . Let  $f: S^1 \times S^2 \rightarrow A^2$  be the map defined by  $f(w, z, t) = \frac{1}{2}(t+3)w$ . It is easy to check that  $f$  is a Werther fibration.  $\square$

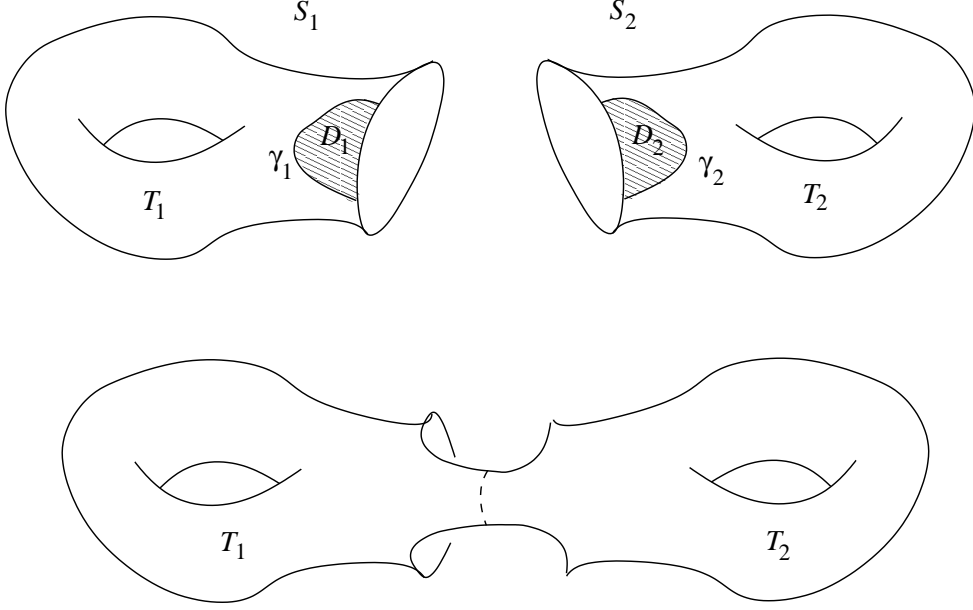


FIGURE 1. The two surfaces  $T_1$  and  $T_2$  are glued together along  $\gamma_1$  and  $\gamma_2$ .

**Lemma 2.5.** *Let  $f_1: M_1 \rightarrow S_1$  and  $f_2: M_2 \rightarrow S_2$  be two Werther fibrations, where  $M_1$  and  $M_2$  are oriented 3-manifolds without boundaries. Suppose that the boundaries  $\partial S_1$  and  $\partial S_2$  are nonempty. Then there is a differentiable surface  $S$  with nonempty boundary and a Werther fibration*

$$f: M_1 \# M_2 \longrightarrow S,$$

where  $M_1 \# M_2$  is the oriented connected sum of  $M_1$  and  $M_2$ .

*Proof.* Let  $\gamma_i \subset S_i$ ,  $i \in \{1, 2\}$  be a simple path having its end points in the same boundary component of  $S_i$ , and whose interior is contained in the interior of  $S_i$ . One may assume that  $\gamma_i$  bounds a closed disc  $D_i$  in  $S_i$ , over the interior of which  $f_i$  is a trivial circle bundle (see Figure 1). Let  $T_i = S_i \setminus \overline{D_i}$  and let  $N_i = M_i \setminus f_i^{-1}(D_i)$ . By construction,  $f_i^{-1}(\gamma_i)$  is a 2-sphere in  $M_i$  bounded by the 3-ball  $f_i^{-1}(D_i)$ . The restriction of  $f_1$  to  $f_1^{-1}(D_1)$  is diffeomorphic to  $f_2^{-1}(D_2)$ . In particular, we have an orientation reversing diffeomorphism between  $f_1^{-1}(\gamma_1)$  and  $f_2^{-1}(\gamma_2)$  compatible with a diffeomorphism between  $\gamma_1$  and  $\gamma_2$ . Therefore, the connected sum  $M$  of  $M_1$  and  $M_2$  is diffeomorphic to the manifold obtained from gluing  $N_1$  and  $N_2$  along the orientation reversing diffeomorphism between  $f_1^{-1}(\gamma_1)$  and  $f_2^{-1}(\gamma_2)$ . Let  $S$  be the manifold obtained from gluing  $T_1$  and  $T_2$  along the diffeomorphism between  $\gamma_1$  and  $\gamma_2$ . One has an induced differentiable map  $f: M \rightarrow S$  that is a Werther fibration.  $\square$

**Theorem 2.6.** *Let  $N_1$  be an oriented connected sum of finitely many lens spaces, and let  $N_2$  be an oriented connected sum of finitely many copies of  $S^1 \times S^2$ . Let  $M$  be the oriented connected sum  $N_1 \# N_2$ . Then, there is a compact connected differentiable surface  $S$  with boundary and a Werther fibration  $f: M \rightarrow S$ .*

*Proof.* The statement follows from Lemmas 2.3, 2.4 and 2.5.  $\square$

*Remark 2.7.* If  $M$  is an oriented connected sum of finitely many lens spaces, then there is a Werther fibration of  $M$  over the closed disc. Indeed, by Lemma 2.3,

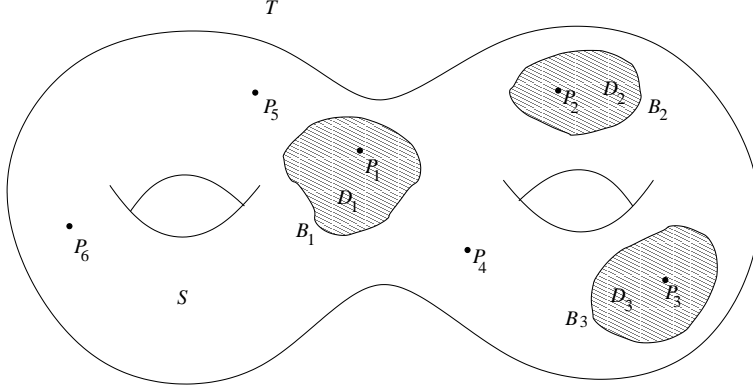


FIGURE 2. The surface  $T$  obtained from  $S$  by gluing closed discs along its boundary, one for each boundary component.

any lens space admits a Werther fibration over a closed disc. By Lemma 2.5, the connected sum of finitely many lens spaces admits a Werther fibration over a topological closed disc, and the statement is proved.

This observation is useful when one wants to construct explicit examples of uniruled real algebraic varieties, one of whose components is diffeomorphic to a given connected sum of lens spaces.

### 3. MAKING A WERTHER FIBRATION LOCALLY TRIVIAL

As for Seifert fibrations [3], we show that a Werther fibration  $f: M \rightarrow S$  is a locally trivial circle bundle over the interior of  $S$  for the finite ramified Grothendieck topology on  $S$ . More precisely, one has the following statement.

**Theorem 3.1.** *Let  $M$  be a manifold that admits a Werther fibration. Then, there is a Werther fibration  $f: M \rightarrow S$  of  $M$  over a compact connected surface  $S$ , and a finite ramified topological covering  $\pi: \tilde{S} \rightarrow S$  such that*

- (1)  $\tilde{S}$  is orientable,
- (2)  $\pi$  is unramified over the boundary of  $S$ ,
- (3)  $\pi$  is Galois, i.e.,  $\pi$  is a quotient map for the group of automorphisms of  $\tilde{S}/S$ ,
- (4) the induced action of  $G$  on the fiber product  $\tilde{M} = \tilde{S} \times_S M$  is free,
- (5) the induced fibration  $\tilde{f}: \tilde{M} \rightarrow \tilde{S}$  is a locally trivial circle bundle over the interior of  $\tilde{S}$ , and
- (6) the restriction of  $\tilde{f}$  over an open neighborhood of any boundary component of  $\tilde{S}$  is diffeomorphic to  $\omega$ .

*Proof.* If  $M$  is a Seifert fibered manifold, i.e., if  $M$  admits a Werther fibration over a surface without boundary then the statement follows from Theorem 1.1 of [3]. Therefore, we may assume that  $M$  admits a Werther fibration  $f: M \rightarrow S$  over a surface with nonempty boundary  $S$ . Let  $B_1, \dots, B_r$  be the boundary components of  $S$ . Let  $T$  be the compact connected surface without boundary obtained from  $S$  by gluing a disjoint union of  $r$  copies of the closed disc along the boundary of  $S$ . Denote by  $D_i$  the closed disc in  $T$  that has  $B_i$  as its boundary and such that  $S \cup \bigcup D_i = T$  (see Figure 2). Now, choose one point  $P_i$  in the interior of each closed disc  $D_i$ , for  $i = 1, \dots, r$ . Let  $P_{r+1}, \dots, P_{r+s}$  be the points of the interior of  $S$  over which  $f$  is not a locally trivial circle bundle. Let  $m_{r+i}$  be the multiplicity of the fiber of  $f$  over  $P_{r+i}$  for  $i = 1, \dots, s$ . By Selberg's Lemma, there is a finite ramified covering  $\rho: \tilde{T} \rightarrow T$  of  $T$ , which is unramified outside the set  $\{P_i\}$ , such that  $\rho$  has ramification index  $m_i$  at each preimage of  $P_i$ , for  $i = r+1, \dots, r+s$ , and has even ramification index

over each preimage of  $P_i$ , for  $i = 1, \dots, r$ . Replacing  $\tilde{T}$  by its orientation double covering, we may assume that  $\tilde{T}$  is orientable. Then, replacing  $\tilde{T}/T$  by its normal closure, we may, moreover, assume that  $\tilde{T}/T$  is Galois. Let  $G$  be the Galois group of  $\tilde{T}/T$ .

Let  $\tilde{S}$  be the inverse image  $\rho^{-1}(S)$  of  $S$ , and let  $\pi: \tilde{S} \rightarrow S$  be the restriction of  $\rho$  to  $\tilde{S}$ . Then,  $\pi$  is a finite ramified topological covering, clearly satisfying conditions (1), (2) and (3). The Galois group of  $\tilde{S}/S$  is  $G$ . Since the map  $\pi$  has ramification index exactly equal to  $m_i$  at each preimage of  $P_i$ , for  $i = r+1, \dots, r+s$ , the action of  $G$  on  $\tilde{S}$  is free. Moreover, the induced map  $\tilde{f}$  is a locally trivial circle bundle over  $\tilde{S}$ . Since  $\pi$  has even ramification index at each preimage of  $P_i$ , for  $i = 1, \dots, r$ , the map  $\tilde{f}$  also satisfies condition (6), according to Remark 2.2.3.  $\square$

#### 4. ALGEBRAIC REALIZATION OF AN EQUIVARIANT PLANE BUNDLE

As noted in the Introduction, the idea of the proof of Theorem 1.1 is to show that the manifold  $\tilde{M}$  of Theorem 3.1 embeds equivariantly into a differentiable plane bundle  $\tilde{V}$  over the surface  $\tilde{T}$  of the proof of Theorem 3.1. At that point, we need to have an equivariant real algebraic model of  $\tilde{V}/\tilde{T}$ . This argument was already used in our earlier paper [3].

Several people, at different occasions, have pointed out to us work of Dovermann, Masuda and Suh [2], and suggested that that would have been useful in realizing algebraically and equivariantly the plane bundle  $\tilde{V}/\tilde{T}$ . However, the results of Dovermann *et al.* apply only to semi-free actions of a group, whereas here, the action of  $G$  is, more or less, arbitrary, in any case, not necessarily semi-free. Therefore, as a by-product of our methods, we can mention the following generalization of [2, Th. B] in the case of a certain finite group actions on a real plane bundle over a surface.

**Theorem 4.1.** *Let  $\tilde{T}$  be an orientable compact connected surface without boundary and let  $G$  be a finite group acting on  $\tilde{T}$ . Let  $(\tilde{V}, p)$  be an orientable differentiable real plane bundle over  $\tilde{T}$ , endowed with an action of  $G$  over the action on  $\tilde{T}$  such that*

- (1)  $\tilde{T}$  contains only finitely many fixed points, and
- (2)  $G$  acts by orientable diffeomorphisms on the total space  $\tilde{V}$ .

*Then there is a smooth projective real algebraic surface  $R$  endowed with a real algebraic action of  $G$ , an algebraic real plane bundle  $(W, q)$  over  $R$ , endowed with a real algebraic action of  $G$  over the action on  $R$ , such that there are  $G$ -equivariant diffeomorphisms  $\phi: \tilde{T} \rightarrow R(\mathbb{R})$  and  $\psi: \tilde{V} \rightarrow W(\mathbb{R})$  making the following diagram commutative.*

$$\begin{array}{ccc} \tilde{V} & \rightarrow & W(\mathbb{R}) \\ \downarrow & & \downarrow \\ \tilde{T} & \rightarrow & R(\mathbb{R}) \end{array}$$

For a proof of Theorem 4.1, we refer to the paper [3], where this statement does not appear explicitly, but its proof does. It makes use of the theory of Klein surfaces, a slight generalization of the theory of Riemann surfaces.

In case the group  $G$  of Theorem 4.1 acts on  $\tilde{T}$  by orientation-preserving diffeomorphisms, the theory of Riemann surfaces suffices to prove that the real plane bundle  $\tilde{V}/\tilde{T}$  can be realized real algebraically. Indeed, thanks to the fact that conditions (1) and (2) are satisfied, there are a structure of a Riemann surface on  $\tilde{T}$ , and a structure of a complex holomorphic line bundle on  $\tilde{V}$  such that  $G$  acts holomorphically on  $\tilde{T}$  and  $\tilde{V}$ . Restriction of scalars with respect to the field extension  $\mathbb{C}/\mathbb{R}$  does the rest. The more general case where  $G$  does not necessarily



act by orientation-preserving diffeomorphisms on  $\tilde{T}$  does really seem to require the theory of Klein surfaces. The reader is referred to [3] for details.

## 5. ALGEBRAIC MODELS

*Proof of Theorem 1.1.* Let  $N_1$  be a connected sum of finitely many lens spaces, let  $N_2$  be a connected sum of finitely many copies of  $S^1 \times S^2$ , and let  $M$  be a connected sum  $N_1 \# N_2$ . One can choose orientations on all lens spaces and all copies of  $S^1 \times S^2$  that are involved in such a way that all connected sums involved are oriented connected sums. By Theorem 2.6,  $M$  admits a Werther fibration. By Theorem 3.1, there is a Werther fibration  $f: M \rightarrow S$  and a finite ramified covering  $\pi: \tilde{S} \rightarrow S$  satisfying the conditions (1) through (6). As before, let  $T$  be the surface without boundary obtained from  $S$  by gluing a finite number of closed discs along the boundary of  $S$ . Similarly, let  $\tilde{T}$  be the surface without boundary obtained from  $\tilde{S}$  by gluing closed discs along the boundary of  $\tilde{S}$ . The map  $\pi$  extends to a ramified covering  $\rho: \tilde{T} \rightarrow T$  having only one ramification point at each closed disc of  $\tilde{T}$  that has been glued in. The action of the Galois group  $G$  of  $\tilde{S}/S$  extends to a differentiable action of  $G$  on  $\tilde{T}/T$ . It is clear that  $\rho$  is a ramified Galois covering of Galois group  $G$ .

Now, we would like to embed the fiber product  $\tilde{M} = M \times_S \tilde{S}$  into a real plane bundle  $\tilde{V}$  over  $\tilde{T}$ , in a  $G$ -equivariant way.

In order to construct the real plane bundle  $\tilde{V}$ , we need to modify  $\tilde{M}$  somewhat. The induced Werther fibration

$$\tilde{F}: \tilde{M} \longrightarrow \tilde{S}$$

satisfies condition (6) of Theorem 3.1, i.e., its restriction over an open neighborhood of any boundary component of  $\tilde{S}$  is diffeomorphic to the model Werther fibration  $\omega$ . Hence, we can “open up” the manifold  $\tilde{M}$  along the boundary of  $\tilde{S}$  and “stretch it out” over  $\tilde{T}$ , and make it into a locally trivial circle bundle over all  $\tilde{T}$ , and not only over  $\tilde{S}^\circ$ . Let us denote by  $\tilde{N}$  the resulting manifold and by  $\tilde{g}$  the locally trivial circle bundle on  $\tilde{N}$ . Observe that  $\tilde{M}$  is the manifold obtained from the submanifold with boundary  $\tilde{g}^{-1}(\tilde{S})$  of  $\tilde{N}$  by contracting each circle  $\tilde{g}^{-1}(P)$  to a point, for  $P \in \partial\tilde{S}$ . It is clear that the action of  $G$  on  $\tilde{M}$  induces an action of  $G$  on  $\tilde{N}$ . As we have shown in [3], it is easy to construct a real plane bundle  $(\tilde{V}, p)$  over  $\tilde{T}$  that comes along with an action of  $G$  and an equivariant differentiable norm  $\nu$ , such that the unit circle bundle of  $U$  is equivariantly diffeomorphic to  $\tilde{N}$ , in such a way that  $\tilde{g}$  corresponds to the restriction of  $p$  to the unit circle bundle.

Let  $r: T \rightarrow \mathbb{R}$  be a differentiable function such that

$$\{P \in T \mid r(P) \geq 0\} = S,$$

and  $r$  takes only regular values on the boundary of  $S$ . Let  $\tilde{r} = r \circ \rho$ . Then  $\tilde{r}$  is a differentiable function on  $\tilde{T}$  such that

$$\{P \in \tilde{T} \mid \tilde{r}(P) \geq 0\} = \tilde{S},$$

and  $\tilde{r}$  takes only regular values on the boundary of  $\tilde{S}$ . Moreover, by construction  $\tilde{r}(gP) = P$  for all  $g \in G$  and  $P \in \tilde{T}$ , i.e.,  $\tilde{r}$  is constant on  $G$ -orbits of  $\tilde{T}$ .

It is now clear that  $\tilde{M}/\tilde{S}$  is equivariantly diffeomorphic to the submanifold  $\{v \in \tilde{V} \mid \nu(v) = \tilde{r}(p(v))\}$  of  $\tilde{V}$  over  $\tilde{S}$ . Since  $M$  is orientable, the group  $G$  acts by orientation-preserving diffeomorphisms on  $\tilde{M}$ . Therefore we can apply Theorem 4.1, and obtain a smooth projective real algebraic surface  $\tilde{R}$  endowed with an algebraic action of  $G$ , a real algebraic plane bundle  $(\tilde{W}, q)$  over  $\tilde{R}$ , such that  $\tilde{V}/\tilde{T}$  is equivariantly diffeomorphic to  $\tilde{W}(\mathbb{R})/\tilde{R}(\mathbb{R})$ .

Let  $\mu$  be a real algebraic norm on  $W$  over some affine open subset  $\tilde{R}'$  of  $\tilde{R}$  containing  $\tilde{R}(\mathbb{R})$  that approximates  $\nu$ . One may assume that  $\tilde{R}'$  is stable for the action of  $G$  on  $\tilde{R}$ , and that  $\mu$  is  $G$ -equivariant. The quotient  $R = \tilde{R}/G$  is a, possibly singular, projective real algebraic variety. The subset  $\tilde{R}(\mathbb{R})/G$  is a semialgebraic subset of  $R(\mathbb{R})$ . After identifying  $T$  with  $\tilde{R}(\mathbb{R})/G$ , the function  $r$  becomes a continuous function on  $\tilde{R}(\mathbb{R})/G$ . Since the set of points where  $r$  vanishes is contained in the smooth locus of  $R(\mathbb{R})$ , there is a real algebraic function  $s$  defined on some affine open subset  $R'$  of  $R$  that contains  $R(\mathbb{R})$  and that approximates  $r$ . In particular,  $s$  has 0 as a regular value on  $\tilde{R}(\mathbb{R})/G$ . Put  $\tilde{s} = s \circ \tau$ , where  $\tau$  is the quotient morphism from  $\tilde{R}$  into  $R$ . The real algebraic function  $\tilde{s}$  is defined on  $\tau^{-1}(R')$ . It approximates  $\tilde{r}$  and is constant on  $G$ -orbits. Replacing  $\tilde{R}'$  by  $\tilde{R}' \cap \tau^{-1}(R')$ , the ruled real algebraic variety  $Y'$  defined by the equation  $\mu(v) = \tilde{s}(q(v))$  over  $\tilde{R}'$  has the property that its set of real points is equivariantly diffeomorphic to  $\tilde{M}$ . Since  $G$  acts freely on  $\tilde{M}$ , it also acts freely on  $Y'(\mathbb{R})$ . It follows that  $Y'(\mathbb{R})/G$  is a connected component of the quotient variety  $X' = Y'/G$  that is diffeomorphic to  $M$ . Let  $X$  be a desingularization of some projective closure of  $X'$ . Then,  $X$  is a uniruled real algebraic variety having a real component diffeomorphic to  $M$ .  $\square$

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