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Stéphane Gerbi, Belkacem Said-Houari

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Abstract

In this paper we consider a multi-dimensional damped semiliear wave equation with dynamic boundary conditions, related to the Kelvin-Voigt damping. We firstly prove the local existence by using the Faedo-Galerkin approximations combined with a contraction mapping theorem. Secondly, the exponential growth of the energy and the L^p norm of the solution is presented.

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1 Introduction

In this paper we consider the following semilinear damped wave equation with dynamic boundary conditions:

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = |u|^{p-2}u, & x \in \Omega,\ t > 0 \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\ u_{tt}(x,t) = -\left[\dfrac{\partial u}{\partial \nu}(x,t) + \dfrac{\alpha \partial u_t}{\partial \nu}(x,t) + r|u_t|^{m-2}u_t(x,t)\right] & x \in \Gamma_1,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in \Omega \quad . \end{cases} \tag{1}$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$, Δ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of $\mathbb{R}^N$, $(N \geq 1)$,

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$\partial\Omega = \Gamma_0 \cup \Gamma_1$, $mes(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \varnothing$ and $\dfrac{\partial}{\partial \nu}$ denotes the unit outer normal derivative, $m \geq 2$, a, α and r are positive constants, $p > 2$ and u_0, u_1 are given functions.

From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such type of boundary conditions are usually called *dynamic boundary conditions.* They are not only important from the theoretical point of view but also arise in several physical applications. In one space dimension, the problem (1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represents the Newton's law for the attached mass, (see [5, 2, 11] for more details). In the two dimension space, as showed in [26] and in the references therein, these boundary conditions arise when we consider the transverse motion of a flexible membrane Ω whose boundary may be affected by the vibrations only in a region. Also some dynamic boundary conditions as in problem (1) appear when we assume that Ω is an exterior domain of $\mathbb{R}^3$ in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [3] for more details). This type of dynamic boundary conditions are known as acoustic boundary conditions.

In the one dimensional case and for $r = 0$, that is in the absence of boundary damping, this problem has been considered by Grobbelaar-Van Dalsen [16]. By using the theory of B-evolutions and the theory of fractional powers developped in [27, 28], the author showed that the partial differential equations in the problem (1) gives rise to an analytic semigroup in an appropriate functional space. As a consequence, the existence and the uniqueness of solutions was obtained. In the case where $r \neq 0$ and $m = 2$, Pellicer and Solà-Morales [25] considered the one dimensional problem as an alternative model for the classical spring-mass damper system, and by using the dominant eigenvalues method, they proved that for small values of the parameter a the partial differential equations in the problem (1) has the classical second order differential equation

$$m_1 u''(t) + d_1 u'(t) + k_1 u(t) = 0,$$

as a limit where the parameter m_1, d_1 and k_1 are determined from the values of the spring-mass damper system. Thus, the asymptotic stability of the model has been determined as a consequence of this limit. But they did not obtain any rate of convergence.

We recall that the presence of the strong damping term $-\Delta u_t$ in the problem (1) makes the problem different from that considered in [15] and widely studied in the litterature [32, 29, 30, 14, 31] for instance. For this reason less results were known for the wave equation with a strong damping and many problems remained unsolved, specially the blow-up of solutions in the presence of a strong damping and nonlinear damping at the same time. Here we will give a partial answer to this question. That is to say, we will prove that the solution is unbounded and grows up exponentially when time goes to infinity.

Recently, Gazzola and Squassina [14] studied the global solution and the finite time blow-up for a damped semilinear wave equations with Dirichlet boundary conditions by a careful study of the stationnary solutions and their stability using the Nehari manifold and a mountain pass energy level of the initial condition.

The main difficulty of the problem considered is related to the non ordinary boundary conditions defined on Γ_1. Very little attention has been paid to this type of boundary conditions. We mention only a few particular results in the one dimensional space and for a linear damping i.e. $(m = 2)$ [18, 25, 12].

A related problem to (1) is the following:

$$\begin{aligned} u_{tt} - \Delta u + g(u_t) &= f && \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial \nu} + K(u)u_{tt} + h(u_t) &= 0, && \text{on } \partial\Omega \times (0,T) \\ u(x,0) &= u_0(x) && \text{in } \Omega \\ u_t(x,0) &= u_1(x) && \text{in } \Omega \end{aligned}$$

where the boundary term $h(u_t) = |u_t|^\rho u_t$ arises when one studies flows of gaz in a channel with porous walls. The term u_{tt} on the boundary appears from the internal forces, and the nonlinearity $K(u)u_{tt}$ on the boundary represents the internal forces when the density of the medium depends on the displacement. This problem has been studied in [12, 13]. By using the Fadeo-Galerkin approximations and a compactness argument they proved the global existence and the exponential decay of the solution of the problem.

We recall some results related to the interaction of an elastic medium with rigid mass. By using the classical semigroup theory, Littman and Markus [21] established a uniqueness result for a particular Euler-Bernoulli beam rigid body structure. They also proved the asymptotic stability of the structure by using the feedback boundary damping. In [22] the authors considered the Euler-Bernoulli beam equation which describes the dynamics of clamped elastic beam in which one segment of the beam is made with viscoelastic material and the other of elastic material. By combining the frequency domain method with the multiplier technique, they proved the exponential decay for the transversal motion but not for the longitudinal motion of the model, when the Kelvin-Voigt damping is distributed only on a subinterval of the domain. In relation with this point, see also the work by Chen et *al.* [9] concerning the Euler-Bernoulli beam equation with the global or local Kelvin-Voigt damping. Also models of vibrating strings with local viscoelasticity and Boltzmann damping, instead of the Kelvin-Voigt one, were considered in [23] and an exponential energy decay rate was established. Recently, Grobbelaar-Van Dalsen [17] considered an extensible thermo-elastic beam which is hanged at one end with rigid body attached to its free end, i.e. one dimensional hybrid thermoelastic structure, and showed that the method used in [24] is still valid to establish an uniform stabilization of the system. Concerning the controllability of the hybrid system we refer to the work by Castro and Zuazua[6], in which they considered flexible beams connected by point mass and the model takes account of the rotational inertia.

In this paper we consider the problem (1) where we have set for the sake of simplity $a = 1$. Section 2 is devoted to the local existence and uniqueness of the solution of the problem (1). We will use a technique close to the one used by Georgiev and Todorova in [15] and Vitillaro in [33, 34]: a Faedo-Galerkin approximation coupled to a fix point theorem.

In section 3, we shall prove that the energy is unbounded when the initial data are large enough. In fact, it will be proved that the L^p-norm of the solutions grows as an exponential function. An essential ingredient of the proof is a lower bound in the L^p norm and the H^1 seminorm of the solution when the initial data are large enough, obtained by Vitillaro in [32]. The other ingredient is the use of an auxillary function L (which is a small perturbation of the energy) in order to obtain a linear differential inequality, that we integrate to finally prove that the energy is exponentially growing. To this end, we use Young's inequality with suitable coefficient, interpolation and Poincaré's inequalities.

Let us recall that the blow-up result in the case of a nonlinear damping ($m \neq 2$) is still an open problem.

2 Local existence

In this section we will prove the local existence and the uniqueness of the solution of the problem (1). We will adapt the ideas used by Georgiev and Todorova in [15], which consists in constructing approximations by the Faedo-Galerkin procedure in order to use the contraction mapping theorem. This method allows us to consider less restrictions on the initial data. Consequently, the same result can be established by using the Faedo-Galerkin approximation method coupled with the potential well method [7].

2.1 Setup and notations

We present here some material that we shall use in order to prove the local existence of the solution of problem (1). We denote

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega) / \; u_{\Gamma_0} = 0 \right\}.$$

By $(.,.)$ we denote the scalar product in $L^2(\Omega)$ i.e. $(u,v)(t) = \displaystyle\int_\Omega u(x,t)v(x,t)dx$. Also we mean by $\|.\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$, and by $\|.\|_{q,\Gamma_1}$ the $L^q(\Gamma_1)$ norm.

Let $T>0$ be a real number and X a Banach space endowed with norm $\|.\|_X$. $L^p(0,T;X)$, $1\le p<\infty$ denotes the space of functions f which are L^p over $(0,T)$ with values in X, which are measurable and $\|f\|_X\in L^p\left(0,T\right)$. This space is a Banach space endowed with the norm

$$\|f\|_{L^p(0,T;X)}=\left(\int_0^T\|f\|_X^p dt\right)^{1/p} \quad .$$

$L^\infty\left(0,T;X\right)$ denotes the space of functions $f:\left]0,T\right[\to X$ which are measurable and $\|f\|_X\in L^\infty\left(0,T\right)$. This space is a Banach space endowed with the norm:

$$\|f\|_{L^\infty(0,T;X)}=\text{ess}\sup_{0<t<T}\|f\|_X \quad .$$

We recall that if X and Y are two Banach spaces such that $X\hookrightarrow Y$ (continuous embedding), then

$$L^p\left(0,T;X\right)\hookrightarrow L^p\left(0,T;Y\right),\ 1\le p\le\infty.$$

We will also use the embedding (see [1, Therorem 5.8]).

$$H^1_{\Gamma_0}(\Omega)\hookrightarrow L^q(\Gamma_1),\ 2\le q\le\bar q \quad \text{where} \quad \bar q=\begin{cases}\dfrac{2(N-1)}{N-2}\,,\ \text{if } N\ge 3\\ +\infty\,,\ \text{if } N=1,2\end{cases} \quad .$$

Let us denote $V=H^1_{\Gamma_0}(\Omega)\cap L^m(\Gamma_1)$.

In this work, we cannot use "directly" the existence result of Georgiev and Todorova [15] nor the results of Vitillaro [33, 34] because of the presence of the strong linear damping $-\Delta u_t$ and the dynamic boundary conditions on Γ_1. Therefore, we have the next local existence theorem.

Theorem 2.1 *Let* $2\le p\le\bar q$ *and* $\max\left(2,\dfrac{\bar q}{\bar q+1-p}\right)\le m\le\bar q$.
Then given $u_0\in H^1_{\Gamma_0}(\Omega)$ *and* $u_1\in L^2(\Omega)$, *there exists* $T>0$ *and a unique solution* u *of the problem (1) on* $(0,T)$ *such that*

$$\begin{aligned} u &\in C\Big([0,T],H^1_{\Gamma_0}(\Omega)\Big)\cap C^1\Big([0,T],L^2(\Omega)\Big),\\ u_t &\in L^2\Big(0,T;H^1_{\Gamma_0}(\Omega)\Big)\cap L^m\Big((0,T)\times\Gamma_1\Big)\end{aligned}$$

We will prove this theorem by using the Fadeo-Galerkin approximations and the well-known contraction mapping theorem. In order to define the function for which a fixed point exists, we will consider first a related problem.
For $u\in C\big([0,T],H^1_{\Gamma_0}(\Omega)\big)\cap C^1\big([0,T],L^2(\Omega)\big)$ given, let us consider the following

problem:

$$\begin{cases} v_{tt} - \Delta v - \alpha \Delta v_t = |u|^{p-2}u, & x \in \Omega,\ t > 0 \\ v(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\ v_{tt}(x,t) = -\left[\dfrac{\partial v}{\partial \nu}(x,t) + \dfrac{\alpha \partial v_t}{\partial \nu}(x,t) + r|v_t|^{m-2}v_t(x,t)\right] & x \in \Gamma_1,\ t > 0 \\ v(x,0) = u_0(x),\ v_t(x,0) = u_1(x) & x \in \Omega \quad . \end{cases} \tag{2}$$

We have now to state the following existence result:

Lemma 2.1 *Let* $2 \le p \le \bar{q}$ *and* $\max\left(2, \dfrac{\bar{q}}{\bar{q}+1-p}\right) \le m \le \bar{q}$. *Then given* $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$ *there exists* $T > 0$ *and a unique solution* v *of the problem (2) on* $(0,T)$ *such that*

$$\begin{aligned} v &\in C\Big([0,T], H^1_{\Gamma_0}(\Omega)\Big) \cap C^1\Big([0,T], L^2(\Omega)\Big), \\ v_t &\in L^2\Big(0,T; H^1_{\Gamma_0}(\Omega)\Big) \cap L^m\Big((0,T)\times\Gamma_1\Big) \end{aligned}$$

and satisfies the energy identity:

$$\begin{aligned} \frac{1}{2}\left[\|\nabla v\|_2^2 + \|v_t\|_2^2 + \|v_t\|_{2,\Gamma_1}^2\right]_s^t &+ \alpha \int_s^t \|\nabla v_t(\tau)\|_2^2 d\tau + r\int_s^t \|v_t(\tau)\|_{m,\Gamma_1}^m d\tau \\ &= \int_s^t \int_\Omega |u(\tau)|^{p-2}u(\tau)v_t(\tau)d\tau dx \end{aligned}$$

for $0 \le s \le t \le T$.

In order to prove lemma 2.1, we first study for any $T > 0$ and $f \in H^1(0,T;L^2(\Omega))$ the following problem:

$$\begin{cases} v_{tt} - \Delta v - \alpha \Delta v_t = f(x,t), & x \in \Omega,\ t > 0 \\ v(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\ v_{tt}(x,t) = -\left[\dfrac{\partial v}{\partial \nu}(x,t) + \dfrac{\alpha \partial v_t}{\partial \nu}(x,t) + r|v_t|^{m-2}v_t(x,t)\right] & x \in \Gamma_1,\ t > 0 \\ v(x,0) = u_0(x),\ v_t(x,0) = u_1(x) & x \in \Omega \quad . \end{cases} \tag{3}$$

At this point, as done by Doronin et *al.* [13], we have to precise exactly what type of solutions of the problem (3) we expected.

Definition 2.1 *A function* $v(x,t)$ *such that*

$$\begin{aligned} v &\in L^\infty\left(0,T; H^1_{\Gamma_0}(\Omega)\right), \\ v_t &\in L^2\left(0,T; H^1_{\Gamma_0}(\Omega)\right) \cap L^m\left((0,T)\times\Gamma_1\right), \\ v_t &\in L^\infty\left(0,T; H^1_{\Gamma_0}(\Omega)\right) \cap L^\infty\left(0,T; L^2(\Gamma_1)\right), \\ v_{tt} &\in L^\infty\left(0,T; L^2(\Omega)\right) \cap L^\infty\left(0,T; L^2(\Gamma_1)\right), \\ v(x,0) &= u_0(x), \\ v_t(x,0) &= u_1(x), \end{aligned}$$

is a generalized solution to the problem (3) if for any function $\omega \in H^1_{\Gamma_0}(\Omega) \cap L^m(\Gamma_1)$ and $\varphi \in C^1(0,T)$ with $\varphi(T)=0$, we have the following identity:

$$\begin{aligned}\int_0^T (f,w)(t)\,\varphi(t)\,dt &= \int_0^T \Big[(v_{tt},w)(t) + (\nabla v,\nabla w)(t) + \alpha(\nabla v_t,\nabla w)(t)\Big]\,\varphi(t)\,dt \\ &+ \int_0^T \varphi(t)\int_{\Gamma_1}\Big[v_{tt}(t) + r|v_t(t)|^{m-2}v_t(t)\Big]w\,d\sigma\,dt.\end{aligned}$$

Lemma 2.2 *Let $2 \le p \le \bar{q}$ and $2 \le m \le \bar{q}$.*
Let $u_0 \in H^2(\Omega) \cap V$, $u_1 \in H^2(\Omega)$ and $f \in H^1(0,T;L^2(\Omega))$, then for any $T>0$, there exists a unique generalized solution (in the sense of definition 2.1), $v(t,x)$ of problem (3).

2.2 Proof of the lemma 2.2

To prove the above lemma, we will use the Faedo-Galerkin method, which consists in constructing approximations of the solution, then we obtain a priori estimates necessary to guarantee the convergence of these approximations. It appears some difficulties in order to derive a second order estimate of $v_{tt}(0)$. To get rid of them, and inspired by the ideas of Doronin and Larkin in [12] and Cavalcanti et *al.* [8], we introduce the following change of variables:

$$\widetilde{v}(t,x) = v(t,x) - \phi(t,x) \text{ with } \phi(t,x) = u_0(x) + t\,u_1(x).$$

Consequently, we have the following problem with the unknown $\widetilde{v}(t,x)$ and null initial conditions:

$$\begin{cases} \widetilde{v}_{tt} - \Delta\widetilde{v} - \alpha\Delta\widetilde{v}_t = f(x,t) + \Delta\phi + \alpha\Delta\phi_t, & x\in\Omega,\ t>0 \\ \widetilde{v}(x,t)=0, & x\in\Gamma_0,\ t>0 \\ \widetilde{v}_{tt}(x,t) = -\Big[\ \dfrac{\partial(\widetilde{v}+\phi)}{\partial\nu}(x,t) + \dfrac{\alpha\partial(\widetilde{v}_t+\phi_t)}{\partial\nu}(x,t)\Big] - & \\ \qquad\qquad \Big(r|(\widetilde{v}_t+\phi_t)|^{m-2}(\widetilde{v}_t+\phi_t)(x,t)\Big) & x\in\Gamma_1,\ t>0 \\ \widetilde{v}(x,0)=0, \qquad \widetilde{v}_t(x,0)=0 & x\in\Omega\quad . \end{cases} \tag{4}$$

Remark 2.1 It is quite clear that if $\widetilde{v}$ is a solution of problem (4) on $[0,T]$, then v is a solution of problem (3) on $[0,T]$. Moreover writing the problem in term of $\widetilde{v}$ shows exactly the regularity needed on the initial conditions u_0 and u_1 to ensure the existence.

Now we construct approximations of the solution $\widetilde{v}$ by the Faedo-Galerkin method as follows.
For every $n \ge 1$, let $W_n = \text{span}\{\omega_1,\dots,\omega_n\}$, where $\{\omega_j(x)\}_{1\le j\le n}$ is a basis in the space V. By using the Grahm-Schmidt orthogonalization process we can take

$\omega = \big(\omega_1, \ldots, \omega_n\big)$ to be orthonormal[1] in $L^2(\Omega) \cap L^2(\Gamma_1)$.
We define the approximations:

$$\widetilde{v}_n(t) = \sum_{j=1}^{n} g_{jn}(t) w_j \tag{5}$$

where $\widetilde{v}_n(t)$ are solutions to the finite dimensional Cauchy problem (written in normal form since ω is an orthonormal basis):

$$\begin{gathered} \int_\Omega \widetilde{v}_{ttn}(t) w_j\, dx + \int_\Omega \nabla\big(\widetilde{v}_n + \phi\big)\nabla w_j + \alpha \int_\Omega \nabla\big(\widetilde{v}_n + \phi\big)_t \nabla w_j\, dx \\ + \int_{\Gamma_1} \big(\widetilde{v}_{ttn}(t) + r|(\widetilde{v}_n + \phi)_t|^{m-2}(\widetilde{v}_n + \phi)_t\big)\, w_j\, d\sigma = \int_\Omega f w_j\, dx \quad . \\ g_{jn}(0) = g'_{jn}(0) = 0,\ j = 1, \ldots, n \end{gathered} \tag{6}$$

According to the Caratheodory theorem, see [10], the problem (6) has solution $(g_{jn}(t))_{j=1,n} \in H^3(0, t_n)$ defined on $[0, t_n)$. We need now to show:

- firstly that forall $n \in \mathbb{N}$,, $t_n = T$,
- secondly that these approximations converge to a solution of the problem (4).

To do this we need the two following a priori estimates: first-order a priori estimates to prove the first point. But we will show that the presence of the nonlinear term $|u_t|^{m-2}u_t$ forces us to derive a second order a priori estimate to pass to the limit in the nonlinear term. Indeed the key tool in our proof is the Aubin-Lions lemma which uses the compactness of the embedding $H^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$.

2.2.1 First order a priori estimates

Multiplying equation (6) by $g'_{jn}(t)$, integrating over $(0,t)\times\Omega$ and using integration by parts we get: for every $n \geq 1$,

$$\begin{aligned} \frac{1}{2} \quad & \big[\|\nabla\widetilde{v}_n(t)\|_2^2 + \|\widetilde{v}_{tn}(t)\|_2^2 + \|\widetilde{v}_{tn}\|_{2,\Gamma_1}^2\big] + \int_0^t \int_\Omega \nabla\phi\nabla\widetilde{v}_n\, dx \\ + \alpha \quad & \int_0^t \int_\Omega \nabla\phi_t \nabla\widetilde{v}_{tn}\, dx + \alpha \int_0^t \|\nabla\widetilde{v}_{tn}(s)\|_2^2\, ds \\ + r \quad & \int_0^t \int_{\Gamma_1} |\,(\widetilde{v}_n + \phi)_t\,|^{m-2}(\widetilde{v}_n + \phi)_t \widetilde{v}_{tn}\, d\sigma ds \\ & = \int_0^t \int_\Omega f(t,x)\widetilde{v}_{tn}(s)\, dx\, ds \end{aligned} \tag{7}$$

[1]Unfortunately, the presence of the nonlinear boundary conditions excludes us to use the spatial basis of eigenfunctions of $-\Delta$ in $H^1_{\Gamma_0}(\Omega)$ as done in [14]

By using Young's inequality, there exists $\delta_1 > 0$, (in fact small enough) such that

$$\alpha \int_0^t \int_\Omega \nabla\phi_t \nabla\widetilde{v}_{tn} dx \leq \delta_1 \int_0^t \int_\Omega |\nabla\widetilde{v}_{tn}|^2 dx + \frac{1}{4\delta_1} \int_0^t \int_\Omega |\nabla\phi_t|^2 dx \tag{8}$$

and

$$\int_0^t \int_\Omega \nabla\phi \nabla\widetilde{v}_n dx \leq \delta_1 \int_0^t \int_\Omega |\nabla\widetilde{v}_n|^2 dx + \frac{1}{4\delta_1} \int_0^t \int_\Omega \nabla\phi|^2 dx. \tag{9}$$

By Young's and Poincaré's inequalities, we can find $C > 0$, such that

$$\int_0^t \int_\Omega f(t,x)\widetilde{v}_{tn}(s) dx ds \leq C \int_0^t \int_\Omega \left(f^2 + |\nabla\widetilde{v}_{tn}(s)|^2\right) dx ds. \tag{10}$$

The last term in the left hand side of equation (7) can be written as follows:

$$\begin{aligned} &\int_0^t \int_{\Gamma_1} |\left(\widetilde{v}_n + \phi\right)_t|^{m-2} \left(\widetilde{v}_n + \phi\right)_t \widetilde{v}_{tn} d\sigma ds \\ = &\int_0^t \int_{\Gamma_1} |\left(\widetilde{v}_n + \phi\right)_t|^m d\sigma ds - \int_0^t \int_{\Gamma_1} |\left(\widetilde{v}_n + \phi\right)_t|^{m-2} \left(\widetilde{v}_n + \phi\right)_t \phi_t d\sigma ds, \end{aligned}$$

Then Young's inequality gives us, for $\delta_2 > 0$:

$$\begin{aligned} &\left| \int_0^t \int_{\Gamma_1} |\left(\widetilde{v}_n + \phi\right)_t|^{m-2} \left(\widetilde{v}_n + \phi\right)_t \phi_t d\sigma ds \right| \\ \leq &\frac{\delta_2^m}{m} \int_0^t \int_{\Gamma_1} |\left(\widetilde{v}_n + \phi\right)_t|^m d\sigma ds + \frac{m-1}{m} \delta_2^{-m/(m-1)} \int_0^t \int_{\Gamma_1} |\phi_t|^m d\sigma ds. \end{aligned} \tag{11}$$

Consequently, using the inequalities (8), (9), (10) and (11) in the equation (7), choosing δ_1 and δ_2 small enough, we may conclude that:

$$\begin{aligned} &\frac{1}{2}\left[\|\nabla\widetilde{v}_n(t)\|_2^2 + \|\widetilde{v}_{tn}(t)\|_2^2 + \|\widetilde{v}_{tn}(t)\|_{2,\Gamma_1}^2\right] \\ &+\alpha \int_0^t \|\nabla\widetilde{v}_{tn}(s)\|_2^2 ds + r \int_0^t \int_{\Gamma_1} |\left(\widetilde{v}_n + \phi\right)_t|^m d\sigma ds \leq C_T \quad , \end{aligned} \tag{12}$$

where C_T is a positive constant independent of n. Therefore, the last estimate (12) gives us, $\forall n \in \mathbb{N}$, $t_n = T$, and:

$$\left(\widetilde{v}_n\right)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty(0,T; H^1_{\Gamma_0}(\Omega)) \quad , \tag{13}$$

$$(\widetilde{v}_{tn})_{n\in\mathbb{N}} \text{ is bounded in } L^\infty(0,T;L^2(\Omega))\cap L^2\left(0,T;H^1_{\Gamma_0}(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Gamma_1)\right). \tag{14}$$

Now, by using the following algebraic inequality:

$$(A+B)^\lambda \le 2^{\lambda-1}(A^\lambda + B^\lambda),\ \ A,B\ge 0\,,\ \lambda \ge 1, \tag{15}$$

we can find $c_1, c_2 > 0$, such that:

$$\int_0^t\int_{\Gamma_1} |(\widetilde{v}_n+\phi)_t|^m d\sigma ds \ge c_1 \int_0^t\int_{\Gamma_1} |\widetilde{v}_{tn}|^m d\sigma ds - c_2 \int_0^t\int_{\Gamma_1} |\phi_t|^m d\sigma ds. \tag{16}$$

Then, by the embedding $H^2(\Omega) \hookrightarrow L^m(\Gamma_1)$ $(2 \le m \le \bar{q})$, we conclude that $u_1 \in L^m(\Gamma_1)$. Therefore, from the inequalities (12) and (16), there exists $C'_T > 0$ such that:

$$\int_0^t\int_{\Gamma_1} |\widetilde{v}_{tn}|^m d\sigma ds \le C'_T.$$

Consequently,

$$\widetilde{v}_{tn} \text{ is bounded in } L^m((0,T)\times\Gamma_1). \tag{17}$$

2.2.2 Second order a priori estimate

In order to obtain a second a priori estimate, we will first estimate $\|\widetilde{v}_{ttn}(0)\|_2^2$ and $\|\widetilde{v}_{ttn}(0)\|_{2,\Gamma_1}^2$. For this purpose, considering $w_j = \widetilde{v}_{ttn}(0)$ and $t=0$ in the equation (6), we get

$$\begin{aligned} &\|\widetilde{v}_{ttn}(0)\|_2^2 + \|\widetilde{v}_{ttn}(0)\|_{2,\Gamma_1}^2 + \int_\Omega \nabla\phi(0)\nabla\widetilde{v}_{ttn}(0)dx \\ &+\alpha\int_\Omega \nabla\phi_t(0)\nabla\widetilde{v}_{ttn}(0)dx + r\int_{\Gamma_1} |\phi_t(0)|^{m-2}\phi_t(0)\widetilde{v}_{ttn}(0)d\sigma ds \\ =& \int_\Omega f(0,x)\widetilde{v}_{ttn}(0)dxds. \end{aligned} \tag{18}$$

Since the following equalities hold:

$$\phi(0) = u_0\,,\ \phi_t(0) = u_1\,,\ \int_\Omega \nabla\phi_t(0)\nabla\widetilde{v}_{ttn}(0) = -\int_\Omega \Delta\phi_t(0)\widetilde{v}_{ttn}(0) + \int_{\Gamma_1} \phi_t \frac{\partial v_{ttn}}{\partial\nu} d\sigma\,,$$

as $f \in H^1\left(0,T;L^2(\Omega)\right)$ and $u_0, u_1 \in H^2(\Omega)$, by using Young's inequality and the embedding $H^2(\Omega) \hookrightarrow L^m(\Gamma_1)$, we conclude that there exists $C>0$ independent of n such that:

$$\|\widetilde{v}_{ttn}(0)\|_2^2 + \|\widetilde{v}_{ttn}(0)\|_{2,\Gamma_1}^2 \le C\ . \tag{19}$$

Differentiating equation (6) with respect to t, multiplying the result by $g''_{jn}(t)$ and summing over j we get:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|\nabla\widetilde{v}_{tn}(t)\|_2^2+\|\widetilde{v}_{ttn}(t)\|_2^2+\|\widetilde{v}_{ttn}(t)\|_{2,\Gamma_1}^2\right]+\int\limits_{\Omega}\nabla\phi_t\nabla\widetilde{v}_{ttn}dx\\
&+\alpha\|\nabla\widetilde{v}_{ttn}(s)\|_2^2+r(m-1)\int\limits_{\Gamma_1}|(\widetilde{v}_n+\phi)_t|^{m-2}(\widetilde{v}_n+\phi)_{tt}\widetilde{v}_{ttn}d\sigma \qquad (20)\\
=&\int\limits_{\Omega}f_t(t,x)\widetilde{v}_{ttn}(s)dxds.
\end{aligned}$$

Since $\phi_{tt}=0$, the last term in the left hand side of the equation (20) can be written as follows:

$$\int\limits_{\Gamma_1}|(\widetilde{v}_n+\phi)_t|^{m-2}(\widetilde{v}_n+\phi)_{tt}\widetilde{v}_{ttn}d\sigma=\int\limits_{\Gamma_1}|(\widetilde{v}_n+\phi)_t|^{m-2}(\widetilde{v}_{ttn}+\phi_{tt})^2d\sigma,$$

But we have,

$$\int\limits_{\Gamma_1}|(\widetilde{v}_n+\phi)_t|^{m-2}(\widetilde{v}_{ttn}+\phi_{tt})^2d\sigma=\frac{4}{m^2}\int\limits_{\Gamma_1}\left(\frac{\partial}{\partial t}\left(|\widetilde{v}_{tn}(t)+\phi_t|^{\frac{m-2}{2}}(\widetilde{v}_{tn}(t)+\phi_t)\right)\right)^2d\sigma.$$

Now, integrating equation (20) over $(0,t)$, using the inequality (19) and Young's and Poincaré's inequalities (as in (11)), there exists $\widetilde{C}_T>0$ such that:

$$\begin{aligned}
&\frac{1}{2}\left[\|\nabla\widetilde{v}_{tn}(t)\|_2^2+\|\widetilde{v}_{ttn}(t)\|_2^2+\|\widetilde{v}_{ttn}(t)\|_{2,\Gamma_1}^2\right]+\alpha\int_0^t\|\nabla\widetilde{v}_{ttn}(s)\|_2^2+\\
&+\frac{4r(m-1)}{m^2}\int\limits_{\Gamma_1}\left(\frac{\partial}{\partial t}\left(|\widetilde{v}_{tn}(t)+\phi_t|^{\frac{m-2}{2}}(\widetilde{v}_{tn}(t)+\phi_t)\right)\right)^2d\sigma\le\widetilde{C}_T.
\end{aligned}$$

Consequently,we deduce the following results:

$$\begin{array}{lll}
(\widetilde{v}_{ttn}(t))_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;L^2(\Omega)) \quad ,\\
(\widetilde{v}_{ttn}(t))_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;L^2(\Gamma_1)) \quad ,\\
(\widetilde{v}_{tn}(t))_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right) \quad .
\end{array} \qquad (21)$$

From (13), (14), (17) and (21), we have $(\widetilde{v}_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)$. Then $(\widetilde{v}_n)_{n\in\mathbb{N}}$ is bounded in $L^2\left(0,T;H^1_{\Gamma_0}(\Omega)\right)$. Since $(\widetilde{v}_{tn})_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, $(\widetilde{v}_{tn})_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(\Omega))$. Consequently $(\widetilde{v}_n)_{n\in\mathbb{N}}$ is bounded in $H^1(0,T;H^1(\Omega))$.

Since the embedding $H^1(0,T;H^1(\Omega))\hookrightarrow L^2(0,T;L^2(\Omega))$ is compact, by using Aubin-Lions theorem, we can extract a subsequence $(\widetilde{v}_\mu)_{\mu\in\mathbb{N}}$ of $(\widetilde{v}_n)_{n\in\mathbb{N}}$ such that

$$\widetilde{v}_\mu\to\widetilde{v}\text{ strongly in }L^2\left(0,T;L^2(\Omega)\right).$$

Therefore,

$$\widetilde{v}_\mu \to \widetilde{v} \text{ strongly and a.e on } (0,T)\times\Omega \quad .$$

Following [19, Lemme 3.1], we get:

$$|\widetilde{v}_\mu|^{p-2}\widetilde{v}_\mu \to |\widetilde{v}|^{p-2}\widetilde{v} \text{ strongly and a.e on } (0,T)\times\Omega.$$

On the other hand, we already have proved in the preceding section that:

$$(\widetilde{v}_{tn})_{n\in\mathbb{N}} \text{ is bounded in } L^\infty\left(0,T;L^2(\Gamma_1)\right)$$

From (13) and (21), since

$$\|\widetilde{v}_n(t)\|_{H^{\frac{1}{2}}(\Gamma_1)} \le C\|\nabla\widetilde{v}_n(t)\|_2 \text{ and } \|\widetilde{v}_{tn}(t)\|_{H^{\frac{1}{2}}(\Gamma_1)} \le C\|\nabla\widetilde{v}_{tn}(t)\|_2$$

we deduce that:

$$\begin{array}{lll} (\widetilde{v}_n)_{n\in\mathbb{N}} & \text{is bounded in} & L^2\left(0,T;H^{\frac{1}{2}}(\Gamma_1)\right) \\ (\widetilde{v}_{tn})_{n\in\mathbb{N}} & \text{is bounded in} & L^2\left(0,T;H^{\frac{1}{2}}(\Gamma_1)\right) \\ (\widetilde{v}_{ttn})_{n\in\mathbb{N}} & \text{is bounded in} & L^2\left(0,T;L^2(\Gamma_1)\right) \end{array} .$$

Since the embedding $H^{\frac{1}{2}}(\Gamma_1)\hookrightarrow L^2(\Gamma_1)$ is compact, again by using Aubin-Lions theorem, we conclude that we can extract a subsequence also denoted $(\widetilde{v}_\mu)_{\mu\in\mathbb{N}}$ of $(\widetilde{v}_n)_{n\in\mathbb{N}}$ such that:

$$\widetilde{v}_{t\mu} \to \widetilde{v}_t \text{ strongly in } L^2\left(0,T;L^2(\Gamma_1)\right) \quad . \tag{22}$$

Therefore from (17), we obtain that:

$$|\widetilde{v}_{t\mu}|^{m-2}\widetilde{v}_{t\mu} \rightharpoonup \chi \text{ weakly in } L^{m'}\left((0,T)\times\Gamma_1\right),$$

It suffices to prove now that $\chi = |\widetilde{v}_t|^{m-2}\widetilde{v}_t$.
Clearly, from (22) we get:

$$|\widetilde{v}_{t\mu}|^{m-2}\widetilde{v}_{t\mu} \to |\widetilde{v}_t|^{m-2}\widetilde{v}_t \text{ strongly and a.e on } (0,T)\times\Gamma_1.$$

Again, by using the Lions's lemma, [19, Lemme 1.3], we obtain $\chi = |\widetilde{v}_t|^{m-2}\widetilde{v}_t$. The proof now can be completed arguing as in [19, Théorème 3.1].

2.2.3 Uniqueness

Let v, w two solutions of the problem (3) which share the same initial data. Let us denote $z = v - w$. It is staightforward to see that z satisfies:

$$\begin{aligned} &\|\nabla z\|_2^2 + \|\nabla z_t\|_2^2 + \|z_t\|_{2,\Gamma_1}^2 + 2\alpha\int_0^t \|\nabla z\|_2^2 ds \\ &+2r\int_0^t\int_{\Gamma_1}\left[|v_t(s)|^{m-2}v_t(s) - |w_t(s)|^{m-2}w_t(s)\right](v_t(s)-w_t(s))dsd\sigma = 0 \quad . \end{aligned} \tag{23}$$

By using the algebraic inequality:

$$\forall\, m \geq 2\,,\ \exists\, c > 0\,,\ \forall\, a\,,\ b \in \mathbb{R}\,,\ \Big[|a|^{m-2}a - |b|^{m-2}b\Big](a-b) \ \geq\ c\,|a-b|^m \tag{24}$$

equation (23) yields to:

$$\|z_t\|_2^2 + \|\nabla z\|_2^2 + \|z_t\|_{2,\Gamma_1}^2 + 2\alpha \int_0^t \|\nabla z_t\|_2^2 ds$$
$$+c \int_0^t \int_{\Gamma_1} |v_t\,(s) - w_t(s)|^m ds d\sigma \leq 0 \quad .$$

Then, this last inequality yields to $z = 0$.

This finishes the proof of the lemma 2.2.

2.3 Proof of lemma 2.1

We first approximate $u \in C\left([0,T], H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right)$ endowed with the standard norm $\|u\| = \max\limits_{t\in[0,T]} \|u_t(t)\|_2 + \|u(t)\|_{H^1(\Omega)}$, by a sequence $(u^k)_{k\in\mathbb{N}} \subset C^\infty([0,T]\times\overline{\Omega})$ by a standard convolution arguments (see [4]). It is clear that $f\left(u^k\right) = |u^k|^{p-2}u^k \in H^1\left(0,T; L^2(\Omega)\right)$. This type of approximation has been already used by Vitillaro in [33, 34]. Next, we approximate the initial data $u_1 \in L^2(\Omega)$ by a sequence $(u_1^k)_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$. Finally, since the space $H^2(\Omega) \cap V \cap H^1_{\Gamma_0}(\Omega)$ is dense in $H^1_{\Gamma_0}(\Omega)$ for the H^1 norm, we approximate $u_0 \in H^1_{\Gamma_0}(\Omega)$ by a sequence $(u_0^k)_{k\in\mathbb{N}} \subset H^2(\Omega) \cap V \cap H^1_{\Gamma_0}(\Omega)$.

We consider now the set of the following problems:

$$\begin{cases} v_{tt}^k - \Delta v^k - \alpha\Delta v_t^k = |u^k|^{p-2}u^k, & x \in \Omega,\ t > 0 \\ v^k(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\ v_{tt}^k(x,t) = -\left[\dfrac{\partial v^k}{\partial \nu}(x,t) + \dfrac{\alpha \partial v_t^k}{\partial \nu}(x,t) + r|v_t^k|^{m-2}v_t^k(x,t)\right] & x \in \Gamma_1,\ t > 0 \\ v^k(x,0) = u_0^k,\ v_t^k(x,0) = u_1^k & x \in \Omega \quad . \end{cases} \tag{25}$$

Since every hypothesis of lemma 2.2 are verified, we can find a sequence of unique solution $(v_k)_{k\in\mathbb{N}}$ of the problem (25). Our goal now is to show that $(v^k, v_t^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the space

$$\begin{aligned} Y_T \ = \Big\{ \ & (v, v_t)/v \in C\left([0,T], H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right), \\ & v_t \in L^2\left(0,T; H^1_{\Gamma_0}(\Omega)\right) \cap L^m\left((0,T)\times\Gamma_1\right) \Big\} \end{aligned}$$

endowed with the norm

$$\|(v, v_t)\|_{Y_T}^2 = \max_{0\leq t\leq T}\Big[\|v_t\|_2^2 + \|\nabla v\|_2^2\Big] + \|v_t\|^2_{L^m\left((0,T)\times\Gamma_1\right)} + \int_0^t \|\nabla v_t(s)\|_2^2\, ds\,.$$

For this purpose, we set $U = u^k - u^{k'}, \ V = v^k - v^{k'}$. It is straightforward to see that V satisfies:

$$\begin{cases} V_{tt} - \Delta V & -\alpha \Delta V_t = |u^k|^{p-2}u^k - |u^{k'}|^{p-2}u^{k'} & x \in \Omega, \ t > 0 \\ V(x,t) = & 0 & x \in \Gamma_0, \ t > 0 \\ V_{tt}(x,t) = & -\Big[\dfrac{\partial V}{\partial \nu}(x,t) + \dfrac{\alpha \partial V_t}{\partial \nu}(x,t)\Big] - & \\ & r\Big(|v_t^k|^{m-2}v_t^k(x,t) - |v_t^{k'}|^{m-2}v_t^{k'}(x,t)\Big) & x \in \Gamma_1, \ t > 0 \\ V(x,0) = & u_0^k - u_0^{k'}, \ V_t(x,0) = u_1^k - u_1^{k'} & x \in \Omega \quad . \end{cases}$$

We multiply the above differential equations by V_t, we integrate over $(0,t) \times \Omega$ and we use integration by parts to obtain:

$$\begin{aligned} \frac{1}{2} \quad & \left(\|V_t\|_2^2 + \|\nabla V\|_2^2 + \|V_t\|_{2,\Gamma_1}^2\right) + \alpha \int_0^t \|\nabla V_t\|_2^2 \, ds \\ + \quad & r \int_0^t \int_{\Gamma_1} \left(|v_t^k|^{m-2}v_t^k - |v_t^{k'}|^{m-2}v_t^{k'}\right) \left(v_t^k - v_t^{k'}\right) d\sigma ds \\ = \quad & \frac{1}{2}\left(\|V_t(0)\|_2^2 + \|\nabla V(0)\|_2^2 + \|V_t(0)\|_{2,\Gamma_1}^2\right) \\ + \quad & \int_0^t \int_\Omega \left(|u^k|^{p-2}u^k - |u^{k'}|^{p-2}u^{k'}\right) \left(v_t^k - v_t^{k'}\right) dx d\tau, \quad \forall t \in (0,T) \quad . \end{aligned}$$

By using the algebraic inequality (24), we get:

$$\begin{aligned} \frac{1}{2} \quad & \left(\|V_t\|_2^2 + \|\nabla V\|_2^2 + \|V_t\|_{2,\Gamma_1}^2\right) + \alpha \int_0^t \|\nabla V_t\|_2^2 ds + c_1 \|V_t\|_{m,\Gamma_1}^m \\ \leq \quad & \frac{1}{2}\left(\|V_t(0)\|_2^2 + \|\nabla V(0)\|_2^2 + \|V_t(0)\|_{2,\Gamma_1}^2\right) \\ & + \int_0^t \int_\Omega \left(|u^k|^{p-2}u^k - |u^{k'}|^{p-2}u^{k'}\right) \left(v_t^k - v_t^{k'}\right) dx d\tau, \quad \forall t \in (0,T) \, . \end{aligned}$$

In order to find a majoration of the term:

$$\int_0^t \int_\Omega \left(|u^k|^{p-2}u^k - |u^{k'}|^{p-2}u^{k'}\right) \left(v_t^k - v_t^{k'}\right) dx d\tau, \quad \forall t \in (0,T)$$

in the previous inequality, we use the result of Georgiev and Todorova [15] (specifically their equations (2.5) and (2.6) in proposition 2.1). The hypothesis on p

ensures us to use exactly the same argument. Thus by applying Young's inequality and Gronwall inequality, there exists C depending only on Ω and p such that:

$$\|V\|_{Y_T} \leq C\left(\|V_t(0)\|_2^2 + \|\nabla V(0)\|_2^2 + \|V_t(0)\|_{2,\Gamma_1}^2\right) + CT\|U\|_{Y_T}.$$

Let us now remark that from the notations used above, we have:

$$V(0) = u_0^k - u_0^{k'} \ , \ V_t(0) = u_1^k - u_1^{k'} \text{ and } U = u^k - u^{k'} \quad .$$

Thus, since $(u_0^k)_{k\in\mathbb{N}}$ is a converging sequence in $H^1_{\Gamma_0}(\Omega)$, $(u_1^k)_{k\in\mathbb{N}}$ is a converging sequence in $L^2(\Omega)$ and $\left(u^k\right)_{k\in\mathbb{N}}$ is a converging sequence in $C\left([0,T], H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right)$ (so in Y_T also), we conclude that $(v^k, v_t^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in Y_T. Thus (v^k, v_t^k) converges to a limit $(v, v_t) \in Y_T$. Now by the same procedure used by Georgiev and Todorova in [15], we prove that this limit is a weak solution of the problem (2). This completes the proof of the lemma 2.1.

2.4 Proof of theorem 2.1

In order to prove theorem 2.1, we use the contraction mapping theorem.
For $T > 0$, let us define the convex closed subset of Y_T:

$$X_T = \{(v, v_t) \in Y_T \text{ such that } v(0) = u_0, v_t(0) = u_1\} \quad .$$

Let us denote:

$$B_R(X_T) = \{v \in X_T; \|v\|_{Y_T} \leq R\} \quad .$$

Then, lemma 2.1 implies that for any $u \in X_T$, we may define $v = \Phi(u)$ the unique solution of (2) corresponding to u. Our goal now is to show that for a suitable $T > 0$, Φ is a contractive map satisfying $\Phi(B_R(X_T)) \subset B_R(X_T)$.
Let $u \in B_R(X_T)$ and $v = \Phi(u)$. Then for all $t \in [0,T]$ we have:

$$\begin{aligned} &\|v_t\|_2^2 + \|\nabla v\|_2^2 + \|v_t\|_{2,\Gamma_1}^2 + 2\int_0^t \|v_t\|_{m,\Gamma_1}^m ds + 2\alpha \int_0^t \|\nabla v_t\|_2^2 ds \\ &= \|u_1\|_2^2 + \|\nabla u_0\|_2^2 + \|u_1\|_{2,\Gamma_1}^2 + 2\int_0^t \int_\Omega |u(\tau)|^{p-2} u(\tau)\, v_t(\tau)\, dx\, d\tau \end{aligned} \quad . \tag{26}$$

Using Hölder inequality, we can control the last term in the right hand side of the inequality (26) as follows:

$$\int_0^t \int_\Omega |u(\tau)|^{p-2} u(\tau)\, v_t(\tau)\, dx d\tau \leq \int_0^t \|u(\tau)\|_{2N/(N-2)}^{p-1} \|v_t(\tau)\|_{2N/\left(3N-Np+2(p-1)\right)} d\tau$$

Since $p \leq \dfrac{2N}{N-2}$, we have $\dfrac{2N}{\big(3N - Np + 2(p-1)\big)} \leq \dfrac{2N}{N-2}$.

Thus, by Young's and Sobolev's inequalities, we get $\forall\, \delta > 0\,, \exists\, C(\delta) > 0$, such that:

$$\forall t \in (0,T)\ ,\ \int_0^t \int_\Omega |u\,(\tau)\,|^{p-2} u\,(\tau)\, v_t\,(\tau)\, dx d\tau \leq C(\delta) t R^{2(p-1)} + \delta \int_0^t \|\nabla v_t\,(\tau)\,\|_2^2 d\tau.$$

Inserting the last estimate in the inequality (26) and choosing δ small enough in order to counter-balance the last term of the left hand side of the inequality (26) we get:

$$\|v\|_{Y_T}^2 \leq \frac{1}{2} R^2 + CTR^{2(p-1)}.$$

Thus, for T sufficiently small, we have $\|v\|_{Y_T} \leq R$. This shows that $v \in B_R\,(X_T)$. Next, we have to verify that Φ is a contraction. To this end, we set $U = u - \bar{u}$ and $V = v - \bar{v}$, where $v = \Phi(u)$ and $\bar{v} = \Phi(\bar{u})$ are the solutions of problem (2) corresponding respectively to u and v. Consequently we have:

$$\begin{cases} V_{tt} - \Delta V & -\alpha \Delta V_t = |u|^{p-2}u - |\bar{u}|^{p-2}\bar{u} & x \in \Omega,\ t > 0 \\ V(x,t) = & 0 & x \in \Gamma_0,\ t > 0 \\ V_{tt}(x,t) = & -\Big[\dfrac{\partial V}{\partial \nu}(x,t) + \dfrac{\alpha \partial V_t}{\partial \nu}(x,t)\Big] - & \\ & r\Big(|v_t|^{m-2}v_t(x,t) - |\bar{v}_t|^{m-2}\bar{v}_t(x,t)\Big) & x \in \Gamma_1,\ t > 0 \\ V(x,0) = & 0,\ V_t(x,0) = 0 & x \in \Omega\ . \end{cases} \tag{27}$$

By multiplying the differential equation (27) by V_t and integrating over $(0,t) \times \Omega$, we get:

$$\begin{aligned} &\frac{1}{2}\ \left(\|V_t\|_2^2 + \|\nabla V\|_2^2 + \|V_t\|_{2,\Gamma_1}^2\right) + \alpha \int_0^t \|\nabla V_t\|_2^2 ds + \\ & r \int_0^t \int_{\Gamma_1} \left(|v_t|^{m-2}v_t - |\bar{v}_t|^{m-2}\bar{v}_t\right)(v_t - \bar{v}_t) d\sigma ds \\ &= \int_0^t \int_\Omega \left(|u|^{p-2}u - |\bar{u}|^{p-2}\bar{u}\right)(v_t - \bar{v}_t) dx d\tau,\ \forall t \in (0,T) \end{aligned}.$$

Again, by using the algebraic inequality (24), we have:

$$\begin{aligned} &\frac{1}{2}\left(\|V_t\|_2^2 + \|\nabla V\|_2^2 + \|V_t\|_{2,\Gamma_1}^2\right) + \alpha \int_0^t \|\nabla V_t\|_2^2 ds + c_1 \|V_t\|_{m,\Gamma_1}^m \\ &\leq \int_0^t \int_\Omega \left(|u|^{p-2}u - |\bar{u}|^{p-2}\bar{u}\right)(v_t - \bar{v}_t) dx d\tau,\ \forall t \in (0,T) \end{aligned}. \tag{28}$$

To estimate the term in the right hand side of the inequality (28), let us denote:

$$I(t) := \int_0^t \int_\Omega \left(|u|^{p-2}u - |\bar{u}|^{p-2}\bar{u}\right)(v_t - \bar{v}_t)dxd\tau \quad .$$

Using the algebraic inequality:

$$\left||u|^{p-2}u - |\bar{u}|^{p-2}\bar{u}\right| \leq c_p|u-\bar{u}|\left(|u|^{p-2} + |\bar{u}|^{p-2}\right),$$

which holds for any $u, \bar{u} \in \mathbb{R}$, where c_p is a positive constant depending only on p, we find:

$$I(t) \leq c_p \int_0^T \int_\Omega |u-\bar{u}|\left(|u|^{p-2} + |\bar{u}|^{p-2}\right)|V_t|dxd\tau \quad .$$

Following the same argument as Vitillaro in [34, eq 77], choosing $p < r_0 < \bar{q}$ such that:

$$\frac{\bar{q}}{\bar{q}-p+1} < \frac{r_0}{r_0-p+1} < m \quad ,$$

let $s > 1$ such that:

$$\frac{1}{m} + \frac{1}{r_0} + \frac{1}{s} = 1 \quad .$$

Using Hölder's inequality we obtain:

$$I(t) \leq c_p \int_0^T \left(\|u-\bar{u}\|_{r_0}\|V_t\|_m\right) . \left(\int_\Omega \left(|u|^{p-2} + |\bar{u}|^{p-2}\right)^s\right)^{1/s} . \tag{29}$$

Therefore, the algebraic inequality (15) gives us:

$$\left(\int_\Omega \left(|u|^{p-2} + |\bar{u}|^{p-2}\right)^s\right)^{1/s} \leq 2^{s-1}\left(\|u\|_{(p-2)s}^{(p-2)s} + \|\bar{u}\|_{(p-2)s}^{(p-2)s}\right)^{1/s} .$$

But since

$$(A+B)^\beta \leq A^\beta + B^\beta \,,\, \forall A\,,\, B \geq 0 \text{ and } 0 < \beta < 1$$

we get

$$\left(\int_\Omega \left(|u|^{p-2} + |\bar{u}|^{p-2}\right)^s\right)^{1/s} \leq 2^{s-1}\left(\|u\|_{(p-2)s}^{(p-2)} + \|\bar{u}\|_{(p-2)s}^{(p-2)}\right) . \tag{30}$$

Consequently, inserting the inequality (29) in (30) and using Poincaré's inequality, we obtain:

$$I(t) \leq c_2 R^{p-2} \int_0^T \|u-\bar{u}\|_{r_0}\|\nabla V_t\|_2 ds.$$

Applying Hölder's inequality in time, we finally get:

$$\begin{aligned} I(t) &\leq c_2 R^{p-2} T^{1/2} \|u-\bar{u}\|_{L^\infty(0,T;L^{r_0}(\Omega))} \left(\int_0^T \|\nabla V_t\|_2^2 \right)^{1/2} \\ &\leq \frac{c_2}{2} R^{p-2} T^{1/2} \left[\|u-\bar{u}\|^2_{L^\infty(0,T;L^{r_0}(\Omega))} + \int_0^T \|\nabla V_t\|_2^2 \right] . \end{aligned} \tag{31}$$

Lastly, by choosing T small enough in order to have:

$$\alpha - \frac{c_2}{2} R^{p-2} T^{1/2} > 0 \quad ,$$

we conclude by inserting the estimate (31) in the estimate (28) that:

$$\begin{aligned} \frac{1}{2} \left(\|V_t\|_2^2 + \|\nabla V\|_2^2 + \|V_t\|^2_{2,\Gamma_1} \right) + \alpha \int_0^t \|\nabla V_t\|_2^2 ds + c_1 \|V_t\|^m_{m,\Gamma_1} \\ \leq \frac{c_2}{2} R^{p-2} T^{1/2} \|u-\bar{u}\|^2_{L^\infty(0,T;L^{r_0}(\Omega))} \quad . \end{aligned} \tag{32}$$

Since $r_0 < \bar{q}$, using the embedding

$$L^\infty \left(0,T; H^1_{\Gamma_0}(\Omega)\right) \hookrightarrow L^\infty \left(0,T; L^{r_0}(\Omega)\right)$$

in the estimate (32), we finally have:

$$\|V\|^2_{Y_T} \leq c_3 R^{p-2} T^{1/2} \|U\|^2_{Y_T}. \tag{33}$$

By choosing T small enough in order to have

$$c_3 R^{p-2} T^{1/2} < 1 \quad .$$

the estimate (33) shows that Φ is a contraction. Consequently the contraction mapping theorem guarantees the existence of a unique v satisfying $v = \Phi(v)$. The proof of theorem 2.1 is now completed.

Remark 2.2 To prove the existence and uniqueness of the solution to the more general problem:

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = f(u), & x \in \Omega, \ t > 0 \\ u(x,t) = 0, & x \in \Gamma_0, \ t > 0 \\ u_{tt}(x,t) = -\left[\dfrac{\partial u}{\partial \nu}(x,t) + \dfrac{\alpha \partial u_t}{\partial \nu}(x,t) + g(u_t) \right] & x \in \Gamma_1, \ t > 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & x \in \Omega \quad . \end{cases}$$

we can use the same method, provided that the functions f and g satisfy respectively the conditions $(H_3) - (H_7)$ and $(H_8) - (H_9)$ of the paper of Calvacanti et *al.* [8].

3 Exponential growth

In this section we consider the problem (1) and we will prove that when the initial data are large enough (in the energy point of view), the energy grows exponentially and thus so the L^p norm.
In order to state and prove the result, we introduce the following notations. Let B be the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ defined by:

$$B^{-1} = \inf\left\{ \|\nabla u\|_2 : u \in H_0^1(\Omega), \|u\|_p = 1 \right\} \quad .$$

We also define the energy functional:

$$E(u(t)) = E(t) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 . \tag{34}$$

Finally we define the following constant which will play an important role in the proof of our result:

$$\alpha_1 = B^{-p/(p-2)}, \text{ and } d = (\frac{1}{2} - \frac{1}{p})\alpha_1^2. \tag{35}$$

In order to obtain the exponential growth of the energy, we will use the following lemma (see Vitillaro [32], for the proof):

Lemma 3.1 *Let u be a classical solution of (1). Assume that*

$$E(0) < d \text{ and } \|\nabla u_0\|_2 > \alpha_1.$$

Then there exists a constant $\alpha_2 > \alpha_1$ such that

$$\|\nabla u(.,t)\|_2 \geq \alpha_2, \quad \forall t \geq 0, \tag{36}$$

and

$$\|u\|_p \geq B\alpha_2, \quad \forall t \geq 0. \tag{37}$$

Let us now state our new result.

Theorem 3.1 *Assume that $m < p$ where $2 < p \leq \bar{q}$. Suppose that*

$$E(0) < d \text{ and } \|\nabla u_0\|_2 > \alpha_1.$$

Then the solution of problem (1) growths exponentially in the L^p norm.

Proof: By setting

$$H(t) = d - E(t) \tag{38}$$

we get from the definition of the energy (34):

$$0 < H(0) \leq H(t) \leq d - \left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p\right], \tag{39}$$

using the fundamental estimate (36) and the equality (35), we get:

$$d - \frac{1}{2}\|\nabla u\|_2^2 < d - \frac{1}{2}\alpha_1^2 = -\frac{1}{p}\alpha_1^2 < 0, \ \forall t \geq 0.$$

Hence we finally obtain the following inequality:

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_p^p, \quad \forall t \geq 0.$$

For ε small to be chosen later, we then define the auxillary function:

$$L(t) = H(t) + \varepsilon \int_\Omega u_t u dx + \varepsilon \int_{\Gamma_1} u_t u d\sigma + \frac{\varepsilon\alpha}{2}\|\nabla u\|_2^2. \tag{40}$$

Let us remark that L is a small perturbation of the energy. By taking the time derivative of (40), we obtain:

$$\begin{aligned} \frac{dL(t)}{dt} &= \alpha\|\nabla u_t\|_2^2 + r\|u_t\|_{m,\Gamma_1}^m + \varepsilon\|u_t\|_2^2 + \varepsilon\alpha\int_\Omega \nabla u_t \nabla u dx \\ &\quad + \varepsilon\int_\Omega u_{tt} u dx + \varepsilon\int_{\Gamma_1} u_{tt} u d\sigma + \varepsilon\|u_t\|_{2,\Gamma_1}^2. \end{aligned} \tag{41}$$

Using problem (1), the equation (41) takes the form:

$$\begin{aligned} \frac{dL(t)}{dt} &= \alpha\|\nabla u_t\|_2^2 + r\|u_t\|_{m,\Gamma_1}^m + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 \\ &\quad + \varepsilon\|u\|_p^p + \varepsilon\|u_t\|_{2,\Gamma_1}^2 - \varepsilon r\int_{\Gamma_1} |u_t|^m u_t u(x,t) d\sigma. \end{aligned} \tag{42}$$

To estimate the last term in the right hand side of the previous equality, let $\delta > 0$ be chosen later. Young's inequality leads to:

$$\int_{\Gamma_1} |u_t|^m u_t u(x,t) d\sigma \leq \frac{\delta^m}{m}\|u\|_{m,\Gamma_1}^m + \frac{m-1}{m}\delta^{-m/(m-1)}\|u_t\|_{m,\Gamma_1}^m.$$

This yields by substitution in (42):

$$\begin{aligned} \frac{dL(t)}{dt} &\geq \alpha\|\nabla u_t\|_2^2 + r\|u_t\|_{m,\Gamma_1}^m + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 \\ &\quad + \varepsilon\|u\|_p^p + \varepsilon\|u_t\|_{2,\Gamma_1}^2 - \frac{\varepsilon r}{m}\delta^m\|u\|_{m,\Gamma_1}^m \\ &\quad - \frac{\varepsilon r(m-1)}{m}\delta^{-m/(m-1)}\|u_t\|_{m,\Gamma_1}^m. \end{aligned} \tag{43}$$

Let us recall the inequality concerning the continuity of the trace operator (here and in the sequel, C denotes generic positive constant which may change from line to line):

$$\|u\|_{m,\Gamma_1} \leq C\|u\|_{H^s(\Omega)},$$

which holds for:

$$m \geq 1 \text{ and } 0 < s < 1, s \geq \frac{N}{2} - \frac{N-1}{m} > 0$$

and the interpolation and Poincaré's inequalities (see [20])

$$\begin{aligned} \|u\|_{H^s(\Omega)} &\leq C \|u\|_2^{1-s} \|\nabla u\|_2^s \\ &\leq C \|u\|_p^{1-s} \|\nabla u\|_2^s \end{aligned}$$

Thus, we have the following inequality:

$$\|u\|_{m,\Gamma_1} \leq C \|u\|_p^{1-s} \|\nabla u\|_2^s .$$

If $s < 2/m$, using again Young's inequality, we get:

$$\|u\|_{m,\Gamma_1}^m \leq C \left[\left(\|u\|_p^p \right)^{\frac{m(1-s)\mu}{p}} + \left(\|\nabla u\|_2^2 \right)^{\frac{ms\theta}{2}} \right] \tag{44}$$

for $1/\mu + 1/\theta = 1$. Here we choose $\theta = 2/ms$, to get $\mu = 2/(2 - ms)$. Therefore the previous inequality becomes:

$$\|u\|_{m,\Gamma_1}^m \leq C \left[\left(\|u\|_p^p \right)^{\frac{m(1-s)2}{(2-ms)p}} + \|\nabla u\|_2^2 \right] . \tag{45}$$

Now, choosing s such that:

$$0 < s \leq \frac{2(p-m)}{m(p-2)},$$

we get:

$$\frac{2m(1-s)}{(2-ms)p} \leq 1. \tag{46}$$

Once the inequality (46) is satisfied, we use the classical algebraic inequality:

$$z^\nu \leq (z+1) \leq \left(1 + \frac{1}{\omega}\right)(z+\omega) \ , \quad \forall z \geq 0 \ , \quad 0 < \nu \leq 1 \ , \quad \omega \geq 0,$$

to obtain the following estimate:

$$\begin{aligned} \left(\|u\|_p^p \right)^{\frac{m(1-s)2}{(2-ms)p}} &\leq D \left(\|u\|_p^p + H(0) \right) \\ &\leq D \left(\|u\|_p^p + H(t) \right) \ , \quad \forall t \geq 0 \end{aligned} \tag{47}$$

where we have set $D = 1 + 1/H(0)$. Inserting the estimate (47) into (44) we obtain the following important inequality:

$$\|u\|_{m,\Gamma_1}^m \leq C \left[\|u\|_p^p + \|\nabla u\|_2^2 + H(t) \right].$$

In order to control the term $\|\nabla u\|_2^2$ in equation (43), we preferely use (as $H(t) > 0$), the following estimate:

$$\|u\|_{m,\Gamma_1}^m \leq C \left[\|u\|_p^p + \|\nabla u\|_2^2 + 2H(t) \right].$$

which gives finally:

$$\|u\|_{m,\Gamma_1}^m \leq C \left[2d + \left(1 + \frac{2}{p}\right) \|u\|_p^p - \|u_t\|_2^2 - \|u_t\|_{2,\Gamma_1}^2 \right]. \tag{48}$$

Consequently inserting the inequality (48) in the inequality (43) we have:

$$\begin{aligned} \frac{dL(t)}{dt} \geq\ & \alpha \|\nabla u_t\|_2^2 + \left(r - \frac{\varepsilon\, r\, (m-1)\, \delta^{-m/(m-1)}}{m} \right) \|u_t\|_{m,\Gamma_1}^m \\ & + \varepsilon \left(1 + \frac{r\, C\, \delta^m}{m} \right) \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 \\ & + \varepsilon \left(1 - \left(1 + \frac{2}{p}\right) \frac{r\, C\, \delta^m}{m} \right) \|u\|_p^p + \varepsilon \left(1 + \frac{r\, C\, \delta^m}{m} \right) \|u_t\|_{2,\Gamma_1}^2 \end{aligned} \tag{49}$$

From the inequality (39) we have:

$$-\|\nabla u\|_2^2 \geq 2H(t) - 2d + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 - \frac{2}{p} \|u\|_p^p .$$

Thus inserting it in (49), we get the following inequality:

$$\begin{aligned} \frac{dL(t)}{dt} \geq\ & \alpha \|\nabla u_t\|_2^2 + \left(r - \frac{\varepsilon\, r\, (m-1)\, \delta^{-m/(m-1)}}{m} \right) \|u_t\|_{m,\Gamma_1}^m \\ & + \varepsilon \left(2 + \frac{r\, C\, \delta^m}{m} \right) \|u_t\|_2^2 + \varepsilon \left(2 + \frac{r\, C\, \delta^m}{m} \right) \|u_t\|_{2,\Gamma_1}^2 \\ & + \varepsilon \left(1 - \frac{2\varepsilon}{p} - \left(1 + \frac{2}{p}\right) \frac{r\, C\, \delta^m}{m} \right) \|u\|_p^p \\ & + 2\,\varepsilon \left(H(t) - d \left(1 + \frac{r\, C\, \delta^m}{m} \right) \right) \end{aligned} \tag{50}$$

Finally, using the definition of α_2 and d (see equation (35) and the lemma 3.1), we obtain:

$$\begin{aligned}
\frac{dL(t)}{dt} \geq\; & \alpha \left\|\nabla u_t\right\|_2^2 + \left(r - \frac{\varepsilon\, r\, (m-1)\, \delta^{-m/(m-1)}}{m}\right) \left\|u_t\right\|_{m,\Gamma_1}^m \\
& + \varepsilon \left(2 + \frac{r\, C\, \delta^m}{m}\right) \left\|u_t\right\|_2^2 + \varepsilon \left(2 + \frac{r\, C\, \delta^m}{m}\right) \left\|u_t\right\|_{2,\Gamma_1}^2 \\
& + \varepsilon \left(\underbrace{1 - \frac{2}{p} - 2d\left(B\alpha_2\right)^{-p}}_{:=c_0} - \left[\left(1+\frac{2}{p}\right) + 4d\left(B\alpha_2\right)^{-p}\right] \frac{r\, C\, \delta^m}{m} \right) \|u\|_p^p \\
& + \varepsilon \left(2H(t) + \frac{r\, C\, \delta^m}{m} d)\right).
\end{aligned} \tag{51}$$

Setting $c_0 = 1 - \dfrac{2}{p} - 2d\left(B\alpha_2\right)^{-p}$, we have $c_0 > 0$ since $\alpha_2 > B^{-p/(p-2)}$.

We choose now δ small enough such that:

$$c_0 - \left[\left(1+\frac{2}{p}\right) + 4d\left(B\alpha_2\right)^{-p}\right] \frac{r\, C\, \delta^m}{m} > 0 \quad .$$

Once δ is fixed, we choose ε small enough such that:

$$r - \frac{\varepsilon r\,(m-1)}{m} \delta^{-m/(m-1)} > 0 \text{ and } L(0) > 0 \quad .$$

Therefore, the inequality (51) becomes:

$$\frac{dL(t)}{dt} \geq \varepsilon\eta \left[H(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p + d\right] \quad \text{for some } \eta > 0 \tag{52}$$

Next, it is clear that, by Young's inequality and Poincaré's inequality, we get

$$L(t) \leq \gamma \left[H(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|\nabla u\|_2^2\right] \quad \text{for some } \gamma > 0. \tag{53}$$

Since $H(t) > 0$, we have:

$$\forall\, t > 0\,,\ \frac{1}{2}\|\nabla u\|_2^2 \leq \frac{1}{p}\|u\|_p^p + d \quad .$$

Thus, the inequality (53) becomes:

$$L(t) \leq \zeta \left[H(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p + d\right] \ , \quad \text{for some } \zeta > 0. \tag{54}$$

From the two inequalities (52) and (54), we finally obtain the differential inequality:

$$\frac{dL(t)}{dt} \geq \mu L(t) \ , \quad \text{for some } \mu > 0. \tag{55}$$

Integrating the previous differential inequality (55) between 0 and t gives the following estimate for the function L:

$$L(t) \geq L(0)\, e^{\mu t}. \tag{56}$$

On the other hand, from the definition of the function L (and for small values of the parameter ε), it follows that:

$$L(t) \leq \frac{1}{p} \|u\|_p^p . \tag{57}$$

From the two inequalities (56) and (57) we conclude the exponential growth of the solution in the L^p-norm.

Remark 3.1 We recall here that the condition $\int_\Omega u_0(x)u_1(x)dx \geq 0$ appeared in [14, Theorem 3.12] is unecessary to our result on the exponential growth.

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