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DANIEL J. BATES, FRÉDÉRIC BIHAN, AND FRANK SOTTILE

Abstract. We use Gale duality for complete intersections and adapt the proof of the fewnomial bound for positive solutions to obtain the bound

$$\frac{e^4+3}{4}2^{\binom{k}{2}}n^k$$

for the number of non-zero real solutions to a system of n polynomials in n variables having $n+k+1$ monomials whose exponent vectors generate a subgroup of $\mathbb{Z}^n$ of odd index. This bound only exceeds the bound for positive solutions by the constant factor $(e^4+3)/(e^2+3)$ and it is asymptotically sharp for k fixed and n large.

Introduction

In [3], the sharp bound of $2n+1$ was obtained for the number of non-zero real solutions to a system of n polynomial equations in n variables having $n+2$ monomials whose exponents affinely span the lattice $\mathbb{Z}^n$. In [4], the sharp bound of $n+1$ was given for the positive solutions to such a system of equations. This last bound was generalized in [7], which showed that the number of positive solutions to a system of n polynomial equations in n variables having $n+k+1$ monomials was less than

$$\frac{e^2+3}{4}2^{\binom{k}{2}}n^k\,,$$

which is asymptotically sharp for k fixed and n large [5]. This dramatically improved Khovanskii's fewnomial bound [8] of $2^{\binom{n+k}{2}}(n+1)^{n+k}$.

We give a bound for all non-zero real solutions. Under the assumption that the exponent vectors $\mathcal{W}$ span a subgroup of $\mathbb{Z}^n$ of odd index, we show that the number of non-degenerate non-zero real solutions to a system of polynomials with support $\mathcal{W}$ is less than

$$\frac{e^4+3}{4}2^{\binom{k}{2}}n^k\ . \tag{1}$$

The novelty is that this bound exceeds the bound for solutions in the positive orthant by a fixed constant factor $(e^4+3)/(e^2+3)$, rather than by a factor of 2^n, which is the number of orthants. By the construction in [5], it is asymptotically sharp for k fixed and n large.

We follow the outline of [7]—we use Gale duality for real complete intersections [6] and then bound the number of solutions to the dual system of master functions. The

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key idea is that including solutions in all chambers in a complement of an arrangement of hyperplanes in $\mathbb{RP}^k$, rather than in just one chamber as in [7], does not increase our estimate on the number of solutions very much. This was discovered while implementing a numerical continuation algorithm for computing the positive solutions to a system of polynomials [1]. That algorithm was improved by this discovery to one which finds all real solutions. It does so without computing complex solutions and is based on [7] and the results of this paper. Its complexity depends on (1), and not on the number of complex solutions.

We state our main theorem in Section 1 and then use Gale duality to reduce it to a statement about systems of master functions, which we prove in Section 2.

1. Gale duality for systems of sparse polynomials

Let $\mathcal{W} = \{w_0 = 0, w_1, \ldots, w_{n+k}\} \subset \mathbb{Z}^n$ be a collection of $n+k+1$ integer vectors ($|\mathcal{W}| = n+k+1$), which correspond to monomials in variables $x_1, \ldots, x_n$. A (Laurent) polynomial f with *support* $\mathcal{W}$ is a real linear combination of monomials with exponents from $\mathcal{W}$,

$$f(x_1, \ldots, x_n) \;=\; \sum_{i=0}^{n+k} c_i x^{w_i} \qquad \text{with } c_i \in \mathbb{R}\,. \tag{2}$$

A *system with support* $\mathcal{W}$ is a system of polynomial equations

$$f_1(x_1, \ldots, x_n) \;=\; f_2(x_1, \ldots, x_n) \;=\; \cdots \;=\; f_n(x_1, \ldots, x_n) \;=\; 0\,, \tag{3}$$

where each polynomial f_i has support $\mathcal{W}$. Since multiplying every polynomial in (3) by a monomial x^α does not change the set of non-zero solutions but translates $\mathcal{W}$ by the vector α, we see that it was no loss of generality to assume that $0 \in \mathcal{W}$.

The system (3) has infinitely many solutions if $\mathcal{W}$ does not span $\mathbb{R}^n$. We say that $\mathcal{W}$ *spans* $\mathbb{Z}^n$ *mod* 2 if the $\mathbb{Z}$-linear span of $\mathcal{W}$ is a subgroup of $\mathbb{Z}^n$ of odd index.

Theorem 1. *Suppose that $\mathcal{W}$ spans $\mathbb{Z}^n$ mod 2 and $|\mathcal{W}| = n+k+1$. Then there are fewer than* (1) *non-degenerate non-zero real solutions to a sparse system* (3) *with support $\mathcal{W}$.*

The importance of this bound for the number of real solutions is that it has a completely different character than Kouchnirenko's bound for the number of complex solutions.

Proposition 2 (Kouchnirenko [2])**.** *The number of non-degenerate solutions in $(\mathbb{C}^\times)^n$ to a system* (3) *with support $\mathcal{W}$ is no more than $n!\mathrm{vol}(\mathrm{conv}(\mathcal{W}))$.*

Here, $\mathrm{vol}(\mathrm{conv}(\mathcal{W}))$ is the Euclidean volume of the convex hull of $\mathcal{W}$.

Perturbing coefficients of the polynomials in (3) so that they define a complete intersection in $(\mathbb{C}^\times)^n$ can only increase the number of non-degenerate solutions. Thus it suffices to prove Theorem 1 under this assumption. Such a complete intersection is equivalent to a complete intersection of master functions in a hyperplane complement [6].

Let $\mathbb{R}^{n+k}$ have coordinates $z_1, \ldots, z_{n+k}$. A polynomial (2) with support $\mathcal{W}$ is the pull-back $\Phi^*_{\mathcal{W}}(\Lambda)$ of the degree 1 polynomial $\Lambda := c_0 + c_1 z_1 + \cdots + c_{n+k} z_{n+k}$ along the map

$$\Phi_{\mathcal{W}} \;:\; (\mathbb{R}^\times)^n \ni x \;\longmapsto\; (x^{w_i} \mid i = 1, \ldots, n+k) \in \mathbb{R}^{n+k}\,.$$

If we let $\Lambda_1, \ldots, \Lambda_n$ be the degree 1 polynomials which pull back to the polynomials in the system (3), then they cut out an affine subspace L of $\mathbb{R}^{n+k}$ of dimension k.

Let $\{p_i \mid i=1,\ldots,n{+}k\}$ be degree 1 polynomials on $\mathbb{R}^k$ which induce an isomorphism between $\mathbb{R}^k$ and L,

$$\Psi_p \;:\; \mathbb{R}^k \ni y \;\longmapsto\; (p_1(y),\ldots,p_{n+k}(y)) \in L \subset \mathbb{R}^{n+k}\,.$$

Let $\mathcal{A}\subset\mathbb{R}^k$ be the arrangement of hyperplanes defined by the vanishing of the $p_i(y)$. This is the pullback along Ψ_p of the coordinate hyperplanes of $\mathbb{R}^{n+k}$.

The image $\Phi_{\mathcal{W}}((\mathbb{R}^\times)^n)$ inside of the torus $(\mathbb{R}^\times)^{n+k}$ has equations

$$z^{\beta_1} \;=\; z^{\beta_2} \;=\; \cdots \;=\; z^{\beta_k} \;=\; 1\,,$$

where the *weights* $\{\beta_1,\ldots,\beta_k\}$ form a basis for the $\mathbb{Z}$-submodule of $\mathbb{Z}^{n+k}$ of linear relations among the vectors $\mathcal{W}$. To these data, we associate a system of *master functions* on the complement $M_{\mathcal{A}}$ of the arrangement $\mathcal{A}$ of $\mathbb{R}^k$,

$$p(y)^{\beta_1} \;=\; p(y)^{\beta_2} \;=\; \cdots \;=\; p(y)^{\beta_k} \;=\; 1\,. \tag{4}$$

Here, if $\beta=(b_1,\ldots,b_{n+k})$ then $p^\beta := p_1(y)^{b_1}\cdots p_{n+k}(y)^{b_{n+k}}$.

A basic result of [6] is that if $\mathcal{W}$ spans $\mathbb{Z}^n$ modulo 2 and either of the systems (3) or (4) defines a complete intersection, then the other defines a complete intersection and the maps $\Phi_{\mathcal{W}}$ and Ψ_p induce isomorphisms between the two solution sets, as analytic subschemes of $(\mathbb{R}^\times)^n$ and $M_{\mathcal{A}}$. Since we assumed that the system (3) is general, these hypotheses hold and the arrangement is *essential* in that the polynomials p_i span the space of all degree 1 polynomials on $\mathbb{R}^k$.

Theorem 3. *A system* (4) *of master functions in the complement of an essential arrangement of $n{+}k$ hyperplanes in $\mathbb{R}^k$ has at most* (1) *non-degenerate real solutions.*

We actually prove a bound for a more general system than (4), namely for

$$p(z)^{2\beta_1} \;=\; p(z)^{2\beta_2} \;=\; \cdots \;=\; p(z)^{2\beta_k} \;=\; 1\,.$$

We write this more general system as

$$|p(z)|^{\beta_1} \;=\; |p(z)|^{\beta_2} \;=\; \cdots \;=\; |p(z)|^{\beta_k} \;=\; 1\,. \tag{5}$$

In a system of this form we may have real number weights $\beta_i\in\mathbb{R}^{n+k}$. We give the strongest form of our theorem.

Theorem 4. *A system of the form* (5) *with real weights β_i in the complement of an essential arrangement of $n{+}k$ hyperplanes in $\mathbb{R}^k$ has at most* (1) *non-degenerate real solutions.*

2. Proof of Theorem 4

We follow [7] with minor, but important, modifications. Perturbing the polynomials $p_i(y)$ and the weights β_j will not decrease the number of non-degenerate real solutions in $M_{\mathcal{A}}$. This enables us to make the following assumptions.

The arrangement $\mathcal{A}^+\subset\mathbb{RP}^k$, where we add the hyperplane at infinity, is general in that every j hyperplanes of $\mathcal{A}^+$ meet in a $(k{-}j)$ dimensional linear subspace, called a *codimension j face of $\mathcal{A}$*. If B is the matrix whose columns are the weights $\beta_1,\ldots,\beta_k$, then the entries of B are rational numbers and no minor of B vanishes. This last technical condition as well as the freedom to further perturb the β_j and the p_i are necessary for the results in [7, Section 3] upon which we rely.

For functions $f_1,\dots,f_j$ on $M_{\mathcal{A}}$, let $V(f_1,\dots,f_j)$ be the subvariety they define. Suppose that $\beta_j=(b_{1,j},\dots,b_{n+k,j})$. For each $j=1,\dots,k$, define

$$\psi_j(y) \ := \ \sum_{i=1}^{n+k} b_{i,j}\log|p_i(y)|\,.$$

Then (5) is equivalent to $\psi_1(y)=\cdots=\psi_k(y)=0$. Inductively define $\Gamma_k,\Gamma_{k-1},\dots,\Gamma_1$ by

$$\Gamma_j \ := \ \mathrm{Jac}(\psi_1,\dots,\psi_j,\ \Gamma_{j+1},\dots,\Gamma_k)\,,$$

the Jacobian determinant of $\psi_1,\dots,\psi_j,\Gamma_{j+1},\dots,\Gamma_k$. Set

$$C_j \ := \ V(\psi_1,\dots,\psi_{j-1},\ \Gamma_{j+1},\dots,\Gamma_k)\,,$$

which is a curve in $M_{\mathcal{A}}$.

Let $\flat(C)$ be the number of unbounded components of a curve $C\subset M_{\mathcal{A}}$. We have the estimate from [7], which is a consequence of the Khovanskii-Rolle Theorem,

$$|V(\psi_1,\dots,\psi_k)| \ \leq \ \flat(C_k)+\cdots+\flat(C_1) \ + \ |V(\Gamma_1,\dots,\Gamma_k)|\,. \tag{6}$$

Here, $|S|$ is the cardinalty of the set S. We estimate these quantities.

Lemma 5.

(1) $|V(\Gamma_1,\dots,\Gamma_k)|\leq 2^{\binom{k}{2}}n^k$.

(2) *C_j is a smooth curve and*

$$\flat(C_j) \ \leq \ \frac{1}{2}2^{\binom{k-j}{2}}n^{k-j}\tbinom{n+k+1}{j}\cdot 2^j \ \leq \ \frac{1}{2}2^{\binom{k}{2}}n^k\cdot\frac{2^{2j-1}}{j!}\,.$$

Proof of Theorem 4. By (6) and Lemma 5, we have

$$|V(\psi_1,\dots,\psi_k)| \ \leq \ 2^{\binom{k}{2}}n^k\Bigl(1+\frac{1}{4}\sum_{j=1}^{k}\frac{4^j}{j!}\Bigr) \ < \ 2^{\binom{k}{2}}n^k\cdot\frac{e^4+3}{4}\,. \qquad \square$$

Proof of Lemma 5. The bound (1) is from Lemma 3.4 of [7]. Statements analogous to (2) for $\widetilde{C}_j$, the restriction of C_j to a single chamber (connected component) of $M_{\mathcal{A}}$, were established in Lemma 3.4 and the proof of Lemma 3.5 in [7]:

$$\flat(\widetilde{C}_j) \ \leq \ \frac{1}{2}2^{\binom{k-j}{2}}n^{k-j}\tbinom{n+k+1}{j} \ \leq \ \frac{1}{2}2^{\binom{k}{2}}n^k\cdot\frac{2^{j-1}}{j!}\,. \tag{7}$$

The bound we claim for $\flat(C_j)$ has an extra factor of 2^j. *A priori* we would expect to multiply this bound (7) by the number of chambers of $M_{\mathcal{A}}$ to obtain a bound for $\flat(C_j)$, but the correct factor is only 2^j.

We work in $\mathbb{RP}^k$ and use the extended hyperplane arrangement $\mathcal{A}^+$, as we will need points in the closure of C_j in $\mathbb{RP}^k$. The first inequality in (7) for $\flat(\widetilde{C}_j)$ arises as each unbounded component of $\widetilde{C}_j$ meets $\mathcal{A}^+$ in two distinct points (this accounts for the factor $\frac{1}{2}$) which are points of codimension j faces where the polynomials

$$F_i(y) \ := \ \Gamma_{k-i}(y) \cdot \Big(\prod_{i=1}^{n+k} p_i(y)\Big)^{2^i}$$

for $i = 0, \ldots, k-j-1$ vanish. (By Lemma 3.4(1) of [7], F_i is a polynomial of degree $2^i n$.) The genericity of the weights and the linear polynomials $p_i(y)$ imply that these points will lie on faces of codimension j but not of higher codimension. The factor $2^{\binom{k-j}{2}} n^{k-j}$ is the Bézout number of the system $F_0 = \cdots = F_{k-j-1}$ on a given codimension j plane, and there are exactly $\binom{n+k+1}{j}$ codimension j faces of $\mathcal{A}^+$.

At each of these points, C_j will have one branch in each chamber of $M_{\mathcal{A}}$ incident on that point. Since the hyperplane arrangement $\mathcal{A}^+$ is general there will be exactly 2^j such chambers. $\square$

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