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Instability results related to compressible Korteweg system

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Abstract This paper presents the study of surface tension effects in compressible mixtures in the framework of diffuse interface models. In the first part, we describe results previously obtained on the so-called compressible Korteweg and shallow water models and we present nonlinear stability using energy estimates and a new entropy equality recently discovered. These diffuse interface models also allow to take account of capillarity effects in turbulent mixtures and plasma flows subject to Rayleigh–Taylor instabilities. The aim of the last part is to study the influence of surface tension on this instability phenomena. More precisely we look at the expression of the growth rate under a small perturbation of wave number k. We prove that for an appropriate choice of the capillary number σ in terms on the surface tension coefficient T_s (that means

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particular pressure laws), we find the same expression as for the two incompressible fluids model with surface tension coefficient on a sharp interface studied for instance by Chandrasekhar (Hydrodynamic and hydromagnetic stability. Dover Publications, Inc. New York, 1981).

Keywords Surface tension effects · Rayleigh–Taylor · Korteweg models · Instabilities · Compressible flows

Mathematics Subject Classification (2000) 35Q30

1 Introduction

In various applications, hydrodynamic instabilities can be observed at the interface between different materials. A refined description of the mixture dynamics by numerical codes is necessary in order to predict and reproduce experiments [15]. In previous papers, we analyzed the stability and well posedness properties of diffuse interface models used to catch the effect of surface tension in a transition zone of finite extension: Korteweg and Shallow water type models, see [7,8].

In order to describe the zone separating two fluids of different properties, various points of view may be adopted:

- A microscopic viewpoint, in which a transition zone of finite extension exists between the two fluids, where the gradient of physical variables are large. Diffusion effect at the molecular level has to be considered.
- A mesoscopic viewpoint, in which the fluids are separated by a zero thickness layer, called "interface". Most of the physics in the layer is contained in suitable boundary conditions.
- A macroscopic viewpoint, where only large scale effects are represented in a transition zone (diffuse interface) containing simultaneously the two fluids.

The instabilities are made of a combination of three basic type instabilities: Kelvin–Helmholtz (induced by shear stress), Richtmyer–Meshkhov (induced by a shock at the interface), and Rayleigh–Taylor (which appears when the gravity and the density gradient are in the opposite sense).

We will describe in this paper a surface tension model published in other physical papers in the context of compressible turbulent mixtures [15] and we will give various mathematical properties. Such a model corresponds to the third description of free boundary interface problem, see for instance [1]. In a first part, we will explain the results obtained in two recent papers regarding the well posedness and energetical consistency of the model. In the second part, we will establish some properties concerning the influence of surface tension on some instabilities phenomena.

These modeling approach of surface tension, which includes a third order derivative term with respect to the density, has good properties in some applications in liquid water-steam mixtures (for instance with respect to the "sharp interface" limit), but has not been studied in the presence of strong amplitude shocks.

We analyze here the influence of the surface tension term on the growth rate of instabilities. We prove that until the first order expansion with respect to the wave

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number, surface tension does not appear in the asymptotic expansion. We follow the lines of the paper [12] where a similar problem has been addressed without surface tension effects. We formally generalize then the Rayleigh equation to the capillary case and establish an asymptotic expansion of the eigenvalue and the eigenvector. Then we put emphasis on the importance of the diffusive term when surface tension is taken into account. We obtain the linear stability and the nonlinear stability for some range regarding surface tension and some other hypothesis. Let us note some experiments in microgravity, where viscosity and surface tension are present, cf. [23,24]. In [23,24], Rayleigh–Taylor instabilities are investigated in the case of two fluids with finite thickness including the effects of viscosity and surface tension terms. The system consists in two horizontal layers of inhomogeneous incompressible fluids of thickness t_1 and t_2 with surface tension T_s at the interface, under the influence of a gravity field of amplitude g, directed from the heavy fluid of density ρ_2 to the light fluid of density ρ_1. See also [22]. A small perturbation of wave number k at the two fluid interface increases exponentially in time in the linear regime with a growth rate γ given by

$$\frac{\gamma^2}{gk} = \frac{\rho_2 - \rho_1 - k^2 T_s/g}{\rho_2 \coth(kt_2) + \rho_1 \coth(kt_1)}.$$

Remark that letting t_1 and t_2, respectively go to $-\infty$, $+\infty$, we get the standard expression that we can find for instance in [10]

$$\frac{\gamma^2}{gk} = \frac{\rho_2 - \rho_1 - k^2 T_s/g}{\rho_2 + \rho_1} = A - \frac{T_s}{g(\rho_2 + \rho_1)} k^2 \tag{1.1}$$

where A is called the Atwood number. As we shall see, it turns out that in case of the Korteweg model, the influence of surface tension on the growth rate γ arises at the same order as in (1.1). This kind of result where surface tension is found at order 3 in k has been found too in [9] in the framework of Richtmyer–Meshkov instabilities at the interface between two incompressible viscous fluids with surface tension. Readers interested by mathematical problems for miscible incompressible fluids with Korteweg stresses is referred to [16]. For hydrodynamical stability results see is [10,20] for justified mathematical results regarding asymptotic methods for the Rayleigh equation for the linearized Rayleigh–Taylor instability.

2 The Korteweg compressible model

In previous mathematical papers, see [7,8], we have established some mathematical properties of plasma junction models very similar to Korteweg type models.

The aim of the two preceding papers was to look at the well posedness of diffuse interface models such as the Korteweg model. The basic hypothesis derived from the mean field theory, is that the volumic free energy F of the system depends not only on the temperature θ and density ρ, but also on its gradient $\nabla\rho$, in a quadratic manner

$$F(\rho, \nabla\rho, \theta) = F_0(\rho, \theta) + \frac{\sigma}{2}|\nabla\rho|^2,$$

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where F_0 corresponds to the free energy per unit volume of the homogeneous material, and σ is the capillarity coefficient of the system.

The thermodynamic and conservation principles allow then to deduce the following model from the expression of F:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0,$$
$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \operatorname{div}(S + K) + \rho \mathbf{f},$$
$$\partial_t\Big(\rho(e + |\mathbf{u}|^2/2) + \frac{\sigma}{2}|\nabla\rho|^2\Big) + \operatorname{div}\big(\rho \mathbf{u}(e + |\mathbf{u}|^2/2)\big)$$
$$= \operatorname{div}(\alpha\nabla\theta) + \operatorname{div}\big((S + K)\cdot \mathbf{u}\big) + \rho \mathbf{f}\cdot \mathbf{u},$$

where $\mathbf{u}$ and ρ respectively denote the velocity and density of the fluid, e the specific internal energy, θ is the temperature, S the stress tensor, K the capillary tensor and $\mathbf{f}$ the external bulk forces. The stress tensor S is given by

$$S_{ij} = \big(\lambda \operatorname{div}\mathbf{u} - P(\rho,\theta)\big)\delta_{ij} + 2\mu D_{ij}(\mathbf{u}),$$

with μ and λ the viscosities, $D(\mathbf{u})$ the strain tensor and P the pressure; the capillary tensor K is expressed as follows

$$K_{ij} = \frac{\sigma}{2}(\Delta\rho^2 - |\nabla\rho|^2)\delta_{ij} - \sigma\partial_i\rho\partial_j\rho.$$

When a barotropic assumption can be made (for instance in the isothermal or in the isentropic case), then the Korteweg model, in absence of forces, reads as

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{2.1}$$
$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - 2\nu \operatorname{div}(\rho D(\mathbf{u})) - \sigma\rho\nabla\Delta\rho + \nabla P(\rho) = 0. \tag{2.2}$$

In the previous work [7], we proved the existence for all times of weak solutions for the above model in the case of barotropic equation of state, i.e. the pressure P only depends on the density ρ. This corresponds to a global in time stability result with respect to perturbations of the initial data $(\rho_0, \rho_0\mathbf{u}_0)$. This stability result assumes that the viscosity μ is a linear function of the density ρ: $\mu = \nu\rho$ (for some positive constant ν). Even though the parabolic system obtained on the velocity u degenerates when ρ tends to 0, this viscous model allows to get some extra conservation law on a velocity $\mathbf{v}$ characterizing the heterogeneities $\mathbf{v} = \nu\nabla\log\rho$, that means the space variability of the density.

In the article [8], we studied the viscous shallow water model, which is obtained from the incompressible Navier–Stokes model with free surface in presence of surface tension, in the limit of large wavelengths. The shallow water model captures at large scale the effects of surface tension, which writes as a tensor of the form (1).

This study showed the crucial importance of drag forces on the stability properties. Drag forces, in the Stokes regime, (proportional to $\mathbf{u}$), or in the Newton-turbulent-regime (proportional to $\mathbf{u}|\mathbf{u}|$), allow to control the oscillations of the solutions when the density gets close to zero.

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The reader interested by recent mathematical results on the homogeneous incompressible Navier–Stokes equations with free surface is referred to [13] and to [18] for inhomogeneous flows. See also [15] for results on the retraction of viscous films in one dimension in space.

3 Stability using energy estimates with surface tension and viscosity

3.1 Linear stability

We prove that the system (2.1)–(2.2) is linearly stable around a constant reference state

$$(\rho_{ref}, u_{ref}) = (\bar{\rho}, 0),$$

provided some condition involving the pressure law and the surface tension is satisfied. For simplicity, we take $\lambda = 0$. The space domain Ω is assumed to be a periodic box $(0, 2\pi L)^d$.

Linearizing around the constant state $(\bar{\rho}, 0)$ $(\bar{\rho} > 0)$, the density and velocity perturbations are still denoted $(\rho, \mathbf{u})$. Using Laplace transform in time, and denoting α the time coefficient, we get

$$\alpha\rho + \bar{\rho}\,\mathrm{div}\mathbf{u} = 0, \tag{3.1}$$

$$\alpha\mathbf{u} - 2\nu\,\mathrm{div}D(\mathbf{u}) - \sigma\nabla\Delta\rho + \frac{P'(\bar{\rho})}{\bar{\rho}}\nabla\rho = 0. \tag{3.2}$$

Then we prove that we get linear stability for σ large enough, more precisely, if we assume $P'(\bar{\rho})L^2 \geq -\bar{\rho}\sigma$.

Let us multiply (3.1) by the conjugate ρ^* of ρ. We get

$$\alpha\int_\Omega |\rho|^2 + \bar{\rho}\int_\Omega \rho^*\mathrm{div}\mathbf{u} = 0.$$

We multiply now the conjugate of (3.2) by $\mathbf{u}$, we get

$$\alpha\int_\Omega|\mathbf{u}|^2 + 2\nu\int_\Omega|D(\mathbf{u})|^2 + \sigma\int_\Omega\Delta\rho^*\,\mathrm{div}\mathbf{u} - \int_\Omega\frac{P'(\bar{\rho})}{\bar{\rho}}\rho^*\mathrm{div}\mathbf{u} = 0.$$

Multiplying now Eq. (3.1) by $\Delta\rho^*$, this gives

$$-\alpha\int_\Omega |\nabla\rho|^2 + \bar{\rho}\int_\Omega \mathrm{div}\mathbf{u}\,\Delta\rho^* = 0.$$

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The three previous equalities give

$$\alpha\int_\Omega |\mathbf{u}|^2 + 2\nu\int_\Omega |D(\mathbf{u})|^2 + \alpha\frac{\sigma}{\bar\rho}\int_\Omega |\nabla\rho|^2 + \alpha\frac{P'(\bar\rho)}{\bar\rho^2}\int_\Omega |\rho|^2 = 0.$$

Then we have

$$\alpha = \frac{-\nu\int_\Omega |\nabla\mathbf{u}|^2 - \nu\int_\Omega |\mathrm{div}\mathbf{u}|^2}{\int_\Omega |\mathbf{u}|^2 + \frac{\sigma}{\bar\rho}\int_\Omega |\nabla\rho|^2 + \frac{P'(\bar\rho)}{\bar\rho^2}\int_\Omega |\rho|^2}.$$

Using the Poincare–Wirtinger Inequality (note that $\int_\Omega \rho = 0$ and $\int_\Omega \mathbf{u} = 0$), we get the linear stability if

$$\frac{P'(\bar\rho)L^2}{\bar\rho\sigma} \geq -1.$$

In other words, we remark that in the case where $P(\rho) = \bar P(\rho/\bar\rho)^\delta$, $\delta \in \mathbb{R}$, we get the linear stability condition $\sigma \geq -\delta L^2\bar P/\bar\rho^2$. Remark that pressure may satisfy such constraints, see for instance [2].

3.2 Nonlinear stability

We will prove in this part that the presence of viscosity and surface tension allow to obtain the exponential stability if ρ is assumed to be uniformly bounded from below and from above.

We begin by a classical monotone stability result.

3.2.1 Monotone stability

Using the direct energy inequality, we get the monotonic stability without any hypothesis on the data, assuming $\sigma > 0$. Indeed

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \Pi(\rho) + \frac{\sigma}{2}|\nabla\rho|^2\right) \leq -\int_\Omega \nu\rho|D(\mathbf{u})|^2$$

where

$$\Pi(s) = s\int_0^s \frac{P(\tau)}{\tau^2}d\tau \geq 0.$$

Let us prove that System (2.1)-(2.2) is monotonically stable if $\Pi''(s) \geq -\sigma/L^2$.

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From [7],we also have the following inequality

$$\frac{d}{dt}\int\limits_{\Omega}\Big(\frac{1}{2}\rho|\mathbf{u}|^2+\frac{1}{2}\rho|\mathbf{u}+\nu\nabla\log\rho|^2+2\Pi(\rho)+\sigma|\nabla\rho|^2\Big)$$
$$\leq -\nu\int\limits_{\Omega}\frac{P'(\rho)}{\rho}|\nabla\rho|^2-\nu\sigma\int\limits_{\Omega}|\Delta\rho|^2.$$

We remark that $s\Pi''(s) = P'(s)$ then if we assume $\Pi''(s) \geq -\sigma/L^2$, then the system is monotonically stable for a norm involving a space derivative for ρ.

Let us remark that without surface tension we would have to assume $\Pi''(s) \geq 0$ that means a convex potential. The presence of surface tension allow to consider some transition zones. See [2] for some forms of $P(\rho)$ such as the Van der Waals equation of state.

3.2.2 Exponential stability

We will look at the nonlinear stability around $(\bar{\rho}, 0)$. We prove that if we assume $\nu > 0$, $c_1 \leq \rho \leq c_2$ and $\Pi''(s) > -\sigma/L^2$, then the basic motion is exponentially stable.

We have

$$\frac{d}{dt}\int\limits_{\Omega}\Big(\rho|\mathbf{u}|^2+\frac{1}{2}\rho|\mathbf{u}+\nu\nabla\log\rho|^2+2\Pi(\rho)+\sigma|\nabla\rho|^2\Big)$$
$$\leq -\nu\int\limits_{\Omega}\frac{P'(\rho)}{\rho}|\nabla\rho|^2-\nu\sigma|\nabla\nabla\rho|^2-\nu\int\limits_{\Omega}\rho|\nabla\mathbf{u}|^2.$$

Thus if $0 < c_1 \leq \rho \leq c_2$ and if $\Pi''(s) > -\sigma/L^2$, then we get the exponential stability of the model without restrictions of the size of the data. This allows to look at the nonlinear stability of the model given in [15]. Let us note that the norm

$$\int\limits_{\Omega}\Big(\rho|\mathbf{u}|^2+\frac{1}{2}\rho|\mathbf{u}+\nu\nabla\log\rho|^2+2\Pi(\rho)+\sigma|\nabla\rho|^2\Big)$$

is equivalent to the norm

$$\int\limits_{\Omega}\Big(|\mathbf{u}|^2+|\rho|^2+|\nabla\rho|^2\Big)$$

if ρ is assumed to be uniformly bounded from above and from below. The reader interested in nonlinear stability of the rest state as basic solution to the full incompressible nonlinear Korteweg model is referred to [17].

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4 Rayleigh–Taylor stability

In this part, we study the influence of the surface tension coefficient on the growth rate of Rayleigh–Taylor instabilities. The gravity field $\mathbf{g}$ is assumed to be constant and directed along the z coordinate $\mathbf{g} = (0, 0, -g)$ for some positive acceleration g. Again, we restrict to the case of barotropic equations of state for simplicity. We consider an inviscid model and we show that the effect of surface tension may be seen only at the order 3 with respect to the wave number k. This result is similar to the one obtained in [24] on a superposition of two fluids with different densities. In addition, we prove that in the presence of viscosity, an exponential stability result can be obtained under the assumption of lower and upper bounds for the density.

4.1 Linear instability result

In this part, we will study the effect of the presence of surface tension term on the instability growth rate of Rayleigh–Taylor type. More precisely, looking at perturbations around $(0, p^0, \rho^0)$ (to be specified later on) under the form

$$\varphi(x, z, t) = \varphi(z)\exp(ikx + \gamma t), \quad \varphi = \rho, u, w, p,$$

we prove that the growth rate γ satisfies the following expansion

$$\frac{gk}{\gamma^2} \approx \lambda_0 + k\lambda_1 + k^2\lambda_2,$$

where λ_0, λ_1 and λ_2 are given by

$$\lambda_0 = \frac{\rho_D^0 + \rho_U^0}{\rho_U^0 - \rho_D^0} = A^{-1}.$$

$$\lambda_1 = \frac{1 - A^2}{2A^3} \int_{-\infty}^{\infty} \frac{A^2 - (\rho^0 - 1)^2}{\rho^0} dz.$$

$$\lambda_2 - \lambda_2(\sigma = 0)$$

$$= \frac{\widetilde{\sigma}\lambda_0}{2A}\left[\begin{array}{l} (\lambda_0 + 1)\displaystyle\int_0^{+\infty} \left|\frac{d\rho^0}{dz}\right|^2 dz + \lambda_0(\lambda_0 - 1)\displaystyle\int_0^{+\infty} \left|\frac{d\rho^0}{dz}\right|^2 \frac{(\rho^0 - (1 + A))}{\rho^0} dz \\ -(\lambda_0 - 1)\displaystyle\int_{-\infty}^{0} \left|\frac{d\rho^0}{dz}\right|^2 dz + \lambda_0(\lambda_0 + 1)\displaystyle\int_{-\infty}^{0} \left|\frac{d\rho^0}{dz}\right|^2 \frac{(\rho^0 - (1 - A))}{\rho^0} dz \end{array}\right]$$

$$+ \frac{\widetilde{\sigma}\lambda_0(\lambda_0^2 - 1)}{2}\left[\begin{array}{l} \displaystyle\int_0^{+\infty} \left|\frac{d\rho^0}{dz}\right|^2 \frac{(\rho^0 - (1 + A))}{(\rho^0)^2}(1 - \lambda_0) dz - \displaystyle\int_0^{+\infty} \left|\frac{d\rho^0}{dz}\right|^2 \frac{1}{\rho^0} dz \\ - \displaystyle\int_{-\infty}^{0} \left|\frac{d\rho^0}{dz}\right|^2 \frac{(\rho^0 - (1 - A))}{(\rho^0)^2}(\lambda_0 + 1) dz + \displaystyle\int_{-\infty}^{0} \left|\frac{d\rho^0}{dz}\right|^2 \frac{1}{\rho^0} dz \end{array}\right].$$

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Remark that since we are interested in the surface tension coefficient on the growth rate, only the terms depending on it are given here for λ_2. The expression of $\lambda_2^{\sigma=0}$ is given later on.

We assume that the density, the velocity $\mathbf{u} = (u, v, w)$ and the pressure p, function of the density ρ satisfy

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0,$$
$$\partial_t (\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \nu \operatorname{div}(\rho \nabla u) - \sigma \rho \nabla \Delta \rho + \nabla p = \rho \mathbf{g}. \tag{4.1}$$

We remark that the diffusive term is a degenerate one as in [8], $(\mu(\rho) = \nu\rho, \lambda(\rho) = 0)$. More general viscosities may be chosen without extra difficulties. Let us consider a hydrostatic profile ρ^0, p^0 associated with $\mathbf{u}^0 \equiv 0$ that means a couple (p^0, ρ^0) such that

$$\nabla p^0 = \sigma \rho^0 \nabla \Delta \rho^0 + \rho^0 \mathbf{g}, \tag{4.2}$$

which writes as an ordinary differential equation on ρ^0 in z assuming barotropic flows. See for instance [2] for such density profiles that means for corresponding pressure laws: Van-der-Waals type laws for instance. This relation is linked to the Maxwell equilibrium points. We consider incompressible perturbations of the basic flow $(0, p^0, \rho^0)$. Let us note that the study of weak stability associated with System (4.1) has been achieved in [2,3] for $u_0 \neq 0$. Extensions of our results to nonvanishing initial velocity profile and/or compressible perturbation could be an interesting open problem. Here we consider a 2D incompressible perturbation.

4.2 Proof of growth rate ansatz

The perturbed density ρ^1, the velocity $\mathbf{u}^1 = (u^1, 0, w^1)$ and the pressure p^1 satisfy the following equations

$$\partial_t \rho^1 + \frac{d\rho^0}{dz} w^1 = 0,$$
$$\partial_t u^1 + \frac{1}{\rho^0}\partial_x p^1 = \sigma \partial_x^3 \rho^1 + \sigma \partial_x \partial_z^2 \rho^1 + \nu \partial_x^2 u^1 + \frac{\nu}{\rho^0}\partial_z(\rho^0 \partial_z u^1),$$
$$\partial_t w^1 + \frac{1}{\rho^0}\partial_z p^1 = \sigma \partial_x^2 \partial_z \rho^1 + \sigma \partial_z^3 \rho^1 + \sigma \frac{\rho^1}{\rho^0}\frac{d^3\rho^0}{dz^3} - \frac{\rho^1}{\rho^0} g + \nu \partial_x^2 w^1 + \frac{\nu}{\rho^0}\partial_z(\rho^0 \partial_z w^1),$$
$$\partial_x u^1 + \partial_z w^1 = 0.$$

Let us forget the indices 1 and look for solutions of normal mode type, namely

$$\varphi(x, z, t) = \varphi(z) \exp(ikx + \gamma t), \quad \varphi = \rho, u, w, p,$$

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where the wave number k is considered as a parameter. This gives the following system

$$\gamma \rho + \frac{d\rho^0}{dz} w = 0,$$

$$\gamma u + \frac{ik}{\rho^0} p = -ik^3 \sigma \rho + ik\sigma \frac{d^2 \rho}{dz^2} - \nu k^2 u + \frac{\nu}{\rho^0} \frac{d}{dz}\left(\rho^0 \frac{d}{dz} u\right), \tag{4.3}$$

$$\gamma w + \frac{1}{\rho^0}\frac{dp}{dz} = -k^2\sigma \frac{d\rho}{dz} + \sigma \frac{d^3\rho}{dz^3} + \sigma \frac{\rho}{\rho^0}\frac{d^3\rho^0}{dz^3} - \frac{\rho}{\rho^0} g - \nu k^2 w + \frac{\nu}{\rho^0}\frac{d}{dz}\left(\rho^0 \frac{d}{dz} w\right),$$

$$iku + \frac{dw}{dz} = 0.$$

By following the steps given in [12] that means by rewriting the equation under a non dimensional form and denoting $\varepsilon = k\ell$, it is easy to see that we can write the system as a modified Rayleigh equation.

More precisely, we prove that if ρ, u, w, p is solution of (4.3) then the following Rayleigh equation is satisfied for the vertical component of the velocity

$$\frac{\nu}{\gamma l^2}\frac{d^2}{dz^2}\left(\rho^0 \frac{d^2}{dz^2} w\right) - \frac{d}{dz}\left[\left(\left(1 + \frac{2\nu\varepsilon^2}{\gamma \ell^2}\right)\rho^0 + \frac{\sigma\varepsilon^2\overline{\rho}}{\gamma^2\ell^4}\left|\frac{d\rho^0}{dz}\right|^2\right)\frac{dw}{dz}\right]$$
$$+\varepsilon^2\left(\left(1 + \frac{\nu\varepsilon^2}{\gamma\ell^2}\right)\rho^0 + \frac{\sigma\varepsilon^2\overline{\rho}}{\gamma^2\ell^4}\left|\frac{d\rho^0}{dz}\right|^2\right) w = \frac{\varepsilon^2}{\gamma^2\ell}\frac{d\rho^0}{dz} g w. \tag{4.4}$$

Note that the modified Rayleigh equation, in its dimensional form, may be written in a form similar to Equation (19) in [1] where the following frequency N and velocity M were introduced

$$N^2 = -\frac{g}{\rho^0}\frac{d\rho^0}{dz}, \qquad M^2 = \frac{\sigma}{\rho^0}\left(\frac{d\rho^0}{dz}\right)^2.$$

4.2.1 Asymptotic limit

Let us now assume that $\nu = 0$ and perform the asymptotic analysis when ε goes to 0. We note $\lambda^\varepsilon = \varepsilon g/\gamma^2\ell$. Then, Equation (4.4) rewrites as

$$-\frac{d}{dz}\left[\left(\rho^0 + \frac{\sigma\varepsilon\lambda^\varepsilon\overline{\rho}}{g\ell^3}\left|\frac{d\rho^0}{dz}\right|^2\right)\frac{dw}{dz}\right] + \varepsilon^2\left(\rho^0 + \frac{\sigma\varepsilon\lambda^\varepsilon\overline{\rho}}{g\ell^3}\left|\frac{d\rho^0}{dz}\right|^2\right) w = \varepsilon\,\lambda^\varepsilon \frac{d\rho^0}{dz} w. \tag{4.5}$$

Assume now that the typical size of the interface scales as ε and that the density profile connects two constant states at infinity ($\rho_U/\overline{\rho}$ for positive z and $\rho_D/\overline{\rho}$ for negative z). We note

$$\widetilde{\sigma} = \frac{\sigma\overline{\rho}}{\ell^3 g}.$$

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Let us consider $\rho^0(z) = \widetilde{\rho}^0(z/\varepsilon)$ and $w(z) = \widetilde{w}(z/\varepsilon)$ Then, the above equation reads

$$-\frac{d}{dz}\left[\left(\widetilde{\rho}^0 + \widetilde{\sigma}\varepsilon^3\lambda^\varepsilon\left|\frac{d\widetilde{\rho}^0}{dz}\right|^2\right)\frac{d\widetilde{w}}{dz}\right] + \left(\widetilde{\rho}^0 + \widetilde{\sigma}\varepsilon^3\lambda^\varepsilon\left|\frac{d\widetilde{\rho}^0}{dz}\right|^2\right)\widetilde{w} = \lambda^\varepsilon\frac{d\widetilde{\rho}^0}{dz}\widetilde{w}. \tag{4.6}$$

Taking the sharp interface limit in the weak formulation associated with (4.4) as in [12], we get

$$-\frac{d}{dz}\left(\widetilde{\rho}^0_*\frac{d\widetilde{w}_*}{dz}\right) + \widetilde{\rho}^0_*\widetilde{w}_* - \lambda_0\frac{d\widetilde{\rho}^0_*}{dz}\widetilde{w}_* = 0,$$

where $\widetilde{\rho}^0_* = \rho^0_D/\overline{\rho}$ if $z < 0$ and $\widetilde{\rho}^0_* = \rho^0_U/\overline{\rho}$ elsewhere with $\rho^0_U > \rho^0_D$. This yields the expression on $(-\infty, 0) \cup (0, +\infty)$

$$\widetilde{w}_*(z) = \widetilde{w}_*(0)\exp(-|z|),$$

and

$$\left[\rho^0_U\frac{d\widetilde{w}_*(0^+)}{dz} - \rho^0_D\frac{d\widetilde{w}_*}{dz}(0^-)\right] + \lambda_0(\rho^0_U - \rho^0_D)\widetilde{w}_*(0) = 0,$$

and then, we get the well known expression of λ_0

$$\lambda_0 = \frac{\rho^0_D + \rho^0_U}{\rho^0_U - \rho^0_D} = A^{-1}.$$

4.2.2 Ansatz

In the following we choose the characteristic density scale equal to

$$\overline{\rho} = (\rho^0_U + \rho^0_D)/2,$$

thus the non dimensional density connects two constants states at infinity ($1 + A = \rho^0_U/\overline{\rho}$ for positive z and $1 - A = \rho^0_D/\overline{\rho}$ for negative z). Let us rewrite equation (4.5) in terms of a^ε where

$$w^\varepsilon(z) = a^\varepsilon(z)\exp(-\varepsilon|z|).$$

We get for $z > 0$

$$-\frac{d}{dz}\left[\left(\rho^0 + \widetilde{\sigma}\varepsilon\lambda^\varepsilon\left|\frac{d\rho^0}{dz}\right|^2\right)\frac{da^\varepsilon}{dz}\right] + 2\varepsilon\frac{d}{dz}\left[\left(\rho^0 + \widetilde{\sigma}\varepsilon\lambda^\varepsilon\left|\frac{d\rho^0}{dz}\right|^2\right)a^\varepsilon\right]$$
$$= \varepsilon(\lambda^\varepsilon + 1)\frac{d\rho^0}{dz}a^\varepsilon + \widetilde{\sigma}\lambda^\varepsilon\varepsilon^2\frac{d}{dz}\left(\left|\frac{d\rho^0}{dz}\right|^2\right)a^\varepsilon, \tag{4.7}$$

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and for $z < 0$

$$-\frac{d}{dz}\left[\left(\rho^0 + \widetilde{\sigma}\varepsilon\lambda^\varepsilon\left|\frac{d\rho^0}{dz}\right|^2\right)\frac{da^\varepsilon}{dz}\right] - 2\varepsilon\frac{d}{dz}\left[\left(\rho^0 + \widetilde{\sigma}\varepsilon\lambda^\varepsilon\left|\frac{d\rho^0}{dz}\right|^2\right)a^\varepsilon\right]$$
$$= \varepsilon(\lambda^\varepsilon - 1)\frac{d\rho^0}{dz}a^\varepsilon - \widetilde{\sigma}\lambda^\varepsilon\varepsilon^2\frac{d}{dz}\left(\left|\frac{d\rho^0}{dz}\right|^2\right)a^\varepsilon. \tag{4.8}$$

Then we use a formal asymptotic expansion of the pair $(\lambda^\varepsilon, a^\varepsilon)$ under the form

$$\lambda^\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots,$$
$$a^\varepsilon = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots.$$

and we will prove that

$$\lambda_0 = A^{-1},$$
$$\lambda_1 = \frac{1 - A^2}{2A^3}\int_{-\infty}^{\infty}\frac{A^2 - (\rho^0 - 1)^2}{\rho^0}dz. \tag{4.9}$$

That means that λ_0 and λ_1 do not depend on σ except by ρ^0.

To derive such expressions, we follow the lines given in [12] plugging the Ansatz in (4.7) and (4.8) and identifying the powers. We get, for $z > 0$

$$\frac{d}{dz}\left(\rho^0\frac{da_0}{dz}\right) = 0,$$
$$\frac{d}{dz}\left(\rho^0\frac{da_1}{dz}\right) + \widetilde{\sigma}\lambda_0\frac{d}{dz}\left(\left|\frac{d\rho^0}{dz}\right|^2\frac{da_0}{dz}\right) - 2\frac{d}{dz}(\rho^0 a_0) = -(\lambda_0 + 1)\frac{d\rho^0}{dz}a_0,$$
$$\frac{d}{dz}\left(\rho^0\frac{da_2}{dz}\right) + \widetilde{\sigma}\lambda_1\frac{d}{dz}\left(\left|\frac{d\rho^0}{dz}\right|^2\frac{da_0}{dz}\right) + \widetilde{\sigma}\lambda_0\frac{d}{dz}\left(\left|\frac{d\rho^0}{dz}\right|^2\frac{da_1}{dz}\right) - 2\frac{d}{dz}(\rho^0 a_1)$$
$$-2\widetilde{\sigma}\lambda_0\frac{d}{dz}\left(\left|\frac{d\rho^0}{dz}\right|^2 a_0\right) = -(\lambda_0 + 1)\frac{d\rho^0}{dz}a_1 - \lambda_1\frac{d\rho^0}{dz}a_0 - \widetilde{\sigma}\lambda_0\frac{d}{dz}\left(\left|\frac{d\rho^0}{dz}\right|^2\right)a_0.$$

As in [12], this gives, asking for da_1/dz to tend to zero at $+\infty$

$$a_0(z) = a_{0,U}, \qquad z > 0$$
$$a_1(z) = a_{1,U} + (\lambda_0 - 1)a_{0,U}\int_z^{+\infty}\frac{(\rho^0 - (1 + A))}{\rho^0}dz, \qquad z > 0.$$

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On the lower part, one has similarly

$$a_0(z) = a_{0,D}, \qquad z < 0$$

$$a_1(z) = a_{1,D} - (\lambda_0 + 1)a_{0,D} \int_{-\infty}^{z} \frac{(\rho^0 - (1 - A))}{\rho^0} dz, \qquad z < 0.$$

Let us look at the second order of the Ansatz, that means a_2. We get

$$\frac{d}{dz}\Big(\rho^0 \frac{da_2}{dz}\Big) + \widetilde{\sigma}\lambda_0 \frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2 \frac{da_1}{dz}\Big) - 2\frac{d}{dz}(\rho^0 a_1)$$
$$= -(\lambda_0 + 1)\frac{d\rho^0}{dz} a_1 - \lambda_1 \frac{d\rho^0}{dz} a_0 + \widetilde{\sigma}\lambda_0 \frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big) a_0.$$

By integrating from z to $+\infty$, we obtain

$$-\rho^0 \frac{da_2}{dz} - \widetilde{\sigma}\lambda_0 \Big|\frac{d\rho^0}{dz}\Big|^2 \frac{da_1}{dz} + 2\rho^0 a_1 - 2(1 + A)a_{1,U}$$
$$= -(\lambda_0 + 1) \int_z^{+\infty} \frac{d(\rho^0 a_1)}{dz} dz + (\lambda_0 + 1) \int_z^{+\infty} \rho^0 \frac{da_1}{dz} dz$$
$$-\lambda_1((1 + A) - \rho^0)a_{0,U} - \widetilde{\sigma}\lambda_0 \Big|\frac{d\rho^0}{dz}\Big|^2 a_{0,U}.$$

By using the expression of a_1, this may be written, for $z > 0$:

$$\begin{aligned} -\rho^0 \frac{da_2}{dz} - \widetilde{\sigma}\lambda_0 \Big|\frac{d\rho^0}{dz}\Big|^2 \frac{da_1}{dz} &= -(\lambda_0 - 1)((1 + A)a_{1,U} - \rho^0 a_1) \\ &\quad + (1 - \lambda_0^2) \int_z^{+\infty} a_{0,U}(\rho^0 - (1 + A))dz' \\ &\quad -\lambda_1((1 + A) - \rho^0)a_{0,U} - \widetilde{\sigma}\lambda_0 \Big|\frac{d\rho^0}{dz}\Big|^2 a_{0,U}. \end{aligned} \tag{4.10}$$

At the lower part, that means $z < 0$:

$$\begin{aligned} \rho^0 \frac{da_2}{dz} + \widetilde{\sigma}\lambda_0 \Big|\frac{d\rho^0}{dz}\Big|^2 \frac{da_1}{dz} &= -(\lambda_0 + 1)(\rho^0 a_1 - (1 - A)a_{1,D}) \\ &\quad -(\lambda_0^2 - 1) \int_{-\infty}^{z} a_{0,D}(\rho^0 - (1 - A))dz' \\ &\quad -\lambda_1(\rho^0 - (1 - A))a_{0,D} - \widetilde{\sigma}\lambda_0 \Big|\frac{d\rho^0}{dz}\Big|^2 a_{0,D}. \end{aligned} \tag{4.11}$$

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Now we use the continuity of the normal stress across the interface at order one in ε

$$\frac{da_2}{dz}(0^+) - \frac{da_2}{dz}(0^-) = 2a_1(0),$$

and the continuity of the vertical component of the velocity

$$\begin{aligned}
a_0(0^+) &= a_0(0^-) = a_0(0), \\
a_{1,U} - a_{1,D} &= a_0\Big((-\lambda_0 + 1)\int_0^{+\infty} \frac{(\rho^0 - (1+A))}{\rho^0}\,dz \\
&\quad - (\lambda_0 + 1)\int_{-\infty}^{0} \frac{(\rho^0 - (1-A))}{\rho^0}\,dz\Big)
\end{aligned}$$

By rewriting $\frac{da_2}{dz}(0^+) - \frac{da_2}{dz}(0^-)$, we get, using (4.10) and (4.11),

$$\begin{aligned}
2\rho^0(0)a_1(0) &= \rho^0(0)\Big(\frac{da_2}{dz}(0^+) - \frac{da_2}{dz}(0^-)\Big) \\
&= -(\lambda_0 - 1)(\rho^0(0)a_1(0) - (1+A)a_{1,U}) \\
&\quad + (\lambda_0^2 - 1)a_{0,U}\int_0^{\infty}(\rho^0 - (1+A))\,dz \\
&\quad - \lambda_1 a_{0,U}(\rho^0(0) - (1+A)) + (\lambda_0 + 1)(\rho^0(0)a_1(0) - (1-A)a_{1,D}) \\
&\quad + (\lambda_0^2 - 1)a_{0,D}\int_{-\infty}^{0}(\rho^0 - (1-A))\,dz + \lambda_1 a_{0,D}(\rho^0(0) - (1-A)) \\
&\quad + \frac{\widetilde{\sigma}\lambda_0}{\rho^0}\Big|\frac{d\rho^0}{dz}\Big|^2\Big(\rho^0(0)a_{0,U} - \rho^0\frac{da_1}{dz}|_{z=0^+} + \rho^0(0)a_{0,D} + \rho^0\frac{da_1}{dz}|_{z=0^-}\Big).
\end{aligned} \tag{4.12}$$

As

$$\begin{aligned}
\rho^0\frac{da_1}{dz}|_{z=0^+} &= -(\lambda_0 - 1)a_{0,U}(\rho^0(0) - (1+A)), \\
\rho^0\frac{da_1}{dz}|_{z=0^-} &= -(\lambda_0 + 1)a_{0,D}(\rho^0(0) - (1-A)),
\end{aligned}$$

then the last quantity in terms of σ vanishes using that $a_{0,U} = a_{0,D}$ and $\lambda_0 = A^{-1}$. Replacing a_1 by its expression and using that $\lambda_0 = A^{-1}$, it gives the same expression

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as in [12]. More precisely, we get

$$(1-\lambda_0^2)\Bigg((1-A)\int_0^{+\infty}\frac{(\rho^0-(1+A))}{\rho^0}\,dz+(1+A)\int_{-\infty}^{0}\frac{(\rho^0-(1-A))}{\rho^0}\,dz\Bigg)$$

$$-(1-\lambda_0^2)\Bigg(\int_0^{+\infty}(\rho^0-(1+A))\,dz+\int_{-\infty}^{0}(\rho^0-(1-A))dz\Bigg)+2A\lambda_1=0,$$

and we obtain the expression of λ_1 given by (4.9).

Let us now look at the second order and prove that

$$\begin{aligned}
&\lambda_2-\lambda_2(\sigma=0)\\
&=\frac{\widetilde{\sigma}\lambda_0}{2A}\left[\begin{array}{l}(\lambda_0+1)\displaystyle\int_0^{+\infty}\left|\frac{d\rho^0}{dz}\right|^2dz+\lambda_0\,(\lambda_0-1)\int_0^{+\infty}\left|\frac{d\rho^0}{dz}\right|^2\frac{(\rho^0-(1+A))}{\rho^0}dz\\ -(\lambda_0-1)\displaystyle\int_{-\infty}^{0}\left|\frac{d\rho^0}{dz}\right|^2dz+\lambda_0\,(\lambda_0+1)\int_{-\infty}^{0}\left|\frac{d\rho^0}{dz}\right|^2\frac{(\rho^0-(1-A))}{\rho^0}dz\end{array}\right]\\
&+\frac{\widetilde{\sigma}\lambda_0(\lambda_0^2-1)}{2}\left[\begin{array}{l}\displaystyle\int_0^{+\infty}\left|\frac{d\rho^0}{dz}\right|^2\frac{(\rho^0-(1+A))}{(\rho^0)^2}(1-\lambda_0)dz-\int_0^{+\infty}\left|\frac{d\rho^0}{dz}\right|^2\frac{1}{\rho^0}dz\\ -\displaystyle\int_{-\infty}^{0}\left|\frac{d\rho^0}{dz}\right|^2\frac{(\rho^0-(1-A))}{(\rho^0)^2}(\lambda_0+1)dz+\int_{-\infty}^{0}\left|\frac{d\rho^0}{dz}\right|^2\frac{1}{\rho^0}dz\end{array}\right]
\end{aligned}\tag{4.13}$$

where $\lambda_2(\sigma=0)$ is the expression of λ_2 when $\sigma=0$. That means λ_2 depends now directly of the parameter σ.

To derive such expression, we look at the third order in ε. We have for $z>0$:

$$\begin{aligned}
&-\frac{d}{dz}\Big(\rho^0\frac{da_3}{dz}\Big)-\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_2}{dz}\Big)-\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz}\Big)\\
&-\widetilde{\sigma}\lambda_2\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_0}{dz}\Big)+2\frac{d}{dz}(\rho^0a_2)+2\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2a_0\Big)\\
&+2\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2a_1\Big)=(\lambda_0+1)\frac{d\rho^0}{dz}a_2+\lambda_1\frac{d\rho^0}{dz}a_1+\lambda_2\frac{d\rho^0}{dz}a_0\\
&+\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)a_1+\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)a_0.
\end{aligned}$$

By using now the expression of ρ^0da_2/dz and a_1, we get

$$\begin{aligned}
&\frac{d}{dz}\Big(\rho^0\frac{da_3}{dz}\Big)+\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_2}{dz}\Big)+\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz}\Big)\\
&=-(\lambda_0-1)\frac{d(\rho^0a_2)}{dz}-(\lambda_0+1)\Big[-(\lambda_0-1)((1+A)a_{1,U}-\rho^0a_1)
\end{aligned}$$

$$
\begin{aligned}
&-(\lambda_0^2-1)\int_z^{+\infty} a_{0,U}(\rho^0-(1+A))dz' - \lambda_1((1+A)-\rho^0)a_{0,U}\\
&-\widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2 a_{0,U} + \widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz}\Big] - \lambda_1\frac{d\rho^0}{dz}\Big(a_{1,U}\\
&+(\lambda_0-1)\int_z^{+\infty}\frac{(\rho^0-(1+A))}{\rho^0}a_{0,U}\,dz\Big) - \lambda_2\frac{d\rho^0}{dz}a_0\\
&+\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)\Big(a_{1,U}+(\lambda_0-1)\int_z^{+\infty}\frac{\rho^0-(1+A)}{\rho^0}a_{0,U}\,dz\Big)\\
&+\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)a_{0,U} + 2\widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz}.
\end{aligned}
$$

By integrating from z to $+\infty$, we get

$$
\begin{aligned}
&\rho^0\frac{da_3}{dz} + \widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_2}{dz} + \widetilde{\sigma}\lambda_1\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz} = \lambda_2((1+A)-\rho^0)a_{0,U}\\
&\quad-(1-\lambda_0)((1+A)a_{2,U}-\rho^0a_2) + (1-\lambda_0^2)\int_z^{+\infty}((1+A)a_{1,U}-\rho^0a_1)\\
&\quad-(\lambda_0+1)(\lambda_0^2-1)\int_z^{+\infty}\int_\xi^{+\infty} a_{0,U}(\rho^0-(1+A))\\
&\quad-\lambda_1(\lambda_0+1)\int_z^{+\infty}((1+A)-\rho^0)a_{0,U} - \widetilde{\sigma}\lambda_0(\lambda_0+1)\int_z^{+\infty}\Big|\frac{d\rho^0}{dz}\Big|^2 a_{0,U}\\
&\quad+\widetilde{\sigma}\lambda_0(1-\lambda_0^2)\int_z^{+\infty}\Big|\frac{d\rho^0}{dz}\Big|^2 a_{0,U}\frac{(\rho^0-(1+A))}{\rho^0} - \lambda_1(\rho^0-(1+A))a_{1,U}\\
&\quad+\lambda_1(\lambda_0-1)a_{0,U}\int_z^{+\infty}\frac{d\rho^0}{dz}\int_\xi^{+\infty}\frac{(\rho^0-(1+A))}{\rho_0} + \widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2 a_{1,U}\\
&\quad-\widetilde{\sigma}\lambda_0(\lambda_0-1)a_{0,U}\int_z^{+\infty}\Big[\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)\int_\xi^{+\infty}\frac{(\rho^0-(1+A))}{\rho^0}\Big]\\
&\quad+2\widetilde{\sigma}\lambda_0(\lambda_0-1)a_{0,U}\int_z^{+\infty}\Big|\frac{d\rho^0}{dz}\Big|^2\frac{(\rho^0-(1+A))}{\rho^0} + \widetilde{\sigma}\lambda_1\Big|\frac{d\rho^0}{dz}\Big|^2 a_{0,U}. \qquad (4.14)
\end{aligned}
$$

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At order 3 at the bottom, we have:

$$\frac{d}{dz}\Big(\rho^0\frac{da_3}{dz}\Big)+\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_2}{dz}\Big)+\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz}\Big)$$
$$+\widetilde{\sigma}\lambda_2\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_0}{dz}\Big)+2\frac{d}{dz}(\rho^0a_2)+2\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2a_0\Big)$$
$$+2\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2a_1\Big)=-(\lambda_0-1)\frac{d\rho^0}{dz}a_2-\lambda_1\frac{d\rho^0}{dz}a_1-\lambda_2\frac{d\rho^0}{dz}a_0$$
$$+\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)a_1+\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)a_0.$$

By using the expression of ρ^0da_2/dz and a_1, we get

$$\frac{d}{dz}\Big(\rho^0\frac{da_3}{dz}\Big)+\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_2}{dz}\Big)+\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz}\Big)$$
$$=-(\lambda_0+1)\frac{d(\rho^0a_2)}{dz}-(\lambda_0-1)\Big[-(\lambda_0+1)((1-A)a_{1,D}-\rho^0a_1)$$
$$+(\lambda_0^2-1)\int_{-\infty}^{z}a_{0,D}(\rho^0-(1-A))-\lambda_1((1-A)-\rho^0)a_{0,D}+\widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2a_{0,D}$$
$$+\widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz}\Big]-\lambda_1\frac{d\rho^0}{dz}\Big(a_{1,D}-(\lambda_0+1)\int_{-\infty}^{z}\frac{(\rho^0-(1-A))}{\rho^0}a_{0,D}\Big)$$
$$-\lambda_2\frac{d\rho^0}{dz}a_0-\widetilde{\sigma}\lambda_0\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)\Big(a_{1,D}-(\lambda_0+1)\int_{-\infty}^{z}\frac{(\rho^0-(1-A))}{\rho^0}a_{0,D}\Big)$$
$$-\widetilde{\sigma}\lambda_1\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)a_{0,D}-2\widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz}.$$

By integrating from $-\infty$ to z, we get

$$-\rho^0\frac{da_3}{dz}-\widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_2}{dz}-\widetilde{\sigma}\lambda_1\Big|\frac{d\rho^0}{dz}\Big|^2\frac{da_1}{dz}$$
$$=(\lambda_0+1)(\rho^0a_2-(1-A)a_{2,D})+(\lambda_0^2-1)\int_{-\infty}^{z}(\rho^0a_1-(1-A)a_{1,D})$$
$$+(\lambda_0-1)(\lambda_0^2-1)\int_{-\infty}^{z}\int_{-\infty}^{\xi}a_{0,D}(\rho^0-(1-A))$$
$$+\lambda_1(\lambda_0-1)\int_{-\infty}^{z}(\rho^0-(1-A))a_{0,D}+\widetilde{\sigma}\lambda_0(\lambda_0-1)\int_{-\infty}^{z}\Big|\frac{d\rho^0}{dz}\Big|^2a_{0,D}$$

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$$-\widetilde{\sigma}\lambda_0(\lambda_0^2-1)\int\limits_{-\infty}^{z}\Big|\frac{d\rho^0}{dz}\Big|^2 a_{0,D}\frac{(\rho^0-(1-A))}{\rho^0}+\lambda_1(\rho^0-(1-A))a_{1,D}$$

$$-\lambda_1(\lambda_0+1)a_{0,D}\int\limits_{-\infty}^{z}\frac{d\rho^0}{dz}\int\limits_{-\infty}^{\xi}\frac{(\rho^0-(1-A))}{\rho^0}+\lambda_2(\rho^0-(1-A))a_{0,D}$$

$$+\widetilde{\sigma}\lambda_0\Big|\frac{d\rho^0}{dz}\Big|^2 a_{1,D}-\widetilde{\sigma}\lambda_0(\lambda_0+1)a_{0,D}\int\limits_{-\infty}^{z}\left[\frac{d}{dz}\Big(\Big|\frac{d\rho^0}{dz}\Big|^2\Big)\int\limits_{-\infty}^{\xi}\frac{(\rho^0-(1-A))}{\rho^0}\right]$$

$$-2\widetilde{\sigma}\lambda_0(\lambda_0+1)a_{0,D}\int\limits_{-\infty}^{z}\Big|\frac{d\rho^0}{dz}\Big|^2\frac{(\rho^0-(1-A))}{\rho^0}+\widetilde{\sigma}\lambda_1\Big|\frac{d\rho^0}{dz}\Big|^2 a_{0,D}. \qquad (4.15)$$

By using the expressions involving λ_2, we obtain what we announced in (4.13).

In the same calculation time, we can also get the σ-independent part of λ_2 which is given by the following relation

$$-2Aa_0\lambda_2(\sigma=0)=\frac{1-A^2}{A}\big(a_{2,U}-a_{2,D}\big)_{\sigma=0}+A\lambda_1(a_{1,U}+a_{1,D}).$$

$$+(1-\lambda_0^2)\left[\int\limits_{0}^{+\infty}\big((1+A)a_{1,U}-\rho^0 a_1\big)-\int\limits_{-\infty}^{0}\big(\rho^0 a_1-(1-A)a_{1,D}\big)\right]$$

$$+a_0(1-\lambda_0^2)\Big[(\lambda_0+1)\int\limits_{0}^{+\infty}\int\limits_{z}^{+\infty}\big(\rho^0-(1+A)\big)$$

$$-(\lambda_0-1)\int\limits_{-\infty}^{0}\int\limits_{-\infty}^{z}\big(\rho^0-(1-A)\big)\Big]+a_0\lambda_1\Big[\int\limits_{0}^{+\infty}\frac{(1+A)-\rho^0}{\rho^0}$$

$$-\int\limits_{-\infty}^{0}\frac{\rho^0-(1-A)}{\rho^0}-(\lambda_0+1)\int\limits_{0}^{+\infty}\big((1+A)-\rho^0\big)$$

$$+(\lambda_0-1)\int\limits_{-\infty}^{0}\big(\rho^0-(1-A)\big)+(\lambda_0-1)\int\limits_{0}^{+\infty}\frac{d\rho^0}{dz}\int\limits_{z}^{+\infty}\frac{\rho^0-(1+A)}{\rho^0}$$

$$-(\lambda_0+1)\int\limits_{-\infty}^{0}\frac{d\rho^0}{dz}\int\limits_{-\infty}^{z}\frac{\rho^0-(1-A)}{\rho^0}\Big]$$

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where

$$
\begin{aligned}
&\left(a_{2,U} - a_{2,D}\right)_{\sigma=0} \\
&= (\lambda_0 - 1)\int_0^{+\infty} \frac{(1+A)a_{1,U} - \rho^0 a_1}{\rho^0} - (\lambda_0 + 1)\int_{-\infty}^{0} \frac{\rho^0 a_1 - (1-A)a_{1,D}}{\rho^0} \\
&+a_0(\lambda_0^2 - 1)\Big[\int_0^{+\infty} \frac{1}{\rho^0}\int_z^{+\infty} \big(\rho^0 - (1+A)\big) - \int_{-\infty}^{0} \frac{1}{\rho^0}\int_{-\infty}^{z} \big(\rho^0 - (1-A)\big)\Big] \\
&+a_0\lambda_1\Big[\int_0^{+\infty} \frac{(1+A) - \rho^0}{\rho^0} - \int_{-\infty}^{0} \frac{\rho^0 - (1-A)}{\rho^0}\Big].
\end{aligned}
$$

5 Low Atwood number limit for linear density profiles

In this part, we address the Rayleigh–Taylor instability in the framework of linear density profiles and we derive the asymptotic expressions of the growth rate when the Atwood number goes to zero.

This analysis is of particular interest in the framework of direct numerical simulation of Rayleigh–Taylor instabilities. As a matter of fact, prior to launching large computations, elementary evaluation of the code's behavior has to be done. More precisely, one important problem is to estimate for a given mesh size the wave number range in which the growth rate is correctly computed. Asymptotically analytical solutions in the limit of small Atwood numbers provide such quantitative references.

We consider a nondimensional continuous density profile connecting two constant densities away from a transition zone located in the neighborhood of $z = 0$, given by

$$
\rho^0 = \begin{cases} 1 + A & \text{if } z \geq A \\ 1 + z & \text{if } |z| \leq A \\ 1 - A & \text{if } z \leq -A \end{cases}
$$

Looking at the behavior when $A \to 0$ we obtain:

$$
\lambda_1 = \frac{2}{3} + o(A)
$$

$$
\lambda_2(\sigma = 0) = \frac{4}{45}A + o(A)
$$

$$
\lambda_2 = \lambda_2(\sigma = 0) + \tilde{\sigma}\Big(\frac{4}{3A} + \frac{4A}{15} + o(A)\Big)
$$

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Let's now come back to the asymptotic behavior of $\frac{\gamma^2}{gk}$ with respect to $k = \frac{\varepsilon}{\ell}$ and see the influence of surface tension.

$$\frac{\gamma^2}{gk} = \frac{1}{\lambda} = \frac{1}{\lambda_0}\Big[1 - \frac{\lambda_1}{\lambda_0}\varepsilon - \Big(\frac{\lambda_2}{\lambda_0} - \frac{\lambda_1^2}{\lambda_0^2}\Big)\varepsilon^2 + o(\varepsilon^2)\Big].$$

Since

$$\widetilde{\sigma} = \frac{\sigma(\rho_U^0 + \rho_D^0)}{2g\ell^3},$$

we obtain

$$\frac{\gamma^2}{gk} \approx A\Big[1 - \frac{2\sigma(\rho_U^0 + \rho_D^0)}{3g\ell}k^2\Big].$$

Choosing

$$\sigma = \frac{3T_s}{2(\rho_U^0 - \rho_D^0)^2}A\ell, \tag{5.1}$$

we get exactly

$$\frac{\gamma^2}{gk} = A - \frac{T_s}{g(\rho_2 + \rho_1)}k^2$$

Finally let us recall that the energy concentrated at the interface is interpreted as the surface tension. It depends on the pressure law that is considered and is found looking at the equation (4.2). The reader interested in a modeling paper on this subject is referred to [21].

We recall that analytic solutions of the Rayleigh equation without surface tension for linear profiles have been studied in [11].

6 Some known results on the compressible Korteweg system

Few works consider the diffuse interface model in the literature as far as Rayleigh–Taylor or Richtmyer–Meshkov instabilities are concerned. We try there to describe briefly different works devoted to stability results. In [1], the problems of internal waves in quasi-critical fluids is addressed. The interface is represented by a transition zone with regular density. The static density profiles, frequencies of the internal waves are computed and compared to experiments. In [3], the author studies the linear stability on a transition phase problem for non viscous capillary fluids of Van der Waals type. Two results are obtained: the capillary profiles are weakly linearly stable in any space dimensions, by using an energy method; the technique of Evans functions

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shows a bifurcation phenomenon close to the origin. In [26], the stability and instability of oscillations of amplitudes $O(1)$ in a Van der Waals fluid of Korteweg type is investigated. The author obtains then some asymptotic models by letting the capillarity and viscosity coefficient go to zero with the same order of magnitude. Solutions with a given profile are considered but no assumptions on the structure of oscillations are made. The analysis is globally formal with some points rigorously justified. The main order is a system of three conservation laws. Indeed, a new variable has to be introduced to close the final system. The other terms are solutions of a linear system. Readers interested by recent mathematical results around Korteweg model is referred to [4–7, 14, 19, 25]. It could be interesting using such recent results to investigate again the stability and instability of oscillations of amplitudes $O(1)$.

Appendix: Ansatz

We need the following integrals appearing in the expressions of λ_1 and λ_2:

$$\int_0^\infty \left|\frac{d\rho^0}{dz}\right|^2 = A\ ; \quad \int_{-\infty}^0 \left|\frac{d\rho^0}{dz}\right|^2 = A;$$

$$\int_0^\infty \left|\frac{d\rho^0}{dz}\right|^2 \frac{1}{\rho^0} = \ln(1+A)\ ; \quad \int_{-\infty}^0 \left|\frac{d\rho^0}{dz}\right|^2 \frac{1}{\rho^0} = -\ln(1-A);$$

$$\int_0^\infty \left|\frac{d\rho^0}{dz}\right|^2 \frac{1}{(\rho^0)^2} = \frac{A}{1+A}\ ; \quad \int_{-\infty}^0 \left|\frac{d\rho^0}{dz}\right|^2 \frac{1}{(\rho^0)^2} = \frac{A}{1-A};$$

$$\int_0^\infty (1+A) - \rho^0 = \frac{A^2}{2}\ ; \quad \int_{-\infty}^0 \rho^0 - (1-A) = \frac{A^2}{2};$$

$$\int_0^\infty \int_z^\infty \rho^0 - (1+A) = -\frac{A^3}{6}\ ; \quad \int_{-\infty}^0 \int_{-\infty}^z \rho^0 - (1-A) = \frac{A^3}{6};$$

$$\int_0^\infty \frac{\rho^0 - (1+A)}{\rho^0} = A - (1+A)\ln(1+A);$$

$$\int_{-\infty}^0 \frac{\rho^0 - (1-A)}{\rho^0} = A + (1-A)\ln(1-A);$$

$$\int_0^\infty (1+A)a_{1,U} - \rho^0 a_1 = \frac{A^2}{2} a_{1,U}$$

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$$-a_0\frac{1-A}{2A}\Big(\frac{A^3}{3}-A+(1+A)\Big(\ln(1+A)-\frac{A^2}{2}\Big)\Big);$$

$$\int_{-\infty}^{0}\rho^0 a_1-(1-A)a_{1,D}=\frac{A^2}{2}a_{1,D}$$

$$-a_0\frac{1+A}{2A}\Big(-\frac{A^3}{3}+A+(1-A)\Big(\ln(1-A)-\frac{A^2}{2}\Big)\Big);$$

$$\int_0^{\infty}\frac{1}{\rho^0}\int_z^{\infty}\rho^0-(1+A)=\frac{3A^2}{4}+\frac{A}{2}-\frac{(1+A)^2}{2}\ln(1+A);$$

$$\int_{-\infty}^{0}\frac{1}{\rho^0}\int_{-\infty}^{z}\rho^0-(1-A)=\frac{3A^2}{4}-\frac{A}{2}-\frac{(1-A)^2}{2}\ln(1-A);$$

$$K^+=\int_0^{\infty}\frac{d\rho^0}{dz}\int_z^{\infty}\frac{\rho^0-(1+A)}{\rho^0}=-\frac{A^2}{2}-A+(1+A)\ln(1+A);$$

$$K^-=\int_{-\infty}^{0}\frac{d\rho^0}{dz}\int_{-\infty}^{z}\frac{\rho^0-(1-A)}{\rho^0}=-\frac{A^2}{2}+A+(1-A)\ln(1-A);$$

$$\int_0^{\infty}\frac{(1+A)a_{1,U}-\rho^0 a_1}{\rho^0}=a_{1,U}\big((1+A)\ln(1+A)-A\big)-a_0\frac{1-A}{A}K^+;$$

$$\int_{-\infty}^{0}\frac{\rho_1^0-(1-A)a_{1,D}}{\rho^0}=a_{1,D}\big((1-A)\ln(1-A)+A\big)-a_0\frac{1+A}{A}K^-.$$

First of all, let's look at λ_1, starting with its integral expression given in the preceding section:

$$\lambda_1=\frac{1-A^2}{2A^3}\int_{-\infty}^{\infty}\frac{A^2-(\rho^0-1)^2}{\rho^0}dz$$

$$=\frac{1-A^2}{2A^3}\int_{-A}^{A}\frac{A^2-z^2}{z+1}$$

$$=\frac{1-A^2}{2A^3}\int_{-A}^{A}\frac{A^2-1-(z+1)^2+2(z+1)}{z+1}$$

$$=\frac{1-A^2}{2A^3}\Big[(A^2-1)\big(\ln(1+A)-\ln(1-A)\big)-\frac{(1+A)^2-(1-A)^2}{2}+4A\Big]$$

$$
\begin{aligned}
&= \frac{1-A^2}{2A^3}\Big[(A^2-1)\big(\ln(1+A)-\ln(1-A)\big)+2A\Big]\\
&= \frac{1-A^2}{2A^3}\Big[\frac{4A^3}{3}+\frac{4A^5}{15}+\mathcal{O}(A^5)\Big]\\
&= \frac{2}{3}-\frac{8A^2}{15}+\mathcal{O}(A^2).
\end{aligned}
$$

For the σ-dependent part of λ_2 we obtain

$$
\begin{aligned}
\lambda_2-\lambda_2(\sigma=0) &= \frac{\tilde{\sigma}}{2A^2}\Big[\Big(\frac{1}{A}+1\Big)A+\frac{1}{A}\Big(\frac{1}{A}-1\Big)\Big(A-(1+A)\ln(1+A)\Big)\\
&\quad -\Big(\frac{1}{A}-1\Big)A+\frac{1}{A}\Big(\frac{1}{A}+1\Big)\Big(A+(1-A)\ln(1-A)\Big)\Big]\\
&\quad +\frac{\tilde{\sigma}(1-A^2)}{2A^3}\Big[\Big(1-\frac{1}{A}\Big)\Big(\ln(1+A)-A\Big)-\ln(1+A)\\
&\quad -\Big(1+\frac{1}{A}\Big)\Big(-\ln(1-A)-A\Big)-\ln(1-A)\Big]\\
&= \frac{\tilde{\sigma}}{A}\Big[1+\frac{1}{A^2}+\frac{1-A^2}{A^3}\big(A+\ln(1-A)-\ln(1+A)\big)\Big]\\
&= \frac{\tilde{\sigma}}{A}\Big[1+\frac{1}{A^2}+\frac{1-A^2}{A^3}\big(-A-\frac{2A^3}{3}-\frac{2A^5}{5}+\mathcal{O}(A^6)\big)\Big]\\
&= \frac{\tilde{\sigma}}{A}\Big[\frac{4}{3}+\frac{4A^2}{15}+\mathcal{O}(A^3)\Big]\\
&= \tilde{\sigma}\Big[\frac{4}{3A}+\frac{4A}{15}+\mathcal{O}(A^2)\Big].
\end{aligned}
$$

And for the part which does not depend on σ:

$$
\begin{aligned}
-2Aa_0\lambda_2(\sigma=0) &= \frac{1-A^2}{A}\Big[\Big(\frac{1}{A}-1\Big)\Big[a_{1,U}\Big((1+A)\ln(1+A)-A\Big)\\
&\quad -a_0\Big(\frac{1}{A}-1\Big)\Big(-\frac{A^2}{2}-A+(1+A)\ln(1+A)\Big)\Big]\\
&\quad -\Big(\frac{1}{A}+1\Big)\Big[a_{1,D}\Big((1-A)\ln(1-A)+A\Big)\\
&\quad -a_0\Big(\frac{1}{A}+1\Big)\Big(-\frac{A^2}{2}+A+(1-A)\ln(1-A)\Big)\Big]\\
&\quad +a_0\Big(\frac{1}{A^2}-1\Big)\Big[\frac{A}{2}+\frac{3A^2}{4}-\frac{(1+A)^2}{2}\ln(1+A)\\
&\quad +\frac{A}{2}-\frac{3A^2}{4}+\frac{(1-A)^2}{2}\ln(1-A)\Big]\\
&\quad +a_0\lambda_1\Big[-A+(1+A)\ln(1+A)-A-(1-A)\ln(1-A)\Big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+A\lambda_1(a_{1,U}+a_{1,D})\\
&+\Big(1-\frac{1}{A^2}\Big)\Big[\frac{A^2}{2}(a_{1,U}-a_{1,D})\\
&-a_0\Big(\frac{1}{A}-1\Big)\Big(\frac{A^3}{6}-\frac{A}{2}+\frac{1+A}{2}\ln(1+A)\Big)\\
&+a_0\Big(\frac{1}{A}+1\Big)\Big(-\frac{A^3}{6}+\frac{A}{2}+\frac{1-A}{2}\ln(1-A)\Big)\Big]\\
&+a_0\Big(1-\frac{1}{A^2}\Big)\Big[-\frac{A^3}{6}\Big(\frac{1}{A}+1\Big)-\frac{A^3}{6}\Big(\frac{1}{A}-1\Big)\Big]\\
&+a_0\lambda_1\Big[-A+(1+A)\ln(1+A)-A-(1-A)\ln(1-A)\\
&-\frac{A^2}{2}\Big(\frac{1}{A}+1\Big)+\frac{A^2}{2}\Big(\frac{1}{A}-1\Big)\\
&+\Big(\frac{1}{A}-1\Big)\Big(-\frac{A^2}{2}+(1+A)\ln(1+A)-A\Big)\\
&-\Big(\frac{1}{A}+1\Big)\Big(-\frac{A^2}{2}+(1-A)\ln(1-A)+A\Big)\Big].
\end{aligned}
$$

And after some calculations we get

$$
\begin{aligned}
&-2Aa_0\lambda_2(\sigma=0)\\
&\quad=\frac{1-A^2}{A}\Big[(a_{1,U}-a_{1,D})\Big(\frac{A}{2}+\frac{1-A^2}{2A}\big(\ln(1+A)+\ln(1-A)\big)\Big)\Big]\\
&\qquad+\frac{1-A^2}{A}a_0\Big[\frac{2}{A}-\frac{A}{3}+\Big(\frac{1}{2}-\frac{1}{A^2}+\frac{A^2}{2}\Big)\big(\ln(1+A)-\ln(1-A)\big)\Big]\\
&\qquad+a_0\lambda_1\Big[\Big(1+\frac{1}{A}-A\Big)\Big(-2A+(1+A)\ln(1+A)-(1-A)\ln(1-A)\Big)\\
&\qquad-2+\frac{1-A^2}{A}\big(\ln(1+A)-\ln(1-A)\big)\Big]
\end{aligned}
$$

with

$$
\begin{aligned}
a_{1,U}-a_{1,D}&=a_0\Big((-\lambda_0+1)\int_0^{+\infty}\frac{(\rho^0-(1+A))}{\rho^0}dz-(\lambda_0+1)\int_{-\infty}^{0}\frac{(\rho^0-(1-A))}{\rho^0}\,dz\Big)\\
&=a_0\Big[\Big(\frac{1}{A}+1\Big)\Big(A-(1+A)\ln(1+A)\Big)-\Big(\frac{1}{A}+1\Big)\Big(A+(1-A)\ln(1-A)\Big)\Big]\\
&=a_0\Big[-2+\frac{1-A^2}{A}\big(\ln(1+A)-\ln(1-A)\big)\Big]\\
&=a_0\Big[-\frac{4A^2}{3}-\frac{4A^4}{15}+\mathcal{O}(A^4)\Big].
\end{aligned}
$$

Putting together all these expressions we finally get the following ansatz:

$$
\begin{aligned}
&-2Aa_0\lambda_2(\sigma=0)\\
&\quad = a_0\frac{1-A^2}{A}\left[\begin{array}{l}\left(-\frac{4A^3}{3}-\frac{4A^4}{15}+o(A^4)\right)\left(\frac{A}{2}-\frac{1-A^2}{2A}\left(A^2+\frac{A^4}{2}+o(A^4)\right)\right)\\ +\frac{2}{A}-\frac{A}{3}+\left(-\frac{1}{A^2}+\frac{1}{2}+\frac{A^2}{2}\right)\left(2A+\frac{2A^3}{3}+\frac{2A^5}{5}+o(A^6)\right)\end{array}\right]\\
&\qquad +a_0\left[\frac{2}{3}-\frac{8A^2}{15}+o(A^2)\right]\left[\begin{array}{l}\left(\frac{1}{A}+1-A\right)\left(-\frac{A^3}{3}+o(A^3)\right)\\ -2+\frac{1-A^2}{A}\left(2A+\frac{2A^3}{3}+o(A^3)\right)\end{array}\right]
\end{aligned}
$$

which gives

$$
\lambda_2(\sigma=0)=\frac{4A}{45}+o(A).
$$

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