The eigenvalue problem with conductivity boundary conditions

Dorin Bucur, Ioana Durus, Edouard Oudet

The conductivity eigenvalue problem

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Abstract

In this paper we study isoperimetric inequalities for the eigenvalues of the Laplace operator with constant and locally constant boundary conditions. Existence and stability results are presented in the frame of the gamma and weak gamma convergencies, together with identification of optimal sets by analytical and numerical methods.

1 Introduction

The main purpose of this paper is to study the eigenvalues associated to the Laplace operator with (locally) constant boundary conditions in relationship with the geometric domain. On the one hand we consider globally constant boundary conditions (simply denoted *const. b.c.*) and on the other hand locally constant boundary conditions, which we call conductivity boundary conditions (denoted *cond. b.c.*).

The isoperimetric inequality for the first non zero eigenvalue for the *const. b.c.* eigenvalues was discussed by Greco and Lucia in [10]. There is proved that the union of two equal balls minimizes the first non-zero eigenvalue among all domains of prescribed measure. Several more properties are also obtained in [10], among which relationships with the Dirichlet eigenvalues of the Laplacian and with the twisted eigenvalues of the Laplacian (see also [8]).

The first purpose of the paper is to investigate the shape stability of the spectrum of *const. b.c* Laplacian in relationship with the γ-convergence (see [3]). The γ-convergence is, roughly speaking, the geometric convergence of shapes for which the solution of the Dirichlet-Laplacian is stable. We prove that the spectrum of the *const. b.c.* Laplacian is stable for the γ-convergence if and only if the Lebesque measure of the moving domains is preserved at the γ-limit (which in general is not the case).

As a second purpose, we discuss the isoperimetric inequality for a general shape functional depending on the *const. b.c.* Laplacian eigenvalues, of the form $F(\Lambda_1(\Omega), ..., \Lambda_k(\Omega))$. We prove that for every bounded design region D the problem

$$\min_{\Omega \subseteq D, |\Omega| = c} F(\Lambda_1(\Omega), ..., \Lambda_k(\Omega))$$

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has a solution in the family of quasi-open sets, provided mild monotonicity and continuity assumptions on F. A similar result was obtained by Buttazzo and Dal Maso in [5] for the Dirichlet eigenvalues of the Laplacian, based on γ continuity of the eigenvalues. In our problem, the mapping

$$\Omega \to \Lambda_k(\Omega)$$

is not γ-continuous so that the main theorem of Buttazzo and Dal Maso does not apply. Our argument is based on the study of a modified version of the weak gamma convergence (see [3]) which preserves the measure. We also refer the reader to [10], where existence solutions for isoperimetric inequalities for the *const. b.c.* Laplacian eigenvalues were studied in the class of convex sets.

The third purpose of the paper is to study the isoperimetric inequalities associated to the conductivity eigenvalues of the Laplacian. This study is motivated by the analysis of the defect identification of a material by electrostatic boundary measurements. Friedman and Vogelius introduced in [9] a dual problem to the defect identification problem, where the perfectly insulating defects become perfectly conductive regions. In this last model, the locally constant boundary conditions are naturally used (see also Alessandrini [1] and [4, 2]). In [2], the following space is introduced for the study of the conductivity problem in an arbitrary two dimensional bounded domain. For a bounded open set of $\mathbb{R}^n$ we introduce

$$H^1_{cond}(\Omega) = cl_{H^1(\Omega)}\{u \in H^1_{loc}(\mathbb{R}^n) \mid \exists\varepsilon > 0 \ \nabla u = 0 \ a.e.(\Omega^c)^\varepsilon\}$$

where K^ε is the Minkowski sum

$$K \oplus B(0,\varepsilon) = \bigcup_{x\in K} B(x,\varepsilon).$$

It turns out that if Ω^c is connected, every function in $H^1_{cond}(\Omega)$ is constant on the boundary, in the sense that there exists $c \in \mathbb{R}$ such that $u - c \in H^1_0(\Omega)$. Consequently, in this case the space $H^1_{cond}(\Omega)$ coincides with the space considered previously by Greco and Lucia (up to the zero mean value). If Ω^c is not connected, on the boundary of the smooth connected components of Ω^c, a function $u \in H^1_{cond}(\Omega)$ is locally constant (the constant may differ from one to another connected component). These regions can be seen as perfectly conductive regions in the frame of the dual defect identification problems of Friedeman and Vogelius.

The eigenvalue problem for the *cond. b.c* Laplacian is well posed, and the same questions (see [10]), as for the *const. b.c.* Laplacian can be raised. Unfortunately, we are not able to perform a complete shape analysis of the problem because of major new difficulties which are related to the (open) problem of the relaxation of the Neumann Laplacian.

The isoperimetric inequality of the first non-zero *cond. b.c* eigenvalue can be treated by a two steps rearrangement. As in [10], by a Schwarz rearrangement of two level sets of a test function u, we reduce the search of the optimum to the family of the unions of two sets of concentric annuli. In a second step we prove by a second rearrangement procedure that the annuli can be streched to balls and finally, that two equal disjoint balls minimize the first non zero eigenvalue among domains of constant volume. We refer the reader to [11] for a recent survey concerning isoperimetric inequalities for eigenvalues.

Shape stability is achieved only in two dimensions provided that the number of connected components of the moving complements is uniformly bounded and the Lebesque measure is stable. Some numerical computations for the solutions of the isoperimetric inequalities for the first three *cond. b.c* eigenvalues are also provided.

2 The Laplacian with constant boundary conditions

Let $D \subseteq \mathbb{R}^N$, $N \geq 2$ be a bounded open set. We denote by Ω an open or quasi-open subset of D (we refer the reader to [3] for a detailed introduction to the shape analysis of quasi-open sets).

We introduce the following space:

$$H^1_{const}(\Omega) = \{u \in H^1(\Omega) \mid \exists c \in \mathbb{R}, u - c \in H^1_0(\Omega)\}$$

and consider the closed subspace:

$$\mathcal{U}_{const}(\Omega) = \{u \in H^1_{const}(\Omega) \mid \int_\Omega u dx = 0\}.$$

It is easy to observe that $\mathcal{U}_{const}(\Omega)$ can be equivalently endowed with the scalar product

$$(u, v)_{\mathcal{U}_{const}(\Omega)} = (\nabla u, \nabla v)_{L^2}.$$

Indeed, the following Poincaré inequality holds. There exists a constant $M > 0$, depending only on the measure of Ω such that for every $u \in \mathcal{U}_{const}(\Omega)$

$$\|u\|_{L^2(\Omega)} \leq M \|\nabla u\|_{L^2(\Omega)}.$$

To prove this assertion, one can apply the Poincaré inequality to $u - c \in H^1_0(\Omega)$ and obtain $\|u - c\|_{L^2(\Omega)} \leq M\|\nabla(u - c)\|_{L^2(\Omega)}$. Since $\nabla u = \nabla(u - c)$ and since $\int_\Omega u dx = 0$, we immediately get

$$\|u\|_{L^2(\Omega)} \leq \|u - c\|_{L^2(\Omega)} \leq M \|\nabla u\|_{L^2(\Omega)}.$$

We consider the Laplace operator acting on the space $\mathcal{U}_{const}(\Omega)$. More precisely we define

$$T_\Omega : L^2_0(\Omega) \longrightarrow L^2_0(\Omega), \quad T_\Omega f = R_\Omega f - \frac{1}{|\Omega|} \int_\Omega R_\Omega f dx.$$

Here $L^2_0(\Omega) = \{f \in L^2(\Omega) \mid \int_\Omega f dx = 0\}$ and $R_\Omega : L^2(\Omega) \longrightarrow L^2(\Omega)$ is the resolvent of the Dirichlet Laplacian on Ω, i.e. $R_\Omega f = u$, where u solves

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u \in H^1_0(\Omega). \end{cases} \tag{2.1}$$

in the weak variational sense. The eigenvalues of T_Ω^{-1} are the inverses of the eigenvalues of the *const. b.c.* Laplacian.

Alternatively, they are defined with the Rayleigh quotient

$$\lambda_k^{ct}(\Omega) = \min_{V^k \subset \mathcal{U}_{const}(\Omega)} \max_{v \in V^k \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx}. \tag{2.2}$$

where V^k stands for an arbitrary subspace of dimension k in $\mathcal{U}_{const}(\Omega)$.

We notice at this point that we can also define the Laplacian on $H^1_{const}(\Omega)$, in which case the first eigenvalue is zero. For this reason we refer to $\lambda_1^{ct}(\Omega)$ as the first non zero eigenvalue.

If $\lambda_k^{ct}(\Omega)$ is an eigenvalue defined by (2.2), there exists a non zero function $u \in \mathcal{U}_{const}(\Omega)$ such that formally

$$\begin{cases} -\Delta u = \lambda_k^{ct}(\Omega) u \\ u \in \mathcal{U}_{const}(\Omega). \end{cases} \tag{2.3}$$

This formal relation is understood in the following sense: $u \in \mathcal{U}_{const}(\Omega)$ and

$$\forall \phi \in \mathcal{U}_{const}(\Omega) \int_\Omega \nabla u \cdot \nabla \phi dx = \lambda_k^{ct}(\Omega) \int_\Omega u \phi dx.$$

Notice that the relationship

$$-\Delta u = \lambda_k^{ct}(\Omega) u \tag{2.4}$$

holds also in the sense of distributions. Indeed, $\forall \phi \in \mathcal{D}(\Omega)$, and a suitable constant c such that $\phi - c \in \mathcal{U}_{const}(\Omega)$ we have

$$\int_\Omega \nabla u \cdot \nabla(\phi - c) dx = \lambda_k^{ct}(\Omega) \int_\Omega u(\phi - c) dx.$$

Since $\int_\Omega u dx = 0$ one can eliminate c on both sides above.

We recall from [10] the following properties of $\lambda_k^{ct}(\Omega)$:

1. if $\Omega_1 \subseteq \Omega_2$ then $\lambda_k^{ct}(\Omega_2) \leq \lambda_k^{ct}(\Omega_1)$,
2. $\lambda_k^{ct}(t\Omega) = t^{-2} \lambda_k^{ct}(\Omega)$,
3. $\lambda_k(\Omega) \leq \lambda_k^{ct}(\Omega) \leq \lambda_k^t(\Omega) \leq \lambda_{k+1}(\Omega)$,

where $\lambda_k(\Omega)$ stands for the k-th Dirichlet eigenvalue of the Laplacian and $\lambda_k^t(\Omega)$ for the k-twisted eigenvalue (see [8]).

Greco and Lucia studied in [10] the isoperimetric inequality for λ_1^{ct}. There is proved that among all domains of given volume (say m), the union of two disjoint balls of volume $\frac{m}{2}$ minimizes λ_1^{ct}. By a Schwarz rearrangement of the level sets $(u-c)^+$, $(u-c)^-$ of a test function $u \in \mathcal{U}_{const}(\Omega)$ which is such that $u - c \in H_0^1(\Omega)$, the optimization problem is reduced to the family of unions of two balls.

In order to study more general isoperimetric inequalities associated to $\lambda_k^{ct}(\Omega)$ we recall some results for the Dirichlet Laplacian. For the convenience of the reader, we recall the definition of the γ-convergence.

Definition 2.1 *Let $\Omega_n, \Omega \subseteq D$ be open (or quasi open) sets. It is said that Ω_n γ-converges to Ω if $R_{\Omega_n} \longrightarrow R_\Omega$ in $\mathcal{L}(L^2(D))$.*

For a detailed study of the γ-convergence, we refer the reader to [3].

From the following general inequality (see for instance [7])

$$|\lambda_k(R_{\Omega_n}) - \lambda_k(R_\Omega)| \leq \|R_{\Omega_n} - R_\Omega\|_{\mathcal{L}(L^2(D))}$$

and from the fact that

$$\lambda_k(\Omega_n) = \frac{1}{\lambda_k(R_{\Omega_n})}$$

we get that

$$\Omega_n \xrightarrow{\gamma} \Omega \implies \lambda_k(\Omega_n) \longrightarrow \lambda_k(\Omega).$$

We also notice that if $\Omega_n \xrightarrow{\gamma} \Omega$, we have

$$1_\Omega(x) \le \liminf_{n\longrightarrow\infty} 1_{\Omega_n}(x) \;\; a.e.\; x \in D$$

and consequently

$$|\Omega| \le \liminf_{n\longrightarrow\infty} |\Omega_n|.$$

If we assume both γ convergence and convergence of measures, then we get L^1 convergence of the characteristic functions $1_{\Omega_n} \longrightarrow 1_\Omega$.

The following general existence result is due to Buttazzo and Dal Maso [5]. Let us denote $\mathcal{A}(D)$ the family of quasi open subsets of D.

Theorem 2.2 *Let $F : \mathcal{A}(D) \longrightarrow \mathbb{R}$ be a γ-lower semi continuous functional which is non increasing to set inclusions (up to zero capacity). Then*

$$\min\{F(\Omega) | \Omega \in \mathcal{A}(D), |\Omega| = m\}$$

has a solution.

If $F(\Omega) = \Phi(\lambda_1(\Omega), ..., \lambda_k(\Omega))$ where $\Phi : \mathbb{R}^k \longrightarrow \mathbb{R}$ is l.s.c. and non decreasing in each variable, then Theorem 2.2 applies for

$$F(\Omega) = \Phi(\lambda_1(\Omega), ..., \lambda_k(\Omega)).$$

In particular, the isoperimetric problem

$$\min_{\Omega\in\mathcal{A}(D), |\Omega|=m} \lambda_k(\Omega)$$

has at least one solution.

In order to answer a similar question for shape functionals depending on $\lambda_1^{ct}(\Omega), ..., \lambda_k^{ct}(\Omega)$ one can not use Theorem 2.2, since the mappings $\Omega \longrightarrow \lambda_k^{ct}(\Omega)$ are not in general γ-continuous. This lack of γ-continuity is due to the possible "loss" of measure in the γ-limit process and of the failure of convergence of the resolvent operators.

We prefer to (re)define the operators T_Ω on the fixed space associated to the design region $D \subseteq \mathbb{R}^n$. In this way, all operators on the moving sets are defined on the same functional space, so that the comparison of their eigenvalues is easier to handle. For every (quasi) open set $\Omega \subseteq D$, we introduce $\widetilde{T}_\Omega : L^2_0(D) \longrightarrow L^2_0(D)$ by

$$\widetilde{T}_\Omega(f)(x) = \begin{cases} T_\Omega(f_{|\Omega} - \frac{1}{|\Omega|}\int_\Omega f dx)(x) \text{ if } x \in \Omega, \\ 0 \text{ if } x \in D\backslash\Omega. \end{cases}$$

We state the following proposition which has a standard proof.

Proposition 2.3 *For every (quasi) open set* $\Omega \subseteq D$, $\widetilde{T}_\Omega$ *is positive, self adjoint and compact.*

In the sequel we establish the relationship between the eigenvalues of T_Ω and $\widetilde{T}_\Omega$.

Proposition 2.4 *Let* Ω *be a (quasi) open set. The operators* T_Ω *and* $\widetilde{T}_\Omega$ *have the same set of eigenvalues, and there exists a natural bijection between eigenspaces.*

Proof Let λ be an eigenvalue of T_Ω. There exists $u \in L^2_0(\Omega)$ such that $T_\Omega u = \lambda u$. One can prove that there exist $u^* \in L^2_0(D)$ such that $\widetilde{T}_\Omega u^* = \lambda u^*$. Indeed, let $u^* = \begin{cases} u \text{ if } x \in \Omega \\ 0 \text{ if } x \in D \setminus \Omega. \end{cases}$ Since $\int_D u^* dx = 0$ then $u^* \in L^2_0(D)$. We obtain

$$\widetilde{T}_\Omega u^* = T_\Omega(u^* - \frac{1}{|\Omega|}\int_\Omega u^* dx) = T_\Omega u^* = \lambda u^*.$$

since $\int_\Omega u^* dx = \int_\Omega u dx = 0$. Consequently λ is an eigenvalue for $\widetilde{T}_\Omega$. Conversely, let λ be an eigenvalue for $\widetilde{T}_\Omega$. There exists $u \in L^2_0(\Omega)$ such that $\widetilde{T}_\Omega u = \lambda u$. Since $u \in Im\widetilde{T}_\Omega$ we get that $u = 0$ in $D\backslash\Omega$ and $\int_\Omega u dx = 0$. Consequently we have $\widetilde{T}_\Omega u = T_\Omega(u - \frac{1}{|\Omega|}\int_\Omega u dx) = T_\Omega(u - \frac{1}{|\Omega|}\int_\Omega u dx) = T_\Omega u = \lambda u$.

□

Here is the shape stability theorem for the eigenvalues of $\widetilde{T}_\Omega$ operator.

Theorem 2.5 *Let* $\Omega_n, \Omega \subseteq D$ *be non empty (quasi) open sets such that* $\Omega_n \xrightarrow{\gamma} \Omega$. *Then* $\widetilde{T}_{\Omega_n} \longrightarrow \widetilde{T}_\Omega$ *in* $\mathcal{L}(L^2_0(D))$ *if and only if* $|\Omega_n| \longrightarrow |\Omega|$.

Proof Assume that $\Omega_n \xrightarrow{\gamma} \Omega$ and $|\Omega_n| \longrightarrow |\Omega|$. There exists f_n in the unit ball of $L^2_0(D)$ which realizes the supremum in

$$\sup_{\|f\|_{L^2_0(D)} \leq 1} \|\widetilde{T}_{\Omega_n} f - \widetilde{T}_\Omega f\|_{L^2_0(D)} = \|\widetilde{T}_{\Omega_n}(f_n) - \widetilde{T}_\Omega(f_n)\|_{L^2_0(D)}. \tag{2.5}$$

We may assume, without restricting the generality that $(f_n)_n$ converges weakly to f. We have

$$\begin{aligned}\|\widetilde{T}_{\Omega_n}(f_n) - \widetilde{T}_\Omega(f_n)\|_{L^2_0(D)} &= \|\widetilde{T}_{\Omega_n}(f_n) - \widetilde{T}_\Omega f + \widetilde{T}_\Omega f - \widetilde{T}_\Omega(f_n)\|_{L^2_0(D)} \\ &\leq \|\widetilde{T}_{\Omega_n}(f_n) - \widetilde{T}_\Omega f\|_{L^2_0(D)} + \|\widetilde{T}_\Omega f - \widetilde{T}_\Omega(f_n)\|_{L^2_0(D)}.\end{aligned}$$

Let $\phi \in L^2_0(D)$. The convergence of $\widetilde{T}_{\Omega_n}\phi$ to $\widetilde{T}_\Omega\phi$ is strong in $L^2_0(D)$, since $\Omega_n \xrightarrow{\gamma} \Omega$. Indeed, $\widetilde{T}_{\Omega_n}\phi = T_{\Omega_n}(\phi - \frac{1}{|\Omega_n|}\int_{\Omega_n}\phi dx) = T_{\Omega_n} g_n$, and g_n converge to g, where $g_n = \phi_{|\Omega_n} - \frac{1}{|\Omega_n|}\int_{\Omega_n}\phi dx$ and $g = \phi_{|\Omega} - \frac{1}{|\Omega|}\int_\Omega \phi dx$. Using the continuity of R_Ω and the hypothesis of γ-convergence, we obtain

$$T_{\Omega_n} g_n = R_{\Omega_n} g_n - \frac{1}{|\Omega_n|}\int_{\Omega_n} R_{\Omega_n} g_n dx \longrightarrow R_\Omega g - \frac{1}{|\Omega|}\int_\Omega R_\Omega g dx = T_\Omega g.$$

Consequently, we have in $L^2_0(D)$

$$< \widetilde{T}_{\Omega_n} f_n, \phi > = < f_n, \widetilde{T}_{\Omega_n}\phi > \longrightarrow < f, \widetilde{T}_{\Omega}\phi > = < \widetilde{T}_{\Omega} f, \phi > .$$

The weak $L^2_0(D)$ convergence of $\widetilde{T}_{\Omega_n}(f_n)$ to $\widetilde{T}_{\Omega}(f)$ implies the strong $L^2_0(D)$ convergence of a subsequence (still denoted using the same index).

To conclude, it remains to discuss $\|\widetilde{T}_{\Omega} f - \widetilde{T}_{\Omega} f_n\|_{L^2_0(D)}$. This term converges to zero as a direct consequence of the continuity/compactness of $\widetilde{T}_{\Omega}$ and the weak $L^2_0(D)$ convergence of $f_n \rightharpoonup f$.

Conversely, we assume that $\Omega_n \xrightarrow{\gamma} \Omega$ and $\widetilde{T}_{\Omega_n} \longrightarrow \widetilde{T}_{\Omega}$. Let $\phi \in L^2_0(D)$. The convergence $\widetilde{T}_{\Omega_n}\phi \longrightarrow \widetilde{T}_{\Omega}\phi$ reads

$$R_{\Omega_n}\phi - \frac{1}{|\Omega_n|}\int_{\Omega_n} R_{\Omega_n}\phi dx - \Big(\frac{1}{|\Omega_n|}\int_{\Omega_n}\phi dx\Big) R_{\Omega_n}1 + \frac{1}{|\Omega_n|^2}\int_{\Omega_n}\phi dx \int_{\Omega_n} R_{\Omega_n} 1 dx$$

$$\longrightarrow R_{\Omega}\phi - \frac{1}{|\Omega|}\int_{\Omega} R_{\Omega}\phi dx - \Big(\frac{1}{|\Omega|}\int_{\Omega}\phi dx\Big) R_{\Omega}1 + \frac{1}{|\Omega|^2}\int_{\Omega}\phi dx \int_{\Omega} R_{\Omega} 1 dx.$$

We denote by $g_n = R_{\Omega_n}1$, $f_n = R_{\Omega_n}\phi$. The hypothesis on the γ convergence yields that g_n and f_n converge strongly $H^1_0(D)$ to $g = R_{\Omega}1$, and $f = R_{\Omega}\phi$, respectively, in $H^1_0(D)$. We obtain

$$-\frac{1}{|\Omega_n|}\int_{\Omega_n}\phi dx g_n - \frac{1}{|\Omega_n|}\int_D f_n dx + \frac{1}{|\Omega_n|^2}\int_{\Omega_n}\phi dx\int_{\Omega_n} g_n dx$$

$$\longrightarrow -\frac{1}{|\Omega|}\int_{\Omega}\phi dx g - \frac{1}{|\Omega|}\int_D f dx + \frac{1}{|\Omega|^2}\int_{\Omega}\phi dx\int_{\Omega} g dx.$$

We can rewrite simply $a_n g_n(\cdot) + b_n \longrightarrow a g(\cdot) + b$, where $a_n = -\frac{1}{|\Omega_n|}\int_{\Omega_n}\phi dx$, $a = -\frac{1}{|\Omega|}\int_{\Omega}\phi dx$, $b_n = -\frac{1}{|\Omega_n|}\int_D f_n dx + \frac{1}{|\Omega_n|^2}\int_{\Omega_n}\phi dx\int_{\Omega_n} g_n dx$ and $b = -\frac{1}{|\Omega|}\int_D f dx + \frac{1}{|\Omega|^2}\int_{\Omega}\phi dx\int_{\Omega} g dx$. Since the function g is non vanishing, we get $a_n \longrightarrow a$.

Assume $|\Omega_n| \longrightarrow \alpha > |\Omega|$ and define ϕ by

$$\phi = \begin{cases} c \text{ in } \Omega \\ -\frac{c|\Omega|}{|D\setminus\Omega|} \text{ in D} \setminus \Omega. \end{cases}$$

where c is a positive constant. The convergence $a_n \longrightarrow a$ implies that $\int_{\Omega_n}\phi dx \longrightarrow \frac{\alpha}{|\Omega|}\int_{\Omega}\phi dx$. On the other hand

$$\int_{\Omega_n}\phi dx = c|\Omega| - c|\Omega|\frac{|\Omega_n\setminus\Omega|}{|D\setminus\Omega|}.$$

We obtain $1 - \frac{|\Omega_n\setminus\Omega|}{|D\setminus\Omega|} \longrightarrow \frac{\alpha}{|\Omega|} > 1$, which is absurd, since $\frac{|\Omega_n\setminus\Omega|}{|D\setminus\Omega|} > 0$. Consequently $\alpha = |\Omega|$. □

A direct consequence of Theorem 2.5 is the following.

Corollary 2.6 *Let $\Omega_n, \Omega \subseteq D$ be such that $\Omega_n \xrightarrow{\gamma} \Omega$ and $|\Omega_n| \longrightarrow |\Omega|$. Then $\lambda^{ct}_k(\Omega_n) \longrightarrow \lambda^{ct}_k(\Omega)$.*

We recall the definition of the weak gamma convergence of quasi open sets. It is said that Ω_n weakly gamma converges to Ω if $R_{\Omega_n}1$ converges weakly $H^1_0(D)$ to a function w, and $\Omega = \{w > 0\}$. We write $\Omega_n \xrightarrow{w\gamma} \Omega$. The main property of the weak gamma convergence is the following: assume that $\phi_n \in H^1_0(\Omega_n)$ is weakly convergent in $H^1_0(D)$ to a function ϕ. Then $\phi \in H^1_0(\Omega)$.

The following result is the key of the optimization problem.

Theorem 2.7 *Assume* $\Omega_n \xrightarrow{w\gamma} \Omega$ *and* $\Omega \subseteq \Omega^*$*, such that* $|\Omega^*| = \liminf_{n\to\infty} |\Omega_n|$*. Then*

$$\lambda_k^{ct}(\Omega^*) \leq \lim_{n\longrightarrow\infty} \lambda_k^{ct}(\Omega_n)$$

Proof Let $\epsilon > 0$ and $S_n^k \subseteq \mathcal{U}_{const}(\Omega_n)$be a k-dimensional space such that:

$$\lambda_k^{ct}(\Omega_n) + \varepsilon \geq \max_{v\in S_n^k\setminus\{0\}} \frac{\int_{\Omega_n} |\nabla v|^2 dx}{\int_{\Omega_n} v^2 dx},$$

and let v_n be a maximizer function such that $\int_{\Omega_n} v_n^2 dx = 1$. Let $v_i^n, i = 1,..,k$ be a L^2_0-orthonormal basis for S_n^k.

We assume that $\liminf_{n\to\infty} \lambda_k^{ct}(\Omega_n) < \infty$, otherwise the conclusion is obvious. Since $\int_{\Omega_n} |v_i^n|^2 dx = 1$ we get that $\int_{\Omega_n} |\nabla v_i^n|^2 dx$ is bounded. Consequently $\int_{\Omega_n} |\nabla(v_i^n - c_i^n)|^2 dx$ is bounded (here $v_i^n - c_i^n \in H^1_0(\Omega_n)$). Up to a subsequence we may assume that $v_i^n - c_i^n$ converges weakly in $H^1_0(D)$ to w_i, so the convergence $\Omega_n \overset{w\gamma}{\to} \Omega$ implies $w_i \in H^1_0(\Omega)$. Let $\Omega^* \subset D$ be such that $|\Omega^*| = \alpha$, where α is such that $|\Omega_n| \longrightarrow \alpha$. Since $H^1_0(\Omega) \subset H^1_0(\Omega^*)$ we obtain $w_i \in H^1_0(\Omega^*)$.

Assume that $|\Omega_n| \to \alpha$ and $c_i^n \longrightarrow c_i$. Then

$$\int_D (v_i^n - c_i^n) dx \to \int_D (w_i) dx \Rightarrow \int_{\Omega_n} v_i^n dx - \int_{\Omega_n} c_i^n dx \to \int_{\Omega^*} w_i dx \Rightarrow -c_i^n|\Omega_n| \to \int_{\Omega^*} w_i dx,$$

and we get $\int_{\Omega^*} w_i dx = -c_i\alpha$. Let $v_i = w_i + c_i \in \mathcal{U}_{const}(\Omega^*)$. Indeed, we have $\int_{\Omega^*} v_i dx = \int_{\Omega^*}(w_i + c_i)dx = -c_i\alpha + c_i|\Omega^*| = 0$.

We introduce the space $S = span(v_1, ..., v_k)$. One can verify that $(v_i, v_j)_{L^2} = \delta_{ij}$ so that S is a k dimensional subspace of $\mathcal{U}_{const}(\Omega^*)$. The construction of S and the convergence $\Omega_n \xrightarrow{w\gamma} \Omega$ implies that for all $v \in S$, $v = \Sigma_{i=1}^k \alpha_i v_i$ the sequence $h_n \in S_n$ defined by $h_n = \Sigma_{i=1}^k \alpha_i v_i^n$ converges weakly to v in $H^1_0(D)$. Moreover,

$$\liminf_{n\longrightarrow\infty} \int_{\Omega_n} |\nabla h_n|^2 dx \geq \int_{\Omega^*} |\nabla v|^2 dx. \tag{2.6}$$

To prove that

$$\liminf_{n\longrightarrow\infty} \frac{\int_{\Omega_n} |\nabla h_n|^2 dx}{\int_{\Omega_n} h_n^2} dx \geq \frac{\int_{\Omega^*} |\nabla v|^2 dx}{\int_{\Omega^*} v^2} dx \tag{2.7}$$

it is enough to prove that

$$\limsup_{n\to\infty} \int_{\Omega_n} h_n^2 dx \leq \int_{\Omega^*} v^2 dx. \tag{2.8}$$

By the definition of v_i we obtain

$$\int_{\Omega^*} v^2 dx = \int_{\Omega^*} (\Sigma_{i=1}^k \alpha_i v_i)^2 dx = \int_{\Omega^*} (\Sigma_{i=1}^k \alpha_i (w_i + c_i))^2 dx$$

$$= \int_{\Omega^*} (\Sigma_{i=1}^k \alpha_i w_i + \Sigma_{i=1}^k \alpha_i c_i)^2 dx = \int_{\Omega^*} (\Sigma_{i=1}^k \alpha_i w_i)^2 dx - 2C^2\alpha + C^2\alpha$$

$$= \int_{\Omega^*} (\Sigma_{i=1}^k \alpha_i w_i)^2 dx - C^2\alpha,$$

where $C = \Sigma_{i=1}^k \alpha_i c_i$. On the other hand we get

$$\int_{\Omega_n} h_n^2 dx = \int_{\Omega_n} (\Sigma_{i=1}^k \alpha_i v_i^n)^2 dx = \int_{\Omega_n} (\Sigma_{i=1}^k \alpha_i (v_i^n - c_i^n) + \Sigma_{i=1}^k \alpha_i c_i^n)^2 dx$$

$$= \int_{\Omega_n} (\Sigma_{i=1}^k \alpha_i (v_i^n - c_i^n))^2 dx + 2\Sigma_{i=1}^k \alpha_i c_i^n \int_{\Omega_n} \Sigma_{i=1}^k \alpha_i (v_i^n - c_i^n) dx + (\Sigma_{i=1}^k \alpha_i c_i^n)^2 |\Omega_n|$$

$$= \int_{\Omega_n} (\Sigma_{i=1}^k \alpha_i (v_i^n - c_i^n))^2 dx - 2C^2|\Omega_n| + C^2\alpha.$$

Inequality (2.8) is equivalent to

$$\limsup_{n\to\infty} (\int_{\Omega_n} (\Sigma_{i=1}^k \alpha_i (v_i^n - c_i^n))^2 dx - 2C^2|\Omega_n| + C^2\alpha) \le \int_{\Omega^*} (\Sigma_{i=1}^k \alpha_i w_i)^2 dx - C^2\alpha \qquad (2.9)$$

Passing to the limit gives

$$-\liminf_{n\to\infty} 2C^2|\Omega_n| + C^2\alpha \le C^2\alpha.$$

Finally,

$$\liminf_{n\to\infty} \lambda_k^{ct}(\Omega_n) + \varepsilon \ge \liminf_{n\to\infty} \max_{v \in S_n^k \setminus \{0\}} \frac{\int_{\Omega_n} |\nabla v|^2 dx}{\int_{\Omega_n} v^2 dx} \ge$$

$$\liminf_{n\to\infty} \frac{\int_{\Omega_n} |\nabla h_n|^2 dx}{\int_{\Omega_n} h_n^2 dx} \ge \max_{v\in S} \frac{\int_{\Omega^*} |\nabla v|^2 dx}{\int_{\Omega^*} v^2 dx} \ge \lambda_k^{ct}(\Omega^*).$$

so $\liminf_{n\to\infty} \lambda_k^{ct}(\Omega_n) + \varepsilon \ge \lambda_k^{ct}(\Omega^*)$. Making $\varepsilon \to 0$, we conclude the proof. □

Theorem 2.8 *Let $\Phi : \mathbb{R}^k \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi continuous function, nondecreasing in each variable. Then*

$$\min_{\Omega \in \mathcal{A}(D), |\Omega| = m} \Phi(\lambda_1^{ct}(\Omega), ..., \lambda_k^{ct}(\Omega)) \qquad (2.10)$$

has at least one solution.

Proof Let us denote $\lambda^{ct}(\Omega) = (\lambda_1^{ct}(\Omega), ..., \lambda_k^{ct}(\Omega))$. From the monotonicity property of λ_k^{ct}, the function $\Phi(\lambda^{ct}(\Omega))$ is non-increasing with respect to set inclusions. We assume that $\Phi(\lambda^{ct}(\Omega))$ is not identically $+\infty$, otherwise every open set of measure m is a solution. We notice that $\Phi(\lambda^{ct}(\Omega))$ is bounded from below by $\Phi(\lambda^{ct}(D))$.

Let Ω_n be a minimizing sequence for the problem (2.10). From the compactness of the $w\gamma$ convergence, we may assume that $\Omega_n \xrightarrow{w\gamma} \Omega$, and we get $|\Omega| \leq m$. If $|\Omega| = m$ than Ω is solution from Theorem 2.7. Otherwise, we find Ω^* such that $|\Omega|^* = m$ and $\Omega \subseteq \Omega^*$. Then, from the monotonicity and the lower semicontinuity of Φ we get $\Phi(\lambda^{ct}(\Omega^*)) \leq \liminf \Phi(\lambda^{ct}(\Omega_n))$, consequently Ω^* is a solution. □

3 The conductivity eigenvalue problem

Let D be a bounded, connected open subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus D$ is connected. For every open set $\Omega \subseteq D$ we introduce the conductivity space

$$H^1_{cond}(\Omega) = cl_{H^1(\Omega)}\{u \in H^1_{loc}(\mathbb{R}^n) \mid \exists \epsilon > 0,\ \nabla u = 0 \text{ a.e. on } (\mathrm{R}^n \backslash \Omega)^\epsilon\}.$$

Here, for a set $K \subseteq \mathbb{R}^n$ and for $\epsilon > 0$ we denote

$$K^\epsilon = \bigcup_{x \in K} B(x, \epsilon).$$

Let $u \in H^1_{cond}(\Omega)$. Clearly, there exists a constant c such that $u - c \in H^1_0(D)$. Indeed, by definition there exists a sequence of functions $u_n \in H^1_{loc}(\mathbb{R}^n)$ such that $\nabla u_n = 0$ on $(\mathbb{R}^n \backslash \Omega)^{\epsilon_n}$ and $u_n \to u$ in $H^1(\Omega)$. Consequently, there exists c_n such that $u_n - c_n \in H^1_0(D)$, and we may see the limit $u_n \to u$ in $H^1(\Omega)$ for the extensions of u_n in $H^1_{loc}(\mathbb{R}^n)$. Since $\nabla u_n = 0$ a.e. on $(\mathbb{R}^n \setminus \Omega)^{\epsilon_n}$ we have that $\nabla u_n \overset{L^2(\mathbb{R}^n)}{\rightharpoonup} \nabla \widetilde{u},\ \widetilde{u} \in H^1_{loc}(\mathbb{R}^n)$. Since $\int_{\mathbb{R}^n} |\nabla u_n|^2 dx = \int_\Omega |\nabla u_n|^2 dx$ we get that $\int_{\mathbb{R}^n} |\nabla u_n - \nabla \widetilde{u}|^2 dx \longrightarrow 0$, where

$$\nabla \widetilde{u} = \begin{cases} \nabla u \text{ on } \Omega \\ 0 \text{ on } \Omega^c. \end{cases} \tag{3.1}$$

Up to a sequence (still denoted using the same index) and since D is bounded and connected, we have $u_n \xrightarrow{L^2(D)} \widetilde{u}$ and $\widetilde{u} = u$ on Ω. Let K be a connected component of $\mathbb{R}^n \setminus \Omega$. We get that $\widetilde{u}$ is quasi everywhere constant on K. In particular, $\widetilde{u}$ equals the constant c on the unbounded component, so that $\widetilde{u} - c \in H^1_0(D)$.

We notice that if Ω_1 and Ω_2 differ on a set of zero capacity, it is possible that $H^1_{cond}(\Omega_1)$ and $H^1_{cond}(\Omega_2)$ are not the same! This is not a real problem for the the study of isoperimetric inequalities, as will be observed in the sequel, but is a hard difficulty for the study of the shape stability question (which remains open).

We introduce the following closed subspace in $H^1_{cond}(\Omega)$

$$\mathcal{U}_{cond}(\Omega) = \{u \in H^1_{cond}(\Omega) | \int_\Omega u dx = 0\}.$$

The mapping $u \longrightarrow \|\nabla u\|_{L^2(\Omega)}$ is a norm on $\mathcal{U}_{cond}(\Omega)$. Moreover, the Poincaré inequality

$$\int_\Omega |u|^2 dx \leq C(D) \int_\Omega |\nabla u|^2 dx$$

holds with a constant $C(D)$ depending only on D. Indeed, we can write for a suitable value c,

$$\int_D (u-c)^2 dx \le C(D) \int_D |\nabla(u-c)|^2 dx,$$

since $u - c \in H^1_0(D)$. Consequently

$$\int_\Omega (u-c)^2 dx \le C(D) \int_\Omega |\nabla u|^2 dx.$$

Since $\int_\Omega u dx = 0$ we get $\int_\Omega u^2 dx \le \int_\Omega (u-c)^2 dx \le C(D) \int_D |\nabla u|^2 dx$. The eigenvalues of the Laplace operator with conductivity boundary condition are defined as the eigenvalues of the operator

$$A_\Omega : L^2_0(\Omega) \longrightarrow L^2_0(\Omega)$$

defined by $A_\Omega f = u$ where u solves the problem

$$u \in \mathcal{U}_{cond}(\Omega) \quad \int_\Omega \nabla u \cdot \nabla \phi dx = \int_\Omega f \phi dx \;\; \forall \phi \in \mathcal{U}_{cond}(\Omega).$$

Alternatively, u minimizes

$$\min_{u \in \mathcal{U}_{cond}(\Omega)} \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega f u dx.$$

Proposition 3.1 *The operator A_Ω is self adjoint, positive and compact.*

Proof The continuity of A_Ω is a consequence of the Poincaré inequality in $\mathcal{U}_{cond}(\Omega)$. Its compactness comes from the compact embedding $\mathcal{U}_{cond}(\Omega) \hookrightarrow L^2_0(\Omega)$, while selfadjointess is inherited from the Laplacian. □

A direct consequence of Proposition 3.1 is that the spectrum of conductivity b.c-Laplacian consists on a sequence of eigenvalues

$$0 < \lambda^{cd}_1(\Omega) \le \lambda^{cd}_2(\Omega) \le ....$$

It is straightforward to observe that

$$\lambda^{cd}_k(t\Omega) = t^{-2} \lambda^{cd}_k(\Omega).$$

The monotonicity property $\Omega_1 \subseteq \Omega_2 \Longrightarrow \lambda^{cd}_k(\Omega_2) \le \lambda^{cd}_k(\Omega_1)$ is a consequence of the min max principle:

$$\lambda^{cd}_k(\Omega) = \min_{V^k \in \mathcal{U}_{cond}(\Omega) \setminus \{0\}} \max_{u \in V^k \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$

For every $V^k \subseteq \mathcal{U}_{cond}(\Omega_1)$, $V^k = span\{v_1, ...v_n\}$ we consider v_k extended to elements v^*_k of $H^1_{cond}(\Omega_2)$ and $v^*_k - c_k \in \mathcal{U}_{cond}(\Omega_2)$. The extension is done as follows: consider a sequence v^ε_k converging in $H^1_{cond}(\Omega_1)$ to v_k, such that $\nabla v^\varepsilon_k = 0$ on $(\mathbb{R}^N \setminus \Omega_1)^\varepsilon$. We extend v^ε_k by suitable constants to elements $(v^\varepsilon_k)^* H^1_{cond}(\Omega_2)$ and define v^*_k as the limit of $(v^\varepsilon_k)^*$.

It is straightforward to observe that

$$\frac{\int_{\Omega_1} |\nabla v_k|^2 dx}{\int_{\Omega_1} v_k^2 dx} \geq \frac{\int_{\Omega_2} |\nabla (v_k^* - c_k)|^2 dx}{\int_{\Omega_2} |v_k^* - c_k|^2 dx}.$$

Indeed, $\int_{\Omega_2} |v_k^* - c_k|^2 dx \geq \int_{\Omega_1} |v_k^1 - c_k|^2 dx \geq \int_{\Omega_1} |v_k|^2 dx$.

Proposition 3.2 *For every $k \in \mathbb{N}$ the following inequality holds true*

$$\lambda_{k-p+1}(\Omega) \leq \lambda_{k-p+1}^{ct}(\Omega) \leq \lambda_k^{cd}(\Omega) \leq \lambda_k^{ct}(\Omega) \leq \lambda_{k+1}(\Omega)$$

where $p \geq 1$ is the number of the connected components of Ω^c

Proof The first and the last inequalities are proved in [10]. The third inequality is a consequence of the min-max formula and the inclusion $\mathcal{U}_{const}(\Omega) \subseteq \mathcal{U}_{cond}(\Omega)$.

For the second inequality, let us consider a k-dimensional subspace $S_k \subset \mathcal{U}_{cond}(\Omega)$. It is easy to notice that S_k has a $k - p + 1$ dimensional subspace $S_{k-p+1} \subset \mathcal{U}_{const}(\Omega)$. Consequently, the min-max formula implies the second inequality. □

The shape stability question for λ_k^{cd} is much more difficult than for λ_k^{ct}. No general continuity result could be obtained, since many small pieces carrying different constants may produce a relaxation process into the limit process, which is not clear how to handle. It turns out that this relaxation process is close to the relaxation of Neumann problems, which is an open question.

Proposition 3.3 *In two dimensions of the space, assume that $\Omega_n \subseteq D$, $\sharp\Omega_n^c \leq M$. If $\Omega_n \xrightarrow{H^c} \Omega$ and $|\Omega_n| \longrightarrow |\Omega|$ then $\lambda_k^{cd}(\Omega_n) \longrightarrow \lambda_k^{cd}(\Omega)$.*

The proof is a consequence of the Mosco convergence $\mathcal{U}_{cond}(\Omega_n)$ to $\mathcal{U}_{cond}(\Omega)$. The difficulty is to handle the constant values of varying functions on merging connected components. We refer the reader to [4] for a discussion of this topic.

4 The isoperimetric inequality for the first conductivity eigenvalue

In [10], the authors proved the following isoperimetric eigenvalue for λ_1^{ct}:

$$\lambda_1^{ct}(B_1 \cup B_2) \leq \lambda_1^{ct}(\Omega)$$

where Ω is a bounded open set of measure m and B_1, B_2 are two disjoint balls of measure $m/2$. The main idea consists in rearranging in the sense of Schwarz $(u - c)^+$ and $(u - c)^-$, for some test function $u \in \mathcal{U}_{const}(\Omega)$, such that $u - c \in H_0^1(\Omega)$. This rearrangement leads to searching the minimizer only among the union of two disjoint ball of total measure equal to m, and an analysis depending on the radius drives to the conclusion that the balls have to be equal.

In the sequel we shall prove a similar isoperimetric inequality for the conductivity eigenvalue. The new difficulty is that on the boundary of Ω, the test function has different constant values, so that several regions with flat parts may appear in the rearrangement procedure. Consequently, a second rearrangement process has to be introduced in order to eliminate the flat parts.

Theorem 4.1 *Let Ω be a bounded open set of $\mathbb{R}^N$. Then*

$$\lambda_1^{cd}(B_1 \cup B_2) \leq \lambda_1^{cd}(\Omega)$$

where B_1 and B_2 are two disjoint balls of measure $\frac{|\Omega|}{2}$.

Proof Let $u \in \mathcal{U}_{cond}(\Omega)$ be a test function such that $\nabla u = 0$ on $(\mathbb{R}^n \setminus \Omega)^\epsilon$. In particular, this implies the existence of a finite number of flat parts of u, corresponding to the (finite number of) connected components of $(\mathbb{R}^n \setminus \Omega)^\epsilon$.

Let $c \in \mathbb{R}$ such that $u - c \in H^1(\mathbb{R}^n)$ (i.e. $u = c$ on the unbounded component of $(\mathbb{R}^n \setminus \Omega)^\epsilon$). We make a Schwarz rearrangement of $(u-c)^+$ and $(u-c)^-$ (see for instance [11, 12]). Let us denote $u^+ = [(u-c)^+]^*$ and $u^- = [(u-c)^-]^*$. If $(u-c)^+ = c_1$ on a connected component K of $(\mathbb{R}^n \setminus \Omega)^\epsilon$, after the rearrangement procedure there is an annulus K_{0,r_1,r_2} with measure at least the measure of $|K|$, where u^+ equals c_1.

The set Ω with the test function u is replaced by the set $\Omega_1 \cup \Omega_2$ with the test function $c - u^-$, $c + u^+$ obtained in the following way. We remove out from the flat part of value c_1 the annulus $K_{0,r_1,r_1'}$ such that $r_1' \leq r_2$, and $|K_{0,r_1,r_1'}| = |K \cap (\mathbb{R}^n \setminus \Omega)|$. Making this procedure for every flat part of u, (without c), we construct the new test function $\widetilde{u}$ consisting on $c - u^-$ on Ω_1 and $c + u^+$ on Ω_2, where Ω_1 and Ω_2 are respectively unions of concentric annuli. In this way, $|\Omega_1 \cup \Omega_2| = |\Omega|$ and $\int_{\Omega_1 \cup \Omega_2} \widetilde{u} dx = 0$, $\int_{\Omega_1 \cup \Omega_2} \widetilde{u}^2 dx = \int_\Omega u^2 dx$. Moreover $\widetilde{u} \in \mathcal{U}_{cond}(\Omega_1 \cup \Omega_2)$ and $\int_{\Omega_1 \cup \Omega_2} |\nabla \widetilde{u}|^2 dx \leq \int_\Omega |\nabla u|^2 dx$.

At this point, we have to search the minimizer only among configurations of the form $(\widetilde{u}, \Omega_1, \Omega_2)$. We make a new rearrangement procedure which will keep constant the measures of the level sets of $\widetilde{u}$ and diminish the Dirichlet energy, in order to create a test function in $\mathcal{U}_{const}$. Consequently, we will be able to use the isoperimetric inequality of Greco and Lucia and conclude the proof. In fact, we will stretch the annuli in order to eliminate the flat parts of $\widetilde{u}$ which are not counted in the measure $|\Omega_1 \cup \Omega_2|$.

It is enough to prove the following result corresponding to one annulus. Let $u \in H_0^1(B(0, r_2))$, $u = 1$ on $B(0, r_1)$ be a positive radial function symmetric in the sense of Schwarz. For $r_2' < r_2$ such that $|B(0, r_2')| \geq |K_{0,r_1,r_2}|$, we introduce the function $c : [r_1, r_2] \longrightarrow [r_1', r_2']$, $c(r) = \sqrt[n]{(r_2')^n - r_2^n + r^n}$, such that $|K_{0,c(r),r_2'}| = |K_{0,r,r_2}|$. We define the rearrangement of u by $\{u^* > c\} = B(0, c(r))$ if c is such that $\{u > c\} = B(0, r)$. Clearly, since the measures of the level sets are constant, we have that

$$\forall\, 1 \leq p < +\infty \quad \int_{K_{0,r_1,r_2}} u^p dx = \int_{K_{0,r_1',r_2'}} (u^*)^p dx.$$

We evaluate the behavior of the gradient. Since u is radial, we can use spherical coordinates to express $\int_{K_{0,r_1,r_2}} |\nabla u|^2 dx$. We notice that $u^* \in H_0^1(B(0, r_2'))$, since $u^*(c(r), \theta_1, ..., \theta_{n-1}) =$

$u(r,\theta_1,...,\theta_{n-1})$, and c is a Lipschitz function. Then, we have (below $\omega(n)$ is the measure of the unit sphere of $\mathbb{R}^N$)

$$\int_{K_{0,r_1,r_2}} |\nabla u|^2 dx = \omega(n) \int_{r_1}^{r_2} (\frac{\partial u}{\partial r})^2 \cdot r^{n-1} dr$$

while

$$\int_{K_{0,r_1',r_2'}} |\nabla u^*|^2 dx = \omega(n) \int_{r_1'}^{r_2'} (\frac{\partial u^*}{\partial r'})^2 \cdot (r')^{n-1} dr'$$

We perform the change of variable

$$r' = c(r) \quad r = c^{-1}(r')$$

and $u^*(r') = u(r) = u(\sqrt[n]{(r')^n + r_2^n - (r_2')^n})$, so that

$$\frac{\partial u^*}{\partial r'} = \frac{\partial u}{\partial r} \cdot \frac{(r')^{n-1}}{\sqrt[n]{[(r')^n + r_2^n - (r_2')^n]^{n-1}}}$$

Consequently

$$\int_{K_{0,r_1',r_2'}} |\nabla u^*|^2 = \omega(n) \int_{r_1'}^{r_2'} (\frac{\partial u}{\partial r})^2 (r') \cdot \frac{(r')^{2(n-1)}}{\sqrt[n]{[(r')^n + r_2^n - (r_2')^n]^{2(n-1)}}} ((r')^{n-1} dr'.$$

and writing $r' = c(r)$ and $dr' = \frac{r^{n-1}}{\sqrt{[(r_2')^n - r_2^n + r^n]^{n-1}}} dr$ so that $(r')^{n-1} dr' = r^{n-1} dr$.

$$\int_{K_{0,r_1',r_2'}} |\nabla u^*|^2 = \omega(n) \int_{r_1}^{r_2} (\frac{\partial u}{\partial r})^2 \cdot \frac{(r')^{2(n-1)} \cdot (r')^{n-1}}{r^{2(n-1)}} \cdot \frac{r^{n-1}}{(r')^{n-1}} dr$$

$$= \omega(n) \int_{r_1}^{r_2} (\frac{\partial u}{\partial r})^2 (\frac{r'}{r})^{2(n-1)} r^{n-1} dr$$

Since $\frac{r'}{r} \leq 1$ we deduce that $\int_{K_{0,r_1',r_2'}} |\nabla u^*|^2 \leq \int_{K_{0,r_1,r_2}} |\nabla u|^2$.

$\square$

5 Numerical results

In this section we address the problem of the numerical minimization of the conductivity eigenvalues with respect to a domain Ω in $\mathbb{R}^2$. We refer the reader to [6, 15] for a detailed exposition of classical shape optimization methods.

We recall that the eigenfunctions have to be constant on each connected component of the boundary of the domain. Since the topology of the optimal sets is unknown, a classical boundary variation optimization is not relevant here. To tackle this difficulty, we propose an optimization procedure based on standard genetic algorithm technics (see [13] for details on stochastic optimization methods). The parametrization of domains and the cost evaluation are the main new features of our approach. Our algorithm has the following principal ingredients:

- description of the unknown sets by a small number of parameters
- identification of the topology of each set (in order to handle the constant levels)
- computation of the conductivity eigenvalues imposing the locally constant boundary conditions (relaying on the topology)

The first point is based on the study of level sets of functions described by Fourier series. Let $(a_{m,n}) \subset [0,1]$ be a family of parameters. We consider the function f defined on $[0,1] \times [0,1]$ by

$$f(x,y) = \sum_{m,n} a_{m,n} \sin \pi n x \cos \pi m y.$$

We associate to the set of parameters $(a_{m,n})$ the open set

$$\Omega := \left\{ (x,y) \in [0,1] \times [0,1], f(x,y) < \frac{\max f}{2} \right\}.$$

By classical linear interpolation technics, it is straightforward to get a complete polygonal approximation of the boundary of Ω. Notice that the boundary of Ω is, by construction, (for almost every family of parameters) a union of none overlapping simple closed curves. We show in figures below two random sets obtained with a family of 30 parameters.

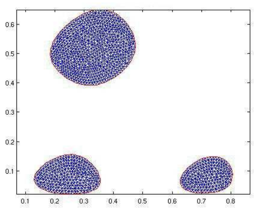

Figure 1: $\lambda_1^{cd} = 45.84$

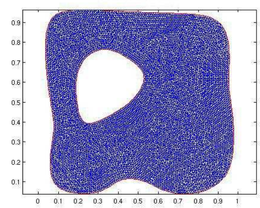

Figure 2: $\lambda_1^{cd} = 53.54$

The next step is to give a complete description of the topology of $\partial\Omega$. We have to determine on which connected components of $\partial\Omega$ the eigenfunction must take the same value. We construct a tree structure whose vertices are the polygonal connected components of $\partial\Omega$ and whose edges represent the "direct" inclusion of those lines. More precisely, two polygonal lines are connected by an edge if one of them contains the other and there is no other polygonal lines contained inbetween. With this tree, it is straightforward to identify the polygonal lines on which the eigenfunctions must have the same value.

Using this information, we can proceed to the computation of eigenvalues. First, we generate a triangular mesh of our set. This has been done with the very performing $2D$ mesh generator *Triangle* (see [14]). Then, we compute the standard stiffness matrix A associated to Neumann boundary condition. Next, we impose the constraints on the boundary by penalization. Imposing the condition $u_i = u_j$ is handled by the matrix penalization indicated below:

	i	...	j
i	$1/\varepsilon$		$-1/\varepsilon$
...			
j	$-1/\varepsilon$		$1/\varepsilon$

In the sequel we present a few examples for the calculus of optimal shapes (together with eigenvalues and eigenfunctions). A good exploration of the space of parameters by genetic algorithms leads to the following numerical results (see figures below) for the shape optimization problems

$$\min_{|\Omega|=m} \lambda_k^{cd}(\Omega), \quad k = 1, 2, 3.$$

We notice that for $k = 1$ we numerically find the optimal solution consisting on two disjoint balls, as Theorem 4.1 states, while for $k = 3$ the optimal shape is a ball.

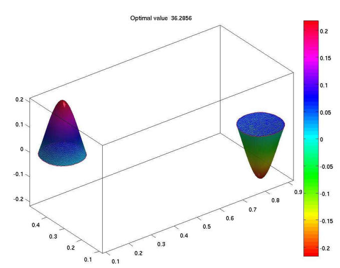

Figure 3: λ_1^{cd}

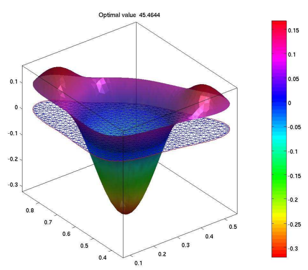

Figure 4: λ_2^{cd}

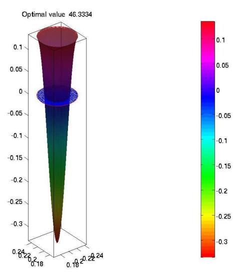

Figure 5: λ_3^{cd}

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