Hilbert domains quasi-isometric to normed vector spaces

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HILBERT DOMAINS QUASI-ISOMETRIC TO NORMED VECTOR SPACES

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Abstract. We prove that a Hilbert domain which is quasi-isometric to a normed vector space is actually a convex polytope.

1. Introduction

A *Hilbert domain* in $\mathbf{R}^m$ is a metric space $(\mathcal{C}, d_{\mathcal{C}})$, where $\mathcal{C}$ is an *open bounded convex* set in $\mathbf{R}^m$ and $d_{\mathcal{C}}$ is the distance function on $\mathcal{C}$ — called the *Hilbert metric* — defined as follows.

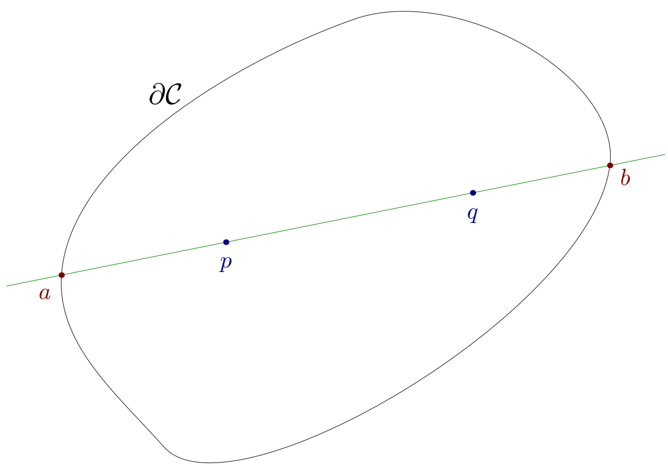
Given two distinct points p and q in $\mathcal{C}$, let a and b be the intersection points of the straight line defined by p and q with $\partial\mathcal{C}$ so that $p = (1-s)a + sb$ and $q = (1-t)a + tb$ with $0 < s < t < 1$. Then

$$d_{\mathcal{C}}(p,q) \coloneqq \frac{1}{2}\ln[a,p,q,b],$$

where

$$[a,p,q,b] \coloneqq \frac{1-s}{s} \times \frac{t}{1-t} > 1$$

is the cross ratio of the 4-tuple of ordered collinear points (a,p,q,b).
We complete the definition by setting $d_{\mathcal{C}}(p,p) \coloneqq 0$.



The metric space $(\mathcal{C}, d_{\mathcal{C}})$ thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of $\mathbf{R}^m$ and in which the affine open segments joining two points of the boundary $\partial\mathcal{C}$ are geodesics that are isometric to $(\mathbf{R}, |\cdot|)$.

For further information about Hilbert geometry, we refer to [4, 5, 9, 11] and the excellent introduction [15] by Socié-Méthou.

The two fundamental examples of Hilbert domains $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbf{R}^m$ correspond to the case when $\mathcal{C}$ is an ellipsoid, which gives the Klein model of m-dimensional hyperbolic geometry (see for

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example [15, first chapter]), and the case when the closure $\overline{\mathcal{C}}$ is a m-simplex for which there exists a norm $\|\cdot\|_{\mathcal{C}}$ on $\mathbf{R}^m$ such that $(\mathcal{C}, d_{\mathcal{C}})$ is isometric to the normed vector space $(\mathbf{R}^m, \|\cdot\|_{\mathcal{C}})$ (see [8, pages 110–113] or [14, pages 22–23]).

Much has been done to study the similarities between Hilbert and hyperbolic geometries (see for example [7], [16] or [1]), but little literature deals with the question of knowing to what extend a Hilbert geometry is close to that of a normed vector space. So let us mention three results in this latter direction which are relevant for our present work.

Theorem 1.1 ([10], Theorem 2)**.** *A Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbf{R}^m$ is isometric to a normed vector space if and only if $\mathcal{C}$ is the interior of a m-simplex.*

Theorem 1.2 ([6], Theorem 3.1)**.** *If $\mathcal{C}$ is an open convex polygonal set in $\mathbf{R}^2$, then $(\mathcal{C}, d_{\mathcal{C}})$ is Lipschitz equivalent to Euclidean plane.*

Theorem 1.3 ([2], Theorem 1.1. See also [17])**.** *If $\mathcal{C}$ is an open set in $\mathbf{R}^m$ whose closure $\overline{\mathcal{C}}$ is a convex polytope, then $(\mathcal{C}, d_{\mathcal{C}})$ is Lipschitz equivalent to Euclidean m-space.*

Recall that a convex *polytope* in $\mathbf{R}^m$ (called a convex *polygon* when $m \coloneqq 2$) is the convex hull of a finite set of points whose affine span is the whole space $\mathbf{R}^m$.

In light of these three results, it is natural to ask whether the converse of Theorem 1.3 — which generalizes Theorem 1.2 in higher dimensions — holds. In other words, if a Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbf{R}^m$ is quasi-isometric to a normed vector space, what can be said about $\mathcal{C}$? Here, by *quasi-isometric* we mean the following (see [3]):

Definition 1.1. Given real numbers $A \geqslant 1$ and $B \geqslant 0$, a metric space (S, d) is said to be (A, B)-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ if and only if there exists a map $f : S \longrightarrow V$ such that

$$\frac{1}{A} d(p, q) - B \leqslant \|f(p) - f(q)\| \leqslant A d(p, q) + B$$

for all $p, q \in S$.

We can now state the result of this paper which asserts that the converse of Theorem 1.3 is actually true:

Theorem 1.4. *If a Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbf{R}^m$ is (A, B)-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geqslant 1$ and $B \geqslant 0$, then $\mathcal{C}$ is the interior of a convex polytope.*

2. Proof of Theorem 1.4

The proof of Theorem 1.4 is based on an idea developed by Förtsch and Karlsson in their paper [10].

It needs the following fact due to Karlsson and Noskov:

Theorem 2.1 ([12], Theorem 5.2)**.** *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a Hilbert domain in $\mathbf{R}^m$ and $x, y \in \partial\mathcal{C}$ such that $[x, y] \not\subseteq \partial\mathcal{C}$. Then, given a point $p_0 \in \mathcal{C}$, there exists a constant $K(p_0, x, y) > 0$ such that for any sequences $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ in $\mathcal{C}$ that converge respectively to x and y in $\mathbf{R}^m$ one can find an integer $n_0 \in \mathbf{N}$ for which we have*

$$d_{\mathcal{C}}(x_n, y_n) \geqslant d_{\mathcal{C}}(x_n, p_0) + d_{\mathcal{C}}(y_n, p_0) - K(p_0, x, y)$$

for all $n \geqslant n_0$.

Now, here is the key result which gives the proof of Theorem 1.4:

Proposition 2.1. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a Hilbert domain in $\mathbf{R}^m$ which is (A,B)-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geqslant 1$ and $B \geqslant 0$.*
Then, if $N = N(A, \|\cdot\|)$ denotes the maximum number of points in the ball $\{v \in V \mid \|v\| \leqslant 2A\}$ whose pairwise distances with respect to $\|\cdot\|$ are greater than or equal to $1/(2A)$, and if $X \subseteq \partial\mathcal{C}$ is such that $[x,y] \nsubseteq \partial\mathcal{C}$ for all $x, y \in X$ with $x \neq y$, we have

$$\mathrm{card}(X) \leqslant N.$$

Proof.
Let $f : \mathcal{C} \longrightarrow V$ such that

$$\frac{1}{A} d_{\mathcal{C}}(p,q) - B \leqslant \|f(p) - f(q)\| \leqslant A d_{\mathcal{C}}(p,q) + B \tag{2.1}$$

for all $p, q \in \mathcal{C}$.

First of all, up to translations, we may assume that $0 \in \mathcal{C}$ and $f(0) = 0$.

Then suppose that there exists a subset X of the boundary $\partial\mathcal{C}$ such that $[x,y] \nsubseteq \partial\mathcal{C}$ for all $x, y \in X$ with $x \neq y$ and $\mathrm{card}(X) \geqslant N+1$. So, pick $N+1$ distinct points $x_1, \ldots, x_{N+1}$ in X, and for each $k \in \{1, \ldots, N+1\}$, let $\gamma_k : [0, +\infty) \longrightarrow \mathcal{C}$ be a geodesic of $(\mathcal{C}, d_{\mathcal{C}})$ that satisfies $\gamma_k(0) = 0$, $\lim\limits_{t \to +\infty} \gamma_k(t) = x_k$ in $\mathbf{R}^m$ and $d_{\mathcal{C}}(0, \gamma_k(t)) = t$ for all $t \geqslant 0$.
This implies that for all integers $n \geqslant 1$ and every $k \in \{1, \ldots, N+1\}$, we have

$$\left\| \frac{f(\gamma_k(n))}{n} \right\| \leqslant A + \frac{B}{n} \tag{2.2}$$

from the second inequality in Equation 2.1 with $p \coloneqq \gamma_k(n)$ and $q \coloneqq 0$.
On the other hand, Theorem 2.1 yields the existence of some integer $n_0 \geqslant 1$ such that

$$d_{\mathcal{C}}(\gamma_i(n), \gamma_j(n)) \geqslant 2n - K(0, x_i, x_j)$$

for all integers $n \geqslant n_0$ and every $i, j \in \{1, \ldots, N+1\}$ with $i \neq j$, and hence

$$\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geqslant \frac{2}{A} - \frac{1}{n}\left(\frac{K(0, x_i, x_j)}{A} + B \right) \tag{2.3}$$

from the first inequality in Equation 2.1 with $p \coloneqq \gamma_i(n)$ and $q \coloneqq \gamma_j(n)$.
Now, fixing an integer $n \geqslant n_0 + AB + \max\{K(0, x_i, x_j) \mid i, j \in \{1, \ldots, N+1\}\}$, we get

$$\left\| \frac{f(\gamma_k(n))}{n} \right\| \leqslant 2A$$

for all $k \in \{1, \ldots, N+1\}$ by Equation 2.2 together with

$$\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geqslant \frac{1}{2A}$$

for all $i, j \in \{1, \ldots, N+1\}$ with $i \neq j$ by Equation 2.3.
But this contradicts the definition of $N = N(A, \|\cdot\|)$.
Therefore, Proposition 2.1 is proved. □

Remark. Given $v \in V$ such that $\|v\| = 2A$, we have $\|-v\| = 2A$ and $\|v - (-v)\| = 2\|v\| = 4A \geqslant 1/(2A)$, which shows that $N \geqslant 2$.

The second ingredient we will need for the proof of Theorem 1.4 is the following:

Proposition 2.2. *Let $\mathcal{C}$ be an open bounded convex set in $\mathbf{R}^2$.*
If there exists a non-empty finite subset Y of the boundary $\partial\mathcal{C}$ such that for every $x \in \partial\mathcal{C}$ one can find $y \in Y$ with $[x, y] \subseteq \partial\mathcal{C}$, then the closure $\overline{\mathcal{C}}$ is a convex polygon.

Proof.
Assume $0 \in \mathcal{C}$ and let us consider the continuous map $\pi : \mathbf{R} \longrightarrow \partial\mathcal{C}$ which assigns to each $\theta \in \mathbf{R}$ the unique intersection point $\pi(\theta)$ of $\partial\mathcal{C}$ with the half-line $\mathbf{R}_+^*(\cos\theta, \sin\theta)$.

For each pair $(x_1, x_2) \in \partial\mathcal{C} \times \partial\mathcal{C}$, denote by $A(x_1, x_2) \subseteq \partial\mathcal{C}$ the arc segment defined by $A(x_1, x_2) \coloneqq \pi([\theta_1, \theta_2])$, where θ_1 and θ_2 are the unique real numbers such that $\pi(\theta_1) = x_1$ and $\pi(\theta_2) = x_2$ with $\theta_1 \in [0, 2\pi)$ and $\theta_1 \leqslant \theta_2 < \theta_1 + 2\pi$.

Before proving Proposition 2.2, notice that adding a point of $\partial\mathcal{C}$ to Y does not change Y's property at all, and therefore we may assume that $\mathrm{card}(Y) \geqslant 2$.

So, write $Y = \{x_1, \ldots, x_n\}$ with $x_1 = \pi(\theta_1), \ldots, x_n = \pi(\theta_n)$, where $\theta_1 \in [0, 2\pi)$ and $\theta_1 < \cdots < \theta_n < \theta_{n+1} \coloneqq \theta_1 + 2\pi$, and let $x_{n+1} \coloneqq \pi(\theta_{n+1}) = x_1$.

Fix $k \in \{1, \ldots, n\}$ and pick an arbitrary $x \in A(x_k, x_{k+1}) \smallsetminus \{x_k, x_{k+1}\}$.

By hypothesis, one can find $y \in Y$ with $[x, y] \subseteq \partial\mathcal{C}$.

Then the convex set $\mathcal{C}$ is contained in one of the two open half-planes in $\mathbf{R}^2$ bounded by the line passing through the points x and y, and hence either $A(x, y) = [x, y]$, or $A(y, x) = [x, y]$.

Since $x_k \in A(y, x)$ and $x_{k+1} \in A(x, y)$, we then have $x_k \in [x, y]$ or $x_{k+1} \in [x, y]$, which yields $A(x_k, x) = [x_k, x]$ or $A(x, x_{k+1}) = [x, x_{k+1}]$.

Conclusion: $A(x_k, x_{k+1}) = S_k \cup S_{k+1}$, where $S_k \coloneqq \{x \in A(x_k, x_{k+1}) \mid A(x_k, x) = [x_k, x]\}$ and $S_{k+1} \coloneqq \{x \in A(x_k, x_{k+1}) \mid A(x, x_{k+1}) = [x, x_{k+1}]\}$.

Now, the set S_k (resp. S_{k+1}) satisfies $[x_k, x] \subseteq S_k$ (resp. $[x, x_{k+1}] \subseteq S_{k+1}$) whenever $x \in S_k$ (resp. $x \in S_{k+1}$).

So, if we consider $\alpha_0 \coloneqq \max\{\theta \in [\theta_k, \theta_{k+1}] \mid A(x_k, \pi(\theta)) = [x_k, \pi(\theta)]\}$, we have $S_k = [x_k, \pi(\alpha_0)]$ and $S_{k+1} = [\pi(\alpha_0), x_{k+1}]$.

Hence, $A(x_k, x_{k+1})$ is the union of the two affine segments $[x_k, \pi(\alpha_0)]$ and $[\pi(\alpha_0), x_{k+1}]$.

Finally, since $\partial\mathcal{C} = \displaystyle\bigcup_{k=1}^{n} A(x_k, x_{k+1})$, this implies that $\partial\mathcal{C}$ is the union of $2n$ affine segments in $\mathbf{R}^2$, and thus $\overline{\mathcal{C}}$ is a convex polygon. $\square$

Before proving Theorem 1.4, let us recall the following useful result, where a convex *polyhedron* in $\mathbf{R}^m$ is the intersection of a finite number of closed half-spaces:

Theorem 2.2 ([13], Theorem 4.7)**.** *Let P be a convex set in $\mathbf{R}^m$ and $p \in \mathring{P}$.*
Then P is a convex polyhedron if and only if all its plane sections containing p are convex polyhedra.

Proof of Theorem 1.4.
Let $(\mathcal{C}, d_{\mathcal{C}})$ be a non-empty Hilbert domain in $\mathbf{R}^m$ that is (A, B)-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geqslant 1$ and $B \geqslant 0$.

According to Theorem 2.2, it suffices to prove Theorem 1.4 for $m \coloneqq 2$ since any plane section of $\mathcal{C}$ gives rise to a 2-dimensional Hilbert domain which is also (A, B)-quasi-isometric to $(V, \|\cdot\|)$.

So, let $m \coloneqq 2$, and consider the set $\mathcal{E} \coloneqq \{X \subseteq \partial\mathcal{C} \mid [x, y] \not\subseteq \partial\mathcal{C}$ for all $x, y \in X$ with $x \neq y\}$.

It is not empty since $\{x, y\} \in \mathcal{E}$ for some $x, y \in \partial\mathcal{C}$ with $x \neq y$ (indeed, $\mathcal{C}$ is a non-empty open set in $\mathbf{R}^2$), which implies together with Proposition 2.1 that $n \coloneqq \max\{\mathrm{card}(X) \mid X \in \mathcal{E}\}$ does exist and satisfies $2 \leqslant n \leqslant N$ (recall that $N \geqslant 2$).

Then pick $Y \in \mathcal{E}$ such that $\text{card}(Y) = n$, write $Y = \{x_1, \ldots, x_n\}$, and prove that for every $x \in \partial\mathcal{C}$ one can find $k \in \{1, \ldots, n\}$ such that $[x, x_k] \subseteq \partial\mathcal{C}$.

Owing to Proposition 2.2, this will show that $\overline{\mathcal{C}}$ is a convex polygon.

So, suppose that there exists $x_0 \in \partial\mathcal{C}$ satisfying $[x_0, x_k] \nsubseteq \partial\mathcal{C}$ for all $k \in \{1, \ldots, n\}$, and let us find a contradiction by considering $Z \coloneqq Y \cup \{x_0\}$.

First, since $x_0 \neq x_k$ for all $k \in \{1, \ldots, n\}$ (if not, we would get an index $k \in \{1, \ldots, n\}$ such that $[x_0, x_k] = \{x_0\} \subseteq \partial\mathcal{C}$, which is false), we have $x_0 \notin Y$. Hence $\text{card}(Z) = n + 1$.

Next, since $Y \in \mathcal{E}$ and $[x_0, x_k] \nsubseteq \partial\mathcal{C}$ for all $k \in \{1, \ldots, n\}$, we have $Z \in \mathcal{E}$.

Therefore, the assumption of the existence of x_0 yields a set $Z \in \mathcal{E}$ whose cardinality is greater than that of Y, which contradicts the very definition of Y.

Conclusion: $\overline{\mathcal{C}}$ is a convex polygon, and this proves Theorem 1.4. □

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