EXPANSIONS OF THE REAL FIELD BY OPEN SETS: DEFINABILITY VERSUS INTERPRETABILITY

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Abstract. An open $U \subseteq \mathbb{R}$ is produced such that $(\mathbb{R}, +, \cdot, U)$ defines a Borel isomorph of $(\mathbb{R}, +, \cdot, \mathbb{N})$ but does not define $\mathbb{N}$. It follows that $(\mathbb{R}, +, \cdot, U)$ defines sets in every level of the projective hierarchy but does not define all projective sets. This result is elaborated in various ways that involve geometric measure theory and working over o-minimal expansions of $(\mathbb{R}, +, \cdot\,)$. In particular, there is a Cantor set $K \subseteq \mathbb{R}$ such that for every exponentially bounded o-minimal expansion $\mathfrak{R}$ of $(\mathbb{R}, +, \cdot\,)$, every subset of $\mathbb{R}$ definable in $(\mathfrak{R}, K)$ either has interior or is Hausdorff null.

The reader is assumed to be familiar with the basics of first-order definability over the real field $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot\,)$, especially o-minimality. Requisite material can be found in van den Dries and Miller [4]. We refer to Kechris [8] and Mattila [10] for basic descriptive set theory and geometric measure theory. We say that a subset of $\mathbb{R}^n$ is **constructible** if it is a boolean combination of open subsets of $\mathbb{R}^n$. By Dougherty and Miller [1], every constructible $E \subseteq \mathbb{R}^n$ is a boolean combination of open sets that are definable in $(\mathbb{R}, <, E)$; we use this fact without further mention.

We begin with a simply-stated question: What can be said about the sets that are definable, allowing arbitrary real parameters, in an expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$ by a collection of constructible subsets of $\mathbb{R}$? First, every quantifier-free definable set is constructible, hence Borel. Next, every existentially definable set is $\mathbf{\Sigma}^1_1$ (also known as Souslin, Suslin, or analytic), and every universally definable set is $\mathbf{\Pi}^1_1$ (also known as co-analytic). Continuing in this fashion, every definable set is projective in the sense of descriptive set theory. All real projective sets are definable in $(\overline{\mathbb{R}}, \mathbb{N})$ [8, 37.6]. As a result, there are now many examples known where $\mathfrak{R}$ defines all real projective sets; see [12, 13] for some non-obvious ones. On the other hand, there are now many examples known where every definable set is Borel; see [5, 6, 12, 15] for some non-o-minimal ones. Heretofore, no other behaviors have been documented. We show in this paper that there is at least one other possibility: $\mathfrak{R}$ can define sets in every $\mathbf{\Sigma}^1_{n+1} \setminus \mathbf{\Sigma}^1_n$, and thus of every projective level, yet not define all real projective sets. As Borel isomorphisms preserve projective level, this is immediate from

Theorem A. *There is a closed $E \subseteq \mathbb{R}$ such that:*

(a) *$(\overline{\mathbb{R}}, E)$ defines a Borel isomorph of $(\overline{\mathbb{R}}, \mathbb{N})$.*
(b) *Every unary (that is, contained in $\mathbb{R}$) set definable in $(\overline{\mathbb{R}}, E)$ either has interior or is nowhere dense.*

Expansions of $\overline{\mathbb{R}}$ in which every unary definable set either has interior or is nowhere dense have a number of good properties [12], but we shall not dwell on this here. On the other hand, this condition is not strong enough to rule out a significant difference between interpretability and definability of interesting algebraic objects.

Corollary. *With E as in Theorem A, $(\overline{\mathbb{R}}, E)$ defines no proper nontrivial subgroups of $(\mathbb{R}, +)$, nor any proper noncyclic subgroups of $(\mathbb{R}^{>0}, \cdot)$, yet $(\overline{\mathbb{R}}, E)$ interprets every real projective set, in particular, every projective subfield of $\overline{\mathbb{R}}$.*

As we shall see (Theorem B, below), we can choose E so that part (b) of Theorem A holds even for $(\overline{\mathbb{R}}, \exp, E)$, in which case neither does $(\overline{\mathbb{R}}, E)$ define any proper nontrivial subgroups of $(\mathbb{R}^{>0}, \cdot)$.

We postpone beginning the proof proper of Theorem A, but we outline some of the main ideas now. In order to satisfy part (b), it suffices by [5, Theorem A] and cell decomposition to produce a closed $E \subseteq \mathbb{R}$ such that:

(b)′ For every $n \in \mathbb{N}$, bounded open semialgebraic cell $U \subseteq \mathbb{R}^n$, and bounded continuous semialgebraic $f: U \to \mathbb{R}$, the image $f(U \cap E^n)$ is nowhere dense, where E^n is the n-th cartesian power of E.

Given any uncountable $\mathbf{\Sigma}^1_1$ set $E \subseteq \mathbb{R}$ such that (b)′ holds, there is a Cantor set[1] K such that both (a) and (b)′ hold with E replaced by K; this is fairly easy modulo known descriptive set theory and some definability tricks (1.14, below). Thus, it suffices to find a Cantor set E such that condition (b)′ holds for E. In order to motivate further developments, we consider a naive approach that we could not make work. Let E be a Cantor set such that every E^n is Hausdorff null (that is, has Hausdorff dimension zero). By cell decomposition, we reduce further to the case that f is C^1 and nowhere locally constant. Write U as the union of the compact sets A_r, $r > 0$, where A_r is the set of $x \in U$ whose distance to the boundary of U is at least $1/r$. Each restriction $f \restriction A_r$ is continuous and Lipshitz, so $f(E^n \cap A_r)$ is compact and Hausdorff null. Since f is nowhere locally constant, $f(E^n \cap A_r)$ is also Cantor for all sufficiently large r. Hence, $f(U \cap E^n)$ is the union of a "semialgebraically parameterized" increasing family of Hausdorff null Cantor sets. But we see no way to conclude from this that $f(U \cap E^n)$ is nowhere dense, which is required in order to employ the aforementioned technology from [5]. The fundamental shortcoming of this approach is that it appears not to account for how limit points of $f(U \cap E^n)$ are formed at the boundary of U. We overcome this by a more careful choice of E based on an analysis of the behavior of bounded semialgebraic functions near their points of discontinuity. It turns out to be just as easy to work in the more general setting of o-minimal expansions of $\overline{\mathbb{R}}$ and prove stronger statements. In doing so, we establish some results in o-minimality (1.1 through 1.8) that seem to be new to the literature even as semialgebraic or subanalytic geometry. We also prove a generalization and some variants of Theorem A, one of which we state now, leaving others for later.

A structure on $\mathbb{R}$ is **exponentially bounded** if for every definable $f: \mathbb{R} \to \mathbb{R}$ there exists $m \in \mathbb{N}$ such that f is bounded at $+\infty$ by the m-th compositional iterate $\exp_m$ of exp. It is an easy consequence of quantifier elimination that $\overline{\mathbb{R}}$ itself is o-minimal and exponentially bounded.

Theorem B. *There is a Cantor set K such that $(\overline{\mathbb{R}}, K)$ defines a Borel isomorph of $(\overline{\mathbb{R}}, \mathbb{N})$ and, for every exponentially bounded o-minimal expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$, every unary set definable in $(\mathfrak{R}, K)$ either has interior or is Hausdorff null.*

[1]For our purposes, a **Cantor set** is a subset of $\mathbb{R}$ that is nonempty, compact, and has neither interior nor isolated points.

For the following reasons, we regard Theorem B as a natural extension of Theorem A and our original motivating question. (i) By cell decomposition, every o-minimal expansion of the real line $(\mathbb{R}, <)$ is interdefinable with a structure on $\mathbb{R}$ generated by a collection of open $U_\alpha \subseteq \mathbb{R}^{n(\alpha)}$, α ranging over some index set. (ii) For any expansion of $(\mathbb{R}, <)$, if every unary definable set either has interior or is Hausdorff null, then every unary definable set either has interior or is nowhere dense. (For every $A \subseteq \mathbb{R}$ and open interval I, at most one of $I \cap A$ and $I \setminus A$ is Hausdorff null.) (iii) By growth dichotomy [11], Pfaffian closure [16], and Lion *et al.* [9], if $\mathfrak{R}$ is an exponentially bounded o-minimal expansion of $\overline{\mathbb{R}}$, then so is $(\mathfrak{R}, \exp)$. In particular, $(\overline{\mathbb{R}}, \exp)$ is o-minimal and exponentially bounded. (iv) There are now many examples of expansions of $\overline{\mathbb{R}}$ that are known to be exponentially bounded and o-minimal.[2]

It is easy to see that a Cantor set is interdefinable over $\overline{\mathbb{R}}$ with the set of midpoints of its complementary intervals (see 2.3), so Theorem B holds with "discrete", even "countable", in place of "Cantor", with similar modifications to Theorem A and its corollary.

As of this writing, every expansion of $\overline{\mathbb{R}}$ known to be o-minimal is exponentially bounded, thus partly justifying our decision to postpone the statement of our most general version (Theorem C in §2). Another reason is simply to avoid for now having to introduce further notation and technical definitions.

Though one might be tempted to regard both Theorems A and B as teratological, we believe that the techniques of the proofs are interesting in their own right, and potentially useful for other settings.

Here is an outline of the remainder of this paper. We begin in Section 1 with some preliminaries, including some results in pure o-minimality that we believe will be useful in other settings. We prove Theorem B in Section 2, as well as some variants and corollaries. We close in Section 3 with discussion and open issues.

1. Preliminaries

We begin by establishing some global conventions and notation. Throughout, "definable" (in some first-order structure) means "definable with parameters", while "$\emptyset$-definable" means "definable without parameters". The variables j, k, m, n range over $\mathbb{N}$, the non-negative integers. Given a set A, its n-fold cartesian power is denoted by A^n, with $A^0 := \{0\}$. Whenever convenient, we identify $A^m \times A^n$ with A^{m+n}, in particular, $A^m \times A^0 \cong A^0 \times A^m \cong A^m$. If A belongs to a topological space, we denote its interior by $\operatorname{int}(A)$, closure by $\operatorname{cl}(A)$, and frontier by $\operatorname{fr}(A) := \operatorname{cl}(A) \setminus A$. If $A \subseteq \mathbb{R}^n$, then all of these sets (in the usual topology) are $\emptyset$-definable in $(\mathbb{R}, <, A)$. Given a set B, we identify a function $f\colon A^0 \to B$ with the constant $f(0) \in B$. Given a function $f\colon A \to B$ and $A' \subseteq A$, we let $f{\restriction}A'$ denote the restriction of f to A'. Limits of functions are always taken with respect to the declared domain of the function, and similarly with limits superior and inferior. Metric notions are taken with respect to the sup norm $|x| := \sup\{|x_1|, \dots, |x_n|\}$. In particular, for $x \in \mathbb{R}^n$ and $r > 0$, put $B(x, r) = \{\, y \in \mathbb{R}^n : |x - y| < r \,\}$.

[2]There are also exponentially bounded expansions of $\overline{\mathbb{R}}$ that are not o-minimal. Such structures are closely related to o-minimality in a certain way, but cannot be obtained by expanding $\overline{\mathbb{R}}$ by collections of constructible sets (of any arities), and thus are irrelevant for present purposes. See [14] for details.

Next are some crucial technical results in o-minimality that seem to be new to the literature even as semialgebraic or subanalytic geometry.[3]

1.1. *Let* $\mathfrak{R}$ *be an o-minimal expansion of* $(\mathbb{R},<)$, $C \subseteq \mathbb{R}^n$ *be a bounded open cell, and* $f\colon C \to \mathbb{R}$ *be definable, continuous, and bounded. Then there is a definable* $X \subseteq \operatorname{fr}(C)$ *such that* $\dim(\operatorname{fr}(C) \setminus X) < n-1$ *and* f *extends continuously to* $C \cup X$.

(Recall that $\dim \emptyset = -\infty$ by convention.)

Note. The resulting extension of f to $C \cup X$ is necessarily definable, as it is given by $y \mapsto \lim_{x\to y} f(x)$.

Proof. As is often the case in o-minimality, we find it convenient to prove simultaneously a related condition. We proceed by induction on $n \geq 1$ to show the following in turn.

- (i_n) There is a definable $Y \subseteq \operatorname{fr}(C)$ such that $\dim(\operatorname{fr}(C) \setminus Y) < n-1$ and C is locally connected at every $y \in Y$.
- (ii_n) There is a definable $X \subseteq \operatorname{fr}(C)$ such that $\dim(\operatorname{fr}(C) \setminus X) < n-1$ and f extends continuously to $C \cup X$.

(A subset A of a topological space X is **locally connected** at $x \in X$ if for every open neighborhood U of x there is an open neighborhood $V \subseteq U$ of x such that $V \cap A$ is connected.)

If $n = 1$, then C is an open interval, so the result is immediate from the monotonicity theorem. Let $n \geq 1$ and assume the result for n. Let $C \subseteq \mathbb{R}^{n+1}$ be a bounded open cell.

(i_{n+1}). Let D be the projection of C on the first n variables. Then D is a bounded open cell and there exist bounded definable continuous functions $g, h\colon D \to \mathbb{R}$ such that $g < h$ and $C = \{\, (x,r) : x \in D \ \ \&\ \ g(x) < r < h(x) \,\}$. Inductively, there exist definable $Z \subseteq \operatorname{fr}(D)$ and definable continuous $G, H\colon D \cup Z \to \mathbb{R}$ such that $\dim(\operatorname{fr}(D) \setminus Z) < n-1$, D is locally connected at every $z \in Z$, $g = G{\restriction}D$, and $h = H{\restriction}D$. Put

$$Y = \operatorname{graph}(g) \cup \operatorname{graph}(h) \cup \{\, (z,r) : z \in Z \ \ \&\ \ G(z) < r < H(z) \,\}.$$

Then

$$\operatorname{fr}(C) \setminus Y \subseteq \operatorname{fr}(\operatorname{graph}(g)) \cup \operatorname{fr}(\operatorname{graph}(h)) \cup [(\operatorname{fr}(D) \setminus Z) \times \mathbb{R}],$$

so $\dim(\operatorname{fr}(C) \setminus Y) < n$. We now show that C is locally connected at every $y \in Y$. As D is open and cells are connected, C is locally connected at every point of $\operatorname{graph}(g) \cup \operatorname{graph}(h)$. Let $z \in Z$ and $r \in \mathbb{R}$ be such that $G(z) < r < H(z)$. Let U be an open set containing (z,r). We must find an open box about the point (z,r) that is contained in U and whose intersection with C is connected. By continuity of G and H, there is an open box $B \times I \subseteq U$ about (z,r) such that $(B \times I) \cap C$ is disjoint from $\operatorname{graph}(g) \cup \operatorname{graph}(h)$. Since D is locally connected at z, we may shrink B so that $B \cap D$ is connected. Then $(B \times I) \cap C$ is connected, because $(B \times I) \cap C = (B \cap D) \times I$.

(ii_{n+1}). Let $f\colon C \to \mathbb{R}$ be definable, continuous, and bounded. Let Z be the set of all $z \in \operatorname{fr}(C)$ such that $\lim_{x\to z} f(x)$ exists. We claim that $\dim(\operatorname{fr}(C) \setminus Z) < n$. Suppose not. Then there is a cell $E \subseteq \operatorname{fr}(C)$ such that $\dim E = n$ and $\liminf_{x\to y} f(x) < \limsup_{x\to y} f(x)$ for every $y \in E$. By (i_{n+1}), we may shrink E so that C is locally connected at every $y \in E$. Then

$$\operatorname{fr}(\operatorname{graph}(f)) \supseteq \{\, (y,r) : y \in E \ \ \&\ \ \liminf_{x\to y} f(x) < r < \limsup_{x\to y} f(x) \,\},$$

[3] We would appreciate any information to the contrary.

yielding the absurdity

$$\dim \operatorname{fr}(\operatorname{graph}(f)) \geq \dim E + 1 = n + 1 = \dim \operatorname{graph}(f) > \dim \operatorname{fr}(\operatorname{graph}(f)).$$

(See van den Dries [3, pp. 65, 68].) Define $g\colon Z \to \mathbb{R}$ by $g(z) = \lim_{x\to z} f(x)$. Let X be the set of points of continuity of g. By cell decomposition, $\dim(Z \setminus X) < \dim Z$, so $\dim(\operatorname{fr}(C) \setminus X) < n$. Finally, define $h\colon C \cup X \to \mathbb{R}$ by $h{\restriction}C = f$ and $h{\restriction}X = g{\restriction}X$. Then h is continuous and extends f. □

Remarks. (i) The result and its proof hold for all abstract o-minimal structures (as defined in [3]) provided that the definition of locally connected is relativized to definable connectedness. (ii) If $C = \{\,(x,y) \in \mathbb{R}^2 : 0 < y < x < 1\,\}$ and $f(x,y) = y/x$, then f is bounded and does not extend continuously to the origin. (iii) If $C = \{\,(x,y,z) \in \mathbb{R}^3 : |x| < 1 \;\&\; 0 < y < 1 \;\&\; -1 < z < \sqrt{|x|/y}\,\}$, then C is bounded and not locally connected at any point of $\{\,(0,0,z) : 0 < z < 1\,\}$.

We need yet more refined results, for which we require some technical definitions and notation.

Define the **corners** in $\mathbb{R}^n$ inductively as follows.[4] (i) $\mathbb{R}^0$ is the only corner in $\mathbb{R}^0$. (ii) If $C \subseteq \mathbb{R}^n$ is a corner and $f\colon C \to (0,\infty)$ is continuous, then

$$\{\,(x,t) \in \mathbb{R}^{n+1} : x \in C \;\&\; 0 < t < f(x)\,\}$$

is a corner in $\mathbb{R}^{n+1}$. We note some easy facts. Every corner in $\mathbb{R}^n$ is an open cell contained in $(0,\infty)^n$. The projection on the first m coordinates of a corner in $\mathbb{R}^{m+n}$ is a corner in $\mathbb{R}^m$. For every cell decomposition $\mathcal{C}$ of $\mathbb{R}^n$ that is compatible with $(0,\infty)^n$ there is a unique $C \in \mathcal{C}$ such that C is a corner.

1.2**.** We now define some special corners. Let Φ be the collection of all homeomorphisms of $[0,\infty)$. Let $\mathcal{S}_n$ be the collection of all nonempty sets of the form

$$\{\, x \in (0,\infty)^n : \phi_n(x_n) < \phi_{n-1}(x_{n-1}) < \cdots < \phi_1(x_1) < b \,\}$$

where $b \in (0,+\infty]$ and $\phi_1,\dots,\phi_n \in \Phi$. Note that this set is an open cell of the structure $(\mathbb{R},<,\phi_1,\dots,\phi_n)$. We interpret $\mathcal{S}_0$ as $\mathbb{R}^0$. Let $S \in \mathcal{S}_n$. Note the easy facts: S is a corner; for every $m \leq n$, the projection of S on the first m coordinates belongs to $\mathcal{S}_m$; and $\operatorname{cl}(S) \setminus [(0,\infty)^{n-1} \times \mathbb{R}] = \{0\}^n$. With $b = +\infty$ and $\phi_i = t \mapsto 2^i t$ for $i = 1,\dots,n$, we obtain the set

$$\mathbb{S}_n := \left\{\, x \in \mathbb{R}^n : 0 < x_n < \frac{x_{n-1}}{2} < \cdots < \frac{x_1}{2^{n-1}} \,\right\}.$$

Note that $\mathbb{S}_n$ is definable in $(\mathbb{R},<,+)$.

1.3**.** *Let $C \subseteq \mathbb{R}^n$ be a corner such that $(\mathbb{R},<,+,C)$ is o-minimal. Then there exists $S \in \mathcal{S}_n$ definable in $(\mathbb{R},<,+,C)$ such that $\operatorname{cl}(S) \cap (0,\infty)^n \subseteq C$.*

Proof. We proceed by induction on $n \geq 1$. (The case $n = 0$ is trivial.) If $n = 1$, then C is an open interval $(0,s)$ for some $s \in (0,\infty]$, so let S be any $(0,r)$ with $r \in (0,s)$.

Let $n \geq 1$ and assume the result for n. Let $C \subseteq (0,\infty)^{n+1}$ be a corner such that $(\mathbb{R},<,+,C)$ is o-minimal. It suffices to consider the case that C is bounded. Let D be the

[4]More precisely, we define the corners "at 0^+". Corners can be defined relative to any point in $[-\infty,\infty]^n$ regarded with a fixed appropriate sign condition, but we shall not need any of these variants in this paper.

projection of C on the first n coordinates. Then D is a bounded definable corner and there is a continuous definable $f\colon D \to (0,\infty)$ such that

$$C = \{\, (x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in D \ \ \& \ \ 0 < x_{n+1} < f(x) \,\}.$$

Inductively, there is a definable $S' \in \mathcal{S}_n$ such that $\mathrm{cl}(S') \cap (0,\infty)^n \subseteq D$. The projection of S' on the last coordinate is an open interval $(0,a)$, and for every $t \in (0,a)$, the set $\{\, x \in \mathrm{cl}(S') : x_n = t \,\}$ is compact. As f is continuous on $\mathrm{cl}(S') \cap (0,\infty)^n$, for all $t \in (0,a)$ we have $0 < \inf\{\, f(x) : x \in \mathrm{cl}(S') \ \ \& \ \ x_n = t \,\} < \infty$. Define $g\colon (0,a) \to \mathbb{R}$ by

$$g(t) = \min(t, \inf\{\, f(x)/2 : x \in \mathrm{cl}(S') \ \ \& \ \ x_n = t \,\}).$$

Note that g is definable. As $g > 0$ and $\lim_{g \to 0^+}(t) = 0$, there exists by the monotonicity theorem some $a' \in (0,a]$ such that $g\restriction(0,a')$ is continuous and strictly increasing. The set $\{\, x \in S' : x_n < a \,\}$ is a corner, so inductively, we may shrink S' so that g is continuous and strictly increasing. Put $S = \{\, (x, x_{n+1}) : x \in S' \ \ \& \ \ x_{n+1} < g(x_n) \,\}$. Now,

$$\mathrm{cl}(S) \cap (0,\infty)^{n+1} \subseteq \{\, (x, x_{n+1}) : x \in \mathrm{cl}(S') \cap (0,\infty)^n \ \ \& \ \ 0 < x_{n+1} \le f(x)/2 \,\} \subseteq C,$$

so it suffices to show that $S \in \mathcal{S}_{n+1}$. Extend g to $\phi \in \Phi$ by setting $\phi(0) = 0$ and $\phi(t) = t - a + g(a)$ for $t > a$. Write

$$S' = \{\, x \in (0,\infty)^n : \phi_n(x_n) < \cdots < \phi_1(x_1) < b \,\}$$

as in 1.2. Then $\phi_n \circ \phi^{-1} \in \Phi$ and

$$S = \{\, (x, x_{n+1}) \in (0,\infty)^{n+1} : \phi_n \circ \phi^{-1}(x_{n+1}) < \phi_n(x_n) < \cdots < \phi_1(x_1) < b \,\} \in \mathcal{S}_{n+1}. \quad \square$$

1.4. *Let $S \in \mathcal{S}_n$, $f\colon S \to \mathbb{R}$ be bounded and continuous, and $(\mathbb{R}, <, +, f)$ be o-minimal. Then there exists $S' \in \mathcal{S}_n$ definable in $(\mathbb{R}, <, +, f)$ such that $S' \subseteq S$ and $f\restriction S'$ extends continuously to $\mathrm{cl}(S')$.*

Proof. We proceed by induction on $n \ge 1$. The case $n = 1$ is immediate from the monotonicity theorem. Let $n \ge 1$ and assume the result for n. Let π denote projection on the first n coordinates. By 1.1 and 1.3, we may shrink πS (hence also S) so that there is a definable continuous $g\colon S \cup (\pi S \times \{0\}) \to \mathbb{R}$ with $f = g\restriction S$. Inductively, we reduce to the case that $g\restriction(\pi S \times \{0\})$ extends continuously and definably to $\mathrm{cl}(\pi S) \times \{0\}$; we denote this extension just by g. By continuity, the set $\{\, x \in S : |f(x) - g(\pi x, 0)| \le x_{n+1} \,\}$ contains a corner, so we reduce by 1.3 to the case that $|f(x) - g(\pi x, 0)| \le x_{n+1}$ for all $x \in S$. Hence, $\lim_{x \to (y,0)} f(x) = g(y,0)$ for all $y \in \mathrm{cl}(\pi S)$, so f extends continuously to $\mathrm{cl}(\pi S) \times \{0\}$. Finally, we shrink S again by 1.3 so that f is continuous on $\mathrm{cl}(S) \cap (0,\infty)^{n+1}$. Then f extends continuously to $\mathrm{cl}(S)$. $\square$

1.5. Let $\mathcal{T}_n$ be the group of symmetries, regarded as linear transformations $\mathbb{R}^n \to \mathbb{R}^n$, of the polyhedron inscribed in the unit ball in $\mathbb{R}^n$ whose vertices are the intersections of the unit sphere in $\mathbb{R}^n$ with the set $\{\, tu : t > 0 \ \ \& \ \ u \in \{-1,0,1\}^n \,\}$. For example, $\mathcal{T}_2$ is the symmetry group of the octagon. With $\mathbb{S}_n$ as in 1.2, we have $\mathbb{R}^n = \bigcup_{T \in \mathcal{T}_n} T(\mathrm{cl}(\mathbb{S}_n)) = \bigcup_{T \in \mathcal{T}_n} \mathrm{cl}(T(\mathbb{S}_n))$. As an immediate consequence of cell decomposition, 1.3, and 1.4, we obtain a key technical lemma:

1.6. *Let $A \subseteq \mathbb{R}^n$ and $f\colon A \to \mathbb{R}$ be such that $(\mathbb{R}, <, +, f)$ is o-minimal. Let f be bounded near $y \in \mathrm{cl}(A)$. Then there exists $S \in \mathcal{S}_n$ definable in $(\mathbb{R}, <, +, f)$ such that for every $T \in \mathcal{T}_n$ and $m \le n$, the restriction of f to $A \cap (y + T(\pi_m S \times \{0\}^{n-m}))$ is continuous and extends continuously to the closure, where π_m denotes projection on the first m coordinates.*

Remark. The result is easily modified to hold for o-minimal expansions of arbitrary ordered groups. As one might imagine, the result can be generalized significantly if $\mathfrak{R}$ also expands an ordered field—recall the triangulation theorem—but it takes a bit of effort to make this precise. As we shall not need any of these of these generalizations in this paper, we leave details to the interested reader.

1.7. We now define some special elements of $\mathcal{S}_n$. First, define $\psi\colon [0,\infty) \to \mathbb{R}$ by

$$\psi(t) = \begin{cases} 0, & t = 0 \\ e^{-1/t}, & 0 < t < 1 \\ t - 1 + e^{-1}, & t \geq 1. \end{cases}$$

Note that $\psi \in \Phi$. For $l \in \mathbb{Z}$, let ψ_l be the l-th compositional iterate of ψ. Every ψ_l is definable in $(\overline{\mathbb{R}}, \exp)$. For $l \geq 0$, $\psi_l(t) = 1/\exp_l(1/t)$ for all sufficiently small $t > 0$, and $\psi_{-l}(t) = 1/\log_l(1/t)$ for all sufficiently small $t > 0$, where $\log_l$ denotes the ultimately defined l-th compositional iterate of log. If $\mathfrak{R}$ is an exponentially bounded expansion of $\overline{\mathbb{R}}$, then for each definable $f\colon (0,b) \to (0,\infty)$ such that $\lim_{t\to 0^+} f(t) = 0$ there is a least $l \in \mathbb{Z}$ such that $f(t) \geq \psi_l(t)$ as $t \to 0^+$; if $\mathfrak{R}$ is also o-minimal, then $f(t) < \psi_{l-1}(t)$ as $t \to 0^+$. Define sets $S_{n,l} \in \mathcal{S}_n$ by

$$S_{n,l} = \{\, x \in \mathbb{R}^n : 0 < x_n < \psi_l(x_{n-1}) < \cdots < \psi_{(n-1)l}(x_1) \,\}.$$

Every $S_{n,l}$ is an open cell of $(\overline{\mathbb{R}}, \exp)$. An easy induction shows that if $S \in \mathcal{S}_n$ and $(\overline{\mathbb{R}}, S)$ is o-minimal and exponentially bounded, then there exist $\delta, l > 0$ such that $B(0,\delta) \cap S_{n,l} \subseteq S$. Hence, by 1.6,

1.8. *Let $A \subseteq \mathbb{R}^n$ and $f\colon A \to \mathbb{R}$ be such that $(\overline{\mathbb{R}}, f)$ is o-minimal and exponentially bounded. Let f be bounded near $y \in \mathrm{cl}(A)$. Then there exist $\delta, j > 0$ such that for every $T \in \mathcal{T}_n$ and $m \leq n$, the restriction of f to $B(y,\delta) \cap A \cap (y + T(S_{m,j} \times \{0\}^{n-m}))$ is continuous and extends continuously to the closure.*

Remark. The above holds with "polynomially" instead of "exponentially" by using the sets

$$\left\{\, x \in \mathbb{R}^m : 0 < x_n < x_{n-1}^l < \cdots < x_1^{l^{m-1}} \,\right\}$$

instead of the $S_{n,l}$.

Our results so far have not required working over $\mathbb{R}$ in that they are easily generalized to abstract o-minimal structures. This now begins to change.

Given $A \subseteq \mathbb{R}^n$ and $r > 0$, let $N(A,r)$ be the infimum of all k such that A is covered by k-many closed cubes of side length r. The set A is **Minkowski null**[5] if $\lim_{r\to 0^+} r^\epsilon N(A,r) = 0$ for all $\epsilon > 0$. While Minkowski nullity is not generally preserved under countable unions or C^1 images, it is much better behaved than Hausdorff nullity in some other ways. We extend this notion relative to expansions $\mathfrak{R}$ of $\overline{\mathbb{R}}$ by defining $A \subseteq \mathbb{R}^n$ to be **$\mathfrak{R}$-null** if $\lim_{r\to 0^+} f(r)N(A,r) = 0$ for every definable $f\colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{r\to 0^+} f(r) = 0$. As $\overline{\mathbb{R}}$ defines all rational power functions, $\mathfrak{R}$-null implies Minkowski null. If $\mathfrak{R}$ is polynomially bounded, then $\mathfrak{R}$-null is the same as Minkowski null. We have some other easy basic facts that will be used often.

1.9. (1) *$\mathfrak{R}$-nullity is preserved under taking subsets, closure, finite unions, finite cartesian products, and images under Lipshitz maps.*

[5] Also known under several other names and equivalent formulations.

(2) *$\mathfrak{R}$-null sets are nowhere dense and Hausdorff null.*
(3) *Countable unions of $\mathfrak{R}$-null sets are Baire meager and Hausdorff null.*
(4) *If every bounded unary definable set either has interior or is $\mathfrak{R}$-null, then every unary definable set either has interior or is Hausdorff null.*
(5) *If $\mathfrak{R}$ is o-minimal and exponentially bounded, then $A \subseteq \mathbb{R}^n$ is $\mathfrak{R}$-null if and only if $\lim_{r\to 0^+} r\exp_m(N(A,r)) = 0$ for all m.*

Proof. We give a sketch for (5) but leave the rest as exercises.

Suppose that A is $\mathfrak{R}$-null. Then $\lim_{r\to 0^+} N(A,r)/\log_m(\epsilon/r) = 0$ for every $m \in \mathbb{N}$ and $\epsilon > 0$. Hence, $N(A,r) < \log_m(\epsilon/r)$ for all sufficiently small $r > 0$. Then $r\exp_m(N(A,r)) < \epsilon$ for all sufficiently small $r > 0$.

Conversely, suppose that $\lim_{r\to 0^+} r\exp_m(N(A,r)) = 0$ for all m. We need only show that $\lim_{r\to 0^+} N(A,r)/\log_m(1/r) = 0$ for all m. By assumption, $\lim_{r\to 0^+} r\exp_{m+1}(N(A,r)) = 0$, so $N(A,r) < \log_{m+1}(1/r)$ for all sufficiently small $r > 0$. Note that $\log_{m+1}(1/r)/\log_m(1/r) \to 0$ as $r \to 0^+$. $\square$

1.10. *Let $\mathfrak{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$. If $A \subseteq \mathbb{R}^m$ is $\mathfrak{R}$-null, $B \subseteq \mathbb{R}^m$ is compact, and $f\colon B \to \mathbb{R}^n$ is continuous and definable, then $f(A \cap B)$ is $\mathfrak{R}$-null.*

Proof. We may take $A \subseteq B$. By generalized Hölder continuity [4, C.15], there is a definable $\phi \in \Phi$ such that $|f(x) - f(y)| \le \phi(|x-y|)$ for all $x, y \in B$. Then $N(f(A), \phi(r)) \le N(A,r)$ for all $r > 0$. Let $g\colon (0,\infty) \to \mathbb{R}$ be definable such that $\lim_{r\to 0^+} g(r) = 0$. Then

$$\lim_{r\to 0^+} g(r)N(f(A),r) = \lim_{r\to 0^+} g(\phi(r))N(f(A),\phi(r)) = \lim_{r\to 0^+} (g\circ\phi)(r)N(A,r) = 0. \qquad \square$$

We next recall a result from [5] and some minor variants. Given a structure $\mathfrak{R}$ on $\mathbb{R}$ and $Y \subseteq \mathbb{R}$, let $(\mathfrak{R}, Y)^{\#}$ denote the expansion $(\mathfrak{R}, (X))$ of $\mathfrak{R}$, where X ranges over all subsets of all cartesian powers of Y.

1.11. *Let $A \subseteq \mathbb{R}$ and $\mathfrak{R}$ be an o-minimal expansion of $(\mathbb{R}, <, +, 1)$. Let $B \subseteq \mathbb{R}$ have no interior and be definable in $(\mathfrak{R}, A)^{\#}$. Then there exists $f\colon \mathbb{R}^n \to \mathbb{R}$ definable in $\mathfrak{R}$ such that $B \subseteq \operatorname{cl}(f(A^n))$. If B is bounded, then f can be taken to be bounded by replacing it with $\max(N, |f|)$ for some N such that $B \subseteq [-N, N]$. This all holds with "$\emptyset$-definable" in place of "definable".*

Thus, as $\mathfrak{R}$-nullity is preserved under taking closure, we have

1.12. *Let $\mathfrak{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$ and $A \subseteq \mathbb{R}$ be such that $f(A^n)$ is $\mathfrak{R}$-null for all bounded definable $f\colon \mathbb{R}^n \to \mathbb{R}$. Then every bounded unary set definable in $(\mathfrak{R}, A)^{\#}$ either has interior or is $\mathfrak{R}$-null.*

We leave the proof of the following amusing result as an exercise.[6]

1.13. *For every $\mathbb{Q}$-linearly independent $A \subseteq \mathbb{R}$, the function $x \mapsto \sum_{i=1}^n 2^{i-1}x_i\colon \mathbb{R}^n \to \mathbb{R}$ is injective on A^n.*

Next is a combination of basic definability and descriptive set theory.

1.14. *Let $A \subseteq \mathbb{R}$ be uncountable and $\boldsymbol{\Sigma}^1_1$. Then there is a Cantor set K such that:*
(1) *$(\mathbb{R}, <, +, 1, K)$ $\emptyset$-defines a Borel isomorph of $(\overline{\mathbb{R}}, \mathbb{N})$.*

[6] We are not aware of this appearing elsewhere. We would appreciate any information to the contrary.

(2) *For every bounded* $f\colon \mathbb{R}^n \to \mathbb{R}$ *there is a finite set* $\mathcal{H}$ *of bounded functions* $\emptyset$*-definable in* $(\mathbb{R}, <, +, 1, f)$ *such that* $f(K^n) \subseteq \bigcup_{h\in\mathcal{H}} h(A^{n(h)})$.

Proof. There exists $l \in \mathbb{Z}$ such that $A \cap [l, l+1]$ is uncountable. By translation, we take $A \subseteq [1, 2]$. Every uncountable analytic set contains a Cantor set, so we take A to be Cantor. Every Cantor set contains a $\mathbb{Q}$-linearly independent Cantor set (use [8, 19.1]), so we also take A to be $\mathbb{Q}$-linearly independent. Let D be the result of removing from A its maximum and minimum, and all left endpoints of the complementary intervals of A. By a classical construction—for example, Gelbaum and Olmstead [7, I.8.14]—there is a strictly increasing bijection $g\colon \mathbb{R} \to D$ whose compositional inverse is continuous. Hence, g is a Borel isomorphism and a closed map. Put

$$X = \{\, \big(g(x), g(y), g(x+y), g(xy), g(\mathrm{d}_{\mathbb{N}}(x))\big) : x, y \in \mathbb{R} \,\}$$

where $\mathrm{d}_{\mathbb{N}}\colon \mathbb{R} \to \mathbb{R}$ denotes the distance function to $\mathbb{N}$. Define $T\colon \mathbb{R}^5 \to \mathbb{R}$ by $T(x) = \sum_{i=1}^{5} 2^{i-1}x_i$. Put $K = A \cup T(\mathrm{cl}(X))$. As X has no isolated points, the same is true of $\mathrm{cl}(X)$, hence also of $T(\mathrm{cl}(X))$. Since $\mathrm{cl}(X) \subseteq A^5$ and $T{\restriction}A^5$ is injective (1.13), $T(\mathrm{cl}(X))$ is compact and has no interior. Thus, $T(\mathrm{cl}(X))$ is Cantor, hence so is K.

(1). We have $T(A^5) \subseteq (2, \infty)$, so $A = K \cap [1, 2]$ and $(\mathbb{R}, <, +, 1, K)$ $\emptyset$-defines A, hence also D. As g is a closed map and $T{\restriction}A^5$ is injective, we have $X = D^5 \cap T^{-1}(K \setminus A)$. Thus, $(\mathbb{R}, < +, 1, K)$ $\emptyset$-defines X. Observe that $(\mathbb{R}, X)$ $\emptyset$-defines the image under g of $(\overline{\mathbb{R}}, \mathrm{d}_{\mathbb{N}})$, hence also that of $(\overline{\mathbb{R}}, \mathbb{N})$.

(2). Straightforward, but tedious to write up in detail. We illustrate the point via the case $n = 2$ and leave the rest to the reader. Since $K \subseteq A \cup T(A^5)$, we have

$$K^2 \subseteq A^2 \;\cup\; A \times T(A^5) \;\cup\; T(A^5) \times A \;\cup\; (T(A^5))^2.$$

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be bounded. Put $\mathcal{H} = \{h_1, \dots, h_4\}$, where

$$\begin{aligned} h_1 &= f(x_1)\\ h_2 &= f(x_1, T(x_2, \dots, x_6))\\ h_3 &= f(T(x_1, \dots, x_5), x_6)\\ h_4 &= f(T(x_1, \dots, x_5), T(x_6, \dots, x_{10})). \end{aligned}$$

Then $f(K^2) \subseteq h_1(A^2) \cup h_2(A^6) \cup h_3(A^6) \cup h_4(A^{10})$. $\square$

Remark. By using distance functions, the set X is easily modified to encode over $(\mathbb{R}, <, +, 1)$ any expansion of $(\mathbb{R}, <, +, 1)$ by finitely many closed sets.

2. Main results

In this section, we prove Theorem B, as well as some variants and corollaries.

Proof of Theorem B. By diagonalization, we fix a sequence (r_k) of positive real numbers such that $r_0 = 1$, $2r_{k+1} < r_k$ for all k, and $\lim_{k\to+\infty} r_{k+1}/\psi_j(r_k) = 0$ for all j (ψ_j as in 1.7). Define sets E_k inductively by $E_0 = [0, 1]$ and $E_{k+1} = E_k \setminus \bigcup_c (c + r_{k+1}, c + r_k - r_{k+1})$, where c ranges over the left endpoints of the connected components of E_k. Then $\bigcap_k E_k$ is Cantor. Let E be a $\mathbb{Q}$-linearly independent Cantor set contained in $\bigcap_k E_k$ (recall the proof of 1.14). Let K be a Cantor set constructed as in 1.14 with $A = E$; then $(\overline{\mathbb{R}}, K)$ defines a Borel isomorph of $(\overline{\mathbb{R}}, \mathbb{N})$. Let $\mathfrak{R}$ be an exponentially bounded o-minimal expansion of $\overline{\mathbb{R}}$. We must show that every unary set definable in $(\mathfrak{R}, K)$ either has interior or is Hausdorff null.

Indeed, with an eye toward applications and further generalization, we show that every bounded unary set definable in $(\mathfrak{R}, K)^{\#}$ either has interior or is $\mathfrak{R}$-null (recall 1.9.4). We use repeatedly that $\mathfrak{R}$-nullity is preserved under finite unions. By 1.12 and 1.14, it suffices to let $f\colon \mathbb{R}^n \to \mathbb{R}$ be bounded and definable in $\mathfrak{R}$, and show that $f(E^n)$ is $\mathfrak{R}$-null. By compactness of E^n, it is enough to show that if $x \in E^n$ then $f(E^n \cap B(x,\delta))$ is $\mathfrak{R}$-null for some $\delta > 0$. By 1.8, there exist $\delta, j > 0$ such that for every $T \in \mathcal{T}_n$ and $m \le n$, the restriction of f to $(x + T(S_{m,j} \times \{0\}^{n-m})) \cap B(x,\delta)$ is continuous and extends continuously to the closure. Thus, it suffices now by 1.10 to show that:

(i) Every E^n is $\mathfrak{R}$-null.

(ii) For every $j > 0$ there exists $\delta > 0$ such that for all n,

$$E^n - E^n \cap (-\delta,\delta)^n \subseteq \bigcup_{m\le n} \bigcup_{T\in\mathcal{T}_n} T(S_{m,j} \times \{0\}^{n-m}),$$

where $E^n - E^n$ denotes the difference set of E^n.

Proof of (i). As $\mathfrak{R}$-nullity is preserved under finite cartesian products, it suffices by 1.9.5 to fix m and show that $\lim_{r\to 0^+} r \exp_m(N(E,r)) = 0$. Let $0 < r < 1$; then there exists k such that $r_{k+1} \le r < r_k$. For every j, the set E_j consists of 2^j disjoint closed intervals each of length r_j, so

$$r \exp_m(N(E,r)) \le r \exp_m(2^{k+1}) < r_k \exp_{m+1}(k+1).$$

If r is sufficiently small, then $r_{k-1} \le 1/(k+1)$ and $\exp_{m+1}(k+1) = 1/\psi_{m+1}(1/(k+1))$, so $r \exp_m(N(E,r)) \le r_k/\psi_{m+1}(r_{k-1})$. By construction, $\lim_{k\to+\infty} r_k/\psi_{m+1}(r_{k-1}) = 0$.

Proof of (ii). As $j > 0$, we have $t - \psi_j(t) > 0$ for all sufficiently small $t > 0$. By exponential bounds and properties of (r_k), we have

$$\lim_{k\to+\infty} \frac{r_{k+1}}{r_k - \psi_j(r_k)} = 0 = \lim_{k\to+\infty} \frac{r_{k+1}}{\psi_j(\psi_j(r_k))}.$$

Thus, there exists $N \in \mathbb{N}$ such that for all $k > N$ we have $r_{k+1} < \psi_j(\psi_j(r_k))$ and $\psi_j(r_k) < r_k - r_{k+1}$, hence also $r_{k+1} < \psi_j(r_k - r_{k+1})$. Put $\delta = r_N$. We now proceed by induction on n. The case $n \le 1$ is trivial. Let $n > 1$ and assume the result for all lower values of n. The argument is routine, but a bit tedious to write up in detail; we give only an outline. By permuting coordinates, it is enough to show that

$$E^n - E^n \cap (0,\delta)^n \subseteq \bigcup_{m\le n} \bigcup_{T\in\mathcal{T}_n} T(S_{m,j} \times \{0\}^{n-m}).$$

By $\mathbb{Q}$-linear independence and symmetry, it is enough to show that $E^n - E^n \cap (0,\delta)^n \cap \mathbb{S}_n \subseteq S_{n,j}$ (recall 1.2 and 1.5). Let $(x, x_{n-1}, x_n), (y, y_{n-1}, y_n) \in E^{n-2} \times E \times E$ be such that $(x, x_{n-1}, x_n) - (y, y_{n-1}, y_n) \in (0,\delta)^n \cap \mathbb{S}_n$. Inductively, $(x, x_{n-1}) - (y, y_{n-1}) \in S_{n-1,j}$, so it suffices to show that $x_n - y_n < \psi_j(x_{n-1} - y_{n-1})$. Let k be such that $r_{k+1} < x_{n-1} - y_{n-1} \le r_k$. It suffices now by choice of δ and monotonicity of ψ_j to show that $x_n - y_n \le r_{k+1}$. By construction of E, we have

$$E^2 \cap [0, r_k]^2 \cap \mathbb{S}_2 \subseteq [0, r_{k+1}]^2 \cup [r_k - r_{k+1}, r_k] \times [0, r_{k+1}].$$

Since $x_{n-1} - y_{n-1} \in [r_k - r_{k+1}, r_k]$, we have $x_n - y_n \in [0, r_{k+1}]$. □

Having established Theorem B, we now proceed to some variants and corollaries.

Following [6], we say that a sequence (a_k) of positive real numbers is **fast** for an expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$, or **$\mathfrak{R}$-fast**, if $\lim_{k\to+\infty} f(a_k)/a_{k+1} = 0$ for every $f\colon \mathbb{R}\to\mathbb{R}$ definable in $\mathfrak{R}$. For (r_k) as in the proof of Theorem B, the sequence $(1/r_k)$ is fast for every exponentially bounded expansion of $\overline{\mathbb{R}}$. An examination of the proof of Theorem B yields the following generalization.

Theorem C. *Let (a_k) be a sequence of positive real numbers. Then there is a Cantor set K such that $(\mathbb{R},<,+,1,K)$ $\emptyset$-defines a Borel isomorph of $(\overline{\mathbb{R}},\mathbb{N})$, and for every o-minimal expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$, if (a_k) is $\mathfrak{R}$-fast, then every bounded unary set definable in $(\mathfrak{R},K)^{\#}$ either has interior or is $\mathfrak{R}$-null.*

Under fairly reasonable assumptions, $\mathfrak{R}$-fast sequences exist.

2.1. *Let $\mathfrak{R}$ be an expansion of $\overline{\mathbb{R}}$.*

(1) *If there is a countable collection $\mathcal{F}$ of functions $\mathbb{R}\to\mathbb{R}$ such that every unary function definable in $\mathfrak{R}$ is bounded at $+\infty$ by a member of $\mathcal{F}$, then there exist $\mathfrak{R}$-fast sequences.*

(2) *If $\mathfrak{R}$ is o-minimal and the language of $\mathfrak{R}$ is countable, then there exist $\mathfrak{R}$-fast sequences.*

Proof. Diagonalization yields (1). For (2), suppose that $\mathfrak{R}$ is o-minimal. By the proof of [4, C.4], every unary function definable in $\mathfrak{R}$ is bounded at $+\infty$ by a unary function $\emptyset$-definable in $\mathfrak{R}$. Since the language is countable, there are only countably many unary functions $\emptyset$-definable in $\mathfrak{R}$. Apply (1). □

When combined with Theorem C,

2.2. *Let $(\mathfrak{R}_k)_{k\in\mathbb{N}}$ be a sequence of o-minimal expansions of $\overline{\mathbb{R}}$, each in a countable language. Then there is a Cantor set K such that $(\overline{\mathbb{R}},K)$ defines a Borel isomorph of $(\overline{\mathbb{R}},\mathbb{N})$ and every bounded unary set definable in any $(\mathfrak{R}_k,K)^{\#}$ either has interior or is Minkowski null.*

2.3. *Theorem C holds with "discrete", hence also "countable", in place of "Cantor".*

Proof. Let E be any Cantor set and M be the set of midpoints of the (bounded) complementary intervals of E. Note that

$$M = \{\, r\in\mathbb{R} : \exists \epsilon>0, E\cap[r-\epsilon,r+\epsilon] = \{r-\epsilon,r+\epsilon\}\,\}$$

and $E = \operatorname{fr}(M)$. Hence, $(\overline{\mathbb{R}},E)$ and $(\overline{\mathbb{R}},M)$ are $\emptyset$-interdefinable. □

We now answer a question raised in [12, §3.1]. A set $A\subseteq\mathbb{R}^n$ **has a locally closed point** if it has nonempty interior in its closure. It is easy to see that if every nonempty unary set definable in $\mathfrak{R}$ has a locally closed point, then every definable unary set either has interior or is nowhere dense. We show that the converse fails.

2.4. *There exist $\emptyset\neq A\subseteq\mathbb{R}$ having no locally closed points such that every unary set definable in $(\overline{\mathbb{R}},A)$ either has interior or is nowhere dense.*

Proof. With K as in Theorem B, let A be the set of left endpoints of the complementary intervals of K. Note that $A\subseteq K$ and $\operatorname{cl}(A)=\operatorname{cl}(K\setminus A)=K$. □

3. Discussion and open issues

It is easy to construct Cantor sets E that do define $\mathbb{N}$ over $\overline{\mathbb{R}}$: Just encode $\mathbb{N}$ by the set of lengths of the complementary intervals. For example, remove successively from $[0,1]$ the middle intervals of length $1/(n!)$ for $n \geq 2$. Then $(\overline{\mathbb{R}}, E)$ defines the set $A := \{\, n! : n \in \mathbb{N} \,\}$, hence also the successor function $\sigma\colon A \to A$, hence also $\mathbb{N} = \{0\} \cup \{\, \sigma(a)/a : a \in A \,\}$.

Though Theorem A answers one question, others arise immediately. Are there closed $E \subseteq \mathbb{R}$ and $N \in \mathbb{N}$ such that $(\overline{\mathbb{R}}, E)$ defines a non-Borel set, yet every set definable in $(\overline{\mathbb{R}}, E)$ is $\mathbf{\Sigma}^1_N$? (If so, then $N \geq 2$ by Souslin's Theorem [8, 14.11].) Evidently, one can generalize this question. For example, allow E to be constructible, or F_σ, or of finite Borel rank. We can go in the other direction as well. In Theorem A, can we take E to be closed and countable? Closed and discrete? (*cf.* 2.3.) Regarding the last, there are some known restrictions. If $E \subseteq (0, \infty)$ is infinite, closed and discrete, then it is the range of a strictly monotone and unbounded-above sequence (a_k) of positive real numbers. If $(\log a_{k+1})/(\log a_k) \to \infty$, that is, if (a_k) is $\overline{\mathbb{R}}$-fast, then every set definable in $(\overline{\mathbb{R}}, E)$ is constructible [6]. On the other hand, if $(a_k) = (f(k))$ for some sufficiently well behaved $f\colon \mathbb{R} \to \mathbb{R}$—in particular, if $(\overline{\mathbb{R}}, f)$ is o-minimal—and $(\log a_k)/k \to 0$, then $(\overline{\mathbb{R}}, E)$ defines $\mathbb{N}$. See [13] for a proof of the latter and some information on behavior between these extremes.

(There are non-Borel $E \subseteq \mathbb{R}$ such that every set definable in $(\overline{\mathbb{R}}, E)$ is $\mathbf{\Sigma}^1_2$, but this is far off the point of this paper, so we give only a hint: By van den Dries [2, Theorem 1], if E is $\mathbf{\Sigma}^1_N$ and a real-closed subfield of $\mathbb{R}$, then every set definable in $(\overline{\mathbb{R}}, E)$ is a boolean combination of $\mathbf{\Sigma}^1_N$ sets.)

Currently, we know of no expansions of $\overline{\mathbb{R}}$ by constructible subsets of $\mathbb{R}$ that define non-constructible sets, yet every definable set is Borel. There are candidates, though, as we explain in the next two paragraphs.

The proof of 2.4 shows that if E is a Cantor set, then $(\overline{\mathbb{R}}, E)$ defines a unary set that is not F_σ (hence not constructible). Are there Cantor sets E such that every set definable in $(\overline{\mathbb{R}}, E)$ is a boolean combination of F_σ sets? Along these lines, the Cantor set K in 1.14 was designed to encode $(\overline{\mathbb{R}}, \mathbb{N})$. But we could omit deliberately encoding $\mathbb{N}$, that is, in the proof of 1.14, replace X with its projection on the first four coordinates. Define E as in the proof of Theorem B, and so on. What can be said about the definable sets of $(\overline{\mathbb{R}}, K)$? (Of course, by the rest of the proof of Theorem B, every unary definable set either has interior or is Hausdorff null.) Or, in the definition of X, replace $\mathrm{d}_{\mathbb{N}}$ with exp. Then $(\overline{\mathbb{R}}, K)$ Borel-interprets $(\overline{\mathbb{R}}, \exp)$. Is $\mathbb{N}$ definable? Is exp? And so on. The technology from [5] appears not to tell us much more about the sets definable in $(\overline{\mathbb{R}}, E)$ than those in $(\overline{\mathbb{R}}, E)^\#$, so it seems that new ideas are needed.

Given $\alpha > 0$, it is known that every set definable in $(\overline{\mathbb{R}}, \alpha^{\mathbb{N}})$ is constructible [14, §4] (and more [5, 12]), where $\alpha^{\mathbb{N}} = \{\, \alpha^n : n \in \mathbb{N} \,\}$. The expansion of $\overline{\mathbb{R}}$ by $\{\, 2^n + 3^n : n \in \mathbb{N} \,\}$ defines both $2^{\mathbb{N}}$ and $3^{\mathbb{N}}$ (see [13]). The multiplicative group $2^{\mathbb{Z}} \cdot 3^{\mathbb{Z}}$ is dense and co-dense in $(0, \infty)$, hence not constructible, and is definable in $(\overline{\mathbb{R}}, 2^{\mathbb{N}}, 3^{\mathbb{N}})$. Aside from this, very little is known about $(\overline{\mathbb{R}}, 2^{\mathbb{N}}, 3^{\mathbb{N}})$, in particular, whether it defines any sets that are not $\mathbf{\Delta}^0_3$. There is nothing special here about the generators 2 and 3: the situation is the same for any $\alpha, \beta > 0$ provided that $\log \alpha$ and $\log \beta$ are $\mathbb{Q}$-linearly independent.

We close with a few words on history and attribution. Recall that our original goal was to find a constructible $E \subseteq \mathbb{R}$ such that $(\overline{\mathbb{R}}, E)$ defines a non-Borel set but does not define

$\mathbb{N}$. Every expansion of $\overline{\mathbb{R}}$ that defines $\mathbb{N}$ also $\emptyset$-defines it, hence also $\mathbb{Q}$, because $\mathbb{N}$ is the unique subset of $[0, \infty)$ that is closed under $x \mapsto x + 1$ and whose intersection with $[0, 1]$ is equal to $\{0, 1\}$. Hence, the following *a priori* weaker version of Theorem A due to Friedman and Miller suffices.

Theorem A$_0$. *There is a closed $E \subseteq \mathbb{R}$ such that $(\overline{\mathbb{R}}, E)$ defines a Borel isomorph of $(\overline{\mathbb{R}}, \mathbb{N})$ and every unary set $\emptyset$-definable in $(\overline{\mathbb{R}}, E)$ either has interior or is nowhere dense.*

For this, it suffices by 1.11 and 1.14 to find a Cantor set E such that $f(E^n)$ is nowhere dense for every $f\colon \mathbb{R}^n \to \mathbb{R}$ that is $\emptyset$-definable in $\overline{\mathbb{R}}$. The naive approach described in the introduction appears to stall for functions $\emptyset$-definable in $\overline{\mathbb{R}}$ just as it does for parametrically definable functions. But in any o-minimal structure in a countable language, there are only countably many $\emptyset$-definable functions, and they are all Borel (by cell decomposition). This prompted Friedman to produce the following result of independent interest, thus establishing an appropriately modified version of 2.2, hence also Theorem A$_0$.

3.1. *For every sequence $(f_k\colon \mathbb{R}^{n(k)} \to \mathbb{R})_{k\in\mathbb{N}}$ of Borel functions there is a Cantor set E such that every image $f_k(E^{n(k)})$ is nowhere dense.*

We shall not prove this here as we no longer need it; the interested reader may wish to attempt verification by amalgamating the proofs of 19.1 and 19.8 from [8]. On the other hand, attempts by Friedman and Miller to derive Theorem A from Theorem A$_0$ were unsuccessful, as were attempts to conclude from 3.1 the existence of a Cantor set E such that $f(E^n)$ is nowhere dense for every semialgebraic $f\colon \mathbb{R}^n \to \mathbb{R}$. Of course, 3.1 is a very blunt hammer in this setting, as it uses nothing about o-minimality (yet relies heavily on countability). Convinced that a more singularity-theoretic approach was in order, Miller approached Kurdyka and Speissegger, who subsequently solved the semialgebraic case, which Miller then refined to its current form. The crucial idea of using Minkowksi rather than Hausdorff nullity is due to Kurdyka. Result 1.1 is due to Speissegger, who had known it for several years but not made prior use of it.

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