Grothendieck ring of semialgebraic formulas and motivic real Milnor fibers

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GROTHENDIECK RING OF SEMIALGEBRAIC FORMULAS AND MOTIVIC REAL MILNOR FIBRES

by

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Abstract. — We define a Grothendieck ring for basic real semialgebraic formulas, that is for systems of real algebraic equations and inequalities. In this ring the class of a formula takes into consideration the algebraic nature of the set of points satisfying this formula and contains as a ring the usual Grothendieck ring of real algebraic formulas. We give a realization of our ring that allows to express a class as a $\mathbb{Z}[\frac{1}{2}]$-linear combination of classes of real algebraic formulas, so this realization gives rise to a notion of virtual Poincaré polynomial for basic semialgebraic formulas. We then define zeta functions with coefficients in our ring, built on semialgebraic formulas in arc spaces. We show that they are rational and relate them to the topology of real Milnor fibres.

Introduction

Let us consider the category $SA(\mathbb{R})$ of real semialgebraic sets, the morphisms being the semialgebraic maps. We denote by $(K_0(SA(\mathbb{R})), +, \cdot)$, or simply $K_0(SA(\mathbb{R}))$, the Grothendieck ring of $SA(\mathbb{R})$, that is to say the free ring generated by all semialgebraic sets A, denoted by $[A]$ as viewed as element of $K_0(SA(\mathbb{R}))$, in such a way that for all objects A, B of $SA(\mathbb{R})$ one has : $[A \times B] = [A] \cdot [B]$ and for all closed semialgebraic set F in A one has : $[A \setminus F] + [F] = [A]$ (this implies that for every semialgebraic sets A, B, one has: $[A \cup B] = [A] + [B] - [A \cap B]$).

When furthermore an equivalence relation for semialgebraic sets is previously considered for the definition of $K_0(SA(\mathbb{R}))$, one has to be aware that the induced quotient ring, still denoted for simplicity by $K_0(SA(\mathbb{R}))$, may dramatically collapse. For instance let us consider the equivalence relation $A \sim B$ if and only if there exists a semialgebraic bijection from A to B. In this case we simply say that A and B are isomorphic. Then for the definition of $K_0(SA(\mathbb{R}))$, starting from classes of isomorphic sets instead of simply sets, one obtains a quite trivial Grothendieck ring, namely $K_0(SA(\mathbb{R})) = \mathbb{Z}$. Indeed, denoting $[\mathbb{R}]$ by $\mathbb{L}$ and $[\{*\}]$ by $\mathbb{P}$, from the fact that $\{*\} \times \{*\} \sim \{*\}$, one gets

$$\mathbb{P}^k = \mathbb{P},\ \forall k \in \mathbb{N}^*,$$

and from the fact that $\mathbb{R} =]-\infty, 0[\cup \{0\} \cup]0, +\infty[$ and that intervals of same type are isomorphic, one gets

$$\mathbb{L} = -\mathbb{P}.$$

On the other hand, by the semialgebraic cell decomposition theorem, we obtain that a real semialgebraic set is a finite union of disjoint open cells, each of them being isomorphic to $\mathbb{R}^k$, with $k \in \mathbb{N}$ (with the convention that $\mathbb{R}^0 = \{*\}$). It follows that $K_0(SA(\mathbb{R})) =< \mathbb{P} >$, the ring generated by $\mathbb{P}$. At this point, the ring $< \mathbb{P} >$ could be trivial. But one knows that the Euler-Poincaré characteristic with compact supports $\chi_c : SA(\mathbb{R}) \to \mathbb{Z}$ is surjective, and since χ_c is additive, multiplicative and invariant under isomorphims, it factors through $K_0(SA(\mathbb{R}))$, giving a surjective morphism of rings, and finally an isomorphism of rings, still denoted for simplicity by χ_c (cf also [**15**])

$$\begin{tikzcd}
SA(\mathbb{R}) \arrow[r, "\chi_c"] \arrow[d] & \mathbb{Z} \\
< \mathbb{P} >= K_0(SA(\mathbb{R})) \arrow[ur, "\chi_c"'] &
\end{tikzcd}$$

The characteristic $\chi_c(A)$ of a semialgebraic set A is in fact defined in the same way that we proceed to obtain the equality $K_0(SA(\mathbb{R})) =< \mathbb{P} >$, that is from a specific cell decomposition of A, where $< \mathbb{P} >$ is replaced by $\chi_c(\{*\}) = 1$. The difficulty in the definition of χ_c is then to show that χ_c is independant of the choice of the cell decomposition of A (it technically consists in showing that the definition of $\chi_c(A)$ does not depend on the isomorphism class of A, see [**7**] for instance).

When one starts from the category of real algebraic varieties $\mathrm{Var}_{\mathbb{R}}$ as well as from the category of real algebraic sets $\mathbb{R}\mathrm{Var}$, as we do not have algebraic cell decompositions, we could expect that the induced Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{R}})$ is no more a trivial one. This is indeed the case, since for instance the virtual Poincaré polynomial morphism factors through $K_0(\mathrm{Var}_{\mathbb{R}})$ and has image $\mathbb{Z}[u]$ (see [**13**]).

The first part of this article is devoted to the construction of some non-trivial Grothendieck ring $K_0(BSA_{\mathbb{R}})$ associated to $SA(\mathbb{R})$, with a canonical inclusion

$$K_0(\mathrm{Var}_{\mathbb{R}}) \hookrightarrow K_0(BSA_{\mathbb{R}}),$$

that gives rise to a notion of virtual Poincaré polynomial for basic real semialgebraic formulas extending the virtual Poincaré polynomial of real algebraic sets and that allows factorization of the Euler-Poincaré characteristic of real semialgebraic sets of points satisfying the formulas.

To be more precise, we first construct $K_0(BSA_{\mathbb{R}}))$, the Grothendieck ring of basic real semialgebraic formulas (that are quantifier free real semialgebraic formulas or simply systems of real algebraic equations and inequalities) where the class of basic formulas without inequality are considered up to algebraic isomorphism of the underlying real algebraic varieties. In general a class in $K_0(BSA_{\mathbb{R}})$ of a basic real semialgebraic formula highly depends on the formula itself rather than the only geometry of the real semialgebraic set of points satisfying this formula. This construction is achieved in Section 2.

In order to make some computation more convenient we present a realization, denoted χ, of the ring $K_0(BSA_{\mathbb{R}})$ in the somewhat more simple ring $K_0(\mathrm{Var}_{\mathbb{R}})\otimes\mathbb{Z}[\frac{1}{2}]$, that is a morphism of rings

$$\chi : K_0(BSA_{\mathbb{R}}) \to K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}],$$

that restricts to the identity map on $K_0(\mathrm{Var}_{\mathbb{R}}) \hookrightarrow K_0(BSA_{\mathbb{R}})$. The morphism χ provides an explicit computation (see Proposition 2.1.2) presenting a class of $K_0(BSA_{\mathbb{R}})$ as a $\mathbb{Z}[\frac{1}{2}]$-linear combination of classes of $K_0(\mathrm{Var}_{\mathbb{R}})$. When one wants to simplify more the computation of a class of a basic real semialgebraic formula, one can shrink a little bit more the original ring $K_0(BSA_{\mathbb{R}})$ from $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ to $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$, where for instance algebraic formulas with empty set of real points have trivial class. However as noted in point 2 of Remark 2.1.4 the class of a basic real semialgebraic formula with empty set of real points may be not trivial in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$. The ring $K_0(BSA_{\mathbb{R}})$ is not defined with a prior notion of isomorphism relation contrary to the ring $K_0(\mathrm{Var}_{\mathbb{R}})$ where algebraic isomorphism classes of varieties are generators. Nevertheless we indicate a notion of isomorphism for basic semialgebraic formulas that factors through $K_0(BSA_{\mathbb{R}})$ (see Proposition 2.2.3). This is done in Section 3.

The realization $\chi : K_0(BSA_{\mathbb{R}}) \to K_0(\mathrm{Var}_{\mathbb{R}})) \otimes \mathbb{Z}[\frac{1}{2}]$ naturally let us define in Section 4 a notion of virtual Poincaré polynomial for basic real semialgebraic formulas: for a class $[F]$ in $K_0(BSA_{\mathbb{R}})$ that is written as a $\mathbb{Z}[\frac{1}{2}]$-linear combination $\sum_{i=1}^{q} a_i[A_i]$ of classes $[A_i] \in K_0(\mathrm{Var}_{\mathbb{R}})$ of real algebraic varieties A_i, we simply defines the virtual Poincaré polynomial of F as the corresponding $\mathbb{Z}[\frac{1}{2}]$-linear combination $\sum_{i=1}^{q} a_i\beta(A_i)$ of virtual Poincaré polynomials $\beta(A_i)$ of the varieties A_i. The virtual Poincaré polynomial of F is thus a polynomial $\beta(F)$ in $\mathbb{Z}[\frac{1}{2}][u]$. It is then shown that the evaluation at -1 of $\beta(F)$ is the Euler-Poincaré characteristic of the real semialgebraic set of points satisfying the basic formula F (Proposition 3.1.4).

These constructions are summed up in the following commutative diagram

$$\begin{tikzcd}
Var_{\mathbb{R}} \arrow[rrr, hook] \arrow[d] \arrow[dr] & & & BSA_{\mathbb{R}} \arrow[dll] \arrow[ddd, "\chi_c"] \\
K_0(\mathrm{Var}_{\mathbb{R}}) \arrow[r, hook] \arrow[dr, hook] \arrow[dd, "\beta"'] & K_0(BSA_{\mathbb{R}}) \arrow[d, "\chi"] \arrow[dr, "\chi"] & & \\
& K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}] \arrow[r] \arrow[d, "\beta"] & K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}] \arrow[dl, "\beta"] & \\
\mathbb{Z}[u] \arrow[r, hook] & \mathbb{Z}[\frac{1}{2}][u] \arrow[rr, "u=-1"] & & \mathbb{Z}
\end{tikzcd}$$

The second and last part of this article concerns the real Milnor fibres of a given polynomial function $f \in \mathbb{R}[x_1, \cdots, x_n]$. As geometrical objects, we consider real semialgebraic Milnor fibres of the following types $f^{-1}(\pm c) \cap \bar{B}(0,\alpha)$, $f^{-1}(]0, \pm c[) \cap \bar{B}(0,\alpha)$, $f^{-1}(]0, \pm\infty[) \cap S(0,\alpha)$, for $0 < |c| \ll \alpha \ll 1$, $\bar{B}(0,\alpha)$ the closed ball of $\mathbb{R}^d$ of centre 0 and radius α and $S(0,\alpha)$ the sphere of centre 0 and radius α. The topological types of these fibres are easily comparable, and in order to present a motivic version of these real semialgebraic Milnor fibres we define appropriate zeta functions with coefficients in $(K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}]$ (the localization of the ring $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ with respect to the multiplicative set generated by $\mathbb{L}$). As it is the case in the complex context (see [**3**], [**4**]) we prove that these zeta functions are rational functions expressed in terms of an embedded resolution of f (see Theorem 4.1.2). For a complex hypersurface f, the rationality of the corresponding zeta function allows the definition of the so-called motivic Milnor fibre S_f, defined as minus the limit at infinity of the rational expression of the zeta function. In the real semialgebraic case, the same definition makes sense but we obtain a class S_f in $K_0(\mathrm{Var}_{\mathbb{R}})) \otimes \mathbb{Z}[\frac{1}{2}]$ having a realization under the Euler-Poincaré characteristic of greater combinatorial complexity in terms of the data of the resolution of f than in the complex case. Actually all the strata of the natural stratification of the exceptional divisor of the resolution of f appear in the expression of $\chi_c(S_f)$ in the real case. Nevertheless we show that the motivic real semialgebraic Milnor fibres have for value under the Euler-Poincaré characteristic morphism the Euler-Poincaré characteristic of the corresponding set theoretic real semialgebraic Milnor fibres (Theorem 4.2.8).

In what follows we sometimes simply call measure the class of some object in a given Grothendieck ring. The term inequation refers to the symbol $\neq$ as the term inequality refers to the symbol $>$.

Contents

1. The Grothendieck ring of basic semialgebraic formulas.

1.1. Affine real algebraic varieties.— By an affine algebraic variety over $\mathbb{R}$ we mean an affine reduced and separated scheme of finite type over $\mathbb{R}$. The category of affine algebraic varieties over $\mathbb{R}$ is denoted by $\mathrm{Var}_{\mathbb{R}}$. An affine real algebraic variety X is then defined by a subset of $\mathbb{A}^n$ together with a finite number of polynomial equations. Namely, there exist $P_i \in \mathbb{R}[X_1, \ldots, X_n]$, for $i = 1, \ldots, r$, so that the real points $X(\mathbb{R})$ of X are given by

$$X(\mathbb{R}) = \{x \in \mathbb{A}^n | P_i(x) = 0,\ i = 1, \ldots, r\}.$$

A Zariski-constructible subvariety Z of $\mathbb{A}^n$ is similarly defined by real polynomial equations and inequations. Namely there exist $P_i, Q_j \in \mathbb{R}[X_1, \ldots, X_n]$, for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, so that the real points $Z(\mathbb{R})$ of Z are given by

$$Z(\mathbb{R}) = \{x \in \mathbb{A}^n | P_i(x) = 0, Q_j(x) \neq 0,\ i = 1, \ldots, p,\ j = 1, \ldots, q\}.$$

As an abelian group, the Grothendieck ring $K_0(\mathrm{Var}_\mathbb{R})$ of affine real algebraic varieties is formally generated by isomorphism classes $[X]$ of Zariski-constructible real algebraic varieties, subject to the additivity relation

$$[X] = [Y] + [X \setminus Y],$$

in case $Y \subset X$ is a closed subvariety of X. Here $X \setminus Y$ is the Zariski-constructible variety defined by combining the equations and inequations that define X together with the equations and inequations obtained by reversing the equations and inequations that define Y. The product of constructible sets induces a ring structure on $K_0(\mathrm{Var}_\mathbb{R})$. We denote by $\mathbb{L}$ the class in $K_0(\mathrm{Var}_\mathbb{R})$ of $\mathbb{A}^1$.

1.2. Real algebraic sets.— The real points $X(\mathbb{R})$ of an affine algebraic variety X over $\mathbb{R}$ form a real algebraic set (in the sense of [**2**]). The Grothendieck ring $K_0(\mathbb{R}\mathrm{Var})$ of affine real algebraic sets [**13**] is defined in a similar way than that of real algebraic varieties over $\mathbb{R}$. Taking the real points of an affine real algebraic variety over $\mathbb{R}$ gives a ring morphism from $K_0(\mathrm{Var}_\mathbb{R})$ to $K_0(\mathbb{R}\mathrm{Var})$. A great advantage of $K_0(\mathbb{R}\mathrm{Var})$ from a geometrical point of view is that the additivity property implies that the measure of an algebraic set without real point is zero in $K_0(\mathbb{R}\mathrm{Var})$.

We already know some realizations of $K_0(\mathbb{R}\mathrm{Var})$ in simpler rings, such as the Euler characteristics with compact supports in $\mathbb{Z}$ or the virtual Poincaré polynomial in $\mathbb{Z}[u]$ (cf. [**13**]). We obtain therefore similar realizations for $K_0(\mathrm{Var}_\mathbb{R})$ by composition with the realizations of $K_0(\mathrm{Var}_\mathbb{R})$ in $K_0(\mathbb{R}\mathrm{Var})$.

1.3. Basic semialgebraic formulas.— Let us now precise the definition of the Grothendieck ring $K_0(BSA_\mathbb{R})$ of basic semialgebraic formulas. This definition is inspired by [**5**]. The ring $K_0(BSA_\mathbb{R})$ will contain $K_0(\mathrm{Var}_\mathbb{R})$ as a subring (Proposition 1.3.3) and will be projected on the ring $K_0(\mathrm{Var}_\mathbb{R}) \otimes \mathbb{Z}[\frac{1}{2}]$ (Proposition 2.1.2.1) by an explicit computational process.

A basic semialgebraic formula A in n variables is defined as a finite number of equations, inequations and inequalities, namely there exist $P_i, Q_j, R_k \in \mathbb{R}[X_1, \ldots, X_n]$, for $i = 1, \ldots, p$, $j = 1, \ldots, q$ and $k = 1, \ldots, r$, so that $A(\mathbb{R})$ is equal to those $x \in \mathbb{A}^n$ such that

$$P_i(x) = 0, Q_j(x) \neq 0, R_k(x) > 0,\ i = 1, \ldots, p,\ j = 1, \ldots, q,\ k = 1, \ldots, r.$$

The relations $Q_j(x) \neq 0$ are called inequations and the relations $R_k(x) > 0$ are called inequalities. We will simply denote a basic semialgebraic formula by

$$A = \{P_i = 0, Q_j \neq 0, R_k > 0,\ i = 1, \ldots, p,\ j = 1, \ldots, q,\ k = 1, \ldots, r\}.$$

Note in particular that A is not only defined by the real points $A(\mathbb{R})$ solutions of the equations, inequations and inequalities, but by these equations, inequations and inequalities themselves.

We will consider basic semialgebraic formulas up to algebraic isomorphisms, when the basic semialgebraic formulas are defined without inequality.

1.3.1 Remark. — In the sequel, we will allow ourselves to use the notation $\{P < 0\}$ for the basic semialgebraic formula $\{-P > 0\}$ and similarly $\{P > 1\}$ instead of $\{P - 1 > 0\}$, where P denotes a polynomial with real coefficients. Furthermore given two basic semialgebraic formulas A and B, the notation $\{A, B\}$ will denote the basic formula with equations, inequations and inequalities coming from A and B together.

We define the Grothendieck ring $K_0(BSA_{\mathbb{R}})$ of basic semialgebraic formulas as the free abelian ring generated by basic semialgebraic formulas $[A]$, up to algebraic isomorphim when the formula A has no inequality, and subject to the three following relations

1. (*algebraic additivity*)

$$[A] = [A, S = 0] + [A, \{S \neq 0\}]$$

where A is a basic semialgebraic formula in n variables and $S \in \mathbb{R}[X_1, \ldots, X_n]$.

2. (*semialgebraic additivity*)

$$[A, R \neq 0] = [A, R > 0] + [A, -R > 0]$$

where A is a basic semialgebraic formula in n variables and $R \in \mathbb{R}[X_1, \ldots, X_n]$.

3. (*product*) The product of basic semialgebraic formulas, defined by taking the conjonction of the formulas with disjoint sets of free variables, induces the ring product on $K_0(BSA_{\mathbb{R}})$. In other words we consider the relation

$$[A, B] = [A] \cdot [B],$$

for A and B basic real semialgebraic formulas with disjoint set of variables.

1.3.2 Remark. — 1. Contrary to the Grothendieck ring of algebraic varieties or algebraic sets, we do not consider isomorphism classes of basic real semialgebraic formulas in the definition of $K_0(BSA_{\mathbb{R}})$. As a consequence the realization we are interested in does depend in a crucial way on the description of the basic semialgebraic set as a basic semialgebraic formula. For instance $\{X - 1 > 0\}$ and $\{X > 0, X - 1 > 0\}$ will have different measures.

2. One may decide to enlarge the basic semialgebraic formulas with large inequalities by imposing, by convention, that the measure of $\{A, R \geq 0\}$, for A a basic semialgebraic formula in n variables and $R \in \mathbb{R}[X_1, \dots, X_n]$, is the sum of the measures of $\{A, R > 0\}$ and of $\{A, R = 0\}$.

1.3.3 Proposition. — *The natural map i from $K_0(\mathrm{Var}_{\mathbb{R}})$ that associates to an affine real algebraic variety its value in the Grothendieck ring $K_0(BSA_{\mathbb{R}})$ of basic real semialgebraic formulas is an injective morphism*

$$i : K_0(\mathrm{Var}_{\mathbb{R}}) \longrightarrow K_0(BSA_{\mathbb{R}}).$$

We therefore identify $K_0(\mathrm{Var}_{\mathbb{R}})$ with a subring of $K_0(BSA_{\mathbb{R}})$.

Proof. — We construct a left inverse j of i as follows. Let $a \in K_0(BSA_{\mathbb{R}})$ be a sum of products of measures of basic semialgebraic formulas. If there exist Zariski constructible real algebraic sets $Z_1, \dots, Z_m$ such that $[Z_1] + \dots + [Z_m]$ is equal to a in $K_0(BSA_{\mathbb{R}})$, then we define the image of a by j to be

$$j(a) = [Z_1] + \dots + [Z_m] \in K_0(\mathrm{Var}_{\mathbb{R}}).$$

Otherwise, the image of a by j is defined to be zero in $K_0(\mathrm{Var}_{\mathbb{R}})$. The map j is well-defined. Actually if $Y_1, \dots, Y_l$ are another Zariski constructible sets such that $[Y_1] + \dots + [Y_l]$ is equal to a in $K_0(BSA_{\mathbb{R}})$, then

$$[Y_1] + \dots + [Y_l] = [Z_1] + \dots + [Z_m]$$

in $K_0(BSA_{\mathbb{R}})$. The equality still holds in $K_0(\mathrm{Var}_{\mathbb{R}})$ by remark 1.3.2 and the fact that j defines a left inverse of i is immediate. □

1.3.4 Remark. — Note however that the map j constructed in the proof of Proposition 1.3.3 is not a group morphism. For instance $j([X > 0]) = j([X < 0]) = 0$ whereas $j([X \neq 0]) = \mathbb{L} - 1$.

2. A realization of $K_0(BSA_{\mathbb{R}})$

An example of a ring morphism from $K_0(BSA_{\mathbb{R}})$ to $\mathbb{Z}$ is given by the Euler characteristic with compact supports χ_c. We construct in this section a realization for elements in $K_0(BSA_{\mathbb{R}})$ with value in the ring of polynomials with coefficient in $\mathbb{Z}[\frac{1}{2}]$. This realization specializes to the Euler characteristic with compact supports. To this aim, we contruct more generally a ring morphism from $K_0(BSA_{\mathbb{R}})$ to the tensor product of $K_0(\mathrm{Var}_{\mathbb{R}})$ with $\mathbb{Z}[\frac{1}{2}]$.

2.1. The realization.— We define a morphism χ from the ring $K_0(BSA_{\mathbb{R}})$ to the ring $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ as follows. Let A be a basic semialgebraic formula without inequality. We assign to A its value $\chi(A) = [A]$ in $K_0(\mathrm{Var}_{\mathbb{R}})$ as a constructible set. We proceed now by induction on the number of inequalities in the description of the basic semialgebraic formulas. Assuming that we have defined χ for basic semialgebraic formulas with at most k inequalities, $k \in \mathbb{N}$, let A be a basic real semialgebraic formula with n variables and $R \in \mathbb{R}[X_1, \dots, X_n]$. Define $\chi([A, R > 0])$ by

$$\chi([A, R > 0]) := \frac{1}{4}\big([A, Y^2 = R] - [A, Y^2 = -R]\big) + \frac{1}{2}[A, R \neq 0],$$

where $\{A, Y^2 = \pm R\}$ is a basic real semialgebraic formula with $n+1$ variables, with at most k inequalities and $\{A, R \neq 0\}$ is a basic semialgebraic formula with n variables with at most k inequalities.

2.1.1 Remark. — The way to define χ may be seen as an average of two different natural ways of understanding a basic semialgebraic formula as a quotient of algebraic varieties. Namely, for a basic semialgebraic formula in n variables of the form $\{R > 0\}$, we may see its set of real points as a projection of $\{Y^2 = R\}$ with fibre two points, or as the complement outside the zero set of R of the projection of $Y^2 = -R$. The algebraic average of these two possible points of view is

$$\frac{1}{2}\Big(\big(\frac{1}{2}[Y^2 = R] - [R = 0]\big) + \big(\mathbb{L}^n - \frac{1}{2}[Y^2 = -R]\big)\Big),$$

which, considering that $\mathbb{L}^n - [R = 0] = [R \neq 0]$, gives for $\chi(R > 0)$ the expression just defined above.

We give below the general formula that computes the measure of a basic semialgebraic formula in terms of the measure of real algebraic varieties.

2.1.2 Proposition. — *Let Z be a constructible set in $\mathbb{R}^n$ and take $R_k \in \mathbb{R}[X_1, \dots, X_n]$, with $k = 1, \dots, r$. For $I \subset \{1, \dots, r\}$ a subset of cardinal $\sharp I = i$ and $\varepsilon \in \{\pm 1\}^i$, we denote by $R_{I,\varepsilon}$ the real constructible set defined by*

$$R_{I,\varepsilon} = \{Y_j^2 = \varepsilon_j R_j(X), j \in I; \quad R_k(X) \neq 0, k \notin I\}.$$

Then $\chi([Z, R_k > 0, \ k = 1, \dots, r])$ is equal to

$$\sum_{i=0}^{r} \frac{1}{2^{r+i}} \sum_{I \subset \{1,\dots,r\}, \sharp I = i} \ \sum_{\varepsilon \in \{\pm 1\}^i} \big(\prod_{j \in I} \varepsilon_j\big)[Z, R_{I,\varepsilon}]$$

Proof. — If $r = 1$ it follows from the definition of χ. We prove the general result by induction on $r \in \mathbb{N}$. Assume $Z = \mathbb{R}^n$ to simplify notations. Take $R_k \in \mathbb{R}[X_1, \dots, X_n]$, with $k = 1, \dots, r+1$. Choose $I \subset \{1, \dots, r\}$ a subset of cardinal $\sharp I = i$ and $\varepsilon \in \{\pm 1\}^i$. Then, we obtain from the definition of χ that

$$[R_{r+1} > 0, R_{I,\varepsilon}] = \frac{1}{4}([R_{I \cup \{r+1\}, \varepsilon^+}] - [R_{I \cup \{r+1\}, \varepsilon^-}]) + \frac{1}{2}[R_{\tilde{I},\varepsilon}]$$

where $\varepsilon^+ = (\varepsilon_1, \dots, \varepsilon_r, 1)$, $\varepsilon^- = (\varepsilon_1, \dots, \varepsilon_r, -1)$ and $\tilde{I}$ denotes I as a subset of $\{1, \dots, r+1\}$. Therefore

$$\frac{1}{2^{r+i}}\big(\prod_{j \in I} \varepsilon_j\big)[R_{r+1} > 0, R_{I,\varepsilon}]$$

is equal to

$$\frac{1}{2^{(r+1)+(i+1)}}(\prod_{j\in I}\varepsilon_j)([R_{I\cup\{r+1\},\varepsilon^+}]-[R_{I\cup\{r+1\},\varepsilon^-}])+\frac{1}{2^{(r+1)+i}}(\prod_{j\in I}\varepsilon_j)[R_{\tilde{I},\varepsilon}]$$

which gives the result. □

The morphism χ is then actually defined on $K_0(BSA_{\mathbb{R}})$.

2.1.2.1 Theorem. — *The map*

$$\chi : K_0(BSA_{\mathbb{R}}) \longrightarrow K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$$

is a ring morphism that is identical on $K_0(\mathrm{Var}_{\mathbb{R}}) \subset K_0(BSA_{\mathbb{R}})$.

Proof. — We must prove that the definition of χ given is compatible with the algebraic and semialgebraic additivity. However the semialgebraic additivity follows directly from the definition of χ. Actually, if A is a basic semialgebraic formula and R a real polynomial, then the sum of $\chi([A,R>0])$ and $\chi([A,-R>0])$ is equal to

$$\begin{aligned}\frac{1}{4}\big(\chi([A,Y^2=R])-\chi([A,Y^2=-R])\big)+\frac{1}{2}\chi([A,R\neq 0])\\+\frac{1}{4}\big(\chi([A,Y^2=-R])-\chi([A,Y^2=R])\big)+\frac{1}{2}\chi([A,-R\neq 0])\\=\chi([A,-R\neq 0]).\end{aligned}$$

The algebraic additivity as well as the multiplicativity follow from Proposition 2.1.2 that enables to express the measure of a basic semialgebraic formula in terms of algebraic varieties for which additivity and multiplicativity hold. We conclude by noting that we may construct a left inverse to χ restricted to $K_0(\mathrm{Var}_{\mathbb{R}})$ in the same way as in the proof of Proposition 1.3.3. □

2.1.3 Example. —

1. *A half-line defined by $X>0$ has measure in $K_0(\mathrm{Var}_{\mathbb{R}})\otimes\mathbb{Z}[\frac{1}{2}]$ half of the value of the line minus one point, as expected, since by definition*

$$\chi([X>0])=\frac{1}{4}(\mathbb{L}-\mathbb{L})+\frac{1}{2}(\mathbb{L}-1)=\frac{1}{2}(\mathbb{L}-1).$$

However, if we add one more inequality, like $\{X>0, X>-1\}$, then the measure has more complexity. By Proposition 2.1.2 we obtain in that case

$$\chi([X>0,X>-1])=\frac{1}{4}(\mathbb{L}-3).$$

2. *Using the multiplicativity, we find the measure of the half-plane and the meseaure of the quarter plane as expected*

$$\chi([X_1>0])=\frac{1}{2}(\mathbb{L}^2-\mathbb{L})$$

and

$$\chi([X_1>0,X_2>0])=\frac{1}{4}(\mathbb{L}-1)^2.$$

2.1.4 Remark. — 1. Let $R \in \mathbb{R}[X_1, \ldots, X_n]$ be odd. Then

$$\chi([R>0]) = \chi([R<0]) = \frac{[R \neq 0]}{2}.$$

Actually, the varieties $Y^2 = R(X)$ and $Y^2 = -R(X)$ are isomorphic via $X \mapsto -X$, and the result follows from the definition of χ.

2. The ring morphism from $K_0(\mathrm{Var}_{\mathbb{R}})$ to $K_0(\mathbb{R}\mathrm{Var})$ gives a realization from the ring $K_0(BSA_{\mathbb{R}})$ to the ring $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$ for which the measure of a real algebraic variety without real point is zero, this is why it is often convenient to push the computations to the ring $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$ rather than staying at the higher level of $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$. However we have to notice that the measure of a basic real semialgebraic formula without real point is not necessarily zero in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$. For instance, let us compute the measure of $X^2+1>0$ in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$. By definition of χ we obtain that $\chi([X^2+1>0])$ is equal to

$$\begin{aligned}\frac{1}{4}\big(\chi([Y^2 = X^2+1]) - \chi([Y^2 = -X^2-1])\big) + \frac{1}{2}\chi([X^2+1 \neq 0])\\ = \frac{1}{4}(\mathbb{L}-1) + \frac{1}{2}\mathbb{L} = \frac{1}{4}(3\mathbb{L}-1).\end{aligned}$$

Again by definition we have

$$\begin{aligned}\chi([X^2+1<0]) &= \chi([X^2+1 \neq 0]) - \chi([X^2+1>0])\\ &= \mathbb{L} - \chi([X^2+1=0]) - \chi([X^2+1>0]).\end{aligned}$$

But since $\chi([X^2+1=0]) = 0$ in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$, we obtain that the measure of $\{X^2+1<0\}$ in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$, whose real points set is empty, is

$$\chi([X^2+1<0]) = \frac{1}{4}(\mathbb{L}+1).$$

3. In a similarly way, the basic semialgebraic formula $\{P>0, -P>0\}$ with $P(X) = 1+X^2$, whose set of real points is empty, has measure

$$\chi([P>0,-P>0]) = \frac{1}{8}(\mathbb{L}+1).$$

2.2. Isomorphism between basic semialgebraic formulas. — In this section we give a condition for two basic semialgebraic formulas to have the same realization by χ. It deals with the complexification of the algebraic liftings of the basic semialgebraic formulas.

Let X be a real algebraic subvariety of $\mathbb{R}^n$ defined by $P_i \in \mathbb{R}[X_1, \ldots, X_n]$, for $i = 1, \ldots, r$. The complexification $X_{\mathbb{C}}$ of X is defined to be the complex algebraic subvariety of $\mathbb{C}^n$ defined by the same polynomials $P_1, \ldots, P_r$. We define similarly the complexification of a real algebraic map.

Let $Y \subset \mathbb{R}^n$ be a Zariski constructible subset of $\mathbb{R}^n$ and take $R_1, \ldots, R_r \in \mathbb{R}[X_1, \ldots, X_n]$. Let A denotes the basic semialgebraic formula of $\mathbb{R}^n$ defined by

Y together with the inequalities $R_1 > 0, \dots, R_r > 0$, and V denotes the Zariski constructible subset of $\mathbb{R}^{n+r}$ defined by

$$V = \{Y, Y_1^2 = R_1, \dots, Y_r^2 = R_r\}.$$

Note that V is endowed with an action of $\{\pm 1\}^r$ defined by multiplication by -1 on the indeterminates $Y_1, \dots, Y_r$.

Let $Z \subset \mathbb{R}^m$ be a Zariski constructible subset of $\mathbb{R}^m$ and take similarly $S_1, \dots, S_r \in \mathbb{R}[X_1, \dots, X_m]$. Let B denotes the basic semialgebraic formula of $\mathbb{R}^m$ defined by Z together with the inequalities $S_1 > 0, \dots, S_r > 0$, and W denotes the Zariski constructible subset of $\mathbb{R}^{m+r}$ defined by

$$W = \{Z, Y_1^2 = S_1, \dots, Y_r^2 = S_r\}.$$

2.2.1 Definition. — We say that the basic semialgebraic formulas A and B are isomorphic if there exists a real algebraic isomorphism $\phi : V \longrightarrow W$ between V and W which is equivariant with respect to the action of $\{\pm 1\}^r$ on V and W, and whose complexification $\phi_{\mathbb{C}}$ induces a complex algebraic isomorphism between the complexifications $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ of V and W.

2.2.2 Remark. — Let us consider first the particular case $Y = \mathbb{R}^n$, $Z = \mathbb{R}^m$ and $r = 1$. Change moreover the notation as follows. Put $V^+ = V$ and $W^+ = W$, and define $V^- = \{Y^2 = -R(X)\}$ and $W^- = \{Y^2 = -S(X)\}$.

Then the complex points $V_{\mathbb{C}}^+$ and $V_{\mathbb{C}}^-$ of V^+ and V^- are isomorphic via the complex (and not real) isomorphism $(x, y) \mapsto (x, iy)$. Now, suppose that the basic semialgebraic formula $\{R > 0\}$ is isomorphic to $\{S > 0\}$. Let $\phi = (f, g) : (x, y) \mapsto (f(x, y), g(x, y))$ be the real isomorphism involved in the definition (that is f and g are defined by real equations, and moreover $f(x, -y) = f(x, y)$ and $g(x, -y) = -g(x, y)$). Then the following diagram

$$\begin{array}{ccc} V_{\mathbb{C}}^+ & \xrightarrow{(f,g)} & W_{\mathbb{C}}^+ \\ {\scriptstyle (x,y)\mapsto(x,iy)}\downarrow & & \downarrow{\scriptstyle (x,y)\mapsto(x,iy)} \\ V_{\mathbb{C}}^- & & W_{\mathbb{C}}^- \end{array}$$

induces a complex isomorphism (F, G) between $V_{\mathbb{C}}^-$ and $W_{\mathbb{C}}^-$ given by

$$(x, y) \mapsto (f(x, -iy), ig(x, -iy)).$$

In fact, this isomorphism is defined over $\mathbb{R}$ since

$$\overline{F(x, y)} = \overline{f(x, -iy)} = f(\overline{x}, \overline{-iy}) = f(\overline{x}, i\overline{y}) = f(\overline{x}, -i\overline{y}) = F(\overline{x}, \overline{y})$$

and

$$\overline{G(x, y)} = \overline{ig(x, -iy)} = -ig(\overline{x}, \overline{-iy}) = -ig(\overline{x}, i\overline{y}) = ig(\overline{x}, -i\overline{y}) = G(\overline{x}, \overline{y}),$$

where the bar denotes complex conjugation. Therefore it induces a real algebraic isomorphism between V^- and W^-.

2.2.3 Proposition. — *If the basic semialgebraic formulas A and B are isomorphic, then $\chi([A]) = \chi([B])$.*

Proof. — Thanks to Proposition 2.1.2, we only need to prove that the real algebraic varieties $R_{I,\varepsilon}$ corresponding to A and B are isomorphic two by two, which is a direct generalisation of Remark 2.2.2. $\square$

3. Virtual Poincaré polynomial

3.1. Polynomial realization. — The best realization known (with respect to the highest algebraic complexity of the realization ring) of the Grothendieck ring of real algebraic varieties is given by the virtual Poincaré polynomial [**13**]. This polynomial, whose coefficients coincide with the Betti numbers with coefficients in $\frac{\mathbb{Z}}{2\mathbb{Z}}$ when the sets are compact and nonsingular, has coefficient in $\mathbb{Z}$. As a corollary of Theorem 2.1.2.1 we obtain the following realization of $K_0(BSA_{\mathbb{R}})$ in $\mathbb{Z}[\frac{1}{2}][u]$.

3.1.1 Proposition. — *There exists a ring morphism*

$$\beta : K_0(BSA_{\mathbb{R}}) \longrightarrow \mathbb{Z}[\frac{1}{2}][u]$$

whose restriction to $K_0(\mathrm{Var}_{\mathbb{R}}) \subset K_0(BSA_{\mathbb{R}})$ *coincides with the virtual Poincaré polynomial.*

The interest of such a realization is that it enables to make concrete computations.

3.1.2 Example. — *The virtual Poincaré polynomial of the open disc* $X_1^2 + X_2^2 < 1$ *is equal to*

$$\frac{1}{4}\big(\beta([Y^2 = 1 - (X_1^2 + X_2^2)]) - \beta([Y^2 = X_1^2 + X_2^2 - 1])\big) + \frac{1}{2}\beta([X_1^2 + X_2^2 \neq 1])$$

$$= \frac{1}{4}(u^2 + 1 - u(u+1)) + \frac{1}{2}(u^2 - u - 1) = \frac{1}{4}(2u^2 - 3u - 1).$$

3.1.3 Remark. — In case the set of real points of a basic semialgebraic formula is a real algebraic set (or even an arc symmetric set [**11, 8**]), its virtual Poincaré polynomial does not coincide in general with the virtual Poincaré polynomial of the real algebraic set. For instance, the basic semialgebraic formula $X^2 + 1 > 0$, considered in Remark 2.1.4, has virtual Poincaré polynomial equal to $\frac{1}{4}(3u - 1)$ whereas its set of points is a real line whose with virtual Poincaré polynomial equals u as a real algebraic set.

Evaluating u at an integer gives another realization, with coefficient in $\mathbb{Z}[\frac{1}{2}]$. The virtual Poincaré polynomial of a real algebraic variety, evaluated at $u = -1$, coincides with its Euler characteristic with compact supports [**13**]. Actually, evaluating the virtual Poincaré polynomial of a basic semialgebraic formula gives also the Euler characteristic with compact supports of its set of real points, and therefore has its values in $\mathbb{Z}$.

3.1.4 Proposition. — *The virtual Poincaré polynomial $\beta(A)$ of a basic semialgebraic formula A is equal to the Euler characteristic with compacts supports of its set of real points $A(\mathbb{R})$ when evaluated at $u=-1$. In other words*

$$\beta(A)(-1)=\chi_c(A(\mathbb{R})).$$

Proof. — As explained in Remark 2.1.1, one may see $\chi(A)$ as an average of two natural 2-coverings of the set of real points $A(\mathbb{R})$ of A. However, for such coverings, the Euler characteristic with compact supports of the total space is twice that of the basis. □

3.2. Homogeneous case. — We propose some computations of the virtual Poincaré polynomial of basic real semialgebraic formulas of the form $\{R>0\}$ where R is homogeneous. Looking at Euler characteristic with compact supports, it is equal to the product of the Euler characteristics with compact supports of $\{X>0\}$ with $\{R=1\}$. We investigate the case of virtual Poincaré polynomial. A key point in the proofs will be the invariance of the virtual Poincaré polynomial of constructible sets under regular homeomorphisms (see [**14**]).

3.2.1 Proposition. — *Let $R\in\mathbb{R}[X_1,\dots,X_n]$ be a homogeneous polynomial of degree d. Assume d is odd. Then*

$$\beta([R>0])=\beta([X>0])\beta([R=1]).$$

Proof. — The algebraic varieties defined by $Y^2=R(X)$ and $Y^2=-R(X)$ are isomorphic since $R(-X)=-R(X)$, therefore

$$\beta([R>0])=\frac{\beta([R\neq 0])}{2}.$$

The map $(\lambda,x)\mapsto\lambda x$ from $\mathbb{R}^*\times\{R=1\}$ to $R\neq 0$ is a regular homeomorphism with inverse $y\mapsto(R(y)^{1/d},\frac{y}{R(y)^{1/d}})$ therefore

$$\beta([R\neq 0])=\beta(\mathbb{R}^*)\beta([R=1]),$$

so that

$$\beta([R>0])=\frac{\beta(\mathbb{R}^*)}{2}\beta([R=1])=\beta([X>0])\beta([R=1]).$$

□

The result is no longer true when the degree is even. However, in the particular case of the square of a homogeneous polynomial of odd degree, the relation of Proposition 3.2.1 remains valid.

3.2.2 Proposition. — *Let $P\in\mathbb{R}[X_1,\dots,X_n]$ be a homogeneous polynomial of degree k. Assume k is odd, and define $R\in\mathbb{R}[X_1,\dots,X_n]$ by $R=P^2$. Then*

$$\beta([R>0])=\beta([X>0])\beta([R=1]).$$

Proof. — Note first that $Y^2 - R$ can be factorized as $(Y-P)(Y+P)$ therefore the virtual Poincaré polynomial of $Y^2 - R$ is equal to

$$\beta(Y-P=0)+\beta(Y+P=0)-\beta(P=0).$$

However the algebraic varieties $Y-P=0$ and $Y+P=0$ are isomorphic to a n-dimensionnal affine space, whereas $Y^2+R=0$ is isomorphic to $P=0$ since $R=P^2$ is positive, so that the virtual Poincaré polynomial of $R>0$ is equal to

$$\frac{1}{4}(2\beta(\mathbb{R}^n)-2\beta([P=0]))+\frac{1}{2}\beta([P\neq 0])=\beta([P\neq 0]).$$

To compute $\beta([P\neq 0]$, note that the map $(\lambda,x)\mapsto \lambda x$ from $\mathbb{R}^*\times[P=1]$ to $[P\neq 0]$ is a regular homeomorphism with inverse $y\mapsto (R(y)^{1/k},\frac{y}{R(y)^{1/k}})$ therefore

$$\beta([P\neq 0])=\beta(\mathbb{R}^*)\beta([P=1]).$$

We achieve the proof by noticing that $R-1=(P-1)(P+1)$ so that $\beta([P=1])=\frac{\{R=1\}}{2}$ because the degree of the homogeneous polynomial P is odd. Finally

$$\beta([R>0])=\frac{\beta(\mathbb{R}^*)}{2}\beta([R=1])$$

and the proof is achieved. □

More generally, for a homogeneous polynomial R of degree twice a odd number, we can express the virtual Poincaré polynomial of $[R>0]$ in terms of that of $[R=1]$ and $[R\neq 0]$ as follows.

3.2.3 Proposition. — *Let $k\in\mathbb{N}$ be odd and put $d=2k$. Let $R\in\mathbb{R}[X_1,\dots,X_n]$ be a homogeneous polynomial of degre d. Then*

$$\beta([R>0])=\frac{1}{4}(\beta([R=1])-\beta([R=-1]))+\frac{1}{2}\beta([R\neq 0]).$$

3.2.4 Example. — *We cannot do better in general as illustrated by the following examples. For $R_1=X_1^2+X_2^2$ one obtain*

$$\beta([R_1>0])=\frac{3}{2}\beta([X>0])\beta([R_1=1])$$

whereas for $R_2=X_1^2-X_2^2$ one has

$$\beta([R_2>0])=\beta([X>0])\beta([R_2=1]),$$

and the proof is achieved.

The proof of Proposition 3.2.3 is a direct consequence of the next lemma.

3.2.5 Lemma. — *Let $k\in\mathbb{N}$ be odd and put $d=2k$. Let $R\in\mathbb{R}[X_1,\dots,X_n]$ be a homogeneous polynomial of degre d. Then*

$$\beta([Y^2=R])=\beta([R\neq 0])+\beta(\mathbb{R}^*)\beta([R=1]).$$

Proof. — Note first that the algebraic varieties $Y^2 = R$ and $Y^d = R$ have the same virtual Poincaré polynomial. Indeed the map $(x, y) \mapsto (x, y^k)$ realizes a regular homeomorphism between $Y^2 = R$ and $Y^d = R$, whose inverse is given by $(x, y) \mapsto (x, y^{1/k})$. However the polynomial $Y^d - R$ being homogeneous, we have

$$\beta([Y^d - R = 0]) = \beta([R \neq 0]) + \beta(\mathbb{R}^*)\beta([R = 1]).$$

□

4. Zeta functions and Motivic real Milnor fibres

We apply in this section the preceding construction of $\chi : K_0(BSA_{\mathbb{R}}) \to K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ in defining, for a given polynomial $f \in \mathbb{R}[X_1, \cdots, X_d]$, zeta functions with coefficients being classes in $(K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}]$ of real semialgebraic formulas in truncated arc spaces. We then show that these zeta functions are deeply related to the topology of some corresponding set theoretic real semialgebraic Milnor fibres of f.

4.1. Semialgebraic zeta functions and real Denef-Loeser formulas.— Let $f : \mathbb{R}^d \to \mathbb{R}$ be a polynomial function with coefficients in $\mathbb{R}$ sending 0 to 0. We denote by $\mathcal{L}$ or $\mathcal{L}(\mathbb{R}^d, 0)$ the space of formal arcs $\gamma(t) = (\gamma_1(t), \cdots, \gamma_d(t))$ on $\mathbb{R}^d$, with $\gamma_j(0) = 0$ for all $j \in \{1, \cdots, d\}$, by $\mathcal{L}_n$ or $\mathcal{L}_n(\mathbb{R}^d, 0)$ the space of truncated arcs $\mathcal{L}/(t^{n+1})$ and by $\pi_n : \mathcal{L} \to \mathcal{L}_n$ the truncation map. More generally, for M a variety and W a closed subset of M, $\mathcal{L}(M, W)$ (resp. $\mathcal{L}_n(M, W)$) will denote the space of arcs on M (resp. the nth jet-space on M) with endpoints in W.

Let ϵ be one of the symbols in the set $\{naive, -1, 1, >, <\}$. For such a symbol ϵ, via the realization of $K_0(BSA_{\mathbb{R}})$ in $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$, we define a zeta function $Z^{\epsilon}(T) \in (K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][[T]]$ by

$$Z_f^{\epsilon}(T) := \sum_{n \geq 1} [X_{n,f}^{\epsilon}] \mathbb{L}^{-nd} T^n,$$

where $X_{n,f}^{\epsilon}$ is defined in the following way

- $X_{n,f}^{naive} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a \neq 0\}$,
- $X_{n,f}^{-1} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a = -1\}$,
- $X_{n,f}^{1} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a = 1\}$,
- $X_{n,f}^{>} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a > 0\}$,
- $X_{n,f}^{<} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a < 0\}$.

Note that $X_{n,f}^{\epsilon}$ is a real algebraic variety for $\epsilon = -1$ or 1, a real algebraic constructible set for $\epsilon = naive$ and a semialgebraic set, given by an explicit description involving one inequality, for ϵ being the symbol $>$ or the symbol $<$. Consequently, $Z_f^{\epsilon}(T) \in K_0(\mathrm{Var}_{\mathbb{R}})[\mathbb{L}^{-1}][[T]]$ for $\epsilon \in \{naive, -1, 1\}$ and $Z_f^{\epsilon}(T) \in (K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][[T]]$ for $\epsilon \in \{>, <\}$.

We show in this section that $Z_f^\epsilon(T)$ is a rational function expressed in terms of the combinatorial data of a resolution of f. To define those data let us consider $\sigma : (M, \sigma^{-1}(0)) \to (\mathbb{R}^d, 0)$ a proper birational map which is an isomorphism over the complement of $f = 0$ in $(\mathbb{R}^d, 0)$, such that $f \circ \sigma$ and the jacobian determinant jac σ are normal crossings. We denote by E_j, for $j \in \mathcal{J}$, the irreducible components of $(f \circ \sigma)^{-1}(0)$ and assume that E_k are the irreducible components of $\sigma^{-1}(0)$ for $k \in \mathcal{K} \subset \mathcal{J}$. For $j \in J$ we denote by N_j the multiplicity $mult_{E_j} f \circ \sigma$ of $f \circ \sigma$ along E_j and for $k \in \mathcal{K}$ by ν_k the number $\nu_k = 1 + mult_{E_k} \text{jac } \sigma$. For any $I \subset \mathcal{J}$, we put $E_I^0 = (\bigcap_{i \in I} E_i) \setminus (\bigcup_{j \in \mathcal{J} \setminus I} E_j)$. The sets E_I^0 are constructible sets and the collection $(E_I^0)_{I \subset \mathcal{J}}$ gives a canonical stratification of the divisor $f \circ \sigma = 0$, compatible with $\sigma = 0$ such that in some Zariski neighborhood U of E_I^0 in M we have $f \circ \sigma(x) = u(x) \prod_{i \in I} x_i^{N_i}$, where u is a unit, that is to say a rational function which does not vanish on U, and $x = (x', (x_i)_{i \in I})$ are local coordinates.

Finally for $\epsilon \in \{-1, 1, >, <\}$ and $I \subset \mathcal{J}$, we define $\widetilde{E}_I^{0,\epsilon}$ as the gluing along E_I^0 of the sets

$$R_U^\epsilon = \{(x,t) \in (E_I^0 \cap U) \times \mathbb{R};\ t^m \cdot u(x)\ ?_\epsilon\ \},$$

where $?_\epsilon$ is $= -1$, $= 1$, > 0 or < 0 in case ϵ is $-1, 1, >$ or $<$ and $m = gcd_{i \in I}(N_i)$.

4.1.1 Remark. — The definition of the R_U^ϵ's is independant of the choice of the coordinates, as well as the gluing of the R_U^ϵ is allowed, up to isomorphism, since when in some Zariski neighborhood of E_I^0 one has in another coordinates $z = z(x) = (z', (z_i)_{i \in I})$ the expression $f \circ \sigma(z) = v(z) \prod_{i \in I} z^{N_i}$, there exists nonvanishing functions α_i so that $z_i = \alpha_i(z) \cdot x_i$. We thus obtain $v(z) \prod_{i \in I} \alpha_i^{N_i}(z) = u(x)$, and the transformation

$$\begin{array}{ccc} \{(x,t) \in (E_I^0 \cap U) \times \mathbb{R}; t^m \cdot u(x)\ ?_\epsilon\} & \to & \{(z,s) \in (E_I^0 \cap U) \times \mathbb{R}; s^m \cdot v(z)\ ?_\epsilon\} \\ (x,t) & \mapsto & (z, s = t \prod_{i \in I} \alpha_i(z)^{N_i/m}) \end{array}$$

is an isomorphism in case $?_\epsilon$ is $= 1$ or $= -1$, and induces an isomorphism between the associate double covers $\mathcal{R}_U^\epsilon = \{(x,t,y) \in (E_I^0 \cap U) \times \mathbb{R} \times \mathbb{R}; t^m \cdot u(x) \cdot y^2 = \eta(\epsilon)\}$ and $\mathcal{R}_U'^\epsilon = \{(z,s,w) \in (E_I^0 \cap U) \times \mathbb{R} \times \mathbb{R}; s^m \cdot v(z) \cdot w^2 = \eta(\epsilon)\}$, with $\eta(\epsilon) = 1$ when ϵ is the symbol $>$ and $\eta(\epsilon) = -1$ when ϵ is the symbol $<$. The induced isomorphism simply being

$$\begin{array}{ccc} \mathcal{R}_U^\epsilon & \to & \mathcal{R}_U'^\epsilon \\ (x,t,y) & \mapsto & (z,s,w = y) \end{array}$$

Also notice that $\tilde{E}_I^{0,\epsilon}$ is a constructible set when ϵ is -1 or 1 and a semialgebraic set with explicit description over the constructible set E_I^0 when ϵ is $<$ or $>$. We can thus define the class $[\tilde{E}_I^{0,\epsilon}] \in \chi(K_0(BSA_\mathbb{R}))$.

With these notations one can give the expression of $Z_f^\epsilon(T)$ in terms of $[\widetilde{E}_I^{0,\epsilon}]$, as, for instance, in [**3**], [**4**], [**6**], [**12**], essentially using the Kontsevitch change of variables formula in motivic integration ([**10**], [**4**] for instance).

4.1.2 Theorem. — *With the notations above, one has*

$$Z_f^\epsilon(T) = \sum_{I\cap\mathcal{K}\neq\emptyset} (\mathbb{L}-1)^{|I|-1}[\widetilde{E}_I^{0,\epsilon}] \prod_{i\in I} \frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1-\mathbb{L}^{-\nu_i}T^{N_i}}.$$

4.1.3 Remark. — Classically, the right hand side of the equality of Theorem 4.1.2 does not depend, as a formal series in $(K_0(\mathrm{Var}_\mathbb{R})\otimes\mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][[T]]$, of the choice of the resolution σ, as the definition of $Z_f^\epsilon(T)$ does not depend itself of any choice of resolution.

To prove this theorem, we first start with a lemma that needs the following notations. We denote by

$$\sigma_* : \mathcal{L}(M,\sigma^{-1}(0)) \to \mathcal{L}(\mathbb{R}^d,0),$$

and for $n\in\mathbb{N}$, by

$$\sigma_{n,*} : \mathcal{L}_n(M,\sigma^{-1}(0)) \to \mathcal{L}_n(\mathbb{R}^d,0)$$

the natural mappings induced by $\sigma : (M,\sigma^{-1}(0)) \to (\mathbb{R}^d,0)$. Let

$$Y_{n,f}^\epsilon = \pi_n^{-1}(X_{n,f}^\epsilon).$$

Then $Y_{n,f\circ\sigma}^\epsilon = \{\gamma\in\mathcal{L}(M,\sigma^{-1}(0));\ f(\sigma(\pi_n(\gamma)))(t) = at^n+\cdots,\ a\ ?_\epsilon\}$, where $?_\epsilon$ is $=-1$, $=1$, >0 or <0 in case ϵ is $-1,1,>$ or $<$, and note also that $Y_{n,f\circ\sigma}^\epsilon = \sigma_*^{-1}(Y_{n,f}^\epsilon)$. Finally for $e\geq 1$, let

$$\Delta_e = \{\gamma\in\mathcal{L}(M,\sigma^{-1}(0));\ mult_t\ (\mathrm{jac}\ \sigma)(\gamma(t)) = e\} \text{ and } Y_{e,n,f\circ\sigma}^\epsilon = Y_{n,f\circ\sigma}^\epsilon \cap \Delta_e.$$

4.1.4 Lemma. — *With the notations above, there exists $c\in\mathbb{N}$ such that*

$$Z_f^\epsilon(T) = \mathbb{L}^d \sum_{n\geq 1} T^n \sum_{e\leq cn} \mathbb{L}^{-e} \sum_{I\neq\emptyset} \mathbb{L}^{-(n+1)d}[\mathcal{L}_n(M,E_I^0\cap\sigma^{-1}(0))\cap\pi_n(\Delta_e)\cap X_{n,f\circ\sigma}^\epsilon].$$

Proof. — As usual in motivic integration, the class of the cylinder $Y_{n,f}^\epsilon = \pi_n^{-1}(X_{n,f}^\epsilon)$, $n\geq 1$, is an element of $(K_0(\mathrm{Var}_\mathbb{R})\otimes\mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}]$, the localization of the ring $K_0(\mathrm{Var}_\mathbb{R})\otimes\mathbb{Z}[\frac{1}{2}]$ with respect to the multiplicative set generated by $\mathbb{L}$, and defined by $[Y_{n,f}^\epsilon] := \mathbb{L}^{-(n+1)d}[X_{n,f}^\epsilon)]$, since the truncation morphisms $\pi_{k+1,k} : \mathcal{L}_{k+1}(\mathbb{R}^d,0)\to\mathcal{L}_k(\mathbb{R}^d,0)$, $k\geq 1$, are locally trivial fibrations with fibre $\mathbb{R}^d$. Hence $Z_f^\epsilon(T) = \mathbb{L}^d\sum_{n\geq 1}[Y_{n,f}^\epsilon]T^n$.

Take now $\gamma\in\sigma_*^{-1}(Y_{n,f}^\epsilon)$, and let $I\subset J$ such that $\gamma(0)\in E_I^0$. In some neighbourhood of E_I^0, one has coordinates such that $f\circ\sigma(x) = u(x)\prod_{i\in I}x_i^{N_i}$ and $\mathrm{jac}(\sigma)(x) = v(x)\prod_{i\in I}x_i^{\nu_i-1}$, with u and v units. If one denotes $\gamma = (\gamma_1,\cdots,\gamma_d)$ in these coordinates, with k_i the multiplicity of γ_i at 0 for $i\in I$, then we have $mult_t(f\circ\sigma\circ\gamma(t)) = \sum_{i\in I}k_iN_i = n$. Now

$$mult_t(\mathrm{jac}\,\sigma)(\gamma(t)) = \sum_{i\in I}k_i(\nu_i-1) \leq \max_{i\in I}(\frac{\nu_i-1}{N_i})\sum_{i\in I}N_ik_i = \max_{i\in I}(\frac{\nu_i-1}{N_i})n.$$

Therefore if one sets $c = \max_{i\in I}(\frac{\nu_i - 1}{N_i})$, one has

$$Y^\epsilon_{n,f\circ\sigma} = \bigcup_{e\geq 1} Y^\epsilon_{e,n,f\circ\sigma} = \bigcup_{e\leq cn} Y^\epsilon_{e,n,f\circ\sigma},$$

as disjoint unions. Now we can apply the change of variables theorem (see [**4**], [**10**]) to compute $[Y^\epsilon_{n,f}]$ in terms of $[Y^\epsilon_{e,n,f\circ\sigma}]$:

$$[Y^\epsilon_{n,f}] = \sum_{e\leq cn} \mathbb{L}^{-e}[Y^\epsilon_{e,n,f\circ\sigma}],$$

and summing over the subsets I of J, as $Y^\epsilon_{e,n,f\circ\sigma}$ is the disjoint union

$$\bigcup_{I\neq\emptyset} Y^\epsilon_{e,n,f\circ\sigma} \cap \pi_0^{-1}(E^0_I \cap \sigma^{-1})(0),$$

we obtain

$$\begin{aligned} Z^\epsilon_f(T) &= \mathbb{L}^d \sum_{n\geq 1}[Y^\epsilon_{n,f}]T^n = \mathbb{L}^d \sum_{n\geq 1} T^n \sum_{e\leq cn} \mathbb{L}^{-e} \sum_{I\neq\emptyset}[Y^\epsilon_{e,n,f\circ\sigma} \cap \pi_0^{-1}(E^0_I\cap\sigma^{-1})(0)] \\ &= \mathbb{L}^d \sum_{n\geq 1} T^n \sum_{e\leq cn} \mathbb{L}^{-e} \sum_{I\neq\emptyset} \mathbb{L}^{-(n+1)d}[\pi_n(Y^\epsilon_{e,n,f\circ\sigma} \cap \pi_0^{-1}(E^0_I\cap\sigma^{-1})(0))] = \\ &= \mathbb{L}^d \sum_{n\geq 1} T^n \sum_{e\leq cn} \mathbb{L}^{-e} \sum_{I\neq\emptyset} \mathbb{L}^{-(n+1)d}[\mathcal{L}_n(M, E^0_I\cap\sigma^{-1}(0)) \cap \pi_n(\Delta_e) \cap X^\epsilon_{n,f\circ\sigma}]. \end{aligned}$$

□

Proof of Theorem 4.1.2. — Considering the expression of $Z^\epsilon_f(T)$ given by Lemma 4.1.4, we have to compute the class of $[\mathcal{L}_n(M, E^0_I\cap\sigma^{-1}(0)) \cap \pi_n(\Delta_e) \cap X^\epsilon_{n,f\circ\sigma}]$. For this we notice that on some neighbourhood U of the end point $\gamma(0) \in E^0_I \cap \sigma^{-1}(0))$, one has coordinates such that

$$f\circ\sigma(x) = u(x)\prod_{i\in I} x_i^{N_i} \text{ and } \mathrm{jac}(\sigma)(x) = v(x)\prod_{i\in I} x_i^{\nu_i - 1},$$

with u and v units. As a consequence $\mathcal{L}_n(M, E^0_I\cap U\cap\sigma^{-1}(0)) \cap \pi_n(\Delta_e) \cap X^\epsilon_{n,f\circ\sigma}$ is isomorphic to

$$\{\gamma \in \mathcal{L}_n(M,\sigma^{-1}(0)); \gamma(0) \in E^0_I\cap U\cap\sigma^{-1}(0), \sum_{i\in I} N_i k_i = n, \sum_{i\in I} k_i(\nu_i - 1) = e,$$
$$f\circ\sigma(\gamma(t)) = at^n + \cdots, a\ ?_\epsilon\},$$

where $?_\epsilon$ is $=-1, =1, >0$ or <0 in case ϵ is $-1, 1, >$ or $<$ and k_i is the multiplicity of γ_i for $i \in I$. Now denoting $A(I,n,e)$ the set

$$A(I,n,e) := \{k = (k_1,\cdots,k_d) \in \mathbb{N}^d; \sum_{i\in I} N_i k_i = n, \sum_{i\in I} k_i(\nu_i - 1) = e\},$$

and identifying for simplicity x and $((x_i)_{i\notin I}, (x_i)_{i\in I})$, the set

$$\mathcal{L}_n(M, E^0_I\cap U\cap\sigma^{-1}(0)) \cap \pi_n(\Delta_e) \cap X^\epsilon_{n,f\circ\sigma}$$

is isomorphic to the product

$$(\mathbb{R}^n)^{d-|I|} \times \bigcup_{k\in A(I,n,e)} \{x \in (E_I^0 \cap U \cap \sigma^{-1}(0)) \times (\mathbb{R}^*)^{|I|}; u((x_i)_{i\notin I}, 0) \prod_{i\in I} x_i^{N_i} \ ?_\epsilon\} \prod_{i\in I} (\mathbb{R}^{n-k_i})$$

Actually, denoting $\gamma = (\gamma_1, \ldots, \gamma_d)$ with $\gamma_i(t) = a_{i,0} + \cdots + a_{i,n}t^n$ for $i \notin I$ and $\gamma_i(t) = a_{i,k_i}t^{k_i} + \cdots + a_{i,n}t^n$ for $i \in I$, an arc of $\mathcal{L}_n(M, E_I^0 \cap U \cap \sigma^{-1}(0))$, the first factor of the product comes from the free choice of the coefficients $a_{i,j}$, $i \notin I$, $j = 1, \cdots, n$, the last factor of the product comes from the free choice of the coefficients $a_{i,j}$, $i \in I$, $j = k_i + 1, \ldots, n$ and the middle factor of the product comes from the choice of the coefficients $a_{i,0} \in E_I^0 \cap U \cap \sigma^{-1}(0)$, $i \notin I$ and from the choice of the coefficients a_{i,k_i}, $i \in I$, subject to $f \circ \sigma(\gamma(t)) = u(\gamma(t)) \prod_{i\in I} \gamma_i^{N_i}(t) = u((a_{i,0})_{i\notin I}, 0)(\prod_{i\in I} a_{i,k_i}^{N_i})t^n + \cdots = at^n + \cdots, a \ ?_\epsilon$.

We now choose $n_i \in \mathbb{Z}$ such that $\sum_{i\in I} n_i N_i = m = gcd_{i\in I}(N_i)$ and consider the two semialgebraic sets

$$W_U^\epsilon = \{x \in (E_I^0 \cap U \cap \sigma^{-1}(0)) \times (\mathbb{R}^*)^{|I|}; u((x_i)_{i\notin I}, 0) \prod_{i\in I} x_i^{N_i} \ ?_\epsilon\}$$

and

$$W_U'^\epsilon = \{(x', t) \in (E_I^0 \cap U \cap \sigma^{-1}(0)) \times (\mathbb{R}^*)^{|I|} \times \mathbb{R}^*; u((x_i')_{i\notin I}, 0)t^m \ ?_\epsilon, \ \prod_{i\in I} x_i'^{N_i/m} = 1\},$$

where $?_\epsilon$ is $= -1$, $= 1$, > 0 or < 0 in case ϵ is $-1, 1, >$ or $<$. In case $?_\epsilon = 1$ or $?_\epsilon = -1$, the mapping

$$\begin{array}{ccl} W'^\epsilon & \to & W^\epsilon \\ (x', t) & \mapsto & x = ((x_i')_{i\notin I}, (t^{n_i} x_i')_{i\in I}) \end{array}$$

is an isomorphism of inverse

$$\begin{array}{ccl} W_U^\epsilon & \to & W_U'^\epsilon \\ x & \mapsto & (x' = ((x_i)_{i\notin I}, ((\prod_{\ell\in I} x_\ell^{N_\ell/m})^{-n_i} x_i)_{i\in I}), t = \prod_{\ell\in I} x_\ell^{N_\ell/m}) \end{array}$$

In the semialgebraic case, this isomorphism induces a natural isomorphism on the double-covers $\mathcal{W}_U^\epsilon$ and $\mathcal{W}_U'^\epsilon$ associated to W_U^ϵ and $W_U'^\epsilon$ and defined by

$$\mathcal{W}_U^\epsilon = \{(x, y) \in (E_I^0 \cap U \cap \sigma^{-1}(0)) \times (\mathbb{R}^*)^{|I|} \times \mathbb{R}; y^2 u((x_i')_{i\notin I}, 0)t^m = \eta(\epsilon)\}$$

and

$$\begin{aligned} \mathcal{W}_U'^\epsilon = \{(x, t, w) \in (E_I^0 \cap U \cap \sigma^{-1}(0)) \times (\mathbb{R}^*)^{|I|} \times \mathbb{R}^* \times \mathbb{R}; \\ w^2 u((x_i')_{i\notin I}, 0)t^m = \eta(\epsilon), \ \prod_{i\in I} x_i'^{N_i/m} = 1\}, \end{aligned}$$

where $\eta(\epsilon) = 1$ when ϵ is the symbol $>$ and $\eta(\epsilon) = -1$ when ϵ is the symbol $<$. Now we observe that W'^ϵ is isomorphic to $R_U^\epsilon \times (\mathbb{R}^*)^{|I|-1}$, since at least one of the integers N_i/m is odd.

We finally obtain

$$[\mathcal{L}_n(M, E_I^0 \cap \sigma^{-1}(0)) \cap \pi_n(\Delta_e) \cap X_{n,f\circ\sigma}^\epsilon] = \sum_{k\in A(I,n,e)} \mathbb{L}^{nd - \sum_{i\in I} k_i}[W_U'^\epsilon] =$$

$$\sum_{k\in A(I,n,e)} \mathbb{L}^{nd-\sum_{i\in I}k_i} \times [R_U^\epsilon] \times (\mathbb{L}-1)^{|I|-1}.$$

Summing over the charts U, the expression of $Z_f^\epsilon(T)$ given by Lemma 4.1.4 is now

$$Z_f^\epsilon(T) = \sum_{I\cap\mathcal{K}\neq\emptyset} \mathbb{L}^d \sum_{n\geq 1} T^n \sum_{e\leq cn} \mathbb{L}^{-e}(\mathbb{L}-1)^{|I|-1}\mathbb{L}^{-(n+1)d}[\tilde{E}_I^{0,\epsilon}] \sum_{k\in A(I,n,e)} \mathbb{L}^{nd-\sum_{i\in I}k_i}$$

$$= \sum_{I\cap\mathcal{K}\neq\emptyset} (\mathbb{L}-1)^{|I|-1}[\tilde{E}_I^{0,\epsilon}] \sum_{n\geq 1} T^n \sum_{e\leq cn} \sum_{k\in A(I,n,e)} \mathbb{L}^{-e-\sum_{i\in I}k_i}$$

Noticing that the $(k_i)_{i\in I}$'s such that $k = ((k_i)_{i\notin I}), (k_i)_{i\in I}) \in \bigcup_{e\leq cn, n\geq 1} A(I,n,e)$ are in bijection with $\mathbb{N}^{|I|}$, we have

$$Z_f^\epsilon(T) = \sum_{I\cap\mathcal{K}\neq\emptyset} (\mathbb{L}-1)^{|I|-1}[\tilde{E}_I^{0,\epsilon}] \sum_{(k_i)_{i\in I}\in\mathbb{N}^{|I|}} \prod_{i\in I} (\mathbb{L}^{-\nu_i}T^{N_i})^{k_i}$$

$$= \sum_{I\cap\mathcal{K}\neq\emptyset} (\mathbb{L}-1)^{|I|-1}[\tilde{E}_I^{0,\epsilon}] \prod_{i\in I} \frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1-\mathbb{L}^{-\nu_i}T^{N_i}}.$$

□

4.2. Motivic real Milnor fibres and their realizations.— We can now define a motivic real Milnor fibre by taking the constant term of the rational function $Z_f^\epsilon(T)$ viewed as a power series in T^{-1}. This process formally consists in letting T going to ∞ in the rational expression of $Z_f^\epsilon(T)$ given by Theorem 4.1.2 and using the usual computation rules as in the convergent case (see for instance [**3**], [**6**]).

4.2.1 Definition. — Let $f : \mathbb{R}^d \to \mathbb{R}$ be a polynomial function and ϵ be one of the symbols $naive, 1, -1, >$ or $<$. Consider a resolution of f as above and let us adopt the same notation $(E_I^0)_I$ for the stratification of the exceptional divisor of this resolution, leading to the notations $\tilde{E}_I^{0,\epsilon}$. The real motivic Milnor ϵ-fibre S_f^ϵ of f is defined as (see [**6**] for the complex case)

$$S_f^\epsilon := -\lim_{T\to\infty} Z_f^\epsilon(T) := -\sum_{I\cap\mathcal{K}\neq\emptyset} (-1)^{|I|}[\tilde{E}_I^{0,\epsilon}](\mathbb{L}-1)^{|I|-1} \in K_0(\mathrm{Var}_\mathbb{R}) \otimes \mathbb{Z}[\frac{1}{2}].$$

It does not depend on the choice of the resolution σ.

For ϵ being the symbol 1 for instance, we have $S_f^1 \in K_0(\mathrm{Var}_\mathbb{R})$. We can consider, first in the complex case, the realization of S_f^1 via the Euler-Poincaré characteristic ring morphism $\chi_c : K_0(\mathrm{Var}_\mathbb{C}) \to \mathbb{Z}$. In the complex case, that is for $f : \mathbb{C}^d \to \mathbb{C}$, since $\chi_c(\mathbb{L}-1) = 0$, we obtain

$$\chi_c(S_f^1) = \sum_{|I|=1, I\subset\mathcal{K}} \chi_c(\tilde{E}_I^{0,1}) = \sum_{|I|=1, I\subset\mathcal{K}} N_I \cdot \chi_c(E_I^0 \cap \sigma^{-1}(0)).$$

Now denoting F the set theoretic Milnor fibre of the fibration $f_{|B(0,\alpha)\cap f^{-1}(D_\eta^\times)}$: $B(0,\alpha)\cap f^{-1}(D_\eta^\times) \to D_\eta^\times$, with $B(0,\alpha)$ the open ball in $\mathbb{C}^d$ of radius α centred at 0, D_η the disc in $\mathbb{C}$ of radius η centred at 0 and $D_\eta^\times = D_\eta \setminus \{0\}$, with $0 < \eta \ll \alpha \ll 1$, comparing the above expression $\chi_c(S_f^1) = \sum_{|I|=1, I\subset\mathcal{K}} N_I \cdot \chi_c(E_I^0)$ with the following A'Campo formula of [**1**] for the first Lefschetz number of the iterates of the monodromy $M : H^*(F,\mathbb{C}) \to H^*(F,\mathbb{C})$ of f, that is for the Euler-Poincaré characteristic of the fibre F

$$\chi_c(F) = \sum_{|I|=1, I\subset\mathcal{K}} N_I \cdot \chi_c(E_I^0 \cap \sigma^{-1}(0))$$

we simply observe that

$$\chi_c(S_f^1) = \chi_c(F).$$

The closure $f^{-1}(c)\cap \bar{B}(0,\alpha)$, $0<|c|\ll\alpha\ll 1$, of the Milnor fibre F being denoted by $\bar{F}$ and the boundary of $\bar{F}$ being the odd dimensional compact manifold $f^{-1}(c)\cap S(0,\alpha)$, $\chi_c(f^{-1}(c)\cap S(0,\alpha)) = 0$ we finally have

$$\chi_c(S_f^1) = \chi_c(F) = \chi_c(\bar{F}).$$

4.2.2 Remark. — There is *a priori* no hint in the definition of $Z_f^\epsilon(T)$ that the opposite of the constant term S_f^1 of the power series in T^{-1} induced by the rationality of $Z_f^\epsilon(T)$ could be the motivic version of the Milnor fibre of f (as well as, for instance, there is no evident hint that the expression of Z_f^ϵ in Theorem 4.1.2 does not depend on the resolution σ). As mentionned above, in the complexe case, we just observe that the expression of $\chi_c(S_f^1)$ is the expression of $\chi_c(F)$ provided by the A'Campo formula. Exactly in the same way there is no *a priori* reason for $\chi_c(S_f^\epsilon)$, regarding the definition of Z_f^ϵ, to be so acurately related to the topology of $f^{-1}(\epsilon|c|)\cap B(0,\alpha)$. Nevertheless we prove that it is actually the case (Theorem 4.2.8).

In order to establish this result we start hereafter by a geometrical proof of the formula in the complexe case (compare with [**1**] when only $\Lambda(M^0)$ is considered, M^k being the kth iterate of the monodromy $M : H^*(F,\mathbb{C}) \to H^*(F,\mathbb{C})$ of f). We then will extend to the reals this computational proof in the proof of Theorem 4.2.8, letting us interpret the complex proof as the first complexity level of its real extension.

4.2.3 Remark. — Note that in the complex case a proof of the fact that $\Lambda(M^k) = \chi_c(X_{k,f}^1)$, for $k\geq 1$, is given in [**9**] without using resolution of singularities, that is to say without help of A'Campo's formulas (see Theorem 1.1.1 of [**9**]). As a direct corollary it is thus proved that $\chi_c(S_f^1) = \chi_c(F)$ in the complex case, without using A'Campo formulas.

Realization of the complex motivic Milnor fibre under χ_c. The fibre $F = \{f = c\}\cap B(0,\alpha)$ is homeomorphic to the fibre $\mathcal{F} = \{f\circ\sigma = c\}\cap\sigma^{-1}(B(0,\alpha))$

viewed as the boundary of a tubular neighbourhood of $\sigma^{-1}(0) = \bigcup_{E_J^0 \subset \sigma^{-1}(0)} E_J^0$, keeping the same notation $(E_J^0)_J$ as before for the natural stratification of the strict transform $\sigma^{-1}(\{f=0\})$ of $f=0$. Now the formula may be established for $\mathcal{F}$ in some chart of $M \cap \sigma^{-1}(B(0,\alpha))$, by additivity. In such a chart, where $f \circ \sigma$ is normal crossing, consider $E_J \subset \sigma^{-1}(0)$, given by $x_i = 0$ for all $i \in J$, a closed small enough tubular neighbourhood V_J in M of $\cup_{J \subset K} E_K^0$, that is a tubular neighbourhood of all the E_k^0's bounding E_J^0, such that $E_J^0 \setminus V_J$ is homeomorphic to E_J^0 and an open neighbourhood $\mathcal{E}_J$ of $E_J^0 \setminus V_J$ in $\sigma^{-1}(B(0,\alpha))$ given by $\pi_J^{-1}(E_J^0 \setminus V_J)$, $|x_j| \leq \eta_J$, $j \in J$, with $\eta_J > 0$ small enough and π_J the projection onto E_J along the x_j's, $j \in J$.

4.2.4 Remark. — For $I = \{i\}$, we remark that $\mathcal{F} \cap \mathcal{E}_I$ is homeomorphic to N_i copies of $E_I^0 \cap \mathcal{E}_I$, and thus to N_i copies of E_I^0. Indeed, assuming $f \circ \sigma = u(x) x_i^{N_i}$ in $\mathcal{E}_I$, we observe that the family $(f_t)_{t \in [0,1]}$, with $f_t = u((x_j)_{j \notin I}, t \cdot x_i) x_i^{N_i} - c$, has homeomorphic fibres $\{f_t = 0\} \cap \mathcal{E}_J$, $t \in [0,1]$, by Thom's isotopy lemma, since

$$\frac{\partial f_t}{\partial x_i}(x) = t \frac{\partial u}{\partial x_i}(x) x_i^{N_i} + u(x) x_i^{N_i - 1} = 0,$$

would implie $t \dfrac{\partial u}{\partial x_i}(x) x_i + u(x) = 0$. But the first term in this sum goes to 0 as x_i goes to 0, since the derivatives of u are bounded on the compact $adh(\mathcal{E}_I)$, although the norm of the second term is bounding from below on $\mathcal{E}_I$ by a non zero constant, since u is a unit. Finally, as $\{f_1 = 0\} \cap \mathcal{E}_I$ is homeomorphic to $\{f_0 = 0\} \cap \mathcal{E}_I$ and $\{f_0 = 0\} \cap \mathcal{E}_I$ is a N_i-graph over $E_I^0 \cap \mathcal{E}_I$, $\mathcal{F} \cap \mathcal{E}_I$ is homeomorphic to N_i copies of E_I^0.

By this remark, $\mathcal{F}$ covers a higher dimensional stratum E_I^0, $|I| = 1$, $I \subset \mathcal{K}$, with N_i copies of a leaf $\mathcal{F}_I$ of $\mathcal{F}$. To be more accurate, with the notation introduced above, $\mathcal{F}_I$ covers the neighborhood $E_I^0 \cap \mathcal{E}_I$ of $E_I^0 \setminus V_I$. Moreover the $\mathcal{F}_I$'s overlap in $\mathcal{F}$ over the open set $E_J^0 \cap \mathcal{E}_J$ of the strata E_J^0 that bound the E_I^0's, for $|I| = 1$, $|J| = 2$ and $I \subset J$, in bundles over the $E_J^0 \cap \mathcal{E}_J$'s of fibre $\mathbb{C}^*$. Those sub-leaves $\mathcal{F}_J$ of $\mathcal{F}$ overlap in turn over the open $E_Q^0 \cap \mathcal{E}_Q$ of the strata E_Q^0, $|Q| = 3, J \subset Q$, that bound the E_J^0's, in bundles over the $E_Q^0 \cap \mathcal{E}_Q$'s of fibres $(\mathbb{C}^*)^2$ and so forth... For instance when $f \circ \sigma = u(x) \prod_{i \in I} x_i^{N_i}$ in $\mathcal{E}_I$, $I = \{i\}$, and $f \circ \sigma = v(x) x_i^{N_i} x_j^{N_j}$ in $\mathcal{E}_J$, $J = \{i,j\}$, the N_i leaves $\mathcal{F}_I$, homeomorphic to the N_i copies $x_i^{N_i} = c/u(x)$ of E_I^0, overlap over $E_J^0 \cap \mathcal{E}_J$ in sub-leaves $\mathcal{F}_J$ of $\mathcal{F}_I$, given by $v(x) x_i^{N_i} x_j^{N_j} = c$, fibering cover E_J^0 with fibre $GCD(\{N_i, N_j\})$ copies of $(\mathbb{C}^*)^{|J|-1}$ and so forth... (see figure 1).

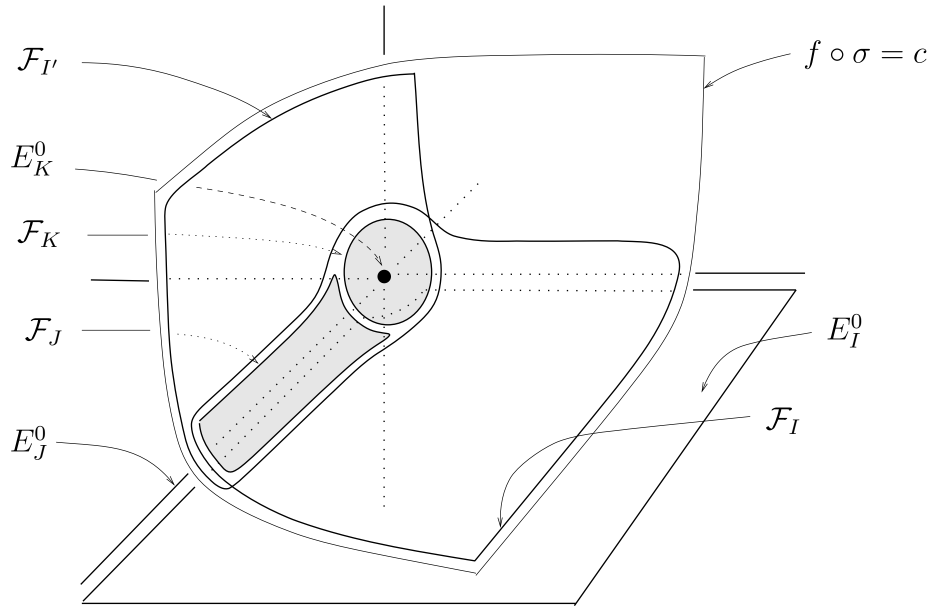

figure 1

4.2.5 Remark. — Note that the topology of $\mathcal{F} = \{f \circ \sigma = c\} \cap \sigma^{-1}(B(0,\alpha))$ is the same as the topology of $\bigcup_{J\cap\mathcal{K}\neq 0} \mathcal{F}_J$ (that is the topology of $\mathcal{F}$ above the strata E_J^0 of $\sigma^{-1}(0)$) since the retraction of $\mathcal{F}$ onto $\bigcup_{J\cap\mathcal{K}\neq\emptyset} \mathcal{F}_J$, as α goes to 0, induces a homeomorphism from $\mathcal{F}$ to $\bigcup_{J\cap\mathcal{K}\neq\emptyset} \mathcal{F}_J$.

From Remark 4.2.5, by additivity, it follows that the Euler-Poincaré characteristic of $\mathcal{F}$ (in our chart) is the sum

$$\sum_{|I|=1, I\subset\mathcal{K}} N_I \cdot \chi_c(E_I^0 \cap \sigma^{-1}(0)) + L, \qquad (*)$$

where L is some $\mathbb{Z}$-linear combination of Euler-Poincaré characteristics of bundles over the open sets $E_J \cap \mathcal{E}_J^0$, $|J| > 1$, of fibre a power of tori $\mathbb{C}^*$. Now the A'Campo formula

$$\chi_c(F) = \sum_{|I|=1, I\subset\mathcal{K}} N_I \cdot \chi_c(E_I^0 \cap \sigma^{-1}(0))$$

for the Milnor number follows from the fact that $\chi_c(\mathbb{C}^*) = 0$ implies $L = 0$.

Realization of the real motivic Milnor fibres under χ_c. The partial covering of $\mathcal{F}$ by the pieces $\mathcal{F}_J$, for $J \cap \mathcal{K} \neq \emptyset$, over the strata of the stratification $(E_J^0)_{J\cap\mathcal{K}\neq\emptyset}$ of $\sigma^{-1}(0)$ allows us to compute the Euler-Poincaré characteristic of the Milnor fibre $\mathcal{F}$ in terms of the Euler-Poincaré characteristic of the strata E_J^0, in the complex as well as in the real case. In the complex case, as noted above, for J with $|J| > 1$, one has $\chi_c(\mathcal{F}_J) = 0$. This cancellation provides a quite simple formula for $\chi_c(F)$: only the higher dimensional strata of the divisor $\sigma^{-1}(0)$ appear in this formula, as expected from the A'Campo formula.

In the real case one does not have such cancellations: on one hand the expression of $\chi_c(F)$ in terms of $\chi_c(\widetilde{E}_J^0)$ is no more trivial (the remaining term L of equation $(*)$ is not zero and consequently terms $\chi_c(\widetilde{E}_J^0)$, for $|J| > 1$ and $E_j \cap \sigma^{-1}(0) \neq \emptyset$, appear), and on the other hand the expression of $\chi_c(S_f^\epsilon)$ given by the real Denef-Loeser formula in Definition 4.2.1 have terms $2^{|J|-1}\chi_c(\widetilde{E}_J^0)$, for $|J| > 1$ and $J \cap \mathcal{K} \neq \emptyset$ (since $\chi_c(\mathbb{L} - 1) = -2$ in the real case).

Nevertheless, in the real case we show that $\chi_c(S_f^\epsilon)$ is again $\chi_c(\bar{F})$, justifying the terminology of motivic real semialgebraic Milnor fibre of f at 0 for S_f^ϵ. The formula stated in Theorem 4.2.8 below is the real analogue of the A'Campo-Denef-Loeser formula for complex hypersurface singularities and thus appears as the extension to the reals of this complex formula, or, in other words, the complex formula is the notably first level of complexity of the more general real formula.

4.2.6 Notations. — Let $f : \mathbb{R}^d \to \mathbb{R}$ be a polynomial function such that $f(0) = 0$ and with isolated singularity at 0, that is $\operatorname{grad} f(x) = 0$ only for $x = 0$ in some open neighbourhood of 0. Let $0 < \eta \ll \alpha$ such that the topological type of $f^{-1}(c) \cap B(0, \alpha)$ does not depend on c and α, for $0 < c < \eta$ or for $-\eta < c < 0$.

- Let us denote, for $\epsilon \in \{-1, 1\}$ and $\epsilon \cdot c > 0$, this topological type by F_ϵ, by $\bar{F}_\epsilon$ the topological type of the closure of the Milnor fibre F_ϵ and by $Lk(f)$ the link $f^{-1}(0) \cap S(0, \alpha)$ of f at the origin. We recall that the topology of $Lk(f)$ is the same as the topology of the boundary $f^{-1}(c) \cap S(0, \alpha)$ of the Milnor fibre $\bar{F}_\epsilon$, when f has an isolated singularity at 0.

- Let us denote, for $\epsilon \in \{<, >\}$, the topological type of $f^{-1}(]0, c_\epsilon[) \cap B(0, \alpha)$ by F_ϵ, and the topological type of $f^{-1}(]0, c_\epsilon[) \cap \bar{B}(0, \alpha)$ by $\bar{F}_\epsilon$, where $c_< \in] - \eta, 0[$ and $c_> \in]0, \eta[$.

- Let us denote, for $\epsilon \in \{<, >\}$, the topological type of $\{f \ \bar{\epsilon}\ 0\} \cap S(0, \alpha)$ by G_ϵ, where $\bar{\epsilon}$ is $\leq$ when ϵ is $<$ and $\bar{\epsilon}$ is $\geq$ when ϵ is $>$.

4.2.7 Remark. — When d is odd, $Lk(f)$ is a smooth odd dimensional submanifold of $\mathbb{R}^d$ and consequently $\chi_c(Lk(f)) = 0$. For $\epsilon \in \{-1, 1, <, >\}$, we thus have in this situation, $\chi_c(F_\epsilon) = \chi_c(\bar{F}_\epsilon)$. This is the situation in the complex setting. When d is even and for $\epsilon \in \{-1, 1\}$ since $\bar{F}_\epsilon$ is a compact manifold with boundary $Lk(f)$, one knows that

$$\chi_c(\bar{F}_\epsilon) = -\chi_c(F_\epsilon) = \frac{1}{2}\chi_c(Lk(f)).$$

For general $d \in \mathbb{N}$ and for $\epsilon \in \{-1, 1, <, >\}$, we thus have

$$\chi_c(\bar{F}_\epsilon) = (-1)^{d+1}\chi_c(F_\epsilon).$$

On the other hand we recall that for $\epsilon \in \{<, >\}$

$$\chi_c(G_\epsilon) = \chi_c(\bar{F}_{\delta_\epsilon}),$$

where $\delta_>$ is 1 and $\delta_<$ is -1.

4.2.8 Theorem. — *With notations 4.2.6, we have, for* $\epsilon \in \{-1, 1, <, >\}$

$$\chi_c(S_f^\epsilon) = \chi_c(\bar{F}_\epsilon) = (-1)^{d+1}\chi_c(F_\epsilon),$$

and for $\epsilon \in \{<, >\}$

$$\chi_c(S_f^\epsilon) = -\chi_c(G_\epsilon).$$

Proof. — Assume first that $\epsilon \in \{-1, 1\}$. We denote by $\mathcal{F}$ the fibre $\sigma^{-1}(F_\epsilon)$ and recall that $\mathcal{F}$ and F_ϵ have the same topological type. Let us denote $\bar{\mathcal{K}}$ the set of multi-indices $J \subset \mathcal{I}$ such that $\bar{E}_J \cap \sigma^{-1}(0) \neq \emptyset$. In what follows only $J \in \bar{\mathcal{K}}$ are concerned, since we study the local Milnor fibre at 0. Note that a connected component of E_J^0 (still denoted E_J^0 for simplicity in the sequel), for $J \subset \mathcal{J}$, is covered by $n_J := M_J \cdot 2^{|J|-1}$ connected components $\mathcal{G}$ of $\mathcal{F}$, where M_J is 0, 1 or 2 depending on the multiplicity $m_J = gcd_{j\in J}(N_j)$ defining $\tilde{E}_J^{0,\epsilon}$ is odd or even and sign condition on c. Note furthermore that M_J is the degree of the covering $\tilde{E}_J^{0,\epsilon}$ of E_J^0. Now expressing a connected component $\mathcal{G}$ of $\mathcal{F}$ as the union $\bigcup_{|I|=1, \mathcal{F}_I \subset \mathcal{G}} \mathcal{F}_I$, where the (connected) leaves $\mathcal{F}_I$ cover (the open subset $E_I^0 \cap \mathcal{E}_I^0$ of E_I^0 homeomorphic to) E_I^0, and using the additivity of χ_c, one has that $\chi_c(\mathcal{G})$ is expressed as a sum of characteristics of the overlapping connected sub-leaves $\mathcal{F}_J$ of the $\mathcal{F}_I$'s, each of them with sign coefficient $s_J := (-1)^{|J|-1}$. Note that (a connected component of) E_J^0 is covered by n_J copies of such a $\mathcal{F}_J$, coming from the n_J connected components of $\mathcal{F}$ above $E_J^0 \cap \mathcal{E}_J^0$, and that a connected sub-leaf $\mathcal{F}_J$ has the topology of $(E_J^0 \cap \mathcal{E}_J^0) \times \mathbb{R}^{|J|-1}$. We denote by t_J the characteristic $t_J := \chi_c(\mathbb{R}^{|J|-1}) = (-1)^{|J|-1}$.

With these notations, summing over all the connected components $\mathcal{G}$ of $\mathcal{F}$, one gets

$$\begin{aligned}
\chi_c(\mathcal{F}) &= \sum_{J \in \bar{\mathcal{K}}} s_J \times n_J \times \chi_c(E_J^0) \times t_J \\
&= \sum_{J \in \bar{\mathcal{K}}} (-1)^{|J|-1} \times 2^{|J|-1} M_J \times \chi_c(E_J^0) \times (-1)^{|J|-1} \\
&= \sum_{J \in \bar{\mathcal{K}}} 2^{|J|-1} \chi_c(\tilde{E}_J^{0,\epsilon}) \\
&= \sum_{J \cap \mathcal{K} \neq \emptyset} 2^{|J|-1} \chi_c(\tilde{E}_J^{0,\epsilon}) + \sum_{J \cap \mathcal{K} = \emptyset, J \in \bar{\mathcal{K}}} 2^{|J|-1} \chi_c(\tilde{E}_J^{0,\epsilon}) \\
&= \chi_c(S_f^\epsilon) + \sum_{J \cap \mathcal{K} = \emptyset, J \in \bar{\mathcal{K}}} 2^{|J|-1} \chi_c(\tilde{E}_J^{0,\epsilon}) \\
&= \chi_c(S_f^\epsilon) + \chi_c(\bigcup_{J \cap \mathcal{K} = \emptyset, J \in \bar{\mathcal{K}}} \mathcal{F}_J).
\end{aligned}$$

Note that the sub-leaves $\mathcal{F}_J$ for $J \cap \mathcal{K} = \emptyset$ and $J \in \bar{\mathcal{K}}$ cover the set $\{f \circ \sigma = c\} \cap \hat{S}(0, \alpha)$, for $\epsilon \cdot c > 0$, where $\hat{S}(0, \alpha)$ is a neighbourhood $\sigma^{-1}(S(0, \alpha) \times]0, \beta[)$ of

$\sigma^{-1}(S(0,\alpha))$, with $0<\beta\ll\alpha$. It follows that

$$\chi_c(\bigcup_{J\cap\mathcal{K}=\emptyset, J\in\bar{\mathcal{K}}}\mathcal{F}_J)=\chi_c(F_\epsilon\cap(S(0,\alpha)\times]0,\beta[))=\chi_c(Lk(f)\times]0,\beta[)=-\chi_c(Lk(f)).$$

We finally obtain

$$\chi_c(F_\epsilon)=\chi_c(S_f^\epsilon)-\chi_c(Lk(f)),$$

and

$$\chi_c(\bar{F}_\epsilon)=\chi_c(F_\epsilon)+\chi_c(Lk(f))=\chi_c(S_f^\epsilon).$$

This proves the first equality of our statement, the equality $\chi_c(\bar{F}_\epsilon)=(-1)^{d+1}\chi_c(F_\epsilon)$ being proved in Remark 4.2.7.

Assume now that $\epsilon\in\{<,>\}$, and denote $\delta_<:=-1$ and $\delta_>:=1$, like in Remark 4.2.7. With this notation $\bar{F}_\epsilon=\bar{F}_{\delta_\epsilon}\times]0,1[$, and by the formula proved above in the case $\epsilon\in\{-1,1\}$, we obtain

$$\chi_c(\bar{F}_\epsilon)=\chi_c(\bar{F}_{\delta_\epsilon})\chi_c(]0,1[)=-\chi_c(\bar{F}_{\delta_\epsilon})=-\chi_c(S_f^{\delta_\epsilon})=-\sum_{J\cap\mathcal{K}\neq\emptyset}2^{|J|-1}\chi(\tilde{E}_J^{0,\delta_\epsilon}).$$

But since $\tilde{E}_J^{0,\epsilon}=\tilde{E}_J^{0,\delta_\epsilon}\times\mathbb{R}_+$, it follows that

$$\chi_c(\bar{F}_\epsilon)=\sum_{J\cap\mathcal{K}\neq\emptyset}2^{|J|-1}\chi(\tilde{E}_J^{0,\delta_\epsilon})\chi_c(\mathbb{R}_+)=\sum_{J\cap\mathcal{K}\neq\emptyset}2^{|J|-1}\chi(\tilde{E}_J^{0,\epsilon})=\chi_c(S_f^\epsilon).$$

This proves the first equality of our statement. The equality $\chi_c(\bar{F}_\epsilon)=(-1)^{d+1}\chi_c(F_\epsilon)$ is the consequence of $\chi_c(\bar{F}_\epsilon)=\chi_c(\bar{F}_{\delta_\epsilon})\chi_c(]0,1[)$, $\chi_c(F_\epsilon)=\chi_c(F_{\delta_\epsilon})\chi_c(]0,1[)$ and $\chi_c(\bar{F}_{\delta_\epsilon})=(-1)^{d+1}\chi_c(F_{\delta_\epsilon})$.

To finish, the equality $\chi_c(S_f^\epsilon)=-\chi_c(G_\epsilon)$ comes from the equality $\chi_c(G_\epsilon)=\chi_c(\bar{F}_{\delta_\epsilon})$ recalled in Remark 4.2.7 and from $\chi_c(\bar{F}_\epsilon)=-\chi_c(\bar{F}_{\delta_\epsilon})$, $\chi_c(S_f^\epsilon)=\chi_c(\bar{F}_\epsilon)$.

□

4.2.9 Remark. — As stated in Theorem 4.2.8, the realization via χ_c of the motivic Milnor fibre S_f^ϵ, for $\epsilon\in\{-1,1,<,>\}$, gives the Euler-Poincaré characteristic of the corresponding set theoretic semialgebraic closed Milnor fibre $\bar{F}_\epsilon$. Nevertheless it is worth noting that this equality is in general not true at the higher level of $\chi(K_0[BSA_\mathbb{R}])$. Even computed in $K_0(\mathrm{Var}_\mathbb{R})\otimes\mathbb{Z}[\frac{1}{2}]$, we may have $S_f^\epsilon\neq[A_{f,\epsilon}]$, for a given semialgebraic formula $A_{f,\epsilon}$ with real points $\bar{F}_\epsilon$. Let's illustrate this remark by the following quite trivial example.

4.2.10 Example. — Let us consider the simple case where $f:\mathbb{R}^2\to\mathbb{R}$ is given by $f(x,y)=xy$. After one blowing-up the situation is as required by Theorem 4.1.2, with $M=S^1\times\mathbb{R}$. We denote by $E_1=S^1\times\{0\}$ the exceptional divisor $\sigma^{-1}(0)$ and by E_2,E_3 the irreducible components of the strict transform $\sigma^{-1}(\{f=0\})$. The induced stratification of E_1 is given by $E_{1,2}^0=E_1\cap E_2$, $E_{1,3}^0=E_1\cap E_3$, and the two connected components $E_1^{\prime 0},E_1^{\prime\prime 0}$ of $E_1\setminus(E_2\cup E_3)$. We consider a chart (X,Y) of M such that $\sigma(X,Y)=(x=Y,y=XY)$. In this chart $(f\circ\sigma)(X,Y)=XY^2$. The multiplicity of $f\circ\sigma$ along E_1 is $N_1=2$, and the multiplicity of $\operatorname{jac}\sigma$ along E_1 is

1, thus $\nu_1 = 2$. Assuming that $E_1'^0$ corresponds to $X > 0$ and $E_1''^0$ corresponds to $X < 0$, it follows that

$$\tilde{E}_1'^{0,\epsilon} = \{(X,t); X \in E_1'^0, t \in \mathbb{R}, Xt^2 ?_\epsilon\} \text{ and } \tilde{E}_1''^0 = \{(X,t); X \in E_1''^0, t \in \mathbb{R}, Xt^2 ?_\epsilon\},$$

where $?_\epsilon$ is $= 1$, $= -1$, $>$ or < 0 in case ϵ is 1, -1, $>$ or $<$. In case $\epsilon = 1$ we obtain

$$[\tilde{E}_1'^{0,1}] = \mathbb{L} - 1 \text{ and } [\tilde{E}_1''^{0,1}] = 0$$

since $\tilde{E}_1'^{0,1}$ has a one-to-one projection onto $\{(X,Y); X = 0, Y \neq 0\})$ and $\tilde{E}_1''^{0,1}$ is empty. Now in a neighbourhood of $E_{1,2}^0$, $f \circ \sigma(X,Y) = XY^2$, giving $N_1 = 1$, $N_2 = 2$ and $m = gcd(N_1, N_2) = 1$. We also have $\nu_1 = 2$ and $\nu_2 = 1$. It follows that

$$\tilde{E}_{1,2}^0 = \{(0,t); t \in \mathbb{R}, t = 1\} \text{ thus } [\tilde{E}_{1,2}^0] = 1.$$

In the same way, using another chart, one finds

$$[\tilde{E}_{1,3}^0] = 1.$$

By Theorem 4.1.2 we then have

$$Z_f^1(T) = (\mathbb{L}-1)^{1-1}(\mathbb{L}-1)\Big(\frac{\mathbb{L}^{-2}T^2}{1-\mathbb{L}^{-2}T^2}\Big) + 2(\mathbb{L}-1)^{2-1}\Big(\frac{\mathbb{L}^{-2}T^2}{1-\mathbb{L}^{-2}T^2}\Big)\Big(\frac{\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T}\Big),$$

$$Z_f^1(T) = \frac{\mathbb{L}-1}{(\mathbb{L}T^{-1}-1)^2} \text{ and } S_f^1 = -(\mathbb{L}-1).$$

Of course we find that $\chi_c(S_f) = \chi_c(\{f = c\} \cap \bar{B}(0,1)) = 2$, $0 < c \ll 1$.

Now let's for instance choose $\{xy = c, 1 - x^2 - y^2 > 0\}$, for $0 < c \ll 1$, as a basic semialgebraic formula to represent the open Milnor fibre and let us compute $\chi([xy = c, 1 - x^2 - y^2 > 0])$. By definition of the realization $\chi : K_0(BSA_{\mathbb{R}}) \to K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$, we have

$$\chi([xy = c, 1 - x^2 - y^2 > 0])$$

$$= \frac{1}{4}[xy = c, z^2 = 1-x^2-y^2] - \frac{1}{4}[xy = c, z^2 = x^2+y^2-1] + \frac{1}{2}[xy = c, 1-x^2-y^2 \neq 0]$$

Projecting the algebraic set $\{xy = c, z^2 = 1 - x^2 - y^2\}$ to the plane $x = -y$ with coordinates $(X = 1/\sqrt{2}(x-y), z)$ one finds twice the quadric $z^2 + 2X^2 = 1 - 2c$ that is, up to isomorphism, two circles. A circle having with class $\mathbb{L} + 1$ in $K_0(\mathrm{Var}_{\mathbb{R}})$, we have

$$[xy = c, z^2 = 1 - x^2 - y^2] = 2(\mathbb{L} + 1).$$

Projecting the algebraic set $\{xy = c, z^2 = x^2 + y^2 - 1\}$ to the plane $x = -y$ with coordinates $(X = 1/\sqrt{2}(x-y), z)$ one finds twice the hyperbola $2X^2 - z^2 = 1 - 2c$. Projecting again the hyperbola onto one of its asymptotic axes we see that this hyperbola has class $\mathbb{L} - 1$. It gives

$$[xy = c, z^2 = x^2 + y^2 - 1] = 2(\mathbb{L} - 1).$$

Finally the constructible set $\{xy = c, 1 - x^2 - y^2 \neq 0\}$ being the hyperbola without 4 points, we have

$$\chi([xy = c, 1 - x^2 - y^2 > 0]) = \frac{1}{2}(\mathbb{L} + 1) - \frac{1}{2}(\mathbb{L} - 1) + \frac{1}{2}(\mathbb{L} - 1) - 4 = \frac{\mathbb{L} - 3}{2}.$$

Of course $\chi_c(\chi([xy = c, 1 - x^2 - y^2 > 0])) = \chi_c(\{f = c\} \cap B(0, 1)) = -2$.

The simple semialgebraic formula representing the set theoretic closed Milnor fibre is $\{xy = c, 1 - x^2 - y^2 \geq 0\}$, it has class $\chi([xy = c, 1 - x^2 - y^2 > 0]) + 4[\{*\}] = \dfrac{\mathbb{L} + 5}{2}$ in $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$. But although

$$\chi_c(\chi([xy = c, 1 - x^2 - y^2 \geq 0])) = \chi_c(S^1_f) = \chi_c(\{f = c\} \cap \bar{B}(0, 1)) = 2$$

as expected from Theorem 4.2.8, we observe that

$$\frac{\mathbb{L} + 5}{2} = \chi([xy = c, 1 - x^2 - y^2 \geq 0]) \neq S^1_f = -(\mathbb{L} - 1).$$

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