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To cite this version:

Nikolay Nikolov, Raymond Stora, Ivan Todorov. Configuration Space Renormalization of Massless QFT as an Extension Problem for Associate Homogeneous Distributions. 1. 2011. <hal-00703717>

HAL Id: hal-00703717

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Configuration Space Renormalization of Massless QFT as an Extension Problem for Associate Homogeneous Distributions

Nikolay M. NIKOLOV, Raymond STORA and Ivan TODOROV

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Avril 2011

IHES/P/11/07

Configuration Space Renormalization of Massless QFT as an Extension Problem for Associate Homogeneous Distributions*

Nikolay M. Nikolov
Institute for Nuclear Research and Nuclear Energy
Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria
mitov@inrne.bas.bg

Raymond Stora
Laboratoire d'Annecy-le-Vieux de Physique Théorique (LAPTH)
F-74941 Annecy-le-Vieux Cedex, France
and
Theory Division, Department of Physics, CERN, CH-1211 Geneva 23, Switzerland

Ivan Todorov
Institut des Hautes Études Scientifiques, F-91440 Bures-sur-Yvette, France
and
Theory Division, Department of Physics, CERN, CH-1211 Geneva 23, Switzerland
permanent address:
Institute for Nuclear Research and Nuclear Energy
Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria
todorov@inrne.bas.bg

April 10, 2011

Abstract

Configuration (x-)space renormalization of euclidean Green functions in a massless quantum field theory is reduced (by generalizing Hörmander's approach [H]) to the study of extensions of associate homogeneous distributions. Primitively divergent graphs are renormalized, in particular, by subtracting the residue of an analytically regularized expression. The renormalized Green functions are again associate homogeneous distributions that transform under indecomposable representations of the dilation group.

*Preliminary version.

Contents

1 Introduction.

Fourier transform is a prime example of the now fashionable notion of duality. It maps a problem of integrating large momenta into one of studying the short distance behaviour of correlation functions. Divergences were first discovered and renormalization theory was developed for momentum space integration. E.C.G. Stueckelberg and A. Petermann, followed by N.N. Bogolubov, a mathematician who set himself to master quantum field theory (QFT), realized that (perturbative) renormalization can be formulated as a problem of extending products of distributions, originally defined for non-coinciding arguments, and that such an extension is naturally restricted by locality or *micro-causality* (a concept introduced in QFT by Ernst Stueckelberg [Stu] and further developed by Bogolubov and collaborators – for a review and references see [BS]). The idea was taken up and implemented systematically by H. Epstein and V. Glaser [EG] (see also parallel work by O. Steinmann [St]). It is conceptually clear and offers a way to develop perturbative QFT and operator product expansions on a curved background [DF, BF, BFV, H07, H08, HW, HW08]. It is therefore not surprising that it attracts more attention now than half a century ago when it was originally put forward – see *e.g.* [G-B, G-BL, FG-B, EGP, FHS, K, N, K10, B10] as well as the recent survey [B] which contains over 80 references. Papers like [BBK] reflect, surely, later developments in both renormalization theory (Kreimer's Hopf algebra structure – see *e.g.* [Kr] – and Connes-Kreimer's reduction to the Riemann-Hilbert's problem [CK]) and the mathematical study of singularities in configuration space [FM, DP]. Recent work on Feynman graphs and motives [BEK, BK] also generated a configuration space development [Ni, N, CM].

A starting point in our work was the observation (cf. [BF], [HW], [G-B], [DF]) that Hörmander's treatment of the extension of homogeneous distributions (Sect. 3.2 of [H]) is tailor-made for treating the ultraviolet (UV) renormalization problem, that is particularly transparent in a massless QFT. In order to explain the main ideas stripped of technicalities, we begin with the study of *dilation invariant euclidean Green's functions* (the only case considered in [BBK]). Furthermore, we concentrate on the UV problem excluding integration in configuration space by considering all vertices as external.The results extend to Minkowski space causal Green functions as sketched in Sect. 4. It is, on the other hand, known that the leading UV singularities in a massive QFT are given by the corresponding massless limit. The full study of the renormalization problem in the massive case requires, however, additional steps and is relegated to future work.

We start with a framework that differs from standard QFT (cf. [Ni]). We separate the renormalization program from concrete (massless) QFT models and state it as a mathematical problem of extension of a class of distributions. In Sects. 2 and 4 we formulate general axiomatic conditions for our construction (corresponding to euclidean and to Minkowski space theories, respectively), such that when combined with a given Lagrangian model it reproduces the result of Epstein-Glaser for the renormalized time ordered products. To this end we introduce a universal algebra of rational translation invariant functions in $\mathbb{R}^{Dn}$, where n runs in $\mathbb{N}$ while D, the space-time dimension, is fixed ($D = 4$ being the case of chief interest). We assume that this algebra is generated by 2-point functions of the type

$$G_{ij}(x_{ij}) = \frac{P_{ij}(x_{ij})}{[x_{ij}^2]^{\mu_{ij}}}\,, \quad x_{ij} = x_i - x_j\,, \quad \mu_{ij} \in \mathbb{N}\,, \tag{1.1}$$

where P_{ij} are homogeneous polynomials in the components of the D-vector x_{ij}. We note that the renormalization of any massless QFT can be reduced to the extension of (a subspace of) rational functions $G = \prod_{i<j} G_{ij}(x_{ij})$ of this algebra to globally defined distributions. The correspondence between the rational functions and the global distributions is called a *renormalization map*. Each expression

$$G_\Gamma = \prod_{(ij)\in\Gamma} G_{ij}(x_{ij})\,, \tag{1.2}$$

can be represented by a decorated graph Γ of n vertices and of lines connecting pairs of different vertices (i,j) whenever there is a (non-constant) factor G_{ij} in the product (1.2). Each $G_{ij} = G_{ij}(x_{ij})$ appears at most once in this expression, so that there are no multiple lines in the graph Γ. The presence of different powers μ and different polynomials P indicates the fact that we give room for composite fields in our theory such as normal products of derivatives of the basic fields. (Matrix valued vertices that enter the Feynman rules can be accounted for by admitting linear combinations of expressions of type (1.2).) A disconnected graph Γ corresponds to the (tensor) product of the distributions associated to its connected components. We shall restrict our attention to connected graphs.

We remark that a quantum field theorist would replace in (1.1) the polynomial in x by a polynomial of derivatives acting on the scalar field propagator. The difference is not accidental: we shall impose the requirement, convenient for the subsequent analysis, that the renormalization map commutes with multiplication by polynomials in x_{ij}. On the other hand, derivatives typically yield anomalies independently of the above requirement (see [N], Sect. 8). Using the renormalization map we achieve the basic property of the time–ordered product: causality. Other constraints compatible with causality and power counting may be imposed - including a description of possible associated anomalies – by adjustment of additional finite renormalizations. An example of such a phenomenon, concerned with the behaviour of renormalized Feynman amplitudes under dilations, is considered in Sect. 3.

Thus, to any graph Γ in a given massless QFT there corresponds a bare Feynman amplitude G_Γ. It is a homogeneous rational function of degree $-d_\Gamma$ which depends on n-1 D-vector differences. We shall denote the arguments of G_Γ by $\vec{x}$, for short, and will introduce a uniform ordering $x^1, ..., x^N$ of their components, where $N = D(n-1)$ (for a connected graph). Then, the homogeneity of G_Γ is expressed as

$$G_\Gamma(\lambda\,\vec{x}) = \lambda^{-d_\Gamma} G_\Gamma(\vec{x})\,. \tag{1.3}$$

We shall call the difference $\kappa := d_\Gamma - N$ the *index of divergence*. It coincides with (minus) the degree of homogeneity of the density form

$$G_\Gamma(\vec{x})\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^N \equiv G_\Gamma(\vec{x})\, \text{Vol.}\,, \tag{1.4}$$

(Whenever the orientation is not relevant we shall skip the wedge sign. The use of densities rather than functions streamlines changes of variables and partial integration.) We say that G_Γ is *superficially divergent* if $\kappa \geq 0; G_\Gamma$ is called *divergent* if it is not locally integrable. The following easy to prove statement justifies the above terminology.

Proposition 1.1. *If the indices of divergence of a connected graph* Γ *and of all its connected subgraphs are negative then* G_Γ *is locally integrable and admits, as a consequence, a unique continuation as a distribution on* $\mathbb{R}^{D(n-1)}$.

The power counting index of divergence of standard renormalization theory is thus replaced by the degree of homogeneity for a (classically dilation invariant) massless QFT.

Abusing the terminology we shall also speak of *(superficially) divergent graphs.* Each function G_Γ defines a tempered distribution (in the sense of Schwartz [Sc]) on test functions f with support

$$\text{supp}\, f \subset \mathbb{R}^{D(n-1)} \backslash \Delta_2\,, \quad \Delta_2 = \{\vec{x}\,;\, \exists\,(i,j)\, i<j,\ s.t.\ x_{ij} = 0\}\,. \tag{1.5}$$

One can, similarly, introduce the partial diagonals Δ_k involving k-tuples of coinciding points; we have $\Delta_n := \{\vec{x}; x_1 = \ldots = x_n\} \subset \Delta_{n-1} \subset \ldots \subset \Delta_2$. We shall be mostly using the *small* or *full* diagonal Δ_n in what follows. The problem of renormalization consists in extending all distributions G_Γ to $\mathcal{S}(\mathbb{R}^{D(n-1)})$ in such a way that a certain recursion relation, which reflects the causality condition, is satisfied. This condition is known as *causal factorization.* We give the precise formulation of its euclidean version in Sect. 2 while the more involved but physically motivated Minkowski space requirement is relegated to Sect. 4. We use an x-space counterpart of Speer's *analytic renormalization* in [Sp] to define the notion of *residue*[1] of G_Γ adapted, in particular, to primitively divergent graphs. It is based on the observation that if $r = r(x_{ij})$ is a norm in the (euclidean) space of coordinate differences and $G(\vec{x})$ is primitively divergent of index κ then the *analytically regularized Feynman amplitude*

$$r^{\kappa+\epsilon} G(\vec{x})\ (\epsilon > 0) \tag{1.6}$$

is locally integrable. It will be proven in Sect. 2 that Eq. (1.6) defines a distribution valued meromorphic function in ϵ which only has simple poles for non-positive integer values of ϵ. This will allow us to define the *renormalized Feynman distribution* G^R of a primitively divergent graph by just subtracting

[1] A notion of residue of a Feynman graph has been introduced in the momentum space approach in terms of the *graph polynomial* [BEK, BK]. More recently, a notion of *Poincaré residue* was considered in the motivic approach to Feynman integrals in configuration space [CM]. It would be interesting to establish the precise relationship between these notions.

the pole term for $\epsilon = 0$. The result will be enforced by one of our main requirements (see (MC1) of Sect. 2, below), namely that G^R is associate homogeneous of the same degree as G (its behaviour for small r only differing from G by log terms). More precisely, we say that G is an *associate homogeneous distribution of degree d and order k* if it obeys the (infinitesimal) indecomposable dilation law

$$(E+d)^{k+1} G(\vec{x}) = 0 \quad \text{where} \quad E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \; (x \frac{\partial}{\partial x} = \sum_{\alpha=1}^{D} x^{\alpha} \frac{\partial}{\partial x^{\alpha}}). \tag{1.7}$$

(For a motivation and for a global characterization of associate homogeneous distributions - see Sect. 3.)

The study of divergent graphs with subdivergences is outlined in Sect. 3 (with examples worked out in Appendix B). It is remarkable that in all cases renormalization is essentially reduced to a 1-dimensional extension problem for associate homogeneous distributions. A construction that provides the solution to this problem is outlined in Appendix A.

One objective of our work is to demonstrate in a systematic fashion that x-space calculations are not only more transparent conceptually but also practical (especially in the euclidean massless case – something noticed long ago by Chetyrkin *et al.* [CKT] (see also [KTV]) but only rarely appreciated afterwards – *cf.* [G-B]). To this end we consider (in Sect. 3 and Appendix B) a number of examples (of 1-, 2- and 3-loop graphs) displaying the basic simplicity of the argument. A primitively divergent n-loop graph whose residue involves $\zeta(2n-3)$ is displayed as Example 2.2.

2 The extension problem for primitively divergent graphs. Analytic regularization and residues.

We shall define ultraviolet (*i.e.* short distance) renormalization by induction with respect to the number of vertices. Assume that all contributions of diagrams with less than n points are renormalized. If then Γ is an arbitrary connected n-point graph its renormalized contribution should satisfy the following inductive *factorization requirement*.

Let the index set $I(n) = \{1, \ldots, n\}$ of Γ be split into any two non-empty non-intersecting subsets

$$I(n) = I_1 \cup I_2 \;(I_1 \neq \emptyset\,,\; I_2 \neq \emptyset)\,,\; I_1 \cap I_2 = \emptyset\,.$$

Let $\mathcal{U}_{I_1,I_2}$ be the open subset of $\mathbb{R}^{Dn} \equiv (\mathbb{R}^D)^{\times n}$ such that $(x_1, \ldots, x_n) \notin \mathcal{U}_{I_1,I_2}$ whenever there is a pair (i,j) such that $i \in I_1$, $j \in I_2$. Let further G_1^R and G_2^R be the renormalized distributions associated with the subgraphs whose vertices belong to the subsets I_1 and I_2, respectively. We demand that for each such splitting our distribution G_Γ^R, defined on all partial diagonals, exhibits the *factorization property* (– see [Ni]):

$$G_\Gamma^R = G_1^R \left(\prod_{\substack{i \in I_1 \\ j \in I_2}} G_{ij} \right) G_2^R \quad \text{on} \quad \mathcal{U}_{I_1,I_2}\,, \tag{2.1}$$

where G_{ij} are factors (of type (1.1)) in the rational function G_Γ and are understood as *multipliers* on $\mathcal{U}_{I_1,I_2}$.

We shall add to this basic physical requirement two more *mathematical conventions* (MC) which will substantially restrict the notion of renormalization used in this paper.

(MC1) *Renormalization maps rational homogeneous functions onto associate homogeneous distributions of the same degree of homogeneity; it extends associate homogeneous distributions defined off the small diagonal to associate homogeneous distributions of the same degree (but possibly of higher order) defined everywhere on* $\mathbb{R}^N$.

(MC2) *The renormalization map commutes with multiplication by polynomials.* If we extend the class of our distributions by allowing multiplication with smooth functions of no more than polynomial growth (in the domain of definition of the corresponding functionals), then this requirement will imply commutativity of the renormalization map with such multipliers.

The induction is based on the following

Proposition 2.1. *The complement* $C(\Delta_n)$ *of the small diagonal is the union of all* $\mathcal{U}_{I_1,I_2}$ *for all pairs of disjoint* I_1, I_2 *with* $I_1 \cup I_2 = \{1, \ldots, n\}$*, i.e.,*

$$C(\Delta_n) \;=\; \bigcup_{I_1 \dot{\cup} I_2 \,=\, \{1,\ldots,n\}} \mathcal{U}_{I_1,I_2}\,.$$

Proof. Let $(x_1, \ldots, x_n) \in C(\Delta_n)$. Then there are at least two different points $x_{i_1} \neq x_{j_1}$. We define I_1 as the set of all indices i of $I = I(n)$ for which $x_i \neq x_{j_1}$ and $I_2 := I \backslash I_1$. Hence, $C(\Delta_n)$ is included in the union of all such pairs. Each $\mathcal{U}_{I_1,I_2}$, on the other hand, is defined to belong to $C(\Delta_n)$. This completes the proof of our statement.

In order to apply and implement the inductive factorization property (2.1) one needs two steps:

(i) to renormalize all *primitively divergent* graphs, *i.e.* all divergent diagrams with no proper subdivergences, in particular, to extend all (superficially) divergent 2-point functions G_{ij} to distributions on $\mathcal{S}(\mathbb{R}^D)$;

(ii) to extend the resulting associate homogeneous distributions defined on the complement of the *full diagonal* $x_1 = x_2 = \ldots = x_n$ to distributions on $\mathcal{S}(\mathbb{R}^{D(n-1)})$.

We shall work out the first step in this section leaving (ii) to Sect. 3.

We begin with the renormalization of the general 2-point function (*cf.* (1.1)) or the corresponding *density* GVol:

$$G_{md}(x) = \frac{P_m(x)}{(x^2)^d}\,,\ G_{md}\,\mathrm{Vol} = G_{md}(x)\,d^Dx\,,\ x \in \mathbb{R}^D\,,\ d, m \in \mathbb{N} \tag{2.2}$$

where $P_m(x)$ is a homogeneous polynomial of degree m. The expression (2.2) is superficially divergent if its index κ is non-negative:

$$\kappa := 2d - m - D \geq 0\,. \tag{2.3}$$

It follows from Hörmander's analysis that the extension problem can be solved by introducing the *analytically regularized* expression[2]

$$G^{\varepsilon}_{md}(x,\ell) := \left(\frac{x^2}{\ell^2}\right)^{\varepsilon} G_{md}(x) \tag{2.4}$$

(the scale parameter ℓ ensuring that G^{ε} has the same (naive) dimension as G). According to Theorem 3.2.3 of Hörmander [H] the homogeneous distribution G^{ε}_{kd} on $\mathcal{S}(\mathbb{R}^D\backslash\{0\})$ has a unique homogeneous extension to $\mathcal{S}(\mathbb{R}^D)$ for non-integer ε. Furthermore, it follows from the analysis in [H] that the resulting distribution is meromorphic in ε and may only have a simple pole at $\varepsilon = 0$ (or, more generally, at integer values of ε).

This can be derived directly from (2.2) by observing that G_{kd} can be restricted to the smooth function $P_k(\omega)$ on the unit sphere

$$\mathbb{S}^{D-1} = \left\{\omega \in \mathbb{R}^D\,;\ \omega^2 := \sum_{\alpha=1}^{D}(\omega^{\alpha})^2 = 1\right\} \tag{2.5}$$

and thus reducing the problem to one (radial) dimension. In particular, the density (1.3) becomes

$$G^{\varepsilon}_{kd}(x,\ell)\,\mathrm{Vol} = G^{\varepsilon}_{kd}(x,\ell)\,d^Dx = \ell^{-2\varepsilon}P_k(\omega)\,d^{D-1}\omega\,r^{2\varepsilon-\kappa-1}\,dr$$
$$\text{for}\quad x = r\omega\,,\ r \geq 0\,. \tag{2.6}$$

Following Hörmander (see [H], Eq. (3.2.17)) we observe that

$$\mathcal{X}^a(r) = \frac{r^a_+}{\Gamma(a+1)}\,,\quad r^a_+ = \begin{cases} r^a & \text{for} \quad r > 0 \\ 0 & \text{for} \quad r < 0 \end{cases}\,, \tag{2.7}$$

defines a distribution valued entire analytic function in a, such that

$$\frac{d^n}{dr^n}\,\mathcal{X}^a(r) = \mathcal{X}^{a-n}(r)\,,\quad r\,\mathcal{X}^a(r) = (a+1)\,\mathcal{X}^{a+1}(r)\,. \tag{2.8}$$

[2]Similar formulas are used by Keller [K], [K10] under the name of "dimensional regularization in position space".

As $\mathcal{X}^0(r)$ is the step function,

$$\mathcal{X}^0(r) \equiv \theta(r) := \begin{cases} 1 & \text{for} \quad r > 0 \\ 0 & \text{for} \quad r < 0 \end{cases},$$

it follows from the first relation (2.8) that

$$\mathcal{X}^{-\nu-1}(r) = \delta^{(\nu)}(r)\,,\ \nu = 0, 1, \ldots \left(\int \delta^{(\nu)}(r)\, f(r)\, dr = (-1)^\nu\, f^{(\nu)}\,(0) \right). \quad (2.9)$$

We then deduce (from the known properties of $\Gamma(a)$) that $r^{2\varepsilon-\kappa-1}$ has indeed a simple pole for $\varepsilon \to 0$, and moreover,

$$r^{2\varepsilon-\kappa-1} - \frac{(-1)^\kappa}{\kappa!}\,\frac{\delta^{(\kappa)}(r)}{2\varepsilon} = \frac{(-1)^\kappa}{\kappa!} \left[\frac{d^{\kappa+1}}{dr^{\kappa+1}} \left(\ell n\, \frac{r}{\ell} \right)_+ + \sum_{j=1}^{\kappa} \frac{1}{j}\, \delta^{(\kappa)}(r) \right]$$

$$\left((f(r))_+ = \begin{cases} f(r) & \text{for} \quad r > 0 \\ 0 & \text{for} \quad r < 0 \end{cases} \right). \quad (2.10)$$

The resulting expression is defined as a distribution by what is sometimes called *differential renormalization* [FJL], [HL]:

$$\begin{aligned} &\lim_{\varepsilon \to 0} \left(r^{2\varepsilon-\kappa-1} - \frac{(-1)^\kappa}{2\varepsilon\kappa!}\, \delta^{(\kappa)}(r)\,,\ f(r) \right) \\ = \ &\frac{1}{\kappa!} \left[\int_0^\infty \ell n\, \frac{\ell}{r}\, f^{(\kappa+1)}(r)\, dr + f^{(\kappa)}(0) \sum_{j=1}^{\kappa} \frac{1}{j} \right] \end{aligned} \quad (2.11)$$

(the second term in the right hand side vanishing for $\kappa = 0$). The parameter ℓ (the length scale of the regularized expression (2.4)) labels the ambiguity in the renormalization. It is the only ambiguity in the 1-dimensional case if we require, following Hörmander, that the renormalized expression is "as close to homogeneous as possible". It is, in fact, an example of what Gelfand and Shilov [GS] call an *associate homogeneous distribution* (a terminology also adopted in [G-B] and recalled in Sect. 3). The parameter ℓ remains the only ambiguity in the D-dimensional 2-point function (the renormalization of (2.2)) provided that we assume – as we shall – full Euclidean covariance.

We define the (distribution valued) *residue* as the coefficient to $\frac{1}{2\varepsilon}$ of the analytically regularized amplitude G^ε. For the density (2.6) we find

$$\operatorname{Res} G_{kd}\, \mathrm{Vol} = P_k(\omega)\, d^{D-1}\, \omega\, \delta^{(\kappa)}(r)\, dr\,. \quad (2.12)$$

This is a rather unconventional way to write a distribution with support at the origin. In particular, for a harmonic $P_k(x)$ and $k > 0$,

$$\int_{\mathbb{S}^{D-1}} P_k(\omega)\, d^{D-1}\, \omega = 0, \quad (2.13)$$

so that, for $\kappa = 0$, the residue of G_{kD+k} actually vanishes (upon integrating with a test function). For $k = 0$ ($P_0 = 1$), $D = 2m$, $d = m$ (a logarithmically divergent graph in a scalar field theory in an even dimensional space time). Eq. (2.12) is equivalent to

$$\mathrm{Res}\, G_{0m}(x)\mathrm{Vol} = |\mathbb{S}^{2m-1}|\,\delta(x)\,d^{2m}\,x \tag{2.14}$$

where $|\mathbb{S}^{2m-1}| = \frac{2\,\pi^m}{(m-1)!}$ is the volume of the $(2m-1)$-dimensional sphere.

Knowing the residue we can define a renormalized (as we shall see - associate homogeneous) density

$$G_{kd}(r\omega,\ell)\,\mathrm{Vol} := \lim_{\varepsilon\to 0}\left\{\left(\frac{x^2}{\ell^2}\right)^{\varepsilon} G_{kd}(x)\,d^D x - \frac{\mathrm{Res}\,G_{kd}}{2\varepsilon}\,\mathrm{Vol}\right\}\ . \tag{2.15}$$

The limit in the right hand can be computed explicitely in terms of the radial coordinate r of Eq. (2.6) (see Eq. (3.2.5) of [H]; it will appear as a special case of a more general limit involving associate homogeneous functions displayed in Appendix A). Here we shall compute it instead in Cartesian coordinates in two examples of 4-dimensional ($4D$) scalar field theory.

Example 2.1. The logarithmically divergent 2-point graph shwon on Fig. 1a

Figure 1.

Logarithmically and quadratically divergent 2*-point graphs.*

is ubiquitous as (sub)divergence in any scalar field theory in $4D$: it appears as a self-energy graph in a φ^3 model and as a contribution to the 4-particle scattering amplitude in the φ^4 theory. The limit (2.15) of this 1-loop graph reads

$$\begin{aligned} G_1(x,\ell) &= \lim_{\varepsilon\to 0}\left[\frac{1}{(x^2)^2}\left(\frac{x^2}{\ell^2}\right)^{\varepsilon} - \frac{2\,\pi^2}{2\varepsilon}\,\delta(x)\right] \\ &= \frac{1}{2}\,\frac{\partial}{\partial x^\alpha}\left[\frac{x^\alpha}{(x^2)^2}\,\ell n\left(\frac{x^2}{\ell^2}\right)\right]\left(= \frac{1}{r^2}\,\frac{\partial}{\partial r^2}\left(\ell n\,\frac{r^2}{\ell^2}\right)_+\right. , \\ & (\ell n\,\rho)_+ = \left\{\begin{matrix} \ell n\,\rho & \text{for} & \rho > 0 \\ 0 & \text{for} & \rho < 0 \end{matrix}\right). \end{aligned} \tag{2.16}$$

The derivatives in (2.16) should be understood in the sense of distributions (after smearing they should be transferred to the test function – see Appendix B1). This is another instance of *differential renormalization* (*cf.* Eq. (2.11) and see [FJL], [Pr]). Renormalized expressions of the type $\frac{\partial}{\partial x^\alpha}\left[\frac{x^\alpha}{(x^2)^2}\,\ell n\,\frac{x^2}{\ell^2}\right]$ (sum over α) are used systematically in [G-B].

Remark 2.1. Note that the double and the triple lines in Fig. 1 should both be viewed as a single line with a different decoration (corresponding to different powers, $\mu = 2$ and $\mu = 3$, in (1.1)). Thus, the self-energy graph on Fig. 1b, which displays overlapping divergences in momentum space, is primitively divergent in x-space according to our definition. Its renormalized expression is additionally restricted by the requirement of full euclidean invariance. (In general, we require the presence of as much of the symmetry of the rational function in the renormalized expression as allowed by the existing anomalies.) Applying further requirement (MC2) which yields the identity $G_1(x,\ell) = x^2\, G_2(x,\ell)$, valid for the original rational functions away from the origin, we find

$$\begin{aligned} G_2(x,\ell) &= \lim_{\varepsilon\to 0}\left\{ \frac{1}{(x^2)^3}\left(\frac{x^2}{\ell^2}\right)^{\varepsilon} - \frac{\pi^2}{8\,\varepsilon}\,\Delta\delta(x)\right\} \\ &= \frac{3\,\pi^2}{16}\,\Delta\delta(x) + \frac{\Delta}{8}\,G_1(x,\ell)\,. \end{aligned} \qquad (2.17)$$

In deriving (2.17) we have used the identities

$$\Delta f = 4\,\frac{\partial^2}{\partial\rho^2}\,(\rho f) + \frac{1}{\rho}\,\Delta_\omega\, f \quad \text{for} \quad \rho = x^2 (= r^2)\,,\ x = r\omega\,;$$

$$\begin{aligned} \frac{1}{\rho^{n+1}}\left(\frac{\rho}{\ell^2}\right)^{\varepsilon} &= \frac{1}{(n-\varepsilon)(n-1-\varepsilon)^2\ldots(1-\varepsilon)^2(-\varepsilon)}\left(\frac{\partial^2}{\partial\rho^2}\,\rho\right)^n \frac{1}{\rho}\left(\frac{\rho}{\ell^2}\right)^{\varepsilon} \\ &= \frac{1}{n!(n-1)!}\left(\frac{\Delta}{4}\right)^{n-1}\left(\frac{\pi^2}{\varepsilon}\,\delta(x) + \pi^2\, s_n\,\delta(x) + G_2(x,\ell)\right) + O(\varepsilon)\,, \end{aligned}$$

where s_n is a sum of partial harmonic series (cf. (A.9)):

$$s_n = \sum_{j=1}^{n-1}\frac{1}{j} + \sum_{j=2}^{n}\frac{1}{j} \qquad \left(s_1 = 0\,,\ s_2 = \frac{3}{2}\,,\ s_3 = \frac{7}{3}\,,\ \ldots\right).$$

We now proceed to define the residue and the renormalized expression for an arbitrary primitively divergent graph. In this case our rational function G_Γ is locally integrable in the neighbourhood of all partial diagonals and defines a distribution on $\mathcal{S}(\mathbb{R}^{D(n-1)}\backslash\{0\})$ and we can use the $D(n-1)$-dimensional radial coordinate to reduce it to an 1-dimensional problem as in (2.6) and then apply (2.9) to obtain the residue. Once we have the residue, the counterpart of (2.15) allows us to compute the renormalized Green function fixed by our requirements (MC1) and (MC2).

One can use a more general (homogeneous, $O(D)$-invariant) norm on the distances x_{ij}^2 instead of the ($O(D(n-1))$-invariant) radial coordinate in order to compute both the residue and the renormalized expression of a primitively divergent graph as illustrated on the following n-loop example.

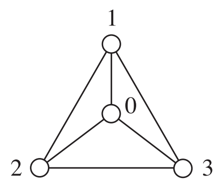
Example 2.2. We consider the $4D$ n-loop ($n+1$-point) primitively divergent Feynman amplitude

$$G_n = (\prod_{i=1}^{n} x_{0i}^2 x_{ii+1}^2)^{-1},\ x_{n+1} \equiv x_1, \qquad (2.18)$$

which we shall parametrize by the spherical coordinates of the n independent 4-vectors x_{0i}:

$$x_{0i} = r_i\, \omega_i\,, \quad r_i \geq 0\,, \quad \omega_i^2 = 1\,, \quad i = 1, 2, ..., n. \tag{2.19}$$

An important special case is given by the complete 4-point graph



$$G_3 = \frac{1}{x_{01}^2\, x_{02}^2\, x_{03}^2\, x_{12}^2\, x_{23}^2\, x_{13}^2}$$

Figure 2.
The tetrahedron graph in the $(\varphi^4)_4$*-theory.*

Setting

$$G_n^{\varepsilon} = \left(\frac{R^2}{\ell^2}\right)^{\varepsilon} G_n,\;\; R = max(r_1, ..., r_n), \tag{2.20}$$

we shall compute its residue by first integrating the corresponding analytically regularized density G_n^{ε}Vol over the angles ω_i using the identification of the propagators $\frac{1}{x_{ij}^2}$ with the generating functions for the Gegenbauer polynomials. Having in mind applications to a scalar field theory in D dimensions (see Example 3.3 below) we shall write down the corresponding more general formulas. The propagator $(x_{12}^2)^{-\lambda}$ of a free massless scalar field in $D = 2\lambda + 2$ dimensional space-time is expanded as follows in (hyperspherical) Gegenbauer polynomials:

$$(x_{ij}^2)^{-\lambda} = (r_i^2 + r_j^2 - 2 r_i r_j \omega_i \omega_j)^{-\lambda} = \frac{1}{R_{ij}^{2\lambda}} \sum_{n=0}^{\infty} \left(\frac{r_{ij}}{R_{ij}}\right)^n C_n^{\lambda}(\omega_i\, \omega_j)\,,$$

$$R_{ij} = \max(r_i, r_j)\,, \quad r_{ij} = \min(r_i, r_j)\,, \quad i \neq j\,,\; i, j = 1, 2, 3. \tag{2.21}$$

We shall also use the integral formula

$$\int_{\mathbb{S}^{2\lambda+1}} d\omega\, C_m^{\lambda}(\omega_1\, \omega)\, C_n^{\lambda}(\omega_2\, \omega) = \frac{\lambda |\mathbb{S}^{2\lambda+1}|}{n + \lambda}\, \delta_{mn}\, C_n^{\lambda}(\omega_1\, \omega_2)\,, \tag{2.22}$$

where $|\mathbb{S}^{2\lambda+1}| = \frac{2\,\pi^{\lambda+1}}{\Gamma(\lambda+1)}$ is the volume of the unit hypersphere in $D = 2\lambda + 2$ dimensions.

Clearly, the expansion (2.21) requires an ordering of the lengths r_i. In general, one should consider separately $n!$ sectors, obtained from one of them, say

$$r_1 \geq r_2 \geq ... \geq r_n \; (\geq 0) \tag{2.23}$$

by permutations of the indices. It is, in fact, sufficient to consider just the sector (2.23) (and multiply the result for the residue by $n!$). (Because of the symmetry of the tetrahedron graph (Fig. 2) this is obvious for $n = 3$ but it is actually true for any $n (\geq 3)$.) The result involves a polylogarithmic function:

$$
\begin{aligned}
\widetilde{G}_n^\varepsilon &:= \int_{\mathbb{S}^3} \dots \int_{\mathbb{S}^3} G_n^\varepsilon (r_1\, \omega_1, \dots, r_n\, \omega_n)\, \mathrm{Vol} \\
&= (2\pi^2)^n \left(\frac{r_1}{\ell}\right)^{2\varepsilon} \frac{dr_1 \wedge \dots \wedge dr_n}{r_1 \dots r_n}\, Li_{n-2}(\frac{r_n^2}{r_1^2}), \\
& \quad Li_{n-2}(\xi) = \sum_{m=1}^{\infty} \frac{1}{m^{n-2}}\, \xi^m \;\; (\xi = \frac{r_n^2}{r_1^2})
\end{aligned}
\tag{2.24}
$$

($r_n = min(r_1, \dots, r_n), r_1 = max(r_1, \dots, r_n)(= R)$). To derive the last equation we have applied once more (2.22) and used

$$\left(C_m^1 (\omega_1^2) = \right) C_m^1(1) = m + 1.$$

The residue distribution corresponding to the (integrated over the angles) density (2.24) is given by

$$\mathrm{Res}\, \widetilde{G}_n^\varepsilon = C(n)\, \delta(r_1) \dots \delta(r_n)\, dr_1 \wedge \dots \wedge dr_n \tag{2.25}$$

where

$$C(n) = n! \lim_{\varepsilon \to 0} 2\varepsilon \int_{r_1=0}^{\infty} \int_{r_2=0}^{r_1} \dots \int_{r_n=0}^{r_{n-1}} \widetilde{G}_n^\varepsilon = n!\, (2\pi^2)^n \int_{K_{n-1}} \dots \int \omega\, ,$$

$$\omega := Li_{n-2}(\xi)\, \frac{r_1\, dr_2 \wedge \dots \wedge dr_n - r_2\, dr_1 \wedge dr_3 \dots \wedge dr_n + \dots (-1)^{n-1} r_n\, dr_1 \wedge \dots \wedge dr_{n-1}}{r_1 \dots r_n}$$

$$(d\omega = 0)\, . \tag{2.26}$$

Here ω is a closed homogeneous form on the compact projective cone

$$K_{n-1} = \left\{ (r_1, \dots, r_n) \in \mathbb{P}_{n-1}\, ;\; r_i \geq 0 \left(\sum_{i=1}^{n} r_i > 0 \right) \right\}\, . \tag{2.27}$$

The integration in (2.26) may be performed over any transverse surface. Choosing $R (= r_1) = 1$ we find

$$
\begin{aligned}
C(n) &= n!\, (2\pi^2)^n \int_0^1 \frac{dr_2}{r_2} \dots \int_0^{r_{n-1}} \frac{dr_n}{r_n} Li_{n-2}(r_n^2) \\
&= n!\, 2\pi^{2n}\, \zeta(2n-3)\, .
\end{aligned}
\tag{2.28}
$$

In particular, for the tetrahedron graph, $n = 3$, we reproduce the known result, $C(3) = 12\pi^6 \zeta(3)$ - see, for instance, [G-B].

Remark 2.2. The closed homogeneous form ω (2.26) of maximal degree has a cohomological interpretation. A prototype of this form appears if we contract the density G_ΓVol (1.3) by the Euler vector field E:

$$E = \vec{x}\,\frac{\partial}{\partial \vec{x}} \left(\equiv \sum_{a=1}^{N} x^a\,\frac{\partial}{\partial x^a} \right),\ i_E\, G_\Gamma\, \mathrm{Vol} = G_\Gamma(\vec{x}) \sum_{a=1}^{N} (-1)^{a-1}\, x^a\, dx^1 \wedge \ldots \widehat{dx^a} \ldots \wedge dx^N\,. \tag{2.29}$$

For $\kappa = 0$, – *i.e.*, for a homogeneous (logarithmically divergent) G_ΓVol the Lie derivative along E of G_ΓVol vanishes,

$$0 = \mathcal{L}_E\, G_\Gamma\, \mathrm{Vol} = d\, i_E\, G_\Gamma\, \mathrm{Vol}\,, \qquad \text{for} \quad \mathcal{L}_E = i_E\, d + d\, i_E\,; \tag{2.30}$$

hence, $i_E\, G_\Gamma$Vol is a closed homogeneous $(N-1)$-form. For $\kappa > 0$ G_ΓVol is expressed in terms of derivatives of a homogeneous form; using $(\vec{\xi}\,\vec{\partial} + N + \kappa)\, G_\Gamma = 0$ we find

$$G_\Gamma = \frac{(-1)^\kappa}{\kappa!}\, \partial_{a_1} \ldots \partial_{a_\kappa} (x^{a_1} \ldots x^{a_\kappa}\, G_\Gamma)\,.$$

The residue (2.26) is a special case of the so called *Wodzicki residue* (see [G-B], [G-BVF] and references therein). It is rewarding to realize that it coincides with the residue in ε under the analytic regularization. The integration technique based on the properties of Gegenbauer polynomials has been introduced in the study of x-space Feynman integrals in [CKT]. The appearance of ζ*-values* in the computation of Feynman integrals has been detected in early work of Rosner and Usyukina [R], [U]. It was related to the non-trivial topology of graphs by Broadhurst and Kreimer (see [BrK], [Kr]).

3 Dilation anomaly. Extension of associate homogeneous distributions.

We now ask what is the behaviour under dilations of a renormalized primitively divergent density $G(\vec{x})$Vol of index κ (≥ 0). By the definition of GVol the *dilation anomaly*

$$A(\vec{x}, \lambda) := \lambda^\kappa\, G(\lambda\, \vec{x})\, \mathrm{Vol} - G(\vec{x})\, \mathrm{Vol} \tag{3.1}$$

is a distribution valued density with support at the small diagonal, $x_1 = x_2 = \ldots = x_n$. Without loss of generality, we can restrict it, following [H], by demanding that it is again homogeneous in $\vec{x}$ of degree $-\kappa$:

$$A(\vec{x}, \lambda) = \sum_{\boldsymbol{\alpha}, |\boldsymbol{\alpha}| = \kappa} a_{\boldsymbol{\alpha}}(\lambda)\, D_{\boldsymbol{\alpha}}\, \delta(\vec{x}) \prod_{i=1}^{n-1} d^D\, x_{in} \tag{3.2}$$

where $\delta(\vec{x})$ is the $D(n-1)$-dimensional δ-function,

$$D_{\boldsymbol{\alpha}} = \prod_{i=1}^{n-1} \prod_{\nu=1}^{D} (\partial_i^\nu)^{\alpha_{i\nu}}\,, \quad |\boldsymbol{\alpha}| = \sum_{i,\nu} \alpha_{i\nu}\,.$$

Repeated application of the dilation law (3.1) yields the cocycle condition

$$a_{\boldsymbol{\alpha}}(\lambda\mu) = a_{\boldsymbol{\alpha}}(\lambda) + a_{\boldsymbol{\alpha}}(\mu)\,. \tag{3.3}$$

The general form of $a_{\boldsymbol{\alpha}}$ satisfying (3.3) is

$$a_{\boldsymbol{\alpha}}(\lambda) = a_{\boldsymbol{\alpha}}(G)\,\ell n\,\lambda \tag{3.4}$$

where $a_{\boldsymbol{\alpha}}(G)$ is a linear functional of the Green function G (or the corresponding density GVol). It is important to note that the coefficient $a_{\boldsymbol{\alpha}}(G)$ in (3.4) is independent of the ambiguity in the definition of the renormalized Green function. Once the problem of renormalizing a primitively divergent graph is reduced to a 1-dimensional one (as in Sect. 2) this follows from the simple observation that the coefficient of $\ell n\, r$ in (2.11) is independent of the ambiguity reflected in the scale parameter ℓ.

In fact, each renormalization of a subdivergence in a given graph increases by one the maximal power of $\ell n\,\lambda$ in the associate homogeneity law. Since $r\,\frac{\partial}{\partial r}\,(\ell n\, r)^j = j(\ell n\, r)^{j-1}$, a general associate homogeneous renormalized Feynman amplitude G will satisfy Eq. (1.7), $(E+d)^{k+1}\,G(\vec{x}) = 0$. We can then characterize G by a (column) vector $\boldsymbol{G} = (G_0 = G, G_1 = (E+d)\,G_0, \ldots, G_k = (E+d)\,G_{k-1})$ of distributions. It carries an *indecomposable representation of the dilation group*[3] *of degree* $-d$ *and order* k such that

$$\boldsymbol{G}(\vec{x}) \to \lambda^d\,\boldsymbol{G}(\lambda\,\vec{x}) = e^{\Delta \ell n \lambda}\,\boldsymbol{G}(\vec{x}) = \sum_{j=0}^{k} \frac{(\ell n\,\lambda)^j}{j!}\,G_j(\vec{x}) \tag{3.5}$$

where Δ is a nilpotent Jordan cell with k units above the diagonal:

$$\Delta = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \qquad \Delta^{k+1} = 0\,. \tag{3.6}$$

The nilpotency condition $\Delta^{k+1} = 0$ remains invariant under an arbitrary non-singular transformation $\boldsymbol{G} \to S\boldsymbol{G}, \Delta \to S^{-1}\Delta S$. We shall encounter in the examples below a particular case of such a change of basis in which just the normalization of G_j is altered.

In the case of causal renormalization it follows from the factorization assumption that the dimension of the support of G_j is decreasing with j and

$$G_k(\vec{x}) = (\vec{x}\,\vec{\partial} + d)^k\,G_0(\vec{x}) = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}\,(G)\,D_{\boldsymbol{\alpha}}\,\delta(\vec{x})\,. \tag{3.7}$$

---

[3] Representations of this type have been considered back in the 1970's [FGG] within a study of a spontaneous breaking of dilation symmetry.

Following the terminology of Gelfand-Shilov [GS] (see the comment after Eq. (2.11)) we shall call $\boldsymbol{G}$ (and its components) *associate homogeneous distributions.* Our task now is to study the extension – *i.e.* the renormalization – of such associate homogeneous distributions. If we assume (as it will be proven in Theorem 3.1 below) that the inductive procedure allows at each step a reduction to a 1-dimensional (radial variable) problem, then we can use the result of Appendix A, that extends Hörmander's treatment of 1-dimensional homogeneous distributions to associate homogeneous ones:

Proposition 3.1. *For each value $\ell > 0$ of the parameter ℓ the family of elementary functions $r^{-a} \frac{\left(\ell n \frac{r}{\ell}\right)_+^n}{n!}$ (non-vanishing for $r > 0$) that are locally integrable for $\operatorname{Re} a < 1$ admit a continuation to tempered distributions*

$$L_n(r, a; \ell) \in \mathcal{S}'(\mathbb{R}) \qquad (\operatorname{supp} L_n \subset \mathbb{R}_{\geq 0})$$

for all complex values of a that are uniquely determined by the relations

$$r\, L_n(r, a+1; \ell) = L_n(r, a; \ell)\,, \tag{3.8}$$

$$(D + a)\, L_n(r, a; \ell) = L_{n-1}(r, a; \ell)\,, \quad D = r\,\frac{d}{dr} \tag{3.9}$$

and the "boundary condition"

$$L_n(r, 0; \ell) = \frac{1}{n!} \left(\ell n\, \frac{r}{\ell}\right)_+^n\,, \tag{3.10}$$

where $(f(r))_+ = \theta(r)\, f(r)$ (see (2.10)).

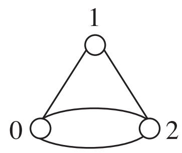
(We note that the right hand side of (3.10) is locally integrable (and bounded by $\frac{r}{\ell}$), so it defines a tempered distribution.)

The requirement (3.8) ensures ultimately that we can multiply our renormalized distributions by regular functions – a property needed in the formulation of the causal factorization requirement (2.1). Proposition 3.1 accounts for a single step in the renormalization procedure and thus involves a single parameter ℓ describing the ambiguity. (This is a property of massless theories with the additional requirement of associate homogeneity. In a massive theory even the scalar field 2-point function (our Example 2.1) involves two parameters – the *mass* and the *wave function renormalization.*)

In fact, the reduction to a 1-dimensional (radial) problem requires an argument. We shall prove at the end of this section a statement (Theorem 3.1) that extends Hörmander's treatment of the n-dimensional homogeneous distributions to associate homogeneous ones. Before that we shall illustrate the applications of the explicit construction displayed in Proposition 3.1 on some low order examples.

To see in more detail what is involved in a graph with subdivergences we consider a simple 3-point graph of this type which displays the role of the factorization condition.

Example 3.1. Renormalization the 3-point two loop diagram displayed on Fig. 3.



$$G_\Delta = \frac{1}{x_{01}^2\, x_{12}^2\, [x_{02}^2]^2} \, .$$

Figure 3.

Logarithmically divergent 3*-point graph with a* 2*-point subdivergence.*

We introduce as independent variables the spherical coordinates of the vectors x_{0i}, $i = 1, 2$

$$x_{01} = r\omega_1 \, , \quad x_{02} = \rho\, \omega_2 \, , \quad r, \rho \geq 0 \, , \quad \omega_i^2 = 1 \ (i.e.\ \omega_i \in \mathbb{S}^3) \ i = 1, 2 \tag{3.11}$$

and set

$$\omega_1 \cdot \omega_2 = \cos\vartheta \, , \quad x_{12}^2 = r^2 + \rho^2 - 2r\,\rho \cos\vartheta \, . \tag{3.12}$$

The renormalized 2-point Green function (2.16), corresponding to the subgraph of vertices $(0, 2)$ is

$$G_2(x_{02}, \ell) = \frac{1}{2} \frac{\partial}{\partial\, x_{02}^\alpha} \left[\frac{x_{02}^\alpha}{(x_{02}^2)^2}\, \ell n \frac{x_{02}^2}{\ell^2} \right]_+ = \frac{1}{\rho^3} \frac{\partial}{\partial\rho} \left(\ell n \, \frac{\rho}{\ell} \right)_+ \, . \tag{3.13}$$

(The last expression only makes sense as a density after multiplying with the volume element $d^4x = \rho^3\, d\rho\, d^3\omega$ that cancels the $\frac{1}{\rho^3}$ factor and permits to transfer the derivative to the test function.)

Next we shall write down the density G_ΔVol with renormalized subdivergence integrated over the six angular variables ω_1 and ω_2 (see Appendix B2):

$$\begin{aligned} G_\Delta \,\mathrm{Vol} \ &:= \ \left[\int d^3\omega_1 \int d^3\omega_2\, G_\Delta(r\omega_1, \rho\,\omega_2; \ell) \right] r^3\, dr\, \rho^3 d\rho \\ &= \ 8\,\pi^3 \int_0^\pi \frac{\sin^2\vartheta\, d\vartheta}{r^2 + \rho^2 - 2r\rho\cos\vartheta} \frac{\partial}{\partial\rho} \left(\ell n \, \frac{\rho}{\ell} \right)_+ r\, dr\, d\rho \\ &= \ 4\,\pi^4\, \frac{r\, dr\, d\rho}{r_\vee^2} \frac{\partial}{\partial\rho} \left(\ell n \, \frac{\rho}{\ell} \right)_+ , \ r_\vee = \max(r, \rho) = \frac{r + \rho + |r - \rho|}{2} \, . \end{aligned} \tag{3.14}$$

Smearing G_ΔVol with a test function $f(r, \rho)$ we find that the *leading term*, $LT\, G_\Delta$Vol, for $r_\vee \to 0$ (the only one that requires overall renormalization) corresponds to $r = \rho$ (Appendix B2):

$$(LT\, G_\Delta^R\, \mathrm{Vol}, f) = -4\,\pi^4 \int_0^\infty dr\, \frac{\ell n^2 \left(\frac{r}{\ell}\right)}{2} \frac{d}{dr} f(r, r) \, . \tag{3.15}$$

Here we have made use of the renormalized associate homogeneous distribution $L_1(r, 1; \ell)$ thus illustrating Proposition 3.1.

Somewhat symbolically we can write

$$G^R_\Delta(r,\rho;\ell)\,\mathrm{Vol} = 4\,\pi^4\,L_1(r,1;\ell)\,\delta(\rho-r)\,dr\,d\rho + G_0(r,\rho)\ \mathrm{Vol} L_0(\rho,1;\ell)\,d\rho \quad (3.16)$$

where $G_0\mathrm{Vol}$ is the regular part of the homogeneous 1-form $4\,\pi^4\,\frac{r\,dr}{r^2_\vee}$ (for $\rho \neq r$). Its associate homogeneity law is determined by the corresponding single variable relations:

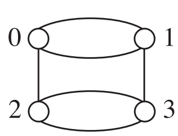
$$\begin{aligned} L_0(\lambda\,\rho,1;\ell)\,d\,\lambda\,\rho &= (L_0(\rho,1;\ell) + \ell n\,\lambda\,\delta(\rho))\,d\rho\,, \\ L_1(\lambda\,r,1;\ell)\,d\,\lambda\,r &= (L_1(r,1;\ell) + \ell n\,\lambda\,L_0(\rho,1;\ell) \\ &+ \frac{1}{2}\,(\ell n\,\lambda)^2\,\delta(r))\,dr\,. \end{aligned} \quad (3.17)$$

We have a manifestation of the general rule: only the coefficient of the highest log term ($\ell n\,\lambda$ for $L_0\,d\rho$ and $(\ell n\,\lambda)^2$ for $L_1\,dr$) is independent of the ambiguity parametrized here by the scale ℓ in the renormalized subdivergence.

Remark 3.1. One could be tempted to replace the renormalization parameter ℓ in the expression (3.14) by the (external to the divergent 2-point subgraph) variable r for $r > \rho$. This would amount to subtracting a local in ρ term, $4\,\pi^4\,\frac{dr}{r}\,\ell n\,\frac{r}{\ell}\,\delta(\rho)\,d\rho$. It is straightforward to observe, however, that neglecting such a term in (3.14) would violate the causal factorization requirement (2.1).

The techniques developed in Example 3.1 also apply to more complicated graphs as illustrated in the following example of a 4-point diagram with two subdivergences.

Example 3.2. Renormalization of subdivergences in the 4-point graph on Fig. 4.



$$x_{0j} = r_j\,\omega_j\,,\ r_j \geq 0\,,\ \omega_j^2 = 1\,,\ j = 1,2,3$$
$$\omega_i\,\omega_3 = \cos\vartheta_i\,,\ i = 1,2$$
$$G_4 = (r_1^4\,r_2^2\,x_{13}^2(x_{23}^2)^2)^{-1}$$

Figure 4.

Logarithmically divergent 4-point graph with two disjoint 2-point subdivergences.

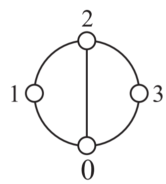
The 4-point density $G_4\mathrm{Vol}$, integrated over seven of the nine angular variables with renormalized 2-point subdivergences is given by (see Appendix B3)

$$\begin{aligned} G_4\,\mathrm{Vol} &= (2\,\pi)^4\,\frac{r_3^2\,dr_3\,\sin^2\vartheta_1\,d\vartheta_1\,d\ell n\,\frac{r_1}{\ell_1}}{r_1^2+r_3^2-2\,r_1\,r_3\cos\vartheta_1}\,\frac{\sin\vartheta_2\,dr_2\,d\vartheta_2}{r_2^2+r_3^2-2\,r_2\,r_3\cos\vartheta_2} \\ &\times\ \frac{\partial}{\partial\,\vartheta_2}\left(\ell n\,\frac{r_2^2+r_3^2-2\,r_2\,r_3\cos\vartheta_2}{\ell^2}\right)_+\,. \end{aligned} \quad (3.18)$$

For $r = \frac{r_2+r_3}{2} > r_1$ we derive (see Appendix B3) the following (analogous to the first term in (3.16)) expression for the leading short distance behaviour of the integrated in the angles renormalized density

$$LT\, G_4^R\, \text{Vol} = 16\, \pi^6\, \delta(r_2 - r_3)\, L_1(r_2, 1; \ell_2)\, L_0(r_1, 1; \ell_1)\, dr_1\, dr_2\, dr_3$$
$$\text{for} \quad r_2\, (= r_3) > r_1\, . \tag{3.19}$$

Example 3.3. As a last example we consider the graph displayed on Fig. 5



$$x_{0j} = r_j\, \omega_j\,,\ \ r_j \geq 0\,,\ \ \omega_j \in \mathbb{S}^5$$
$$G_{\mathbb{O}} = (r_1^2\, r_2^2\, r_3^2\, x_{12}^2\, x_{23}^2)^{-2}$$

Figure 5.

Quadratically divergent diagram in 6*-dimensions.*

which exhibits overlapping divergences in 6-dimensional space-time.

Applying the relations (2.21) (2.22) for $\lambda = 2$, we find the following expression for the analytically regularized integrated with respect to the angles Green function density

$$\widetilde{G}_{\mathbb{O}}^{\varepsilon_1\, \varepsilon_2} = \pi^9\, \frac{r_1\, r_2\, r_3}{(R_{12}\, R_{23})^4} \left(\frac{R_{12}^2}{\ell_1^2}\right)^{\varepsilon_1} \left(\frac{R_{23}^2}{\ell_2^2}\right)^{\varepsilon_2} dr_1\, dr_2\, dr_3\,, \tag{3.20}$$

where $R_{ij} = \max(r_i, r_j)$ (*cf.* (2.21)). The renormalized expression for $G_{\mathbb{O}}$ again depends, as in the preceding examples (see, in particular, Example 2.2) on the inequalities satisfied by the radial variables. For

$$r_1 < r_2 < r_3 \tag{3.21}$$

(and, similarly, for $r_3 < r_2 < r_1$) we have a case of nested singularities. One first renormalizes the logarithmicly divergent triangular subgraph with vertices $(0, 1, 2)$. Integrating first with respect to r_1 in the domain (3.21) we find

$$\begin{aligned} &\lim_{\varepsilon_1 \to 0} \left(\int_0^{r_2} \widetilde{G}_{\mathbb{O}}^{\varepsilon_1\, \varepsilon_2} - \frac{\pi^9}{4\, \varepsilon_1}\, \delta(r_2) \left(\frac{r_3}{\ell_2}\right)^{2\varepsilon_2} \frac{dr_2\, dr_3}{r_3^3} \right) \\ = \;& \frac{\pi^9}{2}\, d\left(\ell n\, \frac{r_2}{\ell_1}\right) \left(\frac{r_3}{\ell_2}\right)^{2\varepsilon_2} \frac{dr_3}{r_3^3}\, . \end{aligned} \tag{3.22}$$

The renormalization of the resulting quadratically divergent in r_3 associate homogeneous distribution follows the lines of Example 3.1. The case $r_1 < r_2 > r_3$, in which $R_{12} = R_{23} = r_2$ and "the divergences overlap", is actually simpler; it is reduced to a single radial renormalization. Setting $\varepsilon_1+\varepsilon_2=\frac{\varepsilon}{2}$ and $\ell_1\,\ell_2=\ell^2$ and integrating in r_1 and r_3, we find

$$\lim_{\varepsilon\to 0}\left(\int_{r_1=0}^{r_2}\int_{r_3=0}^{r_2} G_{\mathbb{O}}^{\varepsilon}-\frac{\pi^9}{8}\,\frac{\delta''(r_2)}{2\,\varepsilon}\,dr_2\right)=\frac{\pi^9}{8}\left(\frac{d^3}{dr^3}\,\ell n\,\frac{r}{\ell}+\frac{3}{2}\,\delta''(r)\right). \quad (3.23)$$

We now proceed to the general treatment of the extension (renormalization) problem for distributions of $N=D(n-1)$ variables.

We start with an associate homogeneous distribution G_0 defined as a linear functional on the subspace $\mathcal{S}_0$ of $\mathcal{S}(\mathbb{R}^N)$ of test functions vanishing (with their derivatives) at the origin and satisfying

$$(\vec{x}\,\vec{\partial}+d)^k\,G_0(\vec{x})=0 \quad \text{on} \quad \mathcal{S}_0\,. \quad (3.24)$$

We will then construct a k-vector $\boldsymbol{G}=(G_0,G_1,\ldots,G_{k-1})$ of distributions on $\mathcal{S}_0$. The following theorem describes their simultaneous associate homogeneous extension to the entire space $\mathcal{S}(\mathbb{R}^N)$.

Theorem 3.1. *Let $\boldsymbol{G}=(G_0,\ldots,G_{k-1})$ be a k-tuple of associate homogeneous distributions of degree d and order $k-1$ on $\mathcal{S}_0$ (i.e. $G_\nu\in\mathcal{S}'(\mathbb{R}^N\backslash 0)$) such that*

$$\lambda^d\,\boldsymbol{G}(\lambda\,\vec{x})=L(\lambda)\,\boldsymbol{G}(\vec{x})\,,\quad L(\lambda)=e^{\Delta_k\ell n\lambda}\,,\ (\Delta_k)^k=0\,, \quad (3.25)$$

or in components

$$\lambda^d\,G_0(\lambda\,\vec{x})=\sum_{\nu=0}^{k-1}G_\nu(\vec{x})\,\frac{(\ell n\,\lambda)^\nu}{\nu!}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}\lambda^d\,G_{k-2}(\lambda\,\vec{x})&=G_{k-2}(\vec{x})+\ell n\,\lambda\,G_{k-1}(\vec{x})\,,\\ \lambda^d\,G_{k-1}(\lambda\,\vec{x})&=G_{k-1}(\vec{x})\quad \textit{on}\quad \mathcal{S}_0\,.\end{aligned} \quad (3.26)$$

For $d\neq N+\kappa$, $\kappa=0,1,\ldots$, the G_ν admits a unique associate homogeneous continuation as distributions in $\mathcal{S}'(\mathbb{R}^N)$. For the exceptional values $d=N+\kappa$ of the scaling dimension they admit an extension as associate homogeneous distributions of the same degree d and order k. In particular,

$$G_k(\vec{x})=D_\kappa\,\delta(\vec{x}) \quad (3.27)$$

where D_κ is a homogeneous polynomial of degree κ in the derivatives $\vec{\partial}=\{\partial/\partial x_i^\alpha\}$. Every two extensions of G_0 differ by a homogeneous distribution of type (3.27). The remaining components G_ν of $\boldsymbol{G}$ are determined uniquely by $G_\nu=(d+\vec{x}\,\vec{\partial})^\nu\,G_0$ and do not depend on the ambiguity in G_0.

Proof. We first prove the existence of the continuation. Let $f = (f_0(\vec{x}), \ldots, f_{k-1}(\vec{x}))$ be any set of test functions in $\mathcal{S}(\mathbb{R}^N)$. We define a set of almost homogeneous functions $\hat{\boldsymbol{f}} = (f_0, \ldots, f_{k-1})$ that are smooth away from the origin:

$$\hat{\boldsymbol{f}}(\vec{x}) = \langle {}^tL(\rho)\, \rho_+^{N-d-1}, \boldsymbol{f}(\rho\,\vec{x})\rangle_\rho \tag{3.28}$$

where ${}^tL$ stands for the transposed of the matrix L and the pairing is defined as the action of a distribution in a single variable ρ (while $\vec{x}$ appears as a vector of parameters):

$$\begin{aligned} \hat{f}_0(\vec{x}) &= \langle L_0(\rho,\kappa;\ell),\ f_0(\rho\,\vec{x})\rangle_\rho = \int_0^\infty \ell n\,\frac{\ell}{\rho}\,\frac{\partial^{\kappa+1}}{\partial\rho^{\kappa+1}}\,f_0(\rho\,\vec{x})\,d\rho\,, \\ \hat{f}_1(\vec{x}) &= \langle L_1(\rho,\kappa;\ell),\ f_0(\rho\,\vec{x})\rangle_\rho + \langle L_0(\rho,\kappa;\ell),\ f_1(\rho\,\vec{x})\rangle_\rho, \ldots \\ \hat{f}_{k-1}(\vec{x}) &= \sum_{\nu=0}^{k-1}\langle L_{k-1-\nu}(\rho,\kappa;\ell),\ f_\nu(\rho\,\vec{x})\rangle_\rho\,, \end{aligned} \tag{3.29}$$

where the tempered distributions $L_\nu(\rho,\kappa;\ell)$ are defined in Proposition 3.1 (and given by Eq. (A.8)). Let now $r = r(\vec{x})$ be any smooth away from the origin (say, $O(D)$-invariant) norm[4] in $\mathbb{R}^N$. Let χ be a smooth function with compact support on the positive semiaxis, $\chi \in D(]0,\infty[)$, satisfying

$$\int_0^\infty \frac{\chi(r)}{r}\,dr = 1\,. \tag{3.30}$$

Then we define the extension G^χ of $\boldsymbol{G}$ to $\mathcal{S}(\mathbb{R}^N)$ by

$$\langle \boldsymbol{G}^\chi, \boldsymbol{f}\rangle := \langle \boldsymbol{G}, \chi\,\hat{\boldsymbol{f}}\rangle = \sum_{\nu=0}^{k-1}\langle G_\nu, \chi\,\hat{f}_\nu\rangle\,. \tag{3.31}$$

To prove that this is indeed an extension of $\boldsymbol{G}$ we have to verify that if f_ν vanish at the origin together with their derivatives, $f_\nu \in \mathcal{S}_0$, then $\langle \boldsymbol{G}^\chi, \boldsymbol{f}\rangle = \langle \boldsymbol{G}, \boldsymbol{f}\rangle$. Indeed, for $f_\nu \in \mathcal{S}_0$, we have

$$\hat{f}_\nu(x) = \sum_{\mu=0}^{\nu}\int_0^\infty f_{\nu-\mu}(\rho\,\vec{x})\,\frac{(\ell n\,\rho)^\mu}{\mu!}\,\frac{d\rho}{\rho^{1+\kappa}}$$

and hence

$$\begin{aligned} \sum_{\nu=0}^{n-1} G_\nu(\vec{x})\,\hat{f}_\nu(\vec{x})\,d^N\vec{x} &= \sum_{\sigma=0}^{n-1}\sum_{\mu=0}^{n-1-\sigma}\rho^{-\kappa}\,G_{\sigma+\mu}(\vec{x})\,\frac{(\ell n\,\rho)^\mu}{\mu!}\,f_\sigma(\rho\,\vec{x})\,d^N x \\ &= \sum_{\sigma=0}^{n-1} G_\sigma(\rho\,\vec{x})\,f_\sigma(\rho\,\vec{x})\,d^N\rho\,\vec{x}\,. \end{aligned}$$

[4] One may (but is not bound to) think, following Hörmander, of the radial norm $r^2 = \sum_{i=1}^{n-1}\sum_{\alpha=1}^{D}(x_i^\alpha)^2$.

Then, it remains to set $\rho\,\vec{x} = \vec{y}$ and to take into account that

$$\int_0^\infty \chi\left(\frac{|\vec{y}|}{\rho}\right)\frac{d\rho}{\rho} = \int_0^\infty \chi(r)\,\frac{dr}{r} = 1\,.$$

Furthermore, if $f_\kappa(\vec{x}) \in \mathcal{S}(\mathbb{R}^N)$ we can also introduce

$$\hat{f}_k(\vec{x}) = \sum_{\nu=0}^{k} \langle L_{k-\nu}(\rho,\kappa;\ell),\ f_\nu(\rho\,\vec{x})\rangle_\rho \tag{3.32}$$

and define the $(k+1)$-vector of associate homogeneous distributions $\boldsymbol{G}^{\rm ext} = (G_0, \ldots, G_k)$ where

$$G_k = (\vec{x}\,\vec{\partial} + d)\,G_{k-1} = \ldots = (\vec{x}\,\vec{\partial} + d)^k\,G_0 \tag{3.33}$$

is a homogeneous distribution of degree $-d$ with support at the origin of $\mathbb{R}^N$; then

$$\langle \boldsymbol{G}^{\rm ext}, \boldsymbol{f}^{\rm ext}\rangle := \sum_{\nu=0}^{k} \langle G_\nu, \chi\,\hat{f}_\nu\rangle \tag{3.34}$$

again coincides with $\langle \boldsymbol{G}, \boldsymbol{f}\rangle$ for $f_\nu \in \mathcal{S}_0$ since then $\langle G_k, f_k\rangle = 0$.

The uniqueness part is based on the observation that any distribution with support at the origin is a linear combination of derivatives of $\delta(\vec{x})$ (see, *e.g.*, Theorem 3.1.16 of [H]). Its proof uses the assumption that the extension is an associate homogeneous distribution of the same degree and the fact that

$$(d + \vec{x}\,\vec{\partial})\,D_\kappa\,\delta(\vec{x}) = 0 \quad \text{for} \quad d = N + \kappa\,. \tag{3.35}$$

This completes the proof of the Theorem.

4 Renormalization in Lorentzian signature. Quadratic configuration space.

The true objects of interest in a relativistic QFT are the Poincaré covariant Green functions $G(x_1, \ldots, x_n)$ on (the $n^{\rm th}$ Cartesian power of) Minkowski space M, for all $n \in \mathbb{N}$. It is on these correlation functions that one imposes the physical requirements of causality, stability (or energy positivity) *etc.* The corresponding (sometimes simpler looking) properties of Euclidean Green functions have to be derived from these basic assumptions. We shall therefore formulate the Minkowski space renormalization problem in some detail, albeit we do not intend to present its full treatment in this paper.

Once we assume that the square distances in the denominators of the expression (1.1) is indefinite

$$x^2 = \boldsymbol{x}^2 - (x^0)^2\,, \quad \boldsymbol{x}^2 = \sum_{i=1}^{D-1} (x^i)^2 \tag{4.1}$$

and hence vanish on a cone in M, it becomes more natural to start with the complex Dn dimensional space $(\mathbb{C}^D)^{\times n}$. We define the *quadratic configuration space* as

$$C_n = \{(z_1, \ldots, z_n) \in (\mathbb{C}^D)^{\times n} \, ; \quad z_{ij}^2 := (z_i - z_j)^2 \neq 0 \quad \text{for} \quad i \neq j\} \, . \tag{4.2}$$

Our objective is to define a map $\mathcal{R}$ from the set of rational functions $\mathcal{O}_n$ spanned by monomials of type (1.2) to the space of (tempered) distributions on $M^{\times n}/M$ (where factoring by M reflects the translation invariance of G), restricted by a number of conditions which we proceed to formulate.

First, we introduce *boundary value operators* with respect to tube domains $\mathcal{T}_{\vec{I}} \subset M_{\mathbb{C}}^I$. Every such tube $\mathcal{T}_{\vec{I}}$ is defined for a finite *ordered* set $\vec{I}$ - that is a set $I = \{j_1, \ldots, j_n\}$ equipped with a line (total) order $j_1 \prec \cdots \prec j_n$ on it, in other words, $\vec{I}$ is the pair $(I, \prec)$. We shall also write it as

$$\vec{I} \, = \, (j_1, \ldots, j_n) \, .$$

Note that if $I \subset \mathbb{N}$ then we consider the order $\prec$ on I as an *independent* structure on it, which may not coincide with the order $<$ induced by $\mathbb{N}$. For every ordered set $\vec{I} = (j_1, \ldots, j_n)$ we have a standard backward tube domain associated to $\vec{I}$,

$$\mathcal{T}_{\vec{I}} := \Big\{ (x_j)_{j \, \in \, I} \, \in \, M^I + iM^I \, : \, x_{j_k} - x_{j_{k+1}} \, \in \, M - iV_+ \text{ for } k = 1, \ldots, n \Big\}$$

(V_+ being the open forward light–cone in M) and then we define a boundary value map with respect to this tube $\mathcal{T}_{\vec{I}}$:

$$\text{b.v.}_{\vec{I}} : \mathscr{O}_I \to \mathscr{D}'_I$$

where we denote for short

$$\mathscr{D}'_I \, := \, \mathscr{D}'\big(M^I/M\big)$$

(the space of translation invariant distributions on M^I). Hence, for G_Γ (1.2) we have:

$$\text{b.v.}_{\vec{I}} \, G_I := \prod_{(j,k) \in \Gamma} \frac{P_{jk}(x_j - x_k)}{\big((x_j - x_k)^2 \pm i0(x_j^0 - x_k^0)\big)^{N_{jk}}} \, , \quad (\pm) = \begin{cases} (+) \text{ if } j \prec k, \\ (-) \text{ if } k \prec j \end{cases} . \tag{4.3}$$

Note that $\text{b.v.}_{\vec{I}} : \mathscr{O}_I \to \mathscr{D}'_I$ *is an injection.* This is because of a theorem stating that if a boundary value of an analytic function vanishes on some open set then the function is zero everywhere. For the same reason the $\text{b.v.}_{\vec{I}}$ maps commute with the action of the differential operators with polynomial coefficients. Another property of the boundary value maps is that they preserve the multiplication,

$$\text{b.v.}_{\vec{I}}\big(G'G''\big) \, = \, \text{b.v.}_{\vec{I}}\big(G'\big) \, \text{b.v.}_{\vec{I}}\big(G''\big) \, ,$$

which includes the statement that distributions, which are $\text{b.v.}_{\vec{I}}$–values (for a fixed $\vec{I}$) can be multiplied.

The renormalization $\mathcal{R}$ is another linear map $\mathcal{R} : \mathscr{O}_I \to \mathscr{D}'_I$ (where $\mathscr{D}'_I$ is the space of distributions with arguments labelled by the index set I). It is assumed to satisfy the axioms listed below.

$\mathcal{R}1$. The image $\mathcal{R}\,\mathscr{O}_I \subset \mathscr{D}'_I$ of a homogeneous rational function G_Γ of degree d_I is an associate homogeneous distributions of the same degree (but, in general, of a higher order).

Remark 4.1. In a massive theory one uses, more generally, the Steinman scaling degree and preservation of filtration (see [St, BF]). One may hope to use the homogeneity degree of the leading short distance singularity, viewing the mass as a perturbation, also in the general case.

$\mathcal{R}2$. $\mathcal{R}$ commutes with permutations σ of vertices in $I : \mathcal{R} \circ \sigma = \sigma \circ \mathcal{R}$.

$\mathcal{R}3$. *Causality:* for every disjoint union $I = I' \dot\cup I''$ we have

$$\mathcal{R}(G_I)\,|_{I' \gtrsim I''} = \mathcal{R}(G_{I'})\,G_{I',I''}\,\mathcal{R}(G_{I''}) \tag{4.4}$$

where

$$I' \gtrsim I'' \Leftrightarrow x_{ij} \notin \overline{V}_-(= -\overline{V}_+) \quad \text{for} \quad i \in I'\,,\ j \in I''\,, \tag{4.5}$$

$$G_{I'I''} = b_{I' \gtrsim I''} \prod_{\substack{i \in I' \\ j \in I''}} G_{ij} \tag{4.6}$$

where $b_{i' \gtrsim I''}$ is the boundary value operator and $\overline{V}_-$ is the closure of V_-. The product (4.4) exists as a distribution for $G_{I'I''}$ given by the boundary value (4.6). This can be seen by examining the wave front sets of the various terms. These are restricted by translation invariance and by the convexity of the light-cone which characterizes the wave fronts of the G_{ij}'s. The statement follows by a straightforward application of Hörmander's argument in Chapter VIII of [H]. We stress that while the euclidean factorization property only required the existence of the distribution (2.1) on the domain $\mathcal{S}_{12}$ here the distribution (4.4) is found to exist everywhere.

$\mathcal{R}4$. $\mathcal{R}$ commutes with multiplication by (homogeneous, and hence, because of linearity, by any) polynomial p :

$$\mathcal{R}(p\,G) = p\,\mathcal{R}(G)\,. \tag{4.7}$$

$\mathcal{R}5$. $\mathcal{R}$ intertwines the natural action of the Lorentz group (and hence also of the Poincaré group, with dilations) on $\mathcal{O}_I$ and on $\mathcal{S}'_I$.

We assert that a renormalization map satisfying conditions $\mathcal{R}1$–5 exists and can be constructed inductively (as in the euclidean case). Rather than presenting the full argument we shall formulate and prove a lemma that serves to state the induction in the number of vertices (and whose counterpart in the euclidean case is given by Proposition 2.1).

Proposition 4.1. *Fix a (unordered) set I of n (decorated) indices* $(1, 2, \ldots, n)$*. The complement $C(\Delta_n)$ of the small diagonal*

$$\Delta_n = \{x_1 = x_2 = \ldots = x_n\} \tag{4.8}$$

is the union of all configurations $I = I' \dot{\cup} I''$ (with non empty I' and I'') entering the causality condition $\mathcal{R}4$.

Proof. Let $(x_1, \ldots, x_n) \in C(\Delta_n)$. Then there are at least two different points, say $x_i \neq x_j$. Given such a pair, there exists a Lorentz frame in which the time components of x_i and x_j are different, say $x_i^0 > x_j^0$. Let then I' be the set of indices k such that $x_k^0 \geq x_i^0$, and let I'' be its complement in I. Clearly, $I = I' \dot{\cup} I''$ is a splitting of the type entering $\mathcal{R}4$ (and neither set is empty as $i \in I'$ and $j \in I''$). This completes (the non-trivial part of) the proof of our proposition.

Postulate $\mathcal{R}3$ also explains the connection of Eq. (2.1) (in the euclidean case) with causality.

Acknowledgments. The authors thank Michel Dubois-Violette, Maxim Kontsevich and Dirk Kreimer for stimulating discussions. N.M.N. and I.T. acknowledge partial support by grant DO 02-257 of the Bulgarian National Science Foundation. I.T. thanks the Theory division of CERN and IHÉS for hospitality during the course of this work.

Appendix A. Radial associate homogeneous distributions

The family of elementary functions

$$L_n(r, a; \ell) = \frac{r^{-a}}{n!} \left(\ell n \, \frac{r}{\ell}\right)_+^n , \quad (f(r))_+ = \begin{cases} f(r) & r > 0 \\ 0 & r < 0 \end{cases} , \tag{A.1}$$

$n = 0, 1, 2, \ldots$, $a \in \mathbb{C}$ satisfies the system of coupled ordinary differential equations

$$(D + a)\, L_n(r, a; \ell) = L_{n-1}(r, a; \ell) \quad \text{for} \quad r > 0 \, , \; D := r \, \frac{d}{dr} \, , \tag{A.2}$$

which provides a differential form of its associate homogeneity property. For $\mathrm{Re}\, a < 1 \quad L_n(r, a; \ell)$ is locally integrable and defines a (tempered) distribution – a continuous functional on Schwartz space $\mathcal{S}(\mathbb{R})$ – with support on the positive (real) semiaxis. For a different from a positive integer $(a \neq 1, 2, \ldots)$ the functions L_n again admit a unique continuation as associate homogeneous distributions satisfying, by definition, Eqs. (A.2) and (3.8),

$$r \, L_n(r, a; \ell) = L_n(r, a - 1; \ell) \, . \tag{A.3}$$

For $\mathrm{Re}\, a \leq \kappa + 1$, $\kappa = 0, 1, \ldots$, they are defined as linear functionals on the subspace $\mathcal{S}_\kappa$ of $\mathcal{S}(\mathbb{R})$ of test functions vanishing at the origin together with their first κ derivatives. The functional $L_n(r, a; \ell)$ is an *associate homogeneous distribution of order* n *and degree* $-a$ *on* $\mathcal{S}_\kappa$. In fact, it satisfies

$$L_n(\lambda\, r, a; \ell) = \sum_{\nu=0}^{n} L_\nu(\lambda, a; \ell)\, L_{n-\nu}(r, a; \ell) \quad \text{on} \quad \mathcal{S}_\kappa\,, \tag{A.4}$$

which implies

$$(D + a)^{n+1}\, L_n(r, a; \ell) = 0 \quad \text{on} \quad \mathcal{S}_\kappa \left(D = r\, \frac{d}{dr} \right) . \tag{A.5}$$

It admits a unique extension to $\mathcal{S}(\mathbb{R})$ at the exceptional points $a = \nu + 1$, $\nu = 0, 1, \ldots, \kappa$ as an associate homogeneous distribution of order $n+1$, satisfying the "boundary conditions" (3.10),

$$L_n(r, 0; \ell) = \frac{1}{n!} \left(\ell n\, \frac{r}{\ell} \right)_+^n = \frac{\theta(r)}{n!} \left(\ell n\, \frac{r}{\ell} \right)^n . \tag{A.6}$$

The distributions $L_n(r, k; \ell)$ are related for different n by (A.2) assumed valid for $n = 0, 1, 2, \ldots$, and $k = \nu + 1 \quad \nu = 0, \ldots, \kappa$ where $L_{-1}(r, k; \ell) \equiv L_{-1}(r, k)$ has support at $r = 0$:

$$(D + \kappa + 1)^{n+1}\, L_n(r, \kappa + 1; \ell) = L_{-1}(r, \kappa + 1) := \frac{\delta^{(\kappa)}(-r)}{\kappa!} \tag{A.7}$$

($\delta^{(\kappa)}(-r) = (-1)^\kappa\, \delta^{(\kappa)}(r)$). This is the assertion of Proposition 3.1 which we proceed to prove.

We first give an explicit construction of the family of distributions $\{L_n(r, \kappa + 1; \ell)\}$ on the real axis that extends the elementary functions (A.1) preserving the properties (A.2) (A.3) (A.6) and (A.7). To this end we shall differentiate in ε Hörmander's relation (2.10) (or, equivalently, (2.11) – see the unnumbered relation between (3.2.4) and (3.2.5) in [H]):

$$\begin{aligned} L_n(r, \kappa + 1; \ell) &= \lim_{\varepsilon \to 0} \left\{ \frac{d^n}{d\varepsilon^n} \left(r^{\varepsilon - \kappa - 1} - \frac{1}{\varepsilon}\, \frac{\delta^{(\kappa)}(-r)}{\kappa!} \right) \right\} \\ &= \frac{(-1)^\kappa}{\kappa!} \left(\frac{d}{dr} \right)^{\kappa+1} \sum_{\nu=0}^{n+1} \sigma_{\nu\kappa}\, L_{n+1-\nu}(r, 0; \ell)\,, \end{aligned} \tag{A.8}$$

where $L_m(r, 0; \ell)$ are (integrable) powers of log (see (A.6)) and the constants $\sigma_{\nu\kappa}$ are, in fact, determined by (A.3) and (A.6):

$$\sigma_{0\kappa} = 1\,, \quad \sigma_{\nu\kappa} = \sum_{1 \leq j_1 \leq \ldots \leq j_\nu \leq \kappa} \frac{1}{j_1 \ldots j_\nu} \quad \text{for} \quad \kappa = 1, 2, \ldots,$$
$$(\sigma_{\nu 0} = 0) \qquad \nu = 1, \ldots, n + 1\,. \tag{A.9}$$

(Eq. (A.8) should again be understood in terms of "differential renormalization" – just as the meaning of (2.10) is spelled out by (2.11).)

Before proceeding to the proof that the expression (A.8) indeed satisfies the above constraints we note that the last term in the sum is a derivative of the δ-function (that is independent of ℓ):

$$\frac{(-1)^\kappa}{\kappa!}\,\sigma_{n+1\kappa}\,\frac{d^{\kappa+1}}{dr^{\kappa+1}}\,\theta(r) = \sigma_{n+1\kappa}\,\frac{\delta^{(\kappa)}(-r)}{\kappa!}\,. \tag{A.10}$$

In verifying (A.3) we use the identities

$$r\,\frac{d^{\kappa+1}}{dr^{\kappa+1}} = \left(\frac{d}{dr}\right)^{\kappa}(D-\kappa)\,, \quad D\,\frac{\left(\ell n\,\frac{r}{\ell}\right)^{k+1}}{(k+1)!} = \frac{\left(\ell n\,\frac{r}{\ell}\right)^{k}}{k!}$$

$$\sigma_{\nu\kappa} = \sigma_{\nu\kappa-1} - \frac{\sigma_{\nu-1\kappa}}{\kappa}\,. \tag{A.11}$$

To check (A.2) we use in addition

$$(D+\kappa+1)\left(\frac{d}{dr}\right)^{\kappa+1} = \left(\frac{d}{dr}\right)^{\kappa+1} D\,. \tag{A.12}$$

To prove the uniqueness of the distributions satisfying (A.2) (A.3) (A.6) we assume that there are two such families, L_n and L_n' and consider their differences,

$$\Delta_{n\kappa} \equiv \Delta_{n\kappa}(r,\ell) = L_n(r,\kappa+1;\ell) - L_n(r,\kappa+1;\ell)\,. \tag{A.13}$$

These differences obey the same linear homogeneous equations (A.2) (A.3) as the L_n. The boundary condition (A.6), on the other hand, implies

$$r\,\Delta_{n0}(r,\ell) = 0\,. \tag{A.14}$$

The general solution of (A.14) in $\mathcal{D}'(\mathbb{R})(\supset \mathcal{S}'(\mathbb{R}))$ is [Sc]

$$\Delta_{n0}(r,\ell) = C_n\,\delta(r)\,.$$

Combining this with (A.2), $(\mathcal{D}+1)\,\Delta_{n0} = \Delta_{n-10}$ we deduce that $C_n = 0 = \Delta_{n0}$. The equations (A.3), implying successively

$$r\,\Delta_{n\kappa+1}(r,\ell) = 0\,, \quad \kappa = 0,1,\ldots, \tag{A.15}$$

have, on the other hand, no non-trivial homogeneous solutions of degree $\kappa+2$ ($\kappa \geq 0$). This completes the (uniqueness part of the) proof of Proposition 3.1.

Appendix B. Calculating renormalized densities in configuration space

B1. The one-loop scalar 2-point function

We shall prove Eq. (2.16) for the renormalized expression of the logarithmically divergent 2-point function corresponding to the graph on Fig. 1a by applying both sides to a test function $f(x) \in \mathcal{S}(\mathbb{R}^4)$. Introducing spherical coordinates

$$d^4x = r^3 dr\, d^3\omega\,, \qquad \int_{\mathbb{S}^3} d^3\omega = 2\,\pi^2 \tag{B.1}$$

we can write

$$\int d^4x \left\{ \frac{1}{(x^2)^2} \left(\frac{x^2}{\ell^2} \right)^{\varepsilon} - \frac{\pi^2}{\varepsilon}\, \delta(x) \right\} f(x) = \frac{\pi^2}{\varepsilon} \left\{ \int_0^\infty \overline{f}(r)\, d \left(\frac{r^2}{\ell^2} \right)^{\varepsilon} - f(0) \right\} \tag{B.2}$$

where we have set

$$\overline{f}(r) = \frac{1}{2\,\pi^2} \int_{\mathbb{S}^3} f(r\omega)\, d^3\omega \, (\Rightarrow \overline{f}(0) = f(0))\,. \tag{B.3}$$

Integrating by parts and using the relation

$$\frac{1}{\varepsilon} \left[\left(\frac{r^2}{\ell^2} \right)^{\varepsilon} - 1 \right] = \ell n\, \frac{r^2}{\ell^2} + O(\varepsilon)$$

we find

$$\frac{\pi^2}{\varepsilon} \left\{ \int_0^\infty \overline{f}(r)\, d \left(\frac{r^2}{\ell^2} \right)^{\varepsilon} - f(0) \right\} = \pi^2 \int_0^\infty \left(\ell n\, \frac{\ell^2}{r^2} \right) \frac{d\overline{f}}{dr}\, dr + O(\varepsilon)\,. \tag{B.4}$$

This yields the following expression for the renormalized 2-point density

$$\int_{\mathbb{S}}^{3} G_1(r,\ell)\, \mathrm{Vol} := \left[\int_{\mathbb{S}^3} d^3\omega\, G_1(r\omega,\ell) \right] r^3\, dr$$

$$= 2\,\pi^2\, \frac{d}{dr} \left(\ell n\, \frac{r}{\ell} \right) dr = \pi^2\, dr^2\, \frac{d}{dr^2} \left(\ell n\, \frac{r^2}{\ell^2} \right)\,. \tag{B.5}$$

Finally we prove that the same density is reproduced from the right hand side of (2.16). To this end we apply the identity

$$\frac{\partial}{\partial x^\alpha} \left(x^\alpha\, g(r) \right) = \left(r\, \frac{\partial}{\partial r} + 4 \right) g(r) = \frac{1}{r^3}\, \frac{\partial}{\partial r} \left(r^4\, g(r) \right)$$

to $g(r) = \frac{1}{r^4}\, \ell n\, \frac{r^2}{\ell^2}$ (for $r > 0$) with the result

$$\int d^3\omega\, \frac{1}{2}\, \frac{\partial}{\partial x^\alpha} \left[\frac{x^\alpha}{(x^2)^2}\, \ell n\, \frac{x^2}{\ell^2} \right] r^3 dr = \pi^2\, \frac{\partial}{\partial r} \left(\ell n\, \frac{r^2}{\ell^2} \right) dr = 2\,\pi^2\, d\,\ell n\, \frac{r}{\ell}\,. \tag{B.6}$$

B2. Three-point graph with a 2-point subdivergence

In order to perform the first five angular integrations in the 3-point density corresponding to the graph on Fig. 3 (yielding the first equation (3.14)) we set

$$\omega_2 = \omega_1 \cos\vartheta + n \sin\vartheta\,, \quad n^2 (= \omega_i^2) = 1\,, \quad \omega_1\, n = 0 \tag{B.7}$$

and integrate in ω_1 over the 3-sphere (with $|\mathbb{S}^3| = 2\,\pi^2$) and n over $\mathbb{S}^2$ ($|\mathbb{S}^2| = 4\,\pi$). To derive the last formula (3.14) we have used the expansion (2.21) of the propagator

$$(x_{12}^2)^{-1} = (r^2 + \rho^2 - 2\, r\rho \cos\vartheta)^{-1} \tag{B.8}$$

into Gegenbauer polynomials and their orthogonality (2.22) as well as the relations

$$C_0^1(\cos\vartheta) = 1\,, \quad \int_0^\pi \sin^2\vartheta\, d\vartheta = \frac{\pi}{2}\,.$$

In order to find the leading term $LT\, G_\Delta$Vol of the density G_ΔVol for small r and ρ we first smear it with a $f_0(r,\rho)$ belonging to the subspace of test functions $\mathcal{S}_0 \subset \mathcal{S}(\mathbb{R}^2)$ vanishing at the origin, $f_0(0,0) = 0$. We split the double integral (G_Δ, f_0)Vol into two parts $\rho > r$ and $r \geq \rho$

$$\begin{aligned}(G_\Delta, f_0)\,\mathrm{Vol} &= 4\,\pi^4 \int_0^\infty d\rho \left(\ell n\, \frac{\ell}{\rho}\right)_+ \frac{\partial}{\partial\rho} \left\{ \frac{1}{\rho^2} \int_0^\rho f_0(r,\rho)\, r\, dr \right\} \\ &+ 4\,\pi^4 \int_0^\infty \!\! \int_0^\infty \frac{dr}{r} \left(\ell n\, \frac{\ell}{\rho}\right)_+ d\rho\, \frac{\partial}{\partial\rho} \left[\theta(r-\rho)\, f_0(r,\rho)\right].\end{aligned} \tag{B.9}$$

It is easy to verify (expanding $f_0(r,\rho)$ in a Taylor series in r around $r=0$) that the first (double) integral converges even without the assumption that $f_0(0,0) = 0$. The same is true for the part of the second integral in the right hand side of (B.9) involving $\frac{\partial f_0}{\partial\rho}$. The leading term that is singular at small distances is the one proportional to

$$\frac{\partial}{\partial\rho}\,\theta(r-\rho) = -\delta(r-\rho)$$

which gives

$$(LT\, G_\Delta, f_0)\,\mathrm{Vol} = 4\,\pi^4 \int_0^\infty \frac{dr}{r} \left(\ell n\, \frac{r}{\ell}\right) f_0(r,r)\,. \tag{B.10}$$

The extension of this term to general test functions f yields the renormalized expression (3.15) in accord with Proposition 3.1.

This calculation illustrates the fact that whenever using a nonsmooth norm, like $\max(r,\rho)$, one should take into account the singular contribution of its derivatives.

B3. Renormalization of a 3-loop 4-point graph with two subdivergences

We shall compute the renormalized density corresponding to the graph on Fig. 4.

In order to reproduce the expression (3.18) valid in the domain

$$r_\vee := \max\left(r_1, \frac{r_2+r_3}{2}\right) > 0 \tag{B.11}$$

(i.e. away of the origin in the 12-dimensional space of coordinate differences x_{0j}, $j=1,2,3$) we set

$$x_{0j} = r_j\,\omega_j\,, \quad j=1,2,3\,; \quad \omega_i = \cos\vartheta_i\,\omega_3 + \sin\vartheta_i\,n_i\,, \quad i=1,2$$
$$\omega_3\,n_i = 0\,, \quad n_i^2 = 1 (= \omega_j^2) \tag{B.12}$$

and integrate over the 3-sphere $\omega_3^2 = 1$ and the 2-spheres $n_i^2 = 1$, $n_i\,\omega_3 = 0$ (in general, $|\mathbb{S}_{2n-1}| = \frac{2\,\pi^n}{(n-1)!}$, $|\mathbb{S}_2| = 4\,\pi$, $|\mathbb{S}_N| = \frac{2\,\pi}{N-1}\,|S_{N-2}|$). Next we use the substitutions (B.8) (B.9) for the variables r_2, r_3 (and ϑ_2):

$$r_2 = r(1+t)\,, \quad r_3 = r(1-t) \qquad \left(r = \frac{r_2+r_3}{2}\,,\ t = \frac{r_2-r_3}{r_2+r_3}\right),$$

$$\frac{dr_2\,dr_3\,\sin\vartheta_2}{r_2^2+r_3^2-2\,r_2\,r_3\cos\vartheta_2} = \frac{sc\,dr\,dt}{r(s^2+c^2t^2)}\,, \quad s = \sin\frac{\vartheta_2}{2}\,, \quad c = \cos\frac{\vartheta_2}{2}\,. \tag{B.13}$$

As suggested by our analysis of the renormalized 3-point function (3.16) the leading $\left(\left(\ell n\,\frac{r}{\ell}\right)^2 - \right)$ term at small distances is the one, coming from $\frac{\partial}{\partial\,\vartheta_2}$ $\left[\left(\ell n\,\frac{r^2}{\ell^2}\right)_+\right]\,\theta(\sin\vartheta_2)$, proportional to

$$\frac{sc\,d\vartheta_2\,dt}{s^2+c^2t^2}\,\delta(\vartheta_2) = \pi\,\delta(t)\,\delta(\vartheta_2)\,dt\,d\vartheta_2\,. \tag{B.14}$$

(Here $\delta(\vartheta_2) = \frac{\partial}{\partial\,\vartheta_2}\,\theta(\sin\vartheta_2)$ is the *periodic δ-function* of period 2π.) It allows to integrate the leading contribution, $LT\,G_4$Vol of (3.18) (B.13) in ϑ_2, t and ϑ_1:

$$\int_0^\pi d\vartheta_1 \int_{-1}^1 dt \int_0^\pi d\vartheta_2\,LT\,G_4\,\mathrm{Vol} = (2\pi)^5 \int_0^\pi \frac{\sin^2\vartheta_1\,d\vartheta_1}{r_1^2+r^2-2\,r r_1\cos\vartheta_1}\left(\ell n\,\frac{r}{\ell_2}\right)_+ r\,dr\,d\ell n\,\frac{r_1}{\ell_1}$$

$$= 16\,\pi^6\,\frac{r}{r_\vee^2}\left(\ell n\,\frac{r}{\ell_2}\right)_+ dr\,d\left(\ell n\,\frac{r_1}{\ell_1}\right)_+\,, \quad r_\vee = \max(r, r_1)\,. \tag{B.15}$$

(Here we again used the expansion in $C_n^1(\cos\vartheta_1)$ as in the derivation of the last equation (3.14).) Eq. (3.19) is now a consequence of (B.14) and (B.15).

References

[B] C. Bergbauer, Notes on Feynman integrals and renormalization, arXiv: 1005.3960 [hep-th].

[BBK] C. Bergbauer, R. Brunetti, D. Kreimer, Renormalization and resolution of singularities, arXiv:0908.0633v2 [hep-th].

[BEK] S. Bloch, H. Esnault, D. Kreimer, On motives and graph polynomials, *Commun. Math. Phys.* **267** (2006) 181-225; math/0510011.

[BK] S. Bloch, D. Kreimer, Mixed Hodge structures and renormalization in physics, *Commun. Number Theory Phys.* **2** (2008) 637-718, arXiv:0804.4399 [hep-th]; Feynman amplitudes and Landau singularities for 1-loop graphs, arXiv:1007.0338 [hep-th].

[BS] N.N. Bogoliubov, D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, 3d edition, Wiley, 1980.

[B10] R.E. Borcherds, Renormalization and quantum field theory, arXiv:1008.0129 [math-ph].

[BrK] D.J. Broadhurst, D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, *Phys. Lett.* **B393** (1997) 403-412; hep-th/9609128.

[BF] R. Brunetti, K. Fredenhagen, Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds, *Commun. Math. Phys.* **208** (2000) 623-661; math-ph/990328.

[BFV] R. Brunetti, K. Fredenhagen, R. Verch, The generally covariant locality principle: a new paradigm for local quantum physics, *Commun. Math. Phys.* **237** (2003) 31-68; math-ph/0112041.

[CM] O. Ceyhan, M. Marcolli, Feynman integrals and motives of configuration space, arXiv:1012.5485 [math-ph].

[CKT] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, New approach to evaluation of multiloop Feynman integarls: the Gegenbauer polynomial x-space technique, *Nucl. Phys.* **B174** (1980) 345-377.

[CK] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem, II: the β-function diffeomorphism and the renormalization group, *Commun. Math. Phys.* **210** (2000) 249-273, 216 (2001) 215-241; hep-th/9912092, hep-th/0003188; Insertion and elimination: the doubly infinite algebra of Feynman graphs, *Ann. Inst. Henri Poincaré* **3** (2002) 411-433; hep-th/0201157.

[DP] C. De Concini, C. Procesi, Wonderful models of subspace arrangements, *Selecta Math.* **1**:3 (1995) 459-494.

[DF] M. Dütsch, K. Fredenhagen, Causal perturbation theory in terms of retarded products, and a proof of the action Ward identity, *Rev. Math. Phys.* **16**:10 (2004) 1291-1348.

[EG] H. Epstein, V. Glaser, The role of locality in perturbation theory, *Ann. Inst. H. Poincaré*, **A19**:3 (1973) 211-295.

[EGP] K. Ebrahimi-Fard, J.M. Gracia-Bondia, F. Patras, A Lie algebra approach to renormalization, *Commun. Math. Phys.* **276** (2007) 519549; hep-th/0609035.

[FHS] S. Falk, R. Häußling, F. Scheck, Renormalization in quantum field theory: an improved rigorous method, *J. Phys. A: Math. Theor.* **43** (2010) 035401 (15p); arXiv:0901.2252 [hep-th].

[FGG] S. Ferrara, R. Gatto, A.F. Grillo, Logarithmic scaling and spontaneous breaking, Phys.Lett. **42B** (1972) 264-266.

[FG-B] H. Figeroa, J.M. Gracia-Bondia, Combinatorial Hopf algebras in quantum field theory, *Rev. Math. Phys.* **17** (2005) 881- ; hep-th/0408145.

[FJL] D.Z. Freedman, K. Johnson, J.I. Latorre, Differential regularization and renormalization: a new method of calculation in quantum field theory, *Nucl. Phys.* **B371** (1992) 353-314.

[FM] W. Fulton, R. MacPherson, A compactification of configuration space, *Ann. Math.* **139** (1994) 183-225.

[GS] I.M. Gelfand, G.E. Shilov, *Generalized Functions* I, Academic Press, New York 1968 (original Russian edition: Moscow 1958).

[G-B] J.M. Gracia-Bondia, Improved Epstein-Glaser renormalization in coordinate space I. Euclidean framework, *Math. Phys. Anal. Geom.* **6** (2003) 55-88; hep-th/0202023.

[G-BL] J.M. Gracia-Bondia, S. Lazzarini, Improved Epstein-Glaser renormalization in coordinate space II. Lorentz invariant framework, *J. Math. Phys.* **44** (2003) 3863-3875; hep-th/0212156.

[G-BVF] J.M. Gracia Bondía, J.C. Várilly, H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, 2000.

[HL] P.E. Haagensen, J.I. Latorre, Differential renormalization of massive quantum field theoris, *Phys. Lett.* **B283** (1992) 293-297; hep-ph/9203207; A comprehensive coordinate space renormalization of quantum electrodynamics, *Annals Phys.* **221** (1993) 77-105; hep-ph/9206252.

[H07] S. Hollands, The operator product expansion for perturbative quantum field theory in curved spacetime, *Commun. Math. Phys.* **273** (2007) 1-36; gr-qc/0605072.

[H08] S. Hollands, Renormalized quantum Yang-Mills fields in curved spacetime, *Rev. Math. Phys.* **20** (2008) 1033-1172; arXiv:0705.3340 [gr-qc].

[HW] S. Hollands, R.M. Wald, Existence of local covariant time ordered products of quantum fields in curved spacetime, *Commun. Math. Phys.* **231** (2002) 309-345; gr-qc/0111108.

[HW08] S. Hollands, R.M. Wald, Quantum field theory in curved spacetime the operator product expansion and dark energy, *Gen. Rel. Grav.* **40** (2008) 2051-2059; arXiv:0805.3419 [gr-qc].

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators, I. Distribution Theory and Fourier Analysis*, 2d edition, Springer, Berlin 1990 (see, in particular, Sect. 3.2 and Chapter VIII).

[K] K.J. Keller, Euclidean Epstein-Glaser renormalization, *J. Math. Phys.* **50** (2009) 103503; arXiv:0902.4789 [math-ph].

[K10] K.J. Keller, Dimensional regularization in position space and a forest formula for regularized Epstein-Glaser renormalization, arXiv:1006.2148 [math-ph].

[Kr] D. Kreimer, *Knots and Feynman Diagrams*, Cambridge Lecture Notes in Physics **13**, Cambridge Univ. Press, Cambridge 2000.

[KTV] A.N. Kuznetsov, F.V. Tkachov, V.V. Vlasov, Techniques of distributions in perturbative quantum field theory I, hep-th/9612037.

[Ni] N.M. Nikolov, Anomalies in quantum field theory and cohomologies in configuration spaces, arXiv:0903.0187 [math-ph]; Talk on anomaly in quantum field theory and cohomologies of configuration spaces, arXiv:0907.3735 [hep-th].

[N] N.M. Nikolov, Renormalization theory of Feynman amplitudes on configuration space, arXiv:0907.3734 [hep-th].

[Pr] D. Prange, Causal perturbation theory and differential renormalization, *J. Phys.* **A32** (1999) 2225-2238; hep-th/9710225.

[R] J. Rosner, Sixth order contribution to Z_3 in finite quantum electrodynamics, *Phys. Rev. Letters* **17**:23 (1966) 1190-1192; Higher order contributions to the divergent part of Z_3 in a model quantum electrodynamics, Ann. of Physics **44** (1967) 11-34.

[Sc] L. Schwartz, *Théorie des distributions* I, II, Hermann, Paris 1950-51.

[Sp] E.R. Speer, On the structure of anlytic renormalization, *Commun. Math. Phys.* **23** (1971) 23-36; *ibid.* **25** (1972) 336.

[St] O. Steinmann, *Perturbative Expansion in Axiomatic Field Theory*, Lecture Notes in Physics, Springer 1971.

[SP] E.C.G. Stueckelberg, A. Petermann, La normalization des constantes dans la théorie des quanta, *Helv. Phys. Acta* **26** (1953) 499-520.

[Stu] J. Lacki, G. Wanders, H. Ruegg (Eds.) *Stueckelberg, an unconventional figure in twentieth century physics*. Birkhäuser Verlag AG, 2009.

[U] N.I. Usyukina, Algorithm for calculating massless Feynman diagrams, *Theor. Math. Phys.* **79** (1989) 385-391 (transl from *Teor. Mat. Fiz.* **79** (1989) 63-71).