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GEOMETRIC PALINDROMIC CLOSURE

Eric Domenjoud — Laurent Vuillon

ABSTRACT. We define, through a set of symmetries, an incremental construction of geometric objects in $\mathbb{Z}^d$. This construction is directed by a word over the alphabet $\{1,\ldots,d\}$. These objects are composed of d disjoint components linked by the origin and enjoy the nice property that each component has a central symmetry as well as the global object. This construction may be seen as a geometric palindromic closure. Among other objects, we get a 3 dimensional version of the Rauzy fractal. For the dimension 2, we show that our construction codes the standard discrete lines and is equivalent to the well known palindromic closure in combinatorics on words.

Communicated by Pierre Liardet

Dedicated to the memory of Gérard RAUZY

1. Introduction

Laurent Vuillon met Gérard Rauzy for the first time in 1992 when he was a student in the DEA "Informatique et Mathématiques" at the University Aix-Marseille Luminy. After some lessons on automata and tilings, Professor Rauzy showed him a dragon that tiles the plane and which is constructed by a numerical system (see the chapter by Ch. Frougny in [17]). Gérard Rauzy told him also about his seminal paper "Nombres algébriques et substitutions" of 1982 [20] which is the starting point of many great developments in number theory, dynamical systems, theory of tilings and discrete geometry [1, 2, 3, 14]. A few months later, Gérard Rauzy gave him a thesis topic on the link between discrete plane, combinatorics and dynamical systems related to the so called Rauzy fractal [6]. During all the years of thesis, Laurent Vuillon noticed that the construction which leads to the Rauzy fractal by renormalisation or to a discrete plane linked to the famous sequence of Tribonacci by substitution on faces was not so far

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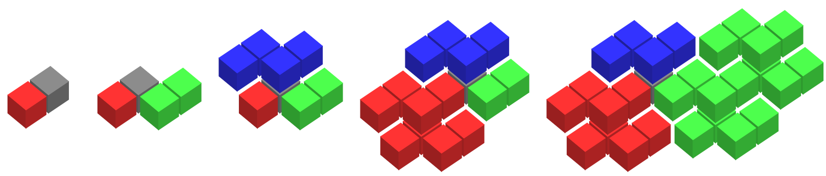
from a geometric palindromic operation. In particular, Rauzy, Ito, Kimura and Arnoux have developed a nice construction by substitution on faces linked to the Jacobi-Perron continued fraction algorithm in dimension 3 [2, 14]. In winter 2012, Eric Domenjoud participated in an annual winter working group in Laurent Vuillon's laboratory in Chambéry. He showed him an incremental construction of discrete hyperplanes by symmetry and translation. Then we realized that it was exactly the good object in order to generalize to higher dimensions the geometric interpretation of the iterated palindromic closure on words. This gave birth to this paper.

Example 1. *The 5 first steps of a 3 dimensional version of the construction of Rauzy for* $\Delta = (123)^\omega$.



In fact, in dimension 2 the geometric construction leads to standard discrete lines and the usual palindromic closure gives the coding to the adjacency of unit squares of these discrete lines. This operation is hidden in the palindromic closure used in combinatorics on words in order to construct the Arnoux-Rauzy sequences and the generalization called episturmian words [3, 9, 11] because the palindromic closure is done only by adding a new letter a on the right of a given word W and by constructing the shortest palindrome with Wa as a prefix. In our construction in 2 dimensions, according to the current letter in some *directive word*, we make either a geometric palindromic closure on the right or a geometric palindromic closure on the left around the origin. This construction is general and not so far from the original one of Rauzy described for the Tribonnacci case [20]. The work of Rauzy is always fascinating and performs mathematics on fundamental objects that gave 30 years later inspiring ideas and a huge range of new works on number theory, tiling theory and discrete geometry. In particular our work is related to the theory of tilings and to number theory [16, 21, 22].

Our construction is general and does not depend on the usual techniques to describe the geometric objects by numerical systems. We show that the geometric object has many nice properties in all dimensions. For example, at each step of the construction for the dimension d, we maintain d distinct unions of hypercubes that are also non-adjacent with the strong property of central symmetry of the whole object and of each union of hypercubes. In fact, we propose with this construction a kind of geometric generalization of Christoffel words in all dimensions [4, 8]. At the end of the paper we focus on the aspect of geometric palindromic transformation when $d = 2$. We show that our construction codes the standard discrete lines and, if we consider the adjacency of unit squares on the standard discrete lines, our construction is equivalent to the well known palindromic closure in combinatorics on words [9, 15].

2. Preliminaries

Let $d \geqslant 2$ be a positive integer and $\mathcal{D} = \{1, \ldots, d\}$. The canonical basis of $\mathbb{R}^d$ is denoted by $(e_1, \ldots, e_d)$ and $\langle ., . \rangle$ denotes the usual scalar product on $\mathbb{R}^d$.

Two distinct points x and y in $\mathbb{Z}^d$ are *neighbours* if $\|x - y\|_1 = 1$, i.e., $x - y = \pm e_i$ for some i in $\mathcal{D}$. A *path* in $\mathbb{Z}^d$ is a sequence $(x_1, \ldots, x_n)$ such that x_{i-1} and x_i are neighbours for each $i = 2, \ldots, n$. A subset S of $\mathbb{Z}^d$ is *connected* if and only if for all pairs of points x and y in S, there exists a path $x_1, \ldots, x_n$ in S such that $x_1 = x$ and $x_n = y$. Two disjoint subsets X and Y of $\mathbb{Z}^d$ are *adjacent* if there exist x in X and y in Y such that x and y are neighbours. A *circuit* is a path $x_1, \ldots, x_n$ such that $x_n = x_1$. It is *simple* if $x_1, \ldots, x_{n-1}$ are all distinct and it is *trivial* if $n = 1$.

The central symmetry with centre $c \in \mathbb{R}^d$ is denoted by sym_c and the translation with vector v is denoted by trans_v.

The set of finite words over an alphabet $\mathcal{A}$ is denoted by $\mathcal{A}^\star$ and the set of right infinite words over $\mathcal{A}$ is denoted by $\mathcal{A}^\omega$. The empty word is denoted by ϵ. Words are usually denoted by capital letters while individual letters in $\mathcal{A}$ are usually denoted by small ones. Thus $W = w_1 w_2 \cdots$. If W is a word in $\mathcal{A}^\star$ then $|W|$ is the length of W, $|W|_a$ is the number of occurrences of a in W and $\lambda_a(W)$ is the index of the last occurrence of a in W. If a does not occur in W then $\lambda_a(W) = 0$. If W belongs to $\mathcal{A}^\omega$ then $\lambda_a(W)$ still denotes the index of the last occurrence of a in W. If a occurs infinitely many times in W then $\lambda_a(W) = \infty$. If $W = w_1 \cdots w_n$ is a word in $\mathcal{A}^\star$ then $\widetilde{W}$ denotes the *reversed word* of W, i.e., $\widetilde{W} = w_n \cdots w_1$. W is a *palindrome* if $\widetilde{W} = W$.

The set of right infinite words over the alphabet $\{0, 1\}$ containing finitely many 1's is denoted by $\{0, 1\}^\star 0^\omega$. For A in $\{0, 1\}^\star \cup \{0, 1\}^\star 0^\omega$, $\overline{A}$ is the bitwise negation of A, that is the word obtained by replacing each 0 with 1 and each 1 with 0. If A and B belong to $\{0, 1\}^n$ with n in $\mathbb{N} \cup \{\omega\}$, then $A \backslash B$ denotes $A \wedge \overline{B}$, where $\wedge$ is the bitwise *and*. $A \backslash B$ is obtained by replacing a_i with 0 if $b_i = 1$.

3. Geometric palindromic closure

We define a sequence $(S_n)_{n\geqslant 0}$ of subsets of $\mathbb{Z}^d$ where each S_n is defined from the previous one through a symmetry operation with respect to some centre. The resulting sets enjoy some nice symmetry and auto-similarity properties.

Let $\Delta = (\delta_n)_{n\geqslant 1} \in \mathcal{D}^\omega$ be an infinite word over $\mathcal{D}$, called the *directive word*. The sequences $(S_n)_{n\geqslant 0}$, $(X_n)_{n\geqslant 0}$ and for each $i \in \mathcal{D}$, $(T_n^{(i)})_{n\geqslant 0}$ and $(Y_n^{(i)})_{n\geqslant 0}$ are defined as follows: $S_0 = \{0\} \subset \mathbb{Z}^d$, $X_0 = 0 \in \mathbb{Z}^d$, for each $i \in \mathcal{D}$, $T_0^{(i)} = \emptyset$, $Y_0^{(i)} = \frac{1}{2}e_i$ and for all $n \geqslant 1$,

$$\begin{aligned}
Y_n^{(i)} &= \begin{cases} \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(X_{n-1}) & \text{if } i = \delta_n; \\ Y_{n-1}^{(i)} & \text{if } i \neq \delta_n. \end{cases} \\
T_n^{(i)} &= \begin{cases} \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1}) & \text{if } i = \delta_n; \\ T_{n-1}^{(i)} & \text{if } i \neq \delta_n. \end{cases} \\
X_n &= Y_{n-1}^{(\delta_n)}. \\
S_n &= S_{n-1} \cup \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1}) \\
&= S_{n-1} \cup \mathrm{sym}_{X_n}(S_{n-1})\,.
\end{aligned}$$

Figure 1 below schematizes one step of this construction with $\delta_n = 2$ and Figure 2 shows a real object obtained for $d = 3$, $\Delta = (123)^\omega$ and $n = 10$. One recognizes an object which is very similar to the Rauzy fractal. Figure 3 shows S_{15} for $d = 3$ and $\Delta = (111222333)^\omega$.

The above construction does build subsets of $\mathbb{Z}^d$. Indeed, this is obvious for $n = 0$ since $S_0 = \{0\}$ and $T_0^{(i)} = \emptyset$. Now, if $c \in \left(\frac{1}{2}\mathbb{Z}\right)^d$ and $x \in \mathbb{Z}^d$ then $\mathrm{sym}_c(x) = 2c - x \in \mathbb{Z}^d$. Hence for $n \geqslant 1$, if $S_{n-1} \subset \mathbb{Z}^d$, $X_{n-1} \in \frac{1}{2}\mathbb{Z}^d$ and for each i in $\mathcal{D}$, $T_{n-1}^{(i)} \in \mathbb{Z}^d$ and $Y_{n-1}^{(i)} \in \left(\frac{1}{2}\mathbb{Z}\right)^d$ then $S_n \subset \mathbb{Z}^d$, $X_n \in \frac{1}{2}\mathbb{Z}^d$ and for each i in $\mathcal{D}$, $T_n^{(i)} \in \mathbb{Z}^d$ and $Y_n^{(i)} \in \left(\frac{1}{2}\mathbb{Z}\right)^d$. Since $X_0 \in \frac{1}{2}\mathbb{Z}^d$ and for each i in $\mathcal{D}$, $Y_0^{(i)} = \frac{1}{2}e_i \in \left(\frac{1}{2}\mathbb{Z}\right)^d$, the property holds for all n by induction.

These objects have nice symmetry properties. Indeed for all $n \geqslant 0$, S_n is symmetric with respect to X_n, and for each i in $\mathcal{D}$, $T_n^{(i)}$ is symmetric with respect to $Y_n^{(i)}$. This is obvious for $n = 0$ since $S_0 = \{0\}$ and $T_0^{(i)}$ is empty for each i. For $n \geqslant 1$, S_n equals $S_{n-1} \cup \mathrm{sym}_{X_n}(S_{n-1})$ which is symmetric with respect to X_n by construction. If $i = \delta_n$ then $T_n^{(i)} = \mathrm{sym}_{Y_{n-1}^{(i)}}(S_{n-1})$ and, since S_{n-1} is symmetric with respect to X_{n-1}, $\mathrm{sym}_{Y_{n-1}^{(i)}}(S_{n-1})$ is symmetric with respect to

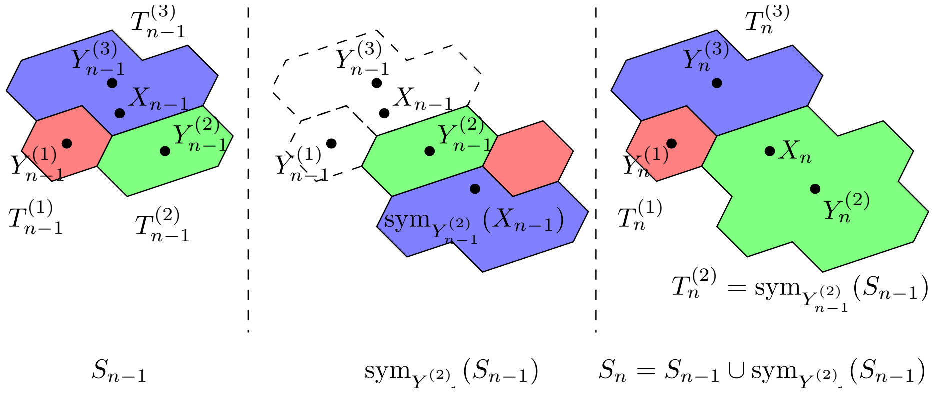


FIGURE 1. Construction of S_n from S_{n-1} when $\delta_n = 2$.

$\mathrm{sym}_{Y^{(i)}_{n-1}}(X_{n-1})$, that is $Y^{(i)}_n$. If $i \neq \delta_n$ then $T^{(i)}_n = T^{(i)}_{n-1}$ and $Y^{(i)}_n = Y^{(i)}_{n-1}$ and the property holds by induction.

The sequences $(S_n)_{n\geqslant 0}$ and $(T^{(i)}_n)_{n\geqslant 0}$ are increasing. Indeed, by construction, we have $S_{n-1} \subset S_n$ for all $n \geqslant 1$. Now, for $n \geqslant 1$ and $i \in \mathcal{D}$, assume $T^{(i)}_{n-1} \subset S_{n-1}$, which is true for $n = 1$. Either $i \neq \delta_n$ and $T^{(i)}_n = T^{(i)}_{n-1} \subset S_{n-1}$ or $i = \delta_n$ and $T^{(i)}_n = \mathrm{sym}_{Y^{(i)}_{n-1}}(S_{n-1})$. Since $T^{(i)}_{n-1}$ is symmetric with respect to $Y^{(i)}_{n-1}$ and $T^{(i)}_{n-1} \subset S_{n-1}$, we have $T^{(i)}_{n-1} = \mathrm{sym}_{Y^{(i)}_{n-1}}(T^{(i)}_{n-1}) \subset \mathrm{sym}_{Y^{(i)}_{n-1}}(S_{n-1}) = T^{(i)}_n$. In addition, in both cases $T^{(i)}_n \subset S_{n-1} \cup \mathrm{sym}_{Y^{(\delta_n)}_{n-1}}(S_{n-1}) = S_n$ hence the result by induction. Since $(S_n)_{n\geqslant 0}$ and $(T^{(i)}_n)_{n\geqslant 0}$ are increasing, it makes sense to consider their limits $S_\infty = \lim_{n\to\infty} S_n$ and $T^{(i)}_\infty = \lim_{n\to\infty} T^{(i)}_n$ for $i \in \mathcal{D}$.

As could be suspected from the previous examples, S_n is more or less composed of $T^{(i)}_n$'s. More precisely:

LEMMA 2. *For all $n \geqslant 0$, we have $S_n = \{0\} \cup T^{(1)}_n \cup \cdots \cup T^{(d)}_n$.*

Proof. The property obviously holds for $n = 0$. For $n \geqslant 1$, if we assume that it holds for $n - 1$ then we get

$$\begin{aligned} S_n &= S_{n-1} \cup \mathrm{sym}_{Y^{(\delta_n)}_{n-1}}(S_{n-1}) \\ &= \{0\} \cup T^{(1)}_{n-1} \cup \cdots \cup T^{(d)}_{n-1} \cup \mathrm{sym}_{Y^{(\delta_n)}_{n-1}}(S_{n-1}) \\ &= \{0\} \cup T^{(1)}_{n-1} \cup \cdots \cup T^{(d)}_{n-1} \cup T^{(\delta_n)}_n \end{aligned}$$

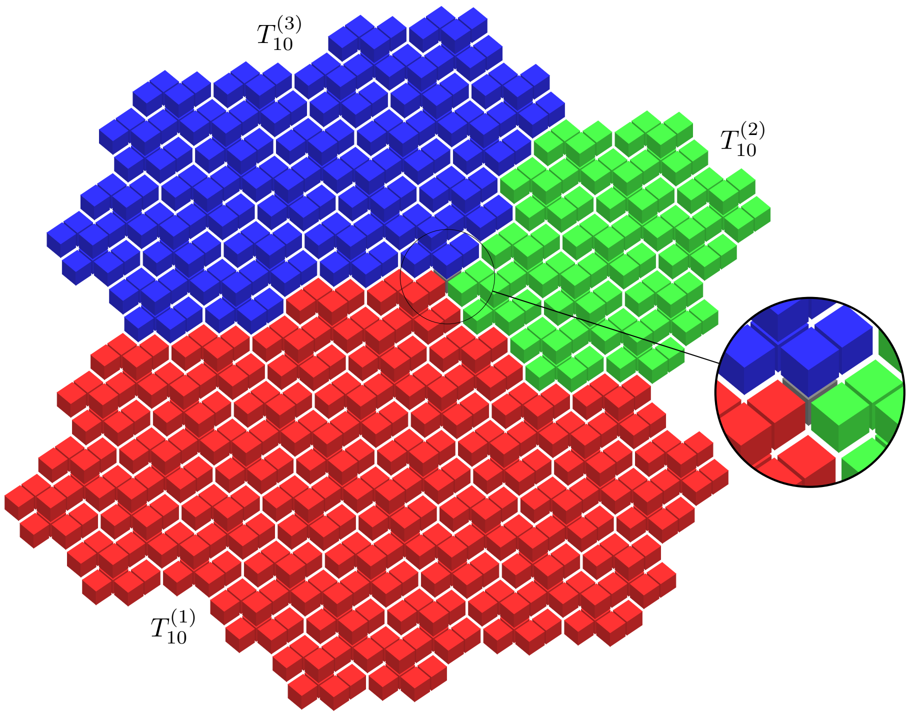


Figure 2. S_{10} for $d = 3$ and $\Delta = (123)^\omega$. The (hardly visible) cube which joins $T_{10}^{(1)}$, $T_{10}^{(2)}$ and $T_{10}^{(3)}$ is the origin.

$$= \quad \{0\} \cup T_{n-1}^{(1)} \cup \cdots \cup \left(T_{n-1}^{(\delta_n)} \cup T_n^{(\delta_n)}\right) \cup \cdots \cup T_{n-1}^{(d)}\,.$$

For $i \neq \delta_n$, we have $T_n^{(i)} = T_{n-1}^{(i)}$ and since $(T_n^{(i)})_{n\geqslant 0}$ is increasing for each i we have $T_{n-1}^{(\delta_n)} \cup T_n^{(\delta_n)} = T_n^{(\delta_n)}$. Thus $S_n = \{0\} \cup T_n^{(1)} \cup \cdots \cup T_n^{(d)}$. □

It will be seen further that these components are actually disjoint but this requires more tools which will be introduced in the next sections. Some topological results may nevertheless already be proven from the definition.

Lemma 3. *For all $n \geqslant 0$, S_n is connected and for each i in $\mathcal{D}$, $T_n^{(i)}$ is connected.*

Proof. If S_{n-1} is connected, then $T_n^{(\delta_n)} = \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1})$ obviously also is. S_0 is obviously connected. For $n \geqslant 1$, assume that S_{n-1} is connected. Since

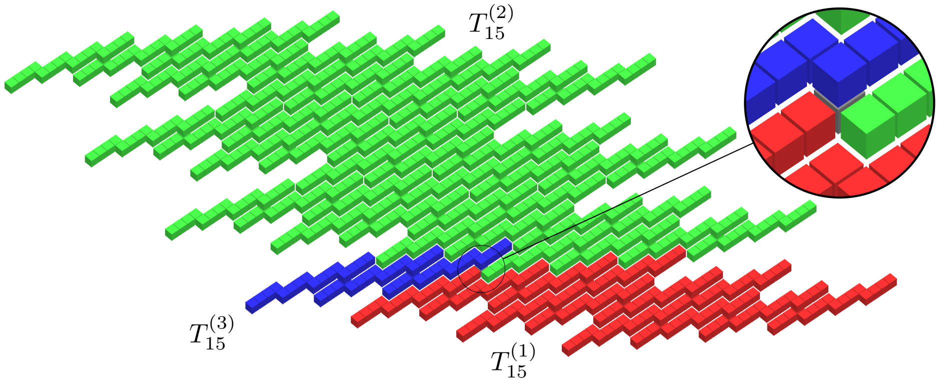


FIGURE 3. S_{15} for $d = 3$ and $\Delta = (111222333)^{\omega}$.

$T_{n-1}^{(\delta_n)}$ is symmetric with respect to $Y_{n-1}^{(\delta_n)}$ and $T_{n-1}^{(\delta_n)} \subset S_{n-1}$, we have $T_{n-1}^{(\delta_n)} = \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(T_{n-1}^{(\delta_n)}) \subset \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1})$. Therefore $T_{n-1}^{(\delta_n)} \subset S_{n-1} \cap \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1})$. If $T_{n-1}^{(\delta_n)} \neq \emptyset$ then $S_n = S_{n-1} \cup \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1})$ is connected. If on the contrary $T_{n-1}^{(\delta_n)} = \emptyset$, this implies $\delta_m \neq \delta_n$ for all $m < n$. In this case, $Y_{n-1}^{(\delta_n)} = Y_0^{\delta_n} = \frac{1}{2}e_{\delta_n}$. We have $0 \in S_{n-1}$ and $\mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(0) = e_{\delta_n}$. Since e_{δ_n} is a neighbour of 0, either S_{n-1} and $\mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1})$ have a non-empty intersection or they are adjacent. In both cases it implies that $S_n = S_{n-1} \cup \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1})$ is connected. $\square$

4. Generation by translations

If a subset E of $\mathbb{R}^d$ is symmetric with respect to c then $\mathrm{sym}_x(E) = \mathrm{trans}_{2(x-c)}(E)$ for all x. Therefore the definition of S_n, X_n, $T_n^{(i)}$ and $Y_n^{(i)}$ may be reformulated in terms of translations.

$$Y_n^{(i)} = \begin{cases} 2Y_{n-1}^{(\delta_n)} - X_{n-1} & \text{if } i = \delta_n\,; \\ Y_{n-1}^{(i)} & \text{if } i \neq \delta_n\,. \end{cases}$$

$$T_n^{(i)} = \begin{cases} \mathrm{trans}_{2\left(Y_{n-1}^{(\delta_n)} - X_{n-1}\right)}(S_{n-1}) & \text{if } i = \delta_n\,; \\ T_{n-1}^{(i)} & \text{if } i \neq \delta_n\,. \end{cases}$$

$$\begin{aligned} X_n &= Y_{n-1}^{(\delta_n)}. \\ S_n &= S_{n-1} \cup \operatorname{trans}_{2\left(Y_{n-1}^{(\delta_n)} - X_{n-1}\right)}(S_{n-1}) \\ &= S_{n-1} \cup \operatorname{trans}_{2(X_n - X_{n-1})}(S_{n-1})\,. \end{aligned}$$

If we set $U_n = 2(X_n - X_{n-1})$ for $n \geqslant 1$ then the recursive definition of S_n becomes $S_n = S_{n-1} \cup \operatorname{trans}_{U_n} S_{n-1}$. This means that S_n contains the elements of S_{n-1} and the same translated by U_n. The sets S_n and $T_n^{(i)}$ may then be characterized as follows.

Theorem 4. *For all $n \geqslant 0$ and all $i \in \mathcal{D}$,*

$$S_n = \left\{ \sum_{j \in J} U_j \mid J \subset \{1, \ldots, n\} \right\} \tag{1}$$

$$T_n^{(i)} = \left\{ \sum_{j \in J} U_j \mid J \subset \{1, \ldots, n\},\ J \neq \emptyset \text{ and } \delta_{\max(J)} = i \right\}. \tag{2}$$

Proof. Since $S_0 = \{0\}$ and for $n \geqslant 1$, $S_n = S_{n-1} \cup \operatorname{trans}_{U_n}(S_{n-1})$, the first equality is immediate by induction on n. Now, let $x = \sum_{j \in J} U_j$ where $J \subset \{1, \ldots, n\}$ and $J \neq \emptyset$ and let $k = \max(J)$. We have $x - U_k \in S_{k-1}$ and therefore $x \in \operatorname{trans}_{U_k}(S_{k-1})$. Since $\operatorname{trans}_{U_k}(S_{k-1}) = T_k^{(\delta_k)}$ and $T_k^{(\delta_k)} \subset T_n^{(\delta_k)}$, we get $x \in T_n^{(\delta_k)}$. Conversely, if $\delta_m \neq i$ for all $m \leqslant n$, then $T_n^{(i)} = T_0^{(i)} = \emptyset$ and the result is trivial. Otherwise, let $x \in T_n^{(i)}$. We have $T_n^{(i)} = T_{n_0}^{(i)}$ where $n_0 = \max\{m \mid m \leqslant n \text{ and } \delta_m = i\}$. Thus $x \in T_{n_0}^{(\delta_{n_0})}$ and since $T_{n_0}^{(\delta_{n_0})} = \operatorname{trans}_{U_{n_0}}(S_{n_0-1})$, we have $x = U_{n_0} + \sum_{j \in J} U_j$ for some $J \subset \{1, \ldots, n_0 - 1\}$. Therefore $x = \sum_{j \in J'} U_j$ with $J' \subset \{1, \ldots, n\}$, $J' \neq \emptyset$ and $\max(J') = n_0$ which implies $\delta_{\max(J')} = i$. □

In the following, we study the properties of $(U_n)_{n \geqslant 1}$.

Lemma 5. *If $\delta_j = \delta_i$ for some $j > i$ and $\delta_k \neq \delta_i$ for all $k = i+1, \ldots, j-1$, then $U_{i+1} + \cdots + U_j = U_i$.*

Proof.
$$\begin{aligned} U_j + \cdots + U_{i+1} &= 2X_j - 2X_{j-1} + \cdots + 2X_{i+1} - 2X_i \\ &= 2X_j - 2X_i \\ &= 2Y_{j-1}^{\delta_j} - 2X_i \\ &= 2Y_{j-1}^{\delta_i} - 2X_i \qquad \text{since } \delta_j = \delta_i \\ &= 2Y_i^{\delta_i} - 2X_i \qquad \text{since } \delta_{i+1}, \ldots, \delta_{j-1} \neq \delta_i \\ &= 2(2Y_{i-1}^{\delta_i} - X_{i-1}) - 2X_i \\ &= 2(2X_i - X_{i-1}) - 2X_i \\ &= 2X_i - 2X_{i-1} \\ &= U_i\,. \end{aligned}$$

□

Lemma 6. *For each k occurring in Δ, if n_k is the smallest index such that $\delta_{n_k} = k$ then $U_1 + \cdots + U_{n_k} = e_k$.*

Proof.
$$\begin{aligned} U_1 + \cdots + U_{n_k} &= 2(X_1 - X_0) + 2(X_2 - X_1) + \ldots. + 2(X_{n_k} - X_{n_k-1}) \\ &= 2X_{n_k} \\ &= 2Y_{n_k-1}^{(k)} \\ &= 2Y_0^{(k)} \qquad \text{since } \delta_1, \ldots, \delta_{n_k-1} \neq k \\ &= e_k\,. \end{aligned}$$

□

We consider now for each $k \in \mathcal{D}$, the linear map σ_k defined by

$$\begin{aligned} \sigma_k : \mathbb{R}^d &\to \mathbb{R}^d \\ (x_1, \ldots, x_d) &\mapsto (x_1 - x_k, \ldots, x_{k-1} - x_k, x_k, x_{k+1} - x_k, \ldots, x_d - x_k) \end{aligned}$$

or equivalently

$$v \mapsto v - \langle v, e_k \rangle \textstyle\sum_{j \neq k} e_j\,.$$

We have for all $i, k \in \mathcal{D}$

$$\sigma_k(e_i) = \begin{cases} e_i - \sum_{j \neq k} e_j & \text{if } i = k\,; \\ e_i & \text{if } i \neq k\,; \end{cases}$$

$${}^t\sigma_k(e_i) = \begin{cases} e_i & \text{if } i = k\,; \\ e_i - e_k & \text{if } i \neq k\,; \end{cases}$$

$$\sigma_k^{-1}(e_i) = \begin{cases} e_i + \sum_{j \neq k} e_j = \sum_{j=1}^d e_j & \text{if } i = k\,; \\ e_i & \text{if } i \neq k\,. \end{cases}$$

Then U_n may be expressed by means of σ_i's as follows:

Lemma 7. *For all $n \geqslant 1$, we have $U_n = {}^t\sigma_{\delta_1} \cdots {}^t\sigma_{\delta_{n-1}}(e_{\delta_n})$.*

Proof. We set for all $n \geqslant 0$ and $i \in \mathcal{D}$, $V_n^{(i)} = 2(Y_n^{(i)} - X_n)$. Then for each i in $\mathcal{D}$, $V_0^{(i)} = e_i$ and for all $n \geqslant 1$,

$$V_n^{(i)} = \begin{cases} V_{n-1}^{(i)} - V_{n-1}^{(\delta_n)} & \text{if } i \neq \delta_n\,; \\ V_{n-1}^{(i)} & \text{if } i = \delta_n\,. \end{cases}$$

We have also $U_n = V_{n-1}^{\delta_n}$ for all $n \geqslant 1$. Let us consider the $d\times d$ matrix the columns of which are $V_n^{(1)}, \ldots, V_n^{(d)}$. We have $(V_0^{(1)} \cdots V_0^{(d)}) = (e_1 \cdots e_d) = Id_d$ and for all $n \geqslant 1$,

$$\begin{aligned} &\big(V_n^{(1)} \cdots V_n^{(d)}\big) \\ &\quad= \big((V_{n-1}^{(1)} - V_{n-1}^{(\delta_n)}) \cdots (V_{n-1}^{(\delta_n-1)} - V_{n-1}^{(\delta_n)})\ V_{n-1}^{(\delta_n)}\ (V_{n-1}^{(\delta_n+1)} - V_{n-1}^{(\delta_n)}) \cdots \\ &\qquad\qquad (V_{n-1}^{(d)} - V_{n-1}^{(\delta_n)})\big) \\ &\quad= \big(V_{n-1}^{(1)} \cdots V_{n-1}^{(d)}\big)\,{}^t\Sigma_{\delta_n} \end{aligned}$$

where Σ_i is the matrix of σ_i in the canonical basis. Thus $(V_n^{(1)} \cdots V_n^{(d)}) = (e_1 \cdots e_d)\,{}^t\Sigma_{\delta_1} \cdots {}^t\Sigma_{\delta_n} = {}^t\Sigma_{\delta_1} \cdots {}^t\Sigma_{\delta_n}$. Hence $U_n = V_{n-1}^{(\delta_n)} = {}^t\sigma_{\delta_1} \cdots {}^t\sigma_{\delta_{n-1}}(e_{\delta_n})$. □

Lemma 8. *Let $\mathcal{C}^+$ be the nonnegative orthant of $\mathbb{R}^d$, i.e., the closed cone generated by $e_1, \ldots, e_d$, and let $\mathcal{C}_n = \sigma_{\delta_1}^{-1} \cdots \sigma_{\delta_{n-1}}^{-1}(\mathcal{C}^+)$. If v is a vector in the interior of $\mathcal{C}_n$ then $\langle v, U_i\rangle > 0$ for all i in $\{1, \ldots, n\}$.*

Proof. We have $\langle v, U_i\rangle = \langle v, {}^t\sigma_{\delta_1} \cdots {}^t\sigma_{\delta_{i-1}}(e_{\delta_i})\rangle = \langle \sigma_{\delta_{i-1}} \cdots \sigma_{\delta_1}(v), e_{\delta_i}\rangle$. For each $i \in \mathcal{D}$, we have $\sigma_i^{-1}(\mathcal{C}^+) \subset \mathcal{C}^+$ so that $\sigma_{\delta_{i-1}} \cdots \sigma_{\delta_1}(v)$ is in the interior of $\mathcal{C}^+$ and the last scalar product is positive. □

Lemma 9. *Let J be a finite non-empty subset of $\mathbb{N}^\star$. Then $\sum_{j\in J} U_j \neq 0$.*

Proof. Let $U = \sum_{j\in J} U_j$, $n = \max(J)$ and v be such as above. We have $\langle v, U\rangle = \sum_{j\in J}\langle v, U_j\rangle > 0$ so that $U \neq 0$. □

As an interesting corollary of these lemmas, we get

Corollary 10. *For each $x \in S_\infty\backslash\{0\}$, there exists a positive vector v such that $\langle v, x\rangle > 0$.*

P r o o f. Let n be such that $x \in S_n$. From Theorem 4, we have $x = \sum_{j \subset J} U_j$ for some $J \subset \{1, \ldots, n\}$. Since $x \neq 0$, from Lemma 9, we have $J \neq \emptyset$. If we take v as defined in Lemma 8, then we get $\langle v, x \rangle = \sum_{j \in J} \langle v, U_j \rangle > 0$. □

5. Normalized representation

We have seen in the previous section that any $x \in S_\infty$ may be written as a finite sum of U_i's. However, Lemma 5 shows that this writing is not unique. This lemma provides us nevertheless with a way to compute for each $x \in S_\infty$ a normalized writing as a sum of U_i's.

Lemma 11. *Any element x of S_∞ may be written as $x = \sum_{i=1}^{n} \epsilon_i U_i$ for some n, with $\epsilon_i \in \{0,1\}$ and if $\delta_j = \delta_i$ with $i < j \leqslant n$, and $\epsilon_i = 0$ then $\epsilon_{i+1} \times \cdots \times \epsilon_j = 0$.*

P r o o f. Any element x of S_∞ belongs to some S_n and may obviously be written as $x = \sum_{i=1}^{n} \epsilon_i U_i$ where $\epsilon_i \in \{0,1\}$. If this writing does not have the property of the lemma then let j_0 be the smallest index strictly greater than i such that $\delta_{j_0} = \delta_i$. We have $j_0 \leqslant j$ and by Lemma 5, $U_{i+1} + \cdots + U_{j_0} = U_i$. In the writing of x, we may replace $U_{i+1} + \cdots + U_{j_0}$ with U_i. Doing so, the word $\epsilon_1 \cdots \epsilon_n$ decreases in the lexicographic ordering induced by $1 < 0$ so that the process terminates and yields a writing of x with the desired property. □

Considering the word $\epsilon_1 \cdots \epsilon_n$ is a convenient way to represent a sum of U_i's as a word in $\{0,1\}^\star$. However it has the drawback that for any $A \in \{0,1\}^\star$, A and $A0^k$ represent actually the same sum hence an ambiguity. Therefore in the sequel, we rather represent such sums by words in $\{0,1\}^\star 0^\omega$, i.e., infinite words on $\{0,1\}$ with finitely many 1's. Then we define a valuation function φ on $\{0,1\}^\star 0^\omega$ by

$$\begin{array}{rccl} \varphi & : & \{0,1\}^\star 0^\omega & \to \mathbb{R}^d \\ & & A & \mapsto \displaystyle\sum_{i=1}^{\infty} a_i U_i = \sum_{i \mid a_i = 1} U_i \end{array}$$

where the sum is actually finite because A contains finitely many 1's. The valuation function has the fairly obvious following properties:

- $\forall A \in \{0,1\}^\star 0^\omega,\ \varphi(A) = 0 \iff A = 0^\omega$;
- $\forall A, B \in \{0,1\}^\star 0^\omega,\ \varphi(A) = \varphi(B) \iff \varphi(A \backslash B) = \varphi(B \backslash A)$.

The first one is an immediate consequence of Lemma 9. The second one follows from the fact that $\varphi(A) = \varphi(A\backslash B) + \varphi(A \wedge B)$ for any A and B. Indeed, $A\backslash B$ is defined as $A \wedge \overline{B}$. If, given a word W, we consider the set I_W of indices for which $w_i = 1$, then $\varphi(W) = \sum_{i\in I_W} U_i$. Moreover, $I_{A\backslash B} = I_A\backslash I_B$ and $I_{A\wedge B} = I_A \cap I_B$. Since for any two sets X and Y we have $X = (X\backslash Y) \cup (X \cap Y)$ and the union is disjoint, we get $\varphi(A) = \sum_{i\in I_A} U_i = \sum_{i\in I_A\backslash I_B} U_i + \sum_{i\in I_A\cap I_B} U_i = \varphi(A\backslash B) + \varphi(A \wedge B)$.

We say that a word A in $\{0,1\}^\star \cup \{0,1\}^\star 0^\omega$ is *normalized* (with respect to Δ) if it does not contain a subword of the form $a_i \cdots a_j = 01^{j-i}$ with $i < j$ and $\delta_i = \delta_j$. Given a word A, we compute as follows a normalized word, called *the normal form of* A and denoted by $A{\downarrow}$, such that $\varphi(A{\downarrow}) = \varphi(A)$. We replace iteratively any subword $a_i \cdots a_j = 01^{j-i}$ such that $i < j$, $\delta_i = \delta_j$ and for all $k \in \{i+1, \ldots, j-1\}$, $\delta_k \neq \delta_i$ with 10^{j-i}. Below is shown the result of applying one step of this normalization process to a word A, yielding A'.

$$\begin{array}{lclccccl} & & & i & & j & & \\ \Delta & = & \cdots & t & \Delta' & t & \cdots & \qquad \text{where } \Delta' \in (\mathcal{D}\backslash\{t\})^\star \\ A & = & \cdots & 0 & 1^{|\Delta'|} & 1 & \cdots & \\ A' & = & \cdots & 1 & 0^{|\Delta'|} & 0 & \cdots & \end{array}$$

The normalization process terminates because at each step, $\lambda_1(A)$ does not increase and A decreases in the lexicographic ordering induced by $1 < 0$. The result of this process does not depend on the order of application of the reductions because two reducible subwords in A are necessarily disjoint. Hence the normal form is well defined. By a slight abuse of notation, we write $AB{\downarrow}C$ to denote the word obtained by applying to ABC all possible reductions in the subword B. Then we have $(ABC){\downarrow} = (AB{\downarrow}C){\downarrow}$.

One important property of the normal form is

Lemma 12. *For all $A \in \{0,1\}^\star 0^\omega$ such that $A \neq 0^\omega$, we have $\delta_{\lambda_1(A\downarrow)} = \delta_{\lambda_1(A)}$.*

Proof. The only case where $\lambda_1(A)$ changes during the normalization process is when $\Delta = \Delta_0 t \Delta_1 t \Delta_2$ with $\Delta_1 \in (\mathcal{D}\backslash\{t\})^\star$ and $A = A_0 0 1^{|\Delta_1|} 1 0^\omega$ with $|A_0| = |\Delta_0|$. Then A is replaced with $A' = A_0 1 0^\omega$ and $\lambda_1(A') = |A_0| + 1$, $\lambda_1(A) = |A_0| + 1 + k$ and $\delta_{\lambda_1(A')} = \delta_{|A_0|+1} = t = \delta_{|A_0|+1+k} = \delta_{\lambda_1(A)}$. This is shown below (trailing 0's are omitted).

$$\begin{array}{lcccccl} \Delta & = & \Delta_0 & t & \Delta_1 & t & \Delta_2 \qquad \text{where } \Delta_1 \in (\mathcal{D}\backslash\{t\})^\star \\ A & = & A_0 & 0 & 1^{|\Delta_1|} & 1 & \\ A' & = & A_0 & 1 & & & \end{array}$$

□

If A represents a sum of U_i's, then $A{\downarrow}$ represents the normalized sum defined in Lemma 11 which means that the following properties hold:

- $\forall A \in \{0,1\}^\star 0^\omega,\ \varphi(A{\downarrow}) = \varphi(A)$;
- $\forall A, B \in \{0,1\}^\star 0^\omega,\ A{\downarrow} = B{\downarrow} \implies \varphi(A) = \varphi(B)$.

The first one is an immediate consequence of Lemma 5 and entails the second one. However, in order to be able to compare effectively two elements of S_∞ we need actually the converse of the second property above. The rest of this section is devoted to proving the following theorem:

Theorem 13. $\forall A, B \in \{0,1\}^\star 0^\omega,\ A{\downarrow} = B{\downarrow} \iff \varphi(A) = \varphi(B)$.

Although this result might seem simple at the first sight, the proof is rather technical. Before giving this proof, let us first explain the ideas behind it. Since it was already established that $A{\downarrow} = B{\downarrow}$ implies $\varphi(A) = \varphi(B)$, we have to prove the converse. To this end, we build two sequences of words $(A_n)_{n\geqslant 0}$ and $(B_n)_{n\geqslant 0}$ such that $B_0 = B{\downarrow}$, $A_0 = A{\downarrow}$ and for each n, on one hand $A_n{\downarrow} = B_n{\downarrow}$ implies $A{\downarrow} = B{\downarrow}$ and on the other hand $\varphi(A_n) = \varphi(B_n)$ is equivalent to $\varphi(A) = \varphi(B)$. The construction has the property that at some point, we must have $\min(\lambda_1(A_n), \lambda_1(B_n)) < \min(\lambda_1(A{\downarrow}), \lambda_1(B{\downarrow}))$ so that eventually, a point is reached where $\min(\lambda_1(A_n), \lambda_1(B_n)) = 0$. This means either $A_n = 0^\omega$ or $B_n = 0^\omega$. Say $A_n = 0^\omega$ which is equivalent to $\varphi(A_n) = 0$ by Lemma 9. Then $\varphi(A) = \varphi(B)$ is equivalent by construction to $\varphi(A_n) = \varphi(B_n)$ which means $0^\omega = B_n$ because $A_n = 0^\omega$. Then we have $A_n{\downarrow} = B_n{\downarrow}$ which, by construction, implies $A{\downarrow} = B{\downarrow}$.

In effect, we compute a word C such that either $\varphi(C) = \varphi(A) - \varphi(B)$ or $\varphi(C) = \varphi(B) - \varphi(A)$. The construction makes a heavy use of the following property

$$\forall A, B \in \{0,1\}^\star 0^\omega,\ (A\backslash B){\downarrow} = (B\backslash A){\downarrow} \implies A{\downarrow} = B{\downarrow} \tag{3}$$

which will be obtained as a corollary of Lemma 14 below. The idea behind this property is to mimic the property of φ that was mentioned earlier, namely

$$\varphi(A) = \varphi(B) \iff \varphi(A\backslash B) = \varphi(B\backslash A)\,.$$

The construction works as follows. If $i_0 = \lambda_1(B) < \lambda_1(A)$ and j_0 is the index of the next occurrence of δ_{i_0} in Δ then B is *denormalized* by replacing its last 1 with $01^{j_0-i_0}$. Then A and B are replaced with $A\backslash B$ and $B\backslash A$. This last step *erases* all the 1's in A between the position i_0 and j_0 while B grows. Repeating this process will eventually erase all 1's in A starting from $\lambda_1(B_0)$, yielding an A *shorter* than B_0.

Lemma 14. *For all $A_1, B_1 \in \{0,1\}^\star$ such that $|A_1| = |B_1|$, $A_2, B_2 \in \{0,1\}^\star 0^\omega$, we have*

$$(A_1 0 A_2){\downarrow} = (B_1 0 B_2){\downarrow} \implies (A_1 1 A_2){\downarrow} = (B_1 1 B_2){\downarrow}\,.$$

Proof. Let $i_0 = |A_1| + 1$, $A = A_1 0 A_2$, $B = B_1 0 B_2$, $A' = A_1 1 A_2$, $B' = B_1 1 B_2$ and assume $A{\downarrow} = B{\downarrow}$ while $A'{\downarrow} \neq B'{\downarrow}$. Since $(XxY){\downarrow} = (X{\downarrow}xY{\downarrow}){\downarrow}$ we may assume that A_1, A_2, B_1, B_2 are normalized. A and B cannot be both normalized because we would have $A_1 = B_1$ and $A_2 = B_2$ hence $A' = B'$ and $A'{\downarrow} = B'{\downarrow}$. Without loss of generality, assume that A is reducible. It cannot contain a reducible subword $a_i \cdots a_j$ with $i < i_0 \leqslant j$ because we would have $a_{i_0} = 1$. Since A_1 and A_2 are normalized, the only possible reduction in A takes place at the position i_0. We consider in A the largest subword $a_i \cdots a_j$ such that $j > i$, $\delta_i = \delta_j = t$, $a_i = 0$, $a_j = 1$ and $a_k = 1$ for all $k \in \{i+1, \ldots, j-1\}$ such that $\delta_k \neq t$. Such a subword always exists since the reducible subword at position i_0 must be of this form. Then Δ and A may be written as follows:

$$\begin{array}{ccccccccc} & & & i & \leqslant & i_0 & < & j & \\ \Delta & = & \Delta' & t & t^{\alpha_0} & \prod_{k=1}^{n}(\Delta_k t^{\alpha_k}) & & t & \Delta'' \\ A & = & P & 0 & \mu_0 & \prod_{k=1}^{n}(1^{|\Delta_k|}\mu_k) & & 1 & S \end{array}$$

where

- $n \geqslant 0$,
- $\Delta_1, \ldots, \Delta_n \in (\mathcal{D}\backslash\{t\})^+$,
- if Δ' ends with t^β then P ends with 1^β and if Δ'' starts with t^γ then S starts with 0^γ,
- $\alpha_0, \alpha_n \geqslant 0$ and $\alpha_1, \ldots, \alpha_{n-1} \geqslant 1$,
- $\mu_k \in \{0,1\}^{\alpha_k}$ for $k = 0, \ldots, n$.

Since $a_{i_0} = 0$, the position i_0 must be in some μ'_m where $\mu'_0 = 0\mu_0$ and $\mu'_m = \mu_m$ if $1 \leqslant m \leqslant n$. Since A_1 and A_2 are normalized, we have

$$\mu_k = \begin{cases} 0^{\alpha_k} & \text{if } 0 \leqslant k < m \\ 0^\theta 1^{\alpha_m - \theta} & \text{if } k = m \text{ with } \begin{cases} 0 \leqslant \theta \leqslant \alpha_m & \text{if } m = 0 \\ 1 \leqslant \theta \leqslant \alpha_m & \text{if } m \geqslant 1 \end{cases} \\ 1^{\alpha_k} & \text{if } m < k \leqslant n\,. \end{cases}$$

If $m > 0$ then

$$\begin{array}{ccccccc} & & & & i_0 & & \\ \Delta & = & \Delta'\ t\ \prod_{k=1}^{m}(t^{\alpha_{k-1}}\Delta_k) & t^{\theta-1} & t & t^{\alpha_m-\theta}\ \prod_{k=m+1}^{n}(\Delta_k t^{\alpha_k}) & t\ \Delta'' \\ A & = & \underbrace{P\ \ 0 \prod_{k=1}^{m}(0^{\alpha_{k-1}}1^{|\Delta_k|})\ 0^{\theta-1}}_{A_1} & & 0 & \underbrace{1^{\alpha_m-\theta}\ \prod_{k=m+1}^{n}(1^{|\Delta_k|}1^{\alpha_k})\ 1\ S}_{A_2} & \\ A{\downarrow} & = & P\ \ 1 \prod_{k=1}^{m}(0^{\alpha_{k-1}}0^{|\Delta_k|}) & & 1^{\alpha_m-\theta}0^\theta & \prod_{k=m+1}^{n}(0^{|\Delta_k|}1^{\alpha_k})\ 0\ S & \end{array}$$

and since $B\!\downarrow\, = A\!\downarrow$, $B \neq A$ and B_1 and B_2 are normalized, the only possible form for B is

$$B \;=\; \underbrace{P\ 1\ \textstyle\prod_{k=1}^{m}(0^{\alpha_{k-1}}0^{|\Delta_k|})\ 1^{\lambda}0^{\theta-1-\lambda}}_{B_1}\ 0\ \underbrace{1^{\alpha_m-\theta-\lambda}0^{\lambda}\ \textstyle\prod_{k=m+1}^{n}(0^{|\Delta_k|}1^{\alpha_k})\ 0\ S}_{B_2}.$$

Indeed, B must be obtained by *denormalizing* $A\!\downarrow$. Any denormalization in P or S yields a B_1 or B_2 which is not normalized. The only possibility on the left of i_0 is to replace $10^{\alpha_0}0^{|\Delta_1|}0$ with $00^{\alpha_0}1^{|\Delta_1|}1$ and iteratively $10^{\alpha_{k-1}-1}0^{|\Delta_k|}0$ with $0^{\alpha_{k-1}}1^{|\Delta_k|}1$. As long as $k < m$, B_1 is not normalized. When $k = m$, we actually replace $10^{\alpha_{m-1}-1}0^{|\Delta_k|}1^{\alpha_m-\theta}0^{\theta}$ with $0^{\alpha_{m-1}}1^{|\Delta_k|}1^{\alpha_m-\theta+1}0^{\theta-1}$. In order for B_1 to be normalized, we must *reorder* the subword $1^{\alpha_m-\theta+1}0^{\theta-1}$ as $0^{\theta-1}1^{\alpha_m-\theta+1}$. At this point, we have

$$B \;=\; P\ 0\ \textstyle\prod_{k=1}^{m}(0^{\alpha_{k-1}}1^{|\Delta_k|})\ 0^{\theta-1}\ \overset{i_0}{1}\ 1^{\alpha_m-\theta}\ \textstyle\prod_{k=m+1}^{n}(0^{|\Delta_k|}1^{\alpha_k})\ 0\ S\,.$$

Then we have $B_1 = A_1$ but $b_{i_0} = 1$. The only way to make a 0 appear at this position is to continue the denormalization on the right of i_0. Since there is no 0 between $0^{|\Delta_k|}$ and $0^{|\Delta_{k+1}|}$ for $k = m+1, \ldots, n-1$, we have to start at the end and first replace $10^{|\Delta_n|}1^{\alpha_n}0$ with $01^{|\Delta_n|}1^{\alpha_n+1}$ and then iteratively $10^{|\Delta_k|}1^{\alpha_k-1}0$ with $01^{|\Delta_k|}1^{\alpha_k}$ until we replace $11^{\alpha_m-\theta}0^{|\Delta_{m+1}|}0$ with $1^{\alpha_m-\theta}01^{|\Delta_{m+1}|}1$. At this point, we have $B = A$. We could perhaps denormalize $P0$ or $1S$ but such denormalization would not propagate and would let B_1 and/or B_2 unnormalized. Therefore we get $B = A$ hence $B'\!\downarrow\, = A'\!\downarrow$ which contradicts the hypothesis.

If on the contrary we start with a denormalization on the right of i_0 then we have to start from the right end of $A\!\downarrow$ and B_1 and B_2 will be normalized only if all denormalizations are performed until eventually we find again $B = A$.

The only remaining possibility to have $B \neq A$ consists in rewriting the central subword $1^{\alpha_m-\theta}0^{\theta}$ so that B_1 and B_2 are normalized. We get the form given above and we check easily that again $A'\!\downarrow\, = B'\!\downarrow$.

If $m = 0$ then

$$\begin{array}{rcl}
\Delta & = & \Delta'\ t^{\theta}\ \overset{i_0}{t}\ t^{\alpha_0-\theta}\ \prod_{k=1}^{n}(\Delta_k t^{\alpha_k})\quad t\ \Delta'' \\
A & = & P\ \ 0^{\theta}\ 0\ 1^{\alpha_0-\theta}\ \prod_{k=1}^{n}(1^{|\Delta_k|}1^{\alpha_k})\ 1\ S \\
A\!\downarrow & = & P\quad 1^{\alpha_0-\theta+1}0^{\theta}\ \prod_{k=1}^{n}(0^{|\Delta_k|}1^{\alpha_k})\ 0\ S
\end{array}$$

and the only possible form for B is

$$B \;=\; P\ 1^{\lambda}0^{\theta-\lambda}\ 0\ 1^{\alpha_0-\theta-\lambda}0^{\lambda}\ \textstyle\prod_{k=1}^{n}(0^{|\Delta_k|}1^{\alpha_k})\ 0\ S\,.$$

Again, we verify that we have $A'\!\downarrow\, = B'\!\downarrow$ which contradicts the hypothesis. ☐

As an immediate corollary, we get

Corollary 15. $\forall A, B \in \{0,1\}^\star 0^\omega,\ (A\backslash B){\downarrow} = (B\backslash A){\downarrow} \implies A{\downarrow} = B{\downarrow}$.

Proof. Let $C = A \wedge B$ and for each $i \geqslant 1$, let $N_i = 0^{i-1}10^\omega$. We have $A = (A\backslash B) \vee \bigvee_{i|C_i=1} N_i$ and $B = (B\backslash A) \vee \bigvee_{i|C_i=1} N_i$ where $\vee$ is the bitwise *or*. The result follows from an iterative application of Lemma 14, *adding* 1's to $A\backslash B$ and $B\backslash A$, one at a time. □

We are now ready to prove the main theorem of this section.

Proof of Theorem 13. If $A{\downarrow} = B{\downarrow}$, then obviously $\varphi(A) = \varphi(B)$ because $\varphi(A{\downarrow}) = \varphi(A)$ for all A.

For the converse, we proceed by induction on $\min(\lambda_1(A), \lambda_1(B))$. If $\min(\lambda_1(A), \lambda_1(B)) = 0$, the result is an immediate consequence of Lemma 9. Otherwise, let A and B be two normalized words such that $A \neq B$ and $0 < \lambda_1(B) \leqslant \lambda_1(A)$. Assume that the result holds for all pairs of normalized words (S, S') such that $\min(\lambda_1(S), \lambda_1(S')) < \lambda_1(B)$. We shall prove that we have $\varphi(A) \neq \varphi(B)$.

We may assume that each letter in $\mathcal{D}$ occurs infinitely many times in Δ. Indeed, $\varphi(A)$ and $\varphi(B)$ depend only on $U_1, \ldots, U_{\lambda_1(A)}$ which do not change if we replace Δ with $\Delta' = \delta_1 \cdots \delta_{\lambda_1(A)}(1 \cdots d)^\omega$. $A{\downarrow}$ and $B{\downarrow}$ do not change either. Thanks to Corollary 15, we may also assume that $a_i \wedge b_i = 0$ for all i. Otherwise, we replace A and B with $(A\backslash B){\downarrow}$ and $(B\backslash A){\downarrow}$ which are still different and $\varphi((A\backslash B){\downarrow}) = \varphi((B\backslash A){\downarrow})$ is equivalent to $\varphi(A) = \varphi(B)$. Thus $0 < \lambda_1(B) < \lambda_1(A)$.

We build recursively two sequences $(A_n)_{n\geqslant 0}$ and $(B_n)_{n\geqslant 0}$ such that $A_n{\downarrow} \neq B_n{\downarrow}$ and $\varphi(A_n) = \varphi(B_n)$ is equivalent to $\varphi(A) = \varphi(B)$ for all $n \geqslant 0$. We set $A_0 = A$, $B_0 = B$ and while $\lambda_1(B_n) < \lambda_1(A_n)$, let j_n be the smallest index strictly greater than $\lambda_1(B_n)$ such that $\delta_{j_n} = \delta_{\lambda_1(B_n)}$. We define

$$\begin{aligned} B_n &= B'_n 10^\omega\,, \\ C_n &= B'_n 01^{j_n-\lambda_1(B_n)}0^\omega\,, \\ A_{n+1} &= A_n\backslash C_n\,, \\ B_{n+1} &= C_n\backslash A_n\,. \end{aligned}$$

This construction is better explained below (trailing 0's are omitted):

$$\begin{array}{lcllllll} & & & \lambda_1(B_n) & & j_n & & \\ \Delta & = & \Delta'_n & t & \Delta''_n & t & \Delta'''_n & \qquad \text{where } \Delta''_n \in (\mathcal{D}\backslash\{t\})^\star \\ A_n & = & A'_n & 0 & A''_n & \alpha_n & A'''_n & \\ B_n & = & B'_n & 1 & & & & \\ C_n & = & B'_n & 0 & 1^{|\Delta''_n|} & 1 & & \\ A_{n+1} & = & A'_n\backslash B'_n & 0 & 0^{|\Delta''_n|} & 0 & A'''_n & \\ B_{n+1} & = & B'_n\backslash A'_n & 0 & \overline{A''_n} & \overline{\alpha_n} & & \end{array}$$

By an easy induction on n, we have $A_n \wedge B_n = 0^\omega$ so that $A'_n \backslash B'_n = A'_n$ and $B'_n \backslash A'_n = B'_n$.

By construction, $C_n{\downarrow} = B_n{\downarrow}$ and $\varphi(A_{n+1}) = \varphi(B_{n+1})$ is equivalent to $\varphi(A_n) = \varphi(C_n)$ which in turn is equivalent to $\varphi(A_n) = \varphi(B_n)$. From Corollary 15, $A_n{\downarrow} \neq B_n{\downarrow}$ implies $A_{n+1}{\downarrow} \neq B_{n+1}{\downarrow}$. We have to prove $\lambda_1(B_{n+1}) > \lambda_1(B_n)$ which holds if and only if $\overline{A''_n\alpha_n} \neq 0^{|\Delta''_n|+1}$ or equivalently $A''_n\alpha_n \neq 1^{|\Delta''_n|+1}$. Let us assume this property for now. Therefore we have eventually $j_{n_0} \geqslant \lambda_1(A_{n_0})$. At this step, $A'''_{n_0} = 0^\omega$ hence $A_{n_0+1} = A'_{n_0}0^\omega$. At each step, $A'_{n+1} = (A'_n \backslash B'_n)0^{\theta_n} = A'_n 0^{\theta_n}$ for some $\theta_n \geqslant 0$ hence $\lambda_1(A'_{n+1}) = \lambda_1(A'_n)$. Therefore, for all n, $\lambda_1(A'_n) = \lambda_1(A'_0) < \lambda_1(B)$ hence $\lambda_1(A_{n_0+1}) < \lambda_1(B_0)$. Since $A_{n_0+1}{\downarrow} \neq B_{n_0+1}{\downarrow}$ and $\lambda_1(S{\downarrow}) \leqslant \lambda_1(S)$ for all S, by the induction hypothesis, we get $\varphi(A_{n_0+1}{\downarrow}) \neq \varphi(B_{n_0+1}{\downarrow})$ which implies $\varphi(A) \neq \varphi(B)$.

Let us now return to prove $A''_n\alpha_n \neq 1^{|\Delta''_n|+1}$. If $A''_n\alpha_n = 1^{|\Delta''_n|+1}$ then A_n contains the reducible subword $01^{|\Delta''_n|+1}$. We prove that this subword was actually already contained in A which is impossible since A is normalized.

Clearly $A''_0\alpha_0 \neq 1^{|\Delta''_0|+1}$ since otherwise the reducible subword $0A''_0\alpha_0 = 01^{|\Delta''_0|+1}$ would be contained in $A_0 = A$. Hence $\lambda_1(B_1) > \lambda_1(B_0)$. Now assume by contradiction $A''_n\alpha_n = 1^{|\Delta''_n|+1}$ and assume also $\lambda_1(B_{m+1}) > \lambda_1(B_m)$ for $m = 0, \dots, n-1$. By construction, we have a 0 in A_n at each position in $\{\lambda_1(B_m), \dots, j_m\}$ for $m = 0, \dots, n-1$. Since we assumed $\lambda_1(B_{m+1}) > \lambda_1(B_m)$ for $m = 0, \dots, n-1$ and by construction $\lambda_1(B_{m+1}) \leqslant \lambda_1(C_m) = j_m$, we have actually a 0 in A_n at each position in $\{\lambda_1(B_0), \dots, \max\{j_0, \dots, j_{n-1}\}\}$. Further positions are unchanged from A. This implies $j_m \leqslant \lambda_1(B_n)$ for $m = 0, \dots, n-1$ otherwise we could not have $A''_n\alpha_n = 1^{|\Delta''_n|+1}$. But we have also $\lambda_1(B_n) \leqslant \lambda_1(C_{n-1}) = j_{n-1}$ hence $j_{n-1} = \lambda_1(B_n)$ and $\delta_{\lambda_1(B_{n-1})} = \delta_{j_{n-1}} = \delta_{\lambda_1(B_n)} = t$. The situation is described below (trailing 0's are omitted).

$$
\begin{array}{lllccccccc}
 & & & & & j_{n-1} & & & \\
 & & & \lambda_1(B_{n-1}) & & \lambda_1(B_n) & & j_n & \\
\Delta & = & \Delta'_{n-1} & t & \Delta''_{n-1} & t & \Delta''_n & t & \Delta'''_n \\
A_{n-1} & = & A'_{n-1} & 0 & A''_{n-1} & \alpha_{n-1} & A''_n & \alpha_n & A'''_n \\
B_{n-1} & = & B'_{n-1} & 1 & & & & & \\
C_{n-1} & = & B'_{n-1} & 0 & 1^{|A''_{n-1}|} & 1 & & & \\
A_n & = & A'_{n-1} & 0 & 0^{|A''_{n-1}|} & 0 & A''_n & \alpha_n & A'''_n \\
B_n & = & B'_{n-1} & 0 & \overline{A''_{n-1}} & \overline{\alpha_{n-1}} & & &
\end{array}
$$

We have necessarily $\overline{\alpha_{n-1}} = 1$, i.e., $\alpha_{n-1} = 0$, which means that $\alpha_{n-1}A''_0\alpha_0 = 01^{|\Delta''_0|+1}$ is a subword of A_{n-1}. We cannot have $j_m = \lambda_1(B_n)$ for $m < n-1$ because we would have $\lambda_1(B_m) = \lambda_1(B_{n-1})$ which contradicts the induction hypothesis. Hence $j_m < \lambda_1(B_n)$ for all $m < n-1$ and the value of α_{n-1} is unchanged from A which is thus reducible. $\square$

This result allows us to define the inverse function of φ as follows.

Definition 16. *Given $x \in S_\infty$, $\varphi^{-1}(x)$ is the unique normalized word $S \in \{0,1\}^\star 0^\omega$ such that $\varphi(S) = x$.*

6. Topological properties of S_∞

Armed with the tools of the previous section, we may now study more precisely the topology of S_∞. We first have an immediate theorem.

Theorem 17. *$T_\infty^{(1)}, \ldots, T_\infty^{(d)}$ are disjoint and each $T_\infty^{(i)}$ is either empty or adjacent to* 0*.*

Proof. If $x \in T_\infty^{(i)}$ then $\lambda_1(\varphi^{-1}(x)) > 0$ and $\delta_{\lambda_1(\varphi^{-1}(x))} = i$. Therefore we cannot have $x \in T_\infty^{(j)}$ with $j \neq i$ and we cannot have $x = 0$ because $\varphi^{-1}(0) = 0^\omega$. Now if $T_\infty^{(i)}$ is not empty then it contains e_i which is a neighbour of 0. $\square$

Lemma 18. *For each i in $\mathcal{D}$, if $T_\infty^{(i)}$ is nonempty then it contains exactly one neighbour of* 0*.*

Proof. The neighbours of 0 are $\pm e_k$. From Corollary 10 there exists for every $x \in S_\infty \backslash \{0\}$ a positive vector v such that $\langle v, x\rangle > 0$. Hence $-e_k$ does not belongs to S_∞ for any k.

From Lemmas 6 and 12, if $e_k \in S_\infty$ then necessarily $e_k \in T_\infty^{(k)}$. Hence the result. $\square$

Theorem 19. *$T_\infty^{(1)}, \ldots, T_\infty^{(d)}$ are pairwise non-adjacent.*

Proof. If $T_\infty^{(i)}$ and $T_\infty^{(j)}$ are adjacent then $T_{n_0}^{(i)}$ and $T_{n_0}^{(j)}$ must be adjacent for some n_0. The sequence $\Delta' = \delta_1 \cdots \delta_{n_0}(1 \cdots d)^\omega$ defines sequences $(S'_n)_{n\geqslant 0}$, and $(T_n'^{(k)})_{n\geqslant 0}$ for $k \in \mathcal{D}$, such that $S'_n = S_n$ and $T_n'^{(k)} = T_n^{(k)}$ for all $n \leqslant n_0$ and all $k \in \mathcal{D}$. Thus $T_{n_0}'^{(i)}$ and $T_{n_0}'^{(j)}$ are adjacent hence $T_\infty'^{(i)}$ and $T_\infty'^{(j)}$ are as well. We shall prove that this is impossible. In the sequel, we assume that each $k \in \mathcal{D}$ occurs infinitely many times in Δ.

We consider the sequence of cones $\mathcal{C}_n$ defined in Lemma 8 and let $\mathcal{C}'_n$ be the interior of $\mathcal{C}_n$. Since $\sigma_i^{-1}(\mathcal{C}^+) \subset \mathcal{C}^+$ for each i, we have $\mathcal{C}'_{n+1} \subset \mathcal{C}'_n$ for all n. Hence $\mathcal{C}'_n$ converges to a limit $\mathcal{C}'_\infty$ which is nonempty provided that Δ is not eventually constant. We consider a vector v in $\mathcal{C}'_\infty$ and the sequence $(v^n)_{n\geqslant 1}$ defined by

$v^n = \sigma_{\delta_{n-1}} \cdots \sigma_{\delta_1}(v)$. By construction, v belongs to the interior of $\mathcal{C}_n$ for all n so that v^n belongs to the interior of $\mathcal{C}^+$. Hence v^n is positive. In addition, the sequence $(v^n)_{n \geqslant 1}$ decreases component-wise. Indeed, $v^{n+1} = \sigma_{\delta_n}(v^n)$ hence $v_i^{n+1} = v_i^n$ if $i = \delta_n$ and $v_i^{n+1} = v_i^n - v_{\delta_n}^n$ if $i \neq \delta_n$. In all cases, we have $v_i^{n+1} \leqslant v_i^n$. Therefore v^n converges to a limit v^∞. For $n \geqslant 2$, we have

$$\begin{aligned}
\|v^n\|_1 &= \|\sigma_{\delta_{n-1}} \cdots \sigma_{\delta_1}(v)\|_1 \\
&= \|\sigma_{\delta_{n-1}}(v^{n-1})\|_1 \\
&= \|v^{n-1}\|_1 - (d-1) v_{\delta_{n-1}}^{n-1} \\
&= \|v^{n-1}\|_1 - (d-1)\langle v^{n-1}, e_{\delta_{n-1}} \rangle \\
&= \|v^{n-1}\|_1 - (d-1)\langle v, U_{n-1} \rangle \,.
\end{aligned}$$

Hence, for all $n \geqslant 1$, $\|v^n\|_1 = \|v\|_1 - (d-1) \sum_{i=1}^{n-1} \langle v, U_i \rangle$ and consequently

$$\sum_{i=1}^{\infty} \langle v, U_i \rangle = \frac{\|v\|_1 - \|v^\infty\|_1}{d-1} = \Omega^\infty \,.$$

Since $\langle v, U_n \rangle = \langle v^n, e_{\delta_n} \rangle > 0$ for all n, we deduce that for all $x \in S_\infty$ we have $0 \leqslant \langle v, x \rangle < \Omega^\infty$.

From now on, in order to simplify the writing, we set $\omega_n = \langle v, U_n \rangle = \langle v^n, e_{\delta_n} \rangle = v_{\delta_n}^n$. For all n, we have $\omega_n = \min_i v_i^n$. Indeed, for each i in $\mathcal{D}$, we have $v_i^{n+1} = v_{\delta_n}^n$ if $i = \delta_n$ and $v_i^{n+1} = v_i^n - v_{\delta_n}^n$ if $i \neq \delta_n$. Since v^{n+1} is positive, we must have $\omega_n = v_{\delta_n}^n \leqslant v_i^n$ for each i in $\mathcal{D}$. Furthermore $\omega_{n+1} = \min_i v_i^{n+1} \leqslant v_{\delta_n}^{n+1} = v_{\delta_n}^n = \omega_n$ so that $(\omega_n)_{n \geqslant 1}$ is decreasing.

We shall prove that $x \in T_\infty^{(i)}$ has no neighbour in $T_\infty^{(j)}$ if $j \neq i$, which means that for all k, $x \pm e_k \notin T_\infty^{(j)}$. It is actually sufficient to consider $x + e_k$ because the other case is symmetric by exchanging x and $x - e_k$. Then it is sufficient to prove that $x \in T_\infty^{(i)}$ implies $x + e_k \in T_\infty^{(i)}$ or $x + e_k = 0$ or $x + e_k \notin S_\infty$. We cannot have $x + e_k = 0$ since it would mean $x = -e_k$ but $-e_k \notin S_\infty$ because $\langle v, e_k \rangle < 0$. For the same reason, it is sufficient to prove that $x + e_k \notin T_\infty^{(i)}$ implies $\langle v, x + e_k \rangle \geqslant \Omega^\infty$.

Let $A = \varphi^{-1}(x)$. We have $\delta_{\lambda_1(A)} = i$. Let ζ_k be the smallest index such that $\delta_{\zeta_k} = k$ and $a_{\zeta_k} = 0$. If $a_j = 0$ for all $j < \zeta_k$ such that $\delta_j \neq k$ then we are in the following situation:

$$\begin{array}{rcllcl}
 & & & & \zeta_k & \\
\Delta & = & \Delta_0 & \prod_{m=1}^{p} (k\, \Delta_m) & k & \Delta' \\
A & = & 0^{|\Delta_0|} & \prod_{m=1}^{p} (1\, 0^{|\Delta_m|}) & 0 & A'
\end{array}$$

where $p \geqslant 0$ and $\Delta_m \in (\mathcal{D}\backslash\{k\})^\star$ for $m = 1, \ldots, p$. Since $(0^{|\Delta_0|+1} 1^{\zeta_k - |\Delta_0| - 1} A')\!\downarrow = A$ and $e_k = \varphi(1^{|\Delta_0|+1})$, we have

$$\varphi(1^{\zeta_k} A') = \varphi(0^{|\Delta_0|+1} 1^{\zeta_k - |\Delta_0| - 1} A') + \varphi(1^{|\Delta_0|+1}) = \varphi(A) + e_k = x + e_k\,.$$

If $A' \neq 0^\omega$ then $\lambda_1(1^{\zeta_k} A') = \lambda_1(A)$. Otherwise, we cannot have $p = 0$ because we would have $A = 0^\omega$ hence $x = \varphi(A) = 0 \notin T^{(i)}_\infty$. Thus $\delta_{\lambda_1(1^{\zeta_k} A')} = \delta_{\lambda_1(A)} = k$. In all cases, we have $\delta_{\lambda_1(1^{\zeta_k} A')} = \delta_{\lambda_1(A)} = i$ hence $x + e_k \in T^{(i)}_\infty$.

If A does not have the above form then there exists $j < \zeta_k$ such that $\delta_j \neq k$ and $a_j = 1$. We shall prove that in this case, $\langle v, x + e_k\rangle \geqslant \Omega^\infty$. It is sufficient to prove that if $j < \zeta_k$ and $\delta_j \neq k$ then $\langle v, \varphi(1^{\zeta_k}) + U_j\rangle \geqslant \Omega^\infty$ or equivalently $\sum_{m=1}^{\zeta_k} \omega_m + \omega_j \geqslant \Omega^\infty$. Since ω_j decreases when j increases, it is sufficient to prove the result for the greatest possible j. Similarly, since $\omega_j > 0$, it is sufficient to consider the case where $\zeta_k = j+1$. Indeed, if $\delta_{\zeta_k - 1} = \delta_{\zeta_k}$ and $\sum_{m=1}^{\zeta_k - 1} \omega_m + \omega_j \geqslant \Omega^\infty$, then we certainly have $\sum_{m=1}^{\zeta_k} \omega_m + \omega_j \geqslant \Omega^\infty$. Finally, it is sufficient to prove $\sum_{m=1}^{j+1} \omega_m + \omega_j \geqslant \Omega^\infty$ for all j such that $\delta_{j+1} \neq \delta_j$. From what precedes

$$\begin{aligned}
\sum_{m=1}^{j+1} \omega_m + \omega_j \geqslant \Omega^\infty \quad &\Longleftrightarrow \quad \frac{\|v\|_1 - \|v^{j+2}\|_1}{d-1} + \omega_j \geqslant \frac{\|v\|_1 - \|v^\infty\|_1}{d-1} \\
&\Longleftrightarrow \quad \|v^{j+2}\|_1 \leqslant (d-1)\,\omega_j + \|v^\infty\|_1\,.
\end{aligned}$$

C. Kraaikamp and R. Meester [16, 19] have proven that $v^\infty = 0$ if each $k \in \mathcal{D}$ occurs infinitely many times in Δ. Thus we are left to prove $\|v^{j+2}\|_1 \leqslant (d-1)\,\omega_j$. They have also proven that $\|v^n\|_1 \geqslant (d-1)\,\|v^n\|_\infty$ for all n. Without loss of generality, let us assume $\delta_j = 1$ and $\delta_{j+1} = 2$ and let $v^j = (x_1, \ldots, x_d)$. Then we have

$$\begin{aligned}
\omega_j &= x_1 \\
v^{j+1} &= (x_1,\, x_2 - x_1,\, \ldots,\, x_d - x_1) \\
v^{j+2} &= (2\,x_1 - x_2,\, x_2 - x_1,\, x_3 - x_2,\, \ldots,\, x_d - x_2) \\
\|v^{j+2}\|_1 &= \|v^j\|_1 - (d-1)\,x_2\,.
\end{aligned}$$

Let $(x'_1, \ldots, x'_d)$ be $(x_1, \ldots, x_d)$ sorted in ascending order. Since $\delta_j = 1$, we have $x_1 = \min(x_1, \ldots, x_d)$ which means $x'_1 = x_1$. Since $\delta_{j+1} = 2$, we have $x_2 - x_1 = \min(x_1, x_2 - x_1, \ldots, x_d - x_1)$ which implies $x_2 = \min(x_2, \ldots, x_d)$ hence $x'_2 = x_2$. We have also $\|v^j\|_1 = x'_1 + \cdots + x'_d$ and $\|v^j\|_\infty = x'_d$. Now assume $\|v^{j+2}\|_1 > (d-1)\,\omega_j$, which means $\|v^j\|_1 - (d-1)\,x_2 > (d-1)\,x_1$ or equivalently $x'_1 + \cdots + x'_d > (d-1)(x'_1 + x'_2)$. Then we prove easily that the set of inequalities

$$\begin{aligned} &0 \leqslant x'_1 \leqslant x'_2 \leqslant \cdots \leqslant x'_d \\ &x'_1 + \cdots + x'_d \geqslant (d-1)\, x'_d \\ &x'_1 + \cdots + x'_d > (d-1)(x'_1 + x'_2) \end{aligned}$$

has no solution. □

As a corollary of this result, we get

Corollary 20. *S_∞ contains no non-trivial simple circuit.*

Proof. If S_∞ contains a non-trivial simple circuit, let n_0 be the smallest index such that S_{n_0} contains such a circuit and let $\mathcal{C}$ be such a circuit of minimal length in S_{n_0}. We must have $n_0 > 0$ because $S_0 = \{0\}$ contains no such circuit. Since $T_{n_0}^{(i)}$'s are pairwise non-adjacent, $\mathcal{C}$ must be contained in some $T_{n_0}^{(i_0)}$, otherwise it would contain at least twice the unique neighbour of 0 in some $T_{n_0}^{(i)}$ and $\mathcal{C}$ would not be simple. If $\delta_{n_0} \neq i_0$, then $T_{n_0}^{(i_0)} = T_{n_0-1}^{(i_0)}$ and $\mathcal{C}$ is contained in S_{n_0-1} which contradicts the minimality of n_0. If $\delta_{n_0} = i_0$ then $T_{n_0}^{(i_0)} = \mathrm{trans}_{U_{n_0}}(S_{n_0-1})$ and $\mathrm{trans}_{-U_{n_0}}(\mathcal{C})$ is a circuit in S_{n_0-1} which again contradicts the minimality of n_0. □

Let us conclude this section with a remark about the relationship between the objects we generate and the Rauzy fractal. When $d = 3$ and $\Delta = (123)^\omega$, we actually build successive approximations of the original Rauzy fractal. Indeed, the original fractal

$$\mathcal{R} = \left\{ \sum_{i=1}^{\infty} \epsilon_i\, \beta^i \mid \forall i,\ \epsilon_i \in \{0,1\} \text{ and } \epsilon_i \epsilon_{i+1} \epsilon_{i+2} = 0 \right\}$$

where β is a complex zero of $X^3 - X^2 - X - 1$ is approximated by

$$\mathcal{R}_n = \left\{ \sum_{i=1}^{n} \epsilon_i\, \beta^i \mid \forall i,\ \epsilon_i \in \{0,1\} \right\}.$$

Because as the sums are finite, if $\epsilon_i \epsilon_{i+1} \epsilon_{i+2} = 1$ for some i by a finite number of rewriting rules a normalize sequence $(\epsilon_i)_{i>1}$ can be reached. Now, $S_n = \{\sum_{i=1}^{n} \epsilon_i\, U_i \mid \forall i,\ \epsilon_i \in \{0,1\}\}$ where $U_1 = e_1$, $U_2 = e_2 - e_1$, $U_3 = e_3 - e_2$ and for $i \geqslant 4$, $U_i = U_{i-3} - U_{i-2} - U_{i-1}$. Consider the linear map $f : \mathbb{R}^3 \to \mathbb{C}$ defined by $f(U_1) = \beta^{-1}$, $f(U_2) = \beta^{-2}$ and $f(U_3) = \beta^{-3}$. An easy induction shows that $f(U_i) = \beta^{-i}$ for all $i \geqslant 1$. Hence

$$f(S_n) = \left\{ \sum_{i=1}^{n} \epsilon_i\, \beta^{-i} \mid \forall i,\ \epsilon_i \in \{0,1\} \right\} = \beta^{-(n+1)}\, \mathcal{R}_n$$

which means that $f(S_n)$ is similar to $\mathcal{R}_n$. In addition, the map $\beta^{n+1}f$ is a bijection from S_n to $\mathcal{R}_n$. Indeed, $\mathrm{Ker}(f) = \mathbb{R}.(1, \alpha, \alpha^2)$ where α is the real zero of $X^3 - X^2 - X - 1$, that is the Tribonacci number. Since α is not rational, this kernel contains no integral point but $(0, 0, 0)$ so that no two elements of S_n have the same image by f.

7. Case $d = 2$: Geometric vs iterated palindromic closure

In dimension 2, the geometric palindromic closure constructs standard discrete segments as in the following example.

Example 21. *We take $d = 2$. Figure 4 shows $S_0, \ldots, S_8$ for $\Delta = 12112211\cdots$.*

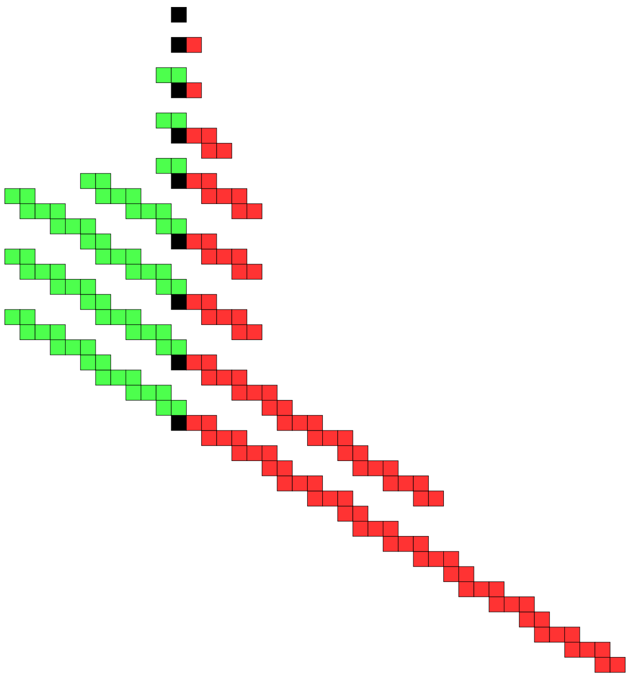


Figure 4. $S_0, \ldots, S_8$ for $d = 2$ and $\Delta = 12112211\cdots$.

We show now that our construction can be directly defined on finite words and that we find exactly for the dimension 2 the palindromic closure defined on words by de Luca and Justin [9, 11, 15].

7.1. Justin's formula

In the field of combinatorics on words, the *(right) palindromic closure* of a word W is the shortest palindrome having W as a prefix. For example, the shortest palindrome having 11221 as a prefix is 112211, while for 112212 we find 112212211. Usually, we would like to make many times the construction by adding letters one at a time to the prefix. Therefore we define inductively the *iterated palindromic closure* by $\psi(\epsilon) = \epsilon$ and $\psi(\Delta a) = \Theta(\psi(\Delta)a)$ where $\Theta(X)$ is the palindromic closure of X.

The iterated palindromic closure for a directive word $\Delta = (\delta_n)_{n\geqslant 1}$ is given by successive applications of the closure using the information of Δ. We start with $W_0 = \epsilon$ and for each $i \geqslant 1$, we compute W_i which is the smallest palindrome with $\psi(W_{i-1})\delta_i$ as a prefix. For example, for the word $\Delta = 1122$, we find $W_0 = \epsilon$, $W_1 = \psi(1) = 1$, $W_2 = \psi(11) = \Theta(\psi(1)1) = \Theta(11) = 11$, $W_3 = \psi(112) = \Theta(\psi(11)2) = \Theta(112) = 11211$ and $W_4 = \psi(1122) = \Theta(\psi(112)2) = \Theta(112112) = 11211211$.

Although the definition of ψ is mathematically satisfying, it is algorithmically very inefficient. In particular, constructing the palindromic closure is not so easy in general because we must find the centre of symmetry of the palindrome, which is not immediate. This is why J. Justin stated and proved his formula [11].

Consider a word $W \in \{1,2\}^\star$ and a letter $a \in \{1,2\}$. If W contains the letter a, then we can write $W = UaV$ with $|V|_a = 0$. Then

$$\psi(Wa) = \begin{cases} \psi(W)a\psi(W) & \text{if } |W|_a = 0\,; \\ \psi(W)\psi(U)^{-1}\psi(W) & \text{if } W = UaV \text{ with } |V|_a = 0\,. \end{cases} \tag{4}$$

The computation is performed in the free group generated by $\{1,2\}$ rather than in the free monoid and X^{-1} denotes the inverse of X. However, one checks that in the second case above, $\psi(U)$ is both a prefix and a suffix of $\psi(W)$ so that the result belongs to $\{1,2\}^\star$ after simplification.

Using Justin's formula, we find for $\Delta = 1122$,

$$\begin{aligned} \psi(1) &= 1\,; \\ \psi(11) &= \psi(1)\psi(\epsilon)^{-1}\psi(1) = 11\,; \\ \psi(112) &= \psi(11)2\psi(11) = 11211\,; \\ \psi(1122) &= \psi(112)\psi^{-1}(11)\psi(112) = 11211(11)^{-1}11211 = 11211211\,. \end{aligned}$$

7.2. Words generated by translation

In the case $d = 2$, we are able to define the construction by symmetry directly on words. We work on finite words with a pointed origin denoted by "|" and we define the map ψ' on these words inductively as follows:

Definition 22 (Definition of ψ' by symmetry)**.**

$$\begin{array}{rcll} \psi'(\epsilon) & = & | \,; & \\ \psi'(W1) & = & U|1\widetilde{UV} & \text{if} \quad \psi'(W) = U|V\,; \\ \psi'(W2) & = & \widetilde{UV}2|V & \text{if} \quad \psi'(W) = U|V\,. \end{array}$$

Lemma 23. *For all words W in $\{0,1\}^\star$, if $U|V = \psi'(W)$ then UV, $2U$ and $V1$ are palindromes.*

P r o o f. By induction on $|W|$. The result is obvious if $W = \epsilon$. Now assume that the result holds for W and let us show that is holds for $W1$ (the case $W2$ is similar).

We have $\psi'(W1) = U|1\widetilde{UV} = U'|V'$ with $U' = U$ and $V' = 1\widetilde{UV}$. Then $U'V' = U1\widetilde{UV} = U1\widetilde{V}\widetilde{U}$ and $U1\widetilde{V}\widetilde{U}$ is a palindrome since $V1$, hence $1\widetilde{V}$, is a palindrome by hypothesis. We have $2U' = 2U$ and $2U$ is a palindrome by hypothesis and finally, $V'1 = 1\widetilde{UV}1$ and $1\widetilde{UV}1$ is a palindrome because UV, hence $\widetilde{UV}$, is. $\square$

Remark: From this lemma, we deduce that either $U = \epsilon$, or $U = p2$ for some palindrome p and either $V = \epsilon$ or $V = 1s$ for some palindrome s.

Thanks to this lemma, we may rewrite the definition of ψ' as follows:

Definition 24 (Definition of ψ' by translation)**.**

$$\begin{array}{rcll} \psi'(\epsilon) & = & | \,; & \\ \psi'(W1) & = & U|1UV & \text{if} \quad \psi'(W) = U|V\,; \\ \psi'(W2) & = & UV2|V & \text{if} \quad \psi'(W) = U|V\,. \end{array}$$

The following theorem gives the equivalence in dimension 2 between Justin's Formula and the construction by translation.

Theorem 25. *For all words W in $\{1,2\}^\star$, if $\psi'(W) = U|V$ then $\psi(W) = UV$.*

P r o o f. By induction on $|W|$. The result is obvious if $W = \epsilon$. Assume that it holds for all W' such that $|W'| \leqslant |W|$ and let us show that it holds for $W1$ (the case $W2$ is similar).

If $|W|_1 = 0$ then $W = 2^\ell$ for some $\ell \geqslant 0$. We check easily that $\psi'(W1) = 2^\ell|12^\ell$ and $\psi(W1) = 2^\ell 12^\ell$.

If $|W|_1 > 0$ then $W = X12^\ell$ for some word X and some $\ell \geqslant 0$. By an easy induction on ℓ, we verify that $\psi'(X12^\ell) = A(1AB2)^\ell|1AB$ and therefore $\psi'(X12^\ell 1) = A(1AB2)^\ell|1A(1AB2)^\ell 1AB$ where $A|B = \psi'(X)$. By the induction hypothesis, $\psi(X) = AB$ and $\psi(X12^\ell) = A(1AB2)^\ell 1A(1AB2)^\ell 1AB$, hence

$$\begin{aligned}
\psi(X12^\ell 1) &= \psi(X12^\ell)\psi(X)^{-1}\psi(X12^\ell) \\
&= A(1AB2)^\ell 1AB(AB)^{-1}A(1AB2)^\ell 1AB \\
&= A(1AB2)^\ell 1A(1AB2)^\ell 1AB\,.
\end{aligned}$$

□

Remarks:

1) In the version of ψ' by translation, we increase the word either to the right or to the left depending on whether the current letter in the directive word is a 1 or a 2. Nevertheless the word constructed by Justin's formula is the same as the word constructed by translation if we remove the information about the origin of the constructed word.

2) The palindromic closure ψ with Justin's formula gives increasing prefixes hence, at the limit, a right infinite word. Our construction by ψ' gives a bi-infinite word provided that the directive word is not eventually constant.

In this case, if $S = \psi(\Delta)$ then we may observe that $\psi'(\Delta) = \widetilde{S}2|1S$. Indeed, if $S_n = \psi(\delta_1 \cdots \delta_n)$ then, for n large enough, we have $\psi'(\delta_1 \cdots \delta_n) = U_n 2|1V_n$ and $S_n = U_n 21V_n = V_n 12U_n$. Thus V_n is a prefix of S_n for all n so that $V = S$ in $\psi'(\Delta) = U2|1V$. Similarly $\widetilde{U_n} = U_n$ is also a prefix of S_n so that U_n is a suffix of $\widetilde{S_n}$ hence $U = \widetilde{S}$.

3) In the original definition of the iterated palindromic closure, we must find at each step a centre of symmetry to construct the shortest palindrome and this is not so trivial. Using Justin's formula we must maintain the list of preceding constructions of $\psi(\Delta_n)$ in order to compute easily $\psi(U)$, that is the overlapping part in the construction of the shortest palindrome. In the construction using translations, we use only the last step in the construction and put it either at the right or at the left of the origin and we do not have to compute explicitly the overlapping part.

7.3. Equivalence in dimension 2 between the construction by symmetry on words and by centres

The equivalence between the two constructions is relatively easy because in fact words in the alphabet $\{1, 2\}$ code exactly discrete segments. Moreover the centres of the set X_n, $Y_n^{(1)}$ and $Y_n^{(2)}$ correspond bijectively to the centres of three palindromes in words $\psi(W) = U|V$, namely the palindrome UV, the palindrome p in $U = p2$ if U is non-empty and the palindrome s in $V = 1s$ if V is non-empty.

We would like to find the definition on the centre of symmetry directly on the words that is we define the operation on centres of symmetry on words (i.e., on centres of palindromes):

$$\begin{aligned} Y_n^{(i)} &= \begin{cases} \mathrm{sym}_{Y_{n-1}^{(i)}}(X_{n-1}) & \text{if } i = \delta_n\,, \\ Y_{n-1}^{(i)} & \text{otherwise,} \end{cases} \\ W_n &= W_{n-1} \diamond \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(W_{n-1})\,, \\ X_n &= Y_{n-1}^{(w_n)} \end{aligned}$$

where the operator $U \diamond V$ builds a word on the union of the domains of U and V (each word is viewed as a function from $\mathbb{Z}$ to the alphabet $\{1,2\}$). For each position i, if i is in the domain of U and not in the domain of V then $(U \diamond V)_i = u_i$; if i is in the domain of V and not in the domain of U then $(U \diamond V)_i = v_i$; if $u_i = v_i$ then $(U \diamond V)_i = u_i$ and otherwise $(U \diamond V)_i = ?$. For example if $U = u_0u_1 = 12$ and $V = v_1v_2 = 21$ then $U \diamond V = u_0u_1v_2 = 121$ and if $U = u_0u_1 = 11$ and $V = v_1v_2 = 22$ then $U \diamond V = u_0w_1v_2 = 1?2$ because of the collision of letter in position 1.

Remark that in our construction, we do not have “?”. Indeed, the overlapping is given by some W_j so that there is no collision of letters.

Lemma 26. *The construction by symmetry on words gives the same formula as the construction by symmetry on discrete segments.*

Proof. The idea is to code geometrical discrete segments by finite words on the alphabet $\{1,2\}$. The letter 1 codes the horizontal position of two adjacent unit squares. The letter 2 codes the vertical position of two adjacent unit squares.

Base case: $\psi'(\epsilon) = W_0 = |$;
we set

$$\begin{aligned} X_0 &= 0\,; \\ Y_0^{(1)} &= 1/2\,; \\ Y_0^{(2)} &= -1/2\,. \end{aligned}$$

The justification of the values is just a coding of the first step in the definition of centres of symmetry in the geometrical object. Recall that in the geometrical object, we consider a unit square at the origin with one centre of symmetry given by $X_0 = 0$ and 2 centres of symmetries in the segments that is in positions $\frac{1}{2}e_i$ with $i = 1, 2$. As we add squares either to the right side or to the upper side, the two other centres of symmetry in the coding word must be in position $-1/2$ and $1/2$. In order to have a symmetry in the indices of the words, we place the letters in positions $n + 1/2$.

To investigate the base case we must verify that the formula applied to the centres of $\psi'(\epsilon)$ gives exactly the centres for the word $\psi'(1)$ and for the word $\psi'(2)$ by the following formula

$$\begin{aligned} Y_n^{(i)} &= \begin{cases} \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(X_{n-1}) & \text{if } i = \delta_n\,; \\ Y_{n-1}^{(i)} & \text{if } i \neq \delta_n\,; \end{cases} \\ X_n &= Y_{n-1}^{(\delta_n)}\,. \end{aligned}$$

For $\psi'(1)$, we code two horizontal adjacent unit squares with 3 centres of symmetry, one in the middle of the two squares and the two others at the centre of empty words. $\psi'(1) = W_1 = |1 = |w_{1/2}\,;$
we set

$$\begin{aligned} X_1 &= 1/2 \quad \text{(centre of symmetry of the palindrome } w_{1/2})\,; \\ Y_1^{(1)} &= 1\,; \\ Y_1^{(2)} &= -1/2\,. \end{aligned}$$

We verify easily that the coding is given by the formula on centres: $X_1 = Y_0^{(1)} = 1/2$, $Y_1^{(2)} = Y_0^{(2)} = -1/2$ and $Y_1^{(1)} = \mathrm{sym}_{Y_0^{(1)}}(X_0) = \mathrm{sym}_{1/2}(0) = 1$.

For $\psi'(2)$, we code two vertical adjacent unit square with 3 centres of symmetry, one in the middle of the two squares and the two others at the centre of empty words. $\psi'(2) = W_1 = 2| = w_{-1/2}|$;
we set

$$\begin{aligned} X_1 &= -1/2 \quad \text{(centre of symmetry of the palindrome } w_{-1/2}); \\ Y_1^{(1)} &= 1/2\,; \\ Y_1^{(2)} &= -1\,. \end{aligned}$$

We verify easily that the coding is given by the formula on centres: $X_1 = Y_0^{(2)} = -1/2$, $Y_1^{(1)} = Y_0^{(1)} = 1/2$ and $Y_1^{(2)} = \mathrm{sym}_{Y_0^{(2)}}(X_0) = \mathrm{sym}_{-1/2}(0) = -1$.

In the general case, the induction is not so difficult. We use the formulas of Definition 24 to construct the words by translation. By construction of $\psi'(W) = U|V$, X_n is the centre of symmetry of the palindrome UV; the point $Y_n^{(1)}$ is either the centre of symmetry of the palindrome s in $V = 1s$ if V is non-empty or the centre of the empty word; the point $Y_n^{(2)}$ is either the centre of symmetry of the palindrome p in $U = p2$ if U is non-empty or the centre of the empty word.

If the current letter of the directive word is 1 then we construct $\psi'(W1) = U|1UV$ that is a word with at least 3 palindromes namely the word $\psi'(W1)$ itself, the palindrome p in $U = p2$ and a new palindrome near the origin that is UV. Now we have just to verify that $X_{n+1} = Y_n^{(1)}$ that is the centre of the palindrome $\psi'(W)$; $Y_{n+1}^{(2)} = Y_n^{(2)}$ that is the centre of the palindrome p and $Y_{n+1}^{(1)} = \mathrm{sym}_{Y_n^{(1)}}(X_n)$ that is the symmetry of the centre of the palindrome $\psi'(W)$ by the centre of the palindrome s. On words if we take $\psi'(W) = U|V = U|1s$ and make a symmetry on the centre of the rightmost s we find $U|1s1U$ that is exactly the desired word $U|1UV$. Thus we use exactly the symmetry formula on centres of $\psi'(W)$ to construct $\psi'(W1)$. It remains to investigate the case $\psi'(W) = U|V$ with $U = \epsilon$. We have then $\psi'(W1) = U|1UV = |1V$ which is also constructed by the formula on centres on $1V$, V and ϵ.

If the current letter of the directive word is 2 then the proof is similar by exchanging the roles of 1 and 2. ☐

7.4. Examples of palindromic closure on words and on discrete segments

To illustrate the construction on an example we use the following directive word $\Delta = 12112211\cdots$. Figure 4 at the beginning of this section shows the geometric palindromic closure in this case. The first words given by the palindromic closure on words given by ψ' are

$$|1$$

$$12|1$$

$$12|1121$$

indeed $W_2 = p_2 2|1s_2$ with $p_2 = 1$ and $s_2 = \epsilon$ hence $W_3 = p_2 2|1W_2$

$$12|1121121$$

indeed $W_3 = p_3 2|1s_3$ with $p_3 = 1$ and $s_3 = 121$ hence $W_4 = p_3 2|1W_3$

$$1211211212|1121121$$

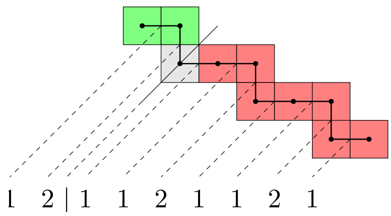
indeed $W_4 = p_4 2|1s_4$ with $p_4 = 1$ and $s_4 = 121121$ hence $W_5 = W_4 2|1s_4$

$$1211211212112112 12|1121121$$

$$121121121211211212|1121121121211211212121121121$$

$$121121121211211212|11211211212112112121121121121211211212121121121$$

At each step the word given by palindromic closure codes the adjacency of unit squares on the discrete segments given by geometric closure. For example 12|1121121 is exactly the coding of the geometric object S_4 with $\Delta = 1211$. Remark that 121121121, 1 and 121121 are palindromes in correspondence in the next figure with the whole object, the upper left and the lower right part respectively.



8. Conclusion and further work

By considering the fundamental objects proposed by G. Rauzy, we construct a geometric palindromic closure that leads to a generalization of palindromic closure in all dimensions. This construction can be seen as a generalization of the construction of Christoffel words in all dimension. Many open questions arise in particular to explore the link between our construction that gives a discrete hyperplane with a given height and the construction by substitutions on faces of Ito and Arnoux that gives the surface of a discrete plane. Our construction deals with discrete hyperplanes in dimension d and the construction of Ito Arnoux can be generalized to the surface of discrete hyperplanes in dimension d. One first interesting question is the link between the construction by the formalism $E_1^\star$ on faces (dimension $d-1$) [1] and our geometric object. For dimension 3, it is interesting to try to generate our object only by considering substitution on faces in order to have the upper surface and the lower surface of our object. Our construction is a generalization of Christoffel words [4, 8] in all dimension and Christoffel words in dimension 2 have many nice properties (balanceness, palindromicity of the central word, link with continued fraction and algebra). Which properties can be extended to all dimensions? Of course in our construction, the fundamental property of palindromicity of the whole object and of its d components is maintained in all dimensions but what about the other properties?

Our work is strongly linked to Rauzy fractal and to the theory of tilings and our proofs do not use any numerical systems. Can we transfer some properties of discrete geometry to the world of numerical systems?

Another interesting question is the link to the problem of connectedness of discrete hyperplanes. Given $v \in \mathbb{R}^d$ and $\mu, \omega \in \mathbb{R}$, the discrete hyperplane $\mathbb{P}(v, \mu, \omega)$ is the set of points of $x \in \mathbb{Z}^d$ which satisfy the double inequalities $0 \leqslant \langle v, x\rangle + \mu < \omega$. Such a hyperplane is connected as soon as ω is greater than some value denoted by $\Omega(v, \mu)$ and called the *connecting thickness* of v with shift μ [7, 10, 13]. Actually, $\Omega(v, \mu) = \Omega(v, 0) + \gamma(v, \mu)$ where $\gamma(v, \mu)$ is easy to compute. Therefore, the question is to determine $\Omega(v, 0)$ that we simply write $\Omega(v)$. We write also $\mathbb{P}(v, \omega)$ instead of $\mathbb{P}(v, 0, \omega)$. A very simple algorithm exists to compute $\Omega(v)$. First take the absolute value of v, which does not change $\Omega(v)$, and initialise Ω with 0. Then repeatedly subtract the smallest coordinate of v from other coordinates and add it to Ω. If a coordinate of v happens to vanish, then simply project v onto $\mathbb{R}^{d-1}$ by removing this coordinate. If v happens to have only one coordinate left, then we are done and Ω is the connecting thickness. If this never happens, then v and Ω converge to v^∞ and Ω^∞. In this case, the connecting thickness is $\Omega^\infty + \|v^\infty\|_\infty$.

Since $\mathbb{P}(v, \omega)$ is disconnected for any $\omega < \Omega(v)$ and connected for any $\omega > \Omega(v)$, the natural question which arises is whether $\mathbb{P}(v, \Omega(v))$ is connected or not. If some coordinate of v vanishes during the computation of $\Omega(v)$, then surely $\mathbb{P}(v, \Omega(v))$ is disconnected. In the other cases, let $\Delta = (\delta_n)_{n \geqslant 1}$ where δ_n is the index of the smallest coordinate of v at the n^{th} step of the algorithm. If some $k \in \{1, \dots, d\}$ does not appear infinitely many times in Δ, then again $\mathbb{P}(v, \Omega(v))$ is disconnected. If we apply our construction to Δ, it turns out that each S_n is actually included in $P(v, \Omega(v))$ and so is S_∞. As a matter of fact, Ω^∞ is exactly the value computed in the proof of Theorem 19 and C. Kraaikamp and R. Meester [16] have proven that $v^\infty = 0$ in this case, so that we actually have $\Omega(v) = \Omega^\infty$. We proved that S_∞ is connected but, although it is reasonable to conjecture it, we do not have a proof that $S_\infty = \mathbb{P}(v, \Omega(v))$. However, using results from β-numeration established by C. Frougny and B. Solomyak [12], this was proven [5] in the particular case where $v = (\alpha,\, \alpha + \alpha^2,\, 1)$ and $1/\alpha$ is the Tribonacci number, i.e., α is the real zero of $x^3 + x^2 + x - 1$. In this case, we have $\Delta = (123)^\omega$. The proof relies on $1/\alpha$ belonging to some specific class of Pisot numbers and could most probably be extended to some other cases where Δ is eventually periodic. However, the problem we consider here is much more general since Δ is arbitrary. If we were able to find an alternative proof that $S_\infty = \mathbb{P}(v, \Omega(v))$ whenever any k appears infinitely many times in Δ, it might have some implications in β-numeration.

Still connected to the same problem is the question of the nature of S_∞ in the general case. We conjecture that it is actually the connected component of 0 in $\mathbb{P}(v, \Omega^\infty)$ but the question of the nature of $\mathbb{P}(v, \Omega^\infty)$ itself is open since in the general case we have $\Omega^\infty < \Omega(v)$. For instance, if we take $v = (1, \sqrt{2}, \sqrt{3})$ then we can show $v^\infty = (11-9\sqrt{2}+\sqrt{3}, 0, 0)$ hence $\Omega^\infty = \frac{1}{2}(\|v\|_1 - \|v^\infty\|_1) = 5\sqrt{2}-5$ and $\Omega(v) = \Omega^\infty + \|v^\infty\|_\infty = 6 + \sqrt{3} - 4\sqrt{2}$.

However, we could well have $v^\infty = 0$, hence $\Omega^\infty = \Omega(v)$, and yet be sure that $S_\infty \neq P(v, \Omega(v)$. Take for instance $v = (1, \phi, 1+\phi)$ where ϕ is the golden ratio $\frac{1}{2}(1+\sqrt{5})$. In this case, we have $\Omega^\infty = \Omega(v) = 1 + \phi$ and $\Delta = (12)^\omega$. Since 3 never occurs in Δ, S_n never grows in the third dimension and thus may not be a plane.

Hence the right condition to have $S_\infty = \mathbb{P}(v, \Omega(v))$ seems to be that each $k \in \{1, \ldots, d\}$ occurs infinitely many times in Δ. The set of vectors v for which this happens has been shown to be negligible [16, 18].

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