On the topology of discrete hyperplanes[*]

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We deal with the connectedness of discrete hyperplanes $\mathrm{H}(\mathbf{v}, \mu, \omega) = \{\mathbf{x} \in \mathbb{Z}^n \mid 0 \leq \langle \mathbf{v}, \mathbf{x} < \omega \rangle\}$ with $\mathbf{v} \in \mathbb{R}^n$, $\mu \in \mathbb{R}$ and $\omega \in \mathbb{R}$.

Given a vector $\mathbf{v}$ and an intersect $\mu \in \mathbb{R}$, the question we investigate is how to calculate $\Omega(\mathbf{v}, \mu) = \inf \{\omega \in \mathbb{R} \mid \mathrm{H}(\mathbf{v}, \mu, \omega) \text{ is connected}\}$ of $\mathbb{R}$

Let S be a subset of $\mathbb{Z}$, let us recall that a *path* in S is a finite sequence $\pi = (\mathbf{x_1}, \ldots, \mathbf{x_k})$ such that $\mathbf{x}_i \in S$ for every $i \in \{1, \ldots, k\}$ and $d(\mathbf{x_i}, \mathbf{x_{i+1}}) = 1$, for every $i \in \{1, \ldots, k\}$, where d denotes Euclidean distance. One says that the path $\pi = (\mathbf{x_1}, \ldots, \mathbf{x_k})$ *links* $\mathbf{x_1}$ to $\mathbf{x_k}$. The set S is *connected* if, for each pair $(\mathbf{x}, \mathbf{y})$, there exists a path $\pi = (\mathbf{x_1}, \ldots, \mathbf{x_k})$ in S linking $\mathbf{x}$ to $\mathbf{y}$.

The first result we present is an algorithm allowing us to compute $\Omega(\mathbf{v}, \mu)$. This algorithm is already known as the *fully subtractive algorithm* [5] and has already been studied in [3, 4] in case of percolations models defined by rotations. It has been proved that the convergence set of the fully subtractive algorithm has Lebesgue-measure zero [3, 4, 2]. A direct consequence is the almost sure finiteness of the algorithm computing $\Omega(\mathbf{v}, \mu)$.

A natural question follows : what about the connectdness of $\mathrm{H}(\mathbf{v}, \mu, \omega)$ at the *critical value* $\omega = \Omega(\mathbf{v}, \mu)$?

Since $\mathrm{H}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ is connected only if the fully subtractive algorithm is convergent on the entry $\mathbf{v}$, it becomes natural to consider such a vector. Let α be the unique real eigenvalue of the matrix

$$\mathcal{M}_3 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and let $\mathbf{v} = \alpha \mathbf{e_1} + (\alpha + \alpha^2)\mathbf{e_2} + \mathbf{e_3}$ be the associated eigenvector. Then one computes, for each $\mu \in \mathbb{R}$, $\Omega(\mathbf{v}, \mu) = \dfrac{1+\alpha}{\alpha}$.

Let $(\mathbf{t_n})_{n \in \mathbb{N}}$ be a sequence of integer vectors satisfying $\langle \mathbf{t_n}, \mathbf{v} \rangle = \alpha^n$ and let P_n be a sequence of subsets of $\mathbb{Z}^3$ defined as follows (see Figure 1):

$$\begin{array}{rcccl} \mathrm{P} & : & \mathbb{Z} & \longrightarrow & \mathfrak{P}\left(\mathbb{Z}^3\right) \\ & & n & \longmapsto & \left| \begin{array}{ll} \mathrm{P}_n = \{(0,0,0)\} & \text{if } n = 0, \\ \mathrm{P}_n = \mathrm{P}_{n-1} \cup (\mathrm{P}_{n-1} + \mathbf{t}_n) & \text{if } n > 0 \end{array} \right. \end{array}$$

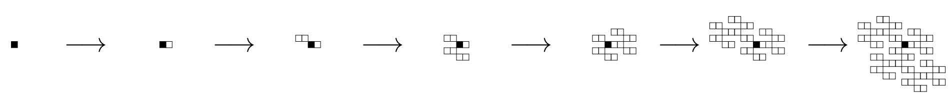

Figure 1: Step by step construction of P_n

It follows that, for all $n \geq 0$, the set P_n is connected and is a tree. Moreover, the set $\mathrm{P}_n \setminus \{\mathbf{0}\}$ has exactly 3 connected components (see Figure 2)

[*]Joint work with J.-L. Toutant (ISIT, Clermont-Ferrand), E. Domenjoud (Loria, Nancy), V. Berthé (Liafa, Paris 7) and X. Provençal (Lama, Chambéry)

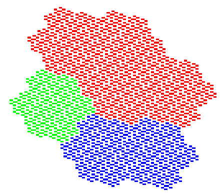

Figure 2: The three connected components of $\mathrm{P}_{14} \setminus \{\mathbf{0}\}$ in three different colors

It is easy to see that, for all $n \in \mathbb{N}$, $\mathrm{P}_n \subseteq \mathrm{H}\left(\mathbf{v}, 0, \dfrac{1+\alpha}{\alpha}\right)$. The other inclusion is a consequence of [1] which states that $\mathrm{Fin}(\alpha^{-1}) = \mathbb{Z}[\alpha]$, since $\alpha^{-3} - \alpha^{-2} - \alpha^{-1} - 1 = 0$. It follows that the set $\mathrm{H}\left(\mathbf{v}, 0, \dfrac{1+\alpha}{\alpha}\right)$ is a connected, is a tree and the set $\mathrm{H}\left(\mathbf{v}, 0, \dfrac{1+\alpha}{\alpha}\right) \setminus \{\mathbf{0}\}$ has exactly 3 connected components.

The last problem we investigate is the role of μ in the connectedness of $\mathrm{H}\left(\mathbf{v}, \mu, \dfrac{1+\alpha}{\alpha}\right)$ and we show that the discrete plane $\mathrm{H}\left(\mathbf{v}, \dfrac{1+\alpha}{\alpha}, \dfrac{1+\alpha}{\alpha}\right)$ is not connected since it is a symmetric of the set $\mathrm{H}\left(\mathbf{v}, \mu, \dfrac{1+\alpha}{\alpha}\right) \setminus \{\mathbf{0}\}$.

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