# An asymptotical method to estimate the parameters of a deteriorating system under condition-based maintenance

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Philippe Briand[1], Edwige Idée[2] and Céline Labart[3]

Laboratoire de Mathématiques, CNRS UMR 5127, Université de Savoie,

Campus Scientifique, 73376 Le Bourget du Lac, France.

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**Abstract**

In this paper, we develop a new method to estimate the parameters of a deteriorating system under perfect condition-based maintenance. This method is based on the asymptotical behavior of the system, which is studied by using the renewal process theory. We obtain a Central Limit Theorem (CLT in the following) for the parameters. We compare the accuracy and the speed of the method with the maximum likelihood one (ML method in the following) on different examples.

## 1 Introduction

Many systems suffer from increasing wear with usage and age and are subject to failures resulting from deteriorations. The deterioration and failures might lead to high costs, then preventive maintenance is necessary. In the past several decades, maintenance, replacement and inspection problems have been widely studied (see the surveys [McC65], [BP96], [PV76], [ON76] and [SS81], [VFF89] among others). If the deterioration of a system can be observed while inspecting, it is more judicious to set up the maintenance policy on the state of the system rather than on its age. Deterioration systems and their optimal maintenance policy have been studied in the literature (see [MK75], [OKM86], [TvdDS85], [BSK96], [Wan00] and [GBD02]). In this paper, we consider a system subject to random deteriorations, which can lead to failures. As long as the system operates, it is monitored by planned inspections. At these inspections, the system can be in two states : sane or damaged. If the system is found out to be damaged, a preventive perfect repairing ("as good as new") is performed. If the system fails, an unplanned inspection is performed immediately and a corrective perfect repairing ("as good as new") is done.

When the deterioration of the system can be continuously measured, the deterioration is usually represented by a stochastic process with stationary and independent increments and the states of the system are fixed by some thresholds on the stochastic process (see [GBD02], [GDBR02] for example). However, some deteriorations can not be easily measured and the state of the system is only known when inspecting. In order to deal with this kind of problem, we assume that the transition time from repairing to damaging and the transition time from damaging to failure are two positive random variables, whose parameters $(\mu, \lambda)$ are unknowns (we consider that failures only ensue from deteriorations). Moreover, we assume that there exists some uncertainty on planned

[1] philippe.briand@univ-savoie.fr.
[2] edwige.idee@univ-savoie.fr.
[3] celine.labart@univ-savoie.fr.

inspection dates (which can be due to physical or financial constraints). The purpose of the present paper is to propose a new method, based on the asymptotic behavior of the system, to estimate the unknown parameters of the transition times. At some time $t$, we assume that we only know the number of inspections before $t$, and the state of the system at the inspection dates. Then, we have at hand $N_t^r$, the number of repairs before time $t$, $N_t^i$, the number of inspections before time $t$, and $N_t^f$, the number of failures before time $t$. By using the renewal processes theory, we write $(\mu, \lambda)$ as the limit of a function of $(\frac{N_t^r}{t}, \frac{N_t^i}{t}, \frac{N_t^f}{t})$. Moreover, we get a CLT for this triplet of variables, which gives us a confidence interval for $(\mu, \lambda)$.

Practically, this method not only applies to long term systems. We can also deal with identical units, repaired at least one time, operating independently and simultaneously in a similar environment and being analogously exploited. Putting repairing times of these units end to end is equivalent to studying a single system on a long period of time. The paper is organized as follows : Section 2 introduces some notations and presents the assumptions on the model, Section 3 recalls standard results on renewal and renewal reward processes. Section 4 states a CLT for our parameters. Sections 5 and 6 present some applications and numerical examples.

## 2 Model Assumptions

In the following, we represent the time from repairing to damaging by a positive random variable (r.v. in the following) $Y^s$ and the time from damaging to failure by a positive r.v. $Y^d$. We modelize the uncertainty of inspection dates (which can be due to physical or financial constraints) by a sequence of random variables. The elapsed time between two inspection dates is a random variable $C$, and if $C_i$ denotes the time spent between the $(i-1)$th and the $i$th inspection, $D_i := C_1 + \cdots + C_i$ is the age of a system at the $i$th inspection (with convention $D_0 = 0$). This enables to define the index of the inspection following the damage

$$K^r = \inf\{n \geq 1 : D_n \geq Y^s\} = 1 + \sum_{n \geq 1} \mathbf{1}_{D_n < Y^s}.$$

and the inter-repairing time

$$X^r = \min\left(D_{K_r}, Y^s + Y^d\right).$$

This means that we repair the system as soon as a damage is detected ($X^r = D_{K_r}$) (see Figure 1) or as soon as the system fails ($X^r = Y^s + Y^d$) (see Figure 2). In the following, we assume that the law of $Y^s$ depends of a parameter $\mu$, the law of $Y^d$ depends of a parameter $\lambda$, and the law of $C$ is known.

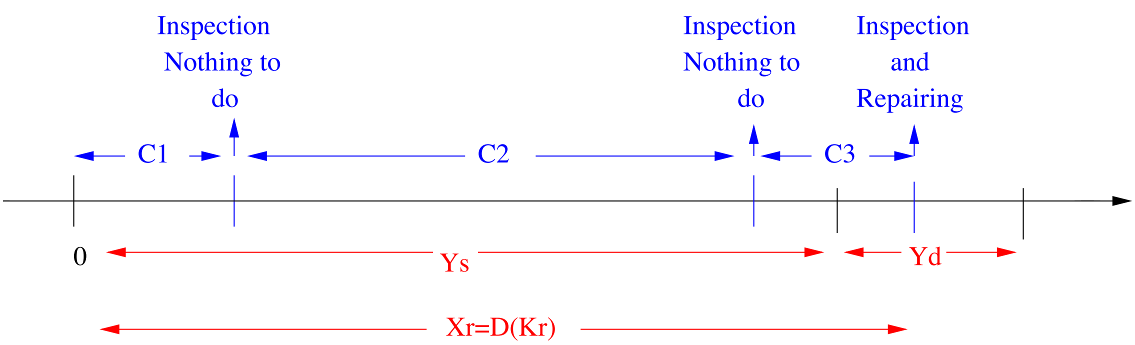


Figure 1: The system is repaired as soon as the damage is detected

We introduce the following notations :

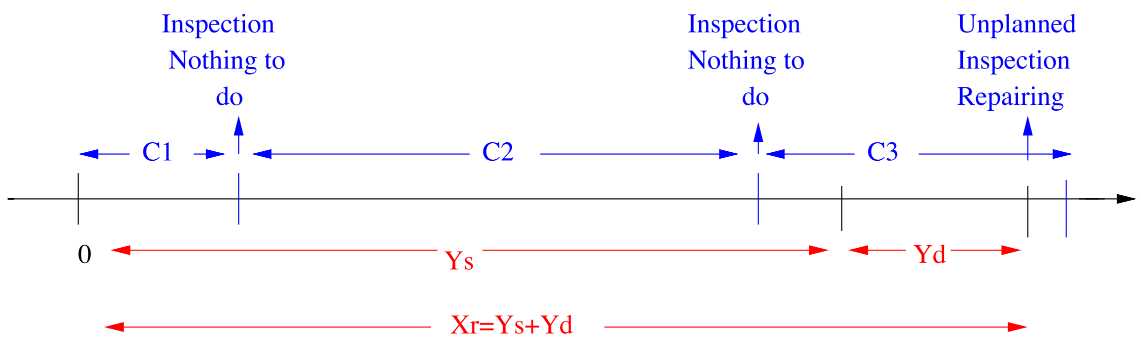


Figure 2: The system is repaired as soon as it fails

**Definition 2.1.**

- $F_s$ *(resp.* $F_d$*) denotes the cumulative distribution function of* $Y^s$ *(resp.* $Y^d$*)*
- $R_s$ *(resp.* $R_d$*) denotes the survival function of* $Y^s$ *(resp.* $Y^d$*)*
- $n_\mu$ *denotes the law of* $Y^s$
- $L$ *denotes the Laplace transform of* $C$ *:* $\forall s \geq 0$, $L(s) := \mathbb{E}(e^{-sC})$
- $\overline{B}(t)$ *represents the index of the inspection following* $t$ *:* $\overline{B}(t) := \inf\{n \geq 1 : D_n \geq t\} = 1 + \sum_{n\geq 1} \mathbf{1}_{D_n<t}$*. Then,* $K^r = \overline{B}(Y^s)$
- $\underline{B}(t)$ *represents the index of the inspection before* $t$ *:* $\underline{B}(t) := \sup\{n \geq 0 : D_n \leq t\}$
- $V^s$ *denotes the age of the system when a damage is detected :* $V^s = D_{K^r}$
- $Z^d$ *denotes the time of failure :* $Z^d = Y^s + Y^d$
- $P_d$ *denotes the probability that the system fails before the inspection following the damage occurs :* $P_d = \mathbb{P}(V^s \geq Z^d) = \mathbb{P}(D_{\overline{B}(Y^s)} - Y^s \geq Y^d)$
- $m_x$ *denotes* $\mathbb{E}(X^r)$ *and* $m_k$ *denotes* $\mathbb{E}(K^r)$

*With these notations,* $X^r = \min(V^s, Z^d)$*.*

**Hypothesis 2.2.** *In the following, we assume that the r.v.* $Y^s$ *and* $Y^d$ *are such that the law of* $Y^s$ *depends on a parameter* $\mu$ *taking values in* $U$ *and the law of* $Y^d$ *depends on a parameter* $\lambda$ *taking values in* $V$*. The r.v.* $(C_i)_{i\geq 1}$ *are independent with common law* $C$*, and such that* $\mathbb{P}(C > 0) = 1$*. We also assume that they are independent of* $Y^s$ *and* $Y^d$*. Moreover, we assume* $\mathbb{E}((Y^s)^2) + \mathbb{E}(C^2) < \infty$*.*

*Remark* 2.3. Under Hypothesis 2.2, $\mathbb{V}(X^r) + \mathbb{V}(K^r) < \infty$. We first prove that $\mathbb{E}((K^r)^2) < \infty$. $\mathbb{E}((K^r)^2) = \int_0^\infty \mathbb{E}(\overline{B}(t)^2) n_\mu(dt) = \int_0^\infty \frac{\mathbb{E}(\overline{B}(t)^2)}{t^2} t^2 n_\mu(dt)$. $(\overline{B}(t))_{t\geq 0}$ is a renewal process with interarrival time $C$, then $\frac{\mathbb{E}(\overline{B}(t)^2)}{t^2} \to \frac{1}{(\mathbb{E}(C))^2}$. Since $\exists c > 0$ such that $\forall t \geq 0$ $\mathbb{E}(\overline{B}(t)^2) \leq ce^t$, we get $\forall \varepsilon > 0$, $\exists t_0$ such that $\mathbb{E}((K^r)^2) \leq ce^{t_0} + (\frac{1}{\mathbb{E}(C)^2} + \varepsilon)\mathbb{E}((Y^s)^2)$. Concerning $\mathbb{E}((X^r)^2)$, we have $\mathbb{E}((X^r)^2) \leq \mathbb{E}((V^s)^2)$ and $V^s \leq Y^s + C_{\overline{B}(Y^s)}$. Then, $\mathbb{E}((V^s)^2) \leq 2(\mathbb{E}((Y^s)^2) + \mathbb{E}(C^2_{\overline{B}(Y^s)}))$. Since we have

$$\mathbb{E}(C^2_{\overline{B}(Y^s)}) = \sum_{k=1}^\infty \mathbb{E}(C_k^2 \mathbf{1}_{\overline{B}(Y^s)=k}) = \sum_{k=1}^\infty \mathbb{E}(C_k^2 \mathbf{1}_{D_{k-1}<Y^s\leq D_k}) \leq \sum_{k=1}^\infty \mathbb{E}(C^2) R^s(D_{k-1}) = \mathbb{E}(C^2)\mathbb{E}(K^r),$$

the result follows (the last equality comes from (1)).

**Proposition 2.4.** *Under Hypothesis 2.2, we have*

$$\mathbb{E}\left[K^r\right] = \sum_{n\geq 0} \mathbb{E}\left[R_s(D_n)\right], \tag{1}$$

$$1 - P_d = \int_0^{+\infty} \mathbb{E}\left[R_d\left(D_{\overline{B}(t)} - t\right)\right] \, n_\mu(dt), \tag{2}$$

$$\mathbb{E}\left[X^r\right] = \mathbb{E}\left[Y^s\right] + \int_0^{+\infty}\int_0^{+\infty} \mathbb{P}\left(D_{\overline{B}(t)} - t > y\right) R^d(y)\, dy\, n_\mu(dt). \tag{3}$$

*If in addition we assume that $Y^d$ follows the exponenial law, we get*

$$1 - P_d = (1 - L(\lambda)) \sum_{k\geq 1} \mathbb{E}\left[e^{-\lambda D_k} \int_0^{D_k} e^{\lambda t} n_\mu(dt)\right] - L(\lambda) n_\mu\left(\{0\}\right), \tag{4}$$

$$\mathbb{E}\left[X^r\right] = \mathbb{E}\left[Y^s\right] + \frac{1}{\lambda}\mathbb{P}\left(V^s \geq Z^d\right) = \mathbb{E}\left[Y^s\right] + \frac{1}{\lambda} P_d. \tag{5}$$

*Proof.* By independence, one gets (1) :

$$\mathbb{E}\left[K^r\right] = 1 + \sum_{n\geq 1} \mathbb{E}\left[\mathbf{1}_{D_n < Y^s}\right] = 1 + \sum_{n\geq 1} \mathbb{E}\left[R_s(D_n)\right] = \sum_{n\geq 0} \mathbb{E}\left[R_s(D_n)\right].$$

Let us prove (2). By independence, we have

$$P_d = \mathbb{P}\left(D_{\overline{B}(Y^s)} - Y^s \geq Y^d\right) = \int_0^{+\infty} \mathbb{P}\left(D_{\overline{B}(t)} - t \geq Y^d\right) n_\mu(dt) = \int_0^{+\infty} \mathbb{E}\left[F_d\left(D_{\overline{B}(t)} - t\right)\right] n_\mu(dt),$$

and the result follows. If $Y^d$ follows an exponential law of parameter $\lambda$,

$$\begin{aligned}
1 - P_d &= \int_0^{+\infty} \mathbb{E}\left[e^{-\lambda\left(D_{\overline{B}(t)} - t\right)}\right] \, n_\mu(dt) = \int_0^{+\infty} \sum_{k\geq 1} \mathbb{E}\left[e^{-\lambda(D_k - t)} \mathbf{1}_{\overline{B}(t) = k}\right] \, n_\mu(dt), \\
&= \int_0^{+\infty} \sum_{k\geq 1} \mathbb{E}\left[e^{-\lambda(D_k - t)} \mathbf{1}_{D_{k-1} < t \leq D_k}\right] \, n_\mu(dt), \\
&= \sum_{k\geq 1} \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda(D_k - t)} \left(\mathbf{1}_{t\leq D_k} - \mathbf{1}_{t \leq D_{k-1}}\right) \, n_\mu(dt)\right], \\
&= \sum_{k\geq 1} \mathbb{E}\left[e^{-\lambda D_k} \int_0^{+\infty} e^{\lambda t} \mathbf{1}_{t\leq D_k}\, n_\mu(dt) - e^{-\lambda C_k} e^{-\lambda D_{k-1}} \int_0^{+\infty} e^{\lambda t} \mathbf{1}_{t\leq D_{k-1}}\, n_\mu(dt)\right].
\end{aligned}$$

By independence,

$$1 - P_d = \sum_{k\geq 1} \left(\mathbb{E}\left[e^{-\lambda D_k} \int_0^{+\infty} e^{\lambda t} \mathbf{1}_{t\leq D_k}\, n_\mu(dt)\right] - L(\lambda)\, \mathbb{E}\left[e^{-\lambda D_{k-1}} \int_0^{+\infty} e^{\lambda t} \mathbf{1}_{t\leq D_{k-1}}\, n_\mu(dt)\right]\right),$$

and (4) follows.

In order to prove (3), we compute the survival function of $X^r$.

$$\begin{aligned}
\mathbb{P}\left(X^r > x\right) &= \mathbb{P}\left(\min\left(V^s, Z^d\right) > x\right) = \mathbb{P}\left(D_{\overline{B}(Y^s)} > x, Y^s + Y^d > x\right), \\
&= \mathbb{P}\left(Y^s > x\right) + \mathbb{P}\left(D_{\overline{B}(Y^s)} > x, Y^s + Y^d > x, Y^s \leq x\right), \\
&= R_s(x) + \mathbb{E}\left[\mathbf{1}_{D_{\overline{B}(Y^s)} > x} \mathbf{1}_{Y^s \leq x} R_d\left(x - Y^s\right)\right], \\
&= R_s(x) + \int_0^{+\infty} \mathbb{P}\left(D_{\overline{B}(t)} - t > x - t\right) R^d(x - t) \mathbf{1}_{x - t > 0}\, n_\mu(dt).
\end{aligned}$$

Then,

$$\begin{aligned}\mathbb{E}\left[X^r\right] &= \int_0^{+\infty} \mathbb{P}\left(X^r > x\right) dx \\ &= \mathbb{E}\left[Y^s\right] + \int_0^{+\infty}\int_0^{+\infty} \mathbb{P}\left(D_{\overline{B}(t)} - t > x - t\right) R^d(x-t)\mathbf{1}_{x-t>0}\, n_\mu(dt)\, dx, \\ &= \mathbb{E}\left[Y^s\right] + \int_0^{+\infty}\int_0^{+\infty} \mathbb{P}\left(D_{\overline{B}(t)} - t > x - t\right) R^d(x-t)\mathbf{1}_{x-t>0}\, dx\, n_\mu(dt),\end{aligned}$$

and the change of variable $y = x - t$ gives the result. If $Y^d$ follows the exponential law with density $f_d$, $R_d(y) = \frac{f^d(y)}{\lambda}$, and (5) follows.

□

# 3 Renewal Theory

## 3.1 Renewal and Renewal Reward processes

In this section, we recall classical results on renewal processes. We refer to [CT97], [Asm03] [AJ99, Appendix B] among others. Let $(X_n)_{n\geq 0}$ be a sequence of nonnegative independent identically distributed (i.i.d.) r.v. with distribution $F$. We assume that $\mathbb{P}(X=0) < 1$. We also define the sequence $(T_n)_n$

$$T_0 = 0, \;\; T_n = \sum_{i=1}^{n} X_i, \; n \in \mathbb{N}^*$$

and

$$N_t = \sup\{j : t_j \leq t\} = \sum_{i=1}^{\infty} \mathbf{1}_{T_j \leq t}$$

$(N_t)_{t\geq 0}$ is a renewal process, and we have

**Theorem 3.1** (Almost sure convergence, $\mathrm{L}^1$ convergence and CLT for $N_t$)**.** *We have*

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}(X)} \, a.s. \text{ and in } \mathrm{L}^1.$$

*Assume that $\mathbb{V}(X) < \infty$. Then, $N_t$ satisfies the following CLT*

$$\lim_{t\to\infty} \sqrt{t}\left(\frac{N_t}{t} - \frac{1}{\mathbb{E}(X)}\right) \overset{law}{=} \mathcal{N}\left(0, \frac{\mathbb{V}(X)}{\mathbb{E}(X)^3}\right).$$

Let $(X_n, Y_n)_{n\geq 1}$ denote a sequence of i.i.d. pairs of r.v.. $Y_i$ can be interpreted as the reward associated with the $j^{th}$ interarrival time $X_j$ ($Y_j$ may depend on $X_j$). The process $Z_t$ defined by

$$Z_t = \sum_{i=1}^{N_t} Y_i$$

satisfies the following theorem

**Theorem 3.2** (Almost sure convergence, $\mathrm{L}^1$ convergence and CLT for $Z_t$)**.** *We have*

$$\lim_{t\to\infty} \frac{Z_t}{t} = \frac{\mathbb{E}(Y)}{\mathbb{E}(X)} \text{ almost surely and in } \mathrm{L}^1.$$

*Assume that $\mathbb{V}(Y) < \infty$ and $\mathbb{V}(X) < \infty$. Then, $Z_t$ satisfies the following CLT*

$$\lim_{t\to\infty} \sqrt{t}\left(\frac{Z_t}{t} - \frac{\mathbb{E}(Y)}{\mathbb{E}(X)}\right) \overset{law}{=} \mathcal{N}\left(0, \frac{\mathbb{V}(Y - \frac{\mathbb{E}(Y)}{\mathbb{E}(X)}X)}{\mathbb{E}(X)^2}\right).$$

The second part of Theorem 3.2 ensues from Rényi's theorem, recalled below, which will be useful in the following.

**Theorem 3.3** ([Rén57], Theorem 1). *Let $(\xi_n)_{n\geq 1}$ be a sequence of i.i.d. square integrable r.v. such that $\sigma^2 = \mathbb{E}[\xi^2]$. Let $(\nu_t)_{t\geq 0}$ be a process taking values in $\mathbb{N}$ such that $\nu_t/t$ converges in probability when $t \to \infty$ to a constant $c > 0$. Then*

$$\frac{\sum_{i=1}^{\nu_t} \xi_i}{\sqrt{\nu_t}} \longrightarrow \mathcal{N}(0,\sigma^2), \quad \frac{\sum_{i=1}^{\nu_t} \xi_i}{\sqrt{t}} \longrightarrow \mathcal{N}(0, c\,\sigma^2).$$

## 3.2 Age and Survival of a renewal process

Let $(A_t, S_t)_{t\geq 0}$ denotes the age and the survival of a renewal process : $A_t := t - T_{N_t}$ and $S_t := T_{N_t+1} - t$. The following result gives the limiting laws of both processes :

**Proposition 3.4.** *Let us consider a renewal process with interarrival time $X$ such that $\mathbb{E}(X) < \infty$. Then*

$$\lim_{t\to\infty} \mathbb{P}(A_t \leq x) = \lim_{t\to\infty} \mathbb{P}(S_t \leq x) = \frac{1}{\mathbb{E}[X]} \int_0^x \mathbb{P}(X > u)du,$$

*and their densities are given by* $p_{A_\infty}(x) = p_{S_\infty}(x) = \dfrac{1}{\mathbb{E}[X]}\mathbb{P}(X > x)$.

**Lemma 3.5.** *The process $(\frac{\underline{B}(A_t)}{\sqrt{t}})_{t\geq 0}$ converges in probability to $0$ when $t$ tends to infinity.*

*Proof.* Let us first prove that $(\frac{A_t}{\sqrt{t}})_{t\geq 0}$ converges in probability to 0 when $t$ tends to infinity. Indeed, by using the Markov inequality, we get $\forall \varepsilon < 1$, $\mathbb{P}\left(\frac{A_t}{\sqrt{t}} > \varepsilon\right) = \mathbb{P}\left(\frac{A_t}{\sqrt{t}} \wedge 1 > \varepsilon\right) \leq \frac{\mathbb{E}\left[\frac{A_t}{\sqrt{t}} \wedge 1\right]}{\varepsilon}$. Since the age of a renewal process converges in law when $t \to +\infty$ (see Proposition 3.4), we get

$$\forall a > 0, \qquad \limsup_{t\to\infty} \mathbb{E}\left[\frac{A_t}{\sqrt{t}} \wedge 1\right] \leq \limsup_{t\to\infty} \mathbb{E}\left[\frac{A_t}{\sqrt{a}} \wedge 1\right] = \mathbb{E}\left[\frac{A_\infty}{\sqrt{a}} \wedge 1\right] \leq \frac{\mathbb{E}[A_\infty]}{\sqrt{a}}.$$

Then, $\forall \varepsilon < 1$, $\forall a > 0$, $\limsup_{t\to\infty} \mathbb{P}\left(\frac{A_t}{\sqrt{t}} > \varepsilon\right) \leq \frac{\mathbb{E}[A_\infty]}{\varepsilon\sqrt{a}}$ and the result follows. Let us now prove the Lemma. $\forall \varepsilon < 1$,

$$\mathbb{P}\left(\frac{\underline{B}(A_t)}{\sqrt{t}} > \varepsilon\right) = \mathbb{P}\left(\underline{B}(A_t) > \varepsilon\sqrt{t}\right) = \mathbb{P}\left(\sum_{i=1}^{\lceil \varepsilon\sqrt{t}\rceil} C_i \leq A_t\right) = \mathbb{P}\left(\frac{\varepsilon}{\varepsilon\sqrt{t}} \sum_{i=1}^{\lceil \varepsilon\sqrt{t}\rceil} C_i \leq \frac{A_t}{\sqrt{t}}\right).$$

Let $\eta > 0$. It remains to split the last probability in two parts by introducing the set $\{\left|\frac{1}{\varepsilon\sqrt{t}} \sum_{i=1}^{\lceil \varepsilon\sqrt{t}\rceil} C_i - \mathbb{E}(C)\right| > \eta\}$ and its complement. The strong law of large numbers and the convergence in probability of $(\frac{A_t}{\sqrt{t}})_{t\geq 0}$ end the proof. □

# 4 Main Results

## 4.1 Number of repairs, number of failures and number of inspections

**Definition 4.1.** *Since the system is repaired as good as new at each repairing date, the number of repairs at time $t$ is a renewal process given by*

$$N_t^r = \sum_{i=1}^{\infty} \mathbf{1}_{T_i^r \leq t},$$

*where $T_i^r = \sum_{j=1}^{i} X_j^r$.*

**Definition 4.2.** *Let $X^f$ be the r.v. representing the time between two consecutive failures :*

$$X^f \stackrel{law}{=} \sum_{i=1}^{\tau} X_i, \quad \textit{where } \tau = \inf\left\{i \geq 1 : V_i^s \geq Z_i^d\right\} \quad \inf \emptyset = +\infty.$$

*The number of failures at time $t$ is given by*

$$N_t^f = \sum_{i=1}^{\infty} \mathbf{1}_{T_i^f \leq t},$$

*where $T_i^f = \sum_{j=1}^{i} X_j^f$. Then, $N_t^f$ is a renewal process with interarrival time $X^f$.*

*Remark* 4.3. $N_t^f$ is also a renewal reward process, since $N_t^f = \sum_{i=1}^{N_t^r} \mathbf{1}_{V_i^s \geq Z_i^d}$.

*Remark* 4.4. The number of damages $N^d$ is a renewal process with interarrival time $X^d \stackrel{law}{=} \sum_{i=1}^{\sigma} X_i, \quad$ where $\sigma = \inf\left\{i \geq 1 : Z_i^d > V_i^s\right\}$. As for $N^f$, we can write $N_t^d = \sum_{i=1}^{N_t^r} \mathbf{1}_{Z_i^d \geq V_i^s}$.

**Definition 4.5.** *The number of inspections at time $t$, denoted $N_t^i$, is given by*

$$N_t^i = \sum_{i=1}^{N_t^r} K_i^r + \underline{B}(t - T_{N_t^r}),$$

*where $K_i^r$ denotes the number of inspections on the interval $]T_{i-1}^r, T_i^r]$.*

**Theorem 4.6** (Almost sure and $\mathrm{L}^1$ convergences for $\frac{N_t^r}{t}$, $\frac{N_t^f}{t}$ and $\frac{N_t^i}{t}$)**.** *The following results hold almost surely and in* $\mathrm{L}^1$

$$\lim_{t\to+\infty} \frac{N_t^r}{t} = \frac{1}{m_x}, \quad \lim_{t\to+\infty} \frac{N_t^f}{t} = \frac{P_d}{m_x}, \quad \textit{and} \quad \lim_{t\to+\infty} \frac{N_t^i}{t} = \frac{m_k}{m_x}. \tag{6}$$

*Proof.* The first and second results ensue from Theorem 3.1 and from Wald's identity, since $\mathbb{E}(X^f) = \mathbb{E}(\tau)\mathbb{E}(X^r)$ and $\tau$ has a geometric law of parameter $P_d$. From the definition of $N_t^i$, we get $\sum_{i=1}^{N_t^r} K_i \leq N_t^i \leq \sum_{i=1}^{N_t^r+1} K_i$. Then,

$$\frac{1}{N_t^r} \sum_{i=1}^{N_t^r} K_i^r \leq \frac{N_t^i}{N_t^r} \leq \frac{N_t^r + 1}{N_t^r} \frac{1}{N_t^r + 1} \sum_{i=1}^{N_t^r+1} K_i^r.$$

Since $\lim_{t\to\infty} N_t^r = \infty$ almost surely, we get $\lim_{t\to\infty} \frac{N_t^i}{N_t^r} = \mathbb{E}(K^r)$. The strong law of large numbers for renewal processes yields the almost sure convergence of $\frac{N_t^i}{t}$.
Let us now deal with the $\mathrm{L}^1$-convergence. We have

$$\mathbb{E}[\sum_{i=1}^{N_t^r} K_i^r] \leq \mathbb{E}[N_t^i] \leq \mathbb{E}[\sum_{i=1}^{N_t^r+1} K_i^r]. \tag{7}$$

We first deal with the right hand side. $N_t^r + 1$ is an $\mathcal{F}$-stopping time ($\{N_t^r + 1 = k\} = \{N_t^r = k - 1\} = \{T_{k-1}^r \leq t < T_k^r\}$), then

$$\mathbb{E}\left[N_t^i\right] \leq \mathbb{E}\left[\sum_{k=1}^{N_t^r+1} K_i^r\right] = \mathbb{E}\left[N_t^r + 1\right] \mathbb{E}\left[K_i^r\right]. \tag{8}$$

Concerning the left hand side, we write

$$\mathbb{E}\left[\sum_{k=1}^{N_t^r} K_i^r\right] = \mathbb{E}\left[\sum_{k\geq 1} K_i^r \mathbf{1}_{T_k \leq t}\right] = \sum_{k\geq 1} \mathbb{E}\left[K_i^r \mathbf{1}_{T_k^r \leq t}\right].$$

By symetry, we get $\mathbb{E}\left[\sum_{k=1}^{N_t^r} K_i^r\right] = \mathbb{E}\left[K_1^r \sum_{k\geq 1} \mathbf{1}_{T_k\leq t}\right] = \mathbb{E}\left[K_1^r N_t^r\right]$. Combining this result with (7) and (8), we obtain

$$\mathbb{E}\left[K_1^r N_t^r\right] \leq \mathbb{E}\left[N_t^i\right] \leq \mathbb{E}\left[N_t^r+1\right]\mathbb{E}\left[K^r\right].$$

Since $\lim_{t\to\infty} N_t^r/t \longrightarrow 1/\mathbb{E}\left[X^r\right]$, Fatou's lemma gives $\liminf_{t\to+\infty} \frac{1}{t}\,\mathbb{E}\left[K_1^r N_t^r\right] \geq \frac{\mathbb{E}[K^r]}{\mathbb{E}[X^r]}$. The elementary renewal theorem gives $\lim_{t\to+\infty} \frac{1}{t}\mathbb{E}\left[N_t^r+1\right]\mathbb{E}\left[K^r\right] = \frac{\mathbb{E}[K^r]}{\mathbb{E}[X^r]}$ and the last result follows. □

**Theorem 4.7** (CLT for $\frac{N_t^r}{t}$, $\frac{N_t^f}{t}$ and $\frac{N_t^i}{t}$)**.** *Let us denote $X := X^r$, $I := \mathbf{1}_{V^s\geq Z^d}$, $K := K^r$. The following result holds*

$$\sqrt{t}\left(\frac{N_t^r}{t}-\frac{1}{m_x}, \frac{N_t^f}{t}-\frac{P_d}{m_x}, \frac{N_t^i}{t}-\frac{m_k}{m_x}\right) \longrightarrow \mathcal{N}(0,R),$$

*where*

$$R = (m_x)^{-3}\begin{pmatrix} \mathbb{V}[X] & \mathrm{Cov}[X, P_dX-m_xI] & \mathrm{Cov}[X, m_kX-m_xK] \\ \mathrm{Cov}[X, P_dX-m_xI] & \mathbb{V}[P_dX-m_xI] & \mathrm{Cov}[P_dX-m_xI, m_kX-m_xK] \\ \mathrm{Cov}[X, m_kX-m_xK] & \mathrm{Cov}[P_dX-m_xI, m_kX-m_xK] & \mathbb{V}[m_kX-m_xK] \end{pmatrix}.$$

*Proof.* Let $(X_n, Y_n, Z_n)$ be a sequence of i.i.d. square integrable r.v. in $\mathbb{R}_+^3$. One denotes $\lambda_x = \mathbb{E}[X]$, $\lambda_y = \mathbb{E}[Y]$ and $\lambda_z = \mathbb{E}[Z]$. We denote $S_n^x = X_1+\ldots+X_n$, $S_n^y = Y_1+\ldots+Y_n$, $S_n^z = Z_1+\ldots+Z_n$ and $N_t^x = \sup\{n\geq 1 : S_n^x \geq t\}$.

We consider the triplet $Q_n = (S_n^x - n\lambda_x, \lambda_x\, S_n^y - \lambda_y\, S_n^x, \lambda_x\, S_n^z - \lambda_z\, S_n^x)$. CLT gives us that $\frac{1}{\sqrt{n}}Q_n \longrightarrow \mathcal{N}(0,V)$ where

$$V = \begin{pmatrix} \mathbb{V}[X] & \mathrm{Cov}[X, \lambda_xY-\lambda_yX] & \mathrm{Cov}[X, \lambda_xZ-\lambda_zX] \\ \mathrm{Cov}[X, \lambda_xY-\lambda_yX] & \mathbb{V}[\lambda_xY-\lambda_yX] & \mathrm{Cov}[\lambda_xY-\lambda_yX, \lambda_xZ-\lambda_zX] \\ \mathrm{Cov}[X, \lambda_xZ-\lambda_zX] & \mathrm{Cov}[\lambda_xY-\lambda_yX, \lambda_xZ-\lambda_zX] & \mathbb{V}[\lambda_xZ-\lambda_zX] \end{pmatrix}$$

By applying Rényi's theorem (see Theorem 4.6) to the real r.v. $u\cdot Q_{N_t^x}/\sqrt{N_t^x}$ we obtain

$$\frac{1}{\sqrt{N_t^x}}Q_{N_t^x} \longrightarrow \mathcal{N}(0,V), \text{ and } \qquad \frac{1}{\sqrt{t}}Q_{N_t^x} \longrightarrow \mathcal{N}(0, \lambda_x^{-1}\,V).$$

Let $A_t^x$ denote the age of the renewal process $N^x$, i.e. $A_t^x := t - S_{N_t^x}^x$. We have

$$\begin{aligned} Q_{N_t^x} &= \left(S_{N_t^x}^x - \lambda_x N_t^x, \lambda_x\, S_{N_t^x}^y - \lambda_y\, S_{N_t^x}^x, \lambda_x\, S_{N_t^x}^z - \lambda_z\, S_{N_t^x}^x\right) \\ &= \left(t-\lambda_xN_t^x, \lambda_x\, S_{N_t^x}^y - \lambda_y t, \lambda_x\left(S_{N_t^x}^z + \underline{B}(A_t^x)\right) - \lambda_z t\right) + (1,\lambda_y,\lambda_z)A_t^x - (0,0,\lambda_x)\underline{B}(A_t^x). \end{aligned}$$

Combining Lemma 3.5 and Slutsky's Lemma yields

$$\frac{1}{\sqrt{t}}\left(t-\lambda_xN_t^x, \lambda_x\, S_{N_t^x}^y - \lambda_y t, \lambda_x\left(S_{N_t^x}^z + \underline{B}(A_t^x)\right) - \lambda_z t\right) \longrightarrow \mathcal{N}(0, \lambda_x^{-1}V)$$

Combining this result and the application $(x,y,z) \longmapsto (-x,y,z)/\lambda_x$ gives

$$\frac{1}{\sqrt{t}}\left(N_t^x - \frac{t}{\lambda_x}, S_{N_t^x}^y - \frac{\lambda_y t}{\lambda_x}, S_{N_t^x}^z + \underline{B}(A_t^x) - \frac{\lambda_z t}{\lambda_x}\right) \longrightarrow \mathcal{N}(0,R),$$

where

$$R = \lambda_x^{-3}\begin{pmatrix} \mathbb{V}[X] & \mathrm{Cov}[X, \lambda_yX-\lambda_xY] & \mathrm{Cov}[X, \lambda_zX-\lambda_xZ] \\ \mathrm{Cov}[X, \lambda_yX-\lambda_xY] & \mathbb{V}[\lambda_yX-\lambda_xY] & \mathrm{Cov}[\lambda_yX-\lambda_xY, \lambda_zX-\lambda_xZ] \\ \mathrm{Cov}[X, \lambda_zX-\lambda_xZ] & \mathrm{Cov}[\lambda_yX-\lambda_xY, \lambda_zX-\lambda_xZ] & \mathbb{V}[\lambda_zX-\lambda_xZ] \end{pmatrix}.$$

To conclude, it remains to choose $X = X^r$, $Y = \mathbf{1}_{V^s > Z^d}$ and $Z = K^r$. We get $S_{N_t^x}^y = N_t^f$ and $S_{N_t^x}^z + \lfloor \frac{A_t^x}{c} \rfloor = N_t^i$ which gives the result. □

## 4.2 Estimation of the parameters

Theorem 4.6 gives us the almost sure convergence of $\left(\frac{N_t^r}{t}, \frac{N_t^f}{t}, \frac{N_t^i}{t}\right)$. Combining these limits yields

$$\lim_{t\to\infty} \frac{N_t^i}{N_t^r} = m_k, \ \lim_{t\to\infty} \frac{N_t^f}{N_t^r} = P_d.$$

$m_k$ depends only on $\mu$. Then, if $m_k = f(\mu)$, where $f$ is a continuous and strictly monotone function, we get that $\mu = f^{-1}(m_k)$. $P_d$ depends on $\mu$ and $\lambda$. Then, if $P_d = g(\mu, \lambda)$, where $\lambda \longmapsto g(\mu, \lambda)$ is a continuous and strictly monotone function for all $\mu$, we get that $\lambda = g_\mu^{-1}(\mu, P_d)$ ($g_\mu^{-1}$ denotes the inverse of $\lambda \longmapsto g(\mu, \lambda)$).

**Lemma 4.8.** *Assume that $g$ is a $C^1$ function from $U \times V$ to $[0,1]$, strictly monotone in $\lambda$ for all $\mu$. Then, $g_\mu^{-1}$, the inverse of $\lambda \longmapsto g(\mu, \lambda)$, is $C^1$ from $U \times [0,1]$ to $V$.*

*Proof.* For all $\mu$ and $\lambda$, we have $g_\mu^{-1}(\mu, g(\mu, \lambda)) = \lambda$. By differentiating this equality w.r.t. $\lambda$, we get $\partial_y g_\mu^{-1}(\mu, g(\mu, \lambda))\partial_y g(\mu, \lambda) = 1$. Since for all $\mu$ $g$ is $C^1$ from $U \times V$ to $[0,1]$ and strictly monotone in $\lambda$, we have $\partial_y g_\mu^{-1}(\mu, p) = \frac{1}{\partial_y g(\mu, g_\mu^{-1}(\mu,p))}$ for all $(\mu, p) \in U \times [0,1]$. By differentiating $g_\mu^{-1}(\mu, g(\mu, \lambda)) = \lambda$ w.r.t. $\mu$, we get $\partial_x g_\mu^{-1}(\mu, g(\mu, \lambda)) + \partial_y g_\mu^{-1}(\mu, g(\mu, \lambda))\partial_x g(\mu, \lambda) = 0$, i.e. $\partial_x g_\mu^{-1}(\mu, p) = -\partial_y g_\mu^{-1}(\mu, p)\partial_x g(\mu, g_\mu^{-1}(\mu, p))$ for all $(\mu, p) \in U \times [0,1]$.

□

Let us introduce

$$\mu_t := f^{-1}\left(\frac{N_t^i}{N_t^r}\right), \ \ \lambda_t := g_\mu^{-1}\left(\mu_t, \frac{N_t^f}{N_t^r}\right) \tag{9}$$

The following Theorem gives a CLT for $\sqrt{t}(\mu_t - \mu, \lambda_t - \lambda)$.

**Theorem 4.9.** *Assume that $m_k = f(\mu)$, $m_x = h(\mu, \lambda)$ and $P_d = g(\mu, \lambda)$, where $f$ is a $C^1$ strictly monotone function from $U$ to $\mathbb{R}_+$, and $g$ is a $C^1$ function from $U \times V$ to $[0,1]$, strictly monotone in $\lambda$ for all $\mu$. Let $(\mu_t, \lambda_t)$ be defined by (9). It holds*

$$\lim_{t\to\infty} \sqrt{t}(\mu_t - \mu, \lambda_t - \lambda) \overset{law}{=} \mathcal{N}(0, \Sigma^2)$$

*where $\Sigma^2 = ARA^T$, $R$ is given in Theorem 4.7 and*

$$A = \frac{h(\mu,\lambda)}{f'(\mu)} \begin{pmatrix} -f(\mu) & 1 & 0 \\ f(\mu)\frac{\partial_\mu g(\mu,\lambda)}{\partial_\lambda g(\mu,\lambda)} - g(\mu,\lambda)\frac{f'(\mu)}{\partial_\lambda g(\mu,\lambda)} & -\frac{\partial_\mu g(\mu,\lambda)}{\partial_\lambda g(\mu,\lambda)} & \frac{f'(\mu)}{h(\mu,\lambda)\partial_\lambda g(\mu,\lambda)} \end{pmatrix}.$$

*Remark* 4.10. If the survival function of $Y^s$ is strictly monotone in $\mu$, $f(\mu)$ (defined by (1)) is strictly monotone. By using (2), we get that $g(\mu, \lambda) = \int_0^\infty \mathbb{E}(F_d(D_{\overline{B}(t)} - t))n_\mu(dt)$. Then, If the survival function of $Y^d$ is strictly monotone in $\lambda$, $\lambda \longmapsto g(\mu, \lambda)$ is strictly monotone.

*Proof.* Let us first consider $\mu_t - \mu$. A Taylor expansion gives

$$\mu_t - \mu = f^{-1}\left(\frac{N_t^i}{N_t^r}\right) - f^{-1}(m_k) = \left(\frac{N_t^i}{N_t^r} - m_k\right)(f^{-1})'(\zeta_t), \tag{10}$$

where $\zeta_t$ belongs to $\left[\min\left(\frac{N_t^i}{N_t^r}, m_k\right), \max\left(\frac{N_t^i}{N_t^r}, m_k\right)\right]$. Moreover, $\lim_{t\to\infty} \zeta_t = m_k$ a.s..
Let us now consider $\lambda_t - \lambda$.

$$\lambda_t - \lambda = g_\mu^{-1}\left(\mu_t, \frac{N_t^f}{N_t^r}\right) - g_\mu^{-1}(\mu, P_d) \tag{11}$$

$$= g_\mu^{-1}\left(\mu_t, \frac{N_t^f}{N_t^r}\right) - g_\mu^{-1}\left(\mu, \frac{N_t^f}{N_t^r}\right) + g_\mu^{-1}\left(\mu, \frac{N_t^f}{N_t^r}\right) - g_\mu^{-1}(\mu, P_d), \tag{12}$$

$$= (\mu_t - \mu)\partial_x g_\mu^{-1}\left(\xi_t, \frac{N_t^f}{N_t^r}\right) + \left(\frac{N_t^f}{N_t^r} - P_d\right)\partial_y g_\mu^{-1}(\mu, \eta_t), \tag{13}$$

where $\xi_t$ belongs to $[\min(\mu_t, \mu), \max(\mu_t, \mu)]$ and $\eta_t$ belongs to $\left[\min\left(\frac{N_t^f}{N_t^r}, P_d\right), \max\left(\frac{N_t^f}{N_t^r}, P_d\right)\right]$. Moreover, $\lim_{t\to\infty} \xi_t = \mu = f^{-1}(m_k)$ and $\lim_{t\to\infty} \eta_t = P_d$ a.s.. Combining (10) and (11) gives

$$\lambda_t - \lambda = \left(\frac{N_t^i}{N_t^r} - m_k\right)(f^{-1})'(\zeta_t)\partial_x g_\mu^{-1}\left(\xi_t, \frac{N_t^f}{N_t^r}\right) + \left(\frac{N_t^f}{N_t^r} - P_d\right)\partial_y g_\mu^{-1}(\mu, \eta_t).$$

Let us introduce $\Gamma_t := \frac{N_t^r}{t} - \frac{1}{m_x}$, $\Pi_t := \frac{N_t^i}{t} - \frac{m_k}{m_x}$ and $\Delta_t := \frac{N_t^f}{t} - \frac{P_d}{m_x}$. From Theorem 4.7, we get $\sqrt{t}(\Gamma_t, \Pi_t, \Delta_t)$ converges to $\mathcal{N}(0, R)$. We rewrite $\frac{N_t^i}{N_t^r} - m_k$ as a function of $\Gamma_t$ and $\Pi_t$.

$$\begin{aligned} \frac{N_t^i}{N_t^r} - m_k &= \frac{N_t^i}{t}\frac{t}{N_t^r} - m_k = \left(\frac{N_t^i}{t} - \frac{m_k}{m_x}\right)\frac{t}{N_t^r} + \frac{m_k}{m_x}\frac{t}{N_t^r} - m_k \\ &= \Pi_t \frac{t}{N_t^r} + \frac{m_k}{m_x}\left(\frac{t}{N_t^r} - m_x\right) = \Pi_t \frac{t}{N_t^r} + m_k \frac{t}{N_t^r}\left(\frac{1}{m_x} - \frac{N_t^r}{t}\right) \end{aligned}$$

Then

$$\sqrt{t}\left(\frac{N_t^i}{N_t^r} - m_k\right) = h\left(\sqrt{t}\Gamma_t, \sqrt{t}\Pi_t, \frac{N_t^r}{t}, m_k\right),$$

where $h : (x, y, z, d) \longmapsto \frac{y}{z} - d\frac{x}{z}$. By the same type of computations, we get that

$$\sqrt{t}\left(\frac{N_t^f}{N_t^r} - P_d\right) = h\left(\sqrt{t}\Gamma_t, \sqrt{t}\Delta_t, \frac{N_t^r}{t}, P_d\right),$$

Then,

$$\sqrt{t}(\mu_t - \mu) = h\left(\sqrt{t}\Gamma_t, \sqrt{t}\Pi_t, \frac{N_t^r}{t}, m_k\right)(f^{-1})'(\zeta_t),$$

$$\sqrt{t}(\lambda_t - \lambda) = h\left(\sqrt{t}\Gamma_t, \sqrt{t}\Pi_t, \frac{N_t^r}{t}, m_k\right)(f^{-1})'(\zeta_t)\partial_x g_\mu^{-1}\left(\xi_t, \frac{N_t^f}{N_t^r}\right) + h\left(\sqrt{t}\Gamma_t, \sqrt{t}\Delta_t, \frac{N_t^r}{t}, P_d\right)\partial_y g_\mu^{-1}(\mu, \eta_t).$$

Since $\sqrt{t}(\Gamma_t, \Pi_t, \Delta_t) \overset{law}{\to} \mathcal{N}(0, R)$ and $\left(\frac{N_t^r}{t}, \zeta_t, \xi_t, \eta_t\right) \overset{\mathbb{P}}{\to} \left(\frac{1}{m_x}, m_k, f^{-1}(m_k), P_d\right)$, Slutsky's Theorem gives

$$\begin{aligned} \begin{pmatrix} \sqrt{t}(\mu_t - \mu) \\ \sqrt{t}(\lambda_t - \lambda) \end{pmatrix} &\overset{law}{\to} \begin{pmatrix} \frac{1}{f'(\mu)} h(G_1, G_2, \frac{1}{m_x}, m_k) \\ \frac{1}{f'(\mu)}\partial_x g_\mu^{-1}(\mu, P_d) h(G_1, G_2, \frac{1}{m_x}, m_k) + \partial_y g_\mu^{-1}(\mu, P_d) h(G_1, G_3, \frac{1}{m_x}, P_d) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{f'(\mu)}(-m_x m_k G_1 + m_x G_2) \\ \frac{1}{f'(\mu)}\partial_x g_\mu^{-1}(\mu, P_d)(-m_x m_k G_1 + m_x G_2) + \partial_y g_\mu^{-1}(\mu, P_d)(-m_x P_d G_1 + m_x G_3) \end{pmatrix} \\ &= AG \end{aligned}$$

where $G = (G_1, G_2, G_3)^T \sim \mathcal{N}(0, R)$ and

$$A = \frac{m_x}{f'(\mu)} \begin{pmatrix} -m_k & 1 & 0 \\ -(m_k \partial_x g_\mu^{-1}(\mu, P_d) + P_d f'(\mu) \partial_y g_\mu^{-1}(\mu, P_d)) & \partial_x g_\mu^{-1}(\mu, P_d) & \frac{f'(\mu)}{m_x} \partial_y g_\mu^{-1}(\mu, P_d) \end{pmatrix}.$$

□

# 5 Applications

**Hypothesis 5.1.** *In this Section, we assume that $Y^s$ follows the gamma law $\Gamma(n, \mu)$, where $n \in \mathbb{N}^*$ and $\mu \in \mathbb{R}_+^*$, and $Y^d$ follows the exponential law of parameter $\lambda \in \mathbb{R}_+^*$.*

**Proposition 5.2.** *Under Hypothesis 5.1, we have*

$$\mathbb{E}[K^r] = \sum_{i=0}^{n-1} \frac{\mu^i}{i!} (-1)^i \left( \frac{1}{1-L} \right)^{(i)} (\mu), \tag{14}$$

$$1 - P_d = \begin{cases} \frac{\mu^n}{(\mu-\lambda)^n} \left( L(\lambda) - (1 - L(\lambda)) \sum_{i=0}^{n-1} \frac{(\mu-\lambda)^i}{i!} (-1)^i \left( \frac{L}{1-L} \right)^{(i)} (\mu) \right) & \text{if } \lambda \neq \mu \\ (1 - L(\mu)) \frac{\mu^n}{n!} (-1)^n \left( \frac{L}{1-L} \right)^{(n)} (\mu) & \text{if } \lambda = \mu \end{cases}$$

We postpone the proof of Proposition 5.2 to the Appendix A.

In order to apply Theorem 4.9, let us check that the assumptions are satisfied. Since the survival function of the law $\Gamma(n, \mu)$ is strictly monotone in $\mu$ for all $n \in \mathbb{N}^*$, Remark 4.10 gives that $\mu \longmapsto f(\mu)$ and $\lambda \longmapsto g(\mu, \lambda)$ are strictly monotone. From Proposition 5.2, we get that $f(\mu)$ is $C^1$. Let us now check that $g$, given by $g(\mu, \lambda) = \int_0^\infty \mathbb{E}(e^{-\lambda(D_{A(t)}-t)}) \frac{\mu^n}{(n-1)!} e^{-\mu t} dt$, is a $C^1$ function on $U \times V$, i.e. we prove that $\partial_\mu g(\mu, \lambda)$ and $\partial_\lambda g(\mu, \lambda)$ exist and are continuous.

- $\forall \mu \in K$, a compact set included in $U$, $\mathbb{E}(e^{-\lambda(D_{A(t)}-t)}) \partial_\mu \left( \frac{\mu^n}{(n-1)!} e^{-\mu t} \right)$ is bounded by a positive and integrable fonction $\phi_K(t)$. Then $\partial_\mu g(\mu, \lambda)$ exist and is given by $\partial_\mu g(\mu, \lambda) = \int_0^\infty \mathbb{E}(e^{-\lambda(D_{A(t)}-t)}) \frac{\mu^{n-1}}{(n-1)!} e^{-\mu t}(n - \mu t) dt$, which is continuous.
- $\forall \lambda \in V$, we have $\mathbb{E}((D_{A(t)}-t)e^{-\lambda(D_{A(t)}-t)}) \leq \mathbb{E}(D_{A(t)}-t)$, and $\int_0^\infty \mathbb{E}((D_{A(t)}-t)) \frac{\mu^n}{(n-1)!} e^{-\mu t} dt = \mathbb{E}(V^s - Y^s) < \infty$. Then $\partial_\lambda g(\mu, \lambda)$ exist and is given by $\partial_\lambda g(\mu, \lambda) = \int_0^\infty \mathbb{E}((D_{A(t)} - t)e^{-\lambda(D_{A(t)}-t)}) \frac{\mu^n}{(n-1)!} e^{-\mu t} dt$, which is continuous.

In view of the application of Theorem 4.9, we need to compute the matrix $R$ defined in Theorem 4.7. The following Proposition gives explicit formulas of its terms.

**Proposition 5.3.** *If $Y^d$ follows an exponential law, we get*

$$\begin{aligned} \mathbb{E}\left((K^r)^2\right) &= \mathbb{E}[K^r] + 2 \sum_{i=0}^{n-1} \frac{\mu^i}{i!} (-1)^i \left( \frac{L}{(1-L)^2} \right)^{(i)} (\mu), \\ \mathbb{E}((X^r)^2) &= \mathbb{E}((Y^s)^2) + \frac{2}{\lambda} \mathbb{E}(Z^d \mathbf{1}_{V^s \geq Z^d}) \\ \mathrm{Cov}(X^r, \mathbf{1}_{V^s \geq Z^d}) &= \mathbb{E}(Z^d \mathbf{1}_{V^s \geq Z^d}) - \mathbb{E}(K^r) P_d \\ \mathrm{Cov}(K^r, \mathbf{1}_{V^s \geq Z^d}) &= (1 - P_d) \mathbb{E}(K^r) - \mathbb{E}(K^r R_d(D_{\overline{B}(Y^s)} - Y^s)) \\ \mathrm{Cov}(K^r, X^r) &= \frac{n}{\mu} \mathbb{E}(K^r)_{n+1} + (\frac{1}{\lambda} - \mathbb{E}(X^r)) \mathbb{E}(K^r) - \frac{1}{\lambda} \mathbb{E}(K^r R_d(D_{\overline{B}(Y^s)} - Y^s)), \end{aligned}$$

*where* $\mathbb{E}(Z^d \mathbf{1}_{V^s \geq Z^d}) = \frac{P_d}{\lambda} + \mathbb{E}(Y^s) - \frac{L'(\lambda)}{1-L(\lambda)}(1 - P_d + L(\lambda) n_\mu(\{0\}))$ *and* $\mathbb{E}(K^r)_{n+1}$ *means that* $n$ *is replaced by* $n+1$ *in* (14). *Under Hypothesis 5.1, we have*

$$\mathbb{E}(K^r R_d(D_{\overline{B}(Y^s)} - Y^s)) =$$

$$\begin{cases} -(1-P_d)\frac{L(\lambda)}{1-L(\lambda)} + \frac{\mu^n}{(\mu-\lambda)^n}\left(\frac{L(\lambda)}{1-L(\lambda)} - (1-L(\lambda))\sum_{i=0}^{n-1}\frac{(\mu-\lambda)^i}{i!}(-1)^i\left(\frac{L}{(1-L)^2}\right)^{(i)}(\mu)\right) & \text{if } \lambda \neq \mu, \\ (1-P_d)\left(\mathbb{E}(K^r) + \frac{L(\lambda)}{1-L(\lambda)}\right) - \frac{\mu^n}{n!}(-1)^n(1-L(\mu))\left(\frac{L}{(1-L)^2}\right)^{(n)}(\mu) & \text{if } \lambda = \mu \end{cases}$$

The proof of Proposition 5.3 requires long but not difficult computations, we leave it to the reader.

## 5.1 Case $n = 1$

Proposition 5.2 and (5) give

$$\mathbb{E}[K^r] = f(\mu) = \frac{1}{1-L(\mu)},$$
$$P_d = g(\mu,\lambda) = \begin{cases} 1 - \frac{\mu}{\mu-\lambda}\frac{L(\lambda)-L(\mu)}{1-L(\mu)} & \text{if } \lambda \neq \mu \\ 1 + \frac{\mu L'(\mu)}{1-L(\mu)} & \text{if } \lambda = \mu \end{cases}$$
$$\mathbb{E}[X^r] = h(\mu,\lambda) = \frac{1}{\mu} + \frac{1}{\lambda}Pd.$$

Then, $f'(\mu) = \frac{L'(\mu)}{(1-L(\mu))^2}$ and

$$\partial_\mu g : (\mu,\lambda) \longmapsto \begin{cases} -\left(\frac{\lambda}{(\mu-\lambda)^2}\frac{L(\mu)-L(\lambda)}{1-L(\mu)} + \frac{\mu}{\mu-\lambda}\frac{L'(\mu)(L(\lambda)-1)}{(1-L(\mu))^2}\right) & \text{if } \mu \neq \lambda \\ \frac{\frac{\mu}{2}L''(\mu)+L'(\mu)}{1-L(\mu)} + \frac{\mu(L'(\mu))^2}{(1-L(\mu))^2} & \text{if } \mu = \lambda \end{cases}$$

$$\partial_\lambda g : (\mu,\lambda) \longmapsto \begin{cases} -\left(\frac{\mu}{(\mu-\lambda)^2}\frac{L(\lambda)-L(\mu)}{1-L(\mu)} + \frac{\mu}{\mu-\lambda}\frac{L'(\lambda)}{1-L(\mu)}\right) & \text{if } \mu \neq \lambda \\ \frac{\mu L''(\mu)}{2(1-L(\mu))} & \text{if } \mu = \lambda \end{cases}$$

## 5.2 Second case : $n = 2$

**Hypothesis 5.4.** *In this Section, we assume that* $Y^s$ *follows the Gamma law* $\Gamma(2, \frac{1}{\mu})$, *where* $\mu \in \mathbb{R}_+^*$, *and* $Y^d$ *follows the exponential law of parameter* $\lambda \in \mathbb{R}_+^*$.

Proposition 5.2 gives

$$\mathbb{E}[K^r] = f(\mu) = \frac{1-L(\mu)-\mu L'(\mu)}{(1-L(\mu))^2},$$
$$P_d = g(\mu,\lambda) = \begin{cases} 1 - \frac{\mu^2}{(\mu-\lambda)^2}\left(\frac{L(\lambda)-L(\mu)}{1-L(\mu)} + (\mu-\lambda)L'(\mu)\frac{1-L(\lambda)}{(1-L(\mu))^2}\right) & \text{if } \lambda \neq \mu \\ 1 - \frac{\mu^2 L'(\mu)}{(1-L(\mu))^2} & \text{if } \lambda = \mu \end{cases}$$
$$\mathbb{E}[X^r] = h(\mu,\lambda) = \frac{2}{\mu} + \frac{1}{\lambda}P_d$$

Then $f'(\mu) = -\mu\frac{L''(\mu)(1-L(\mu))+2(L'(\mu))^2}{(1-L(\mu))^3}$ and

$\partial_\mu g : (\mu,\lambda) \longmapsto$

$$\begin{cases} -\frac{\mu}{(1-L(\mu))^3(\mu-\lambda)^3}\big(-2\lambda(1-L(\mu))\left[(L(\lambda)-L(\mu))(1-L(\mu))+(\mu-\lambda)L'(\mu)(1-L(\lambda))\right] \\ \qquad +\mu(\mu-\lambda)^2(1-L(\lambda))(L''(\mu)(1-L(\mu))+2(L'(\mu))^2)\big) & \text{if } \mu\neq\lambda \\ -\frac{\mu}{(1-L(\mu))^3}\big(2(1-L(\mu))(\mu L'(\mu)L''(\mu)+(L'(\mu))^2)+(1-L(\mu))^2(L''(\mu) \\ \qquad +\frac{\mu}{3}L^{(3)}(\mu))+2\mu(L'(\mu))^3\big) & \text{if } \mu=\lambda \end{cases}$$

$\partial_\lambda g : (\mu,\lambda) \longmapsto$

$$\begin{cases} -\frac{2\mu^2}{(\mu-\lambda)^3(1-L(\mu))^2}\Big((1-L(\mu))(L(\lambda)-L(\mu))+\frac{\mu-\lambda}{2}(L'(\lambda)(1-L(\mu)) \\ \qquad +L'(\mu)(1-L(\lambda)))-\frac{(\mu-\lambda)^2}{2}L'(\lambda)L'(\mu)\Big) & \text{if } \mu\neq\lambda \\ -\frac{\mu^2}{2(1-L(\mu))^2}(L''(\mu)L'(\mu)+\frac{1}{3}(1-L(\mu))L^{(3)}(\mu)) & \text{if } \mu=\lambda \end{cases}$$

# 6 Numerical examples

In this Section we compare the asymptotical method (AM) and the ML one. We generate datas $(Y_i^s, Y_i^d)_{i\geq 1}$ with a set of parameters $(\mu,\lambda)$ until $\sum_{i\geq 1} X_i^r$ becomes bigger than a fixed time $T$. We consider two cases : deterministic inspections (see Section 6.1.1) and uniform random inspections (see Section 6.2). In all cases, we consider that $Y^d$ follows an exponential law, and $Y^s$ follows either an exponential law or a gamma law. From the sample $(Y_i^s, Y_i^d)_{i\geq 1}$, we get $(N_T^r, N_T^i, N_T^f)$, and (9) gives $(\mu_T, \lambda_T)$. Moreover, Theorem 4.9 gives a confidence interval for this approximation of $(\mu,\lambda)$. We compare these results with the ones given by the maximum likelihood method.

## 6.1 Deterministic inspections

We assume that the random variable $C$ is constant, equal to $c$. In this case we have $L(s) = e^{-sc}$.

### 6.1.1 Exponential law for $Y^s$

We assume that $Y^s$ follows the law $\mathcal{E}(\mu)$. We have generated datas with the following parameters

$$\mu = 10^{-3}, \;\; \lambda = 5.10^{-4}, \;\; c = 1000. \tag{15}$$

We have obtained $N_T^r = 33501$, $N_T^f = 8255$, and $N_T^i = 53116$ at time $T = 50001908$. Table 1 compares both estimators (ML and AM) and their 95% confidence intervals for $\mu$ and $\lambda$. Both methods are very precise and give almost the same values for the estimators and confidence intervals. However, the asymptotical method is much faster than the likelihood one : the computational time of the ML method is 62.23s, whereas the computational time of the asymptotical method is less than $10^{-4}$.

*Remark* 6.1. The relative error on $\mu$ is about 0.3% and the one on $\lambda$ is about 0.8%. Indeed, we have at our disposal more datas to calibrate $\mu$ than to calibrate $\lambda$.

| Method | $\mu$ and CI | $\lambda$ and CI |
|---|---|---|
| ML | 0.000996215 [0.0009850, 0.0010074] | 0.000504164 [0.0004928, 0.0005155] |
| AM | 0.000996184 [0.0009532, 0.0010392] | 0.000504197 [0.0004870, 0.0005214] |

Table 1: Comparison of AM and ML when C is constant and $Y^s \sim \mathcal{E}$

Figures 3 and 4 plot the evolution of the parameters with respect to time $t$, when $t$ varies in $[0,T]$ ($(\hat{\mu}_t, \hat{\lambda}_t)_t$ represents the evolution of the ML estimators). Both estimators evolve in the same way. Moreover, the convergence is quite fast. As said in Remark 6.1, we observe that $(\mu_t)_t$ is more precise than $(\lambda_t)_t$.

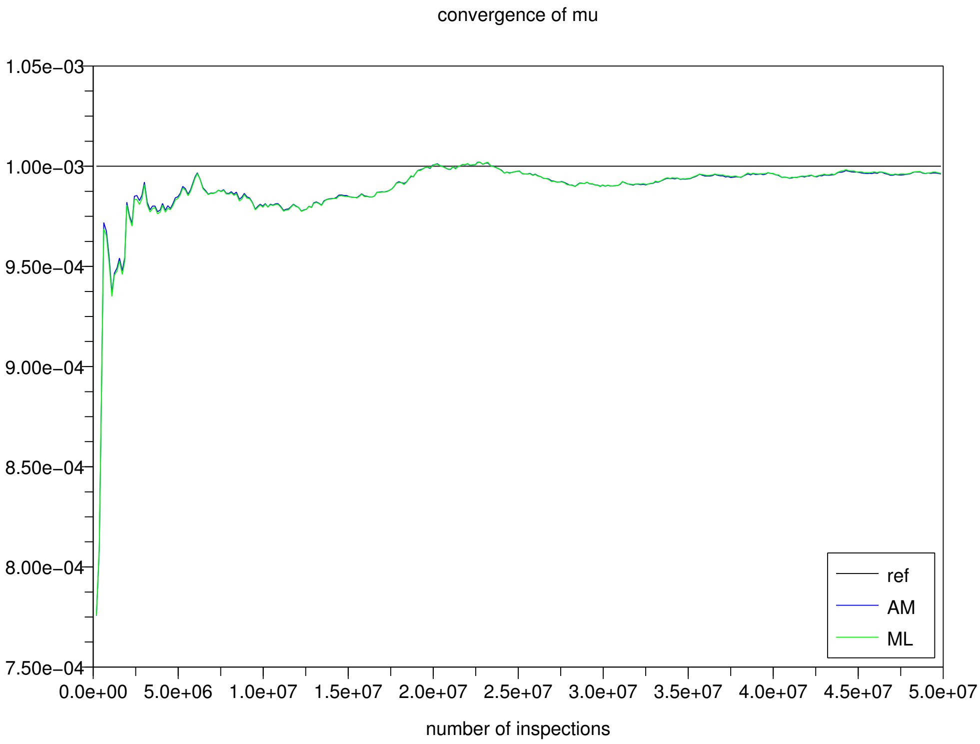


Figure 3: Convergence of $\mu_t$ and $\hat{\mu}_t$

### 6.1.2 Gamma law for $Y^s$

We assume that $Y^s$ follows the law $\Gamma(2, \mu)$ and parameters are given by (15). We obtain $N_T^r = 20668$, $N_T^f = 4369$, and $N_T^i = 51503$ at time $T = 50002058$. Table 2 compares both estimators (ML and AM) and their 95% confidence intervals for $\mu$ and $\lambda$. As in the case of an exponential law for $Y^s$, both methods are very precise and give almost the same values for the estimators and confidence intervals.

| Method | $\mu$ and CI | $\lambda$ and CI |
|---|---|---|
| ML | 0.0010052 $[0.0009953, 0.0010151]$ | 0.0004992 $[0.0004784, 0.0005144]$ |
| AM | 0.0010054 $[0.0009742, 0.0010366]$ | 0.0004918 $[0.0004789, 0.0005047]$ |

Table 2: Comparison of AM and ML when C is constant and $Y^s \sim \Gamma(2, \mu)$

## 6.2 Random inspections

We assume that $C$ follows a uniform law on $[c-h, c+h]$. In this case, we have $L(s) = e^{-sc}\frac{\sinh(sh)}{sh}$.

### 6.2.1 Exponential law for $Y^s$

We assume that $Y^s$ follows the law $\mathcal{E}(\mu)$. We have generated datas with the following parameters

$$\mu = 10^{-3}, \quad \lambda = 5.10^{-4}, \quad c = 1000, \quad h = 100. \tag{16}$$

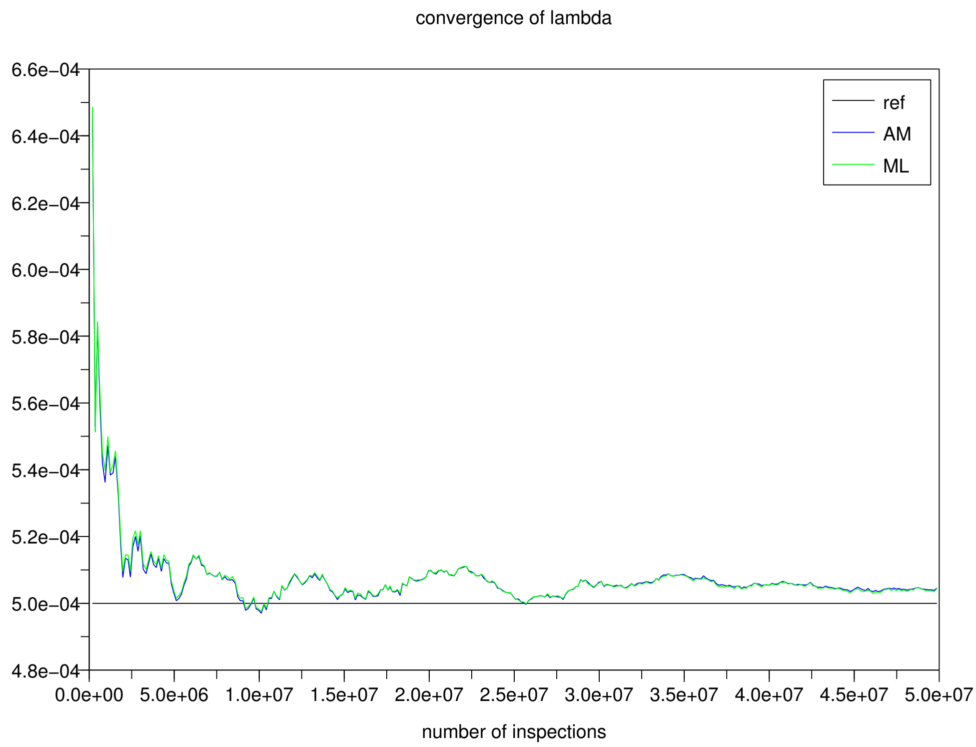


Figure 4: Convergence of $\lambda_t$ and $\hat{\lambda}_t$

We obtain $N_T^r = 33613$, $N_T^f = 8278$, and $N_T^i = 53133$ at time $T = 50001271$. Table 3 compares both estimators (ML and AM) and their 95% confidence intervals for $\mu$ and $\lambda$. As in case of deterministic inspections, both methods are very precise and give almost the same values for the estimators and confidence intervals.

| Method | $\mu$ and CI | $\lambda$ and CI |
|---|---|---|
| ML | 0.0010030 $[0.0009918, 0.0010142]$ | 0.000502 $[0.0004907, 0.0005133]$ |
| AM | 0.0010030 $[0.0009602, 0.0010458]$ | 0.0005021 $[0.0004849, 0.0005193]$ |

Table 3: Comparison of AM and ML when C is random and $Y^s \sim \mathcal{E}$

### 6.2.2 Gamma law for $Y^s$

We assume that $Y^s$ follows the law $\Gamma(2, \mu)$, with parameters given by (16). We obtain $N_T^r = 20470$, $N_T^f = 4452$, and $N_T^i = 51522$ at time $T = 50000355$. Table 4 compares both estimators (ML and AM) and their 95% confidence intervals for $\mu$ and $\lambda$. As in case of deterministic inspections, both methods are very precise and give almost the same values for the estimators and confidence intervals.

## 7 Conclusion

We have presented a new method to estimate the parameters of the laws of the transition times of a 3-state deteriorating system. Each law depends on one parameter. However, Theorem 4.6

| Method | $\mu$ and CI | $\lambda$ and CI |
|---|---|---|
| ML | $0.000996\ [0.0009862, 0.0010058]$ | $0.000505\ [0.0004895, 0.0005205]$ |
| AM | $0.0009964\ [0.0009661, 0.0010267]$ | $0.0005064\ [0.0004934, 0.0005194]$ |

Table 4: Comparison of AM and ML when C is random and $Y^s \sim \Gamma(2, \mu)$

gives the almost sure convergence of three quantities of interest, which allows to estimate three unknown parameters. Then, our method can be extended to a more general setting. This topic, as well as possible extensions to $k$-state deteriorating systems ($k \geq 4$) will be treated in a future work.

# A Proof of Proposition 5.2

Firstly we give the following lemma, useful to show Propositions 5.2 and 5.3. Since its proof requires long but not difficult computations, we leave it to the reader.

**Lemma A.1.** *Under Hypothesis 5.1, we have*

$$\sum_{k\geq 1} \mathbb{E}\left[e^{-\lambda D_k}\int_0^{D_k} e^{\lambda t} n_\mu(dt)\right] = \begin{cases} \frac{\mu^n}{(\mu-\lambda)^n}\left(\frac{L(\lambda)}{1-L(\lambda)} - \sum_{i=0}^{n-1}\frac{(\mu-\lambda)^i}{i!}(-1)^i\left(\frac{L}{1-L}\right)^{(i)}(\mu)\right) & \text{if } \lambda \neq \mu, \\ \frac{\mu^n}{n!}(-1)^n\left(\frac{L}{1-L}\right)^{(n)}(\mu) & \text{if } \lambda = \mu \end{cases}$$

*and*

$$\sum_{k\geq 1} k\mathbb{E}\left[e^{-\lambda D_k}\int_0^{D_k} e^{\lambda t} n_\mu(dt)\right] = \begin{cases} \frac{\mu^n}{(\mu-\lambda)^n}\left(\frac{L(\lambda)}{(1-L(\lambda))^2} - \sum_{i=0}^{n-1}\frac{(\mu-\lambda)^i}{i!}(-1)^i\left(\frac{L}{(1-L)^2}\right)^{(i)}(\mu)\right) & \text{if } \lambda \neq \mu, \\ \frac{\mu^n}{n!}(-1)^n\left(\frac{L}{(1-L)^2}\right)^{(n)}(\mu) & \text{if } \lambda = \mu \end{cases}$$

Let us first prove (14). From (1), we have

$$\mathbb{E}\left[K^r\right] = \sum_{k\geq 0}\mathbb{E}\left[R^s(D_k)\right] = \sum_{k\geq 0}\sum_{i=0}^{n-1}\frac{\mu^i}{i!}\mathbb{E}\left[D_k^i e^{-\mu D_k}\right] = \sum_{k\geq 0}\sum_{i=0}^{n-1}\frac{\mu^i}{i!}(-1)^i L_k^{(i)}(\mu).$$

Then

$$\mathbb{E}\left[K^r\right] = \sum_{i=0}^{n-1}\frac{\mu^i}{i!}(-1)^i\sum_{k\geq 0}L_k^{(i)}(\mu) = \sum_{i=0}^{n-1}\frac{\mu^i}{i!}(-1)^i\left(\sum_{k\geq 0}L_k\right)^{(i)}(\mu),$$

and the result follows. The second result ensues from (4) and Lemma A.1.

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